AN ABSTRACT OF THE THESIS OF

Jaw-Fang Lee for the degree of Doctor of Philosophy in Civil Engineering presented on December 16, 1986.

Title: FINITE ELEMENT ANALYSIS OF WAVE-STRUCTURE INTERACTIONS IN THE TIME DOMAIN

Abstract approved: Redacted for privacy

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The time-domain simulation of wave-structure interactions using the finite element method is presented in this study. Combined initial and boundary value problems for diffraction and radiation of waves by large structures are posed. Linearized free surface boundary conditions are used, but motions of the structural boundaries compared to the wave length are allowed to be large: adaptive meshes are used in the fluid adjoining the structures to accommodate structural motions without excessive distortions of the fluid elements. An extrapolated version of Orlanski's condition is developed for use for an artificial boundary to truncate the discretized region. Galerkin's method is used to formulate integral expressions for isoparametric, quadratic finite elements in the fluid domain and along its boundaries. Multiple fixed or floating rigid bodies are included which may be moored or interconnected by
linear springs.

Dynamic analysis procedures for problems of fluid-structure interactions are presented. A partitioned, staggered integration scheme is developed for the fluid-structure system. The implicit Newmark's method is used for the structural equations, and a nodal partitioned explicit-implicit integration scheme is derived for the fluid.

Test examples are included to demonstrate the validity of the finite element fluid model, the extrapolated time dependent Orlanski condition for the artificial boundary, and the dynamic solution procedures developed to describe the interactions between fluids and structures for initial value problems or problems with nonlinear structural motions.
FINITE ELEMENT ANALYSIS OF WAVE-STRUCTURE INTERACTIONS
IN THE TIME DOMAIN

by

Jaw-Fang Lee

A THESIS
submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of
Doctor of Philosophy

Completed December 16, 1986
Commencement June 1987
ACKNOWLEDGEMENT

The author wishes to express his sincere gratitude and appreciation to his advisor, Dr. John W. Leonard, for his support, encouragement and guidance. The author would also like to thank the fellows in the Ocean Engineering for their helpful discussions. Special appreciation is also extended to his families for their support.

The financial support provided by the Sea Grant Program through Grant No. NA81AA-D-00086 to Oregon State University is also gratefully acknowledged.
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FINITE ELEMENT ANALYSIS OF WAVE-STRUCTURE INTERACTIONS
IN THE TIME DOMAIN

1.0 INTRODUCTION

1.1 Motivation

There are a wide variety of offshore structures used in the ocean environment. Among them are jacket type platforms (64)*, hybrid towers (95), caissons (43), concrete gravity platforms (19), storage tanks (41), and semisubmersibles (54). In the design against wave-induced loads, the structures are classified into two regimes according to the ratio of the characteristic length of the structure, D, to the wave length, L. When $D/L \geq 0.2$, the structure is considered large and the diffraction and radiation of waves by the structure needs to be considered. On the other hand, if $D/L < 0.2$, the structure is treated as small and Morison's equation is used to calculate wave loads. In analyzing the interactions between waves and large structures, a frequency-domain approach has been extensively used to pose and solve a linear boundary value problem describing the interactions under steady state conditions.

* Numbers in parentheses refer to numbered references in BIBLIOGRAPHY.
Representative problems of wave diffraction and radiation in the frequency domain can be found in Refs. (67, 87, 83, 98, 97, 100, 106, 105).

For problems of unsteady motions, a combined initial and boundary value problem has to be considered (87, 68). Due to transient motions of the structure, the radiated waves in the fluid possess unsteady features. The time-dependent wave loadings induced by unsteady flow in turn affect the structural motions. To account for such time-dependent behaviors, it is then necessary to use a time-domain simulation for unsteady wave-structure interactions. Typical examples of unsteady motions include vibrations of a floating structure for which an initial displacement is specified (65), interactions between structures and a finite number of waves (63). Transient problems may also occur during tow out and upending or launching of an offshore structure (95). An analysis in the time-domain of the dynamic response of such a structure during installation has several advantages, not only in determining the probable behavior during the critical stages of upending and sinking, but also as a simulation exercise for training launch operators and evaluating real-time control systems (95).
The wave motions of the fluid and the corresponding structural motions can be categorized into small and large motions. Since magnitudes of the structural motions are of the same order as the motions of the free surface, the same criterion of determining large or small motion can be used for both fluid and structures. Ursell's criterion (91) for surface waves is adopted here. For $H/L \leq 0.03$, the wave motions of the fluid and the corresponding structural motions are considered small relative to the wave length, $L$, and the boundary conditions on the structural surface and on the free surface boundary can be calculated at the initial equilibrium position and at the still water level, respectively. For $H/L > 0.03$, the wave motions of the fluid and the corresponding structural motions are considered large relative to the wave length, and changes of the structural positions and the free surface boundary have to be considered (10, 21, 31, 49, 69, 70, 76, 96).

In the description of nonlinear interactions of waves and highly deformable bodies, time domain simulations must be used. The research described in this study represents the first step toward such a nonlinear time domain simulation. The use of the finite element method in the analysis of interactions of wave with large
structures in the time domain is the subject of the present study. Linear wave theory is used but the structural motions relative to the wave length are allowed to be large. Ideal flow, combined initial and boundary value problems are posed to solve the corresponding diffraction and radiation problems. A combined explicit and implicit scheme is used to numerically integrate the equations in the time domain. A modified Orlanski's condition is used for the artificial boundary of the fluid domain to ensure that only outgoing waves are present in the truncated fluid domain. Unsteady wave interference effects between multiple rigid or flexible structures with and without external and inter-structural constraints, are simulated in the time domain using a direct interference method for each time step, where the structures involved are embedded in an inner adaptive domain and the wave field variables determined using a single generalized matrix for the system.

1.2 Previous Studies

The calculation of wave forces on large offshore structures of arbitrary shape is often performed in the frequency domain by using linear wave diffraction theory.
There are several analytical solutions available, but they are restricted to problems with specific structural geometries (17, 29, 35, 62, 61, 102). As to numerical solutions, the integral equation methods and finite element methods have been applied extensively to the prediction of wave diffraction and radiation by large structures. Both solution procedures have many variants and each possesses certain merits and limitations. Integral equations of several forms have been reviewed (67, 87, 98, 97, 100). Examples of their applications to wave diffraction and radiation problems may be found in Refs. (30, 37, 38, 72, 99). Adaptations of the finite element method for fluid problems have been reviewed by Shen (84) and applications to wave diffraction and radiation problems by Mei (67), by Zienkiewicz, et al. (105) and by Huang (44).

There has been little attention devoted to the analysis of interactions between water waves and large structures in the time domain. Only a few analytical solutions are available, as described below, and are limited to problems with special structural geometries. It is necessary to use numerical methods to investigate wave-structure interactions in the time-domain. The review of analytical and numerical approaches will be
followed by a description of the available finite element solution procedures.

Analytical Approaches in the Time Domain

The generation of surface waves by a moving partition has been solved in an analytical form by Kennard (55) as a combined initial and boundary value problem. A source integral method was used to solve the problem. Madsen (63) studied Kennard's steady state solution and applied it to a sinusoidal piston-type wavemaker which runs for a finite length of time. Madsen concluded that Kennard's solution agrees well with experimental results.

The transient motion of a floating body due to an initial displacement or an initial velocity has been solved by Ursell (91). A Fourier transform was used to transform the combined initial and boundary value problem to a form which had been solved previously by Ursell (90) and by Havelock (42). Since the solutions were complicated to calculate, only the asymptotic behavior of the solution was discussed. Maskell and Ursell (65) repeated Ursell's solutions and made a numerical study of the kernels of the response functions of the structure. However, they concluded "We made a brief attempt at a
Numerical Approaches in the Time Domain

An exact nonlinear model of a wave generator has been reported by Multer (71). An instantaneous mixed boundary value problem has been solved using a Lagrangian formulation for the free surface boundary. Numerical estimate of the solution of the linearized combined initial and boundary value problem tends in the limit to the solution of the linearized steady state problem.

A nonlinear wavemaker problem has been solved by O'Brien (77) using a finite element method. A Galerkin weighted residual method was used to formulate the instantaneous mixed boundary value problem. Adaptive finite-element meshes were used near moving boundaries, the free surface and the wavemaker. Linear finite elements were used to model the domain of the problem. An explicit scheme of time integration was used to solve the field variables on the free surface independently of the rest of the domain. Hence, it appears that the coupling terms between the free surface and the rest of the fluid domain were neglected. A demonstration of the
transients of water waves was made. However, numerical results for the linear wavemaker problem did not show good agreement with the analytical results from linear wavemaker theory.

A boundary element method has been used by Nakayama (73) to solve a wave-making problem in the time domain. A Green's formula was used to formulate the integral equation for the corresponding combined initial and boundary value problem. Excellent agreements were obtained between numerical results and experimental data and analytical solutions. However, only cases of long waves, e.g. tsunami and solitary waves, were demonstrated. Nakayama and Washizu (74) used the same technique to solve a nonlinear sloshing problem in the time domain.

A time-domain simulation of the dynamic response of a semi-submersible platform in severe sea conditions has been reported by Matsuura and Ikegami (66). No wave diffraction/radiation problems were considered. A relative Morison's equation was used to estimate the hydrodynamic loadings.

A time-domain simulation of two dimensional, nonlinear free surface flows has been reported by Jagannathan (53). Wave diffraction and radiation
problems were considered. Vinje and Brevig's method (94) was used as the basic technique to solve the combined initial and boundary value problem for complex velocity potentials. Free surface waves were considered unsteady in time, but a spatial periodicity was assumed. Since a complex velocity potential was used, the method cannot be generalized to three dimensional cases.

The time-domain simulation of the interactions between waves and structures can also be achieved by means of convolution integrals of the hydrodynamic force coefficients in the frequency domain based on Cumming's method (27). Koman (56) has applied this solution technique to the design of an open sea shipping berth. Van Oortmerssen (93) has investigated the time-domain simulation of the interference phenomena of two floating vertical cylinders. However, practical restrictions involved are such that the hydrodynamic force coefficients must be known for all frequencies and the calculations of the convolution integrals have to be carried out over a large time interval.

Nonreflecting Boundaries

In discrete numerical computations, such as conducted using finite difference or finite element methods, it is usually necessary to curtail the infinite
domain and replace it by a finite domain with an artificial boundary placed at a finite distance from the disturbances. A boundary condition is thus needed to ensure that only outgoing waves are present so that all energy generated by the disturbances is radiated outward. Several different methods for the treatment of this boundary have been proposed and employed with varying success. The resulting techniques are variously known as radiation boundaries (107), transmitting boundaries (22), absorbing boundaries (59), nonreflecting boundaries (86), silent boundaries (24) or open boundaries (53). A literature review has been given by Zienkiewicz (107) and by Cohen (24).

For frequency domain analyses, there have been several alternatives: boundary dampers (3, 75, 85), boundary solutions (exterior analytical or series solutions) (36, 101), exterior boundary integral formulations (106, 32), and infinite elements (14, 16, 15, 104). A discussion on these methods can be found in Zienkiewicz (107) and in Huang (44). Since these boundaries are frequency dependent, they are not suitable for general transient analyses.

For transient analyses of soil-structure interactions, Lysmer and Kuhlmeier (59) proposed an
absorbing boundary using viscous damping forces along the boundary as a means of absorbing, rather than reflecting, the radiated energy. The frequency-dependent boundary condition was tuned to minimize the reflections from the boundary. Smith (86) proposed a nonreflecting boundary in which two wave solutions having different boundary conditions, one with a Dirichlet boundary condition and the other with a Neumann boundary condition, are added so as to eliminate reflections. Cundall, et al. (28) applied this idea to a nonlinear problem where Smith's method was implemented in the time marching scheme only in boundary regions. Based on Keiss and Pearson's idea (82), Orlanski (79) suggested the use of a modified Sommerfeld radiation condition at the boundary of a finite domain in hyperbolic flows. Instead of using a constant phase velocity, a propagating phase velocity was numerically calculated at each time step for use in the boundary condition. Recently, Jagannathan (53) used Orlanski's condition for unsteady, two dimensional surface flows wherein the average value of the phase velocities at a number of points near the open boundary was used. So far, each of the proposed boundary schemes has been shown to be effective for selected wave problems.
Dynamic Coupled Field Analyses

There have been three general solution schemes adopted in dynamic fluid-structure analyses: field elimination solutions (39, 40, 81), simultaneous solutions (7, 81), and partitioned (or staggered) solutions (12, 47, 48, 46, 80). In the scheme of field elimination solutions, one field variable of primary interest is chosen and the rest of the field variables are eliminated through the use of governing differential equations of the coupled fields. The elimination process then yields a higher-order differential equation in one field variable. Although, the use of field elimination solutions was moderately successful, it was characterized as a poor strategy due to the following reasons: 1. The raising of the order of the differential equation can be the source of many numerical difficulties. 2. Proper treatment of initial conditions is complicated by the increased order. 3. Sparseness and symmetry attributes of the original matrices are adversely affected by the elimination process.

In the scheme of simultaneous solutions, the field equations are solved simultaneously. This approach removes many of the objections to the field elimination technique. Inasmuch as the order of the differential equations is not raised, difficulties with initial
conditions do not arise. However, due to the presence of matrix coupling terms, the coefficient matrices of the global equations of motion pose enormous computational demands.

