Behavior of Large Waterdrops in Shear Flow

D. R. Caldwell and W. P. Elliott

School of Oceanography, Oregon State University, Corvallis, Oregon 97331

Nondimensionalized equations of motion for waterdrops, not necessarily obeying Stokes's law, are derived and used to calculate the response of various size drops to changes in horizontal wind speed. We find that (1) cloud drops respond almost instantaneously to changes in wind speed, whereas raindrops require considerable time to adjust, (2) one can ignore the dependence of drag coefficient on Reynolds number in small shear, and (3) the probability of collision between raindrops and cloud drops is likely to be increased very slightly by the presence of wind shear.

Waterdrops of different sizes falling through a fluid will respond differently to vertical shears in the horizontal mean flow, and their subsequent trajectories will differ from those in a uniform flow. The problem of determining the responses is fairly simple if the drop size is in the Stokes law region but becomes more complicated for larger drops. *Fuchs* [1964] presented some equations for aerosol motion at high Reynolds number in still air that require graphical solution. We have reformulated the problem and solved the nonlinear equations numerically. The equations show that vertical and horizontal motions are coupled unless the drag is a linear function of air speed, as it is in the Stokes law regime.

The shear flow problems examined in this report are confined to linear wind speed gradients. The special case of raindrops falling through a logarithmic wind profile near the ground was examined in another paper [*Caldwell and Elliott*, 1971]. In the present study we include drops ranging in size from the Stokes law region (maximum radius of ∼0.004 cm) through large raindrops, and we consider specifically the Reynolds number dependence of the drag coefficient. All the pertinent physical data are from *List* [1966], and thus it is assumed that the drag coefficients are not affected by accelerations. *Ogden and Jayaweera* [1971] have found experimentally that for very large accelerations the drag coefficient of a waterdrop appears to be about 20% less than the unaccelerated value. This decrease in drag would affect

these calculations only quantitatively and would be significant only for shears much larger than those likely to be found in natural atmospheric conditions. We have ignored the effects of the transfer of momentum from drop to air that can make small changes in the assumed profiles, as shown by *Caldwell and Elliott* [1972].

We present numerical solutions to the general equations for several velocity profiles and an approximate solution in which the variation of drag coefficient with air speed is neglected. The shears chosen for these examples are those that might be found in the mean flow but do not represent the extreme shears over short distances that might be encountered in turbulence. The same method is applicable to those larger shears.

Model

We consider a drop of mass m, radius r, density ρ_d, and cross-sectional area a to be falling in the z direction with speed w. The air is moving in the positive x direction with speed v, and the drop is moving in the x direction with speed u: thus the drop is moving horizontally through the air with speed $u - v$, and the total speed of the drop through the air is

$$c = [(u - v)^2 + w^2]^{1/2} \qquad (1)$$

The forces on the drop consist of the gravity force mg and the drag force

$$d_f = 0.5\rho_a C_D c^2 a \qquad (2)$$

where ρ_a is the air density and C_D is the drag coefficient, which is a function of Reynolds

number and so of c. Figure 1 shows these speeds and forces.

The equations of motion for the drops are, in Cartesian form,

$$m \, du/dt = -0.5\rho_a C_D ac^2 \sin \theta \qquad (3)$$

$$m \, dw/dt = -0.5\rho_a C_D ac^2 \cos \theta + mg$$

where $\sin \theta$ and $\cos \theta$ are $(u - v)/c$ and w/c, respectively. By dividing both sides of (3) by mg and noting that

$$mg = (0.5)\rho_a C_D' aw_t^2$$

where C_D' is the drag coefficient at the terminal speed w_t, (3) can be put into nondimensional form:

$$dU/dT = -\beta(U - V)C \qquad (4)$$

$$dW/dT = 1 - \beta CW$$

where $W = w/w_t$, $U = u/w_t$, $V = v/w_t$, $C = c/w_t$, and $T = gt/w_t$. The quantity β repre-
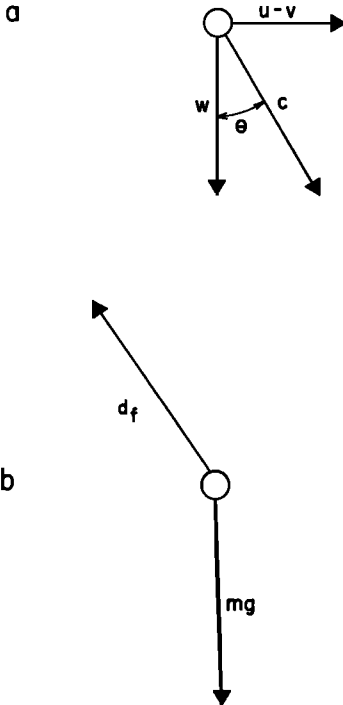
a

b

Fig. 1. Speeds and forces: (a) $u - v$ represents the horizontal speed of the drop relative to the air, w represents the vertical velocity, and c represents the total air speed; (b) mg represents the force due to gravity, and d_f represents the force due to drag of air on the drop.

