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Title: HEAT CONDUCTION IN A CONE WITH A RING SOURCE

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The solution of the heat conduction equation

$$\kappa \Delta u = \frac{\partial u}{\partial t}$$

is found for the case of a finite and semi-infinite cone with a ring source placed symmetrically about the axis of the cone. The solution is obtained by determining the modified first and second Green's functions using the normalized eigenfunctions and eigenvalues first, and then the regular Green's function by the inverse Laplace transform. Through the use of the modified Green's functions as an intermediate, the solution of the semi-infinite cone is obtained as a special case of the finite cone. The need for the modified Green's function as an intermediate is demonstrated by an example for which the solution is known.

Heat Conduction in a Cone with a Ring Source

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HEAT CONDUCTION IN A CONE WITH A RING SOURCE

I. INTRODUCTION

The problem considered and solved in this thesis is that of the conduction of heat inside a homogeneous solid; a cone of slant height a and terminated by a spherical cap of radius a. Also the semi-infinite cone as a particular case is considered. An initial temperature distribution $u(r, \theta, \varphi, 0)$ throughout the interior, together with certain conditions for the temperature u at the boundaries are prescribed. We will consider only the isothermic case which occurs when the boundary is kept at the constant temperature zero at all times. Our heat source is a ring of unit strength placed symmetrically about the axis of the cone.

Not much work has been done with regard to the above problem. Probably the first investigation of the problem was done by H. S. Carslaw. In 1913 he determined the Green's function for $\Delta u + k^2 u = 0$ in the cone by contour integration and gave the results in the form of infinite series. Later, in 1921, the solution to the heat conduction problem in the finite cone appeared in his book "Introduction to the Mathematical Theory of the Conduction of Heat in Solids," which came out in print in 1940, and was completely revised in 1947 and 1959. In the 1947 and 1959 editions the Green's function for the infinite cone is found also by the method of Laplace transformation.

A more recent publication related to the above problem is the one by R. Muki and E. Sternberg [7]. They investigated the problem of steady-state heat conduction in the semi-infinite cone by the use of the Mellin transform and the results are given both in the form of an infinite series and in the form of a contour integral. Also some numerical results are obtained by numerical integration.

The method used here is that of determining the Green's function using eigenvalues and eigenfunctions. This could be done using a well known expansion formula [4, p. 267], but we use the modified Green's function as an intermediate, because that enables us to find the Green's function for the semi-infinite cone as a particular case. The use of the eigenvalues and eigenfunctions does not work very well in the adiabatic case, which occurs when the boundary is insulated so that no heat flows through into the surrounding medium, because the eigenvalues are obtained as the roots of a transendental equation (Appendix B, Equation (B. 2)) and they are hard to locate.

The primary result obtained in this paper is the first Green's function for both the finite and semi-infinite cone with an axi-symmetric source ring. Then the temperature inside the finite or semi-infinite cone in time t due to an initial temperature distribution independent of φ is easily obtained. However, it was felt that an example demonstrating the advantages of the method used should be done first. A lot of the formulas in this example are used again in obtaining the solution to the main problem. Those relations used later in the main problem are developed with reasonable detail and for that reason this example might seem a little longer than what one would normally expect in a simple demonstration.

II. DEFINITION OF GREEN'S FUNCTION FOR THE HEAT CONDUCTION EQUATION. MODIFIED GREEN'S FUNCTION

Our problem consists of finding the solution $u = u(x, y, z, t)$ to the heat conduction equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u \quad \text{in} \quad V; \quad t > 0$$

satisfying the homogeneous boundary condition

$$u = 0 \quad \text{on} \quad S; \quad t \geq 0$$

and the initial condition

$$u = f(x, y, z) \quad \text{in} \quad \overline{V}; \quad t = 0$$

where V is the interior of the volume of the conical structure described in the Introduction, S is the surface area of that structure, $\overline{V} = V \cup S$, and κ is the thermal conductivity of the solid.

The so called Green's function of the heat conduction equation in free space is [8, p. 59]

$$u(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \, e^{-\frac{(\overrightarrow{PQ})^2}{4\kappa t}} \tag{1}$$

Here $\overrightarrow{PQ}$ is the distance of a point of observation P from the location Q of a thermic point source which is usually referred to as a heat pole [8, p. 59].

The function u, as it is represented by Equation (1), has the following properties

(a) u satisfies the heat conduction equation

(b) $u(x, y, z, 0) = 0$ if $\overrightarrow{PQ} \neq 0$

(c) $u(x, y, z, 0) \to \infty$ as $P \to Q$

(d) $\iiint_{-\infty}^{+\infty} u(x, y, z, t)dxdydz = 1$, regardless of t and (x', y', z').

In other words, u, as given by Equation (1), has the character of a δ-function [8, p. 27].

We denoted here (x, y, z) as the Cartesian coordinates of the point of observation P, while (x', y', z') are the Cartesian coordinates of the location of the source point Q.

The Green's function for the interior of a volume bounded by a closed surface σ is defined as

$$G = G(P, Q, t) \tag{2}$$

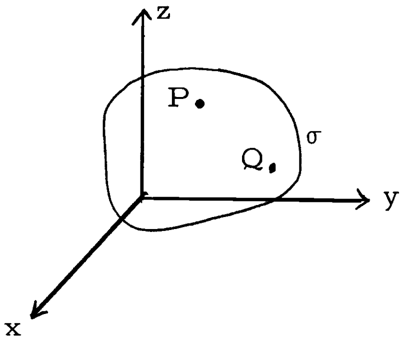
(i) $\dfrac{\partial G}{\partial t} = \kappa \Delta G$ everywhere inside σ for $t > 0$.

(ii) For $t = 0$, $G = 0$ everywhere inside σ except when P is at Q where it has the character of a δ-function (heat pole).

(iii) $G_1 = 0$ if P is on σ (first Green's function).

$\frac{\partial G_2}{\partial n} = 0$ if P is on σ (second Green's function).

Here $\frac{\partial}{\partial n}$ is the normal derivative with respect to the surface σ.



Physically, the expressions (1) and (2) represent a temperature field produced by a heat pole located at a point Q (source point) in the case of Equation (1) in free space, in the case of Equation (2) inside a closed surface σ, according as the boundary condition $u = 0$ (isothermic case) or $\frac{\partial u}{\partial n} = 0$ (adiabatic case) is imposed.

If, for instance, Cartesian coordinates are considered, then, with $P(x, y, z)$ and $Q(x', y', z')$

$$G = G(x, y, z; x', y', z', t).$$

Furthermore, if at $t = 0$ a temperature distribution $f(x, y, z)$ is imposed, then the temperature field inside σ is

$$u(x, y, z, t) = \iiint_V f(x', y', z')G(x, y, z; x', y', z', t)dx'dy'dz' \qquad (3)$$

where G is one of the Green's functions as defined in (2). The integration has to be performed over the whole volume bounded by σ.

Let

$$u(P, Q, t) = g(P, Q)T(t)$$

Then $g(P, Q)$ is a solution of Helmholtz equation

$$\Delta g + k^2 g = 0$$

where k is a separation parameter.

Using the eigenvalues and the normalized eigenfunctions, $g(P, Q)$ is given by [5, p. 267]

$$g(P, Q) = \sum_n \frac{u_n(P)u_n^*(Q)}{k^2 - k_n^2} \tag{4}$$

where the summation is single, double, or triple depending only on the geometry of the problem, k_n are the eigenvalues, $u_n(P)$ are the corresponding normalized eigenfunctions evaluated at the point of observation P, and $u_n^*(Q)$ are the complex conjugate normalized eigenfunctions evaluated at the source point Q.

Hence, we have the following expansion formula for a particular solution of the heat conduction equation.

$$u(P, Q, t) = \sum_n \frac{u_n(P)u_n^*(Q)}{k^2 - k_n^2} e^{-k^2 \kappa t} \tag{5}$$

A well known formula giving the Green's function in terms of the eigenvalues and the normalized eigenfunctions is [4, p. 288]

$$G(P, Q, t) = \sum_{n=1}^{\infty} u_n(P)u_n^*(Q)e^{-k_n^2 \kappa t} \tag{6}$$

where again the summation is single, double, or triple depending on the geometry of the problem, and $u_n(P)$, $u_n^*(Q)$, and k_n are as in Equation (4).

To determine $g(P, Q)$ or $G(P, Q, t)$ using eigenvalues and eigenfunctions one could use Equations (4) and (6), respectively. We are going to use a method that uses the modified Green's function, denoted by $\overline{G}$, as an intermediate.

Let $k = -i\gamma$ in Equation (4). Then, denoting by $\overline{G}(P, Q, \gamma)$ the resulting value of $g(P, Q)$, we have

$$-\overline{G}(P, Q, \gamma) = \sum_{n} \frac{u_n(P)u_n^*(Q)}{\gamma^2+k_n^2} \tag{7}$$

The function $\overline{G}(P, Q, \gamma)$ is a solution of

$$\Delta g - \gamma^2 g = 0$$

which is referred to as the "modified Helmholtz equation."

