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Title ON EXTENSION OF THE GENERALIZED SAMPLING THEOREM

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Abstract approved ______________________________
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The relation of the sampling theorem, developed by Shannon for Information Theory, to the Whittaker interpolatory functions is investigated and further extensions in this direction are suggested. The theory of the Shannon sampling theorem with its application in Information Theory is reviewed. The Kramer generalization of this theorem is illustrated for many more functions of interest in Mathematical Physics, so the use of this theorem in other fields besides communications is suggested.

Campbell's question concerning the possibility of reducing Kramer's sampling theorem to that of Whittaker is investigated and illustrated for more functions of interest. Some useful theorems in this direction are proposed and proved.

It is shown that although most illustrative functions satisfy Campbell's conditions, the advantage of Kramer's sampling theorem

over that of Whittaker is removed only in the mind of the communications engineer, who is mainly interested in no integral transforms other than the celebrated Fourier Transform.

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Chapter I

INTERPOLATORY FUNCTIONS AND SAMPLING THEOREMS

In this chapter we review the theory of interpolatory functions, since this is where the Shannon [34] sampling theorem [1] had originated. As such we intend to show that it is also here that the Kramer [26] generalization of the above sampling theorem emerged as a natural extension.

1.1 The Cardinal Series

E. T. Whittaker [51] set out to find an analytic expression for a function when the values of the function are known for equidistant values a, $a+w, \cdots$, $a+nw$, of its argument, and such that this expression is free of periodic components with period less than $2w$. This function was called the Cardinal Function. As such he showed that this analytic expression is not only an interpolatory expression but a representative one as well. Hence, we may say that the sampling theorem of Shannon had originated. The first thing we notice is that Whittaker's problem is concerned with equally spaced values of

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[1] For the exact statements of the Shannon sampling theorem and the Kramer generalization see Sections 2.1 and 3.1 respectively.

the argument so a periodic function is expected. Whittaker considered the tabulated values of the function $f(t)$, i.e., $f(a)$, $f(a+w), \cdots, f(a+nw)$, then defined the set of Cotabular Functions as all functions $F(t) = f(t) + g(t)$, where $g(t)$ is analytic and vanishes at the sampling points. He also showed that even if the choice of $f(t)$ is an analytic function with a finite number of singularities these singularities may be removed when the function is replaced by another function of the cotabular set. This was demonstrated with a choice of $g(t)$ as

$$- \frac{r \sin \frac{\pi(t-a)}{w}}{(t-c) \sin \frac{\pi(c-a)}{w}} \tag{1.1.1}$$

for the case of $f(t)$ with a simple pole at $t = c$. As a result one can always find an analytic function $f(t)$ as a member of the cotabular set.

For the problem of removing from $f(t)$ the periodic constituents of period less than $2w$ Whittaker resorted to Fourier analysis and substituted for them from the cotabular components with periods greater than $2w$. For example, $\sin \lambda t$ has a period $\frac{2\pi}{\lambda}$ while $\sin[(\lambda - \frac{4\pi}{w})t + \frac{4\pi a}{w}]$ is cotabular with $\sin \lambda t$ at $t = a \mp n\pi$ yet has a larger period $\frac{2\pi}{\lambda - \frac{4\pi}{w}}$. By using this, and assuming the use of the Fourier integral formulas, the final form of the cardinal function is

$$\sum_{n=-\infty}^{\infty} f(a+nw) \frac{\sin \frac{\pi}{w}(t-a-nw)}{\frac{\pi}{w}(t-a-nw)} . \qquad (1.1.2)$$

We note that this cardinal series is the one Shannon used for his Sampling Theorem and is what is sometimes called the Whittaker Sampling Theorem. There are two references here, to E. T. Whittaker [51] and J. M. Whittaker [52]. This may be due to the fact that the final statement of the above sampling theorem in terms of band-limited signals is very close to the more refined statements of J. M. Whittaker [53, p. 68] concerning the relation between the cardinal series and the finite Fourier integral. In fact we can show that E. T. Whittaker had the same statement since, when using Fourier theorems he considered

$$g(t,k) = \frac{1}{\pi}\int_{-\infty}^{\infty} du\, f(u)\Big[\int_0^{\frac{\pi}{w}} + \int_{\frac{\pi}{w}}^{\frac{3\pi}{w}} + \cdots\Big]\, e^{-\lambda k}\cos\lambda(t-u)\,d\lambda , \qquad (1.1.3)$$

then replaced this by the following cotabular function which is free of components with periods less than $2w$:

$$G(t,k) = \frac{1}{\pi}\int_{-\infty}^{\infty} du\, f(u)\, [\, \int_{0}^{\frac{\pi}{w}} e^{-\lambda k} \cos \lambda(t-u) d\lambda$$

$$+ \int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} e^{-k(\lambda + \frac{2\pi}{w})} \cos [\lambda (t-u) + \frac{2\pi}{w}(a-u)]\, d\lambda \qquad (1.1.4)$$

$$+ \int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} e^{-k(\lambda + \frac{4\pi}{w})} \cos [\lambda (t-u) + \frac{4\pi}{w}(a-u)]\, d\lambda + \cdots] \,,$$

and we can show that this is equivalent to the Fourier theorem representation of $G(t)$ by a finite Fourier transform. If we use (1. 1. 4) before integrating with respect to λ we find

$$\lim_{k \to 0} G(t,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} du\, f(u) \int_{-\frac{\pi}{w}}^{\frac{\pi}{w}} \sin \lambda(t-u) \cot \frac{\pi}{w}(a-u) d\lambda \,, \qquad (1.1.5)$$

but the Fourier theorem gives $f(t)$ as

$$f(t) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} d\lambda \int_{-\infty}^{\infty} e^{i\lambda(t-u)} f(u)\, du, \qquad (1.1.6)$$

so if we use the complex form, let $b = \frac{\pi}{w}$, and allow the interchange of the order of integration we get

$$G(t) = \frac{1}{2\pi} \int_{-b}^{b} e^{i\lambda t} d\lambda \int_{-\infty}^{\infty} e^{-i\lambda u} f(u) \cot b(a-u)\, du. \qquad (1.1.7)$$

Hence,

$$G(t) = f(t) \cot b(a-t) \qquad (1.1.8)$$

and is represented by a truncated Fourier transform.

It is noted here that E. T. Whittaker used the Fourier theorem as a tool for constructing his cardinal series representation for the function. But the present sampling theorems need only a truncated Fourier transform representation for the function to guarantee a cardinal series representation which is free of the fine structure. Also, E. T. Whittaker's purpose in constructing the cardinal function was to have it analytic and represented by the following expression:

$$f_0 + n\delta f_{\frac{1}{2}} + \frac{n(n-1)}{2!} \delta^2 f_0 + \frac{n(n+1)(n-1)}{3!} \delta^3 f_{\frac{1}{2}} + \cdots \qquad (1.1.9)$$

where

$$\delta f_{\frac{1}{2}} = f_1 - f_0, \quad \delta^2 f_0 = \delta f_{\frac{1}{2}} - \delta f_{-\frac{1}{2}}, \quad \cdots . \qquad (1.1.10)$$

In addition, he gave an example of a cotabular function which is not cardinal:

$$\sum_{n=-\infty}^{\infty} e^{-c(t-a-nw)^{2k}} f(a+nw) \left\{ \frac{\sin \frac{\pi}{w}(t-a-nw)}{\frac{\pi}{w}(t-a-nw)} \right\}^{m}, \qquad (1.1.11)$$

where $c \neq 0$ and k and m are positive integers. For $c = 0$ and $m = 1$ this is a cardinal function.

1.2 Suggestions for Other Series

At this point it is not surprising that we raise the question, "Is it possible to consider some expression resembling the cardinal expression but which will sample a function with its tabulated values for non-equidistant values of its argument, say $\{t_n\}$?" To follow the same procedure we know that $\sin \lambda t$ is the simplest periodic function with period $\frac{\pi}{\lambda}$, so for our case we avoid it and try $K(\lambda, t)$, where $\lambda \in [a, b]$ and

$$K(\lambda, t_n) = 0 \qquad (1.2.1)$$

but $K(\lambda, t)$ is not necessarily periodic. For this interpolating function we propose

$$S_n(t) = S(t; t_n, a, b) \qquad (1.2.2)$$

where $S(t_n; t_n, a, b)$ is unity. The explicit expression for such an $S_n(t)$ is given by Kramer [26] for his generalized sampling theorem for any choice of $K(\lambda, t_n)$ as an orthogonal set on $[a,b]$. So we can regard Kramer's generalization as a natural extension of Whittaker's work and the popular sampling theorem.

1.3 The Cardinal Series and the Fourier Integral

In this section we will discuss J. M. Whittaker's [52] important development toward what we now know as the Shannon Sampling Theorem. In particular, his explicit theorem involves the cardinal series and Fourier and Fourier–Stieltjes integrals. Hence, he came the closest to the present statement of the sampling theorem as it is given in terms of a band-limited signal (i. e., a truncated Fourier transform). J. M. Whittaker's [52,th.2] theorem is

Theorem 1.3.1. " If the series

$$\sum_{n=1}^{\infty} \frac{|a_n| + |a_{-n}|}{n} \tag{1.3.1}$$

converges, the cardinal series

$$C(x) = \frac{\sin \pi x}{\pi} \left\{ \frac{a_0}{x} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{a_n}{x-n} + \frac{a_{-n}}{x+n} \right] \right\} \tag{1.3.2}$$

is absolutely convergent, and its sum is of the form

$$\int_0^1 [\cos \pi xt \, dF(t) + \sin \pi xt \, dG(t)] \,, \tag{1.3.3}$$

F, G continuous functions. Given any function $f(x)$ of the form of (1.3.3) the series

$$\frac{\sin \pi x}{\pi} \left\{ \frac{f(0)}{x} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{f(n)}{x-n} + \frac{f(-n)}{x+n} \right] \right\} \tag{1.3.4}$$

is summable [2] (C, 1) to $f(x)$".

Previously Ferrar [16] gave the following theorem, which we consider to be even closer to Shannon's original statement of the sampling theorem:

Theorem 1.3.2. " If $\sum_{n=-\infty}^{\infty} |a_n|^p$ is convergent, $p > 1$, and $C(x)$ is defined by

$$C(x) = \frac{\sin \pi x}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a_n}{x-n} \,, \tag{1.3.5}$$

then

$$C(x) = \frac{\sin \pi(x-b)}{\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n C(b+n)}{x-b-n} \,, \tag{1.3.6}$$

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2 By Hardy's theorem (1.3.4) converges if $f(n)$ is bounded.

where $\{a_n\} \epsilon \ell_p$ implies that the series in (1.3.5) and (1.3.6) are convergent[3]". Ferrar called this the consistency of the cardinal series. This corresponds to the representation of the sampling theorem as compared to the interpolation only in the case of interpolatory theory. Again J. M. Whittaker asserted that, given a sequence $a_0, a_., \cdots, a_n, \cdots$ of real numbers, then the series (cardinal) of type (1.3.4), convergent or (C, 1) summable, affords a means of defining the trigonometric integrals associated with the Fourier and Fourier-Stieltjes series respectively. For example,

$$a(x) = \int_{-\pi}^{\pi} f(x) \cos xt \, dt, \qquad (1.3.7)$$

where $f(x)$ is represented by the Fourier series and $a(x)$ by the cardinal series. Here, we are led to the finite Fourier integral in (1.3.7). At this point we note that the above statement is another, more precise statement of what E. T. Whittaker had started, with almost everything centered around the cardinal series.

1.4 The Cardinal Series and Other Finite Transforms

This section deals with a question of different nature but is

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[3] Note that, by Hardy's theorem, for $C(x)$ as $(C, 1)$ summable to be convergent we need $a_n/(x-n) = 0(1/n)$, i.e., if $C(b+n)$ is bounded.

still aimed at tying the Kramer generalization of the sampling theorem to a common origin with the Shannon sampling theorem and, hence, as a natural extension of the latter. This question is, "What kind of integral representation would a series other than the cardinal series offer?" It is sufficient to consider the Bessel function $J_m(xt)$ instead of $\sin xt$, and we first write the Fourier-Bessel expansion for $J_m(xt)$,

$$J_m(xt) = \sum_{n=1}^{\infty} b_n J_m(xt_n), \qquad J_m(t_n) = 0, \quad n = 1, 2, \cdots, \tag{1.4.1}$$

then consider $f(t)$ with a Fourier-Bessel expansion,

$$f(x) = \sum_{n=1}^{\infty} a_n J_m(xt_n), \qquad J_m(t_n) = 0, \quad n = 1, 2, \cdots, \tag{1.4.2}$$

where

$$a_n = \int_0^1 x f(x) J_m(xt_n) dx. \tag{1.4.3}$$

The orthogonality property of the Bessel functions leads to

$$\int_0^1 x f(x) J_m(xt) dx = \sum_{n=1}^{\infty} a_n \int_0^1 x J_m(xt) J_m(xt_n) dx. \tag{1.4.4}$$

If $\int_0^1 x J_m(xt) J_m(xt_n) dx$ is taken to be the interpolating function for the series (1.4.4), as is asserted by the Kramer [26] generalized sampling theorem (as compared to $\frac{\sin(x-n\pi)}{x-n\pi}$ for the cardinal series), we obtain

$$a(x) = \int_0^1 x f(x) J_m(xt) dx . \tag{1.4.5}$$

That is, $a(x)$ is represented by a finite Hankel transform. In general, with the help of Kramer's theorem one might consider any orthogonal expansion for $f(x)$ with its corresponding finite transform for $a(x)$. We mention here that in moving from $\sin nx$ to $K(x, t_n)$ we have at least gained the liberty to sample at points $\{t_n\}$, the zeros of $K(x, t_n)$ which are not necessarily equidistant. The other point remains with the advantage of other orthogonal expansions over the Fourier sinusoidal expansions, a matter which is closely related to the nature and geometry of the problem, and as such it might not be of importance to the communication engineer. J. M. Whittaker [53, p. 71] came close to touching this question when he considered the general partial fraction series [53, p. 64]

$$H(z) \left\{ \frac{f(0)}{z} + \sum_{n=1}^{\infty} \left[\frac{f(c_n)}{H'(c_n)(z-c_n)} + \frac{f(-c_n)}{H'(c_n)(z+c_n)} \right] \right\}, \tag{1.4.6}$$

where the $c_1, c_2, \cdots,$ is a strictly increasing sequence of positive numbers such that $\sum_{n=1}^{\infty} c_n^{-2}$ converges and

$$H(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{c_n^2}\right) . \tag{1.4.7}$$

Provided the series in (1.4.6) is convergent the divided difference series

$$f(0) + zf(0, c_1) + z(z-c_1)f(0, c_1, -c_1) + z(z^2-c_1^2)f(0, c_1, -c_1, c_2)$$

$$+ z(z^2-c_1^2)(z-c_2)f(0, c_1, -c_1, c_2, -c_2) + \cdots \tag{1.4.8}$$

converges to the same sum. In addition, he noted that Theorem (1.3.1) does not apply to (1.4.7) in general, but to the special case $H(z) = \sin \pi z$ and $c_n = n\pi$, $z = cx$, as the cardinal series is in terms of $\{\sin n\pi x\}$, an orthogonal set of functions relative to its zeros in $[0, 1]$. At this point he hinted [53, p. 71] that a theorem similar to (1.3.1) holds if $c_n = t_n$, the zeros of $J_0(z)$ and $H(z) = z J_0(z)$. So, $H(xc_n)$ is the orthogonal set relative to its zeros with a weight function $\rho(x) = \frac{1}{x}$. It is no surprise to find the Bessel functions among the first examples of the Kramer generalized sampling theorem, where we accept the theorem as the natural extension of the work of Ferrar and both Whittakers, and away from their cardinal series.

CHAPTER II

REVIEW OF THE SAMPLING THEOREM AND ITS APPLICATIONS

The sampling theorem that is about to be discussed in detail was introduced by Shannon [34] to information theory. As we have seen in Chapter I this theorem was originated by both Whittakers [51, 52, 53] and Ferrar [16], even though some attribute it to Cauchy (see Section 2.6). In the Russian literature this theorem was introduced to communication theory by Kotel'nikov [24], and took its name from him as opposed to Shannon's, Whittakers' or popular sampling theorems in the English literature. In what follows we will use either one of the above references or, in brief, we will use WKS sampling theorem after both Whittakers, Kotel'nikov and Shannon. We will do this with every sampling theorem that involves a band-limited signal, i.e., represented by a finite Fourier transform. WKSK may stand for Kramer's [26] generalization of the sampling theorem which involves a much more general kernel than the earlier Fourier kernel.

2.1 The Original Shannon Sampling Theorem

Shannon's original statement [34] of the WKS sampling theorem is the following:

Theorem 2.1.1. "If a function $f(t)$ contains no frequencies higher than W cps it is completely determined by giving its ordinates at a series of points spaced $\frac{1}{2W}$ seconds apart".

