UTILIZATION OF THE ELECTROLYTIC TANK IN ENGINEERING ANALYSIS AND SYNTHESIS

by

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I. INTRODUCTION

The analog computer has been rapidly gaining favor as a convenient approach to the solution of many types of engineering problems. In certain types of problems assumptions can be made that greatly reduce the complexity of the problem so that it will yield readily to a straightforward mathematical solution. Often, however, the problem is of such a nature that standard mathematical techniques are effective only for elementary situations and prove inadequate for practical engineering problems. An example is the field problem, which can be readily solved for simple geometrical configurations such as cylinders and spheres, but for most shapes the solution is available only as an infinite series of Bessel functions, Hankle functions, etc., which are quite difficult to interpret physically. The use of digital computers for the numerical evaluation of mathematical solutions has done much for this method.

Graphical methods can often be used to give reasonable results; however, each solution is independent of others and for problems of synthesis the time involved often becomes prohibitive.

Of the many types of analogs, the electric analogs are by far the most popular because of the wide variety of problems that can be solved. The various other analogs;

mechanical, thermal, rubber membranes, soap films and fluid mappers, are usually special-purpose devices useful only for very limited types of problems.

There are several types of electric analogs; electronic analogs, resistance and reactance analogs, conductive-solids analogs, and conductive liquids or the electrolytic analog. The latter one is the subject of this thesis and lends itself particularly well to problems governed by Laplace's and Poisson's equations, which include problems of electrostatics, electron ballistics, electromagnetic waves, fields in electric conductors, magnetic fields, elasticity, heat transfer and fluid dynamics with viscous and non-viscous flow (12, p. 405-426).

Discovery of the principles involved in the electrolytic analogy are attributed to Kirchhoff. In 1845 he expressed the original idea of tracing equipotential lines (13). The analogy between current flow and the functions of a complex variable was first noted by Maxwell in 1874 and published in 1881 (15, article 647-651). The first practical application seems to have been in 1875 by Adams (1) and the next recorded use in 1913 by Fortescue (6). Since about 1950 the utilization of the electrolytic tank has increased very rapidly as evidenced by the literature.

The purpose of this thesis is to derive the mathematical analogies which exist and show some results utilizing these analogies. A description of the equipment used is accompanied by general design considerations and discussion of sources of error.

II. THE ELECTROLYTIC ANALOG

An analogy is said to exist when there is a direct correspondence between the elements of two systems such that the responsesof the two are similar. A simple example of an analogy of this type is that of a scale model, wherein the actual device is reduced or enlarged in direct proportion to the original. A different type of analogy can be made between two completely different types of systems when the characteristic equations of the systems are similar. Both of these types of analogies are used to establish the electrolytic analog.

1. Analogy for Fields

A fundamental equation that applies to the major fields of engineering and physics is Laplace's equation. This equation is stated simply as:

$$\nabla^2 \phi = 0 \qquad (1)$$

where ϕ is the variable; gravitational potential, electrostatic potential or whatever is determined by the problem, and ∇^2 is the Laplacian operator which signifies "the sum of the second derivatives with respect to all cartesian space variables of interest" (12, p. 27). If the electrolytic tank exhibits this same characteristic equation then the mathematical analogy exists.

For simplicity, consider a two-dimensional situation in which a shallow horizontal rectangular tank contains a solution with uniform resistivity in the x-y plane. The bottom and sides of the tank are an insulating material. Assume also that the electrolyte is excited by applying constant voltages to conducting metal strips placed in the electrolyte. The incremental voltage drop across each side of a differential element is equal to the net current flowing across the side, times a resistance per square which is a function of the electrolyte conductivity and depth, times the differential length of the side and divided by the differential width through which the current flows. For a current flowing in the x direction this could be written:

$$\Delta v = -iR \frac{\Delta x}{\Delta y} \qquad (2)$$

For a complete differential element with currents assumed flowing into each side of the element the expressions for the gradients at each of the four sides appear as:

$$\left(\frac{\partial v}{\partial x}\right)_1 = \frac{-i_1 R}{\Delta y} \qquad (3)$$

$$\left(\frac{\partial v}{\partial x}\right)_3 = \frac{-i_3 R}{\Delta y} \qquad (4)$$

$$\left(\frac{\partial v}{\partial y}\right)_2 = \frac{-i_2R}{\Delta x} \tag{5}$$

$$\left(\frac{\partial v}{\partial y}\right)_4 = \frac{-i_4R}{\Delta x} \tag{6}$$

where partial derivatives are used because more than one independent variable is involved and the subscripts 1, 2, 3 and 4 refer to the left, top, right, and bottom sides of the differential element respectively.

The rate of change of gradients can be found by adding the gradients on opposite sides and dividing by the differential length between them:

$$\frac{\partial^2 v}{\partial x^2} = \frac{\frac{-i_1R}{\Delta y} - \frac{i_3R}{\Delta y}}{\Delta x} \quad \text{as } \Delta x \rightarrow 0 \tag{7}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\frac{-i_2R}{\Delta x} - \frac{i_4R}{\Delta x}}{\Delta y} \quad \text{as } \Delta y \rightarrow 0 \tag{8}$$

Addition of these two equations gives:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\frac{R}{\Delta x \Delta y}\,(i_1+i_3+i_2+i_4) \tag{9}$$

Since the sum of the currents flowing into the differential element must equal zero, equation (9) becomes:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{10}$$

It can be seen that this is Laplace's equation and with specified boundary conditions will completely describe the potential in the tank. A similar analysis in one or three dimensions will again yield Laplace's equation. It is this mathematical similarity that gives the electrolytic tank its usefulness as an analog for field problems.

2. Analogy for Complex Variables

In the field of control there is another characteristic equation that the electrolytic tank can duplicate. The transfer function of a linear lumped-parameter feedback system can be expressed as a ratio of two polynomials in s, the complex variable $\sigma + jw$. In its factored form the transfer function is the ratio of the zero factors to pole factors times a constant. The natural log of the transfer function can be written in general terms as:

$$\ln G(s) = \ln \frac{K\ (1+ST_a)(1+ST_b)\ldots(1+ST_z)}{S^x(1+ST_1)(1+ST_2)\ldots(1+ST_r)} \qquad (11)$$

where K is the zero frequency gain and G has a multiple pole of order x at the origin. Any of the T's may be equal to give multiple poles or zeros at other positions. Equation (11) can also be written:

$$\ln G(s) = \ln K_2 \frac{\pi(s - s_z)}{s^x(s - s_n)} \qquad (12)$$

K_2 is obtained from K by multiplying by the distances of the poles to the origin, excluding those at the origin, and dividing by the distances of the zeros. Since the transfer function is a complex number, equation (12) can be written:

$$\ln G(s) = \ln |G| + j \arg G \tag{13}$$

$$= \ln K_2 + \Sigma \ln |S-S_z| - x \ln |S| - \Sigma \ln |S-S_n| + j\,(\Sigma \arg(s-s_z) - x \arg s - \Sigma \arg(s-s_n)) \tag{14}$$

It will now be convenient to separate this expression and consider first the magnitude alone, which can be written as:

$$\ln \left|\frac{G}{K_2}\right| = \ln |S-S_z| - x \ln |S| - \ln |S-S_n| \tag{15}$$

To see the mathematical analogy of this to the electrolytic tank it is again necessary to consider a shallow horizontal tank containing a uniform conducting solution. If a current I is flowing from a small vertical conductor in the center of the tank to a collecting ring around the edge of the tank the current in the horizontal angle $d\theta$ is:

$$di = I d\theta / 2\pi \tag{16}$$

The voltage drop across a square area element with sides equal to $r d\theta$ and dr, where r is the distance to the current source, is:

$$dv = \rho\, di = \rho I d\theta / 2\pi \tag{17}$$

where ρ is the resistivity of the electrolyte per square. Since for a square element $d\theta = dr/r$, equation (17) becomes:

$$dv = \rho I dr/2\pi r \qquad (18)$$

The potential difference between r and the collector ring is the integral of equation (18).

$$V_r = \int_r^{r_c} \rho I dr/r = \rho I(\ln r_c - \ln r)/2\pi \qquad (19)$$

If a number of current sources were added in different places the resultant potential difference would be the result of superposition of the effects of all of the sources. Let 1, 2, ...n represent positive current sources and a, b, ...z represent negative current sources of equal magnitudes. The potential difference at point r could be written:

$$(2\pi/\rho I)V_r = (n-z)\ln|r_c| - \ln|r_1| - \ln|r_2| - \dots \ln|r_n| + \ln|r_a| + \ln|r_b| + \dots \ln|r_z| \qquad (20)$$

A comparison of this and equation (15) will show the mathematical similarity, if the electrodes carrying positive currents are placed at pole locations and negative current sources placed at zero locations. For the case of multiple poles or zeros, the current to the point can be increased by the order of the multiplicity. In the electrolytic tank points of equal potential represent contours of constant magnitude and these can be plotted in the normal manner of tracing equipotential lines (see section III-6).