sents the ratio $C_D(c)/C_D(w_t)$. The independent variable can be changed to the dimensionless length of fall Z, defined as gz/w_t^2 $(=WT)$. Equations 4 then become

$$W dU/dZ = -\beta(U - V)C \qquad (5)$$

$$W dW/dZ = 1 - \beta CW$$

The above scaling for z by w_t^2/g is most convenient unless there is a length scale in the velocity profile, say h. In that case we make $Z' = z/h$, and equations 5 become

$$\left(\frac{w_t^2}{gh}\right) W \frac{dU}{dZ'} = -\beta C(U - V)$$

$$\qquad\qquad (5')$$

$$\left(\frac{w_t^2}{gh}\right) W \frac{dW}{dZ'} = 1 - \beta CW$$

Equations 5 or 5' are sufficient to find the motion of a drop when the velocity profile is specified and β is given as a function of C.

The quantity gh/w_t^2 is the profile length scale divided by the natural length scale for drop fall (it is also the inverse of the particle parameter as defined by *Davies* [1966]). If gh/w_t^2 is small, a shear over distance h looks like a step change in velocity to the drop. If gh/w_t^2 is large, the motion of the drop reaches equilibrium with the shear.

If C deviates far from W, we must take variations in C_D with C into account. The variations of C_D with $Re = 2rc/v$ can be approximated by

$$C_D = 0.4[1 + (60/Re)^{1/2}]^2 \qquad (6)$$

where the constants have been chosen to fit Stokes's law for small Re. For $Re > 500$, C_D changes very little with Re, and a constant value can be used (i.e., $\beta = 1$). For smaller drops, where β may be different from 1, we use the approximation

$$\beta = [1 + (\gamma/C)^{1/2}]^2[1 + \gamma^{1/2}]^{-2} \qquad (7)$$

where $\gamma = 30v/rw_t$.

Equations 5 or 5' and 7, together with the distribution of V and the appropriate initial conditions, can be used to solve for the motion of a drop. Consider the distribution

$$V = V_h(1 - z/h) \qquad 0 \le z \le h$$

$$V = 0 \qquad\qquad z \ge h$$

The initial conditions at $T = 0$ are $V = U = V_h$, $z/h = 0$, and $W = 1$.

The solutions involve three dimensionless parameters: V_h, the dimensionless magnitude of the wind speed change; gh/w_t^2, the depth of the shear layer divided by the natural length scale of the falling drop; and the Reynolds number. Since w_t is a known function of r, one needs then only to choose the drop size, the distance h, and the velocity change over distance h to specify the problem.

The procedure is to follow the drop downward by using the equations ($\Delta Z'$ is positive)

$$U(Z' + \Delta Z') = U(Z')$$
$$- \frac{\beta \cdot C(Z') \cdot [U(Z') - V(Z')] \cdot \Delta Z'}{W(Z') \cdot w_t^2/gh} \quad (8)$$

$$W(Z' + \Delta Z') = W(Z')$$
$$+ \frac{[1 - \beta \cdot C(Z') \cdot W(Z')] \cdot \Delta Z'}{W(Z') \cdot w_t^2/gh}$$

where β is calculated at Z'.

AN APPROXIMATION

If we assume that $U - V$ is much less than W so that $C \cong W$ and $\beta \cong 1$, (5) can be written as

$$dU/dZ = -(U - V)$$
$$dW/dZ = (1 - W^2)/W \simeq 0 \quad (9)$$

so

$$d(U - V)/dZ$$
$$= -(U - V) - dV/dZ \quad (10)$$

If we further assume that dV/dZ is constant,

$$U - V = (w_t^2/gh) V_h (1 - e^{-Z}) \quad (11)$$

if $U - V = 0$ at $Z = 0$ and V_h is velocity change over distance h. From (11) we can see that after the drop has fallen far enough that $Z \gg 1$, then $U - V \cong (w_t^2/gh) \cdot V_h = (w_t^2/g)/(V_h/h)$, and so in dimensional form $u - v = (w_t^2/g) \, dv/dz$.