Proof: Let

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega) e^{i\omega t} d\omega, \qquad (2.1.1)$$

since $F(\omega)$, the spectrum of $f(t)$, is assumed to be zero outside the band. When the Fourier series expansion of $F(\omega)$ is written with the fundamental period, $-W < \omega < W$, we recognize

$$f(\frac{n}{2W}) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} F(\omega)\, e^{\frac{i\omega n}{2W}} d\omega \qquad (2.1.2)$$

as the sampled values of $f(t)$ and the nth Fourier coefficient, and so they determine $F(\omega)$ when $F(\omega)$ is zero for $|\omega| \geq 2\pi W$. By the uniqueness property of the Fourier transform $f(t)$ is determined. Shannon then constructed $f(t)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n}{2W}) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} . \qquad (2.1.3)$$

We note that the outline of this proof and the method of constructing $f(t)$ as in (2.1.3) is parallel to the work of J. M. Whittaker [52].

In fact, Shannon introduced the physics of time and frequency to the second part of Theorem 1.3.1, where (2.1.3) is Whittaker's cardinal series. This celebrated theorem, with some variations from the above-mentioned Shannon statement, is discussed in a number of texts [2, 30, 33, 55] in the field of communications with some detailed illustrations. The variations in the proofs center around different methods of manipulation in Fourier analysis.

2.2 Physical Interpretation

Reza [33, p. 305] gave the following physical interpretation to Shannon's (WKS) sampling theorem. Suppose that $f(t)$ represents a continuous band-limited voltage signal. Then $f(t)$ can be quantized at times $\{\frac{n}{2W}\}$, $n = 0, \mp 1, \mp 2, \cdots$, and $\frac{\sin Wt}{\pi t}$ is known to be the impulse response of an ideal low-pass filter with frequency cut-off at W. Then $f(t)$ of (2.1.3) will be the output of such a filter with input taken to be the pulse train defined by $f(\frac{n}{2W})$. As we will see in Section 2.8 Papoulis [31] later extended the WKS sampling theorem in such a way that he obtained a physical interpretation with more relaxed conditions on the filter and with a recognizable pulse as input rather than the inattainable impulse. The price of this relaxation is paid in terms of a higher sampling rate.

2.3 Sampling Theorem and Interpolation

Jagerman and Fogel [19] considered the WKS sampling theorem as an interpolation formula, then stated and proved a number of interesting extensions. They first considered the Lagrange interpolation polynomial

$$P_n(t) = g_n(t) \sum_{j=0}^{n} \frac{f(t_j)}{(t-t_j)g'(t_j)}, \quad g_n(t_j) = 0. \tag{2.3.1}$$

They extended the real variable t to a complex variable z. Note that (2.3.1) is the partial fraction expansion of J. M. Whittaker's equation (1.4.6). Here $\frac{P_n(z)}{g_n(z)}$ is analytic except at the zeros of $g_n(z)$, the sampling points, and $P_n(z)$ is entire. This was generalized to include an infinite number of sampling points. The choice for $g(z)$ was obviously $g(z) = \sin \frac{\pi z}{h}$, so

$$P(z) = \sin \frac{\pi z}{h} \sum_{j=-\infty}^{\infty} \frac{(-1)^j f(jh)}{z-jh}, \tag{2.3.2}$$

is the entire cardinal series. The sample points are uniformly spaced on the complex plane. We remark here that a more general choice for $g(z)$ would be a function such as $J_n(z)$, where the sample point distribution would be asymptotically uniform. For their choice of $g_n(z)$ they stated and proved the following versions and extensions of the WKS sampling theorem, using the method of contour integration.

<u>Theorem 2.3.1.</u> "If the entire function $f(z)$ is such that there exists a K such that

$$e^{-\frac{\pi}{|h|}|y|} A(y) \leq \frac{K}{|y|} \tag{2.3.3}$$

as $|y| \to \infty$, and where

$$A(y) = \max_{-\infty < x < \infty} |f(z\, e^{i\phi})|, \tag{2.3.4}$$

then

$$f(z) = \sum_{j=-\infty}^{\infty} f(jh) \frac{\sin \frac{\pi}{h}(z-jh)}{\frac{\pi}{h}(z-jh)}, \tag{2.3.5}$$

in which the cardinal series is uniformly convergent in any finite domain for the z plane and $h = |h| e^{i\phi}$".

We remark that the uniform convergence of the series above is not obvious. We observe that (2.3.3) and (2.3.4) express the fact that $f(z) = o(e^{A|z|})$, a condition which is used for the Paley and Wiener theorem [29, p. 13] concerning finite Fourier transforms. This we will quote and use to prove Corollary 2 of Theorem 4.2.1. Also, the following theorems were stated and proved, which will bring WKS sampling theorem very close to Whittaker's theorem 1.3.1. First they introduced the definition:

Definition 2. 3. 1. "A function $f(z)$ is said to be band-limited if there exists a constant $W > 0$ and a function $g(\omega)$ of bounded variation over the interval $(-2\pi W, 2\pi W)$ so that

$$f(z) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} e^{i\omega z} dg(\omega) . \tag{2.3.6}$$

The $g(\omega)$ is the Fourier-Stieltjes spectrum of $f(z)$. In case $g(\omega)$ is absolutely continuous over $(-2\pi W, 2\pi W)$

$$f(z) = \frac{1}{2\pi} \int_{-2\pi W}^{2\pi W} e^{i\omega z} g(\omega)\, d\omega, \tag{2.3.7}$$

and $g(\omega)$ is the Fourier spectrum of $f(z)$. Clearly, a band-limited function is entire".

Theorem 2. 3. 3. "A band-limited function $f(z)$ with maximum frequency (angular) W is represented by

$$f(z) = \sum_{j=-\infty}^{\infty} f(jh) \frac{\sin \frac{\pi}{h}(z-jh)}{\frac{\pi}{h}(z-jh)} , \tag{2.3.8}$$

provided that the sampling interval h satisfies $|h| < \frac{1}{2W}$".

Corollary 1. "If $zf(z)$ is band-limited and $\lim_{z \to 0} z f(z) = 0$ then $f(z)$ is represented by (2. 3. 8), provided that $|h| \leq \frac{1}{2W}$".

Corollary 2. "If $f(z)$ is band-limited with maximum frequency W and Fourier spectrum of bounded variation over the interval $(-2\pi W, 2\pi W)$, then $f(z)$ is representable by the infinite series (2.3.8), provided that $|h| \leq \frac{1}{2W}$".

Also, Theorem 2.3.3 and its corollaries were proved for the case of Theorem 2.4.2 of the next section, where the samples are $f(jh)$ and $f'(jh)$.

2.4 Sampling with the Values of the Function and Its Derivatives

When Shannon introduced the sampling theorem he also remarked that the value of $f(t)$ can be reconstructed from the knowledge of the function and its derivative at every other sample point, then extended his remarks to the higher derivatives. Fogel [17] considered this question without reference to the above remark, stated and proved the following theorem.

Theorem 2.4.1. "If a function $f(t)$ contains no frequency higher than W (cps) it is determined by giving M function derivative values at each of a series of points extending throughout the time domain, sampling interval $T = \frac{M}{2W}$ being the time interval between instantaneous observations".

Later Jagerman and Fogel [19] were able to incorporate the above theorem with Theorem 2. 3. 1 when they realized that all that was needed in the case of $f(jh)$ and $f'(jh)$ samples is to have double zeros for $g(z)$. Their choice was $g(z) = \sin^2 \frac{\pi}{h} z$, to give

Theorem 2. 4. 2. "If $f(z)$, the entire function, satisfies the conditions in Theorem 2. 3. 1 then

$$f(z) = \sum_{j=-\infty}^{\infty} [f(jh) + (z-jh)f'(jh)] \left[\frac{\sin \frac{\pi}{h}(z-jh)}{\frac{\pi}{h}(z-jh)}\right]^2 , \qquad (2.4.1)$$

in which the series is uniformly convergent in any closed domain of the z-plane, $h = |h| e^{i\phi}$".

Theorem 2. 4. 3. "A band-limited function $f(z)$ with maximum frequency W is representable by Equation (2. 4. 1) provided that $|h| < 1/W$". Corollaries of this theorem follow in exactly the same way as those of Theorem 2. 3. 3.

The importance of the last theorem lies in its application. For example, within an aircraft estimated velocity as well as position is used to determine a continuous course plot of the path with half the sampling rate.

As a generalization to the above results and as an explicit answer to Shannon's remark concerning the reconstruction of a function $f(t)$ when the value of the function and its first R

derivatives are given at equi-distant sampling points $(R+1)/2W$ seconds apart, Linden and Abramson [27] gave this result:

Theorem 2.4.3. "Let $f(t)$ be a continuous function with Fourier transform $F(\omega)$ such that $F(\omega) = 0$ for $|\omega| > 2\pi W$. Then

$$\sum_{k=-\infty}^{\infty} \left[f(kh)+(t-kh)f'(kh) + \frac{(t-kh)^2}{2!} f''(kh) + \cdots + \frac{(t-kh)^R}{R!} f^{(R)}(kh) \right] \left[\frac{\sin \frac{\pi}{h}(t-kh)}{\frac{\pi}{h}(t-kh)} \right]^{R+1} \quad (2.4.2)$$

is equal to $f(t)$ where $h = \frac{R+1}{2W}$ ".

In addition, they showed that for large R this R-derivative expansion approaches the Taylor series weighted by a Gaussian density function centered about each sample point.

We remark that the last result is in agreement with the physical interpretation since, for $R = 0$, $\frac{\sin ax}{x}$ is the impulse response to an ideal low-pass filter and, for $R = 1$, $\left(\frac{\sin ax}{x}\right)^2$ is the impulse response of a filter with idealized triangular form. Then $\lim_{R \to \infty} \left(\frac{\sin ax}{x}\right)^{R+1}$ would be the impulse response of a filter whose function approaches the Gaussian density as $R \to \infty$, and the impulse response itself is of the Gaussian form.

Kahn and Liu [22] treated the problem of the representation and construction of signals, not from one set of data $\{f(\frac{n\pi}{a})\}$ but from several sets of sampled values obtained by using a multiple channel sampling scheme. They showed that with the optimum combination of pre-filters and post-filters, in the case where two sets of sample values are taken, the frequency range of the input signal is limited by the pre-filters to a total width of $4a$. This is in the stead of the usual total width of $2a$ when a single channel is used, which makes it stand as a natural extension of the latter case.

2.5 Sampling at the Zeros of the Function

Bond and Cahn [6] considered extending the WKS sampling theorem for the case when $\{t_n\}$, the sampling points, are not independent of the sampled signal $f(t)$. Their justification was that such a procedure had proved valuable in minimizing the error caused by infinite clipping, which means that one can transmit a continuous signal over a discrete channel if the zero crossings of $f(t)$ are preserved. For $f(t)$ a band-limited function on $(0, W)$ they extended t to a complex variable z and used the Titchmarsh [44] result that

$$F(z) = \int_{-W}^{W} e^{2\pi i f z} V(f)\, df \qquad (2.5.1)$$

is a real, entire function, described by the location of its zeros which are either real or occur as complex conjugate pairs. In general, the zeros tend to cluster near the real axis. Furthermore, the aggregate of the zeros occur at the Nyquist rate. Thus,

$$f(z) = f(0) \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right), \tag{2.5.2}$$

where $f(0) \neq 0$, $z_n = R_n e^{i\theta_n}$, $R_n \leq R_{n+1}$ and $\lim_{n\to\infty} \frac{2WR_n}{n} = 1$.

Note that the formula of (2.5.2) needs all the past and future zeros, both real and complex, which makes it not practicable. Instead, they suggested another, more practicable problem with specified interval $(-\frac{T}{2}, \frac{T}{2})$ and zeros inside this interval occurring at slightly less than the Nyquist rate, zeros outside real and occurring at the Nyquist rate. Let N be the largest integer not exceeding WT; then there are a maximum of $2N$ real or complex zeros, $z_n = t_n + iu_n$, $|t_n| < T/2$. Outside this interval the zeros occur at $t_n = \pm\frac{n}{2W}$, for $n = N+1, N+2, \cdots$. Using this in (2.5.2) and referring to the infinite product representation of the sine function they obtained

$$f(t) = \sum_{n=-N}^{N} (-1)^n A_n \frac{\sin(2\pi Wt - n\pi)}{2\pi Wt - n\pi}, \tag{2.5.3}$$

where A_n is expressed in terms of the values of the zeros inside the interval. So, it appears that a band-limited function $f(t)$ can be represented by a finite sum when the amplitude A_n at these sampling points $\{\frac{n}{2W}\}$ is determined in terms of the finite number of the real and complex zeros for $|t_n| < \frac{T}{2}$.

2.6 Sampling with Non-uniformly Spaced Sampling Points

For the case of a band-limited function $f(t)$ with all the sample points outside the interval $(-T, T)$ being exactly zero, Shannon [34] remarked, as did others before him, that only then can $f(t)$ be specified by $2WT$ sample points. He also remarked that these $2WT$ sample points need not be equally spaced, an idea that cannot be covered by his version of the WKS sampling theorem and its cardinal series. We review here some of the work which was done in this direction. The first is a statement which was attributed to Cauchy by Black [5, p. 41],

> If a signal is a magnitude-time function, and if time divided into equal intervals such that each subdivision comprises an interval T seconds long, where T is less than half the period of the highest significant frequency component of the signal, and if one instantaneous sample is taken from each sub-interval in any manner, then a knowledge of the instantaneous magnitude of each sample plus a knowledge of the instant within each sub-interval at which the sample is taken, contains all the information of the original signal.

Yen [56] considered the case where a finite number of uniform sample points migrate in a uniform distribution to new distinct positions. He proved that the band-limited signal $f(t)$ remains uniquely defined, then reconstructed $f(t)$. When the number of migrated points increases without limit he called it a gap and proved a similar theorem. Yen also considered the case of a "recurrent, non-uniform sampling". That is, when the sampling points are divided into groups of N points each, and the groups have a recurrent period of $N/2W$ seconds. Here W is the maximum frequency of the band-limited function $f(t)$. He determined $f(t)$ uniquely and reconstructed it in terms of its values at $t = t_p + \frac{mN}{2W}$, $p = 1, 2, \cdots, N$ and $m = \cdots, -1, 0, 1, \cdots$. We note that Yen's first result answered the remark of Shannon in that the $2WT$ sample points, necessary for constructing the time-limited, band-limited signal $f(t)$, need not be equally spaced.

In addition, Yen proved the "minimum energy signal" theorem for constructing the above $f(t)$ without specifying the time interval explicitly.

Theorem 2. 6. 1. " If the sample values at a finite set of arbitrarily distributed sample points $t = t_p$, $p = 1, 2, \cdots, N$, are given, then a signal $f(t)$ with no frequency components above W (cps) is defined uniquely under the condition that the energy of the signal,

$\int_{-\infty}^{\infty} f^2(t)dt$, is a minimum".

2.7 Application of the Sampling Theorem for Signal of a Continuous Time Parameter

Another extension of the WKS sampling theorem was considered by Balakrishnan [3] where he proved that the WKS sampling theorem can be used to represent a process of a continuous time parameter. One of the theorems in this direction is

Theorem 2.7.1. "Let $x(t)$, $-\infty < t < \infty$, be a real or complex valued stochastic process, stationary in the "wide sense" (or second order stationary), possessing a spectral density which vanishes outside the interval $[-2\pi W, 2\pi W]$. Then $x(t)$ has the representation

$$x(t) = \text{l.i.m.} \sum_{n=-\infty}^{\infty} x\left(\frac{n}{2W}\right) \frac{\sin \pi(2Wt-n)}{\pi(2Wt-n)} \tag{2.7.1}$$

for every t, where l.i.m. stands for limit in the mean square". The proof consists of using the WKS sampling theorem for the covariance function of the process, since it is assumed to have a truncated Fourier transform. Then $x^{*}(t)$, the optimal estimate of $x(t)$, was constructed by using the cardinal series to show that the mean square error is zero.

In a more recent paper Balakrishnan [4] considered the question, "that a stationary stochastic process that is band-limited is not physically realizable". As a solution he chose to speak of "essentially band-limited stochastic processes", a notion that Slepian and Pollak [36] had given for the deterministic signals by allowing time-limited and band-limited signals at the same time and in apparent violation of the uncertainty principle.

Let us consider now the case of no cross-correlation between m(t), the signal, and n(t), the noise, [48, p. 14]; then $S_{mf}(\omega) = S_m(\omega)$ and $S_f(\omega) = S_m(\omega) + S_n(\omega)$, where $S_f(\omega)$ is the spectral density of f(t). For an optimum filter the transfer function [48, p. 14]

$$K(\omega) = \int_{-\infty}^{\infty} e^{-i\omega\tau} k(\tau) d\tau \qquad (2.7.2)$$

(where k(t) is the impulse response of the filter), must satisfy

$$K(\omega) = \frac{S_{mf}(\omega)}{S_f(\omega)} = \frac{S_m(\omega)}{S_m(\omega) + S_n(\omega)} \; . \qquad (2.7.3)$$

Since $S_f(\omega)$ is band-limited it is obvious that $K(\omega)$ is band-limited. Let $K(\omega) = 0$, $\omega < \omega_2$, $\omega > \omega_2'$, then

$$k(t) = \frac{1}{2\pi} \int_{\omega_2}^{\omega_2'} e^{i\omega t} K(\omega) d\omega, \qquad (2.7.4)$$

the impulse response of the optimum filter. Now the ouptut of the optimum filter is

$$x(t) = \int_{-\infty}^{\infty} k(\tau) f(t-\tau) d\tau . \qquad (2.7.5)$$

But $k(t) \leftrightarrow K(\omega)$, $f(t) \leftrightarrow F(\omega)$, where $\leftrightarrow$ implies Fourier transform mates. Then

$$\int_{-\infty}^{\infty} k(\tau) f(t-\tau) d\tau \leftrightarrow K(\omega) F(\omega), \qquad (2.7.6)$$

and

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} K(\omega) F(\omega) = \int_{\omega_2}^{\omega_2'} e^{i\omega t} K(\omega) F(\omega) d\omega . \qquad (2.7.7)$$

Thus $x(t)$ can be sampled using the WKS sampling theorem. So, we have proved the following:

<u>Corollary 1.</u> The output of any optimum time-invariant filter, with input a stationary stochastic process with band-limited spectral density, is band-limited and is samplable by the WKS sampling theorem.