It can also be shown (4, p. 165-166) that phase contours are analogous to current flow lines, which can then be treated in the same way as field lines, and are everywhere orthogonal to the equipotential lines.

3. Scaling

Once the mathematical similarity has been established it is then necessary to scale the various elements of the analog to those of the actual problem. In the simpler types of problems this is quite straightforward. For example, in a problem where it is desired to find the voltage gradients that exist in a certain piece of apparatus, it is only necessary to make a scale model for the tank and then trace equipotential lines as a percent of the total applied voltage. This allows a rapid correlation to any applied voltage on the actual equipment. In obtaining a plot of field lines, or current flow lines in a problem of heat or fluid flow, use must be made of certain known initial conditions. For example, lines of equal significance leave conductors at evenly spaced points.

For the case of electron trajectories, scaling becomes more difficult. As always, it is necessary to build a scale model to be placed in the electrolyte. The equations of the solution are:

$$x = x_o + \int [v_o + \frac{e}{m} \int \mathcal{E}_x dt] \, dt \tag{21}$$

where x_o is the initial position,

v_o is the initial velocity,

e/m is the charge to mass ratio,

$\mathcal{E}_x$ is the gradient in the x direction.

The equation for y is of the same form. The transfer function of the analog has this same form also; however, the channel gain, time constants and conversion factors to convert the output voltage to inches on the plotting board are all included. If equation (21) is solved in terms of the actual problem and the corresponding analog equation written, by equating the various constants, the values of voltages for the various initial conditions can be readily found.

An alternate way of determining the initial velocities to be set in is as follows: When, as in an electron lens problem, the electron moves from a region of voltage V_1 to a region essentially V_2 as measured from the point at which the electron is at rest, it is the ratio of V_2 to V_1 that determines the trajectory. If the magnitudes of both are increased by the same factor, the trajectory will remain the same although the velocity will be increased. The energy relations at these two voltages are:

$$V_1 e = \tfrac{1}{2} m \, (\dot{X}_1^2 + \dot{Y}_1^2) \tag{22}$$

and

$$V_2 e = \tfrac{1}{2} m \, (\dot{X}_2^2 + \dot{Y}_2^2) \tag{23}$$

The ratio of these two equations shows that

$$V_2/V_1 = (\dot{X}_1^2 + \dot{Y}_1^2)/(\dot{X}_2^2 + \dot{Y}_2^2) \tag{24}$$

Furthermore, for a fixed value of V_1 and a fixed ratio V_2/V_1 the change in energy of the electron between V_1 and V_2 is constant, or

$$(\dot{X}_2^2 + \dot{Y}_2^2) - (\dot{X}_1^2 + \dot{Y}_1^2) = K \tag{25}$$

Another known relation is $\dot{Y}_1/\dot{X}_1$, which is determined by the initial direction of the electron motion. By making an initial run with any arbitrary initial velocities, the value of K in equation (25) can be determined by observing the voltages proportional to velocity when the probe moves in the vicinity of V_2. The initial setting desired can be calculated from equations (24), (25) and the initial direction and gives for $\dot{X}_1$,

$$\dot{X}_1 = \left(\frac{K}{(\frac{V_2}{V_1} - 1)(1 + m)} \right)^{1/2} \tag{26}$$

where $m = \dot{Y}_1/\dot{X}_1$

The initial velocity of the device can be checked in terms of applied voltages by setting in the velocity and placing the probe array in a rectangular tank set up to have a 2-dimensional retarding electric field equal to twice the equivalent voltage of the initial velocity. The probe should then move exactly half the distance between the electrodes before reversing its direction of motion.

For plotting magnitude and phase loci on the s-plane, the initial calibration must be obtained from known points on the plane. For example, in order to plot magnitude contours, which correspond to equipotential lines in the analogy, the magnitude must be known at some point in the plane. The probe will automatically trace out the contour if it is set at the known point. In order to trace another magnitude contour with, for example, twice the magnitude of the known point, the probe could be moved to a point where it would read twice the potential of the calibration point and this would then correspond to the new magnitude position.

Phase contours must each be found separately by knowing one point of the desired locus. Although this is not too convenient there are a number of short cuts in evaluating the position of the root locus, or the 180° phase line. One point easily found is the value at which the locus crosses the imaginary axis. Also, the asymptotes are easily obtained. For details see Truxal (23, p. 221-277).

III. EQUIPMENT

1. The Electrolytic Tank

The size and shape of the electrolytic tank depend on several factors. It must be large enough so that models of a convenient size can be used to secure the desired accuracy. For instance, if the nominal dimensions of the model were to be 10 inches, the construction and placing of the model should be carried out to better than 0.10 inches if 1% maximum error is to be achieved. Since this is not difficult in most cases the model size could be reduced or the accuracy would be increased.

The tank must be large enough so that the edges of the tank do not adversely affect the problem. Often the problem can be set up entirely with either conducting or insulating boundaries, thereby eliminating these edge effects. In cases where an infinite conducting layer is required, such as for use with complex variables, a special double layer tank can be constructed to approximate this condition.

The principle involved in the double layer electrolytic tank is as follows. The upper layer of a circular tank is connected at the periphery to the lower part of the tank. A single conductor at the center of the lower tank acts as infinity and it can be shown that the lower part electrically terminates the upper part so that it appears as part of an infinite conducting sheet (4). A current introduced at any point on this plane will then flow out equally in all directions through the electrolyte and out the electrode corresponding to infinity.

Another factor to be considered is one of temperature rise in the tank due to current flow. A large tank will tend to stabilize rapid room temperature variations as well. Some electrolytic tanks incorporate a control system to maintain constant temperature (2).

Although some tanks are as large as several feet in length, width and depth (10), the tank used here is 18 inches by 22 inches by 3 inches deep. The apparatus will accommodate tanks with depths of up to 8 inches. This tank size will easily accommodate models with nominal dimensions of from one to six inches, requiring placement within ±0.01 inches to be within ±1% error. Another tank with a double layer is also used and this is 16.5 inches in diameter and 2.5 inches deep. The upper layer is connected at the periphery by 0.25 inches of electrolyte. The depth of both layers is 0.5 inches and the separator is 0.125 inches. An edge view of this tank is shown in Figure 1. Provision is made for leveling the tanks and also for tilting the rectangular tank for problems having cylindrical symmetry.

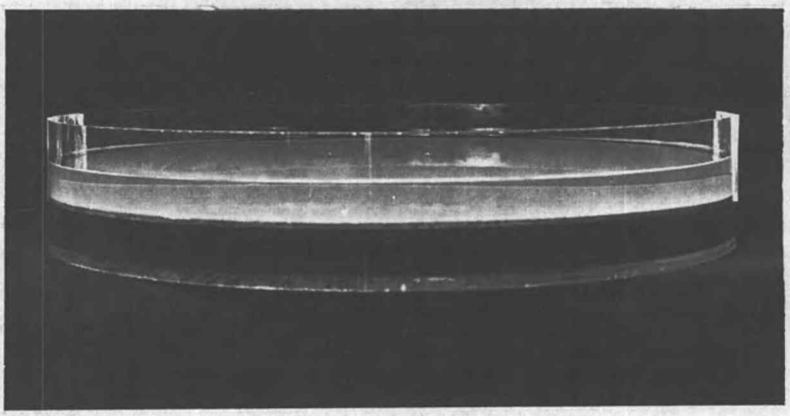

FIGURE 1
EDGE VIEW OF DOUBLE LAYER TANK

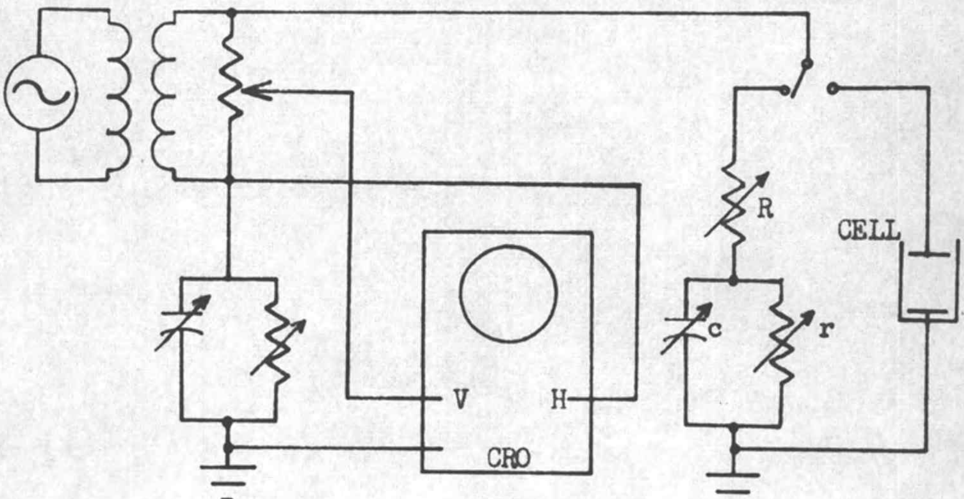

FIGURE 2
SUBSTITUTION BRIDGE USED FOR
DETERMINATION OF POLARIZATION EFFECTS

2. Electrodes and Electrolyte

The mathematical analogies given in section II assume a purely resistive conducting medium. Considerable difficulty is encountered in trying to achieve this with an electrolytic tank because of a phenomenon known as the polarization effect (5). Although the electrolyte by itself appears resistive, the junction of the electrodes and electrolyte constitute an impedance that is capacitive. An equivalent circuit of the combination is a resistance representing the electrolyte resistance in series with a parallel combination of resistance and capacitance representing the contact or polarization effect.