Now replace γ by $\gamma^{1/2}$ and find the inverse Laplace transform of the resulting equation with respect to γ. Since neither $u_n(P)$, nor $u_n^*(Q)$ contain γ we only have to obtain the inverse Laplace transform of $\frac{1}{\gamma+k_n^2}$. Using Equation (A.1) in Appendix A we find that

$$\mathcal{L}_\gamma^{-1}\{\overline{G}(P, Q, \gamma^{1/2})\} = \sum_n u_n(P)u_n^*(Q)e^{-k_n^2 t} \tag{8}$$

Now replacing t by κt in Equation (8) we see that

$$\mathcal{L}_\gamma^{-1}\{\overline{G}(P, Q, \gamma^{1/2})\}_{t \to \kappa t} = G(P, Q, t) \tag{9}$$

Here "$t \to \kappa t$" means t is replaced by κt.

For instance, the point source solution of the Helmholtz equation which applies to time harmonic wave propagation phenomena is [9, Vol. 2, p. 57]

$$u = -\frac{1}{4\pi\overrightarrow{PQ}} e^{-ik\overrightarrow{PQ}} \tag{10}$$

Replace k by $-i\gamma$ in Equation (10).

$$u = -\frac{1}{4\pi\overrightarrow{PQ}} e^{-\gamma\overrightarrow{PQ}} \tag{11}$$

Equation (11) is the point source solution of the modified

Helmholtz equation.

Using Equation (A. 4) in Appendix A we see that

$$\frac{1}{(4\pi\kappa t)^{3/2}} e^{-\frac{(\overrightarrow{PQ})^2}{4\kappa t}} = \mathcal{L}_\gamma^{-1}\left\{-\frac{1}{4\pi\overrightarrow{PQ}} e^{-\gamma^{1/2}\overrightarrow{PQ}}\right\}_{t \to \kappa t}$$

It seems a waste of time, once you have found the eigenvalues and the corresponding eigenfunctions, to find $\overline{G}(P, Q, \gamma)$ as described above, instead of finding g and G directly from Equations (4) and (6).

If we want to examine g and G with a semi-infinite domain, that is with $a = \infty$, the seemingly easiest way to do it, using eigenvalues and eigenfunctions, is to find g and the Green's function using Equations (4) and (6) for finite a and then let $a \to \infty$. In doing so, however, we get an infinite series that does not converge or in other cases it might converge very slowly. By using the modified Green's function as an intermediate step with the help of Equation (7) we get a solution which is monotonic in nature for the finite case and hence converges more rapidly than the oscillatory solution that we would have obtained, if we had used Equations (4) and (6). At the same time we are able to examine the case with infinite domain, something which we would not probably have been able to do, due to the ill behavior of the infinite series in Equation (4) and (6). Furthermore, since $u_n(P)$ and $u_n^*(Q)$ do not involve γ, finding the inverse Laplace transform of Equation (7) is a relatively simple problem in the case of a finite domain.

To demonstrate the need for the modified Green's function as an intermediate step we consider the following example in which we derive Equation (1).

III. FREE SPACE GREEN'S FUNCTION

We obtain the solution by solving the classical problem of finding the Green's function of a sphere of arbitrary radius a and center at the origin and then, as mentioned in the analysis before, we let $a \to \infty$. We let the point source be the arbitrary point Q inside the sphere.

As mentioned earlier first we must find the eigenvalues and the corresponding eigenfunctions for the sphere. That is, we want to find the functions which are continuous, single valued solutions of the scalar Helmholtz equation

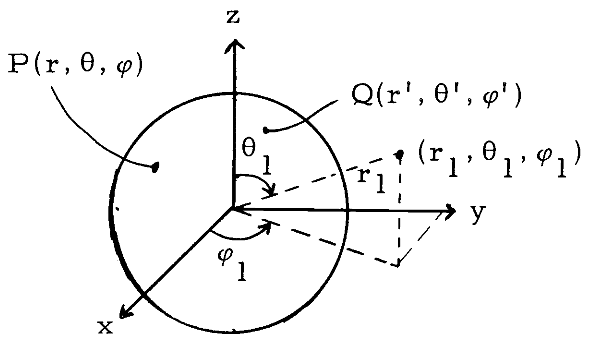
$$\Delta u + k^2 u = 0 \tag{12}$$

in the sphere which satisfy the homogeneous boundary condition

$$u = 0 \quad \text{at the surface of the sphere.} \tag{13}$$

Let

$$x = r \sin\theta \cos\varphi, \quad y = r \sin\theta \sin\varphi, \quad z = r\cos\theta$$



Then Equation (12) expressed in spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}+k^2u=0 \tag{14}$$

and the boundary condition (13) becomes

$$u = 0 \quad \text{at} \quad r = a \tag{15}$$

Since we will be seeking solutions of Equation (14) in solving the problem for the cone later, let us derive the general solution of that equation here.

Assume a solution of the form $u = f_1(r)f_2(\theta)f_3(\varphi)$. Substitution into the Helmholtz equation yields the following three ordinary differential equations:

$$\frac{d^2f_1}{dr^2}+\frac{1}{r}\frac{df_1}{dr}+\left[k^2-\frac{(\nu+\frac{1}{2})^2}{r^2}\right]f_1=0$$

$$\frac{d^2f_2}{d\theta^2}+\cot\theta\frac{df_2}{d\theta}+\left[\nu(\nu+1)-\frac{\mu^2}{\sin^2\theta}\right]f_2=0 \tag{16}$$

$$\frac{d^2f_3}{d\varphi^2}+\mu^2f_3=0 \tag{17}$$

where μ, ν are arbitrary separation parameters.

The first of these equations is recognized as the Bessel's differential equation of order $\nu+\frac{1}{2}$ whose general solution is [4, p. 217]

$$f_1(r) = \frac{1}{r^{1/2}} [AJ_{\nu+\frac{1}{2}}(kr)+BY_{\nu+\frac{1}{2}}(kr)]$$

Here $J_{\nu+\frac{1}{2}}$ and $Y_{\nu+\frac{1}{2}}$ denote the Bessel functions of the first and second kind, respectively, of order $\nu+\frac{1}{2}$ and A, B are arbitrary constants.

Equation (16) is recognized as the Legendre differential equation. Its general solution is given by [4, p. 117]

$$f_2(\theta) = CP_\nu^\mu(\cos\theta) + DQ_\nu^\mu(\cos\theta)$$

where C, D are arbitrary constants and P_ν^μ and Q_ν^μ denote the associated Legendre functions.

Finally, the general solution of Equation (17) is given by

$$f_3(\varphi) = E\cos\mu\varphi + F\sin\mu\varphi$$

where again E, F are arbitrary constants.

Therefore, the general solution of the Helmholtz equation in spherical coordinates is

$$u = \frac{1}{r^{1/2}} [AJ_{\nu+\frac{1}{2}}(kr)+BY_{\nu+\frac{1}{2}}(kr)][CP_\nu^\mu(\cos\theta)+DQ_\nu^\mu(\cos\theta)] \times [E\cos\mu\varphi + F\sin\mu\varphi] \tag{18}$$

We want the solution u to be bounded and single-valued in the sphere. Therefore, we must set $\nu = n = 0, 1, 2, \ldots,$

$\mu = m = 0, 1, 2, \ldots,$ and $B = D = 0$ in Equation (18), because $Y_{n+\frac{1}{2}}(kr)$ has a singularity at $r = 0$ and, while both $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are single-valued, only $P_n^m(\cos\theta)$ is bounded for $0 \leq \theta \leq \pi$ with $Q_n^m(\cos\theta)$ possessing singularities on both the axes $\theta = 0, \pi$ [5, p. 367]. Hence, the eigenfunctions for the sphere are

$$u_{m,n} = \frac{1}{r^{1/2}} J_{n+\frac{1}{2}}(kr) P_n^m(\cos\theta) e^{\pm im\varphi}, \quad m, n = 0, 1, 2, \ldots$$

The boundary condition (15) implies that

$$J_{n+\frac{1}{2}}(ka) = 0$$

Let $\tau_{n,\ell}$ be the ℓth root of this equation. Then the eigenvalues will be given by

$$k_{n,\ell} = \frac{\tau_{n,\ell}}{a} \qquad n = 0, 1, 2, \ldots, \quad \ell = 1, 2, \ldots,$$

and the corresponding eigenfunctions by

$$u_{m,n,\ell} = \frac{1}{r^{1/2}} J_{n+\frac{1}{2}}\left(\tau_{n,\ell} \frac{r}{a}\right) P_n^m(\cos\theta) e^{\pm im\varphi},$$

$$m, n = 0, 1, 2, \ldots, \quad \ell = 1, 2, 3, \ldots$$

In order to find the normalizing constant N we must evaluate the following triple integral

$$N^2 = \int_{r=0}^{a} r[J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})]^2 dr \int_{\theta=0}^{\pi} [P_n^m(\cos\theta)]^2 \sin\theta\, d\theta$$

$$\times \int_{\varphi=0}^{2\pi} \begin{cases} \cos^2 m\varphi \\ \sin^2 m\varphi \end{cases} d\varphi$$