<u>2.8 Error Analysis for Sampling Theory and Other Extensions</u>

The most recent extension of the WKS sampling theorem is due to Papoulis [31]. The first attempt is to move away from the ideal low-pass filter which is associated with physical interpretation of the sampling series. As we mentioned earlier the price of this freedom is paid in terms of a higher sampling rate. Papoulis considered a band-limited function $f(t)$ but he constructed it in a more general way as

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_0(t-nT)}{\omega_2(t-nT)} , \qquad (2.8.1)$$

where $\omega_2 \equiv \frac{\pi}{T} \geq \omega_1$, $\omega_1 \leq \omega_0 \leq 2\omega_2 - \omega_1$. The proof is considered to be a particularly elegant version of the proof of the popular sampling theorem [31] and [30, p. 50]. It is clear now that the sampling rate for (2.8.1) is higher than that of the usual cardinal series (which corresponds to the special case $\omega_1 = \omega_0 = \omega_2$).

Papoulis tried to prove the following converse of the above theorem:

<u>Theorem 2.8.1.</u> "Given an arbitrary sequence of numbers $\{a_n\}$, if we form the sum

$$x(t) = \sum_{n=-\infty}^{\infty} a_n \frac{\sin \omega_0(t-nT)}{\omega_2(t-nT)} , \qquad (2.8.2)$$

then $x(t)$ is band-limited by ω_0". We note that Papoulis' proof of this theorem needs a condition on $\{a_n\}$ so his term by term integration is valid. A sufficient condition is that $\sum_{n=-\infty}^{\infty} |a_n| < \infty$. We will prove this theorem in Chapter IV as Lemma 2 for a different purpose and by a different method.

The result (2.8.1) made it possible to interpret the sampling result

$$f(t) = \sum_{n=-\infty}^{\infty} T\, f(nT) k(t-nT), \qquad (2.8.3)$$

where $k(t)$ is the Fourier transform of

$$K(\omega) = \begin{cases} 1, & -\omega_1 \le \omega \le \omega_1 \\ 0, & 2n\omega_2 - \omega_1 \le \omega \le 2n\omega_2 + \omega_1, \quad n \neq 0, \\ \text{arbitrary elsewhere.} & \end{cases} \qquad (2.8.4)$$

This $K(\omega)$ is a more general system function of a filter than the one for the ideal low-pass filter, although it requires higher sampling rate since $\omega_2 \equiv \frac{\pi}{T} \ge \omega_1$. So, in this way, the sharp cut-off is avoided although the vanishing condition is not allowed exactly for the case of $K(\omega) \in L_2$ and with impulse response as a causal function. A causal function is defined to be zero for negative values of the argument. This is due to the Paley-Wiener condition [30, p. 215]. As a result there will be an error due to the impossibility of eliminating the higher frequencies of $F^*(\omega)$, the periodic extension of

$F(\omega)$. This error can be minimized when the part of $K(\omega)F^*(\omega)$ which does not vanish in the second range of (2. 8. 4) can be neglected as ω_2 becomes large compared to ω_1. This means a higher sampling rate.

To solve this problem and to use more realistic impulses Papoulis considered two filters. The first is chosen with impulse response $h_0(t)$ so that its output for the input pulses $f(nT)$ becomes

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(nT)h_0(t-nT). \qquad (2.\,8.\,5)$$

This $f_0(t)$ will serve as an input to a second filter with a system function $H_1(\omega)$ such that

$$H_1(\omega)H_0(\omega) = F(\omega), \qquad (2.\,8.\,6)$$

where $H_0(\omega)$ is the system function of the first filter and $F(\omega)$ is the spectrum of $f(t)$ with a cut-off at ω_1. Hence $f(t)$, the output of the two filters in cascade, is a band-limited function with a sampling function that can be interpreted in terms of more realistic pulses as input and in terms of filters with more realistic impulse response.

Papoulis also gave the sampling series of the function $f^2(t)$ as

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$$f^2(t) = \sum_{n=-\infty}^{\infty} f^2(nT) \frac{\sin \omega_0(t-nT)}{\omega_2(t-nT)}, \tag{2.8.7}$$

where $\omega_2 \equiv \frac{\pi}{T}$ and $\omega_2 \geq 2\omega_1$ (instead of $\omega_2 \geq \omega_1$ in the case for $f(t)$) and ω_0 is such that $2\omega_1 \leq \omega_0 \leq 2\omega_2 - 2\omega_1$.

In his study of the error analysis for the sampling theorem Papoulis applied this theorem to the round-off error

$$\epsilon_n = f(nT) - \bar{f}(nT), \tag{2.8.8}$$

where $\bar{f}(nT)$ is the recorded or tabulated sampled values which differ from the exact sampled values by ϵ_n. Using the cardinal series with sampled values $\bar{f}(nT)$ he constructed the function $f_r(t)$, which differs from $f(t)$ by the total round-off error $e_r(t)$. Combined with the above results in (2.8.7) he showed that this error $e_r(t)$ is bounded by its own total energy E_e. That is,

$$|e_r(t)| \leq \left(\frac{\omega_1 E_e}{\pi}\right)^{\frac{1}{2}}. \tag{2.8.9}$$

He then treated the truncation error $e_N(t)$ that results when $f_N(t)$ is constructed as a finite sum. This immediately reduced to a truncated Fourier series for $F(\omega)$, its Fourier transform. A bound on $e_N(t)$ was given and was attributed to Jagerman.

Another important error that Papoulis considered is the

"Aliasing Error",

$$e_a = f_a(t) - f(t) , \qquad (2.8.10)$$

which results from constructing $f_a(t)$ as

$$f_a(t) = \sum_{n=-\infty}^{\infty} f(nT) \frac{\sin \omega_2(t-nT)}{\omega_2(t-nT)} , \qquad (2.8.11)$$

where $\omega_2 \equiv \frac{\pi}{T} < \omega_1$ and ω_1 is the band limit of $f(t)$. He gave a bound on this error in terms of the area B of its spectrum as

$$|e_a(t)| \leq \frac{B}{2\pi} |\sin \omega_2 t| . \qquad (2.8.12)$$

Papoulis also considered the jitter problem, which arises when the sample values are not exactly at the sampling points nT but are at some other instants $nT - u_n$, where $\{u_n\}$ is the set of deviations of the sampling points from nT. He considered

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} u_n \frac{\sin \omega_2(\tau - nT)}{\omega_2(\tau - nT)} \qquad (2.8.13)$$

to be band-limited, using Theorem 2.8.1. We again remark that this theorem is correct for $\sum_{n=-\infty}^{\infty} |u_n| < \infty$, a condition which is not obviously satisfied by the set of numbers in the above application.

Assume that this condition is satisfied and that $\theta(\tau)$ is small. He used the transformation

$$t = \tau - \theta(\tau), \qquad \tau = \gamma(t), \tag{2.8.14}$$

to show that $f(t)$ can still be sampled in terms of $f(nT - u_n)$ as

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT - u_n) \frac{\sin \omega_2[\gamma(t) - nT]}{\omega_2[\gamma(t) - nT]}, \tag{2.8.15}$$

provided that $\theta(\tau)$ is such that

$$f[\tau - \theta(\tau)] \simeq f(\tau) - \theta(\tau) f'(\tau). \tag{2.8.16}$$

Note that the above Taylor series expansion, if adequate, results in a band-limited function with cut-off at $\omega_1 + \omega_2$, where ω_2 and ω_1 are the band limits for $F(\omega)$ and $\Theta(\omega)$ respectively. So, unless $\theta(\tau)$ is very small, the (2.8.15) series will suffer from the aliasing error.

We remark that if a second derivative is needed in the above Taylor expansion then the sampled function is band-limited with cut-off at $\omega_1 + 2\omega_2$, which will increase the above error.

From the above discussion one may conclude that it is not clear that the jitter problem is solved. At this point we may propose a safe but less general treatment of the problem. This will consist of a restriction that the set $\{u_n\}$ be restricted to a finite number, say M, of non-zero sampling points.

2.9 Sampling Theorem in n-Dimensional Space

The sampling theorem was extended to include sampling for n variables. The following is the statement given in [33, p. 453] and its proof follows the same method used for the one-dimensional WKS sampling theorem given by Parzen [32].

Theorem 2.9.1. "Let $f(t_1, t_2, \cdots, t_n)$ be a function of n real variables, whose n-dimensional Fourier integral exists and is identically zero outside an n-dimensional rectangle and is symmetrical about the origin; that is,

$$g(y_1, y_2, \cdots, y_n) = 0, \qquad |y_k| > |\omega_k|, \quad k = 1, 2, \cdots, n. \tag{2.9.1}$$

Then

$$f(t_1, t_2, \cdots, t_n) = \sum_{m_1=-\infty}^{\infty} \cdots\cdots \sum_{m_n=-\infty}^{\infty} f\left(\frac{\pi m_1}{\omega_1}, \cdots, \frac{\pi m_n}{\omega_n}\right) \cdot \frac{\sin(\omega_1 t_1 - m_1\pi)}{\omega_1 t_1 - m_1\pi} \cdots\cdots \frac{\sin(\omega_n t_n - m_n\pi)}{\omega_n t_n - m_n\pi}. \tag{2.9.2}$$

2.10 Other Extensions of the WKS Sampling Theorem

One important extension of the sampling theorem was due to

Kohlenberg [23], and that is to consider the function $f(t)$ as band-limited in (w_0, w_0+w) instead of the usual interval $(0, w)$. He proved that in this case $f(t)$ is completely determined by its values at a properly chosen set of points.

As an application to the WKS sampling theorem we introduce the following example which leads to interesting results. From Titchmarsh [45, p. 186] we have

$$\frac{\pi\,\Gamma(a-1)}{2^{a-2}\Gamma(\frac{a+x}{2})\Gamma(\frac{a-x}{2})} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos t)^{a-2} e^{ixt} dt, \quad a > 1. \qquad (2.10.1)$$

We recognize that this is a finite Fourier transform and so by the use of the sampling theorem we can immediately write down the infinite series representation of the left hand quantity:

$$\frac{1}{\Gamma(\frac{a+x}{2})\,\Gamma(\frac{a-x}{2})} = \sum_{n=-\infty}^{\infty} \frac{1}{\Gamma(\frac{a+2n}{2})\Gamma(\frac{a-2n}{2})} \, \frac{\sin(\frac{\pi}{2}x-n\pi)}{(\frac{\pi}{2}x-n\pi)}, \qquad (2.10.2)$$

a result which is apparently not in the literature. Setting $a = 2$ in (2.10.1) leads to the well-known special case

$$\frac{1}{\Gamma(\frac{x}{2})\,\Gamma(1-\frac{x}{2})} = \frac{\sin\frac{\pi}{2}x}{\pi}. \qquad (2.10.3)$$

This method may be extended to many other functions with finite

Fourier transform representation. Finite Fourier transform tables are found in many places [10, 14, 15].

CHAPTER III

THE GENERALIZED SAMPLING THEOREM

3.1 Kramer's Generalization of the Sampling Theorem

Kramer's generalized (WKSK) sampling theorem [26] was originally stated as the following lemma, and we will give his proof with the necessary details.

Lemma "Let I be an interval and $L_2(I)$ the class of functions $\phi(x)$ for which $\int_I |\phi(x)|^2 dx < \infty$. Suppose that for each real t

$$f(t) = \int_I K(t,x)g(x)dx, \tag{3.1.1}$$

where $g(x) \epsilon L_2(I)$. Suppose that for each real t, $K(t,x) \epsilon L_2(I)$, and that there exists a countable set $E = \{t_n\}$ such that $\{K(t_n, x)\}$ is a complete orthogonal set on $L_2(I)$. Then

$$f(t) = \lim_{N \to \infty} \sum_{|n| \leq N} f(t_n) S_n(t), \tag{3.1.2}$$

where

$$S_n(t) = \frac{\int_I K(t,x)\overline{K(t_n,x)}dx}{\int_I |K(t_n,x)|^2 dx} \quad [4] \ . \tag{3.1.3}$$

Proof. Let

$$f_N(t) = \sum_{|n| \leq N} f(t_n) S_n(t) \ ; \tag{3.1.4}$$

from (3. 1. 1)

$$f(t_n) = \int_I K(t_n, x) g(x) dx \tag{3.1.5}$$

and from (3. 1. 1) and (3. 1.4) we get

$$f(t) - f_N(t) = \int_I K(t,x) g(x) dx - \sum_{|n| \leq N} f(t_n) S_n(t). \tag{3.1.6}$$

Using (3. 1.5) this becomes

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[4] This is Kramer's original form but later we will introduce a weighting function $\rho(x)$ to the integral, as was suggested by Campbell [7] .

$$f(t) - f_N(t) = \int_I K(t,x)g(x)dx - \sum_{|n|\le N} [\int_I K(t_n,x)g(x)dx]\, S_n(t)$$

$$= \int_I K(t,x)g(x)dx - \int_I [\sum_{|n|\le N} K(t_n,x)S_n(t)]\, g(x)dx \qquad (3.1.7)$$

$$= \int_I [[K(t,x) - \sum_{|n|\le N} K(t_n,x)S_n(t)]]\, g(x)dx\,.$$

The triangular and the Schwarz inequalities lead to

$$|f(t) - f_N(t)| \le \sqrt{\int_I |[K(t,x) - \sum_{|n|\le N} K(t_n,x)S_n(t)]|^2 dx \int_I |g(x)|^2 dx}. \qquad (3.1.8)$$

Since $\{K(t_n,x)\}$ is a complete orthogonal set

$$K(t,x) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{|n|\le N} c_n K(t_n,x), \qquad (3.1.9)$$

where

$$c_n = \frac{\int_I K(t,x)K(t_n,x)dx}{\int_I |K(t_n x)|^2 dx} \equiv S_n(t). \qquad (3.1.10)$$

Hence

$$\lim_{N\to\infty} \int_I \left| K(t,x) - \sum_{|n|\leq N} K(t_n, x) S_n(t) \right|^2 dx = 0 \,, \tag{3.1.11}$$

and, from (3.1.8) and (3.1.11) we get

$$\lim_{N\to\infty} \left| f(t) - f_N(t) \right| = 0, \tag{3.1.12}$$

which means

$$f(t) = \lim_{N\to\infty} \sum_{|n|\leq N} f(t_n) S_n(t). \tag{3.1.13}$$

3.2 Illustration for the Generalized Sampling Theorem

Kramer showed that the conditions for his lemma of Section 3.1 on the kernel $K(x,t)$ in (3.1.1) is exhibited by the solutions of nth order,[5] self-adjoint[6] differential equations. His theorem now goes as

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[5] This theorem was arrived at independently by P. Weiss [50], for the case $n = 2$.

[6] An excellent treatment of this subject is given in Coddington and Levenson [8, p. 188 and p. 284], and its application to a fourth order differential equation [40] is here given in Appendix A.1.

Theorem 3.2.1. "Let

$$Lu = tu, \quad B_1(u) = B_2(u) = \cdots\cdots = B_n(u) = 0 \tag{3.2.1}$$

be a self-adjoint boundary value problem for an nth order differential operator L on the finite interval (a, b). Suppose that there exists a solution $u(t, x)$ of the differential equation $Lu = tu$ such that the set of zeros E_i of $B_i[u(t, x)]$ is independent of i, then $u(t, x)$ meets the condition in the lemma on $K(t, x)$".

Kramer gave the following two examples as illustration to his theorem. The first will show that the WKS sampling theorem is a special case of the WKSK (generalized) sampling theorem. We mention here for future reference that Campbell [7] also illustrated this WKSK sampling theorem, in addition to raising the question of possibly no advantage of the generalized theorem over the popular theorem.

Example 1. Let

$$Lu = -\frac{i}{2\pi}\frac{du}{dx}, \quad B_1(u) = u(a) - u(b) = 0 . \tag{3.2.2}$$

The solution of $Lu = tu$ is at once

$$u(t, x) = C \exp(2\pi i t x). \tag{3.2.3}$$

The boundary conditions require that

$$u(t_n, x) = C \exp\left(\frac{2\pi inx}{b-a}\right), \quad t_n = \frac{n}{b-a}. \tag{3.2.4}$$

If we use this result in (3. 1. 3) we get

$$S_n(t) = \frac{\exp\left[2\pi ib\left(t - \frac{n}{b-a}\right)\right] - \exp\left[2\pi ia\left(t - \frac{n}{b-a}\right)\right]}{2\pi i(b-a)\left(t - \frac{n}{b-a}\right)}. \tag{3.2.5}$$

Replace a by -a, set b = a, final form is

$$S_n(t) = \frac{\sin 2\pi a\left(t - \frac{n}{2a}\right)}{2\pi a\left(t - \frac{n}{2a}\right)}, \tag{3.2.6}$$

so (3. 1. 2) may be written as

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2a}\right) \frac{\sin 2\pi a\left(t - \frac{n}{2a}\right)}{2\pi a\left(t - \frac{n}{2a}\right)}. \tag{3.2.7}$$

This is the form of the usual sampling series of the band-limited function of Shannon.