In the partitioned integration scheme, the solution state is advanced over each of the subsystems in a staggered or sequential fashion. Interaction terms are treated as "forcing" actions that are judiciously extrapolated. Compared to other methods, the key advantages of partitioned analysis procedures are computational efficiency and modular implementation (80). Park, Felippa, and DeRuntz (81) introduced implicit-implicit staggered partitions for acoustic fluid-structure interactions, wherein an implicit scheme of time integration was used for both the fluid and the structure. Several stabilized formulations were introduced to circumvent severe time step size limitations. Belytschko and Mullen (12) developed the concept of implicit-explicit partitions of finite element analyses of acoustic fluid-structure interaction. An explicit scheme of time integration was used for the fluid and an implicit scheme was used for the structure. A stability-accuracy theory for partitions of second-order systems was given by Park, et al. (80).
1.3 Thesis Objectives and Scope

The objective of this research is to develop a finite element model for the analysis of interactions between waves and structures in the time domain. In order to investigate the transients of water waves on the free surface and the corresponding time dependent structural motions, a dynamic finite element model of the fluid is developed with which the wave field variables can be calculated for each time step; a dynamic structure model is used to predict the structural motions due to unsteady fluid loadings. The structural motions relative to the wave length are allowed to be large. Finite element meshes around the structures thus distorted are adapted at each time step according to the change of the structural positions with time. In order to treat the transient behavior of surface waves, Orlanski's condition (79) is modified for use at the artificial boundary of the finite element domain. Isoparametric, quadratic elements are used in the finite element analysis.

For dynamic analyses of the fluid motions, a nodal-based partitioning scheme is used for the fluid where the finite element nodes are divided into two groups: an explicit group which contains finite element nodes on the free surface boundary and an implicit group which
consists of the rest of the nodes in the fluid domain. Since the partitioning scheme is not a good self-starting method, an implicit solution is used during the first time step as a starting scheme. For the dynamic structural analyses, Newmark's method of implicit time integration is applied. In the numerical time integration of the combined fluid-structure dynamic responses, the field variables of the explicit group of the fluid are calculated from the previous time steps; the implicit group of the fluid and the implicit structural motions are modified into iterative forms which are then solved by iterations for each time step.

Time-domain simulation of the flap wavemaker problem is used to validate the proposed dynamic finite element model for the fluid. Comparisons are made with existing analytical solutions (55) and experimental data from OSUWRF. The study of the flap wavemaker problems provides the guideline in selecting modelling criteria, such as mesh size, time-step size, etc. as well as guidance in the use of the modified Orlanski's condition for unsteady wave problems. Extensive studies are made of the applicability and accuracy of the modified Orlanski's condition in single-frequency as well as multi-frequency wave problems.
Time-domain simulations of wave diffraction and radiation by a single structure are studied; the steady state solutions are compared with existing analytical frequency-domain solutions and experimental results (17, 29, 100). Examples of structures with motions specified as large relative to the wave length are considered to determine the effects of the adaptive structural positions on the generated waves.

The study of the transient motions of a floating structure is used to validate the proposed model of dynamic fluid-structure interactions. Comparisons are made with existing analytical solutions (65, 91). Parametric studies are performed to investigate the effect of water depth on the transient motions of the structures.

Time-domain simulations of interactions between waves and a moored floating structure are studied, the steady state solutions are compared with existing numerical frequency-domain solutions (50). Time-domain simulations of the hydrodynamics of two closely spaced multiple moored, floating structures are studied. Considerations are given to the interference effects and the effects from inter-structural constraints. A model of an actual twin floating bridge is studied wherein design wave conditions are applied.
In Chapter 2, the governing equations of the fluid and the structures are formulated. A finite element representation for the fluid is derived based on Galerkin's weighted residual method. In Chapter 3, a partitioned solution scheme is developed for dynamic fluid analyses. An iterative explicit-implicit solution procedure is presented for dynamic analyses of fluid-structure iterations. Test examples are presented in Chapter 4 to validate of the finite element model and demonstrate the capabilities of the numerical analysis scheme developed. Chapter 5 contains the summary, discussion and possible extensions of the present study. In APPENDIX I, the expressions and calculation procedures of Kennard's analytical solution in the time-domain of the flap wavemaker problem are given. Maskell and Ursell's analytical solutions of the transient motions of a floating circular cylinder are summarized in APPENDIX II.
2.0 FORMULATION OF THE PROBLEM

In this chapter, the problem of wave-structure interactions in the time-domain is formulated. A finite element method is used as the basis to study this problem. A dynamic model of the unsteady, diffraction and radiation of waves by large structures is developed to predict the transient flow fields around the structures and the hydrodynamic loadings on the structures. A dynamic structural model is used to predict the structural motions due to hydrodynamic loadings. Galerkin's weighted residual method is exploited to formulate an integral expression corresponding to the combined initial and boundary value problem of the fluid. Orlanski's condition is modified for use at the artificial boundary which is incorporated into the finite element fluid model. Since structural motions relative to the wave length might be large, an adaptive mesh is used for the region surrounding the structures to avoid excessive deformations of the finite elements.

2.1 General Description of the Problem

Large offshore structures may be floating or submerged; free, moored or interconnected by structural
constraints. The definition sketch of the problem is shown in Fig. 2.1. Figure 2.1 is intended to show general deployments of ocean structures. An inertial, Cartesian coordinate system \((x_1, x_2, x_3)\) with its origin located at the still water level is used as the reference coordinate system. Under the actions of hydrodynamic loadings which are induced by incident disturbances or by unsteady structural or boundary motions, the structures move from initial positions to dynamic equilibrium positions. Due to dynamic structural motions, the radiated waves and the corresponding unsteady flow field, in turn, affect the structural motions. The coupling of fluid-structure interactions takes place through pressure loadings from the fluid and the structural motions on the fluid-structure interfaces. From Fig. 2.1, it can be seen that the boundary, \(B\), of the fluid domain consists of a free surface, \(B_1\), a bottom boundary, \(B_2\), an artificial boundary, \(B_3\), and the wet structural surfaces, \(S_1, S_2, \ldots, S_N\), \(N\) being the number of the structures. The artificial boundary, \(B_3\), is a fixed cylindrical surface with a radius defined large enough to contain the whole domain of interest. Since the infinite fluid domain is curtailed at the artificial boundary, special attention has to be given to implementing
Fig. 2.1. Definition sketch for wave-structure interactions in the time domain
boundary conditions for $B_3$. A time-dependent boundary condition for $B_3$ will be proposed in a later section. The structural surfaces, $S_1, S_2, \ldots, S_N$ as well as the free surface, $B_1$, may change positions with time. To facilitate the finite element analysis, the fluid domain, $D$, is divided into a fixed region $D_f$, and adaptive regions, $D_1, D_2, \ldots, D_M$, $M$ being the number of adaptive regions. Each adaptive region is defined such that the excursions of the structural boundary in that particular region will not be beyond that region. Depending on the motion of the boundary, the finite element mesh in each adaptive region is redefined with time such that finite elements do not have excessive deformations.

2.2 Formulation of the Fluid

The descriptions of the problem given in Sec. 2.1 are general. The free surface and structural boundaries of the fluid domain may change positions with time; the motions of the free surface boundary and of structural boundaries may be large relative to the wave length. However, for simplicity, in this study a less general description is considered. A combined initial and boundary value problem for the fluid is formulated which retains a linearized free surface boundary but satisfies
the exact structural boundary conditions. Similar formulations can be found in Refs. (21, 57). An Eulerian description of the fluid motions is used. Galerkin's weighted residual method is used to derive an integral form for the fluid problem.

2.2.a Combined Initial and Boundary Value Problem

Assuming that the fluid is inviscid and incompressible, and the flow is irrotational, one can describe the fluid motion by a scalar velocity potential, $\phi_T(x_1,x_2,x_3,t)$ (87, 83). The velocity of the flow field can be defined as

$$\vec{V}_T(x_1,x_2,x_3,t) = -\nabla \phi_T(x_1,x_2,x_3,t).$$

If we consider the problem as illustrated in Fig. 2.1, the total velocity potential, $\phi_T$, can be expressed as the sum of an incident potential, $\phi_I$, and a perturbed potential, $\phi$.

$$\phi_T = \phi_I + \phi$$

For the case of wave radiation, the incident wave potential does not exist.

The governing equation for $\phi$ is the Laplace equation (87, 83).

$$\nabla^2 \phi(x_1,x_2,x_3,t) = 0, \text{ in } D = D_F U D_L U ... U D_N,$$

$$\theta \leq t < \infty, \quad (2.3)$$
where $\nabla^2(.) = \partial^2(.)/\partial x_1^2 + \partial^2(.)/\partial x_2^2 + \partial^2(.)/\partial x_3^2$. The initial conditions of the problem are $\phi(x_1, x_2, x_3, 0) = \phi_0(x_1, x_2, x_3)$ and $\dot{\phi}(x_1, x_2, x_3, 0) = \dot{\phi}_0(x_1, x_2, x_3)$, where $\phi_0(x_1, x_2, x_3)$ and $\dot{\phi}_0(x_1, x_2, x_3)$ are specified initial conditions.

The boundary conditions for the fluid domain are described as follows. For the free surface boundary, $B_1$, the linearized, combined kinematic and dynamic free surface boundary condition (87, 83) can be written as

$$\frac{\partial \phi}{\partial n_f} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} = 0, \text{ on } B_1,$$

(2.4)

in which $g$ is the gravitational constant. Since a linear boundary condition is adopted, Eq. (2.4) is calculated at the still water level, $x_2=0$ and $n_f$ is in the positive $x_2$ direction. On the bottom boundary, $B_2$, the kinematic boundary condition is

$$\frac{\partial \phi}{\partial n_b} = 0, \text{ on } B_2, x_2 = h(x_1, x_3),$$

(2.5)

where $n_b$ is the local unit normal of the bottom boundary and is directed outwardly from the fluid; $h$ is the local water depth relative to the still water level. For the artificial boundary, $B_3$, Orlanski's condition (79) is modified for use and can be expressed as

$$\frac{\partial \phi}{\partial n_r} + \frac{1}{C} \frac{\partial \phi}{\partial t} = 0, \text{ on } B_3; C = C(t)$$

(2.6)
in which \( \hat{n}_m \) is the unit normal to \( S_m \) and is directed outwardly from the fluid; \( \mathbf{C} \) is the time-varying phase velocity of waves passing \( B_3 \), and will be calculated for each time. The motivation and discussion of Eq. (2.6) will be given in Sec. 2.3.

For the structural surfaces, \( S_1, \ldots, S_N \), the kinematic boundary condition on the structural boundary can be expressed as

\[
\frac{\partial \phi}{\partial n_m} + \frac{\partial \phi}{\partial t} + \hat{v}_m \cdot \hat{n}_m = 0, \quad \text{on } S_m, \quad m=1, \ldots, N
\]  

(2.7)

in which \( \hat{n}_m \) is the unit normal of \( S_m \) and is directed into structure; \( \hat{v}_m \) is the velocity vector of a material point on \( S_m \). For structural motions large relative to the wave length, structural positions, \( S_m \), as well as \( \hat{v}_m \) and \( \hat{n}_m \) change with time. For wave diffraction problems, since the structures are fixed in space, \( \hat{v}_m = 0 \), and Eq. (2.7) is reduced to

\[
\frac{\partial \phi}{\partial n_m} + \frac{\partial \phi}{\partial t} = 0, \quad \text{on } S_m, \quad m=1, \ldots, N
\]  

(2.7a)

For wave radiation problems, since there are no incident waves, \( \hat{v}_m \) and \( \hat{n}_m \) are determined by the specified structural motions, and Eq. (2.7) becomes

\[
\frac{\partial \phi}{\partial n_m} + \hat{v}_m \cdot \hat{n}_m = 0, \quad \text{on } S_m, \quad m=1, \ldots, N
\]  

(2.7b)
Dynamic boundary conditions on $S_m$ (or the equations of motion of the structures) are required in order to determine the structural motions and accordingly, $\ddot{v}_m$ and $S_m$ in Eq. (2.7). The equations of motion for rigid structures are given in Sec. 2.4.

2.2.b Galerkin Weighted Residual Formulation

In finite element analysis, it is necessary to develop an integral formulation for the problem from which the governing differential equations and boundary conditions could be obtained as the Euler equations (34) for that integral. For the linear wave diffraction and radiation problems, the integral expression has been given by Zienkiewicz (105), by Bai (4) and by Huang (44). An identical expression can also be derived from the Galerkin weighted residual method (103, 26, 34, 77).

Using the Galerkin weighted residual method, one can obtain the integral formulation equivalent to Eqs. (2.3-2.7) for the fluid as

$$
- \int \int \int_D 7^2 \delta \phi \, dD + \int \int_{B_1} \left( \frac{\partial \phi}{\partial n_f} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \right) \delta \phi \, dB
+ \int \int_{B_2} \frac{3 \phi}{\partial n_b} \delta \phi \, dB + \int \int_{B_3} \left( \frac{\partial \phi}{\partial n_r} + \frac{1}{C} \frac{\partial \phi}{\partial t} \right) \delta \phi \, dB
+ \sum_{m=1}^{N} \int \int_{S_m} \left( \frac{\partial \phi}{\partial n_m} + \frac{\partial \phi}{\partial n_m} + \dot{v}_m \cdot \hat{n}_m \right) \delta \phi \, dB = 0,
$$

(2.10)
where $\delta\phi$ is the weighting function.

Integrating by parts and applying Green's theorem, one can write the first integral term in Eq. (2.10) as

$$\int \int \int_{D} \nabla^{2} \phi \ \delta\phi \ dD$$

$$= - \int \int_{B_{1}+B_{2}+B_{3}+\sum_{m=1}^{N} S_{m}} \frac{\delta\phi}{\partial n} \ dB + \int \int \int_{D} \nabla \phi \cdot \nabla \delta\phi \ dD.\ \ (2.11)$$

Using Eq. (2.11) in Eq. (2.10), one obtains

$$\int \int \int_{D} \nabla \phi \cdot \nabla \delta\phi \ dD + \int \int_{B_{1}} \frac{1}{g \beta t^{2}} \ \delta\phi \ dB$$

$$+ \int \int_{B_{3}} \frac{\delta\phi}{\partial t} \ dB + \sum_{m=1}^{N} \int \int_{S_{m}} \left( \frac{\delta\phi_{I}}{\partial n_{m}} + \hat{v}_{m} \cdot \hat{n}_{m} \right) \delta\phi \ dB = 0 \ \ (2.12)$$

Equation (2.12) is the integral formulation for the fluid motion in Eulerian coordinates to be used in developing the approximate finite element models of fluid-structure interaction problems.