RESULTS

Figure 2 displays the result of integrating (8) for a total linear wind speed change of 1 m/sec over distances $h = 1$ meter (Figure 2a) and $h = 10$ meters (Figure 2b). The wind is presumed to be constant below $z/h = 1$ in both cases. The abscissa is u, the horizontal speed of the drop through the air; the ordinate is the distance fallen by the drop. For the conditions of Figure 2a (a shear of 1 m/sec/m), drops with a radius less than 0.01 cm (100 μm) follow the wind almost exactly save for a small effect at the beginning. If the shear is much less, as it is in Figure 2b, drops smaller than 0.03 cm follow the wind. We conclude from this finding that cloud droplets almost always follow the wind with very little lag even for large shears. However, large raindrops ($r > 0.1$ cm) still would have half their speed left after falling a distance $4 \cdot h$ for Figure 2a conditions and about h for Figure 2b conditions.

To display the results in a simple fashion, equations 8 are integrated with $\beta = 1$ for a range of linear wind changes. Figure 3 shows the results. The abscissa is w_t^2/gh, and the ordinate is V_h, the magnitude of the change of wind speed. Isopleths of W are shown in Figure 3a and of $(U - V)/V_h$ in Figure 3b. Although a wide range of values of W is shown in Figure 3a, simple calculations show that it would be a very unusual situation for W to be less than 0.95 in the free air.

In Figure 3a we see that drops for which w_t^2/gh is a little more than 1 have their relative vertical motion slowed the most. Small drops ($w_t^2/gh \ll 1$) adjust so rapidly that their air speed is never much different from w_t. They experience little additional drag, and so w changes little. Large drops ($w_t^2/gh \gg 1$) carry so much momentum that only a small relative change in terminal velocity occurs. (This conclusion is in agreement with *Davies* [1966], who used a quite different method.)

Large values of w_t^2/gh require large drops and small h. A 0.19-cm drop with $h = 1$ meter would give $w_t^2/gh = 4$, but this would require V_h of about 1 to achieve $W = 0.95$. The $V_h = 1$ implies a shear of over 6 m/sec/m. A drop of 0.025 cm would require a shear of 5 m/sec/m to achieve $W = 0.95$. Thus one can conclude that most drops will have vertical velocity essentially equal to their terminal speed.

Figure 3b shows the deviation of the horizontal speed of the drop from the wind speed. As is true in Figure 3a, most cases will fall in the lower left-hand part.
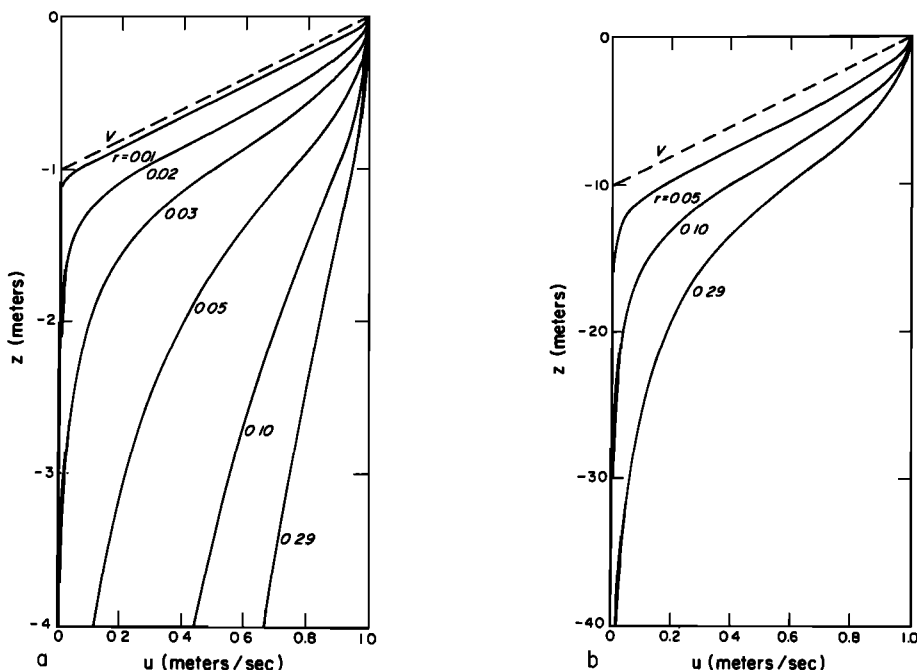
Fig. 2. Change in velocity of drops caused by linear velocity gradients. The horizontal air velocity is reduced by 1 m/sec in vertical distances of (a) 1 meter and (b) 10 meters.