Example 2, Bessel functions. Let

$$Lu = -\frac{d^2u}{dx^2} + \frac{n^2 - \frac{1}{4}}{x^2} = tu, \quad B_1(u) = u(t, 0) = 0, \quad B_2(u) = u(t, 1) = 0. \tag{3.2.8}$$

For $u = \sqrt{x}\,v$ this reduces to the familiar form of the Bessel differential equation in v,

$$x^2 \frac{d^2v}{dx^2} + x\frac{dv}{dx} + (tx^2 - n^2)v = 0, \tag{3.2.9}$$

so the solution which is finite at the origin is $v(x,t) = J_n(xt)$ and

$$u(x,t) = \sqrt{x}\; J_n(x\sqrt{t}) . \tag{3.2.10}$$

Clearly the first boundary condition is satisfied for $n \geq 0$, the second yields a set of numbers,

$$J_n(\sqrt{t_k}) = 0, \quad k = 1, 2, \cdots . \tag{3.2.11}$$

According to Kramer's theorem if

$$f(t) = \int_0^1 \sqrt{x}\; J_n(x\sqrt{t})g(x)dx \tag{3.2.12}$$

then

$$f(t) = \sum_{k=1}^{\infty} f(t_k)S_k(t), \tag{3.2.13}$$

where

$$S_k(t) = \frac{\int_0^1 x\, J_n(x\sqrt{t})J_n(x\sqrt{t_k})dx}{\int_0^1 x[J_n(x\sqrt{t_k})]^2 dx} = -\frac{2\sqrt{t_k}\; J_n(\sqrt{t})}{(t-t_k)J_{n+1}(\sqrt{t_k})} \tag{3.2.14}$$

(see Appendix A. 2).

Campbell [7] illustrated the WKSK sampling theorem for the case of the kernel $K(t, x)$ taking on a form of the Legendre function $P_t(x)$.

Example 3, Legendre functions. Consider the Legendre differential equation,

$$\frac{d}{dx}\left[(1-x^2)\frac{du}{dx}\right] + (t^2 - \tfrac{1}{4})u = 0\;; \tag{3.2.15}$$

the solution is taken to be $P_{t-\frac{1}{2}}(x)$, which is finite at the endpoints. If

$$f(t) = \int_{-1}^{1} P_{t-\frac{1}{2}}(x)g(x)dx \tag{3.2.16}$$

then

$$f(t) = \sum_{n=0}^{\infty} f(t_n)S_n(t), \tag{3.2.17}$$

where

$$S_n(t) = \frac{\int_{-1}^{1} P_{t-\frac{1}{2}}(x) P_n(x) dx}{\int_{-1}^{1} |P_n(x)|^2 dx} . \tag{3.2.18}$$

Using [11, p. 170, Eq. 3.12(17)] we get

$$S_n(t) = \frac{2n+1}{\pi} \frac{\sin(t-n-\frac{1}{2})}{(t-n-\frac{1}{2})(t+n+\frac{1}{2})} . \tag{3.2.19}$$

3.3 Comparison of the Generalized and the Popular Sampling Theorems

Campbell [7] illustrated the WKSK sampling theorem in a different manner. He considered as kernels of the generalized sampling theorem the solution of a regular first order differential equation, the solution of a regular second order differential equation with separated boundary conditions, and the solutions of the singular Bessel and Legendre equations. For all these cases he showed that if a function with such a kernel can be expanded by the use of the WKSK sampling theorem then it can also be expanded by the use of the WKS sampling theorem. In addition, he mentioned that the asymptotic spacing for the WKSK sampling theorem is the same as that for the WKS sampling theorem. He then gave some suggestions concerning the solution of the nth order self-adjoint boundary value problems. In sum, for the above mentioned functions he concluded that the WKSK sampling theorem has no advantage over the popular WKS theorem. The following is a summary of his results with some details.

1. First Order Equations. Consider the self-adjoint boundary value problem,

$$ip(x)\frac{du}{dx}+\left[q(x)+\frac{i}{2}p'(x)+\lambda\rho(x)\right]u = 0, \sqrt{p}(a)u(a) = e^{i\phi}\sqrt{p}(b)\,u(b), \tag{3.3.1}$$

where $p(x)$, $p'(x)$, $q(x)$ and $\rho(x)$ are real-valued, continuous functions on the closed interval $[a,b]$. Then, by a change of variables, $x = \gamma(\xi)$, $\Omega = \Omega(a,b)$, and $v(x) = \sqrt{p}(x)u(x)$, and Campbell showed that the solution may be represented as

$$u(\xi,\lambda) = \frac{C(\lambda)}{\sqrt{p}}\, e^{i[\lambda\xi + M(\xi)]}; \tag{3.3.2}$$

then

$$S_n(\lambda) = \frac{\int_a^b u(x,\lambda)\overline{u(x,\lambda_n)}\rho(x)dx}{\int_a^b |u(x,\lambda_n)|^2\rho(x)dx} \tag{3.3.3}$$

becomes

$$S_n(\lambda) = \frac{C(\lambda)}{C(\lambda_n)}\;\frac{\sin\Omega(\lambda-\lambda_n)}{\Omega(\lambda-\lambda_n)}, \tag{3.3.4}$$

which is the cardinal function of the Whittaker sampling theorem (aside from $\frac{C(\lambda)}{C(\lambda_n)}$) which may be obtained when the $\exp(i\lambda\xi)$ part of (3.3.2) is considered to be the kernel associated with the popular sampling theorem.

2. Second Order Equations. A. Consider the Sturm-Louiville problem

$$\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + [\lambda\rho(x) - q(x)]u = 0, \tag{3.3.5}$$

with boundary conditions

$$\begin{aligned} u(a)\cos\alpha - p(a)u'(a)\sin\alpha &= 0, \\ u(b)\cos\alpha_1 - p(b)u'(b)\sin\alpha_1 &= 0, \end{aligned} \tag{3.3.6}$$

where $p(x)$, $q(x)$ and $\rho(x)$ are positive and $p'(x)$, $q(x)$ and $\rho'(x)$ are continuous on the closed interval $[a,b]$. Let $\lambda = t^2$ and let the characteristic values t_n^2 be restricted to simple and positive.

Campbell also considered a partial differential equation of second order in x and y,

$$\frac{\partial}{\partial x}\left[p(x)\frac{\partial w}{\partial x}\right] - \rho(x)\frac{\partial^2 w}{\partial y^2} - q(x)w = 0. \tag{3.3.7}$$

Under the above restriction on $p(x)$, $q(x)$ and $\rho(x)$ this will be a hyperbolic equation. He integrated it by Riemannian methods; after a number of changes of variables he obtained the following representation for the solution of the above Sturm-Louiville problem:

$$u(x,t) = C(t)[G_1(x)e^{i\beta t} + G_2(x)e^{-i\beta t} + \int_{-\beta}^{\beta} G_3(x,\eta)e^{i\eta t}d\eta] , \tag{3.3.8}$$

where $\eta(\xi)$ is the characteristic curve used in the Riemannian method. It is the solution of the differential equation

$$\frac{d\eta}{d\xi} = \pm \sqrt{\frac{\rho(\xi)}{p(\xi)}} \tag{3.3.9}$$

and

$$\beta(x) = \int_a^x \sqrt{\frac{\rho(y)}{p(y)}}\, dy . \tag{3.3.10}$$

For

$$f(t) = \int_a^b u(x,t)g(x)\rho(x)dx \tag{3.3.11}$$

Campbell used (3.3.8) with

$$\Omega_2 = \int_a^b \sqrt{\frac{\rho(x)}{p(x)}}\, dx ; \tag{3.3.12}$$

an interchange and rearrangement led to

$$f(t) = C(t) \int_{-\Omega_2}^{\Omega_2} H(\gamma) e^{i\gamma t} d\gamma , \qquad (3.3.13)$$

and he concluded that the integral in (3. 3. 13) may be expanded by the WKS sampling theorem, obtaining

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{\Omega_2}\right) \frac{C(t)}{C\left(\frac{n\pi}{\Omega_2}\right)} \frac{\sin(\Omega_2 t - n\pi)}{(\Omega_2 t - n\pi)} . \qquad (3.3.14)$$

B. Equations with singular points. In contrast to the above solution of the second order differential equation with continuous coefficients Campbell treated the Bessel and Legendre functions as solutions of differential equations of the singular type. In Chapter IV we will extend these results to other functions.

Bessel Functions: We refer to (3. 2. 9) and (3. 2. 12) and note that Campbell used the integral representation of $J_n(z)$ [14, p. 11],

$$J_n(z) = \frac{\left(\frac{z}{2}\right)^n}{\sqrt{\pi}\, \Gamma(n+\frac{1}{2})} \int_{-1}^{1} e^{izy} (1-y^2)^{n-\frac{1}{2}} dy. \qquad (3.3.15)$$

If we set $z = tx$ and $\omega = xy$ we get

$$J_n(tx) = \frac{\left(\frac{t}{2x}\right)^n}{\sqrt{\pi}\, \Gamma(n+\frac{1}{2})} \int_{-x}^{x} e^{i\omega t} (x^2 - \omega^2)^{n-\frac{1}{2}} d\omega. \qquad (3.3.16)$$

If we let $\sqrt{t'} = t$ in (3.2.12) and use (3.3.16) for $J_n(tx)$, then interchange the order of integration in a manner similar to that of (3.3.13), we arrive at (see Appendix A.3)

$$f(t) = t^n \int_{-1}^{1} H(\omega) e^{i\omega t} d\omega , \tag{3.3.17}$$

where $H(\omega)$ is given in (A.3.1). So, the integral in (3.3.17) may now be expanded using the WKS sampling theorem and the $f(t)$ sampling series is obtained in a manner similar to (3.3.14).

Legendre Functions: We refer to (3.2.15) and (3.2.16) and use the integral representation for $P_{t-\frac{1}{2}}(\cos\theta)$ [14, p. 22] :

$$P_{t-\frac{1}{2}}(\cos\theta) = \frac{\sqrt{2}}{\pi} \int_0^{\theta} \frac{\cos \alpha t \, d\alpha}{\sqrt{\cos\alpha - \cos\theta}} = \frac{\sqrt{2}}{2\pi} \int_{-\theta}^{\theta} \frac{e^{i\alpha t} d\alpha}{\sqrt{\cos\alpha - \cos\theta}} . \tag{3.3.18}$$

Substituting this in (3.2.16) and interchanging the order of integration (see Appendix A.3) we get

$$f(t) = \int_{-\pi}^{\pi} e^{i\alpha t} H(\alpha) d\alpha , \tag{3.3.19}$$

where $H(\alpha)$ is given in (A.3.2). So, $f(t)$ may be sampled by using the WKS sampling theorem.

CHAPTER IV

PRESENT EXTENSIONS OF THE TWO SAMPLING THEOREMS

4.1 Further Illustrations of the Kramer Generalized (WKSK) Sampling Theorem

In this section we will consider more functions which are solutions of second order differential equations with singular coefficients, plus one of fourth order with continuous coefficients [21]. This section may be considered as an extension of the work of Kramer [26] and Campbell [7], which was discussed in Section 3.2.

1. Associate Legendre Functions. We consider the Legendre equation [11, p. 121]:

$$(1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}]u = 0 \qquad (4.1.1)$$

where z, ν, μ are unrestricted. The solution is defined in terms of the associated Legendre function and in terms of the indicated hypergeometric function:

$$u(z) = p_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} F(-\nu, \nu+1; 1-\mu; \frac{1}{2} - \frac{z}{2}), \qquad (4.1.2)$$

$$|1-z| < 2.$$

To obtain $S_n(\nu)$, the sampling function in (3.1.3), we need the integral $\int_{-1}^{1} P_\nu^m(x) P_n^m(x)\,dx$, n an integer and ν unrestricted. While we have the integral

$$\int_{-1}^{1} |P_n^m(x)|^2 dx = \frac{(n+m)!}{(n-m)!\,(n+\frac{1}{2})} \tag{4.1.3}$$

the evaluation of the integral requires the expansion

$$P_\nu^{-\mu}(\cos\theta) = \frac{\sin \pi\nu}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{\nu-n} - \frac{1}{\nu+n+1}\right] P_n^{-\mu}(\cos\theta), \tag{4.1.4}$$

$$-\pi < \theta < \pi, \quad \mu \geq 0,$$

and [11, p. 140]

$$P_\nu^{-m}(z) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(z) . \tag{4.1.5}$$

Now if we let $\mu = m$ in (4.1.4) and use (4.1.5), then substituting the resulting infinite series for $P_\nu^m(x)$ in the integral and integrate term by term, using the orthogonality property of the associated Legendre polynomials, we get (for the details of these calculations see Appendix B.1)

$$\int_{-1}^{1} P_\nu^m(x)\,P_n^m(x)dx = \frac{2}{\pi}\,\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}\,\frac{\sin \pi(\nu-n-m)}{(\nu-n)(\nu+n+1)}, \tag{4.1.6}$$

For the special case $m = 0$ and $\nu = t - \frac{1}{2}$ this will reduce to the known result [11, p. 170]. Use of (4.1.6) and (4.1.3) in (3.1.3) leads to

$$S_n(\nu) = (-1)^m \frac{2n+1}{\pi} \frac{(n-m)!}{(n+m)!} \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \frac{\sin \pi(\nu-n)}{(\nu-n)(\nu+n+1)}, \quad (4.1.7)$$

and so

$$f(\nu) = \sum_{n=0}^{\infty} f(n) S_n(\nu). \quad (4.1.8)$$

2. Gegenbauer Functions. Consider the differential equation [11, p. 178]

$$(z^2 - 1)u''(z) + (2\nu+1)zu'(z) - a(a+2\nu)u(z) = 0 \quad (4.1.9a)$$

or

$$\frac{d}{dz}\left[(1-z^2)(1-z^2)^{\nu-\frac{1}{2}} \frac{du}{dz}\right] + \lambda_a (1-z^2)^{\nu-\frac{1}{2}} u = 0, \lambda_a = a(a+2\nu). \quad (4.1.9b)$$

The suitable solution is the set of Gegenbauer functions,

$$C_a^{\nu}(z) = \frac{\Gamma(a+2\nu)}{\Gamma(a+1)\Gamma(2\nu)} F(a+2\nu, -a; \nu+\tfrac{1}{2}; \frac{1}{2} - \frac{z}{2}). \quad (4.1.10)$$

For $S_n(a)$ in (3.1.3) we need $\int_{-1}^{1} C_a^{\nu}(x) C_n^{\nu}(x)(1-x^2)^{\nu-\frac{1}{2}} dx$, a an unrestricted real parameter, while from [11, p. 177, Eq. (17)], we have

$$\int_{-1}^{1} |C_n^{\nu}(x)|^2 (1-x)^{\nu-\frac{1}{2}} dx = \frac{\pi\, 2^{1-2\nu}\Gamma(n+2\nu)}{n!\,(\nu+n)\Gamma^2(\nu)}, \quad \nu > 0. \qquad (4.1.11)$$

If we use [11, p. 159, Eq. (27)]

$$P_{\nu}^{\mu}(\cos\theta) = \sqrt{\frac{2}{\pi}}\, \frac{(\sin\theta)^{\mu}}{\Gamma(\frac{1}{2}-\mu)} \int_0^{\theta} (\cos u - \cos\theta)^{-\mu-\frac{1}{2}} \cos[(\nu+\tfrac{1}{2})u]\, du, \qquad (4.1.12)$$

$$0 < \theta < \pi, \quad \operatorname{Re}\mu < \tfrac{1}{2},$$

and [11, p. 178, Eq. (23)]

$$C_a^{\nu}(\cos\phi) = \frac{2^{\nu}\pi^{-\frac{1}{2}}\Gamma(a+2\nu)\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(2\nu)\Gamma(a+1)} (\sin\phi)^{1-2\nu} \int_0^{\phi} (\cos u - \cos\phi)^{\nu-1} \cos[(\nu+a)u]\, du, \qquad (4.1.13)$$

$$\operatorname{Re}\nu > 0, \quad 0 < \phi < \pi.$$

If we use (4.1.5) we arrive at

$$C_a^{m+\frac{1}{2}}(z) = \frac{2^m m!\,(z^2-1)^{-\frac{m}{2}}}{(2m)!} P_{a+m}^{m}(z), \qquad (4.1.14)$$

$$\operatorname{Re}(m+\tfrac{1}{2}) > 0.$$

Substitute (4.1.14) into the desired integral and use the result of (4.1.6), arriving at

$$\int_{-1}^{1} C_a^{m+\frac{1}{2}}(z)\, C_n^{m+\frac{1}{2}}(z)(1-z^2)^m dz = 2\left[\frac{2^m m!}{(2m)!}\right]^2 \frac{\Gamma(a+2m+1)\sin\pi(a-n)}{\Gamma(a+1)(a+2m+n+1)\pi(a-n)} \qquad (4.1.15)$$

$$\operatorname{Re}(m+\tfrac{1}{2}) > 0.$$

(See the details in Appendix B. 2.) If we now use (4. 1. 15) and (4. 1. 11) in (3. 1. 3) we obtain

$$S_n(t) = \frac{2}{\pi}\left[\frac{2^{2m}\, m!}{(2m)!}\right]^2 \frac{n!\,(n+m+\frac{1}{2})\Gamma^2(m+\frac{1}{2})\Gamma(t+2m+1)}{(2m+n)!\,(t+n+2m+1)\Gamma(t+1)} \frac{\sin \pi(t-n)}{\pi(t-n)}, \tag{4.1.16}$$

$$\mathrm{Re}\,(m+\tfrac{1}{2}) > 0 .$$

3. Tchebichef Functions. Consider the differential equation

$$\frac{d}{dx}\left[(1-x^2)(1-x^2)^{\mp\frac{1}{2}}\frac{du}{dx}\right] + \lambda_a(1-x^2)^{\mp\frac{1}{2}} u = 0 , \tag{4.1.17}$$

which is a special case of (4. 1. 9b) where the ambiguous $\mp\frac{1}{2}$ corresponds to $\nu = 1$ and 0 respectively. That is, the two solutions are

$$T_a(x) = C_a^0(x), \tag{4.1.18}$$

for $-\frac{1}{2}$ in (4. 1. 17) and

$$U_a(x) = C_a^1(x) \tag{4.1.19}$$

for $\frac{1}{2}$ in (4. 1. 17). These two Tchebichef functions will reduce to the Tchebichef polynomials, $T_n(x)$ and $U_n(x)$ respectively, when $a = n$, an integer. We note that we cannot use our (4. 1. 15) result for evaluating $S_n(t)$ for either one of the Tchebichef functions since (4.1.15) is restricted to $\mathrm{Re}(m+\frac{1}{2}) > 0$, which is not satisfied in the present case.