### 2.3 Modified Orlanski's Condition

In finite element analyses, a finite fluid domain with artificial boundaries is used to simulate the infinite fluid domain. Certain boundary conditions are
needed for the artificial boundaries. In the time-domain simulation of wave diffraction and radiation problems, the waves scattered or generated by the structures in the fluid domain initially are stretched long waves and the wave forms develop gradually into "steady" state waves. These waves are propagating away from the structures and eventually will encounter the artificial boundary. These phenomena have been observed by Kennard (55), by Multer (71), by O'Brien (77), and by Jagannathan (53). It is necessary to develop a suitable boundary condition which incorporates the transient behaviors of waves passing through the artificial boundaries.

Orlanski's condition was developed in the analysis of unsteady hyperbolic flows (79) and is a modified form of the Sommerfeld radiation condition (87). However, instead of using a predetermined frequency-dependent phase velocity, Orlanski numerically calculated phase velocities for each time step. Recently, Jagannathan (53) has applied Orlanski's condition successfully for an unsteady two-dimensional free surface flow wherein Vinje and Brevig's method (94) was used as the solution basis for the problems. In the present study, Orlanski's condition will be modified for use on the artificial boundaries in the time-domain simulation of wave-structure interactions using the finite element method.
The equation of Orlanski's condition has been given in Eq. (2.6). The unknown phase velocity, \( C \), is a function of time. Similar to the use of the Sommerfeld radiation condition in frequency-domain analyses, the artificial boundaries have to be positioned at least three times the water depth away from the disturbances in order to exclude the effects from evanescent modes of the disturbances (44, 45). In Eq. (2.12), Orlanski's condition on \( B_3 \) and the governing equation of the fluid along with other boundary conditions have been incorporated into one integral equation.

Equation (2.6) is utilized to calculate the phase velocity \( C \) for each time step. From Eq. (2.6), one can obtain

\[
C = - \left( \frac{\partial \phi}{\partial t} \right)_{\phi} \cdot \frac{\partial \phi}{\partial n_r}, \text{ on } B_3
\]  

(2.13)

Equation (2.13) is used to calculate phase velocity along \( B_3 \) for each time step. However, there is a numerical difficulty in using Eq. (2.13). At zero crossings of free surface waves with the still water level, both \( \frac{\partial \phi}{\partial t} = 0 \) and \( \frac{\partial \phi}{\partial n_r} = 0 \) and Eq. (2.13) is indefinite. One could try to use L'Hospital's rule to resolve this indefinite problem. However, in numerical calculations, it is difficult to "catch" the
instantaneous time when a zero crossing is exactly at the artificial boundary. This problem has also been reported by Jagannathan (53) in using Orlanski's condition. Jagannathan used an average value of the phase velocities at a number of points near the artificial boundary. In this study, an extrapolation scheme is developed and used to avoid this numerical difficulty.

To establish an extrapolation equation for the phase velocity, an understanding of the time-dependent behavior of $C$ is essential. There have been no analytical studies of unsteady behavior of $C$. One can make observations of the qualitative form of the time-dependent behavior of $C$ from existing numerical results (53, 55, 77). In the time-domain simulation of wave diffraction and radiation problems, the perturbed waves at initial times are stretched long waves. The corresponding wave celerity can be considered as the long wave celerity. As time increases, the perturbed waves will tend to "steady" state condition and the corresponding phase velocity to the steady propagating wave celerity. During the transition between the above two limiting situations, a continuous, slowly varying behavior will be assumed.

The phase velocity has been assumed to be an exponentially decaying function which asymptotically
approaches the steady propagating wave celerity. The equation for $C$ can then be expressed as

$$C = C_p + \exp \left( b - at \right)$$

(2.14)

where $C_p$ is the steady propagating wave celerity which can be calculated from linear wave theory; $a$ and $b$ are undetermined coefficients. Given wave celerities $C_1$ and $C_2$ at two different time steps $t_1$ and $t_2$, the coefficients $a$ and $b$ can be determined from Eq. (2.14) as

$$a = \frac{1}{t_2 - t_1} \ln \left( \frac{C_1 - C_p}{C_2 - C_p} \right)$$

(2.15)

$$b = \frac{1}{t_2 - t_1} \left[ t_2 \ln(C_1 - C_p) - t_1 \ln(C_2 - C_p) \right]$$

(2.16)

Equations (2.14)-(2.16) can be used to calculate the phase velocity provided that a steady propagating wave celerity $C_p$ exists for the problem.

In the case where $C_p$ is not known a priori, a rational estimate of $C_p$ would be necessary. For example, in a problem with a multiple-frequency wave field, there exist more than one wave celerity. In such a case, the wave celerity corresponding to the principal wave frequency is chosen to minimize the reflections from the artificial boundary. Another example is the free vibration of a floating structure with initial
displacements in which the structural motions are damped due to radiation of free surface waves. In this case, a wave celerity corresponding to the natural frequency of the structural motion can be used as an approximation for $C_p$. As an alternative, $C_p$ can also be determined directly from Eq. (2.14) by using three values of phase velocity $C_1$, $C_2$, and $C_3$ from three equally spaced previous time steps $t_1$, $t_2$, and $t_3$. The $C_p$ thus determined can be expressed as

$$C_p = \frac{C_2^2 - C_1 C_3}{2C_2 - C_1 - C_3}$$

(2.17)

and the coefficients $a$ and $b$ in Eq. (2.14) are the same as Eqs. (2.15) and (2.16). The above mentioned extrapolation techniques of calculating phase velocity will be verified and further discussed in test examples.

### 2.4 Equations of Motion of Rigid Structures

The total pressure at any point in the fluid relative to the pressure on the free surface is given by

$$p(x_1, x_2, x_3, t) = \rho \frac{\partial \phi}{\partial t} - \rho g x_2$$

$$= \rho \left( \frac{\partial \phi}{\partial t} + \frac{\partial \phi}{\partial t} \right) - \rho g x_2$$

(2.18)

The $j$-th component of hydrodynamic force acting on the $m$-th body about the center of rotation can be expressed as
\[
F_{jm}(x_1, x_2, x_3, t) = \rho \int \int_{S_m} \left( \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial t} \right) n_{jm} dB
\]
\[
j = 1, 2, 3, 4, 5, 6, \quad m = 1, \ldots, N \quad (2.19)
\]
in which \( n_{jm} \) is the \( j \)-th component of the unit inward normal on the \( m \)-th structure for \( j = 1, 2, 3 \), respectively; for \( j = 4, 5, 6 \), the pseudo-unit normal (83) is defined as
\[
\begin{align*}
n_{4m} &= (x_{2m} - x_{2m_0}) n_{3m} - (x_{3m} - x_{3m_0}) n_{2m} \\
n_{5m} &= (x_{3m} - x_{3m_0}) n_{1m} - (x_{1m} - x_{1m_0}) n_{3m} \\
n_{6m} &= (x_{1m} - x_{1m_0}) n_{2m} - (x_{2m} - x_{2m_0}) n_{1m}
\end{align*}
\]
where \((x_{1m}, x_{2m}, x_{3m})\) are the coordinates of a material point on the \( m \)-th structure; \((x_{1m_0}, x_{2m_0}, x_{3m_0})\) are the coordinates of the center of rotation of the \( m \)-th structure.

The equations of motion of structures can be found in Refs. (44, 83, 105) and can be written as
\[
\begin{align*}
\sum_{i=1}^{6} \{ M_{jmim} \ddot{U}_{im} + (k_{jmim} + \tilde{k}_{jm} + Q_{jmim}) U_{im} \\
- \sum_{l=1}^{N} Q_{jmil} U_{il} \} &= F_{jm} \\
j &= 1, 2, 3, 4, 5, 6, \quad m = 1, \ldots, N \quad (2.21)
\end{align*}
\]
in which \( M_{jmim}, k_{jmim}, \) and \( Q_{jmim} \) are the mass, mooring restoring coefficient and inter-structural restoring coefficient of the \( m \)-th body in the \( j \)-th component due to motion in the \( i \)-th mode; \( \tilde{k}_{jm} \) is the hydrostatic restoring
coefficient of the m-th body in the j-th mode;
\( \tilde{Q}_{jm} \) is the inter-structural restoring coefficient between the m-th and 1-th body; \( \ddot{U}_{im} \) and \( U_{im} \) are the acceleration and the displacement of the m-th body in the i-th direction at the center of rotation.

2.5 Finite Element Representations
In the following, repeated indices will indicate summations over the range of the corresponding variable.

2.5.a Finite Element Equations of Fluid
For details of the finite element method, one can refer to several texts (26, 52, 78, 103). Isoparametric finite elements (26, 44, 103) are used in this study to model the fluid. For a volume element, the coordinates and the velocity potential can be described by

\[
x_i(\tau_1, \tau_2, \tau_3) = \tilde{H}^J(\tau_1, \tau_2, \tau_3) \ x_i^J
\]

\[
\phi(\tau_1, \tau_2, \tau_3, t) = \tilde{H}^J(\tau_1, \tau_2, \tau_3) \ \phi^J(t)
\]

\[i = 1, 2, 3
\]
\[J = 1, \ldots, N_E
\]

(2.22)
in which \( N_E \) is the number of nodal points for each element; \( \tau_i \) are the natural coordinates of the element which range from -1.0 to 1.0, and \( \tilde{H}^J \) is the shape function for node \( J \) of a volume element; \( x_i^J \) are the coordinates of the \( J \)-th node; \( \phi^J(t) \) is the nodal value of
the perturbed velocity potential and is a function of
time only. For a surface element, \( H^J \) is used to denote
the shape function \( \tilde{H}^J \) of the volume element evaluated on
a surface of the element.

With Eq. (2.22), Eq. (2.12) can be rewritten as

\[
\left\{ \sum_{B_1} L_{1}^{JQ} \phi^J + \sum_{B_3} L_{2}^{JQ} \phi^J + \sum_{D} L_{3}^{JQ} \phi^J + \sum_{m=1}^{N} \sum_{S_m} L_{4}^{Q} \right\} \phi^Q = 0
\]

(2.23)
in which \( \Sigma_{B_1}, \Sigma_{B_3}, \Sigma_{D}, \) and \( \Sigma_{S_m} \) represent
global assemblage of finite elements in \( B_1, B_3, D, \) and \( S_m \), respectively;

\( \phi^I = \) 1-th velocity component of I-th structural
mode

\[
L_{1}^{JQ} = \frac{1}{g} \int_{-1}^{1} \int_{-1}^{1} H^J \tilde{H}^Q \left| J \right| \mathrm{d} \tau_1 \mathrm{d} \tau_2
\]

(2.24)

\[
L_{2}^{JQ} = \frac{1}{c} \int_{-1}^{1} \int_{-1}^{1} H^J \tilde{H}^Q \left| J \right| \mathrm{d} \tau_1 \mathrm{d} \tau_2
\]

(2.25)

\[
L_{3}^{JQ} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \frac{H^J \tilde{H}^Q}{\partial x_1 \partial x_1} \left| J \right| \mathrm{d} \tau_1 \mathrm{d} \tau_2 \mathrm{d} \tau_3
\]

(2.26)

\[
L_{4}^{Q} = \int_{-1}^{1} \int_{-1}^{1} \frac{\partial \phi^I}{\partial x_1} \varepsilon_{1mn} \frac{\partial H^R}{\partial \tau_1} \frac{\partial H^J}{\partial \tau_2} \tilde{H}^Q \mathrm{d} \tau_1 \mathrm{d} \tau_2 x_m x_n
\]

(2.27)
\[
L_4^{1Q} = \int_{-1}^{1} \int_{-1}^{1} H^I \varepsilon_{1mn} \frac{\partial H^R}{\partial \tau^1} \frac{\partial H^J}{\partial \tau^2} H^Q d\tau^1 d\tau^2 \; x^R_m x^J_n
\] (2.28)

where \( \varepsilon_{1mn} \) is the permutation symbol which has a value of 1 for even permutations of \( l,m,n, -1 \) for odd permutations and zero if any of \( l,m,n \) are equal; \( |\tilde{J}| \) and \( |J| \) are the determinants of Jacobian transformations for volume and surface elements, respectively;

\[
|\tilde{J}| = \det \left( \frac{\partial x_i}{\partial \tau^j} \right); \; i,j = 1,2,3
\] (2.29)

\[
|J| = \frac{\partial x^i}{\partial \tau^1} \frac{\partial x^j}{\partial \tau^2}
\] (2.30)

The \( \frac{\partial H}{\partial x^1} \) term in Eq. (2.26) can be calculated by

\[
\frac{\partial H^J}{\partial x^1} = \left( \frac{\partial x^1}{\partial \tau^j} \right) -1 \left( \frac{\partial H^J}{\partial \tau^j} \right); \; i,j = 1,2,3
\] (2.31)

and the \( \frac{\partial \phi_I}{\partial x^1} \) term in Eq. (2.27) can be expressed explicitly by using the linear wave theory for the incident velocity potential \( \phi_I \). Isoparametric finite elements will be used. Gauss quadrature (26, 103) is used in the integrations of Eqs. (2.24)-(2.28).