In order to fulfil the requirements of being pure resistance, an electrolyte-electrode combination must be used such that the polarization impedance is very small compared to the resistance of the electrolyte. A figure of merit, F, as defined by Einstein (5) is the resistance per cubic centimeter of the electrolyte divided by the surface impedance of one square centimeter of electrode surface. This has the dimensions of 1/cm and the reciprocal of F is the maximum apparent shrinkage of the electrodes, or the equivalent extension of the length of the electrolyte required to give the observed impedance.

A small cell was constructed to study different electrode-electrolyte combinations. A polystyrene tube was formed to hold a fixed electrode at one end and a second electrode that could be moved a known amount by means of a lead screw. The cell was filled with electrolyte and the electrodes placed in contact. By means of the calibrated screw the separation of the electrodes could then be set to any desired spacing up to 1 inch.

By means of a substitution bridge (Figure 2), the components of the equivalent circuit of the cell were determined. Although many combinations of R, r, and c would balance the bridge, the correct setting was found by doubling the electrode spacing and again balancing the bridge. If the polarization impedance values did not change appreciably and if the electrolyte equivalent resistance doubled, then the circuit was equivalent to the cell. These values were arrived at by a trial and error procedure.

The frequency of operation of the tank is another important factor in minimizing the errors due to polarization effects. Operation at d-c is not feasible because the gasses generated by electrolysis continually change the resistivity of the solution. At low frequencies the effects of polarization are much more pronounced than at higher frequencies, as shown in Figure 3. At much higher frequencies, although effects of polarization are reduced, stray capacitances in the system make these frequencies more difficult to deal with. Frequencies commonly used range from about 400 cycles per second to about 1500 cps.

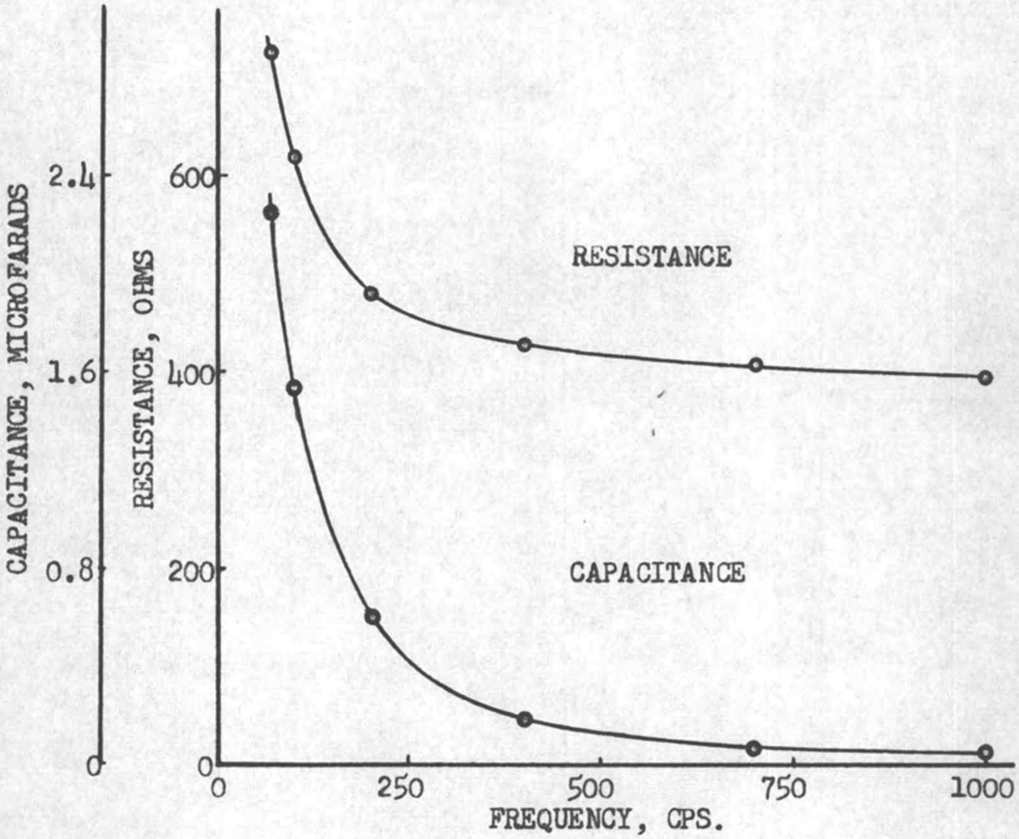

FIGURE 3

RESISTANCE AND CAPACITANCE COMPONENTS OF POLARIZATION FOR DISTILLED WATER AND POLISHED BRASS ELECTRODES

After comparing the relative figures of merit of brass and copper electrodes with tap and distilled water it was found that polished brass and distilled water gave very good results with an F of $100/_{cm}$. Cleaning the brass with hydrochloric acid raised F to over $300/_{cm}$. The poorest combination was tap water and copper, with F equal to $7/_{cm}$. These results were obtained at 400 cps. The choice of frequency can be justified by observing Figure 3, which was obtained for etched and polished brass and distilled water. As frequency is increased above 400 cps very little advantage in polarization effects is obtained. With the parameters chosen to give a figure of merit over $300/_{cm}$, the maximum apparent shrinkage of the electrodes is less than 1.3 mils.

3. Probes

The probes necessary to measure potentials and gradients in the electrolytic tank have been given many shapes and sizes by numerous authors. Of the many types, the four pin probe seems to combine a minimum of error due to the finite spacing of the probes in the x and y directions with a maximum of versatility. The absolute potential of the point in question can be closely approximated by the average potential of the four probes. The x and y gradients are also independently available.

In order to measure the true gradients it would be necessary to have probes of infinitesimal diameters spaced infinitesimally close together. The nearest approach to this is a compromise. The spacing of the probes must be consistent with the voltage amplitude available to the models as well as the average spacing of the electrodes. That is, the gradient in the tank must be such that when multiplied by the spacing of the probes, a voltage results that can be dealt with conveniently. Another consideration is that the probes must be rigid in order to maintain proper positioning with use and also present enough surface area to the electrolyte so that the effects of polarization are not too severe.

The probe array used with this system consists of four #20 brass wires (32 mils in diameter) mounted to form the corners of a square. Although the actual spacing of the diagonal pairs is slightly larger, the apparent electrically determined spacing is about 40 mils. This was obtained by calibrating and balancing the probe preamplifiers and then placing the probe array in a uniform field with a known gradient, and observing the voltage difference between the pairs of probes.

The probe system is one of the largest potential sources of error. For one thing, the mere presence of the probes in the electrolytic model must necessarily distort the field in the vicinity of the probes. Although an exact determination of the effects of the presence of the probe array is quite difficult, an approximation of the perturbation has been made and found to be negligible (18). Examination of the finite spacing has also shown this error to be small (22). Many types of probes have been tried to minimize errors. These include liquid probes (19) and probes mounted at the tank bottom (20). It may be that proper choice of the probe diameter would lead to a minimum distortion of the field. If the radius of the probe were made equal to the equivalent shrinkage of the electrolyte-electrode combination (1/F, see section III-2), the equivalent diameter of the probe would approach zero. The effects remaining to distort the field would be that of the polarization capacitance and any current that flows out the probes. The latter effect is minimized by using a cathode follower to match each pin of the probe to a preamplifier.

One major difficulty is associated with the depth of penetration of the probe. Due to the meniscus, the surface of the electrolyte can be dented and extended, and both of these effects greatly change the gradient measured by the probes (See section IV-1 for details).

It was found to be rather difficult to reproduce readings of the gradient obtained from a known uniform field. This was apparently because each time the probe array was raised and lowered into the electrolyte a larger part of the probes had become wetted. This meant that each time a different penetration was required to minimize the meniscus, resulting in different readings. An attempt was made to eliminate this wetting action by coating the sides of the wires with a thin layer of paraffin, leaving only the lower end exposed to the electrolyte. This permits the array to be set successively by eye with less than 1% error. However, it is then necessary that the traverse mechanism be properly leveled so that as the probe is moved about, the depth of penetration does not change appreciably.

4. Servo System

The servo system, although not a necessary part of an electrolytic analog, can reduce hours of laborious plotting by hand to a few minutes of actual running time. With an automatic electrolytic analog the major effort in solving problems is delegated to the construction of models, scaling and perhaps a numerical check on the answer, with little regard for the solution time on the analog. Early work on automatic features was done to obtain electron trajectories by Gabor and Langmuir in 1937 (7, 14) and have since been followed by many others (3, 8, 9, 16, 18, 22). Although many types of control systems are applicable to the electrolytic analog, this discussion will be limited to details of the servo employed here.