Another explicit but seemingly not as practical a result as (4.1.15) is found in [11, p. 169, Eq. (1)], which is restricted to $\mathrm{Re}(\nu) > 0$ and so can only be used for (4.1.19) to obtain (see details in Appendix B.3)

$$S_n(t) = \frac{(t+1)(n+1)}{2(t-n)(t+n+2)}\Big[x(t-n)P^{-\frac{1}{2}}_{t+\frac{1}{2}}(x)\,P^{-\frac{1}{2}}_{n+\frac{1}{2}}(x)+nP^{-\frac{1}{2}}_{t+\frac{1}{2}}(x)P^{-\frac{1}{2}}_{n-\frac{1}{2}}(x) \tag{4.1.20}$$
$$-tP^{-\frac{1}{2}}_{t-\frac{1}{2}}(x)\,P^{-\frac{1}{2}}_{n+\frac{1}{2}}(x)\Big]^{1}_{-1}$$

4. <u>Spheroidal Wave Functions.</u> Consider the differential equation [13, p. 134, Eq. (1)]

$$(1-z^2)\frac{d^2y}{dz^2} - 2z\frac{dy}{dz} + [\lambda + 4\theta(1-z^2) - \frac{\mu^2}{1-z^2}]\,y = 0, \tag{4.1.21}$$

where λ, θ, μ are given real or complex parameters, z a complex variable and

$$\lambda^{\mu}_{\nu}(\theta)\Big|_{\theta=0} = \nu(\nu+1)\ . \tag{4.1.22}$$

For $\mu = m$, an integer, the solution of (4.1.21) which remains bounded at $z = 1$ is the prolate spheroidal function, $Ps^{m}_{\nu}(x,\theta)$;

this is bounded at $z = -1$ only if $\nu = n$, an integer. For calculating $S_{n'}(\nu)$ in (3.1.3) we need the value of

$\int_{-1}^{1} Ps_{\nu}^{m}(x,\theta) Ps_{n'}^{m}(x,\theta)dx$. First we put [13, p. 138, Eq. (22)],

$$Ps_{\nu}^{m}(x,\theta) = \sum_{r=-\infty}^{\infty} (-1)^{r} a_{\nu,r}^{m}(\theta) P_{\nu+2r}^{m}(x) \tag{4.1.23}$$

in the desired integral, then integrate term by term, using (B1.1) (or (4.1.4) and (4.1.5)) and [13, p. 158, Eq. (11)],

$$\int_{-1}^{1} Ps_{n'}^{m}(x) P_{n}^{m}(x)dx = \begin{cases} (-1)^{r} a_{n',r}^{m}(\theta) \dfrac{(n+m)!}{(n+\frac{1}{2})(n-m)!}, & n-n' = 2r, \\ \\ 0, & n-n' \text{ negative or odd}, \end{cases} \tag{4.1.24}$$

to get

$$\int_{-1}^{1} Ps_{n'}^{m}(x,\theta) Ps_{\nu}^{m}(x,\theta)dx = 2\sum_{r=0}^{\infty} a_{n',r}^{m} a_{\nu,r}^{m} \frac{\Gamma(\nu+2r+m+1)}{\Gamma(\nu+2r-m+1)(\nu+n'+4r+1)} \frac{\sin\pi(\nu-n')}{\pi(\nu-n')} \tag{4.1.25}$$

(see the details in Appendix B.4).

We note that this new result reduces easily to the special case quoted in [13, p. 147, Eq. (6)], since if we let $\nu = n = n'$ in (4.1.25) we arrive at

$$\int_{-1}^{1} Ps_{n'}^{m}(x,\theta)Ps_{n}^{m}(x,\theta)dx = \delta_{nn'} \sum_{r=0}^{\infty} (a_{n',r}^{m})^2 \frac{(n'+2r+m)!}{(n'+2r-m)!} \cdot \frac{1}{n'+2r+\frac{1}{2}} , \tag{4.1.26}$$

$$2r \geq m-n'$$

and when we use [13, p. 147, Eq. (7)] this becomes

$$\int_{-1}^{1} [Ps_{n}^{m}(x)]^2 dx = \frac{1}{n+\frac{1}{2}} \frac{(n+m)!}{(n-m)!} . \tag{4.1.27}$$

To evaluate $S_n(t)$ in (3.1.3) we merely substitute (4.1.27) and (4.1.25).

5. Bessel Functions. The following differential equation is of interest in signal detection. It appeared in the work of Stone and Brock [40, p. 28; 41]. It was obtained after differentiation of an integral equation with a kernel that corresponds to their first order filter:

$$h''(u) + \omega_2 h'(u) + [\frac{2\omega_1^2\omega_2}{\lambda} e^{-\omega_2 u} - \omega_1(\omega_1-\omega_2)] h(u) = 0 . \tag{4.1.28}$$

After simple transformations this reduces easily to the Bessel equation (see the details in Appendix B.5)

$$\frac{d}{dx}[x\frac{dh}{dx}] - \frac{4\gamma(\gamma-1)}{\lambda} h(x) = t^2 xh(x), \tag{4.1.29}$$

so

$$h(x) = J_{2\gamma-1}(tx), \quad J_{2\gamma-1}(t_n) = 0 ,$$

$$n = 1, 2, \cdots . \tag{4.1.30}$$

By methods similar to those of Section 3.2 the sets of functions and numbers take the form

$$S_n(t) = \frac{2t_n J_{2\gamma-1}(t)}{(t_n^2 - t^2) J_{2\gamma}(t_n)}, \quad J_{2\gamma-1}(t_n) = 0,$$

$$n = 1, 2, \cdots . \tag{4.1.31}$$

The real parameter γ has the physical significance of band-width ratio and may well take on quite large values.

<u>6. Fourth Order Differential Equations.</u> A fourth order differential equation appeared in the same manner as (4.1.28) but is related to a second order filter [40, p. 31] (see Appendices A.1 and B.6):

$$[D^4 - (m^2 + \overline{m}^2)D^2 + m^2\overline{m}^2]\, y(te^{-x}) = t^2 e^{-2x} y(te^{-x}),$$

$$y(t) \equiv 0, \quad y'(t_n) = 0, \quad n = 1, 2, \cdots , \tag{4.1.32}$$

In Appendix A.1 we show that the boundary value problem here is self-adjoint. The solution which is admissible is defined in terms of m and its complex conjugate,

$$y(te^{-x}) = y_m(te^{-x})y_{\overline{m}}(t) - y_{\overline{m}}(te^{-x})y_m(t), \qquad (4.1.33)$$

so the identity boundary condition is satisfied. Let

$$H_m(te^{-x}) = C_0 y(te^{-x}), \qquad (4.1.34)$$

be the solution of (4.1.32) which is orthogonal with respect to the zeros $\{t_n\}$ of $y'(t)$ on $[0, \infty]$; it may be referred to as the Bessel-like function where

$$\int_0^{\infty} e^{-2x} H_m^2(t_n e^{-x})dx = \frac{1}{2} t_n^2 H''_m(t_n), \quad n = 1, 2, \cdots. \qquad (4.1.35)$$

If we set $u = e^{-x}$ and $\rho(u) = u$, $H_m(tu)$ is orthogonal with respect to $\{t_n\}$, hence satisfies the condition of the Kramer generalized WKSK sampling theorem (3.2.1) for the finite interval.

To obtain $S_n(t)$ for (3.1.3) we need $\int_0^1 uH_m(tu)H_m(t_n u)du$. For this we consider (4.1.32) with t replaced by t_n,

$$[D^4 - (m^2 + \overline{m}^2)D^2 + m^2\overline{m}^2]\, y(t_n e^{-x}) = t_n^2 y(t_n e^{-x}). \qquad (4.1.36)$$

We eliminate the constant coefficient terms in (4.1.32) and (4.1.36) in the usual manner, arriving at

$$(t_n^2 - t^2) \int_0^{\infty} e^{-2x} y(t_n e^{-x})\, y(te^{-x})dx = -t t_n^2\, y'(t) y''(t_n) \qquad (4.1.37)$$

(see details in Appendix B. 6). Now if we use (4.1.37) and (4.1.35) we obtain

$$S_n(t) = \frac{2t\, H'_m(t)}{(t^2 - t_n^2) H''_m(t_n)} \quad . \qquad (4.1.38).$$

4.2 More on the Comparison of the Two Sampling Theorems.

In this section we will consider functions in the last section and others that have the same type of integral representation as the ones Campbell used for Bessel and Legendre functions. In the light of all these cases we will attempt a simple generalization.

1. Associated Legendre Functions. If

$$f(t) = \int_{-1}^{1} K(t,x)g(x)dx \qquad (4.2.1)$$

and

$$K(t,x) = P_t^m(x), \qquad (4.2.2)$$

we use the integral representation of $P_\nu^m(\cos\theta)$ in (4.1.12),

$$P_\nu^\mu(\cos\theta) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(\sin\theta)^\mu}{\Gamma(\frac{1}{2}-\mu)} \int_0^\theta (\cos v - \cos\theta)^{-\mu-\frac{1}{2}} \cos[(\nu+\tfrac{1}{2})v]\, dv, \tag{4.2.3}$$

$$\operatorname{Re}\mu < \tfrac{1}{2}, \quad 0 < \theta < \pi .$$

We let $\mu = -m$, m a non-negative integer, then refer to (4.1.5) to obtain

$$P_\nu^m(\cos\theta) = \tag{4.2.4}$$

$$(-1)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(\sin\theta)^{-m}}{\Gamma(\frac{1}{2}+m)} \int_0^\theta (\cos v - \cos\theta)^{m-\frac{1}{2}} \cos[(\nu+\tfrac{1}{2})v]\, dv$$

which may be written as

$$P_\nu^m(\cos\theta) = \tag{4.2.5}$$

$$\frac{(-1)^m\,\Gamma(\nu+m+1)}{2\Gamma(\nu-m+1)} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{(\sin\theta)^{-m}}{\Gamma(\frac{1}{2}+m)} \int_{-\theta}^{\theta} (\cos v - \cos\theta)^{m-\frac{1}{2}} e^{i(\nu+\frac{1}{2})v} dv.$$

If we substitute (4.2.5) for (4.2.2) in (4.2.1), then interchange the order of the integration we get

$$f(t) = \frac{\Gamma(t+m+1)}{\Gamma(t-m+1)} \int_{-\pi}^{\pi} e^{itv} F(v)dv \tag{4.2.6}$$

(see the details of this calculation and the $F(v)$ expression in

Appendix B. 7).

We see that the new result (4. 2. 6) is easily reduced to the special case (3. 3. 17) of Campbell [7], where m is zero, by comparing (B7. 2) and (A 3. 2). We conclude that the integral in (4. 2. 6) can be expanded by the WKS sampling theorem and so we can obtain a sampling series for $f(t)$.

2. Gegenbauer Functions. If

$$K(x, t) = C_t^{\nu}(x) \tag{4. 2. 7}$$

in (4. 2. 1) we use the integral representation of $C_t^{\nu}(\cos \phi)$ from (4. 1. 13) in (4. 2. 1) and, after interchanging the order of integration in a manner similar to that used for $P_t^m(\cos \theta)$ above, we obtain

$$f(t) = \frac{\Gamma(t+2\nu)}{\Gamma(t+1)} \int_{-\pi}^{\pi} e^{itv} G(v) dv \tag{4. 2. 8}$$

(see details in Appendix B. 8). So, the integral in (4. 2. 8) may be expanded by the WKS sampling theorem and we can obtain a sampling series for $f(t)$.

3. Tchebichef Functions of the Second Kind. We note here that only $U_a(x) = C_a^1(x)$ and not $T_a(x) = C_a^0(x)$ will lend itself to our calculation of (4. 2. 8) since the restriction of (4. 1. 13) of the integral

representation of $C_t^{\nu}(x)$ is $\mathrm{Re}\,\nu > 0$. Hence, for

$$K(x, t) = U_t(x) \tag{4.2.9}$$

in (4.2.1) we can use (4.1.13) with $\nu = 1$ to obtain

$$f(t) = (t+1) \int_{-\pi}^{\pi} e^{itv} G(v)dv. \tag{4.2.10}$$

By observing the method of (3.3.17), (3.3.19), (4.2.6), (4.2.8), (4.2.10) we will attempt the following simple generalization:

Theorem 4.2.1. If

$$f(t) = \int_{-a}^{a} K(x, t)g(x)dx \tag{4.2.11}$$

and if

$$K(x, t) = h_1(x)h_2(t) \int_{-C(x)}^{C(x)} k(x, \eta)e^{it\eta} d\eta,$$

$$h_2(t) \neq 0, \tag{4.2.12}$$

where $C(x) \geq 0$ is monotonically increasing or decreasing in $[-a, a]$, either $C(a)$ or $C(-a)$ is zero, then $f(t)/h_2(t)$ may be expressed by the WKS sampling theorem.

Proof: If we substitute (4.2.12) into (4.2.11) we obtain

$$f(t) = \int_{-a}^{a} h_1(x) h_2(t) \int_{-C(x)}^{C(x)} k(x,\eta) e^{it\eta} d\eta\, g(x)\rho(x) dx, \qquad (4.2.13)$$

where $\rho(x)$ is a weighting function. Interchange the order of integration

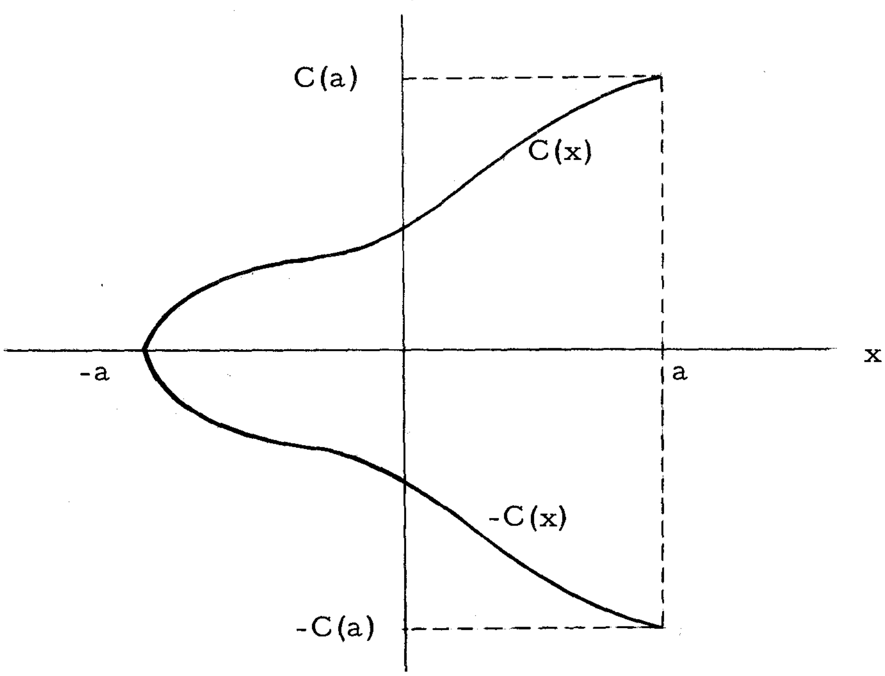

Figure 1.

$$f(t) = h_2(t) \int_{-C(a)}^{C(a)} e^{it\eta} \int_{x=C^{-1}(|\eta|)}^{a} h_1(x) k(x,\eta) g(x)\rho(x) dx d\eta . \qquad (4.2.14)$$

Hence,

$$\frac{f(t)}{h_2(t)} = \int_{-C(a)}^{C(a)} e^{it\eta} H(\eta) d\eta , \qquad (4.2.15)$$

where $H(\eta)$ is defined above. Note that $2a$ is the bandwidth of $g(x)$ but $2C(a)$ is the bandwidth for $H(\eta)$.