2.5.b Adaptive Mesh

In the finite element analysis, the problem domain is discretized into a number of elements. The general
guide of the layout of the finite element mesh can be found in Refs. (26, 103). For cases where motions of the boundaries relative to the wave length are large, the fluid elements adjoining the structural boundaries will have excessive deformations due to accumulated boundary motions. The element mesh has to accommodate the motions of the boundaries. There have been three different approaches to resolve the problems involving large motions in fluid-structure interactions: Eulerian approaches, Lagrangian approaches and arbitrary Lagrangian Eulerian approaches.

In the Eulerian approaches (89, 11), the element mesh is fixed in space. As the boundaries deform the material interfaces move across mesh lines and thus complex topological logic is necessary; the solutions tend to be numerically less well-behaved because of the difficulties in discretizing convective terms. In the Lagrangian approaches (21, 71, 77, 96), the element mesh moves with the material boundaries. However, due to subsequent boundary motions, the Lagrangian mesh becomes so distorted that computations fail or become inaccurate. To extend the range of applicability of Lagrangian approaches, effort has been devoted to techniques of rezoning (26) or remeshing (77). In these techniques, a
severely deformed element mesh is replaced by a more regular mesh and the field variables are then assigned so that the solution computations can be continued. In the arbitrary Lagrangian Eulerian approaches (10, 31, 49), the element mesh is not fixed in space and can be moved in an arbitrary way. Due to the relative motions between the material boundary and the finite element mesh, the convected terms remain in the governing equations. These methods also have drawbacks; the solutions tends to be ill-behaved unless considerable damping is used.

In this study, linearized boundary conditions are used for the free surface boundary, but the motions of the structural boundaries relative to the wave length might be large. To resolve this problem, as shown in Fig. 2.1, the fluid domain is divided into a fixed region and adaptive regions. The Lagrangian approach is adopted in which the finite element meshes on the structural boundaries are set to follow the structural motions and the element meshes inside the adaptive region are adjusted to avoid excessive deformations of element shapes.
3.0 NUMERICAL SOLUTION PROCEDURES

The finite element representation and the corresponding discretized equations for the fluid motion and the equations of motion for the structures have been presented in Chapter 2. The equations of motion of the structures and fluid, Eqs. (2.21) and (2.23), are two systems of ordinary differential equations of second order and numerical solutions are necessary (8). In the fluid equation, Eq. (2.23), the velocity potentials are coupled with structural velocities and the surface and volume integrals involving structural boundaries are dependent on the instantaneous structural positions. A geometrical nonlinearity (8, 58) is associated with the fluid equation because the displacements of the structural boundaries relative to the wave length are assumed to be large. On the other hand, in the structural equation, Eq. (2.21), the structural motions are coupled with hydrodynamic loadings induced by the dynamic flow field surrounding the structures. A nonlinearity of nonconservative loading (58) is involved in the structural equations because of the rotating normal: the hydrodynamic forces may be nonlinearly dependent on the structural displacements if their magnitudes and directions vary with structural orientations. From Eqs. (2.23) and (2.21), it can be seen that the fluid flow and structural motions are
coupled through kinematic and dynamic boundary conditions on the fluid-structural interfaces. Since the fluid-structure interfaces are not known a priori, further manipulation of these equations is necessary in order for Eqs. (2.23) and (2.25) to be solved numerically. In this chapter, numerical solution procedures for fluid-structure interactions are presented. In Sec. 3.1, iterative forms of the equations of motion for fluid and structures are derived. In Sec. 3.2, the characteristics of fluid and structural equations are discussed and numerical time integrations are considered. The numerical coupling algorithm of dynamic fluid-structure interactions are developed in Sec. 3.3.

3.1 Iterative Forms of Equations of Motion

The equations of motion for fluid and structures are two systems of coupled equations. The coupling is introduced at the fluid-structure interfaces; if motions of the interfaces are admitted to be large relative to the wave length, the coupling between fluid and structures becomes nonlinear. In the following, a brief review of solution techniques of nonlinear algebraic equations is made. For detailed descriptions and derivations, one can refer to Refs. (2, 5, 8, 58, 103).
There are a number of approaches to solving nonlinear equations; the first is by iteration, the second is by incremental approximation, and the third is by combined incremental and iterative solutions. In the iteration methods, the results from the previous iteration are used for calculations in the current iteration. This method is simple and straightforward. There are several iterative methods which provide different degree of convergence and computational efficiency. Examples are (2, 8) the direct iteration method, the Newton-Raphson method, the Quasi-Newton method, etc.

The basic idea in the incremental scheme is to apply the "external loads" in small increments and assume the response of the system to be linear during each load increment. This scheme gives reasonably good results but the major difficulties in applying the scheme are in the initiation of the incremental process (the initial dynamic equilibrium state of the system must be known) and in cumulative errors in each step leading to erroneous results in long term analysis. The combined incremental and iterative approach, with a judicious combination of incremental and iterative techniques, is most effective in dealing with most nonlinear finite element problems.
The incremental process eliminates the need for initial approximations required for the iteration scheme (except the initial state), which keeps the residual errors under control. Errors accumulated in each step are reduced by iterations. A combined approach for the coupled fluid and structural equations is described in Sec. 3.2. Appropriate iterative forms for the structural and the fluid equation are derived in Secs. 3.1.a and 3.1.b.

In the following the left hand superscripts will denote time step and the left hand subscripts denote iteration numbers.

3.1.a Structures

The equations of motion of the structure, Eq. (2.21), at time $t+\Delta t$ and the $(r+1)$-th iteration step can be written as

$$
\sum_{i=1}^{6} \left( M_{jmim} t+\Delta t U_{im} + (k_{jmim} + \tilde{k}_{jmim} + Q_{jmim}) t+\Delta t U_{im} \right)_{r+1} + \sum_{l=1}^{N} Q_{jml} t+\Delta t U_{il} \right)_{r+1} = t+\Delta t F_{jm} \\
\left[ \right]_{r+1} ; \ j=1,2,3,4,5,6 \\
\left[ \right]_{r+1} ; \ m=1,\ldots,N
$$

The external force component, $t+\Delta t F_{jm}$, in Eq. (3.1) is calculated from Eq. (2.19), which can be expressed as
\[ t+\Delta t_{r+1} F_{jm} = \rho \iint (\frac{t+\Delta t_{r+1} I}{\partial t} + \frac{t+\Delta t_{r+1} \phi}{\partial t}) t+\Delta t_{r+1n_{jm}} dB \]

where the integration is calculated on the structural surface at time \( t+\Delta t \) and for the \((r+1)\)-th iteration step; \( t+\Delta t_{r+1n_{jm}} \) is the unit normal component of the \( m \)-th structure in \( j \)-th direction which may change with time step and iteration number.

The inertia term in Eq. (3.1) is linear in the accelerations and the stiffness terms are linear in the displacements; there is no need to reformulate linear terms. The only nonlinear term is in the external load: the velocity potentials of the fluid are coupled with the structural unit normals which are dependent on the structural spatial orientations. In linearizing nonlinear terms such as occur in Eq. (3.2), a Taylor series expansion can be used and only linear terms retained. However, the Jacobian thus obtained is non-symmetric, which is computationally inconvenient (5, 58). In this study, such a Jacobian method is not used and the effects of nonconservative loads are accounted for by using approximations for the external load. Equation (3.2) is approximated from the previous iteration as

\[ t+\Delta t_{r+1} F_{jm} \approx t+\Delta t_{r} F_{jm} \]
With Eq. (3.3), Eq. (3.1) can be written as

$$
\sum_{i=1}^{6} \{M_{jmim} + \frac{t+\Delta t^r}{r+1}U_{im} + (k_{jmim} + Q_{jmim}) \frac{t+\Delta t^r}{r+1}U_{im} \}
$$

$$
\sum_{l=1}^{N} Q_{jmil} \frac{t+\Delta t^r}{r+1}U_{il} = \frac{t+\Delta t^r}{r}F_{jm}
$$

; \quad j=1,2,3,4,5,6

; \quad m=1,\ldots,N

(3.4)

Equation (3.4) is the iterative form of the equations of motion for the structures. The matrix expression of Eq. (3.4) is given by

$$
[M_s] \{t+\Delta t^r U\} + [K_s] \{t+\Delta t^r U\} = \{ t+\Delta t^r F_s \}
$$

(3.5)

in which

$$
[M_s] = \text{global mass matrix}
$$

$$
[M_s] = \begin{bmatrix}
\sum_{i=1}^{6} M_{jmim}
\end{bmatrix}
$$

(3.6)

$$
[K_s] = \text{global stiffness matrix}
$$

$$
[K_s] = \begin{bmatrix}
\sum_{i=1}^{6} (k_{jmim} + Q_{jmim} - \sum_{l=1}^{N} Q_{jmil}) + k_{jm}
\end{bmatrix}
$$

(3.7)

$$
\{ t+\Delta t^r F_s \} = \text{global load vector}
$$

$$
\{ t+\Delta t^r U \} = \text{global acceleration vector}
$$

$$
\{ t+\Delta t^r U \} = \text{global displacement vector}
$$
3.1. Fluid

The dynamic equation for the fluid motion, Eq. (2.23), at time $t + \Delta t$ and the $(r+1)$-th iteration step can be written as

$$
\sum_{i=1}^{\text{r+1}} t^\Delta t_{1i} JQ_{1i} t^\Delta t_{3i} JQ_{3i} + \sum_{j=1}^{\text{r+1}} t^\Delta t_{2j} L_2^{j} r^\Delta t_{4i} + \sum_{n=1}^{N} t^\Delta t_{r+1} L_4^{n} + \sum_{m=1}^{\text{r+1}} t^\Delta t_{r+1} S_{r+1}^{m} + \sum_{n=1}^{\text{r+1}} t^\Delta t_{r+1} L_5^{n} = 0 \tag{3.8}
$$

in which $t^\Delta t_{1i} JQ_{1i}$, $t^\Delta t_{3i} JQ_{3i}$, $t^\Delta t_{2j} L_2^{j}$, $t^\Delta t_{4i}$, $t^\Delta t_{r+1} L_4^{n}$, and $t^\Delta t_{r+1} L_5^{n}$ are calculated from Eqs. (2.24), (2.26), (2.27) and (2.28), respectively, with structural surfaces $t^\Delta t_{r+1} S_{r+1}^{m}$, $m=1,...,N$, denoting the locations of the $N$ structures at time $t + \Delta t$ and the $(r+1)$-th iteration number.

From Eq. (3.8), it can be seen that the fluid flow and the structural velocities are coupled through motions of the structural surfaces described by the term $\sum_{n=1}^{\text{r+1}} t^\Delta t_{r+1} L_5^{n} r^\Delta t_{1i}$. If large structural motions are admitted nonlinearities are possible in the terms involving structural surfaces, $t^\Delta t_{1i} JQ_{1i}$, $t^\Delta t_{3i} JQ_{3i}$, $t^\Delta t_{2j} L_2^{j}$, $t^\Delta t_{4i}$, $t^\Delta t_{r+1} L_4^{n}$ and $t^\Delta t_{r+1} L_5^{n}$. Since the coupled fluid and structure equations will be solved with a partitioned staggered
scheme, the fluid equation is written at the prior $r$-th iteration step. Equation (3.8) then becomes

$$
\sum_{r=1}^{r} t + \Delta t JQ + \sum_{r=1}^{r} t + \Delta t \phi J + \sum_{r=1}^{r} t + \Delta t L_2 Q + \sum_{r=1}^{r} t + \Delta t B_1 + \sum_{r=1}^{r} t + \Delta t B_3 + \sum_{r=1}^{r} t + \Delta t L_3 Q + \sum_{m=1}^{m} t + \Delta t L_4 Q + \sum_{m=1}^{m} t + \Delta t L_5 Q + \sum_{m=1}^{m} t + \Delta t S_m = 0
$$

(3.9)

Equation (3.9) is the iterative form for the dynamic fluid motions.

On assembling the element matrices, following standard procedures (26, 103), one obtains the global matrix equation of motion for the fluid as

$$
\begin{align*}
[t + \Delta t M_f] \{ t + \Delta t^{' \sigma} \} &+ [t + \Delta t C_f] \{ t + \Delta t^{' \phi} \} \\
+ [t + \Delta t K_f] \{ t + \Delta t^{' \phi} \} &= \{ t + \Delta t F_s \}
\end{align*}

(3.10)

in which

$$
[t + \Delta t M_f] = \text{global mass matrix}
$$

$$
= [ \sum_{r=1}^{r} t + \Delta t JQ ]

(3.11)

[t + \Delta t C_f] = \text{global damping matrix}

$$
= [ \sum_{r=1}^{r} t + \Delta t L_2 Q ]

(3.12)
\[ [ t^+ \Delta t_{rK_f} ] = \text{global stiffness matrix} \]
\[ = [ \sum_{r=3}^{t+\Delta t} L_J Q ] \]
\[ t^+\Delta t_{rD} \]  \hspace{1cm} (3.13)

\[ [ t^+\Delta t_{rF_f} ] = \text{global load vector} \]
\[ = - \{ \sum_{r=4}^{N} \sum_{m=1}^{t+\Delta t} L_{rL} Q \} \]
\[ m=1 \]
\[ \text{(3.14)} \]

\[ [ t^+\Delta t_{r\varnothing} ] = \text{global perturbed potential vector at time } t^+\Delta t \text{ and } r\text{-th iteration step} \]

3.2 Numerical Time Integration

Procedures for dynamic transient analysis of fluid-structure interactions are described in this section.
Equations (3.5) and (3.10) are two sets of second order, linear, ordinary differential equations with constant coefficients. Given appropriate initial conditions, Eqs. (3.5) and (3.10) can be solved by a variety of methods (2, 20).