The gradients measured by the probes are the inputs used by the electronic analog computer (See Section III-6) to arrive at a solution for the problem being solved. This solution is in terms of voltages proportional to the x and y coordinates of the probe position. It is then up to the servo to position the probe at this point. A separate and identical control system is used for each coordinate.

The output of the electronic analog computer is a voltage between zero and plus eighty volts d-c. In series with this is a voltage selected from a ten-turn potentiometer with eighty volts impressed across the extremes. By proper positioning of the center tap, the position voltage can be exactly cancelled. If it is not cancelled, the error voltage is amplified and used to control a d-c motor that drives the sliding arm of the potentiometer in a direction to make the error zero. The same drive chain controlling the sliding arm of the potentiometer is used to position the probe carriage. The conversion factor between voltage and position of the probe and pen carriage is approximately 2.5 volts per inch. Considering the line width of the pen and mechanical errors the coordinates of the locus of the pen cannot be read closer than about 0.01 inches. To fully utilize this precision the electronics would have to be zeroed and accurate to within ±25 millivolts, which would be extremely difficult to achieve.

As an attempt to minimize the errors due to static friction, the d-c control motor drives one side of a differential and a synchronous motor drives the other side. The output, which is the difference between the two inputs, is used to position the probe carriage. This requires that the d-c motor be turning at the same rate as the a-c motor for the carriage to be at rest. Therefore, to move the probe it is only necessary to speed up or slow down the d-c motor.

The servo is a second order system; that is, one having zero positional error and zero velocity error in the steady state. An experimental investigation showed zero detectable acceleration error for a step of velocity equal to over half of the maximum capability of the system. For larger steps of velocity a small acceleration error was noticeable. However, due to the fact that most problems will have smoothly changing velocities, and can always be scaled down in time, it is believed that the servo is entirely adequate for this system.

5. Integrators

Some form of differential analyzer is a necessary part of an automatic device to solve Laplace's equation. The early automatic trajectory tracers used mechanical differential analyzers. More common recently are the electronic analog integrators or numerical integration performed by digital computers. The mechanical devices were not very satisfactory as the time required was considerable and errors were excessive. The other extreme, digital integrators, can be made as accurate as desired. However, the analog-to-digital converter along with the digital computer comprises considerable equipment. One existing unit is also quite time consuming, taking from 5 to 20 minutes for a single trajectory (17).

This system employs electronic analog integrators with d-c gain of several thousand and time constants of 30 seconds. The first integrators in both x and y channels are linear within 1% from -80 to +80 volts. The second integrators have a range of zero to 80 volts. Initial conditions are supplied to both integrators from the console. The initial voltages on the first integrators correspond to the initial velocity when connected for electron trajectories. The initial conditions on the second integrators set the initial position in x and y coordinates. Relays are used to switch from initial conditions to the problem when the problem is started.

The use of chopper stabilization would be a worthwhile improvement on these integrators as the present ones are plagued with drift, and it is probable that a large part of the system error comes from this source. With the other system constants these integrators permit a maximum plotting speed of 1 inch per second.

6. System Details

The arrangement of the electrolytic tank and associated equipment for use in solving electron trajectories is shown in the block diagram, Figure 4. The operation of the computer can be described as follows.

A 400cps oscillator generates a sine wave with very little distortion. This is then amplified and will supply about ten watts to the console where six controls select taps on the output transformer to provide voltages from zero to 100 volts in one volt steps for use in setting up the problem in the electrolyte. These controls are shown in Figure 5 at the left center of the photograph.

An x-y traverse mechanism moves the probe array in the electrolyte and a pen on the plotting board. Each of the probes feeds a preamplifier, shown at the top of Figure 5, which presents a high impedance to the probe and a low impedance to the subsequent circuitry. Also in the preamplifier circuitry the average voltage of all the probes is obtained and fed to the ground amplifier. The function of this amplifier is to maintain the average potential of the probes at ground potential. This is accomplished by amplifying the average level and applying it out of phase to the secondary of the 400cps output transformer. This allows the probes to operate at low a-c voltages to ground and therefore minimizes the effects of stray capacitances