Remark. If $K(x,t)$ in (4. 2. 11) is band-limited as a function of t, then $f(t)$ is a band-limited function. Furthermore, $f(t)$ has the same band limits.

Proof: Let

$$K(x,t) = \int_{-A}^{A} k(x,\eta)e^{it\eta}d\eta \; ; \tag{4.2.16}$$

then, as for (4. 2. 15) with $C(x) = A$ we get

$$f(t) = \int_{-A}^{A} e^{it\eta}H_1(\eta)d\eta , \tag{4.2.17}$$

where

$$H_1(\eta) = \int_{-a}^{a} k(x,\eta)g(x)\rho(x)dx \; . \tag{4.2.18}$$

From (4. 2. 16) and (4. 2. 17) we note that the bandwidth for both $k(x,\eta)$ and $f(t)$ is $2A$.

For the next corollary we will make use of Paley and Weiner's theorem [29, p. 13] concerning the representation of a function by a finite Fourier transform.

Theorem 4.2.2. "The two following classes of entire functions are identical: (1) The class of entire functions $F(z)$ satisfying a condition $F(z) = o(e^{A|z|})$ and belonging to L_2 on the real axis, (2) the class of entire functions of the form

$$F(z) = \int_{-A}^{A} f(u)e^{iuz}du, \qquad (4.2.19)$$

where $f(u)$ belongs to L_2 on $(-A, A)$".

Corollary 2. If $K(x,t)$ in (4.2.11) belongs to the class $L_2(t)$, is entire and is of exponential order, then $f(t)$ is a band-limited function".

Proof: The proof follows from the Paley and Wiener theorem and Corollary 1.

A more general theorem that will include Theorem 4.2.1 as a special case is

Theorem 4.2.3. If

$$f(t) = \int_a^b K(x,t)g(x)\rho(x)dx \qquad (4.2.20)$$

and if

$$K(x,t) = h_1(x)h_2(t)\int_{-C(x)}^{C(x)} k(x,\eta)e^{it\eta}d\eta, \quad h_2(t) \neq 0 \qquad (4.2.12)$$

where $C'(x)$ exists and vanishes only at a finite number of points on $[a,b]$, then $f(t)/h_2(t)$ is a sum of band-limited funtions.

Proof: From (4.2.12) and (4.2.20) we get

$$f(t) = \int_a^b h_1(x)h_2(t) \int_{-C(x)}^{C(x)} k(x,\eta)e^{it\eta} d\eta\, g(x)\rho(x)dx \,. \qquad (4.2.21)$$

The closed region $\{(x,\eta)\,|\,a \leq x \leq b,\ |\eta| \leq |C(x)|\}$ can be divided into some n closed regions, each of which can be considered similar to the one used for Theorem 4.2.1. Hence, the order of the integration may be interchanged in the same way to arrive at

$$f(t) = h_2(t) \sum_{j=0}^{n-1} \int_{a_j}^{a_j'} H_j(\eta)\, e^{i\eta t}\, d\eta \,. \qquad (4.2.22)$$

The details of an example of the above calculation is given in Appendix B.9.

Corollary 1. If $C(x)$ is non-negative and monotonic for $a \leq x \leq b$ then $f(t)$ is the sum of three band-limited functions, and one band-limited function if there exists an ξ, $a \leq \xi \leq b$, such that $C(\xi) = 0$.

Proof: The first part follows easily from the above theorem and its detailed calculation in Appendix B.9. The second part follows in the same way and is the statement of Theorem 4.2.1.

4.3 On the Equivalence of the Two Sampling Theorems

In this section we will introduce a few definitions necessary for a more precise method of comparing the WKS and the WKSK sampling theorems. We will attempt to find conditions under which the two sampling theorems are equivalent. Hence, in this sense only, we speak of the " No advantage of using the generalized (WKSK) sampling theorem over the Shannon (WKS) sampling theorem". On our way to this goal we shall have to prove a few simple results that may be of general interest for understanding and applying either one of the two sampling theorems. We shall also quote a lemma of Wiener [54, p. 80] concerning the finite Fourier transform, and the Lebesgue (dominated) convergence theorem.

<u>Lemma.</u> " Let $H(x)$ be a continuous function defined over $(-\infty, \infty)$, and vanishing over $(-\infty, -\pi+\varepsilon)$ and $(\pi-\varepsilon, \infty)$. Let

$$h(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) e^{-iux} dx . \qquad (4.3.1)$$

Then the three following statements are equivalent :

(1) $$\sum_{n=-\infty}^{\infty} |h(n)| < \infty, \tag{4.3.2}$$

(2) $$\sum_{n=-\infty}^{\infty} \max_{n \leq u \leq n+1} |h(u)| < \infty, \tag{4.3.3}$$

(3) $$\int_{-\infty}^{\infty} |h(u)|\, du < \infty" . \tag{4.3.4}$$

Lebesgue Convergence Theorem, [18, p. 2]. "Let $f_1, f_2, \cdots,$ be integrable on $(-\infty, \infty)$. If $|f_n(x)| \leq F(x)$ a.e., $(-\infty < x < \infty;\ n = 1, 2, \cdots)$ for some integrable F, and if $\lim_{n \to \infty} f_n(x) = f(x)$ a.e. $(-\infty < x < \infty)$, then f is integrable and

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)\, dx" . \tag{4.3.5}$$

Now we state some definitions.

Definition 4.3.1. Whittaker Samplable Functions belong to the class as defined by

$$W_1(a) = \{f_a(t) \,|\, f_a(t) = \int_{-a}^{a} g(x)e^{ixt} dx, \tag{4.3.6}$$

$g(x) \in L_2(-a, a)$ and $g(x)$ vanishes for $|x| \geq |a|$ }.

Definition 4.3.2. Whittaker Sampled Functions belong to the class as defined by

$$W_2(a) = \{ f_a(t) \,|\, f_a(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{a}) \frac{\sin(at-n\pi)}{(at-n\pi)} \}. \qquad (4.3.7)$$

Definition 4.3.3. The class of Whittaker Sampling Functions is defined as the intersection,

$$W = W_1 \cap W_2 \;. \qquad (4.3.8)$$

Definition 4.3.4. The class of Kramer Samplable Functions is defined by

$$K_1(a,b) = \{ f_{a,b}(t) \,|\, f_{a,b}(t) = \int_a^b K(x,t) g(x) dx, \qquad (4.3.9)$$

$g(x) \in L_2(a,b)$, $K(x,t) \in L_2(a,b)$ and there exists a set of numbers $E = \{t_n\}$ such that $\{K(x,t_n)\}$ is a complete orthogonal set on $[a,b]$ }.

Definition 4.3.5. The class of Kramer Sampled Functions is defined by

$$K_2(a,b) = \{ f_{a,b}(t) \,|\, f_{a,b}(t) = \sum_{n=-\infty}^{\infty} f(t_n) S_n(t) \} \qquad (4.3.10)$$

where $S_n(t)$ is defined by (3.1.3).

<u>Lemma 1.</u> If $f(t) \epsilon W_1(a)$ then $f(t)$ is bounded.

<u>Proof:</u> From Definition 4.3.1

$$f_a(t) = \int_{-a}^{a} e^{ixt} g(x)dx, \quad g(x) \epsilon L_2(-a, a). \tag{4.3.11}$$

Hence,

$$|f_a(t)| \leq \int_{-a}^{a} |e^{ixt} g(x)| dx . \tag{4.3.12}$$

By using the Schwartz inequality and the condition $g(x) \epsilon L_2(-a, a)$ wearrive at

$$|f_a(t)| \leq [2a \int_{-a}^{a} |g(x)|^2 dx]^{\frac{1}{2}} < \infty . \tag{4.3.13}$$

<u>Lemma 2.</u> If $f(t) \epsilon W_2$ such that $f(\frac{n\pi}{a}) \epsilon \ell_1$ then $f(t) \epsilon W_1$.

<u>Proof:</u> Since $f(t) \epsilon W_2$ Definition 4.3.2 establishes that

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at - n\pi} . \tag{4.3.14}$$

So, the triangular inequality yields

$$|f(t)| \leq \sum_{n=-\infty}^{\infty} | f(\frac{n\pi}{a})| \; |\frac{\sin(at-n\pi)}{at-n\pi}| . \tag{4.3.15}$$

The second factor in each term is clearly ≤ 1; since $f(\frac{n\pi}{a}) \epsilon \ell_1$ it is clear that

$$|f(t)| < \infty . \tag{4.3.16}$$

We have

$$\frac{\sin(at-n\pi)}{at-n\pi} = \frac{1}{2a}\int_{-a}^{a} e^{i\omega(t-\frac{n\pi}{a})} d\omega \tag{4.3.17}$$

and

$$\int_{-\infty}^{\infty} \frac{\sin(at-n\pi)}{at-n\pi} \frac{\sin(at-m\pi)}{at-m\pi} dt = \begin{cases} 0, & m \neq n, \\ \frac{\pi}{a}, & m = n . \end{cases} \tag{4.3.18}$$

Multiplying both sides of (4.3.14) by $\frac{\sin(at-m\pi)}{at-m\pi}$ and integrating with respect to t we get

$$\int_{-\infty}^{\infty} f(t) \frac{\sin(at-m\pi)}{at-m\pi} dt = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{a}) \int_{-\infty}^{\infty} \frac{\sin(at-n\pi)}{at-n\pi} \frac{\sin(at-n\pi)}{at-n\pi} dt. \tag{4.3.19}$$

The term by term integration is justified by $f(\frac{n\pi}{a}) \epsilon \ell_1$ and the use of the Lebesgue Convergence Theorem. The orthogonality condition of (4.3.18) yields

$$\int_{-\infty}^{\infty} f(t) \frac{\sin(at-n\pi)}{at-n\pi} dt = f(\frac{n\pi}{a}) . \tag{4.3.20}$$

The convolution theorem gives

$$f\left(\frac{n\pi}{a}\right) = \int_{-a}^{a} e^{\frac{in\pi x}{a}} g(x)dx \; ; \tag{4.3.21}$$

by the uniqueness property of the Fourier transform

$$f(t) = \int_{-a}^{a} e^{ixt} g(x)dx \tag{4.3.22}$$

so

$$f(t) \in W_1(a). \tag{4.3.23}$$

Theorem 4. 3. 1. If $f(\frac{n\pi}{a}) \in \ell_1$, $f(t) \in W_1$ if and only if $f(t) \in W_2$,

so

$$f(t) \in W\ . \tag{4.3.24}$$

Proof: (i) $f(t) \in W_1$ implies $f(t) \in W_2$ follows from Theorem 2. 1. 1, which is the Shannon Sampling Theorem. (ii) $f(t) \in W_2$ implies $f(t) \in W_1$ follows from the above Lemma 2 since $f(\frac{n\pi}{a}) \in \ell_1$. Hence

$$f(t) \in W_1 \cap W_2 = W \tag{4.3.25}$$

and is a Whittaker sampling function.

Corollary 1. If $f(t) \in L_1(-\infty, \infty)$ and $f(t) \in W_1$ with continuous $g(x)$, then

$$f(t) \in W . \tag{4.3.26}$$

Proof: From the above Wiener's Lemma we get $f(\frac{n\pi}{a}) \in \ell_1$, so by using Theorem 4.3.1 we conclude that $f(t) \in W$.

Theorem 4.3.2. If

$$f_j(t) = \int_{-a}^{a} e^{ixt} g_j(x)dx, \quad j = 1, 2, \cdots, N, \tag{4.3.27}$$

(i.e., if $f_j(t) \in W_1(a)$ for $j = 1, 2, \cdots, N$) then

$$f(t) = \sum_{j=1}^{N} f_j(t) \tag{4.3.28}$$

is also in $W_1(a)$ and hence in $W_2(a)$.

Proof: From (4.3.28) and (4.3.27) we have

$$f(t) = \int_{-a}^{a} e^{ixt} \sum_{j=1}^{N} g_j(x)dx . \tag{4.3.29}$$

Hence, by the condition on $g_j(x)$, we conclude that $f(t) \in W_1$. Then, by using Theorem 2.1.1 we get $f(t) \in W_2$.

Corollary 1. If $f_j(\frac{n\pi}{a}) \epsilon \ell_1$, for $j = 1, 2, \cdots, N$, then

$$f(t) \epsilon W. \tag{4.3.30}$$

Proof: Since $f_j(\frac{n\pi}{a}) \epsilon \ell_1$ for $j = 1, 2, \cdots, N$, $f(\frac{n\pi}{a})$ from (4.3.28) is also in ℓ_1 and by Theorem 4.3.1 and Theorem 4.3.2 $f(t) \epsilon W$.

Corollary 2. If $f_j(t) \epsilon L_1(-\infty, \infty)$ for $j = 1, 2, \cdots, N$ and $g_j(x)$ is continuous for all j then

$$f(t) \epsilon W(a) . \tag{4.3.31}$$

Proof: This result follows when we use the Wiener Lemma.

Corollary 3. If

$$f_j(t) = \int_{-a_j}^{a_j} e^{ixt} g_j(x) dx, \quad j = 1, 2, \cdots, N, \tag{4.3.32}$$

then

$$f(t) \epsilon W_1(a) \tag{4.3.33}$$

where $a = \underset{j}{\text{Max}}\ a_j$.

Proof: By definition

$$f(t) = \int_{-a_1}^{a_1} e^{ixt} g_1(x)dx + \int_{-a_2}^{a_2} e^{ixt} g_2(x)dx + \cdots + \int_{-a_N}^{a_N} e^{ixt} g_N(x)dx. \quad (4.3.34)$$

Since $g_j(x) = 0$ for $|x| > |a_j|$ we may write (4. 3. 34) as

$$f(t) = \int_{-a}^{a} e^{ixt} \sum_{j=1}^{N} g_j(x)dx, \quad (4.3.35)$$

where $a = \underset{j}{\mathrm{Max}}\ a_j$. From Theorem 4. 3. 2 we get

$$f(t) \in W_1(a), \quad a = \underset{j}{\mathrm{Max}}\ a_j . \quad (4.3.36)$$

Theorem 4. 3. 3. If $f_j(t) \in W_2(a)$ for $j = 1, 2, \cdots, N$ then

$$f(t) = \{\sum_{j=1}^{N} f_j(t)\} \in W_2(a). \quad (4.4.37)$$

Proof: From the hypothesis it is clear that Definition 4. 3. 2 leads to

$$f(t) = \sum_{n=-\infty}^{\infty} \sum_{j=1}^{N} f_j(\frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at-n\pi} . \quad (4.3.38)$$

By definition

$$f(\frac{n\pi}{a}) = \sum_{j=1}^{N} f_j(\frac{n\pi}{a}) \quad (4.3.39)$$

so

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at-n\pi} . \qquad (4.3.40)$$

According to Definition 4.3.2 the theorem is proved.

Corollary 1. If $f_j(\frac{n\pi}{a}) \epsilon \ell_1$ for $j = 1, 2, \cdots, N$ then $f(t) \epsilon W_1(a)$ and hence

$$f(t) \epsilon W(a). \qquad (4.3.41)$$

Proof: From (4.3.39) we see that $f(\frac{n\pi}{a}) \epsilon \ell_1$ since $f_j(\frac{n\pi}{a}) \epsilon \ell_1$ for all j. By Theorem 4.3.1 we get $f(t) \epsilon W_1$ since $f(\frac{n\pi}{a}) \epsilon \ell_1$ and $f(t) \epsilon W_2$, so the corollary is proved.

Corollary 2. If $f_j(t) \epsilon W_2(a_j)$ for $j = 1, 2, \cdots, N$ and $f_j(\frac{n\pi}{a_j}) \epsilon \ell_1$ for $j = 1, 2, \cdots, N$, then

$$f(t) \epsilon W_1(a) \qquad (4.3.42)$$

where $a = \underset{j}{\text{Max}}\, a_j$.

Proof: Since $f_j(\frac{n\pi}{a_j}) \epsilon \ell_1$ then by Theorem 4.3.1 $f_j(t) \epsilon W_1(a_j)$, all j. By Corollary 3 of Theorem 4.3.2 we have $f(t) \epsilon W_1(a)$, where $a = \underset{j}{\text{Max}}\, a_j$.

Theorem 4.3.4. For $K(x, \frac{n\pi}{a}) \epsilon \ell_1$ and $f(\frac{n\pi}{a}) \epsilon \ell_1$, if $K \epsilon W$ then $f \epsilon K_1$ if and only if $f \epsilon W$. That is, $K \epsilon W \Rightarrow (f \epsilon K_1 \Leftrightarrow f \epsilon W)$.