Two classes of methods are currently used in dynamic analyses, implicit schemes and explicit schemes. The most important difference between the two is that the "displacements" (field variables without time derivative)
and "velocities" (first derivatives of field variables) at time $t+\Delta t$ are assumed to be independent of the "accelerations" (second derivatives of field variables) at time $t+\Delta t$ in explicit schemes, while in implicit schemes the displacements and velocities at time $t+\Delta t$ are assumed to be dependent on the accelerations at time $t+\Delta t$. The advantage of explicit scheme is that no assemblage of the global stiffness matrix is required if a lumped mass matrix is used. Thus solution of simultaneous equations is not necessary. However, this advantage is sometimes offset by the stability requirement that the size of time step must be a fraction of the smallest natural period of the system, which could be prohibitively small. On the other hand, many implicit schemes are unconditionally stable so that larger time steps can be used, but assemblage of the global stiffness matrix and solution of a system of simultaneous equations are mandatory (2, 5, 10, 80, 103).

In the coupled dynamic analysis, the use of a combination of operators for the integration of dynamic response of two systems raised the questions of which operators to choose and how to couple them. There are a large number of possibilities, but in general the selection of the operators depends on their stability and
accuracy characteristics, including the effects due to the operator coupling and the overall effectiveness of the resulting time integration. In the analyses of fluid-structure systems, in which the fluid is very flexible in comparisons to the stiffness of the structure, an explicit, conditionally stable time integration of the fluid response and an implicit, unconditionally stable time integration of the structural response may be a natural choice (7, 5). Successful applications have been reported in nuclear reactor problems wherein acoustic waves were considered in the fluid media (10, 31, 40, 46).

In cases involving nonlinear surface waves, the explicit scheme has been chosen for the calculation of dynamic equations on the free surface boundary (53, 71, 73, 74, 77). The advantages of using explicit scheme in this case are that, firstly, the iterations in the implicit schemes of nonlinear equations are avoided, and secondly, free surface elevations calculated from the explicit scheme can be used to define the free surface of the fluid domain and thus resolve the problem associated with undetermined boundary location.

In Sec. 3.2.a, a dynamic analysis procedures for the structures using implicit Newmark's method are described.
In Sec. 3.2.b, the characteristics of the dynamic equation of fluid motions, Eq. (3.10), are discussed and a nodal partitioned explicit-implicit scheme (80) is developed. An explicit time integration (8, 77) is used for the free surface boundary and an implicit scheme is implemented for the rest of the fluid domain.

3.2.a Structures (Implicit Method)

In the present study, the implicit Newmark's method is chosen for the integrations of the equations of motion of the structures, Eq. (3.5). Newmark's method (8) has been shown to have good accuracy characteristics and is unconditionally stable in linear dynamic analysis. Theoretical analysis of the method in a nonlinear setting is extremely complicated and although there has been some progress in this aspect (13), there is no conclusive proof of unconditional stability for nonlinear systems. Nevertheless, Newmark's method has been applied successfully in several nonlinear dynamic transient analyses (6) and appears to be very reliable while showing good convergence characteristics.

In the Newmark's method, it is assumed that

\[
\{ t_{r+1}^+ \} = \{ t_{r+1}^- \} + \{ t_r^\circ \} \Delta t
+ \left[ \frac{1}{2} - \beta \right] \{ t_r^\circ \} + \{ t_{r+1}^\circ \} \Delta t^2
\]

(3.15a)
\[
\{ t^+ \Delta t U \}_{r+1} = \{ t^* U \} + [(1 - \alpha) \{ t'' U \} + \{ t^+ \Delta t^* U \}] \Delta t
\]

(3.15b)

In which

\{ t^* U \}, \{ t'' U \}, \{ t^* U \} = global vectors of displacement, velocity and acceleration at time \( t \).

\{ t^+ \Delta t U \}_{r+1}, \{ t^+ \Delta t U \}_{r+1}, \{ t^+ \Delta t'' U \}_{r+1} = global vectors of displacement, velocity and acceleration at time \( t + \Delta t \) and \( (r+1) \)-th iteration step.

\( \delta, \alpha \) = Newmark's constants.

Equation (3.15a) can be rearranged so that

\[
\{ t^+ \Delta t'' U \}_{r+1} = \frac{1}{\beta \Delta t^2} \{ t^+ \Delta t U \}_{r+1} - \frac{1}{\beta \Delta t^2} \{ t^* U \} - \frac{1}{\beta \Delta t} \{ t^* U \} - \left( \frac{1}{2} \alpha - 1 \right) \{ t'' U \}
\]

(3.16)

If Eq. (3.16) is substituted into Eq. (3.5), one then obtains

\[
[K_s] \{ t^+ \Delta t U \}_{r+1} = \{ t^+ \Delta t F \}_{r+1}
\]

(3.17)

where

\[
[K_s] = modified\ stiffness\ matrix\ 
= [K_s] + \frac{1}{\beta \Delta t^2} [M_s]
\]

(3.18a)
\[
\begin{align*}
[t_{r+F}^+ \Delta t]_{r,F} &= \text{modified load vector} \\
&= [t_{r+F}^+ \Delta t]_{r,F} + [M_s] \left( \frac{1}{\Delta t^2} \{t_U\} + \frac{1}{\Delta t} \{t_r^+\} \right) \\
&\quad + \left( \frac{1}{2\beta} - 1 \right) \{t_r^{++}\} \\
\end{align*}
\]

Equation (3.17) can be used to solve for the displacements, \(\{t_{r+1}^+ U\}\), directly, provided that the modified stiffness matrix \([t_{r+F}^+ \Delta t]_{r,F}\) and the modified load vector \([t_{r+F}^+ \Delta t]_{r,F}\) have been calculated from Eqs. (3.18a) and (3.18b). After the displacements are computed, the velocities and accelerations can be computed from Eqs. (3.15b) and (3.16), respectively. The Newmark's constants, \(\beta\) and \(\alpha\), are chosen to be 1/2 and 1/4 in this study, respectively. This has proved to be a successful combination although other combinations are possible (5).
3.2.b Fluid (Partitioned, Explicit-Implicit Scheme)

In this section, a partitioned, explicit-implicit scheme for the dynamic equations of the fluid, Eq. (3.10), is developed. A typical finite element mesh shown in Fig. 3.1 is utilized to illustrate the partitioning scheme. In Fig. 3.1, \( \phi_s \) stands for velocity potential at the nodes on the structural surface, \( S_1 \); \( \phi_d \) stands for velocity potential inside the fluid domain other than \( S_1, B_1, B_2 \) and \( B_3 \); \( \phi_{rb}, \phi_{fr} \) and \( \phi_{fs} \) are the velocity potentials at the nodes of the intersection of \( B_2 \) and \( B_3 \), \( B_1 \) and \( B_3 \) and \( B_1 \) and \( S_1 \), respectively. Considering the problem in Fig. 3.1, the full matrix expression of the global finite element equation for the fluid, Eq. (3.10), can be written as
Fig. 3.1. Typical finite element mesh for the illustrations for partitioned explicit-implicit scheme for the fluid
\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
M_f M_f \\
M_f M_f M_f \\
M_f M_f \\
\end{bmatrix} & + \\
\begin{bmatrix}
\rho_s \rho_d \rho_d \rho_d \\
\rho_d \rho_s \rho_d \rho_d \\
\rho_d \rho_d \rho_s \rho_d \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
C_f C_f C_f \\
C_f C_f C_f \\
C_f C_f C_f \\
\end{bmatrix} & + \\
\begin{bmatrix}
\rho_s \rho_d \rho_d \rho_d \\
\rho_d \rho_s \rho_d \rho_d \\
\rho_d \rho_d \rho_s \rho_d \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
K_f K_f K_f \\
K_f K_f K_f \\
K_f K_f K_f \\
\end{bmatrix} & + \\
\begin{bmatrix}
\rho_s \rho_d \rho_d \rho_d \\
\rho_d \rho_s \rho_d \rho_d \\
\rho_d \rho_d \rho_s \rho_d \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
K_f K_f K_f \\
K_f K_f K_f \\
K_f K_f K_f \\
\end{bmatrix} & + \\
\begin{bmatrix}
\rho_s \rho_d \rho_d \rho_d \\
\rho_d \rho_s \rho_d \rho_d \\
\rho_d \rho_d \rho_s \rho_d \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
K_f K_f K_f \\
K_f K_f K_f \\
K_f K_f K_f \\
\end{bmatrix} & + \\
\begin{bmatrix}
\rho_s \rho_d \rho_d \rho_d \\
\rho_d \rho_s \rho_d \rho_d \\
\rho_d \rho_d \rho_s \rho_d \\
\end{bmatrix}
\end{align*} \]

\[ \begin{align*}
\text{t+\Delta t} & \quad \text{t+\Delta t} \\
\begin{bmatrix}
F_f \\
F_f \\
0 \\
\end{bmatrix} & = \\
\begin{bmatrix}
\rho_f \rho_s \rho_f \\
\rho_f \rho_d \rho_f \\
\rho_f \rho_d \rho_f \\
0 \\
0 \\
0 \\
\end{bmatrix}
\end{align*} \]

(3.19)
Equation (3.19) is obtained by following the standard assembling procedures of the finite element method (26, 8, 103). From inspection of Eqs. (2.23)-(2.28), it can be seen that in the dynamic fluid equation, the inertia effect (twice time derivative term) shows only on the free surface boundary, the damping effect (once time derivative term) shows only on the artificial boundary, and the loading shows only on the structural surfaces. These characteristics are also reflected in Eq. (3.19). The nonzero elements in the mass matrix are associated with nodal values on B₁, nonzero elements in the damping matrix with nodal values on B₃ and nonzero loadings with S₁.

Due to the "soft" nature of the dynamic fluid equations, explicit schemes have been traditionally adopted in the dynamic analysis for the fluid (7, 5). However, in Eq. (3.19), because of zeros on the diagonals of the mass and damping matrices, the use of an explicit scheme is restricted to the free surface and the artificial boundary. The idea of using a partitioned explicit-implicit scheme has thus initiated. Using an explicit scheme for the free surface boundary has advantages in the case of nonlinear surface waves as mentioned earlier in this section. Although only
linearized free surface boundary conditions are considered in this study, an explicit scheme still offers advantage. By using Eq. (2.13), the results of the free surface from an explicit scheme can be utilized to calculate the phase velocity $C$ at intersection node of the artificial truncation boundary and the free surface for the next time step, and can be used for the entire artificial boundary if one assumes that $C$ is invariant along the boundary. An explicit scheme could also be applied on the artificial boundary. However, in cases where one does not assume invariance of $C$ along the artificial boundary, an implicit scheme for the artificial boundary is preferrable, in that the Orlanski's condition can be adapted using implicit iterations on the phase velocity. In the present study, the solution algorithm for the fluid is developed intentionally to be suitable for more general problems, e.g. the problems with a nonlinear free surface boundary and with $C$ variable along the artificial boundary. Therefore, an explicit scheme is applied only to the free surface boundary and an implicit scheme to elsewhere in the fluid domain.

The $\{ \emptyset \}$ associated with nodal points in the finite element mesh of the fluid domain are divided into two groups: an explicit nodal group, $\{ \emptyset^E \}$, which contains
nodes on the free surface, and an implicit nodal group, \( \{ \emptyset^I \} \), which contains the rest of the nodal points. Referring to Eq. (3.19), the matrix form of the fluid equation, Eq. (3.10), formulated at time \( t+\Delta t \) and the \( r \)-th iteration step can be expressed in partitioned form as

\[
\begin{bmatrix}
0 & 0 \\
0 & t+\Delta t M_f^r \\
\end{bmatrix}
\begin{bmatrix}
t+\Delta t^* I \\
t+\Delta t^* E \\
\end{bmatrix}
\begin{bmatrix}
t+\Delta t^* I \\
t+\Delta t^* E \\
\end{bmatrix}
+ 
\begin{bmatrix}
t+\Delta t_{C_f}^I \\
t+\Delta t_{C_f}^E \\
\end{bmatrix}
\begin{bmatrix}
t+\Delta t^* I \\
t+\Delta t^* E \\
\end{bmatrix}
+ 
\begin{bmatrix}
t+\Delta t_{K_f}^I \\
t+\Delta t_{K_f}^E \\
\end{bmatrix}
\begin{bmatrix}
t+\Delta t^* I \\
t+\Delta t^* E \\
\end{bmatrix} = 
\begin{bmatrix}
t+\Delta t_F^I \\
t+\Delta t_F^E \\
\end{bmatrix}
\]

(3.20)

in which the right hand superscripts I, E and IE indicate implicit, explicit and coupling between implicit and explicit groups. The partitioning in Eq. (3.20) is also shown in Eq. (3.19) which may provide clearer understanding of the forms of the components in Eq. (3.20). For example, referring to Eq. (3.19), it can be seen that in \( [t+\Delta t_{C_f}^I] \), some of diagonal elements are zeros.
From Eq. (3.20), the equation at time $t+\Delta t$ for the explicit nodal group can be written as

$$
\begin{align*}
\{ t+\Delta t \} E \{ r \} + \{ t+\Delta t \} E \{ r \} + \\
\{ t+\Delta t \} E \{ r \} + \\
\{ t+\Delta t \} E \{ r \}
\end{align*}
$$

$$
= \{ t+\Delta t \} E \{ r \} - \{ t+\Delta t \} E \{ r \} - \{ t+\Delta t \} E \{ r \}
$$

$$
(3.21)
$$

The counterpart of Eq. (3.21) formulated at time $t$ can be expressed as

$$
\begin{align*}
\{ t \} E \{ r \} + \{ t \} E \{ r \} + \\
\{ t \} E \{ r \} + \\
\{ t \} E \{ r \}
\end{align*}
$$

$$
= \{ t \} E \{ r \} - \{ t \} E \{ r \} - \{ t \} E \{ r \}
$$

$$
(3.22)
$$

Since an explicit time integration will be used for Eq. (3.22), the left hand subscripts indicating the iteration step of the explicit variables have been dropped.