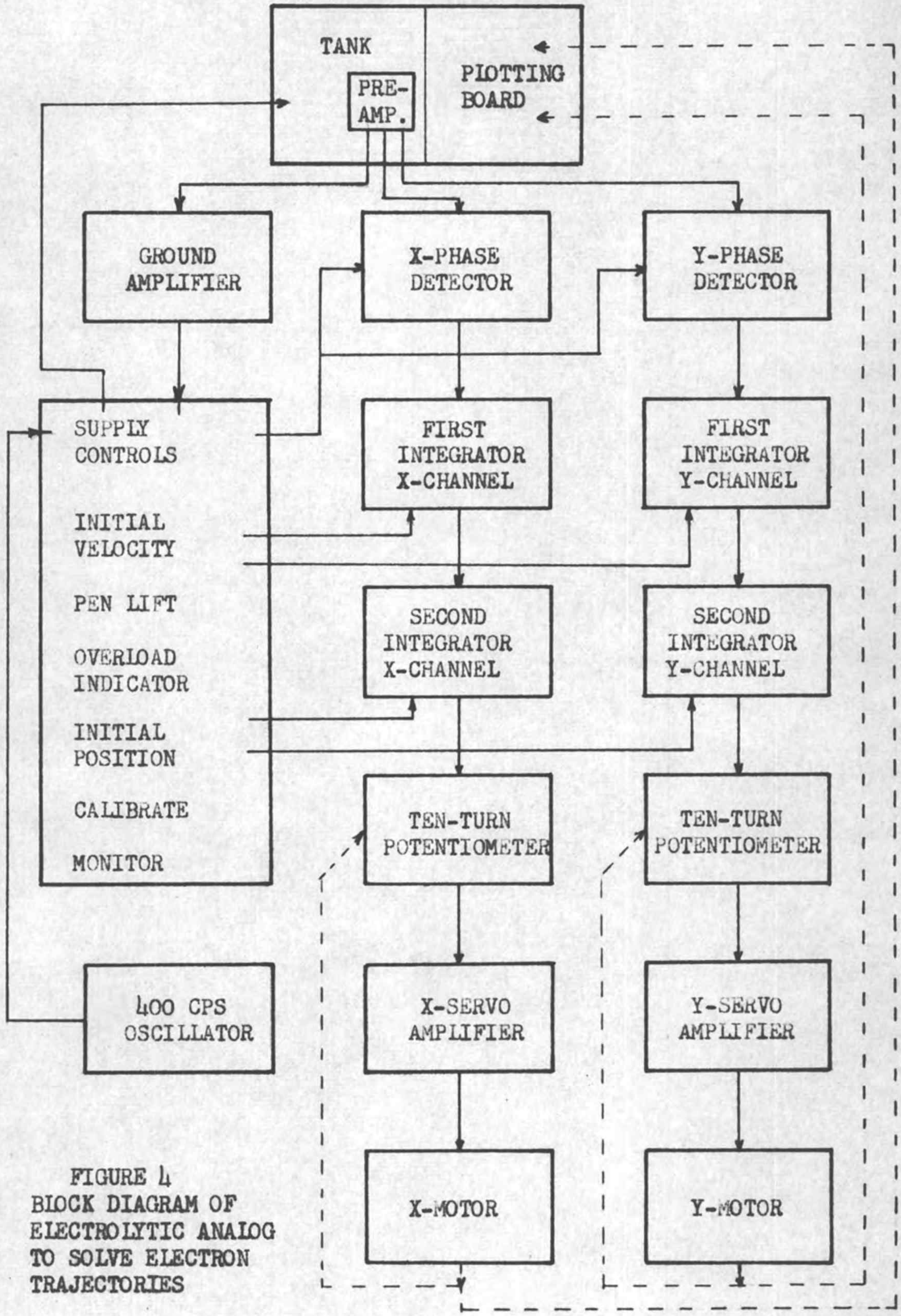

FIGURE 4
BLOCK DIAGRAM OF ELECTROLYTIC ANALOG TO SOLVE ELECTRON TRAJECTORIES

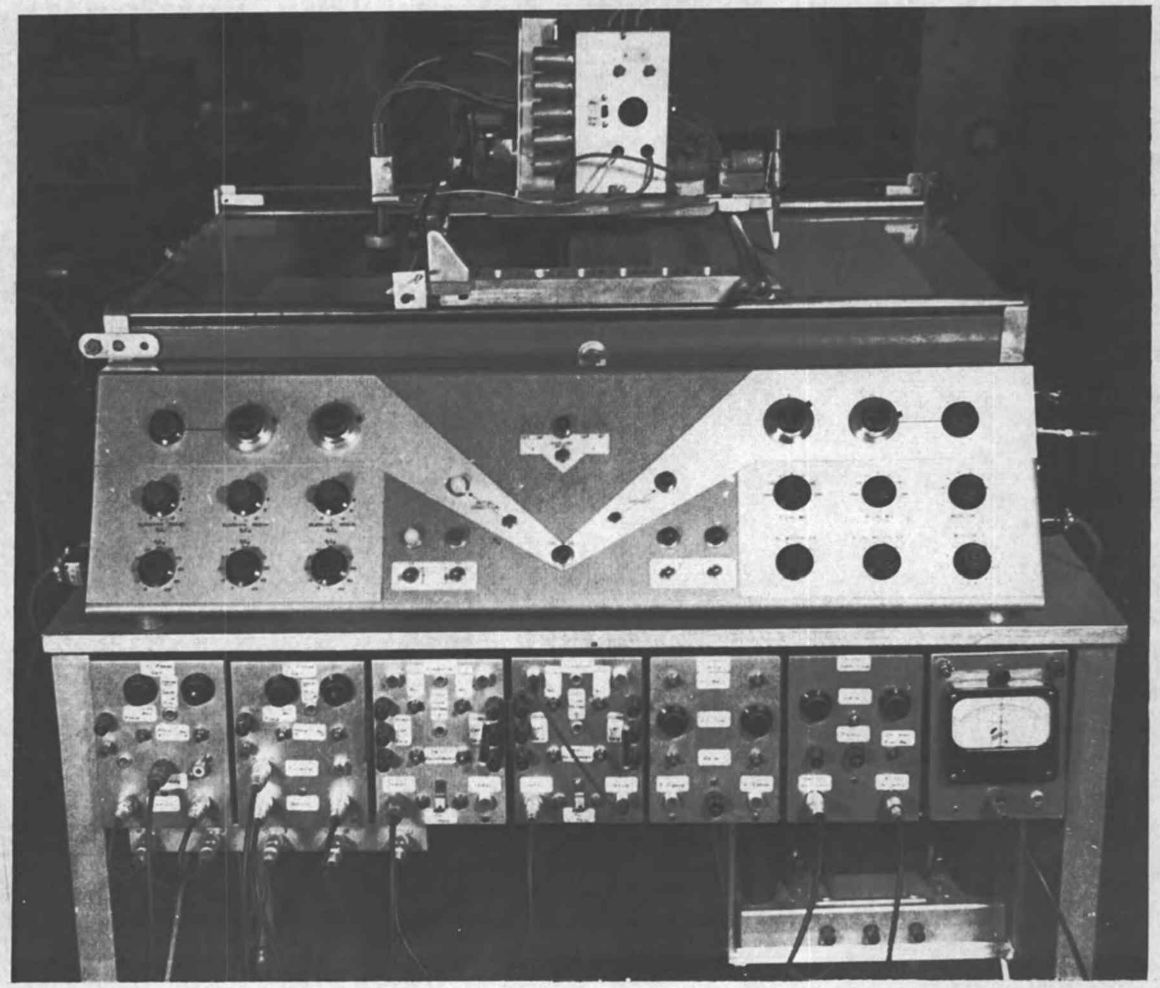

FIGURE 5

FRONT VIEW OF ELECTROLYTIC ANALOG

associated with the probes. A photograph of the probe assembly is shown in Figure 6.

From the preamplifier the probe voltages are fed to their respective x and y phase detectors where the differences between the probe a-c potentials are converted to d-c potentials proportional to the voltage gradient in the direction being considered.

The resulting x and y gradients are then used to solve the desired problem by feeding them into the necessary integrators and amplifiers. For the case shown in the block diagram in which it is desired to plot an electron trajectory, it is necessary to perform a double integration on both the x and the y gradients and also to provide the necessary initial conditions. Initial velocity and position for both channels are provided from the console. The x and y coordinates of the position of the electron are proportional to the voltages coming from the second integrators. These voltages are compared to the voltages coming from precision 10-turn potentiometers connected to their respective traverse mechanisms and the error voltages are fed to the servo drive circuitry which is arranged so that for zero error, the traverse mechanism is stationary. A positive error voltage will move the probe in one direction and a negative error will cause motion in the opposite direction. In this way, the probe and pen will

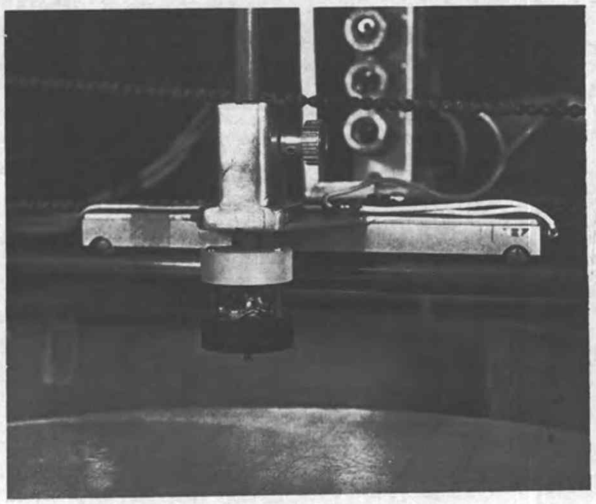

FIGURE 6
PROBE ASSEMBLY

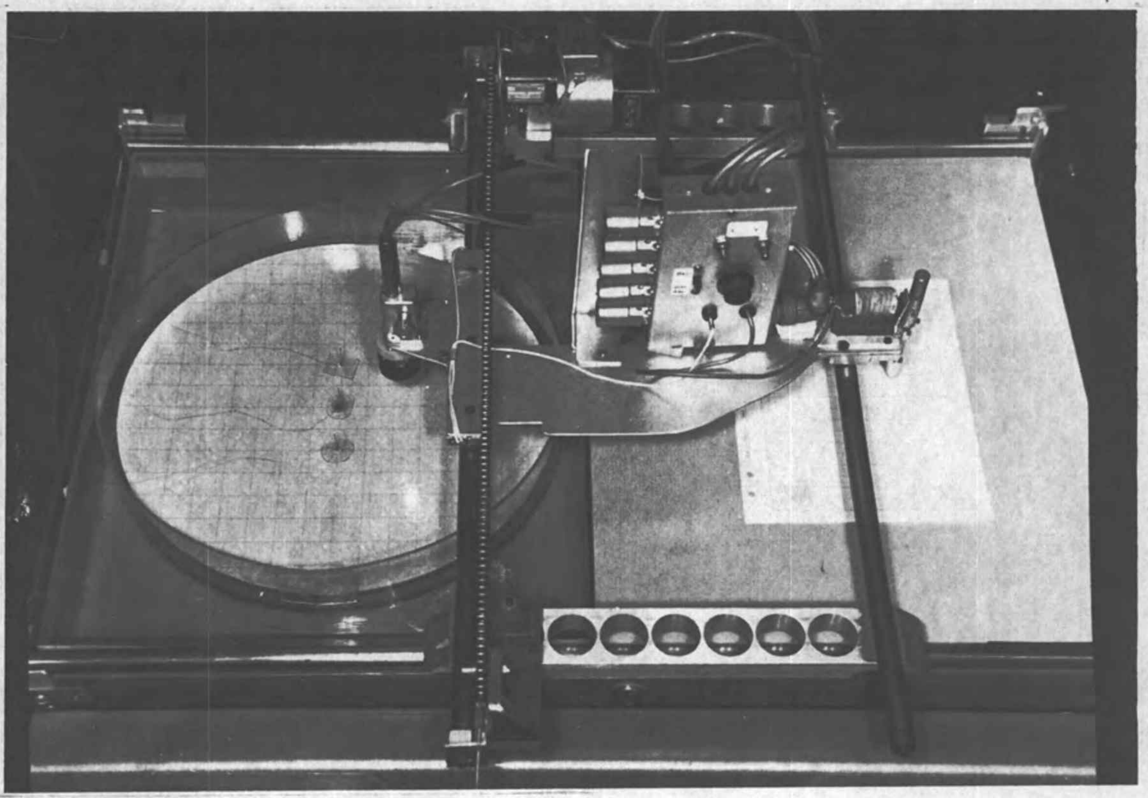

FIGURE 7
TOP VIEW OF ELECTROLYTIC ANALOG

follow the path of an electron. The plotting table and traverse mechanism is shown in Figure 7.

In order to have field lines traced automatically it is merely necessary to bypass the first integrators. This

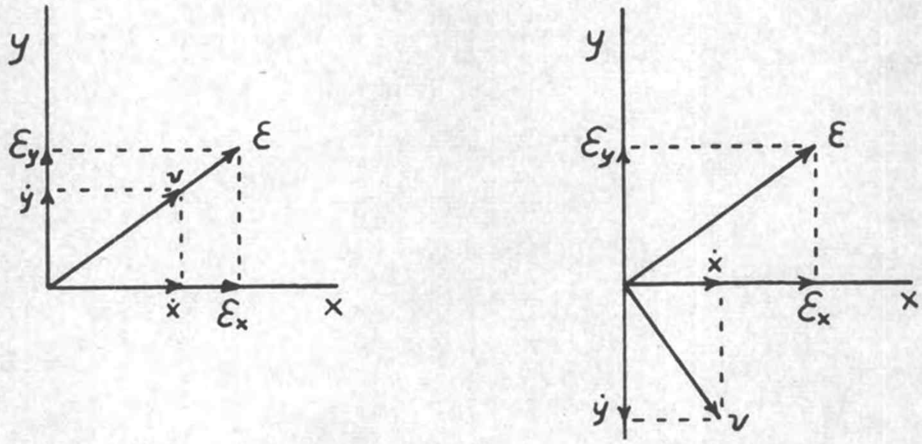

Fig. 8. Vector diagrams showing gradients and velocities.