From the theory of Bessel functions [5, p. 308] we know that

$$\int_{r=0}^{a} r[J_\nu(kr)]^2 dr = \frac{1}{2k^2}\{k^2a^2[J_\nu'(ka)]^2 + (k^2a^2-\nu^2)[J_\nu(ka)]^2\} \tag{19}$$

Hence

$$\int_{r=0}^{a} r[J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})]^2 dr = \frac{a^2}{2}[J'_{n+\frac{1}{2}}(\tau_{n,\ell})]^2$$

Using the recurrence formula [10, p. 45]

$$zJ_\nu'(z) - \nu J_\nu(z) = -zJ_{\nu+1}(z) \tag{20}$$

we get

$$\int_{r=0}^{a} r[J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})]^2 dr = \frac{a^2}{2}[J_{n+\frac{3}{2}}(\tau_{n,\ell})]^2$$

The second integral is known [11, p. 193]

$$\int_{\theta=0}^{\pi} [P_n^m(\cos\theta)]^2 \sin\theta d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

Finally, for the third integration we get

$$\int_{\varphi=0}^{2\pi} \begin{Bmatrix} \cos^2 m\varphi \\ \sin^2 m\varphi \end{Bmatrix} d\varphi = \frac{2\pi}{\varepsilon_m}$$

where ε_m is Neumann's number defined by

$$\varepsilon_m = \begin{cases} 1 & m = 0 \\ 2 & m \neq 0 \end{cases} \tag{21}$$

Therefore, the normalized eigenfunctions are

$$u_{m,n,\ell} = \frac{1}{a}\left[\frac{(2n+1)(n-m)!\varepsilon_m}{2\pi(n+m)!\,r}\right]^{1/2} \frac{J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})P_n^m(\cos\theta)\begin{Bmatrix}\cos m\varphi \\ \sin m\varphi\end{Bmatrix}}{J_{n+\frac{3}{2}}(\tau_{n,\ell})}$$

$$m, n = 0, 1, 2, \ldots, \quad \ell = 1, 2, 3, \ldots .$$

Substituting these eigenvalues and eigenfunctions into Equation (7) we get the modified Green's function

$$\overline{G}(P, Q, \gamma) = -\frac{1}{2a^2\pi(rr')^{1/2}} \sum_{n=0}^{\infty}\sum_{\ell=1}^{\infty}\sum_{m=0}^{\infty} \frac{(2n+1)(n-m)!\varepsilon_m}{(n+m)!}$$

$$\times \frac{J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r'}{a})}{[J_{n+\frac{3}{2}}(\tau_{n,\ell})]^2} \frac{P_n^m(\cos\theta)P_n^m(\cos\theta')}{\gamma^2 + (\frac{\tau_{n,\ell}}{a})^2}$$

$$\times \cos m(\varphi - \varphi') \tag{22}$$

Using the addition theorem [6, Vol. 1, p. 168] in Equation (22) we get

$$\overline{G}(P, Q, \gamma) = - \frac{1}{2a^2\pi(rr')^{1/2}} \sum_{\ell=1}^{\infty} \sum_{n=0}^{\infty} \frac{(2n+1)J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})}{[J_{n+\frac{3}{2}}(\tau_{n,\ell})]^2}$$

$$\times \frac{J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r'}{a})P_n(\cos\theta_1)}{\gamma^2 + (\frac{\tau_{n,\ell}}{a})^2} \tag{23}$$

where $\cos\theta_1 = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$.

Also from the theory of Bessel functions [6, Vol. 2, p. 104] we have the following formula

$$\frac{\pi J_\nu(xz)}{4J_\nu(z)}[J_\nu(z)Y_\nu(Xz)-J_\nu(Xz)Y_\nu(z)]$$

$$= \sum_{n=1}^{\infty} J_\nu(x\gamma_{\nu,n})J_\nu(X\gamma_{\nu,n})[J_{\nu+1}(\gamma_{\nu,n})]^{-2}(z^2-\gamma_{\nu,n}^2)^{-1},$$

$$0 \leq x \leq X \leq 1, \quad \nu \neq -1, -2, \ldots, \tag{24}$$

with x and X interchanged when $X \leq x$.

This formula is also given by Watson [10, p. 499], but his has a mistake. Buchholz was the first to point out and correct this mistake [1, p. 245].

In Equation (24) let $z = ik$. Using the fact that

$$J_\nu(iz) = e^{i\nu\frac{\pi}{2}}I_\nu(z)$$

and

$$Y_\nu(iz) = \frac{e^{i\nu\frac{\pi}{2}} I_\nu(z)\cos\nu\pi - e^{-i\nu\frac{\pi}{2}} I_{-\nu}(z)}{\sin\nu\pi}$$

we find that

$$\sum_{n=1}^{\infty} \frac{J_\nu(x\gamma_{\nu,n})J_\nu(X\gamma_{\nu,n})}{[J_{\nu+1}(\gamma_{\nu,n})]^2[k^2+\gamma_{\nu,n}^2]} = \frac{1}{2}\frac{I_\nu(xk)}{I_\nu(k)}[I_\nu(k)K_\nu(Xk)-K_\nu(k)I_\nu(Xk)],$$

$$0 \leq x \leq X \leq 1 \qquad (25)$$

with x and X interchanged when $X \leq x$.

Here I_ν, K_ν are the modified Bessel functions of the first and second kind respectively.

Applying Equation (25) to Equation (23) we find that the modified Green's function is

$$\overline{G}(P, Q, \gamma) = -\frac{1}{4\pi(rr')^{1/2}} \sum_{n=0}^{\infty} (2n+1)$$

$$\times [I_{n+\frac{1}{2}}(\gamma r')K_{n+\frac{1}{2}}(\gamma r) - I_{n+\frac{1}{2}}(\gamma r')I_{n+\frac{1}{2}}(\gamma r)\frac{K_{n+\frac{1}{2}}(\gamma a)}{I_{n+\frac{1}{2}}(\gamma a)}P_n(\cos\theta_1)) \qquad (26)$$

In Equation (26) we want to let $a \to \infty$. When $|z|$ is large enough the following asymptotic expressions are valid [5, p. 330]

$$I_\nu(z) \sim \frac{1}{(2\pi z)^{1/2}} [e^z + e^{-z+i(\nu+\frac{1}{2})\pi}], \qquad -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$$

and

$$K(z) \sim (\frac{\pi}{2z}) e^{-z} \qquad |\arg z| < \frac{3\pi}{2}$$

Therefore

$$\frac{K_\nu(z)}{I_\nu(z)} \sim \frac{\pi}{e^{2z} + e^{i(\nu+\frac{1}{2})\pi}}$$

which shows that

$$\lim_{z \to \infty} \frac{K_\nu(z)}{I_\nu(z)} = 0 \tag{27}$$

Using Equation (27) in Equation (26) we see that the modified Green's function for the whole space is

$$\overline{G}(P, Q, \gamma) = -\frac{1}{4\pi(rr')^{1/2}} \sum_{n=0}^{\infty} (2n+1) I_{n+\frac{1}{2}}(\gamma r') K_{n+\frac{1}{2}}(\gamma r) P_\nu(\cos\theta_1),\ r' \leq r \tag{28}$$

with r and r' interchanged when $r \leq r'$.

By definition [6, Vol. 2, p. 5]

$$I_\nu(z) = e^{-\frac{i\nu\pi}{2}} J_\nu(iz)$$

and

$$K_\nu(z) = \frac{1}{2} i\pi e^{\frac{i\nu\pi}{2}} H_\nu^{(1)}(iz)$$

Hence, Equation (28) may be written in the following form

$$\overline{G}(P, Q, \gamma) = -\frac{i}{8(rr')^{1/2}} \sum_{n=0}^{\infty} (2n+1) J_{n+\frac{1}{2}}(i\gamma r') H^{(1)}_{n+\frac{1}{2}}(i\gamma r) P_n(\cos\theta_1), \quad r' \leq r$$

This sum now is known [5, p. 359] and so regardless of the relation between r and r' the modified Green's function for the whole space is

$$\overline{G}(P, Q, \gamma) = -\frac{e^{-\gamma R}}{4\pi R} \tag{29}$$

where $R = (r^2 + r'^2 - 2rr' \cos\theta_1)^{1/2}$.