DROP COLLISION

Raindrops falling through a population of cloud droplets will collide with some of them and grow larger. It is pertinent to ask whether the opportunities for collisions might be enhanced if the drops experience large shears. We are here considering velocity changes over meters of vertical fall and not the extreme shears that may be encountered over small distances in turbulence. (*Jonas and Goldsmith* [1972] have presented some evidence that small drops (Stokes's law) can collide more readily than current theory predicts for the large shears that might be present in a turbulent cloud.)

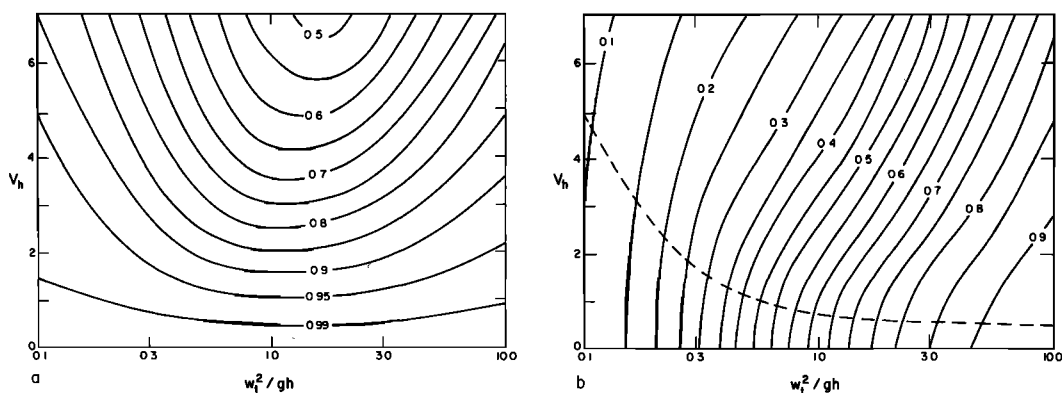The question is whether shear flow can result in drop speeds appreciably different from termi-

Fig. 3. (a) Isopleths of W at the bottom of a linear shear layer of depth h and horizontal wind velocity change V_h of drops of terminal velocity w_t. (b) Isopleths of $(U - V)/V_h$ for the same conditions. The dashed line is the locus of points for which $C = 1.1$.

nal speeds, i.e., whether C is different from 1. The following considerations will show that this difference is unlikely to occur in most cases.

We can use Figure 3 to determine the conditions under which C might be expected to be, say, 1.1. For this to be true, if we assume that $W = 1$, $U - V$ must equal 0.46. The dashed line in Figure 3b shows this particular condition. The changes in wind speed over depths of a meter necessary to attain these values for large drops would seldom be encountered in the atmosphere, and small drops follow the wind so quickly that they will almost never reach large values of $U - V$. Furthermore, small drops have such small terminal speeds that large percentage deviations are unlikely to be important.

As a large drop falls into a region of shear containing a number of small droplets moving with the wind speed, the large drop will have a relative horizontal motion through the droplets. Thus the effective path length of the large drop will be greater than the vertical distance through which it falls, and it will encounter more small droplets than it would in still air. However, the relative path length of the large drop will rarely be increased by more than 10% if total path lengths of more than ½ meter are considered. Thus it appears unlikely that the differential air speeds developed within a population of different drop sizes will greatly increase the number of collisions except possibly in the first few centimeters of an abrupt change. (It is well to note in this connection, however, that Jonas and Goldsmith were not able to account for the increase in collision efficiency

that they found by calculations of relative drop trajectories.)

One possible effect on collision-coalescence processes should be pointed out. A large drop falling into a shear zone with smaller droplets will approach them at a slight angle rather than vertically. It seems possible that the angle between the axis of symmetry of the wake of the drops and gravity may have some effect on the capture of the smaller droplet in the wake of the larger one. Whether this has any significant effect on the growth of clouds and raindrops will have to be investigated further.

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