(a) For plotting fields. (b) For plotting equipotentials.

can be explained by reference to Figure 8-a. With this connection:

$$\dot{X} = K\mathcal{E}_x \quad \text{and} \quad \dot{Y} = K\mathcal{E}_y \tag{27}$$

and it can be seen that the velocity is in the same direction as the gradient, so the probe will follow the field line on which it is placed initially.

To plot equipotential lines it is necessary to bypass the first integrator in one channel, make a unity gain amplifier of the first integrator in the other channel, and interchange the x and y inputs. With this connection, assuming the unity gain amplifier in the y channel:

$$\dot{X} = K\mathcal{E}_y \qquad \text{and } \dot{Y} = -K\mathcal{E}_x \qquad (28)$$

In Figure 8-b it can be seen that this causes the resultant velocity of the probe to be at right angles to the gradient and the probe will then follow the equipotential line on which it is initially placed.

Some of the other features of this equipment that are not necessary for the operation of the analog, but do add much in the way of convenience are listed below.

Panel lights indicate whether the system is in the "initial condition" position or in "operate". Push buttons are provided to put the equipment in either position. If the probe carriage reaches a set limit of its excursion the system will automatically return to "initial conditions". When in the "operate" position the pen is automatically lowered onto the plotting board.

Overload lights on the control panel warn of voltage exceeding a set limit in the x and y phase detectors and the first integrators. Lights indicating the location of the overload remain on until the trouble is corrected and a reset button is depressed. There is also an audible signal while the overload is present.

A calibration switch on the preamplifier chassis will apply a calibration voltage to the preamplifiers for use in checking the adjustment of the equipment. The amplitudes of the calibration signals are adjustable at the console from 0.005 to 5 volts in ten steps.

Monitor controls allow any two of the following voltages to be metered at two pair of terminals; acceleration, velocity and position for both the x and y channels.

IV. SOURCES OF ERROR

In any complex system the overall accuracy is very difficult to predict. If the error of each component were completely random and independent of the others, the problem would be a simple one of statistics. However, in this case these conditions are not satisfied and about all that can be said is that the accuracy of each component must be several times better than the desired overall accuracy.

1. Probe errors

As already mentioned in section III-3 the probes are one of the main points at which errors can arise. The very fact that the probes are present in the electrolyte distorts the field in the vicinity of the probes. Also, the finite size and spacing of the individual probes in an array gives rise to errors. However, these errors have been approximately evaluated and found to be very small.

A problem of much greater magnitude is that of the combined effects of meniscus, probe penetration depth and the effects of capillary action which tends to draw the electrolyte up into the center of the probe array. A four-pin probe with close spacing is particularly susceptible to this last effect. In Figure 9 is shown a curve of measured gradient versus depth of penetration. It can be seen in the hysteresis-like figure there is considerable

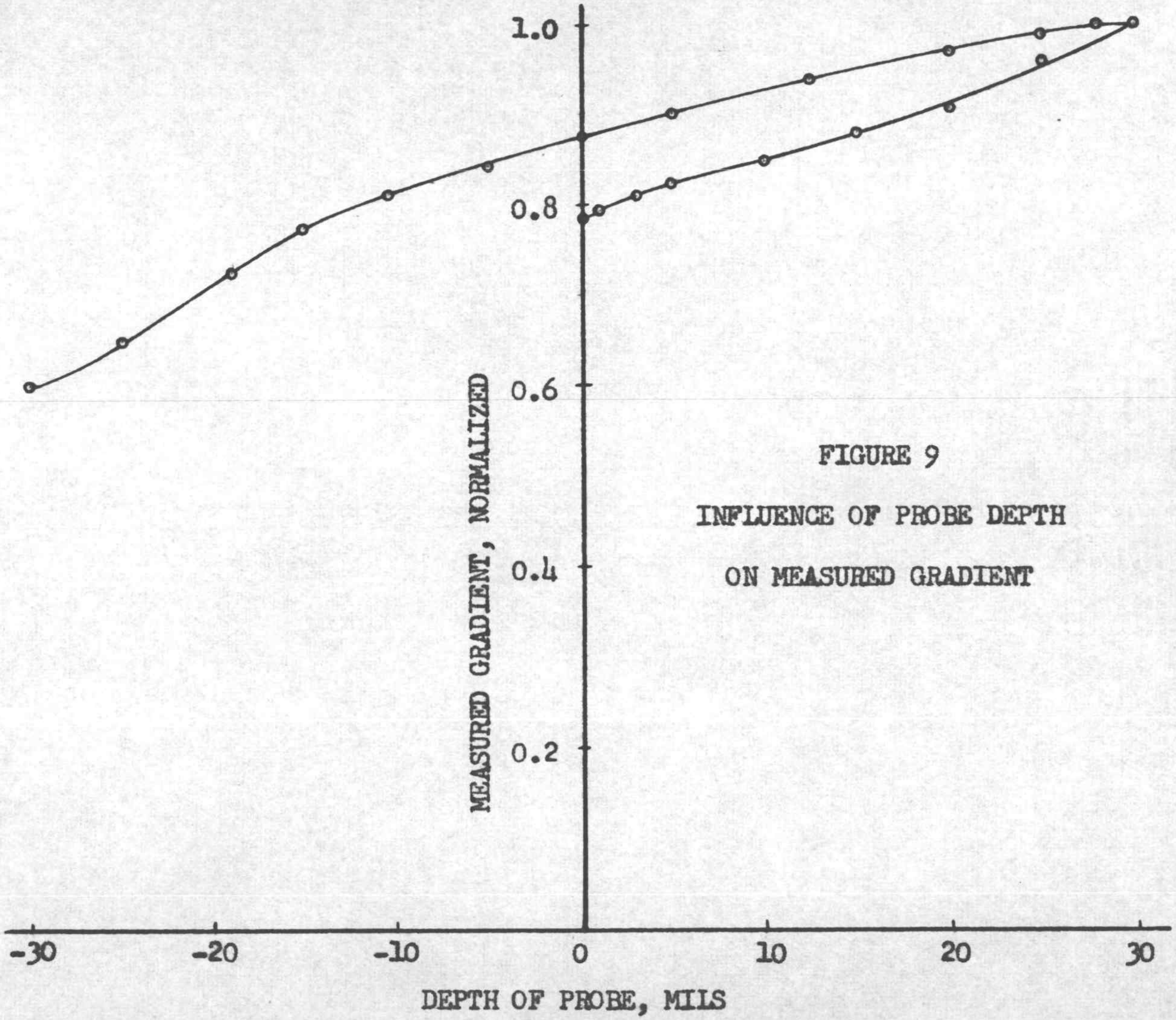

FIGURE 9

INFLUENCE OF PROBE DEPTH

ON MEASURED GRADIENT

dependence on the depth of penetration as well as the past history. This curve was taken with the four-pin probe described in section III-3, with a paraffin coating on the sides of the probe wires. A reasonable place to operate on this curve would be on the horizontal portion just after the probe has reached the maximum depth and started back up. Here, the gradient is essentially constant for a change in depth of several mils. Actually, it was found that more consistent results could be obtained by lowering the probes below the surface only a few mils and then backing up until the meniscus disappears. With this technique the gradient measurements could be reproduced with a precision of better than 1%.

Leveling of the probe carriage mechanism is very important in minimizing probe errors. If the carriage is not level, as the probe array is moved around the tank the depth of penetration will vary. This may result in erroneous gradient measurements.

2. <u>Tank Errors</u>

There are a number of errors associated primarily with the tank. Polarization effects distort the basic mathematical analogy and need to be minimized. Surface effects, such as dust particles on the surface and the meniscus around the edges of the model can introduce errors. This latter error can be minimized by filling the tank exactly to the upper edge of the model. Addition of a wetting agent has also been used to minimize the meniscus.

Evaporation of the electrolyte is another source of error. The magnitude depends on the relative humidity and the time required for solution of the problem. Using distilled water at a relative humidity of about 50%, the level remained essentially constant for more than an hour.

Ripple on the electrolyte caused by vibration of the motors has caused some concern. However, it is believed that the long time constants of the integrators tend to make this error small.

Improper leveling of the tank can also cause erroneous results. This error was calculated for the case of parallel electrodes and was found to be:

$$\text{Error} = Ex/L - (EX/L)\left(\frac{(1 - KX/2 + \frac{(KX)^2}{3} \ldots)}{(1 - \frac{KL}{2} + \frac{(KL)^2}{3} \ldots)}\right) \quad (29)$$

where E is the applied potential

X is the distance from one electrode toward the other

L is the electrode spacing

KL is the error in depth at one edge.