Substituting Equation (29) into Equation (9) we find (Appendix A, Equation (A. 4)) that the free space Green's function is

$$G(P, Q, t) = \frac{1}{4\pi R} \mathcal{L}_{\gamma}^{-1}\{e^{-\gamma^{1/2} R}\}_{t \to \kappa t} = \frac{e^{-\frac{R^2}{4\kappa t}}}{(4\pi\kappa t)^{3/2}} \tag{30}$$

agreeing with what has already been established in the literature [8, p. 59].

If we used Equation (4) instead of Equation (7) we would have

$$g(P, Q) = \frac{1}{2a^2\pi(rr')^{1/2}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2n+1)(n-m)!\varepsilon_m}{(n+m)!}$$

$$\times \frac{J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r'}{a})}{[J_{n+\frac{3}{2}}(\tau_{n,\ell})]^2} \frac{P_n^m(\cos\theta)P_n^m(\cos\theta')}{k^2-(\frac{\tau_{n,\ell}}{a})^2}$$

$$\times \cos m(\varphi-\varphi') \tag{31}$$

In order to see what happens if we let $a \to \infty$, let us use first the relation (24) and the addition theorem [6, Vol. 1, p. 168]. Equation (31) becomes

$$g(P, Q) = \frac{1}{8(rr')^{1/2}} \sum_{n=0}^{\infty} (2n+1)[J_{n+\frac{1}{2}}(kr')Y_{n+\frac{1}{2}}(kr) - J_{n+\frac{1}{2}}(kr)J_{n+\frac{1}{2}}(kr')$$

$$\times \frac{Y_{n+\frac{1}{2}}(ka)}{J_{n+\frac{1}{2}}(ka)}]P_n(\cos\theta_1) \tag{32}$$

Let us examine the behavior of $Y_\nu(z)$ and $J_\nu(z)$ as $z \to \infty$, z real. From the theory of Bessel function [5, p. 321] we have

$$Y_\nu(z) \sim (\frac{2}{\pi z})^{1/2}\sin[z-(\nu+\tfrac{1}{2})\frac{\pi}{2}] \qquad z \to \infty$$

and

$$J_\nu(z) \sim (\frac{2}{\pi z})^{1/2}\cos[z-(\nu+\tfrac{1}{2})\frac{\pi}{2}] \qquad z \to \infty$$

From these two expressions we see that the Bessel functions

$J_\nu(z)$ and $Y_\nu(z)$ behave when z is large like damped trigonometric functions. Furthermore

$$\frac{Y_\nu(z)}{J_\nu(z)} - \tan[z-(\nu+\tfrac{1}{2})\tfrac{\pi}{2}] \qquad z \to \infty$$

Therefore, g in Equation (32) does not approach a limit as $a \to \infty$ and hence the infinite sum will not converge, whereas using the modified Green's function we got not only a solution for the finite sphere, but also a solution for the whole space as a special case which converges very rapidly as it may be seen from Equation (30).

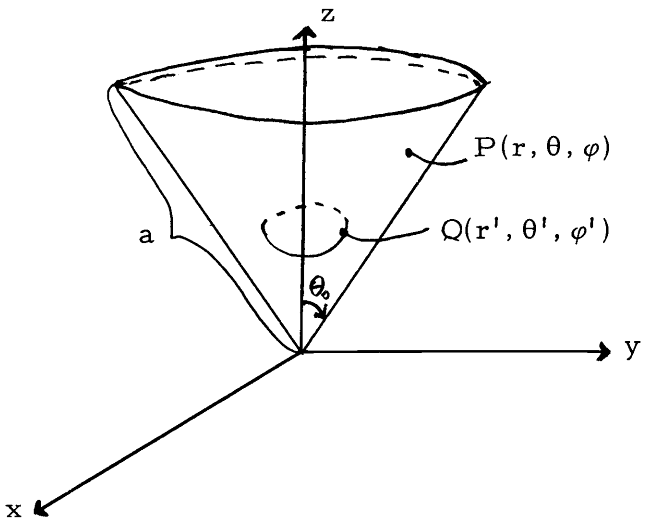
Furthermore direct substitution into Equation (6) and use of the addition theorem [6, Vol 1, p. 168] yields

$$G(P, Q, t) = \frac{1}{2a^2\pi(rr')^{\frac{1}{2}}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{(2n+1)\, J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r}{a})}{[J_{n+\frac{3}{2}}(\tau_{n,\ell})]^2} \times$$

$$\times\ J_{n+\frac{1}{2}}(\tau_{n,\ell}\frac{r'}{a})\, P_n(\cos\theta_1)\, e^{-(\frac{\tau_{n,\ell}}{a})^2 \kappa t}$$

The double sum above is not known. Hence, its behavior as $a \to \infty$ is hard to determine, and so Equation (6) is of little use for the case $a = \infty$.

IV. EIGENVALUES AND EIGENFUNCTIONS FOR THE CONE



The eigenfunctions are the continuous, single-valued solutions of the scaler Helmholtz equation (12) in the cone which satisfy the homogeneous boundary condition

$$B(u) = 0 \quad \text{on the surface of the cone} \tag{33}$$

We will consider only the isothermic condition. So, the boundary condition (33) expressed in sperhical coordinates is

$$\begin{aligned} u = 0 \quad &\text{at} \quad r = a, \quad 0 \leq \theta \leq \theta_0, \quad 0 \leq \varphi \leq 2\pi \\ u = 0 \quad &\text{at} \quad \theta = \theta_0, \quad 0 \leq r \leq a, \quad 0 \leq \varphi \leq 2\pi \end{aligned} \tag{34}$$

Equation (12) expressed in spherical coordinates is given by Equation (14) whose general solution is Equation (18). We want the solution u of the Helmholtz equation to be continuous and single-valued for $0 \leq r \leq a$, $0 \leq \theta \leq \theta_0$, $0 \leq \varphi \leq 2\pi$. The Bessel functions of the second kind and the associated Legendre functions of the second kind, as remarked earlier, become singular at $r = 0$ and $\theta = 0, \pi$, respectively. Therefore, we must set $B = D = 0$ in Equation (18). The solution u must not only be single-valued, it must also be periodic in φ. So we must set $F = 0$ and $\mu = n = 0, 1, 2, \ldots$ in Equation (18). Furthermore, for reasons stated later, we restrict ν to real values, and, since [6, Vol. 1, p. 144] $P^{\mu}_{-\nu-1}(x) = P^{\mu}_{\nu}(x)$, we may take $\nu > -1$. So the general solution of the Helmholtz equation for the geometry considered here is

$$u = \frac{A \cos n\varphi}{r^{1/2}} J_{\nu+\frac{1}{2}}(kr) P^{n}_{\nu}(\cos\theta), \quad n = 0, 1, 2, \ldots, \quad \nu > -1 \tag{35}$$

where A is an arbitrary constant which we may choose to set equal to one for simplicity. Substituting ν by $\nu - \frac{1}{2}$ and $A = 1$ in Equation (35) we find that

$$u = \frac{\cos n\varphi}{r^{1/2}} J_{\nu}(kr) P^{n}_{\nu-\frac{1}{2}}(\cos\theta), \quad n = 0, 1, 2, \ldots, \quad \nu > -\tfrac{1}{2} \tag{36}$$

Now we are ready to impose the boundary conditions (34). The first of the boundary conditions (34) implies that

$$J_\nu(ka) = 0$$

Let $\tau_{m,\nu}$ be the mth root of

$$J_\nu(z) = 0 \tag{37}$$

Then the eigenvalues are

$$k_{m,\nu} = k = \frac{\tau_{m,\nu}}{a}$$

From the theory of eigenfunctions and eigenvalues [8, p. 170] we know that $k_{m,\nu}$ are real. Therefore, $\tau_{m,\nu}$ must be real and by Lommels theorem [10, p. 482] this is true only if ν is real. This justifies our earlier restriction of ν to real values. Furthermore, since $J_{-\nu}(x)$ has the same roots as $J_\nu(x)$ we may restrict $\tau_{m,\nu}$ to be the positive mth root of Equation (37).

Similarly, the second of the boundary conditions (34) implies that

$$P^n_{\nu-\frac{1}{2}}(\cos\theta_0) = 0 \quad n = 0, 1, 2, \ldots, \quad \nu > -\tfrac{1}{2} \tag{38}$$

Equation (38) is a function of ν. Let $\alpha_{n,\ell}$ be the ℓth root of this equation. Then the eigenvalues of our boundary value problem are

$$k_{m,n,\ell} = \frac{\tau_{m,\nu}}{a}, \quad \nu = \alpha_{n,\ell}, \quad n = 0, 1, 2, \ldots, \quad m,\ell = 1,2,3,\ldots$$

Substituting these eigenvalues in Equation (36) we find that the corresponding eigenfunctions for our problem are

$$u_{m,n,\ell} = \frac{\cos n\varphi}{r^{1/2}} J_\nu(\tau_{n,\nu} \frac{r}{a}) P^n_{\nu-\frac{1}{2}}(\cos\theta)$$

with $n = 0, 1, 2, \ldots,$ $m, \ell = 1, 2, 3, \ldots,$ $\nu = \alpha_{n,\ell}$, and $\alpha_{n,\ell}$ being the ℓth root of Equation (38).