Proof: From Corollary 1 of Theorem 4.2.1 we have $K \epsilon W_1 \Rightarrow (f \epsilon K_1 \Rightarrow f \epsilon W_1)$. Since $f(\frac{n\pi}{a}) \epsilon \ell_1 \Rightarrow f(t) \epsilon W$ by Lemma 2. Since $W_1 \subset K_1$ (e^{ixt} is a special case of $K(x, t)$, as Kramer pointed out in Section 3.2) then $f(t) \epsilon W_1 \Rightarrow f(t) \epsilon K_1$, and so $K \epsilon W_1 \Rightarrow (f \epsilon K_1 \Leftrightarrow f \epsilon W)$. But since $K(\frac{n\pi}{a}) \epsilon \ell_1$ then, by Lemma 2, $K \epsilon W$, hence $K \epsilon W \Rightarrow (f \epsilon K_1 \Leftrightarrow f \epsilon W)$.

Theorem 4.3.5. For $K(x, \frac{n\pi}{a}) \epsilon \ell_1$ and $f(\frac{n\pi}{a}) \epsilon \ell_1$, if $f \epsilon K_1$ is equivalent to $f \epsilon W$ then $K \epsilon W$. That is, $(f \epsilon K_1 \Leftrightarrow f \epsilon W) \Rightarrow K \epsilon W$.

Proof: From $f(t) \epsilon W_1(a)$ we have

$$f(t) = \int_{-a}^{a} e^{ixt} g(x) dx , \tag{4.3.43}$$

so

$$f(t) = \sum_{n=-\infty}^{\infty} f(\frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at-n\pi} . \tag{4.3.44}$$

But $f(t) \epsilon K_1(c, d)$ so

$$f(\frac{n\pi}{a}) = \int_{c}^{d} K(x, \frac{n\pi}{a}) G(x) dx . \tag{4.3.45}$$

Therefore

$$f(t) = \sum_{n=-\infty}^{\infty} \int_c^d K(x, \frac{n\pi}{a}) G(x)dx \frac{\sin(at-n\pi)}{at-n\pi}$$

$$= \int_c^d \left[\sum_{n=-\infty}^{\infty} K(x, \frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at-n\pi} \right] G(x)dx. \tag{4.3.46}$$

The interchange of the order of integration and summation is justified by the property $K(x, \frac{n\pi}{a}) \in \ell_1$ and the Lebesgue Convergence Theorem. But $f(t) \in K_1(c, d)$ so, from (4.3.46)

$$K(x,t) = \sum_{n=-\infty}^{\infty} K(x, \frac{n\pi}{a}) \frac{\sin(at-n\pi)}{at-n\pi} . \tag{4.3.47}$$

That is, $K(x,t) \in W_2$. But $K(x, \frac{n\pi}{a}) \in \ell_1$, hence by Lemma 2 we have $K(x,t) \in W_1$ and $K(x,t) \in W$. Finally,

$(f \in K_1 \Leftrightarrow f \in W) \Rightarrow K(x,t) \in W$.

Theorem 4.3.6. For $K(x, \frac{n\pi}{a}) \in \ell_1$ and $f(\frac{n\pi}{a}) \in \ell_1$, $f(t) \in K_1$ is equivalent to $f(t) \in W$ if and only if $K \in W$.

Proof. The proof follows from Theorems 4.3.4 and 4.3.5.

4.4 Some Suggested Applications for the Kramer Sampling Theorem

The communication engineer uses the Fourier transform mainly. So, it is no surprise that the WKS sampling theorem statement appears in terms of a signal represented by a finite Fourier transform. Also, the ideal low-pass filter, with its nice time invariant impulse response, is used to give the physical interpretation. In Chapter 2 we have reviewed a number of interesting extensions and applications that followed the WKS sampling theorem. At the same time we noticed one important extension of this theorem that was neglected, especially in applications. This extension was Kramer's generalization of the Fourier kernel e^{ixt} to $K(x, t)$, as was discussed in Chapter 3. This section is devoted to the possible reason for this neglect and at the same time will offer some suggestions in this direction. We will attempt to raise some questions, similar to the ones raised in Sections 1.2 and 1.4, to show that the Kramer generalization of the sampling theorem might be a very handy tool. First, this generalization takes us from a signal represented by a finite Fourier transform to the same signal represented by another and more general finite integral transform. Hence, the results that have already been obtained in Chapter 3 and 4 for Kramer's theorem may prove to be of use in the field of finite integral transforms. In addition, introducing such transforms might simplify some communication problems as it did in other fields. This last question may deal with the possibility of using the generalized sampling theorem for the case of a signal which is the output of a time variant filter. We now note that if such integral transforms are introduced then Kramer's theorem will play a role similar to the one played by Shannon's theorem in terms of the finite Fourier transform. We should be reminded that even if such development proves useful the road to it will not necessarily be an easy one, since many of the tools needed may not be available. For example, consider a convolution theorem for a Legendre transform.

Also, we introduce two finite integral transforms as they appear in the literature [37, 38, 46, 47, 54] , namely the finite Hankel and Legendre transforms, then try to extend their definition with the help of Kramer's theorem.

1. Finite Hankel Transform [37, 47] .

$$\bar{f}(t_p) = \int_0^1 f(r)rJ_n(t_p r)dr, \qquad J_n(t_p) = 0, \quad p = 1, 2, \cdots, \tag{4.4.1}$$

with the inverse transform

$$f(r) = \sum_{p=1}^{\infty} \frac{2\bar{f}(t_p)}{J_{n+1}^2(t_p)} J_n(t_p r). \tag{4.4.2}$$

Let us extend $\bar{f}(t_p)$ to $\bar{f}(t)$ in (4.4.1) with unrestricted t. We quickly realize that $\bar{f}(t)$ can be automatically calculated in terms of $\bar{f}(t_p)$ by the use of Kramer's theorem, which gives

$$\bar{f}(t) = \sum_{p=1}^{\infty} \bar{f}(t_p) S_p(t), \tag{4.4.3}$$

where $S_p(t)$ is given in (3.2.14).

2. Legendre Transform [46, 47].

$$\bar{f}(n) = \int_{-1}^{1} f(u) P_n(u) du \tag{4.4.4}$$

and

$$f(u) = \sum_{n=0}^{\infty} (n+\tfrac{1}{2}) \bar{f}(n) P_n(u). \tag{4.4.5}$$

We extend $\bar{f}(n)$ to $\bar{f}(\nu)$ and in the same way as above we get $\bar{f}(\nu)$ in terms of $\bar{f}(n)$, with the aid of Kramer's theorem, to be

$$\bar{f}(\nu) = \sum_{n=-\infty}^{\infty} \bar{f}(n) S_n(\nu), \tag{4.4.6}$$

where $S_n(\nu)$ is given by (3.2.19).

Next we take the above two examples of finite integral transforms to show their possible advantage for the system function analysis of filters. In (4.4.1) let the inverse Hankel transform of $\bar{f}(t)$ with $n = 2$ be $f(r) = r$. In contrast to this the inverse Fourier transform to the same $\bar{f}(t)$ is found from (A.3.1) to be

$$f(r) = \frac{\sqrt{1-r^2}\,(2-3r-r^2)}{15\pi} . \tag{4.4.7}$$

This, obviously, is a more complicated system function than $f(r) = r$.

Now in (4.4.4) let $f(u) = 1$, $u = \cos\theta$, be the inverse Legendre transform of $\bar{f}(\nu)$, then using (A.3.2) we can show that the corresponding inverse Fourier transform of $\bar{f}(\nu)$ is

$$f(u) = 2^{\frac{3}{2}} \cos\frac{u}{2} \tag{4.4.8}$$

again a more complicated system function than $f(u) = 1$.

The third question deals with a problem of different nature, concerning the fact that most linear filters are treated as stationary ones [48, p. 8]. That is, their impulse response depends on $(t-t')dt'$. For example, the output

$$x(t) = \int_{-\infty}^{\infty} k(t,t')f(t')dt' \tag{4.4.9}$$

with

$$k(t,t') \equiv k(t-t'). \qquad (4.4.10)$$

A more specific example is the impulse response of an ideal low-pass filter

$$k(t-t') = \frac{\sin a(t-t')}{t-t'} . \qquad (4.4.11)$$

Here we notice the spirit of the Fourier Transform and its corresponding convolution theorem. The question here is that Kramer's generalization considers $K(t,t')$ in general and not only $k(t-t')$. As such it might very well be found to be of interest in the field of time variant [48, p. 8] linear filter analysis or its output signal sampling.

Another example that might show an advantage of Kramer's sampling theorem is given that

$$f(t_1,t_2) = \frac{1}{2\pi} \int_{-a_2}^{a_2} \int_{-a_1}^{a_1} g(x_1,x_2) e^{i(x_1 t_1 + x_2 t_2)} dx_1 dx_2, \qquad (4.4.12)$$

where

$$g(x_1,x_2) = 0, \quad |x_1| > |a_1|, \quad |x_2| > |a_2|, \qquad (4.4.13)$$

is a two-dimensional finite Fourier transform. By (2.9.2) we need a product of two infinite series to represent $f(t_1,t_2)$ in terms of its sample points, $f(\frac{n_1\pi}{a_1}, \frac{n_2\pi}{a_2})$. But if we have $f(t)$ with $\rho = \sqrt{t_1^2 + t_2^2}$ and such that $r = \sqrt{x_1^2 + x_2^2}$ then from Sneddon [38, p. 62] we can write (4.4.12) as

$$\bar{f}(\rho) = \int_0^b r\, f(r)\, J_0(\rho r) dr, \qquad (4.4.14)$$

a finite Hankel transform. Then, using Kramer's theorem, $\bar{f}(\rho)$ can be represented by an infinite series in terms of its sample points ρ_n.

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APPENDIX A

A.1 The Self-Adjoint Eigenvalue Problem

To illustrate the treatment of the self-adjoint differential equations we choose the following problem [40], with fourth order differential equation. This problem was used in Section 4.1 to illustrate the Kramer generalized sampling theorem:

$$[D^4 - (m^2 + \overline{m}^2)D^2 + m^2\overline{m}^2]\, y(te^{-x}) = t^2 e^{-2x} y(te^{-x}), \tag{A.1.1}$$

$$y(t) = 0, \quad y'(t_n) = 0, \quad n = 1, 2, \cdots, \quad \lim_{x\to\infty} y(te^{-x}) = \lim_{x\to\infty} y(t_n e^{-x}) = 0.$$

To show that this problem is self-adjoint we note that the operator

$$L = D^4 - (m^2 + \overline{m}^2)D^2 + m^2\overline{m}^2 \tag{A.1.2}$$

has constant coefficients. The weight function $\rho(x)$ is taken to be e^{-2x}. So, this is a special case and we will not need all the details of the general treatment in [8]. For the problem stated in (A.1.1) to be self-adjoint we need to show that

$$(Lu, v) = (u, Lv), \tag{A.1.3}$$

for all u and v that satisfy the boundary conditions. That is, it is required to show that

$$\int_0^\infty Lu\,\bar{v}\;dx = \int_0^\infty u\,\overline{Lv}\,dx\;. \tag{A.1.4}$$

Note that if $f, g \in L_2(a, b)$ then (f, g) is defined as

$$(f, g) = \int_a^b f\,\bar{g}\;dt\,. \tag{A.1.5}$$

In this case it is clear that $\bar{L} = L$, so we have to show that

$$\int_0^\infty Lu\,\bar{v}\;dx = \int_0^\infty u\;L\bar{v}\;dx\;. \tag{A.1.6}$$

We set

$$\int_0^\infty Lu\,\bar{v}\,dx - \int_0^\infty u\,L\bar{v}\,dx = \int_0^\infty (\bar{v}\,D^4u - uD^4\,\bar{v})dx - (m^2 + \bar{m}^2)\int_0^\infty (\bar{v}\,D^2u - uD^2\,\bar{v})dx$$

$$+\; m^2\bar{m}^2 \int_0^\infty (uv - uv)dx. \tag{A.1.7}$$

But

$$\bar{v}\,D^4u - uD^4\,\bar{v} = \frac{d}{dx}\,(\bar{v}\,D^3u - uD^3\,\bar{v}) + \frac{d}{dx}\,(Du\,D^2\,\bar{v} - D\bar{v}D^2u), \tag{A.1.8}$$

and

$$\bar{v}\,D^2u - uD^2\,\bar{v} = \frac{d}{dx}\,(\bar{v}\,Du - uD\bar{v}\;). \tag{A.1.9}$$

Substituting from Equations (A.1.8) and (A.1.9) into (A.1.7) we get

$$\int_0^\infty Lu\bar{v}\,dx - \int_0^\infty u\,L\bar{v}\,dx = [\bar{v}D^3u - uD^3\bar{v} + DuD^2\bar{v} - D\bar{v}D^2u]_0^\infty - (a^2+b^2)[\bar{v}Du - uD\bar{v}]_0^\infty . \quad \text{(A. 1. 10)}$$

So, if we use the boundary conditions in (A. 1. 1) all the integrated terms above vanish, $(Lu, v) = (u, Lv)$, and the problem of (A. 1. 1) is self-adjoint. This problem was also used to demonstrate the general and more detailed treatment [8] .

A. 2 Integrals for Bessel Functions

For calculating (3. 2. 14) we use the well-known properties of the Bessel functions:

$$\int x\,J_n^2(ax)dx = \frac{x^2}{2}[J_n^2(ax) - J_{n-1}(ax)J_{n+1}(ax)] \; ; \quad \text{(A. 2. 1.)}$$

$$\int xJ_n(ax)J_n(bx)dx = \frac{bxJ_n(ax)J_{n-1}(bx) - axJ_{n-1}(ax)J_n(bx)}{a^2-b^2} ; \quad \text{(A.2. 2)}$$

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x) \; ; \quad \text{(A. 2. 3)}$$

$$J_{n-1}(x) = 2J_n'(x) + J_{n+1}(x) \; ; \quad \text{(A. 2. 4)}$$

$$J_{n-1}(x) = \frac{n}{x}J_n(x) + J_n'(x). \quad \text{(A. 2. 5)}$$

Use of (A. 2. 2), (3. 2. 11) and (A. 2. 3) yields

$$\int_0^1 xJ_n(x\sqrt{t})J_n(x\sqrt{t_k})dx = \frac{\sqrt{t_k}J_n(\sqrt{t})J_{n-1}(\sqrt{t_k})}{t-t_k}$$

$$= -\frac{\sqrt{t_k}J_n(\sqrt{t})J_{n+1}(\sqrt{t_k})}{t-t_k}, \quad J_n(\sqrt{t_k}) = 0 . \tag{A. 2. 6}$$

The other half of the orthogonality statement is obtained by use of (A. 2. 2), (3. 2. 11) and (A. 2. 3),

$$\int_0^1 x J_n^2(x\sqrt{t_k})dx = -\frac{1}{2} J_{n-1}(\sqrt{t_k})\, J_{n+1}(\sqrt{t_k}) = \frac{1}{2} J_{n+1}^2(\sqrt{t_k}). \tag{A. 2. 7}$$

Hence,

$$S_k(t) = -\frac{2t_k J_n(\sqrt{t})}{(t-t_k)J_{n+1}(\sqrt{t_k})}, \tag{A. 2. 8}$$

and from (A. 2. 5) and (3. 2. 11) this may be written as

$$S_k(t) = \frac{2(n+1) J_n(\sqrt{t})}{(t-t_k)J'_{n+1}(\sqrt{t_k})} . \tag{A. 2. 9}$$

A. 3 Bessel and Legendre Functions

For getting (3. 3. 17) we had

$$f(t) = \int_0^1 \frac{x\,g(x)}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})} \left[\left(\frac{t}{2x}\right)^n \int_{-x}^{x} e^{i\omega t}(x^2-\omega^2)^{n-\frac{1}{2}} d\omega \right] dx$$

$$= t^n \int_{-1}^{1} e^{i\omega t}\left[\frac{1}{2^n\Gamma(n+\frac{1}{2})\sqrt{\pi}} \int_{|\omega|}^{1} x^{1-n}(x^2-\omega^2)^{n-\frac{1}{2}} g(x)dx\right] d\omega \tag{A.3.1}$$

$$= t^n \int_{-1}^{1} e^{i\omega t}\, H(\omega)d\omega \, .$$

For getting (3.3.19) we had

$$f(t) = \int_{-\pi}^{\pi} \frac{1}{\pi\sqrt{2}} \int_{-\theta}^{\theta} \frac{e^{i\alpha t}d\alpha}{\sqrt{\cos\alpha - \cos\theta}}\; g(\cos\theta)\sin\theta\, d\theta$$

$$= \int_{-\pi}^{\pi} e^{i\alpha t}\left[\frac{1}{\pi\sqrt{2}} \int_{|\alpha|}^{\pi} \frac{g(\cos\theta)\sin\theta\, d\theta}{\sqrt{\cos\alpha - \cos\theta}}\right] d\alpha \tag{A.3.2}$$

$$= \int_{-\pi}^{\pi} e^{i\alpha t} H(\alpha)d\alpha \, .$$

APPENDIX B

B. 1 Associated Legendre Functions

To get (4. 1. 6) the infinite series resulting from (4. 1. 4) and (4. 1. 5) is

$$P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu+m+1)\sin\pi\nu}{\pi\,\Gamma(\nu-m+1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1}\right) P_n^m(x) \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)}, \tag{B. 1. 1}$$

$$m \geq 0, \quad x > -1 .$$

Now

$$\int_{-1+\epsilon}^{1} P_\nu^m(x) P_n^m(x)\,dx \tag{B. 1. 2}$$

$$= (-1)^m \frac{\Gamma(\nu+m+1)\sin\pi\nu}{\pi\,\Gamma(\nu-m+1)} \int_{-1+\epsilon}^{1} P_{n'}^m(x) \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1}\right) P_n^m(x) \frac{(n-m)!}{(n+m)!}$$

$$= (-1)^m \frac{\Gamma(\nu+m+1)\sin\pi\nu}{\pi\,\Gamma(\nu-m+1)} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{\nu-n} - \frac{1}{\nu+n+1}\right) \frac{(n-m)!}{(n+m)!} \int_{-1+\epsilon}^{1} P_{n'}^m(x) P_n^m(x)\,dx.$$