Equating the derivative of this equation to zero will give the point of maximum error as approximately

$$X_m = L/2 - KL^2/3 \quad (30)$$

This was evaluated for L = 10 inches, E = 1 volt and KL = 10 mils and the maximum error was about 0.25%. With spirit levels it is not unreasonable to expect leveling to within about one mil per foot, so that the error from this source can be made insignificant.

With care in choosing the various parameters for use in the tank, such as a large figure of merit for the electrode-electrolyte combination (see section III-2) sufficient size to allow accurate models and good leveling, the error can be held between 0.1% (19) and 0.2% (5).

3. Electronic Errors

The electronic circuitry necessary for an automatic electrolytic analog can easily be a source of considerable error. The use of a-c voltages for the tank requires difference amplifiers and phase detectors to convert the information to a form that can be integrated. This required good control over linearity and phase shift in the various parts of the circuitry. The use of square-wave excitation and sampling the mid-points of the resulting gradient voltages is one method of minimizing these errors at the expense of greater complexity. Any error developed in these early stages will be amplified many times by the following components of the system.

The natural tendencies of d-c amplifiers to drift must be controlled to the point that the drift is very small over at least the problem time. Variations in line voltage cause similar effects. This requires well regulated d-c power supplies and possibly regulated filament supplies.

Careful attention to balancing channel gains and zeroing all units after a warm-up time is essential in obtaining good results. Even so, the electronics account for a good share of the total error. In Figure 10, from left to right, are shown: y and x phase detectors, y and x integrators, servo-amplifiers, ground amplifier, 400cps source.

4. Mechanical Errors

Good mechanical design can keep error from this source well below 1%. These errors arise primarily from backlash and friction in the drive mechanism and mechanical tolerances in general. The carriage must move only in a horizontal plane in order that the probe penetration is constant. Rigidity of the whole supporting structure is important in keeping down ripples on the electrolyte surface.

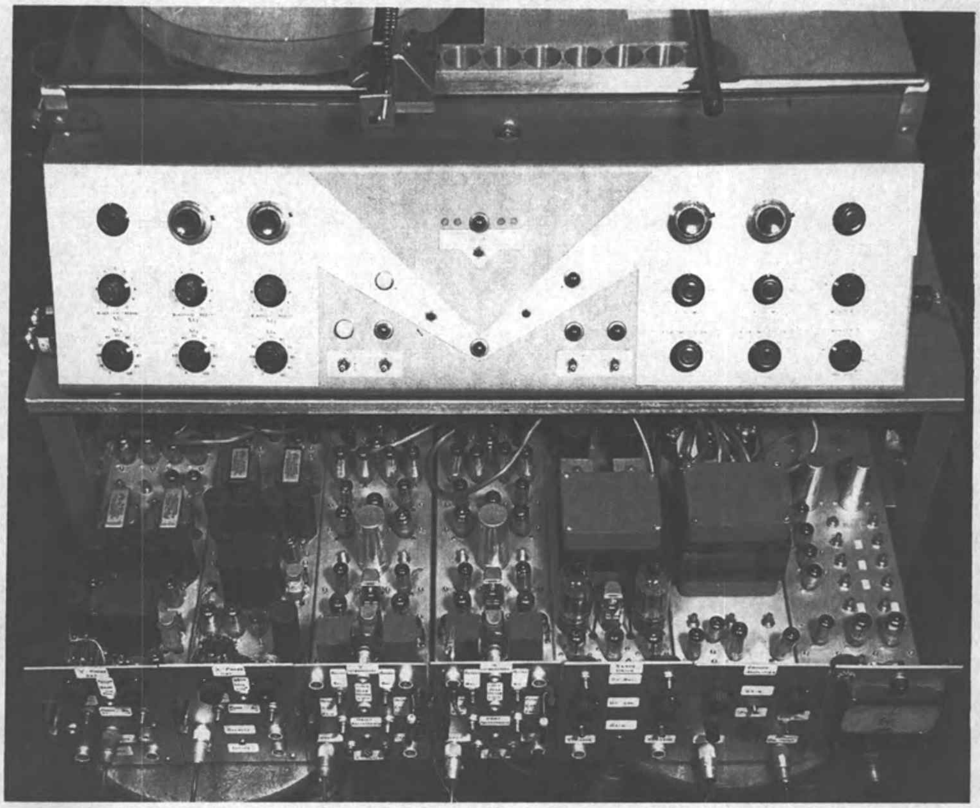

FIGURE 10

ELECTRONIC CHASSIS OF THE ELECTROLYTIC ANALOG

V. EXPERIMENTAL RESULTS

1. Electron Trajectories

A family of parabolas are shown in Figure 11. These were obtained by setting up a uniform field in the electrolyte and setting the probe in with an initial velocity perpendicular to the field. Several different values of initial velocity were used as indicated on the curves. The points were obtained theoretically from the conditions set up and the accuracy at all the calculated points is better than ±3% of the correct coordinates. On this type of problem the spread of several runs can be held between 0.5% and 1%. However, the error depends greatly on the type of problem being solved.

Errors of this magnitude are far in excess of that required for a study of electron lens aberrations. This is adequate to study the effects on the focal point of different lens designs. According to reference 17, even with the use of digital integrators, the accuracy is barely sufficient to observe aberrations. The errors are then due primarily to the probes.