In order to determine the modified Green's function for our boundary problem we need the normalized eigenfunctions. The normalizing constant is given by

$$N^2 = \iiint_V u^2_{m,n,\ell} dV$$

In this case we have

$$N^2 = \int_{r=0}^{a} \int_{\theta=0}^{\theta_0} \int_{\varphi=0}^{2\pi} [\frac{1}{r^{1/2}} J_\nu(\tau_{m,\nu} \frac{r}{a})]^2 [P^n_{\nu-\frac{1}{2}}(\cos\theta)]^2$$

$$\times \cos^2 n\varphi r^2 \sin\theta d\varphi d\theta dr \qquad (39)$$

Considering first the integration with respect to r we have to evaluate the following definite integral

$$\int_{r=0}^{a} r[J_\nu(\tau_{m,\nu} \frac{r}{a})]^2 dr$$

We met this integral in the example considered earlier. Using Equations (19) and (20) we find that

$$\int_{r=0}^{a} r[J_\nu(\tau_{m,\nu}\frac{r}{a})]^2 dr = \frac{a^2}{2}[J_{\nu+1}(\tau_{m,\nu})]^2$$

The integration with respect to φ yields

$$\int_{\varphi=0}^{2\pi} \cos^2 n\varphi d\varphi = \begin{cases} 2\pi & n = 0 \\ \pi & n \neq 0 \end{cases} \tag{40}$$

or using Neumann's number, ε_n, defined by Equation (21) we may write Equation (40) in the compact form

$$\int_{\varphi=0}^{2\pi} \cos^2 n\varphi d\varphi = \frac{2\pi}{\varepsilon_n}, \quad n = 0, 1, 2, \ldots$$

Finally, in order to perform the integration which involves only the associated Legendre functions, we must evaluate the definite integral

$$\int_{\theta=0}^{\theta_0} [P^n_{\nu-\frac{1}{2}}(\cos\theta)]^2 \sin\theta d\theta = \int_{\cos\theta_0}^{1} [P^n_{\nu-\frac{1}{2}}(t)]^2 dt \tag{41}$$

where we have made the change of variable $t = \cos\theta$.

Let ρ, ν, and σ be any arbitrary parameters and $W^\rho_\sigma(x)$, $W^\rho_\nu(x)$ be any solutions of Legendre's differential equation.

Then [6, Vol. 1, p. 169]

$$\int_a^b W_\sigma^\rho W_\nu^\rho dx = \left[x \frac{W_\nu^\rho W_\sigma^\rho}{\nu+\sigma+1} + \frac{(\sigma+\rho)W_\nu^\rho W_{\sigma-1}^\rho - (\nu+\rho)W_{\nu-1}^\rho W_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)} \right]_a^b \tag{42}$$

Letting $\nu \to \sigma$ in formula (42), we find that

$$\int_a^b [W_\nu^\rho]^2 dx = \left\{ x \frac{[W_\nu^\rho]^2}{2\nu+1} + \lim_{\nu \to \sigma} \frac{(\sigma+\rho)W_\nu^\rho W_{\sigma-1}^\rho - (\nu+\rho)W_{\nu-1}^\rho W_\sigma^\rho}{(\nu-\sigma)(\nu+\sigma+1)} \right\} \Bigg|_a^b$$

W_ν^ρ is a function of ν. Therefore, we may use L'Hopital's rule to evaluate the above limit. We find that

$$\int_a^b [W_\nu^\rho]^2 dx = \frac{1}{2\nu+1} \left\{ x[W_\nu^\rho]^2 - W_\nu^\rho W_{\nu-1}^\rho + (\nu+\rho)[W_{\nu-1}^\rho \frac{d}{d\nu} W_\nu^\rho - W_\nu^\rho \frac{d}{d\nu} W_{\nu-1}^\rho] \right\} \Bigg|_a^b \tag{43}$$

Using Equation (43) in Equation (41) we see that

$$\int_{\cos\theta_0}^1 [P_{\nu-\frac{1}{2}}^n(t)]^2 dt = \frac{2\nu+2n-1}{4\nu} P_{\nu-\frac{3}{2}}^n(\cos\theta_0)$$

$$\times \frac{d}{d\nu} P_{\nu-\frac{1}{2}}^n(\cos\theta_0) = M_\nu^n(\theta_0) \tag{44}$$

where we have used the result of Equation (38) and the fact that $P_{\nu-\frac{1}{2}}^n(1) = 0$ [6, Vol. 1, p. 146].

Therefore

$$N^2 = \frac{a^2\pi}{\varepsilon_n}[J_{\nu+1}(\tau_{m,\nu})]^2 M_\nu^n(\theta_0)$$

and

$$U_{m,n,\ell} = \frac{1}{N} u_{m,n,\ell} = \frac{1}{a}\left(\frac{\varepsilon_n}{\pi r}\right)^{1/2} \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})P_\nu^n(\cos\theta)}{[J_{\nu+1}(\tau_{m,\nu})][M_\nu^n(\theta_0)]^{1/2}} \tag{45}$$

where $n = 0, 1, 2, \ldots,$ $m, \ell = 1, 2, \ldots,$ $\nu = \alpha_{n,\ell}$, $\alpha_{n,\ell}$ and $M_\nu^n(\theta_0)$ are determined by Equations (38) and (44) respectively.

Now we are ready to find the Green's function for our conical structure. We will consider two cases: the finite case $(0 < a < \infty)$ and the infinite case $(a = \infty)$.

V. GREEN'S FUNCTION FOR THE CONE

In each of the cases considered in this chapter the observation point P is assumed to be an arbitrary point in the interior of the cone with coordinates (r, θ, φ) and Q is an arbitrary point on the source ring with coordinates (r', θ', φ').

"a" Finite

From Equations (7) and (45) we find that the modified Green's function for this case is

$$\overline{G}_1(P, Q, \gamma) = -\frac{1}{a^2\pi(rr')^{1/2}} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\varepsilon_n \cos n\varphi \cos n\varphi'}{[J_{\nu+1}(\tau_{m,\nu})]^2 M_\nu^n(\theta_0)}$$

$$\times \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})J_\nu(\tau_{m,\nu}\frac{r'}{a})}{\gamma^2 + (\frac{\tau_{m,\nu}}{a})^2} P^n_{\nu-\frac{1}{2}}(\cos\theta) P^n_{\nu-\frac{1}{2}}(\cos\theta') \tag{46}$$

In Equation (46) replace γ by $\gamma^{1/2}$ and then substitute the resulting equation into Equation (9). In doing that we have to find the inverse Laplace transform of $\frac{1}{\gamma + k^2_{m,n,\ell}}$ with respect to γ, where $k_{m,n,\ell} = \frac{\tau_{m,\nu}}{a}$.

From Appendix A, Equation (A. 1), we find that

$$\mathcal{L}_{\gamma}^{-1}\left\{\frac{1}{\gamma+k_{m,n,\ell}^{2}}\right\} = e^{-k_{m,n,\ell}^{2}t}$$

Therefore

$$G_1(P,Q,t) = -\mathcal{L}_{\gamma}^{-1}\{\bar{G}_1(P,Q,\gamma^{1/2})\}_{t\to\kappa t}$$

$$= \frac{1}{a^2\pi(rr')^{1/2}} \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{\ell=1}^{\infty} \varepsilon_n \cos n\varphi \cos n\varphi'$$

$$\times \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})J_\nu(\tau_{m,\nu}\frac{r'}{a})}{[J_{\nu+1}(\tau_{m,\nu})]^2} \; \frac{P^n_{\nu-\frac{1}{2}}(\cos\theta)P^n_{\nu-\frac{1}{2}}(\cos\theta')}{M^n_\nu(\theta_0)}$$

$$\times e^{-k_{m,n,\ell}^{2}\kappa t} \tag{47}$$

In order to find the first Green's function due to the whole ring we must integrate over all the source points. That is, integrate Equation (47) with respect to φ' from $\varphi' = 0$ to $\varphi' = 2\pi$. Since the only term in Equation (47) involving φ' is the term $\cos n\varphi'$, and noting that

$$\int_0^{2\pi} \cos n\varphi' d\varphi' = \begin{cases} 2\pi & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{48}$$

the triple sum in Equation (47) reduces to a double sum

$$G_1(P, Q, t) = \frac{2}{a^2(rr')^{1/2}} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})J_\nu(\tau_{m,\nu}\frac{r'}{a})}{[J_{\nu+1}(\tau_{m,\nu})]^2}$$

$$\times \frac{P_{\nu-\frac{1}{2}}(\cos\theta)P_{\nu-\frac{1}{2}}(\cos\theta')}{M_\nu(\theta_0)} e^{-k^2_{m,\ell}\kappa t} \tag{49}$$

where the index ℓ is determined by Equation (38).