Moreover, in Eq. (3.22), the implicit variables such as $\{ t \} E \{ r \}, \{ t \} E \{ r \}, \{ t \} E \{ r \}$ and $\{ t \} E \{ r \}$ are quantities from the last best iteration of the implicit group at time $t$.

The equation for the implicit nodal group at time $t+\Delta t$ can be expressed as
\[
\begin{align*}
\left[ t+\Delta t^I_C f \right] \left\{ t+\Delta t^* I \right\} &+ \left[ t+\Delta t^I_K f \right] \left\{ t+\Delta t^* I \right\} \\
&= \left\{ t+\Delta t^I_f \right\} - \left[ t+\Delta t^I_C f \right] \left\{ t+\Delta t^* E \right\} \\
&- \left[ t+\Delta t^I_K f \right] \left\{ t+\Delta t^* E \right\}
\end{align*}
\]

(3.23)

in which \( \{ t+\Delta t^* \phi \} \) and \( \{ t+\Delta t^* E \} \) are quantities from the explicit nodal group.

The central difference method (2, 5) is an explicit method of integration which has been shown to be very effective in many problems (25) and is chosen here for the time integration of the equations of the explicit nodal group, Eq. (3.32). It is assumed that

\[
\begin{align*}
\{ t^* E \} &= \frac{1}{\Delta t} \left( \{ t-\Delta t E \} - 2 \{ t E \} + \{ t+\Delta t E \} \right) \\
(3.24a) \\
\{ t^* \phi \} &= \frac{1}{2\Delta t} \left( - \{ t-\Delta t \phi \} + \{ t+\Delta t \phi \} \right) \\
(3.24b)
\end{align*}
\]

Substitution of Eqs. (3.24a) and (3.24b), Eq. (3.22) yields

\[
\left[ t^I E \right] \left\{ t+\Delta t^* \phi \right\} = \left\{ t^* E \right\}
\]

(3.25)
in which
\[ [t_{K_f}^E] = \text{modified stiffness matrix} \]

\[ = \frac{1}{\Delta t^2} [t_{M_f}] + \frac{1}{2\Delta t} [t_{C_f}^E] \quad (3.26a) \]

\[ [t_{F_f}^E] = \text{modified load vector} \]

\[ = [t_{F_f}^E] - [t_{C_f}^E] \{t\phi^E\} - [t_{K_f}^E] \{t\phi^I\} \]

\[ + [t_{M_f}] \frac{1}{\Delta t^2} \left( - \{t-\Delta t\phi^E \} + 2 \{t\phi^E\} \right) \]

\[ + [t_{C_f}^E] \frac{1}{2\Delta t} \{t-\Delta t\phi^E\} - [t_{K_f}^E] \{t\phi^E\} \]

\[ (3.26b) \]

From Eq. (3.25), one can solve for \( \{t+\Delta t\phi^E\} \) of the explicit nodal group. If \( [t_{K_f}^E] \) is a diagonal matrix Eq. (3.25) is solved without factorizing a matrix, and the solution scheme is very effective (26, 5). From Eq. (3.26a), \( [t_{K_f}^E] \) consists of a mass matrix, \( [t_{M_f}] \), and a damping matrix, \( [t_{C_f}^E] \). Referring to Eq. (3.19), \( [t_{C_f}^E] \) is a diagonal matrix and \( [t_{M_f}] \) can also be a diagonal matrix if a lumped mass approach (26, 103) is used in the calculation of the mass matrix. The drawback of using the lumped mass approach is that it normally takes a finer element mesh to obtain a required accuracy (5). In this study, a consistent mass approach (26, 103) is adopted in order to retain the solution accuracy with the use of a relatively coarse mesh for the large fluid.
domain. In Eq. (3.26b), two terms from the implicit nodal group, \( \{ t^C I \} \) \{ \( t^I \) \} and \( \{ t^K I \} \) \{ \( t^I \) \} are included. The term \( \{ t^I \} \) can be calculated by using a backward difference as

\[
\{ t^I \} = \frac{1}{\Delta t} ( \{ \Delta t t^I \} - \{ -\Delta t t^I \} ) \quad (3.27)
\]

As to the equation for the implicit nodal group, Eq. (3.23), the left hand side contains a stiffness term and a damping term (once time derivative term) but no inertia term (twice time derivative term). Therefore a simple backward difference method is adopted.

\[
\{ t^+ \Delta t t^I \} = \frac{1}{\Delta t} ( \{ \Delta t t^I \} - \{ -t^I \} ) \quad (3.28)
\]

in which \( \{ t^I \} \) also indicates the value of the term from the last iteration at time \( t \). Substitution of Eq. (3.28) into Eq. (3.23) leads to

\[
\{ t^+ \Delta t t^I \} \{ t^+ \Delta t I \} = \{ t^+ \Delta t t^I \} \quad (3.29)
\]

in which

\[
\{ t^+ \Delta t t^I \} = \text{modified stiffness matrix} = \frac{1}{\Delta t} \{ t^+ \Delta t C^F \} + \{ t^+ \Delta t K^F \} \quad (3.30a)
\]
\[ \{ t + \Delta t_ I \} = \text{modified load vector} \]
\[ = \{ t + \Delta t^{IE} \} - \{ t + \Delta t^{IE} \} \cdot \{ t + \Delta t^{CE} \} \]
\[\text{and} \quad K{t + \Delta t^{IE}} \cdot \{ t + \Delta t^{IE} \} + \frac{1}{\Delta t} \{ t + \Delta t^{IE} \} \cdot \{ t^{IE} \} \]

Equation (3.29) is solved iteratively for \( \{ t + \Delta t^{IE} \} \) of the implicit nodal group at the r-th iteration number. Two terms from the explicit nodal group, \( \{ t + \Delta t^{CE} \} \cdot \{ t + \Delta t^{IE} \} \) and \( \{ t + \Delta t^{IE} \} \cdot \{ t + \Delta t^{IE} \} \) are contained in Eq. (3.30b); the term \( \{ t + \Delta t^{IE} \} \) can be calculated by a backward difference as

\[ \{ t + \Delta t^{IE} \} = \frac{1}{\Delta t} ( \{ t + \Delta t^{IE} \} - \{ t^{IE} \} ) \quad (3.31) \]

Dynamic transient analysis procedures for fluid-structure interactions will be described in the following section wherein a staggered coupling solution scheme (81) is developed for dynamic analysis between implicit structures and the explicit-implicit fluid.
3.3 **Dynamic fluid-structure coupling algorithm**

In this section, the dynamic analysis of the coupling between the implicit structural equation, Eq. (3.17), and the partitioned fluid equations (explicit fluid equation, Eq. (3.25), and implicit fluid equation, Eq. (3.29)) are described.

Let us examine the procedure during the first time step for starting the numerical integration. From the explicit fluid equations, Eqs. (3.25)-(3.26), with t's replaced by 0's, the \( \{ \Delta t \phi^E \} \) of the explicit nodal group can be solved. However, inspection of Eqs. (3.26a) and (3.26b) shows that \( \{ \Delta t \phi^E \} \) is determined only from initial conditions, \( \{ 0^0 \} \) and \( \{ 0^0 \} \) where left hand superscripts 0 indicates time \( t=0 \), and is independent to structural motions at time \( t=\Delta t \). For example, in a case of an fluid initially at rest, \( \{ 0^0 \} = \{ 0 \} \), \( \{ 0^0 \} = \{ 0 \} \), and from Eqs. (3.25)-(3.26), one gets \( \{ \Delta t \phi^E \} = \{ 0 \} \). This solution is not physical rational. Further use of \( \{ \Delta t \phi^E \} \) in the implicit fluid equation, Eq. (3.29), and in the implicit structural equation, Eq. (3.17) will cause cumulative errors. It was concluded that the explicit central difference scheme for the explicit nodal group of the fluid is not a good self-starting method in the fluid-structure coupling problems.
To provide a better self-starting scheme for the fluid, the implicit Newmark's method (8) is exploited for the entire fluid domain at the first time step. Using Newmark's expressions, Eqs. (3.15)-(3.16), for the velocity potentials at time \( t = \Delta t \) and with initial conditions, \( \{ \dot{\phi}_0 \} \) and \( \{ \ddot{\phi}_0 \} \), one can write the dynamic fluid equation, Eq. (3.10), at time \( t = \Delta t \) and the \( r \)-th iteration step as

\[
\begin{bmatrix}
\Delta t_{rK_f} \\
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
\begin{bmatrix}
\Delta t \phi \\
\Delta t \phi \\
\Delta t \phi
\end{bmatrix}
= 
\begin{bmatrix}
\Delta t \phi \\
\Delta t \phi \\
\Delta t \phi
\end{bmatrix}
\]  \hspace{1cm} (3.32)

in which

\[
\begin{bmatrix}
\Delta t_{rK_f} \\
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
= \frac{1}{\beta \Delta t^2} \begin{bmatrix}
\Delta t_{rM_f} \\
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
+ \frac{\alpha}{\beta \Delta t} \begin{bmatrix}
\Delta t_{rC_f} \\
\Delta t_{rC_f} \\
\Delta t_{rC_f}
\end{bmatrix}
+ \begin{bmatrix}
\Delta t_{rK_f} \\
\Delta t_{rK_f} \\
\Delta t_{rK_f}
\end{bmatrix}
\]  \hspace{1cm} (3.33a)

\[
\begin{bmatrix}
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
= \begin{bmatrix}
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
\]  \hspace{1cm} (3.33b)

\[
+ \begin{bmatrix}
\Delta t_{rM_f} \\
\Delta t_{rC_f}
\end{bmatrix}
\left( \frac{1}{\beta \Delta t^2} \begin{bmatrix}
\phi_0 \\
\phi_0
\end{bmatrix}
+ \frac{1}{\beta \Delta t} \begin{bmatrix}
\dot{\phi}_0 \\
\dot{\phi}_0
\end{bmatrix}
+ \left( \frac{1}{2\beta} - 1 \right) \begin{bmatrix}
\ddot{\phi}_0 \\
\ddot{\phi}_0
\end{bmatrix} \right)
\]

\[
+ \begin{bmatrix}
\Delta t_{rC_f} \\
\Delta t_{rC_f}
\end{bmatrix}
\left( \frac{1}{\beta \Delta t} \begin{bmatrix}
\phi_0 \\
\phi_0
\end{bmatrix}
+ \left( \frac{\alpha}{\beta} - 1 \right) \begin{bmatrix}
\dot{\phi}_0 \\
\dot{\phi}_0
\end{bmatrix}
+ \left( \frac{\alpha}{2\beta} - 1 \right) \begin{bmatrix}
\ddot{\phi}_0 \\
\ddot{\phi}_0
\end{bmatrix} \Delta t \right)
\]
The term \( \{\varnothing^*\} \) in Eq. (3.33b) can be calculated from Eq. (3.10) as

\[
\{\varnothing^*\} = [\{\varnothing^*_{M} \}]^{-1} (\{\varnothing^*_{F} \} - [\{\varnothing^*_{C} \}] [\{\varnothing^*_{C} \} - [\{\varnothing^*_{K} \}] [\{\varnothing^* \}])
\]  

(3.33c)

From Eq. (3.17), the structural equation evaluated at time \( t=\Delta t \) and for the \((r+1)\)-th iteration can be written as

\[
[\bar{K}_S] [\Delta t \bar{U} ] = [\Delta t F]  
\]  

(3.34)

in which \([\bar{K}_S]\) is given by Eq. (3.18) and \( [\Delta t F] \) is obtained from Eq. (3.18b) as

\[
[\Delta t F] = [\varnothing^*_F] \\
+ [M_S] \left( \frac{1}{\beta \Delta t^2} \{\varnothing U\} + \frac{1}{\beta \Delta t} \{\varnothing^* U\} \right) \\
+ \left( \frac{1}{2\beta} - 1 \right) \{\varnothing^* U\}  
\]  

(3.35a)

where \( \{\varnothing U\} \) and \( \{\varnothing^* U\} \) are initial conditions and the term \( \{\varnothing^* U\} \) can be calculated from Eq. (3.5) as

\[
\{\varnothing^* U\} = [\{\varnothing^*_M \}]^{-1} (\{\varnothing^*_F \} - [\{\varnothing^*_K \}] [\{\varnothing U\}])  
\]  

(3.35b)

Equations (3.32) and (3.34) represent implicit equations for the fluid and structure at time \( t=\Delta t \) and can be solved in a staggered recursive fashion. For the fluid, with initial conditions on \( \{\varnothing^* \} \) and \( \{\varnothing \} \), and
with structural boundaries specified at time \( t = \Delta t \) and the \( r \)-th iteration step to satisfy the boundary conditions on the structural surfaces, the modified stiffness matrix and load vector can be calculated from Eqs. (3.33a) and (3.33b), and the velocity potential, \( \{ \Delta t \phi \} \), at time \( t = \Delta t \) and the \( r \)-th iteration step is determined from Eq. (3.32). Using \( \{ \Delta t \phi \} \) from the fluid equation and the initial conditions for the structures, the load vector is calculated from Eqs. (3.35a) and (3.35b), and the structural displacement, \( \{ \Delta t \mathbf{U} \} \), at time \( t = \Delta t \) and the \( (r+1) \)-th iteration step is determined from Eqs. (3.34).