2. S-Plane Loci

In Figure 12 are shown some phase and magnitude loci on the s-plane for

$$G(s) = \frac{K}{s(s+1)(s+2)} \qquad (31)$$

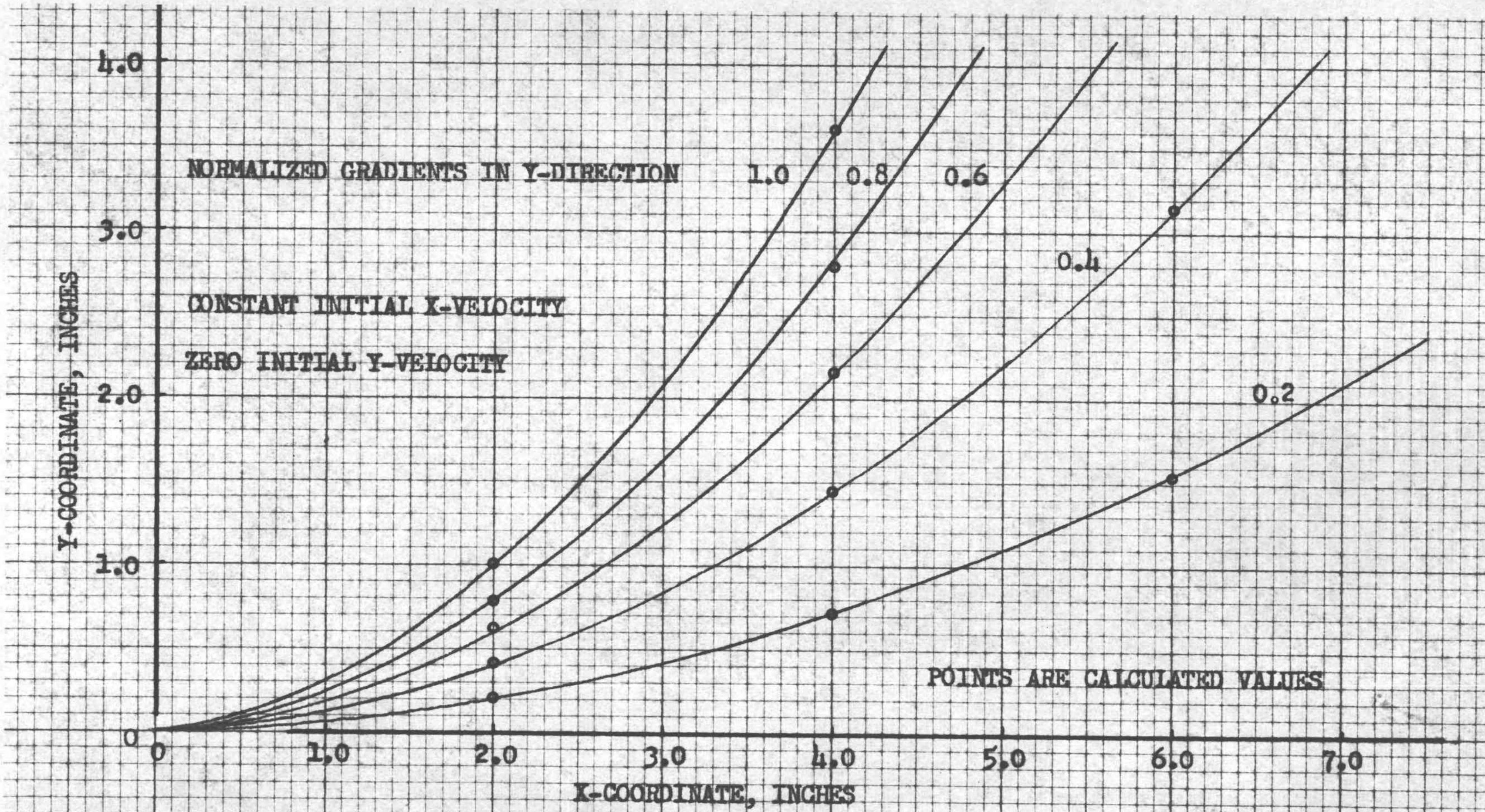

FIGURE 11

FAMILY OF PARABOLIC TRAJECTORIES

OBTAINED WITH THE ELECTROLYTIC ANALOG

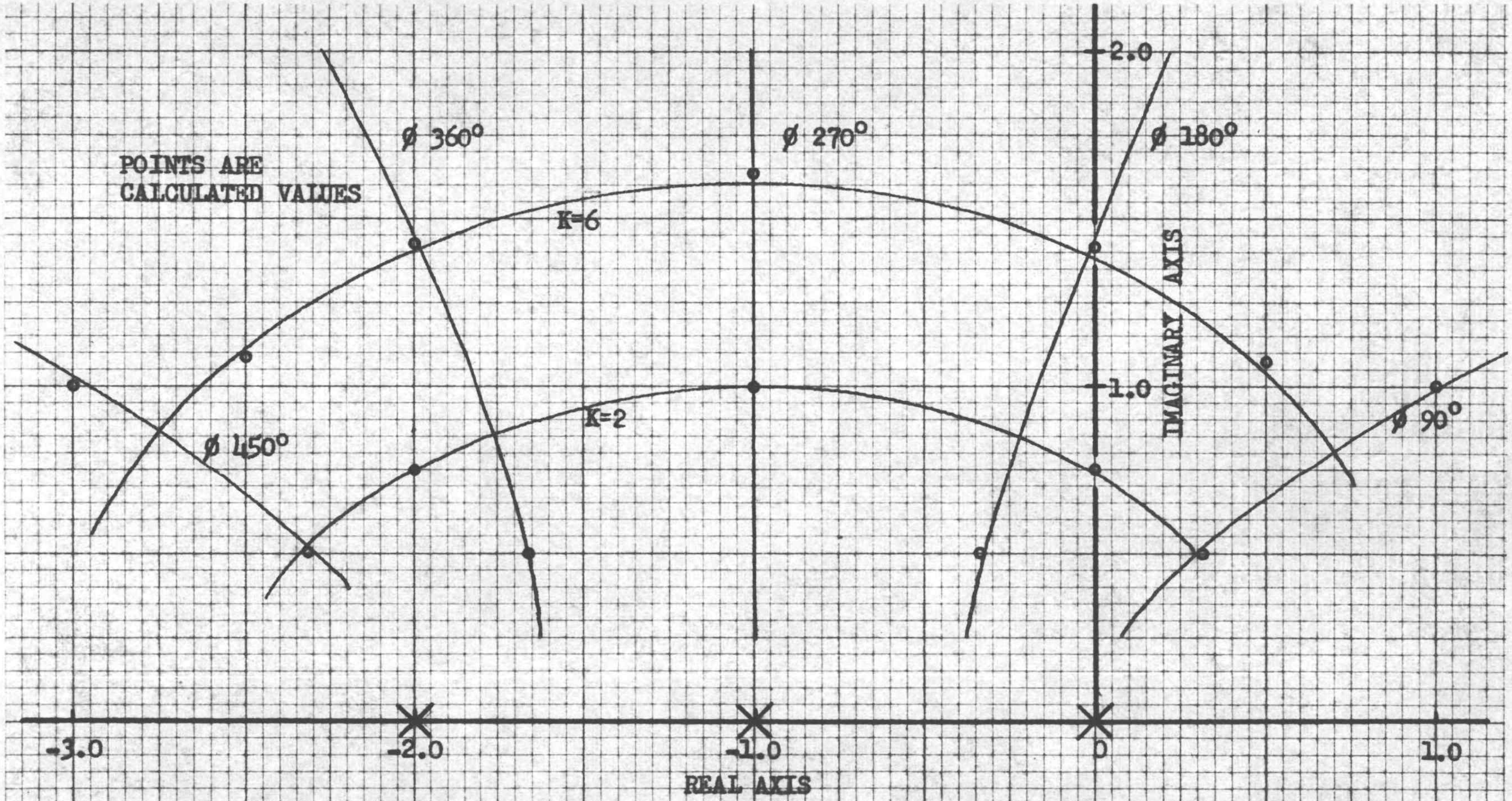

FIGURE 12

S-PLANE LOCI FOR $G(S)=\frac{K}{S(S+1)(S+2)}$

OBTAINED WITH THE ELECTROLYTIC ANALOG

The points are calculated values and should all be cut by the various loci. Interpreting the error of these loci is more difficult than for the trajectories, and perhaps less meaningful. A fixed error on a 90° phase line will be a much larger percentage than the same angle error on a 270° line, and the percent error would be infinite for a zero degree line. A similar situation exists with the magnitude contours. However, for this example the errors were calculated in this way.

The plot shown is typical of the results currently being obtained with this equipment for the s-plane loci. The maximum error at any of the calculated points shown was 4.8%. A fixed error shifted the loci to the left enough to be noticeable on the magnitude contours. The real axis of the s-plane was not crossed in this plot because the plane was divided in half with an insulator and only the upper half plane was plotted.

It was found that a change as small as 1% in the current applied to one of the electrodes caused a noticeable shift in the loci. Since the currents were not initially set with accuracy better than this, the results are not unreasonable.

VI. CONCLUSIONS

The electrolytic analogies have been presented and a particular equipment configuration has been described that will automatically fulfil the necessary requirements to give a solution to Laplace's equation. Results were presented showing electron trajectories and s-plane loci and although the errors run as high as 3 to 5 percent at present it is believed that refinement of techniques and certain of the electronics will yield results with errors of the order of plus and minus 1%.

Some of the unique features of this device which were not mentioned but which are quite important are as follows. The tank can provide 3-dimensional solutions to certain types of problems (10). Space charge can be simulated by addition of current sources in the area where the charge exists (11). Conductivity of the electrolyte can be varied in any desired manner by changing the depth of the electrolyte by inserts placed on the tank bottom. This allows simulation of varying dielectric constants or permittivity.

The electrolytic tank is versatile enough to provide the engineer in nearly every field with a means for obtaining a fast, reasonably accurate, analog solution.

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APPENDIX

APPENDIX

Detailed Operating Instructions for the Electrolytic Analog

1. Allow at least 1 hour warm-up after turning on filaments and power switches before attempting any adjustments.

2. Place models in the tank and fill to top of model with distilled water.

3. Connect operational amplifiers to solve the desired problem.

4. Set the 400 cps signal at the 100 volt output terminal (at upper left end of the console) to 100 volts rms by adjusting the gain of the 400 cps amplifier.

5. Turn switch on the preamplifier chassis to the Calibrate position.

6. Connect a high impedance low range d-c voltmeter to one set of monitor terminals (at the right end of the console) and turn the monitor switch to A_y, acceleration in the y-direction.

7. With the calibrate controls on the console set to zero, the Y-phase detector chassis should be adjusted to zero on the monitor voltmeter.

8. Place the slide switch on the preamplifier chassis to the Differential Adjust position and with a calibrate signal applied, adjust the "Y" control on the preamplifier chassis to zero the monitor voltmeter.

9. With the slide switch in the Balance position and a calibrate voltage applied, the gain of the phase detector chassis can be adjusted to any desired value by observing the monitor voltmeter.

10. Follow steps 5 through 9 for the X channel.

11. With the monitor control in the "V_y" position and with the initial condition controls for V_y set to zero adjust the balance control on the first Y integrator to read zero on the monitor voltmeter. (A coarse balance is accessible by pulling the chassis out about half way).

12. Depress the Operate button on the console and set the Drift Adjust of the first Y integrator to hold the output to zero. Increased sensitivity can be obtained by pushing the red button on the integrator chassis which decreases the time constant of the integrator.

13. Depress the Initial Condition button and repeat step 12 until drift has been reduced adequately.

14. Repeat steps 11, 12 and 13 for the first X integrator.

15. With the Y initial position set to zero and the monitor control set to Y, adjust the Balance control on the second Y integrator to zero on the monitor voltmeter. (A coarse control is accessible by pulling the chassis out about half way).

16. Depress the Operate button and adjust drift control as for the first integrators. Return to initial conditions when properly adjusted.

17. Repeat steps 15 and 16 for the second X integrator.

18. Set the initial X and Y position controls to mid-range and turn on the motor switch. This should approximately center the probe over the tank.

19. After allowing the motors about 5 minutes to warm up, ground the Y error terminal and adjust the D-C Center control so that grounding and ungrounding the error terminal causes no motion of the probe carriage.

20. Repeat step 19 for the X motor.

21. The analog is now ready for problem solution.

22. Set in the desired initial positions and velocities.

23. Lower the probe until it makes a slight depression on the surface of the electrolyte and then raise the probe until there is no meniscus visible.

24. Place the pen switch in the Automatic position.

25. Press the operate button.