Using (4. 1. 3) we get

$$\int_{-1}^{1} P_\nu^m(x) P_{n'}^m(x)\,dx = (-)^m \frac{\Gamma(\nu+m+1)\sin\pi\nu}{\pi\,\Gamma(\nu-m+1)} \cdot \frac{(n'-m)!}{(n'+m)!} (-1)^{n'} \cdot \tag{B. 1. 3}$$

$$\left(\frac{1}{\nu-n'} - \frac{1}{\nu+n'+1}\right) \cdot \frac{(n'+m)!}{(n'-m)!} \cdot \frac{1}{n'+\frac{1}{2}}$$

$$= \frac{2\,\Gamma(\nu+m+1)\sin\pi(\nu-n'-m)}{\pi\,\Gamma(\nu-m+1)(\nu-n')(\nu+n'+1)} . \tag{4. 1. 6}$$

B. 2 Gegenbauer Functions

To get (4. 1. 15) we substitute (4. 1. 14) in the desired integral,

$$\int_{-1}^{1} C_a^{m+\frac{1}{2}}(z) C_n^{m+\frac{1}{2}}(z)(1-z^2)^m dz = \left[\frac{2^m m!}{(2m)!}\right]^2 (-1)^m \int_{-1}^{1} P_{a+m}^m(z) P_{n+m}^m(z) dz . \tag{B. 2. 1}$$

Use of (4. 1. 6) for the unrestricted a leads to

$$\int C_a^{m+\frac{1}{2}}(z) C_n^{m+\frac{1}{2}}(z)(1-z^2)^m dz = \left[\frac{2^m m!}{(2m)!}\right]^2 \frac{2}{\pi} \frac{\Gamma(a+2m+1)\sin\pi(a-n)}{\Gamma(a+1)(a-n)(a+2m+n+1)} . \tag{4. 1. 15}$$

$$\mathrm{Re}(m+\tfrac{1}{2}) > 0 .$$

B. 3 Tchebichef Functions

To arrive at (4. 1. 20) we use [11, p. 169, Eq. (1)]

$$\int_{-1}^{1} P_t^{\mu}(x) P_\sigma^{\mu}(x) dx = \frac{1}{(t-\sigma)(t+\sigma+1)} \Big[x(t-\sigma) P_t^{\mu}(x) P_\sigma^{\mu}(x) + (\sigma+\mu) P_t^{\mu}(x) P_{\sigma-1}^{\mu}(x) \tag{B. 3. 1}$$

$$- (t+\mu) P_{t-1}^{\mu}(x) P_\sigma^{\mu}(x) \Big]_{-1}^{1}$$

If, for (B. 3. 1) we use

$$C_t^{\mu}(x) = \frac{2^{\mu-\frac{1}{2}}\Gamma(t+2\mu)\Gamma(\mu+\frac{1}{2})}{\Gamma(2\mu)\Gamma(t+1)} (1-x^2)^{\frac{1-2\mu}{4}} P_{t+\mu-\frac{1}{2}}^{-\frac{1}{2}-\mu}(x), \mathrm{Re}\mu > 0 , \tag{B. 3. 2}$$

and

$$U_a(x) \equiv C_a^1(x) \tag{B.3.3}$$

we get

$$\int_{-1}^{1} U_t(x)U_n(x)(1-x^2)^{\frac{1}{2}}dx = \int_{-1}^{1} C_t^1(x)C_n^1(x)(1-x^2)^{\frac{1}{2}}dx \tag{B.3.4}$$

$$= \frac{1}{(t-n)(t+n+2)} \cdot \frac{2\Gamma(t+2)\Gamma(n+2)[\Gamma(\frac{3}{2})]^2}{[\Gamma(2)]^2\ \Gamma(t+1)n!}\Big[x(t-n)P_{t+\frac{1}{2}}^{-\frac{1}{2}}(x)P_{n+\frac{1}{2}}^{-\frac{1}{2}}(x)$$

$$+ n P_{t+\frac{1}{2}}^{-\frac{1}{2}}(x)P_{n-\frac{1}{2}}^{-\frac{1}{2}}(x) - t P_{t-\frac{1}{2}}^{-\frac{1}{2}}(x)P_{n+\frac{1}{2}}^{-\frac{1}{2}}(x)\Big]_{-1}^{1}.$$

This plus the orthogonal property

$$\int_{-1}^{1} U_n^1(x)^2 (1-x^2)^{\frac{1}{2}} dx = \pi$$

yields (4.1.20).

B.4 Prolate Spheroidal Wave Functions

To obtain (4.1.25) we first use (4.1.23) for the desired integral to get

$$\int_{-1}^{1} Ps_\nu^m(x,\theta)Ps_{n'}^m(x,\theta)dx = \sum_{r=-\infty}^{\infty}(-1)^r a_{\nu,r}^m(\theta)\int_{-1}^{1} Ps_{n'}^m(x,\theta)P_{\nu+2r}^m(x)dx. \tag{B.4.1}$$

We use the infinite series expansion (B.1.1) of $P_{\nu+2r}^m(x)$ and integrate term by term, arriving at

$$\int_{-1}^{1} Ps_{n'}^{m}(x,\theta)P_{\nu+2r}^{m}(x)dx =$$

$$\frac{\sin \pi\nu}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{\Gamma(\nu+2r+m+1)(2n+1)\Gamma(n-m+1)}{\Gamma(\nu+2r-m+1)(\nu+2r-n)(\nu+2r+n+1)\Gamma(n+m+1)} \tag{B.4.2}$$

$$\cdot \int_{-1}^{1} Ps_{n'}^{m}(x,\theta)P_{n}^{m}(x)dx .$$

If we use (4.1.24) in the above it becomes

$$\int_{-1}^{1} Ps_{n'}^{m}(x,\theta)P_{\nu+2r}^{m}(x)dx = \frac{2\sin\pi(\nu-n')(-1)^r\Gamma(\nu+2r+m+1)a_{n',r}^{m}}{\pi(\nu-n')(\nu+n'+4r+1)\Gamma(\nu+2r-m+1)}, \tag{B.4.3}$$

$2r = n-n'$, r a positive integer.

If we substitute (B.4.3) into (B.4.1) we get (4.1.25),

$$\int_{-1}^{1} Ps_{n'}^{m}(x,\theta)Ps_{\nu}^{m}(x,\theta)dx = 2\sum_{r=0}^{\infty} a_{n',r}^{m}(\theta)a_{\nu,r}^{m}(\theta)\frac{\Gamma(\nu+2r+m+1)}{\Gamma(\nu+2r-m+1)(\nu+n'+4r+1)} \tag{B.4.4}$$

$$\cdot \frac{\sin\pi(\nu-n')}{\pi(\nu-n')} .$$

B.5 Bessel Functions

To obtain (4.1.29) from (4.1.28) we let $x = e^{-\frac{1}{2}\omega_2 u}$ and $\gamma = \omega_1/\omega_2$ to get

$$x^2 h''(x) - xh'(x) + \left[\frac{8\omega_1 \gamma x^2}{\lambda} - 4\gamma(\gamma-1)\right] h(x) = 0, \quad 0 \le x \le 1. \tag{B.5.1}$$

If we let $h(x) = xv(x)$ and $t^2 = \frac{8\omega_1 \gamma}{\lambda}$ we get

$$x^2 v''(x) + xv'(x) + [x^2 t^2 - (2\gamma-1)^2]\, v(x) = 0, \tag{B.5.2}$$

which is the Bessel differential equation of (4.1.29). Then (4.1.31) is obtained using (3.2.14) for the case of $K(t,x) = x^{-\frac{1}{2}} h(t,x)$ in (3.1.3).

B.6 Fourth Order Differential Equations (Bessel-like Functions)

The important definition is

$$y(te^{-x}) = y_m(te^{-x})\, y_{\overline{m}}(t) - y_{\overline{m}}(te^{-x})\, y_m(t) \tag{B.6.1}$$

and it is clear that $y(t)$ is identically zero. The solution of the fourth order differential equation of interest here may be written as

$$y_m(te^{-x}) = \sum_{n=0}^{\infty} \frac{\left(\frac{t}{4} e^{-x}\right)^{m+2n}}{n!\,\Gamma(m+1+n)\,\Gamma\!\left(\frac{m+\overline{m}}{2}+1+n\right)\Gamma\!\left(\frac{m-\overline{m}}{2}+1+n\right)}. \tag{B.6.1}$$

To obtain (4.1.37) we eliminate the $\overline{m}^2 \overline{m}^2 y(t_n e^{-x}) y(te^{-x})$ from the two differential equations in the usual way, arriving at

$$(t_n^2-t^2)\int_0^\infty e^{-2x}y(t_ne^{-x})dx = [\,y(te^{-x})D^3(t_ne^{-x})-y(t_ne^{-x})D^3y(te^{-x}) \tag{B. 6. 2}$$

$$-Dy(te^{-x})D^2y(t_ne^{-x})+Dy(t_ne^{-x})D^2y(te^{-x})]_0^\infty$$

$$-(m^2+\overline{m}^2)[y(te^{-x})Dy(t_ne^{-x})-y(t_ne^{-x})Dy(te^{-x})]_0^\infty$$

$$= Dy(te^{-x})D^2y(t_ne^{-x})\Big|_{x=0} = -tt_n^2y'(t)y''(t_n)\,. \tag{4. 1. 37}$$

B.7 Associated Legendre Functions (Continued)

From (4. 2. 5) and (4. 2. 1) we get

$$f(t) = \frac{\Gamma(t+m+1)(-1)^m}{\sqrt{2\pi}\,\Gamma(t-m+1)\Gamma(m+\frac{1}{2})}\int_{-\pi}^{\pi} d\theta g(\cos\theta)\sin^{1-m}\theta\,\cdot \tag{B. 7. 1}$$

$$\int_{-\theta}^{\theta}(\cos v-\cos\theta)^{m-\frac{1}{2}}e^{i(t+\frac{1}{2})v}dv.$$

Now interchange of the order of integration leads to

$$f(t) = \frac{\Gamma(t+m+1)(-1)^m}{\sqrt{2\pi}\,\Gamma(t-m+1)\Gamma(m+\frac{1}{2})}\int_{-\pi}^{\pi} e^{itv}e^{\frac{iv}{2}}\int_{|v|}^{\pi} g(\cos\theta)\sin^{1-m}\theta(\cos v-\cos\theta)^{m-\frac{1}{2}}d\theta dv \tag{B. 7. 2}$$

$$= \frac{\Gamma(t+m+1)}{\Gamma(t-m+1)}\int_{-\pi}^{\pi} e^{itv}F(v)dv,$$

where $F(v)$ is defined as above.

B.8 Gegenbauer Functions (Continued)

We can write (4.2.7) as

$$C_t^{\nu}(\cos\phi) = \frac{2^{\nu-1}\Gamma(t+2\nu)\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)\Gamma(2\nu)\Gamma(t+1)}\sin^{1-2\nu}\phi\int_{-\phi}^{\phi} e^{iv(t+\nu)}(\cos v-\cos\phi)^{\nu-1}dv. \tag{B.8.1}$$

We use (B.8.1) for $K(t, \cos\phi)$ in (4.2.1) to get (after change in the order of integration)

$$f(t) = \frac{\Gamma(t+2\nu)}{\Gamma(t+1)}\int_{-\pi}^{\pi} e^{itv}\frac{2^{\nu-1}\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(\nu)\Gamma(2\nu)}e^{iv\nu}\int_{|v|}^{\pi} g(\cos\phi)\sin^{2-2\nu}\phi(\cos v-\cos\phi)^{\nu-1}d\phi dv. \tag{B.8.2}$$

Hence,

$$f(t) = \frac{\Gamma(t+2\nu)}{\Gamma(t+1)}\int_{-\pi}^{\pi} e^{itv}H(v)dv, \tag{B.8.3}$$

where $H(v)$ is defined in (B.8.2).

B. 9 Illustration of Theorem 4. 2. 3., p. 69

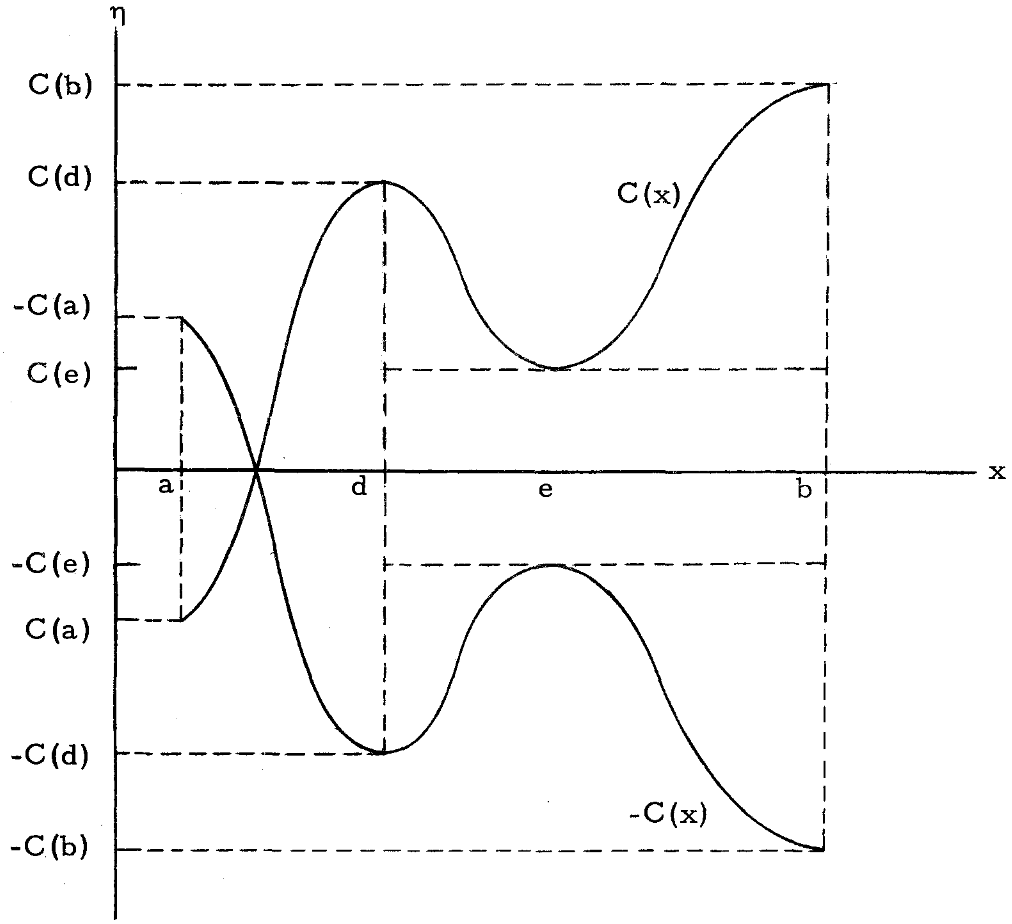

Figure 2.

The function in (4. 2. 21),

$$f(t) = \int_a^b h_1(x)h_2(t) \int_{-c(x)}^{c(x)} k(x, \eta)e^{it\eta}d\eta g(x)\rho(x)dx, \qquad (4.2.12)$$

for the case of $c(x)$ in Figure 2 can be written, after interchange of order of integration, as

$$f(t) = h_2(t) \int_{c(a)}^{-c(a)} e^{it\eta} \left[\int_{x=a}^{c^{-1}(|\eta|)} h_1(x) k(x, \eta) g(x) \rho(x) dx \right] d\eta$$

$$+ \int_{-c(d)}^{c(d)} e^{it\eta} \left[\int_{c^{-1}(|\eta|)}^{d} \quad - - \quad dx \right] d\eta$$

$$+ \int_{-c(d)}^{-c(e)} e^{it\eta} \left[\int_{d}^{c^{-1}(|\eta|)} \quad - - \quad dx \right] d\eta$$

$$+ \int_{-c(e)}^{c(e)} e^{it\eta} \left[\int_{d}^{b} \quad - - \quad dx \right] d\eta \qquad \text{(B. 9. 1)}$$

$$+ \int_{c(e)}^{c(d)} e^{it\eta} \left[\int_{d}^{c^{-1}(|\eta|)} \quad - - \quad dx \right] d\eta$$

$$+ \int_{-c(b)}^{-c(e)} e^{it\eta} \left[\int_{c^{-1}(|\eta|)}^{b} \quad - - \quad dx \right] d\eta$$

$$+ \int_{c(e)}^{c(b)} e^{it\eta} \left[\int_{c^{-1}(|\eta|)}^{b} \quad - - \quad dx \right] d\eta ,$$

a sum of seven different band-limited functions of the type

$$f_j(t) = h_2(t) \int_{a_j}^{a_j'} e^{it\eta} H(\eta) d\eta . \qquad \text{(B. 9. 2)}$$