Substituting Equation (49) into Equation (3) we find the temperature u at time t due to the initial temperature $f(r, \theta, \varphi)$

$$u = \int_{r'=0}^{a} \int_{\theta'=0}^{\theta_0} \int_{\varphi'=0}^{2\pi} G_1(P, Q, t) f(r', \theta', \varphi') r'^2 \sin\theta' d\varphi' d\theta' dr' \tag{50}$$

Example

For a simple example, let the initial temperature in the finite cone be constant. That is $u(r, \theta, \varphi, 0) = f(r, \theta, \varphi) = u_0$, where u_0 is a constant.

The Green's function is given by Equation (49) and the temperature at the time t is given by Equation (50). Then

$$u = \frac{2u_0}{a^2} \int_{r'=0}^{a} \int_{\theta'=0}^{\theta_0} \int_{\varphi'=0}^{2\pi} \frac{1}{(rr')^{1/2}} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})J_\nu(\tau_{m,\nu}\frac{r'}{a})}{[J_{\nu+1}(\tau_{m,\nu})]^2}$$

$$\times \frac{P_{\nu-\frac{1}{2}}(\cos\theta)P_{\nu-\frac{1}{2}}(\cos\theta')}{M_\nu(\theta_0)} e^{-k^2_{m,\ell}\kappa t} r'^2 \sin\theta' d\varphi' d\theta' dr' \tag{51}$$

Interchanging integration and summation in Equation (51) we get

$$u = \frac{2u_0}{a^2} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_\nu(\tau_{m,\nu}\frac{r}{a})}{r^{1/2} J_{\nu+1}(\tau_{m,\nu})} \int_{r'=0}^{a} r'^{3/2} J_\nu(\tau_{m,\nu}\frac{r'}{a}) dr'$$

$$\times \frac{P_{\nu-\frac{1}{2}}(\cos\theta)}{M_\nu(\theta_0)} \int_{\theta'=0}^{\theta_0} P_{\nu-\frac{1}{2}}(\cos\theta')\sin\theta' d\theta' \int_{\varphi'=0}^{2\pi} d\varphi' e^{-k^2_{m,\ell}\kappa t} \tag{52}$$

For the first integration we use the formula [6, Vol. 2, p. 90]

$$\int z^\mu J_\nu(z) dz = (\mu+\nu-1) z J_\nu(z) S_{\mu-1,\nu-1}(z) - z J_{\nu-1}(z) S_{\mu,\nu}(z) \tag{53}$$

where $S_{\mu,\nu}(z)$ is the Lommel's function [6, Vol. 2, p. 40].

Using formula (53) we find that

$$\int_{r'=0}^{a} r'^{3/2} J_\nu(\tau_{m,\nu}\frac{r'}{a}) dr' = (\frac{a}{\tau_{m,\nu}})^{5/2} \Big[(\nu+\tfrac{1}{2})(\tau_{m,\nu}\frac{r'}{a}) J_\nu(\tau_{m,\nu}\frac{r'}{a})$$

$$\times S_{\frac{1}{2},\nu-1}(\tau_{m,\nu}\frac{r'}{a}) - (\tau_{m,\nu}\frac{r'}{a})$$

$$\times J_{\nu-1}(\tau_{m,\nu}\frac{r'}{a}) S_{\frac{3}{2},\nu}(\tau_{m,\nu}\frac{r'}{a}) \Big]_{r'=0}^{a}$$

$$= -\frac{a^{5/2}}{(\tau_{m,\nu})^{3/2}} J_{\nu-1}(\tau_{m,\nu}) S_{\frac{3}{2},\nu}(\tau_{m,\nu}) \tag{54}$$

For the second integration we will use formula (42). Keeping in mind that $P_0(\cos\theta) = 1$ and $P_n^m(\cos\theta) = 0$, if $m > n$, we let

$\rho = \nu = 0$, $\sigma = \nu - \frac{1}{2}$ and $W_{\nu-\frac{1}{2}}(x) = P_{\nu-\frac{1}{2}}(\cos\theta')$ in that formula.

$$\int_{\theta'=0}^{\theta_0} P_{\nu-\frac{1}{2}}(\cos\theta')\sin\theta' d\theta' = \frac{2}{2\nu+1}\left[\cos\theta' P_{\nu-\frac{1}{2}}(\cos\theta') - P_{\nu-\frac{3}{2}}(\cos\theta')\right]_{\theta'=0}^{\theta_0}$$

$$= -\frac{2P_{\nu-\frac{3}{2}}(\cos\theta_0)}{2\nu+1} \tag{55}$$

Finally

$$\int_{\varphi'=0}^{2\pi} d\varphi' = 2\pi \tag{56}$$

Substitute the value of $M_\nu^n(\theta_0)$ obtained by setting $n = 0$ in Equation (44) and the results of Equations (54), (55), and (56) in Equation (52). We get

$$u = -32\pi u_0\left(\frac{a}{r}\right)^{1/2} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\nu S_{\frac{3}{2},\nu}(\tau_{m,\nu}) J_{\nu-1}(\tau_{m,\nu}) J_\nu(\tau_{m,\nu}\frac{r}{a})}{(\tau_{m,\nu})^{3/2}(4\nu^2-1) J_{\nu+1}(\tau_{m,\nu})}$$

$$\times \frac{P_{\nu-\frac{1}{2}}(\cos\theta)\, e^{-k_{m,\ell}^2 \kappa t}}{\frac{d}{d\nu} P_{\nu-\frac{1}{2}}(\cos\theta_0)}$$

Using the recurrence relation [6, Vol. 2, p. 12]

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu z^{-1} J_\nu(z)$$

we get the formula

$$\frac{J_{\nu-1}(z)}{J_{\nu+1}(z)} + 1 = \frac{2\nu J_\nu(z)}{zJ_{\nu+1}(z)}$$

for $J_{\nu+1}(z) \neq 0$.

Hence

$$\frac{J_{\nu-1}(\tau_{m,\nu})}{J_{\nu+1}(\tau_{m,\nu})} = -1$$

and the temperature in the finite cone due to the constant initial temperature u_0 in time t is

$$u = \frac{32\pi u_0}{a^2 r^{1/2}} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{\nu S_{\frac{3}{2},\nu}(\tau_{m,\nu}) J_\nu(\tau_{m,\nu}\frac{r}{a}) P_{\nu-\frac{1}{2}}(\cos\theta)}{(4\nu^2-1)(\tau_{m,\nu})^{3/2}\frac{d}{d\nu}P_{\nu-\frac{1}{2}}(\cos\theta_0)} e^{-k^2_{m,\ell}\kappa t}$$

where the index of summation ℓ is determined by Equation (38).

This expression represents the equilization of temperature inside the cone when the outside temperature is kept at zero.

"a" Infinite

Let

$$L_\nu(\gamma r, \gamma r') = \sum_{m=1}^{\infty} \frac{J_\nu(\tau_{m,\nu}\frac{r}{a}) J_\nu(\tau_{m,\nu}\frac{r'}{a})}{[J_{\nu+1}(\tau_{m,\nu})]^2[(a\gamma)^2+(\tau_{m,\nu})^2]} \qquad (57)$$

Apply Equation (25) to Equation (57).

$$L_\nu(\gamma r, \gamma r') = \frac{1}{2}\frac{I_\nu(\gamma r')}{I_\nu(\gamma a)}[I_\nu(\gamma a)K_\nu(\gamma r) - K_\nu(\gamma a)I_\nu(\gamma r)], \qquad r > r'$$

with r and r' interchanged when $r < r'$.

Equation (46) may be written in the following form

$$\overline{G}_1(P, Q, \gamma) = -\frac{1}{\pi(rr')^{1/2}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \varepsilon_n \cos n\varphi \cos n\varphi'$$

$$\times \frac{P^n_{\nu-\frac{1}{2}}(\cos\theta)P^n_{\nu-\frac{1}{2}}(\cos\theta')}{M^n_\nu(\theta_0)} L_\nu(\gamma r, \gamma r'), \qquad r > r' \qquad (58)$$

with r and r' interchanged when $r < r'$.

We want to let $a \to \infty$ in Equation (58). Since $L_\nu(\gamma r, \gamma r')$ is the only factor that contains terms involving the quantity a, we want to find the limit of $L_\nu(\gamma r, \gamma r')$ as $a \to \infty$.