With \( \{ \Delta t \mathbf{U} \} \) and the calculated structural kinematics, the structural boundaries are updated to the \( (r+1) \)-th iteration step. The iteration number is advanced and the above procedures are repeated. Convergence of the above iterative procedures is defined by the vanishing of the residual errors in the structural displacements between two consecutive iteration steps. The description shown above is used as a starting scheme in the dynamic analysis of a fluid-structure system.

In the subsequent time steps the coupling between fluid and structures at time \( t + \Delta t \) are described. It is assumed that all the quantities regarding fluid and structures at time \( t \) and \( t - \Delta t \) have been calculated from previous time steps. With the fluid domain defined at
At time t, the partitioned explicit fluid equation, Eq. (3.25), is used to solve for \( \{ t+\Delta t_0^E \} \) of the explicit nodal group of the fluid. In addition to the quantities at time t for the fluid and structures, the \( \{ t+\Delta t_0^E \} \) is used to calculate the modified stiffness matrices and load vectors for the partitioned implicit fluid from Eqs. (3.30a) and (3.30b) and for the implicit structures from Eqs. (3.18a) and (3.18b). The partitioned implicit fluid equation, Eq. (3.29), and the implicit structural equation, Eq. (3.17), are then solved in a staggered, iterative fashion as described for the first time step.

At the r-th iteration step, Eq. (3.29) is used to solve for \( \{ t+\Delta t_0^I \} \) of the implicit nodal group of the fluid. Then \( \{ t+\Delta t_0^I \} \) is used to calculate the load vector in the structural equation. From Eq. (3.17), structural displacement, \( \{ t+\Delta t_{r+1}^U \} \), at the \((r+1)\)-th iteration step, are determined. With \( \{ t+\Delta t_{r+1}^U \} \) and the calculated structural kinematics, the structural boundaries are then updated to redefine the fluid domain for the next \((r+1)\)-th iteration step. This completes the r-th iteration step at time \( t+\Delta t \) between the partitioned implicit fluid and implicit structural equations. Upon convergence of the iterations at time \( t+\Delta t \), the partitioned explicit fluid equation, Eq. (3.25), is then used to advance the solution one step forward in time.
In this chapter, test examples are presented to validate the solution algorithms of the finite element analysis for the wave-structure interactions presented in previous chapters.

In Sec. 4.1, the time-domain simulations of several flap wavemaker problems are studied. In Sec. 4.1.a, a flap wavemaker problem with a long wave channel and a total reflecting boundary, is studied in order to evaluate the accuracy of the partitioned, explicit-implicit finite element algorithm for dynamic fluid analyses. In Sec. 4.1.b, the results from the long-channel flap wavemaker considered in Sec. 4.1.a are utilized to evaluate the use of the extrapolated Orlanski condition at artificial boundary truncating the domain. In Sec. 4.1.c, the extrapolated Orlanski condition imposed at an artificial boundary is further evaluated for flap wavemaker problems with shorter channel sections. Results are compared to analytical and experimental results for a long channel to verify the accuracy and the applicability of the extrapolated Orlanski condition. In Sec. 4.1.d, multiple-frequency wavemaker problems are studied to demonstrate applicability of the extrapolated Orlanski condition for a multiple-frequency wave field.
In Sec. 4.2, time-domain simulations of wave diffraction by fixed structures are presented. Both horizontal circular and rectangular cylinders are considered.

In Sec. 4.3, time-domain simulations of wave radiation due to heave, sway and roll motions of structures are studied. The study of large roll motion of the rectangular cylinder is presented in Sec. 4.3.d, wherein structural motion relative to generated wave length is considered large and is resolved by using the adaptive mesh in the region around the structure. The purpose is to show the effect of the adaptive motion of the structure on the generated surface waves. In addition to adaptive structural motions, an example is considered in which the updating of the free surface boundary is included. In Sec. 4.3.e, time-domain wave radiations by a flexible cylindrical membrane vibrating with multiple frequencies and mode shapes are studied.

In Sec. 4.4, the transient motion of a horizontal circular cylinder with initial vertical displacement is studied to validate the accuracy of the dynamic analysis procedures for coupled fluid-structure interactions. A parametric study is given of the effect of the water depth on the transient structural motions.
In Sec. 4.5, the interactions between incident waves and a cross-moored horizontal rectangular structure is studied; the analysis of structural motions, wave attenuation and mooring forces in the time domain are considered.

In Sec. 4.6, time-domain simulations of wave interference phenomena between waves and multiple moored structures with or without inter-structural constraints are considered. As a final example, an actual model of two floating bridges is simulated wherein design wave conditions are used.

4.1 Flap Wave Maker Problems

4.1.a Long Channel with Total Reflection Boundary

This example is intended to show the validity of the partitioned, explicit-implicit integration scheme for the fluid. The results will also aid in the study of approximate phase velocities in the next subsections. A flap wavemaker problem with a 30 meter long wave channel and a water depth of 3 meter, with a flap wave generator at one end, and with a vertical total reflection wall at the other end is simulated. The motion of the wavemaker is periodic with period, $T=2.0$ sec, and stroke $2S=0.1$
meter. It starts from rest at its extreme left position and moves forward. The fluid is also initially at rest. From linear wavemaker theory (92, 77, 45), the generated wave length and wave height are expected to be 6.216m and 0.1356m, respectively. The finite element mesh used in the present finite element analysis is shown in Fig. 4.1.1 wherein quadratic elements are used. In the numerical time marching scheme, the size of the time step is chosen in accordance with Multer's (71) recommandation that there be as many nodes per wave length along the free surface as number of time steps in a wave period. In the present study, with a mesh of 8 elements, i.e., 18 nodes per wave length, the size of the time step, $Dt = T/20$, is used.

The free surface elevations calculated from the present finite element analysis at different times are plotted in Figs. 4.1.2a - 4.1.2.h. Inspection of these figures shows that the generated waves are transforming from unsteady into steady waves; the first wave is a long stretched wave and the wave form reaches steady state after the third wave.

Kennard's analytical solution (55) for the same wavemaker problem, but with a semi-infinite flume is shown in several of Figs. 4.1.2. Kennard's solution and the numerical calculation procedures are given in Appendix I.
Fig. 4.1.1. Finite element mesh of the flap wavemaker problem
Good agreement between the two results can be observed in Figs. 4.1.2. Since a total reflecting boundary is used at the end of the wave channel in the finite element model, reflected waves from the boundary are expected. In Fig. 4.1.2.h, one can observe differences between the finite element solution and Kennard's semi-infinite solution; reflected waves are superposed on the propagating waves, and can be seen to be moving back toward the wavemaker.

4.1.b Extrapolation Equation of C

In this section, the results obtained in Sec. 4.1.a are used to investigate the phase velocity, C, in the Orlanski condition, Eq. (2.6), and the proposed extrapolation equation for C, Eq. (2.14), is evaluated. Replacing the derivative in Eq. (2.13) by their finite difference expressions, and adopting the notations of Chapter 3, one can calculate the phase velocity, C, at time $t + \Delta t$ of points on the free surface as

$$C = \frac{t + \Delta t \varphi_E - t \varphi_E}{\Delta t} \left/ \frac{t + \Delta t \varphi_{E+1} - t + \Delta t \varphi_E}{\Delta x_1} \right.$$

(4.1)

where $\Delta x_1$ is the distance between element nodes (fr) and (fr+1) on the free surface. Using Eq. (4.1) and the
results from Sec. 4.1.a, the C's for each time step at positions $x_1=6m$, $9m$, $12m$ and $15m$, away from the wavemaker are calculated. In Figs.(4.1.3.a)-(4.1.3.d), it can be seen that the magnitude of C is initially large and decreases asymptotically to the propagating wave celerity. Also, numerical instabilities are shown as discussed in Sec. 2.3. The extrapolation equation, Eq. (2.14), can be used to avoid these numerical difficulties. The coefficients $a$ and $b$ in Eq. (2.14) are calculated at a time $t = T/2$ by using Eqs. (2.15) and (2.16). The values of $a$ and $b$ are then used to calculate C for the rest of the time. In Figs.(4.1.3.a) - (4.1.3.d), it can be seen that the extrapolated C-curve can approximate the numerical C's very well and avoids the instabilities. This extrapolation scheme is adopted to approximate the modified Orlanski condition in the rest of the study.

4.1.c Short Channel with modified Orlanski's Boundary

To validate the use of the modified/extrapolated Orlanski condition, as described in Sec. 4.1.b for shorter truncated region, the flap wavemaker problem studied in Sec. 4.1.a is examined. However, instead of using a 30 meter channel with a total reflecting boundary, a shorter channel with an extrapolated Orlanski's boundary is used.
The restriction of installing the extrapolated Orlanski's boundary is that the boundary has to be placed at least 3 times water depth away from the wavemaker, as described in Sec. 2.3. Here, two cases are tested: one with the extrapolated Orlanski's boundary placed at a distance of 3 times water depth, 9m, and the other at 4 times water depth, 12m. The extrapolated Orlanski condition and the extrapolation equation described in 4.1.b are implemented on the artificial boundary.

In Figs. (4.1.4.a)-(4.1.4.b), the free surface elevations along the wave channel at time $t=4.0$ secs and $8.0$ secs obtained from 9m channel are compared to the results from the 30m channel. The comparisons show very good agreement. There is little reflection from the artificial boundary. Since a total reflection boundary is used in the 30m channel, the reflected waves from the boundary are expected to propagate back into the domain. It is understood that the comparisons between results from the 30m and the shorter channels are valid only when the results from the 30m channel without reflected waves are used. The comparisons between results obtained from the 12m channel and 30m channels are shown in Figs.(4.1.5.a)-(4.1.5.b). Since the artificial boundary is placed at a even further position away from the wavemaker, the comparisons show even better agreement.
The present finite element model for the flap wavemaker problem is further validated by comparison to experimental data taken from Oregon State University's O.H. Hinsdale Wave Research Facility (OSU-WRF). The geometry of the problem and the finite element mesh used in this study is shown in Fig. 4.1.6. The water depth at the wavemaker is 14.5' and decreases to a constant depth of 11.5'. The extrapolated Orlanski's condition developed in Sec. 4.1.b is used for the artificial boundary. The motion of the wave generator is specified to be periodic with period, $T$, and stroke, $2S$. It starts from rest at its vertical position and moves backward. The fluid is initially at rest. Two cases are tested: in the first case, $T=2.50$ sec and $S=0.443'$; in the second case, $T=3.70$ sec and $S=0.806'$.

In Figs. 4.1.7 and 4.1.8, time histories of the free surface elevations at $x_1=160'$ for the first case and at $x_1=112'$ for the second case are compared to the experimental results. The comparisons show that the phases predicted by the present method are in close agreement with the experimental results, but the amplitudes are larger. Tabulated comparisons of wave heights between results obtained from present study and experiments are shown in Table 4.1.1. For the first case,
Fig. 4.1.2.a. Free surface elevations at $t=1.0\text{s}$ of the flap wavemaker problem

Fig. 4.1.2.b. Free surface elevations at $t=2.0\text{s}$ of the flap wavemaker problem
Fig. 4.1.2.c. Free surface elevations at $t=3.0s$ of the flap wavemaker problem

Fig. 4.1.2.d. Free surface elevations at $t=4.0s$ of the flap wavemaker problem
Fig. 4.1.2.e. Free surface elevations at $t=5.0\,\text{s}$ of the flap wavemaker problem

Fig. 4.1.2.f. Free surface elevations at $t=6.0\,\text{s}$ of the flap wavemaker problem
Fig. 4.1.2.g. Free surface elevations at $t=7.0s$ of the flap wavemaker problem.

Fig. 4.1.2.h. Free surface elevations at $t=8.0s$ of the flap wavemaker problem.
Fig. 4.1.3.a. Numerical C's at $x_1=6m$ calculated from 30m channel

Fig. 4.1.3.b. Numerical C's at $x_1=9m$ calculated from 30m channel
Fig. 4.1.3.c. Numerical C's at $x_1=12m$ calculated from 30m channel

Fig. 4.1.3.d. Numerical C's at $x_1=15m$ calculated from 30m channel
Fig. 4.1.4.a. Comparisons of free surface elevations between 9m and 30m channels at t=4s.

Fig. 4.1.4.b. Comparisons of free surface elevations between 9m and 30m channels at t=8s.
Fig. 4.1.5.a. Comparisons of free surface elevations between 12m and 30m channels at t=4s

Fig. 4.1.5.b. Comparisons of free surface elevations between 12m and 30m channels at t=8s
Fig. 4.1.6. Finite element mesh of the flap wavemaker of OSU-WRF
Fig. 4.1.7. Comparisons of free surface elevations between present and experimental results for the wavemaker problem (T=2.5 sec, S=0.443')
Fig. 4.1.8. Comparisons of free surface elevations between present and experimental results for the wavemaker problem (T=3.7sec, S=0.806')
Table 4.1.1.  Comparisons of wave heights for flap wavemaker problems of OSU-WRF

<table>
<thead>
<tr>
<th>CASE1 (T=2.50sec, S=0.443')</th>
<th>WAVE HEIGHT (ft)</th>
<th>DIFFERENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEM</td>
<td>EXPR</td>
</tr>
<tr>
<td>0.4143</td>
<td>0.3565</td>
<td></td>
</tr>
<tr>
<td>0.6377</td>
<td>0.6276</td>
<td></td>
</tr>
<tr>
<td>0.9700</td>
<td>0.8938</td>
<td></td>
</tr>
<tr>
<td>1.2657</td>
<td>1.1380</td>
<td></td>
</tr>
<tr>
<td>1.3058</td>
<td>1.0354</td>
<td></td>
</tr>
<tr>
<td>0.9928</td>
<td>0.9549</td>
<td></td>
</tr>
<tr>
<td>AVERAGE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>CASE2 (T=3.70sec, S=0.806')</th>
<th>WAVE HEIGHT (ft)</th>
<th>DIFFERENCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>FEM</td>
<td>EXPR</td>
</tr>
<tr>
<td>1.2735</td>
<td>1.0418</td>
<td></td>
</tr>
<tr>
<td>1.3260</td>
<td>1.1760</td>
<td></td>
</tr>
<tr>
<td>1.2330</td>
<td>1.0926</td>
<td></td>
</tr>
<tr>
<td>1.2020</td>
<td>1.1174</td>
<td></td>
</tr>
<tr>
<td>1.4119</td>
<td>1.1109</td>
<td></td>
</tr>
<tr>
<td>1.3182</td>
<td>1.0992</td>
<td></td>
</tr>
<tr>
<td>1.1846</td>
<td>1.0888</td>
<td></td>
</tr>
<tr>
<td>1.3441</td>
<td>1.1031</td>
<td></td>
</tr>
<tr>
<td>AVERAGE</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
the average difference is 11.28 % and for the second case 15.68 %.