From Equation (27) we may conclude that

$$\lim_{a \to \infty} L_\nu(\gamma r, \gamma r') = \frac{1}{2} I_\nu(\gamma r')K_\nu(\gamma r) = \overline{L}_\nu(\gamma r, \gamma r'), \qquad r' < r$$

with r and r' interchanged when $r < r'$.

Using this result in Equation (58) we see that the modified Green's function for the semi-infinite cone is

$$\overline{G}_1(P, Q, \gamma) = -\frac{1}{\pi(rr')^{1/2}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \varepsilon_n \cos n\varphi \cos n\varphi'$$

$$\times \frac{P^n_{\nu-\frac{1}{2}}(\cos\theta) P^n_{\nu-\frac{1}{2}}(\cos\theta')}{M^n_\nu(\theta_0)} \overline{L}_\nu(\gamma r, \gamma r'), \quad r' < r \qquad (59)$$

The only factor that involves γ in Equation (59) is $\overline{L}_\nu(\gamma r, \gamma r')$. Hence, substitution of that equation into Equation (9) would require finding

$$\mathcal{L}^{-1}_\gamma \{\overline{L}_\nu(\gamma^{1/2} r, \gamma^{1/2} r')\}_{t \to \kappa t}$$

From Appendix A, Equation (A. 7), we find that

$$\mathcal{L}^{-1}_\gamma \{\overline{L}_\nu(\gamma^{1/2} r, \gamma^{1/2} r)\} = \frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{4\pi\kappa t(rr')^{1/2}} I_\nu\left(\frac{rr'}{2\kappa t}\right)$$

Therefore, the Green's function due to one source point for the infinite cone is

$$G_1(P, Q, t) = \frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{4\pi\kappa t(rr')^{1/2}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \varepsilon_n \cos n\varphi \cos n\varphi' I_\nu\left(\frac{rr'}{2\kappa t}\right)$$

$$\times \frac{P^n_{\nu-\frac{1}{2}}(\cos\theta) P^n_{\nu-\frac{1}{2}}(\cos\theta')}{M^n_\nu(\theta_0)} \qquad (60)$$

To find the Green's function for the semi-infinite cone with the whole ring acting as a source we must integrate Equation (60) with respect to φ' from $\varphi' = 0$ to $\varphi' = 2\pi$. The only term that contributes is the term with $n = 0$. Hence, the Green's function for the semi-infinite cone is

$$G_1(P, Q, t) = \frac{e^{-\frac{r^2+r'^2}{4\kappa t}}}{2\kappa t(rr')^{1/2}} \sum_{\ell=1}^{\infty} \frac{P_{\nu-\frac{1}{2}}(\cos\theta)P_{\nu-\frac{1}{2}}(\cos\theta')}{M_\nu(\theta_0)} I_\nu\left(\frac{rr'}{2\kappa t}\right) \qquad (61)$$

with the summation index being determined by the roots of Equation (38).

Therefore, the temperature u at time t due to the initial temperature $f(r, \theta, \varphi)$ is

$$u = \int_{r'=0}^{\infty} \int_{\theta'=0}^{\theta_0} \int_{\varphi'=0}^{2\pi} G_1(P, Q, t) f(r', \theta', \varphi') r'^2 \sin\theta' d\varphi' d\theta' dr'$$

where $G_1(P, Q, t)$ is given by Equation (61).

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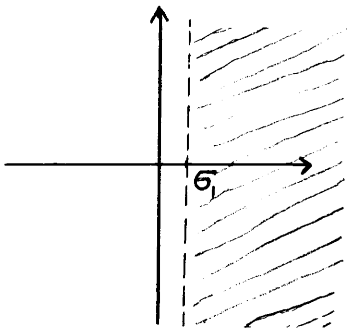
APPENDIX

APPENDIX A

The Laplace transform of $F(t)$ is

$$f(\gamma) = \int_0^{\infty} F(t)e^{-\gamma t}dt, \quad \mathrm{Re}\,\gamma > \sigma_1$$



Half plane of absolute convergence

Then by the Mellin inversion theorem [11, p. 348] we have the following inversion formula, known as the inverse Laplace transform of $f(\gamma)$ with respect to γ

$$\mathcal{L}_{\gamma}^{-1}\{f(\gamma)\} = F(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(\gamma)e^{\gamma t}d\gamma, \quad \mathrm{Re}\,\gamma = c > \sigma_1$$

Consider the following integral

$$\int_0^{\infty} e^{-k_n^2 t} e^{-\gamma t}dt = \lim_{b \to \infty} \int_0^{b} e^{-(k_n^2+\gamma)t} dt$$

which is the Laplace transform of $e^{-k_n^2 t}$. Elementary calculations yield

$$\int_0^\infty e^{-k_n^2 t} e^{-\gamma t} dt = \frac{1}{k_n^2+\gamma}$$

Hence, by the Mellin inversion theorem

$$\mathcal{L}_\gamma^{-1}\left\{\frac{1}{\gamma+k_n^2}\right\} = e^{-k_n^2 t} \tag{A. 1}$$

The second inverse Laplace transform needed was that of $e^{-R\gamma^{1/2}}$. It is derived here using the inversion formula mentioned above.

A well known formula from the theory of Bessel functions [6, Vol. 2, p. 82] is the following

$$K_\nu(az) = \frac{1}{2} a^\nu \int_0^\infty e^{-\frac{z}{2}(t+\frac{a^2}{t})} t^{-\nu-1} dt, \quad \text{Re}z > 0, \quad \text{Re}a^2 z > 0$$

In this formula set $\beta = \frac{a^2 z}{2}$, $\gamma = \frac{z}{2}$. Then

$$K_\nu(2(\beta\gamma)^{1/2}) = \frac{1}{2}\left(\frac{\beta}{\gamma}\right)^{\nu/2} \int_0^\infty e^{-\gamma t} e^{-\beta/t} dt, \quad \text{Re}\gamma > 0, \quad \text{Re}\beta > 0 \tag{A. 2}$$

In Equation (A. 2) let $\nu = \frac{1}{2}$. Then

$$2\left(\frac{\gamma}{\beta}\right)^{1/4} K_{\frac{1}{2}}(2(\beta\gamma)^{1/2}) = \int_0^\infty e^{-\gamma t} e^{-\beta/t} t^{-3/2} dt \qquad (A.3)$$

For $n = 0, 1, 2, \ldots,$ $K_{n+\frac{1}{2}}$ may be expressed as an infinite integral [6, Vol. 2, p. 9]

$$K_{n+\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} \frac{e^{-z}}{n!} \int_0^\infty e^{-t}\left(1+\frac{t}{2z}\right) t^n dt$$

So, letting $n = 0$, we get

$$K_{\frac{1}{2}}(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \int_0^\infty e^{-t} dt = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}$$

Using this result in Equation (A.3) we get

$$\left(\frac{\pi}{\beta}\right)^{1/2} e^{-2(\beta\gamma)^{1/2}} = \int_0^\infty e^{-\gamma t} e^{-\beta/t} t^{-3/2} dt$$

and by the Mellin inversion theorem

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{\pi}{\beta}\right)^{1/2} e^{-2(\beta\gamma)^{1/2}} e^{\gamma t} = e^{-\beta/t} t^{-3/2}$$

Finally, letting $\beta = \frac{R^2}{4}$ we find the inverse Laplace transform of $e^{-R\gamma^{1/2}}$ with respect to γ.

$$\mathcal{L}_{\gamma}^{-1}\{e^{-R\gamma^{1/2}}\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{-\gamma^{1/2}R} e^{\gamma t} dt$$

$$= \frac{Re^{-\frac{R^2}{4t}} t^{3/2}}{2\pi^{1/2}} \qquad (A.4)$$

In Chapter III we also needed the inverse Laplace transform of

$$\overline{L}_{\nu}(\gamma^{1/2}r, \gamma^{1/2}r') = \frac{1}{2} I_{\nu}(\gamma^{1/2}r')K_{\nu}(\gamma^{1/2}r), \quad r' < r,$$

with r and r' interchanged when $r < r'$.

Macdonald's formula says [6, Vol. 2, p. 53]

$$\int_0^{\infty} e^{-\frac{s}{2}} e^{-\frac{x^2+X^2}{2s}} I_{\nu}(xX/s)s^{-1}ds = \begin{cases} 2I_{\nu}(x)K_{\nu}(X) \\ 2K_{\nu}(x)I_{\nu}(X) \end{cases}$$

according as $X > x$ or $X < x$.

Set in this formula $x = 2\alpha\gamma^{1/2}$, $X = 2\beta\gamma^{1/2}$. Then

$$\int_0^{\infty} e^{-\frac{s}{2}} e^{-\frac{2\gamma}{s}(\alpha^2+\beta^2)} I_{\nu}(\frac{4\alpha\beta\gamma}{s})\frac{ds}{s}$$

$$= \begin{cases} 2I_{\nu}(2\alpha\gamma^{1/2})K_{\nu}(2\beta\gamma^{1/2}) \\ 2K_{\nu}(2\alpha\gamma^{1/2})I_{\nu}(2\beta\gamma^{1/2}) \end{cases} \qquad (A.5)$$

according as $\beta > \alpha$ or $\beta < \alpha$.

Finally, in Equation (A. 5) let $s = 2t\gamma$, t being the new parameter.

$$\int_0^\infty e^{-\gamma t} e^{-\frac{\alpha^2+\beta^2}{t}} I_\nu\left(\frac{2\alpha\beta}{t}\right) \frac{dt}{t}$$

$$= \begin{cases} 2I_\nu(2\alpha\gamma^{1/2})K_\nu(2\beta\gamma^{1/2}) & \beta > \alpha \\ 2K_\nu(2\alpha\gamma^{1/2})I_\nu(2\beta\gamma^{1/2}) & \alpha > \beta \end{cases}$$

Hence, by the Mellin inversion theorem we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} I_\nu(2\alpha\gamma^{1/2})K_\nu(2\beta\gamma^{1/2})e^{\gamma t} dt = \frac{1}{2} e^{-\frac{\alpha^2+\beta^2}{t}} I_\nu\left(\frac{2\alpha\beta}{t}\right)t^{-1} \tag{A. 6}$$

regardless of the relation between α and β.

Therefore, if we let $\alpha = \frac{r'}{2}$ and $\beta = \frac{r}{2}$ in Equation (A.6) we get

$$\mathcal{L}_\gamma^{-1}\{\overline{L}_\nu(\gamma^{1/2}r, \gamma^{1/2}r')\} = \frac{1}{4} e^{-\frac{r^2+r'^2}{4t}} I_\nu\left(\frac{rr'}{2t}\right)t^{-1} \tag{A. 7}$$

APPENDIX B

As stated in the Introduction the method used in this thesis does not work very well in the adiabatic case, especially for the semi-infinite cone. In this appendix we see why this is so.

The adiabatic case is similar with the isothermic case, except the boundary condition (33) now becomes

$$\frac{\partial u}{\partial n} = \begin{cases} 0 \text{ at } r = a, & 0 \leq \theta \leq \theta_0, \quad 0 \leq \varphi \leq 2\pi \\ 0 \text{ at } \theta = \theta_0, & 0 \leq r \leq a, \quad 0 \leq \varphi \leq 2\pi \end{cases} \tag{B. 1}$$

Taking the partial derivative with respect to r in Equation (36) and imposing the first of the boundary conditions (B. 1) we see that we must have

$$2akJ_\nu'(ka) - J_\nu(ka) = 0, \quad \nu > -\tfrac{1}{2}$$

Let $\bar{\tau}_{m,\nu}$ be the mth root of

$$2xJ_\nu'(x) - J_\nu(x) = 0, \quad \nu > -\tfrac{1}{2} \tag{B. 2}$$

Then the eigenvalues in this case are

$$\bar{k}_{m,\nu} = \frac{\bar{\tau}_{m,\nu}}{a}$$

It has been proven [10, p. 482] that the zeros of Equation (B. 2) are all real and positive, if $\nu > -1$, except for two that are pure imaginary. The eigenvalues $\bar{k}_{m,\nu}$ are real. Hence, $\bar{\tau}_{m,\nu}$ must be the mth real, positive root of Equation (B. 2). The disadvantage of the method of eigenvalues and eigenfunctions is with Equation (B. 2), for its roots are hard to locate.

Now take the partial derivative of Equation (36) with respect to θ and impose the second of the boundary conditions (B. 1). This implies that

$$\frac{d}{d\theta} P^n_{\nu-\frac{1}{2}}(\cos\theta)\Big|^{\theta=\theta_0} = 0 \qquad \text{(B. 3)}$$

Equation (B. 3) is a function of ν. Let $\bar{a}_{n,\ell}$ be the ℓth root of that equation. Then the eigenvalues of the second boundary value problem are

$$\bar{k}_{m,n,\ell} = \frac{\bar{\tau}_{m,\nu}}{a}, \quad \nu = \bar{a}_{n,\ell}, \quad n = 0, 1, 2, \ldots, \quad m, \ell = 1, 2, 3, \ldots$$

with $\bar{a}_{m,\ell}$ being the root of Equation (B. 3).

Substitution of these eigenvalues in Equation (36) gives the corresponding non-normalized eigenfunctions for the adiabatic case of our problem.

$$u_{m,n,\ell} = \frac{\cos n\varphi}{r^{1/2}} J_\nu(\bar{\tau}_{m,\nu} \frac{r}{a}) P^n_{\nu-\frac{1}{2}}(\cos\theta)$$

The normalizing factor, N, is given by

$$N^2 = \int_{r=0}^{a} \int_{\theta=0}^{\theta_0} \int_{\varphi=0}^{2\pi} [\frac{1}{r^{1/2}} J_\nu(\bar{\tau}_{m,\nu} \frac{r}{a})]^2 [P^n_{\nu-\frac{1}{2}}(\cos\theta)]^2$$

$$\times \cos^2 n\varphi r^2 \sin\theta d\varphi d\theta dr$$

This triple integral can be evaluated in the same way as the integral of Equation (39). Thus we find that

$$N^2 = \frac{\pi}{\varepsilon_n} A_\nu(\bar{\tau}_{m,\nu}) M^n_\nu(\theta_0)$$

where

$$A_\nu(\bar{\tau}_{m,\nu}) = a^2\{[J'_\nu(\bar{\tau}_{m,\nu})]^2 + [1-(\frac{\nu}{\bar{\tau}_{m,\nu}})^2][J_\nu(\bar{\tau}_{m,\nu})]^2\}$$

and so the normalized eigenfunctions for the adiabatic case are

$$U_{m,n,\ell} = \frac{1}{N} u_{m,n,\ell} = (\frac{\varepsilon_n}{\pi r})^{1/2} \frac{J_\nu(\bar{\tau}_{m,\nu} \frac{r}{a}) P^n_{\nu-\frac{1}{2}}(\cos\theta)}{[A_\nu(\bar{\tau}_{m,\nu})]^{1/2} [M^n_\nu(\theta_0)]^{1/2}} \cos n\varphi \tag{B. 4}$$

where $\nu = \bar{a}_{n,\ell}$, $n = 0, 1, 2, \ldots,$ $m, \ell = 1, 2, 3, \ldots,$ $\bar{a}_{n,\ell}$ and $M^n_\nu(\theta_0)$ are determined by Equations (B. 3) and (44) respectively.

Now we are ready to find the second Green's function for our conical structure. The case of the finite cone is similar to the corresponding case with the isothermic boundary condition. The second

Green's function for the finite cone is

$$G_2(P, Q, t) = \frac{2}{(rr')^{1/2}} \sum_{m=1}^{\infty} \sum_{\ell=1}^{\infty} \frac{J_\nu(\overline{\tau}_{m,\nu} \frac{r}{a}) J_\nu(\overline{\tau}_{m,\nu} \frac{r'}{a})}{A_\nu(\overline{\tau}_{m,\nu})}$$

$$\times \frac{P_{\nu-\frac{1}{2}}(\cos\theta) P_{\nu-\frac{1}{2}}(\cos\theta') e^{-\overline{k}^2_{m,\ell} \kappa t}}{M_\nu(\theta_0)}$$

where the index ℓ is determined by Equation (B. 3).

But if we try to find the second Green's function for the semi-infinite cone as a special case of the finite cone, we get an infinite series of the Dini kind [6, Vol. 2, p. 71] whose sum is not known. We would have to find the limit as $a \to \infty$ of

$$\overline{G}_2(P, Q, \gamma) = -\frac{1}{\pi(rr')^{1/2}} \sum_{n=0}^{\infty} \sum_{\ell=1}^{\infty} \frac{\varepsilon_n P^n_{\nu-\frac{1}{2}}(\cos\theta) P^n_{\nu-\frac{1}{2}}(\cos\theta')}{M^n_\nu(\theta_0)}$$

$$\times \cos n\varphi \cos n\varphi' E_\nu(\gamma r, \gamma r')$$

where

$$E_\nu(\gamma r, \gamma r') = \sum_{m=1}^{\infty} \frac{J_\nu(\overline{\tau}_{m,\nu} \frac{r}{a}) J_\nu(\overline{\tau}_{m,\nu} \frac{r'}{a})}{A_\nu(\overline{\tau}_{m,\nu})[\gamma^2 + (\frac{\overline{\tau}_{m,\nu}}{a})^2]} \qquad \text{(B. 5)}$$

The Dini series is Equation (B. 5) and having an unknown sum its behavior as $a \to \infty$ is hard to determine.

Concluding, we may remark that this case may be done by use of the Laplace transformation and contour integration.