4.1.d Multiple-Frequency Wavemaker

To study the applicability of the proposed modified Orlanski's boundary to a multiple-frequency wave field, a multiple-frequency wavemaker problem was studied. The same wave basin for the problem described in Sec. 4.1.a is used. The fluid is assumed to be initially at rest. The stroke of the wavemaker is specified to have two components of different frequencies: 

\[ S = -S_1 \cos \omega_1 t - S_2 \cos \omega_2 t. \]

Equations (2.6)-(2.16) are used as the extrapolated Orlanski condition. The propagating wave celerity corresponding to the principal frequency \( \omega_1 \) of the wavemaker is calculated by linear wave theory and is assigned as \( C_p \) in Eq. (2.14). Again, the results from the 30m channel with a total reflection boundary is used to provide a comparison basis for the shorter channels.

Three different frequency ratios, \( \omega_2/\omega_1 = 1.03, 2.3, 5.3 \), were considered, where \( \omega_1 \) and \( \omega_2 \) are the principal and higher frequencies, respectively. An amplitude ratio of \( S_2/S_1 = 0.2 \) was used for all three cases. \( \omega_1 = \pi \) and \( S_1 = 0.05 \text{m} \) are used here. In Figs. (4.1.9)-(4.1.11), the free surface elevations along the wave channel at selected
times obtained from the 12m channel with the extrapolated Orlanski's boundary are compared to the results obtained from the 30m channel for each of the three frequencies. The comparisons show very good agreement. The long, stretched rising wave can be seen with the higher-frequency waves superposed. The comparisons of the time variations of the free surface elevations at the artificial boundary of the short channel, x=12m, to the results of long channel for different frequency ratios are shown in Figs. (4.1.12.a), (4.1.12.b), and (4.1.12.c).

Since only the principal frequency is used in the extrapolated Orlanski condition, the higher-frequency waves are reflected from the boundary and propagate back into the domain. However, the higher frequency waves propagate slower than the principal waves, and thus in the time required for the principal waves to become fully developed, the higher frequency waves have not yet reached the boundary and returned to the domain of principal concern.

4.2 Diffraction Problem

Time-domain simulations of the diffraction of water waves by a fixed horizontal structure are considered. Both circular and rectangular cylinders are studied. Dean and Ursell's results (29) for circular cylinders and Black
Fig. 4.1.9.a. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 1.03\,\omega$ at $t=4s$.

Fig. 4.1.9.b. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 1.03\,\omega$ at $t=8s$. 
Fig. 4.1.10.a. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 2.3 \times \omega$ at $t=4s$

Fig. 4.1.10.b. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 2.3 \times \omega$ at $t=8s$
Fig. 4.1.11.a. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 5.3 \omega$ at $t=4s$

Fig. 4.1.11.b. Comparisons of free surface elevations between short and long channels for multiple-frequency wavemaker $\omega_2 = 5.3 \omega$ at $t=8s$
Fig. 4.1.12.a. Comparisons of free surface elevations at the artificial boundary of short and long channels for multiple-frequency wavemaker $\omega_2 = 1.03\omega$

Fig. 4.1.12.b. Comparisons of free surface elevations at the artificial boundary of short and long channels for multiple-frequency wavemaker $\omega_2 = 2.3\omega$
Fig. 4.1.12.c. Comparisons of free surface elevations at the artificial boundary of short and long channels for multiple-frequency wavemaker $\omega_2 = 5.3\omega$. 

Free Surface elevations (m)

$\omega_2 = 5.3\omega$

$12m$

$30m$

$t$ (sec) at $x_1 = 12.0m$
and Mei's results (17) for rectangular cylinders are used for comparisons. The fluid is assumed to be initially at rest. In this study, the incident waves are specified to be applied simultaneously at t=0 everywhere on the free surface but with amplitude ramped up to full scale in a quarter of the wave period. A hyperbolic tangent function is used as the ramp function.

4.2.a **Horizontal Circular Cylinder**

The diffraction problem of a semi-submerged circular cylinder is considered. The finite element mesh of the problem is shown in Fig. 4.2.1. A test case with incident wave amplitude A=0.05m and wave period T=2sec ( L=6.24m, Ka=0.81, h/L=1.6, where L=incident wave length; a=radius of the circular cylinder; K=2π/L) is studied. The free surface elevations at different times are shown in Figs. 4.2.2.a - 4.2.2.g. The superposition of the incident waves and the perturbed waves is the resulting total waves. It can be seen that the perturbed waves as well as the total waves tend to a steady state conditions. In Figs. 4.2.3 and 4.2.4, the time histories of the reflected waves at positions x_{1}=-1.28m and x_{1}=-9.34m are shown. The transmitted waves at x_{1}=1.28m and x_{1}=9.34m are shown in Figs. 4.2.5 and 4.2.6. The waves are shown to develop
into steady state. Solutions predicted by Dean and Ursell using frequency domain analysis are shown as horizontal dashed lines in Figs. 4.2.3 to 4.2.8. The comparisons to Dean and Ursell's results (29) show very good agreement. From Figs. 4.2.3 and 4.2.5, one can also observe the near-field effect on the free surface elevations in contrast to far-field results in Figs. 4.2.4 and 4.2.6. The dimensionless hydrodynamic forces, $f_1 = \frac{F_1}{\rho gaAl}$ and $f_2 = \frac{F_2}{\rho gaAl}$ ($l$ = unit length in $x_3$-direction), are shown in Figs. 4.2.7 and 4.2.8. The steady state results also show good agreement with Dean and Ursell's results. The comparisons of reflection and transmission coefficients and the check on the energy conservations obtained in this study to the results obtained by Dean and Ursell's (29) and by Sulisz's (88) are shown in Table 4.2.1. A series expansion method implemented by a numerical solution solving for the coefficients of the series was used by Dean and Ursell. Sulisz used a boundary integral method. Both solutions are in the frequency domain. From Table 4.2.1, it can be seen that the overall error in the present finite element analysis of the time domain simulation is about 5%.
Fig. 4.2.1. Finite element mesh of a horizontal circular cylinder
Fig. 4.2.2.a. Free surface elevations at $t = 1.0 \, T$ for a fixed horizontal circular cylinder

Fig. 4.2.2.b. Free surface elevations at $t = 1.25 \, T$ for a fixed horizontal circular cylinder
Fig. 4.2.2.c. Free surface elevations at $t = 1.50 \, T$ for a fixed horizontal circular cylinder

Fig. 4.2.2.d. Free surface elevations at $t = 1.75 \, T$ for a fixed horizontal circular cylinder
Fig. 4.2.2.e. Free surface elevations at $t = 2.0 \, T$ for a fixed horizontal circular cylinder

Fig. 4.2.2.f. Free surface elevations at $t = 4.0 \, T$ for a fixed horizontal circular cylinder
Fig. 4.2.2.g. Free surface elevations at $t = 8.0 \, T$ for a fixed horizontal circular cylinder
Fig. 4.2.3. Near-field reflected waves from a fixed horizontal circular cylinder

Fig. 4.2.4. Far-field reflected waves from a fixed horizontal circular cylinder
Fig. 4.2.5. Near-field transmitted waves from a fixed horizontal circular cylinder

Fig. 4.2.6. Far-field transmitted waves from a fixed horizontal circular cylinder
Fig. 4.2.7. Dimensionless Hydrodynamic force $f_1$ on a fixed horizontal circular cylinder

Fig. 4.2.8. Dimensionless Hydrodynamic force $f_2$ on a fixed horizontal circular cylinder
Table 4.2.1. Comparisons of reflection and transmission coefficients for a fixed horizontal circular cylinder
4.2.b Horizontal Rectangular Cylinder

The diffraction problem of a semi-submerged rectangular cylinder is considered. The finite element mesh of the problem is shown in Fig. 4.2.9. The test case of incident wave amplitude, $A=0.05m$, wave period, $T=2.0\text{sec}$ and water depth, $h=1.0m$ ($K_a = 0.603$, $h/L = 0.192$, $a=$ half width of the rectangular cylinder) is studied.

The time histories of the reflected waves at positions $x_1 = -0.75m$ and $x_1 = -3.1m$ are shown in Figs. 4.2.10 and 4.2.11. The transmitted waves at positions $x_1 = 0.75m$ and $x_1 = 3.1m$ are plotted in Figs. 4.2.12 and 4.2.13. The waves are developing into steady state waves. The comparisons of steady state waves to Black and Mei's results (shown as dashed lines) show very good agreement. The near-field effects on the free surface elevations can be observed in Figs. 4.2.10 and 4.2.12 in contrast to far-field free surface elevations in Figs. 4.2.11 and 4.2.13.

The nondimensional hydrodynamic forces, $f_1$, $f_2$ and moment, $f_3=F_3/cga^2A_1$, acting on the structure are shown in Figs. 4.2.14 - 4.2.16.
Fig. 4.2.9. Finite element mesh of a horizontal rectangular cylinder
Fig. 4.2.10. Near-field reflected waves from a fixed horizontal rectangular cylinder.

Fig. 4.2.11. Far-field reflected waves from a fixed horizontal rectangular cylinder.
Fig. 4.2.12. Near-field transmitted waves from a fixed horizontal rectangular cylinder

Fig. 4.2.13. Far-field transmitted waves from a fixed horizontal rectangular cylinder
Fig. 4.2.14. Dimensionless Hydrodynamic force $f_1$ on a fixed horizontal rectangular cylinder

Fig. 4.2.15. Dimensionless Hydrodynamic force $f_2$ on a fixed horizontal rectangular cylinder
Fig. 4.2.16. Dimensionless Hydrodynamic force $f_3$ on a fixed horizontal rectangular cylinder.
4.3 Radiation Problems

4.3.a Horizontal Circular Cylinder in Heave Motion

The time-domain simulation of the radiation of waves by a horizontal circular cylinder in heave motion is studied. The geometry of the problem and the finite element mesh used in this study are shown in Fig. 4.2.1. The structure is semi-submerged in the fluid with its center of gravity lying at the still water level. The structure is specified to have a vertical periodic motion by \( S \cos \omega t \), where \( S=0.05 \text{m} \) is the amplitude of the motion; \( \omega=2\pi/T \) is the angular frequency and \( T=2.0 \text{sec} \) is the period of the structural motion. The fluid is assumed to be initially at rest. The spatial distributions of waves generated along the free surface at different times are shown in Figs. 4.3.1.a - 4.3.1.j. From Figs. 4.3.1.a - 4.3.1.d, the evolution of the unsteady free surface waves can be observed. Since the structure has only vertical motion, the generated waves are symmetric about the vertical axis. The fully developed wave forms on the free surface at different times are shown in Figs. 4.3.1.g - 4.3.1.j. The time histories of the free surface elevations at different positions, \( x_1=1.28 \text{m} \) and \( 10.88 \text{m} \), are plotted in Figs. 4.3.2.a and 4.3.2.b. The wave forms
develop to steady state conditions in four wave periods. The comparisons of the steady state wave forms to results (shown as dashed lines) obtained by Yu and Ursell (102) show very good agreement. The differences are about 5%. A series expansion expression implemented by a numerical method solving for the coefficients of the series was used by Yu and Ursell (102). Since the fluid motions are symmetric with respect to the vertical axis, the hydrodynamic horizontal force and moment are zero. The nondimensional hydrodynamic vertical force, $f_2 = F_2/2\rho g a S$, is shown in Fig. 4.3.3.

4.3.b Horizontal Circular Cylinder in Sway Motion

The radiation of waves by a horizontal circular cylinder in sway motion in the time-domain is simulated. The geometrical set-up of the problem and the finite element mesh used in the present analysis are the same as the one used in the heave problem. The structural motion is specified to be periodic in the horizontal direction by $-S \cos \omega t$, where $S=0.05m$ is the amplitude of the motion; $\omega = 2\pi/T$ is the angular frequency and $T=2.0$sec is the period of the structural motion. The fluid is assumed to be initially at rest. The waves generated along the free surface at selected times are plotted in Figs. 4.3.4.a -
4.3.4.j. The evolution of the generated waves on the free surface can be observed; the waves grow and develop into steady state. Since the structural motion is only in the horizontal direction, the waves generated are anti-symmetric with respect to the vertical axis. The developing unsteady wave forms are shown in Figs. 4.3.4.a - 4.3.4.e, and the steady-state wave forms at different times are shown in Figs. 4.3.4.f - 4.3.4.j. The time histories of the free surface elevations at positions $x_1=1.28\,m$ and $x_1=10.88\,m$ are shown in Figs. 4.3.5.a and 4.3.5.b, respectively. The wave forms become steady in about four wave periods. Since the fluid motion is anti-symmetric about the vertical axis, it is understood that the free surface elevations on two sides of the structure are $180^\circ$ out of phase. The hydrodynamic forces acting on the structure are shown in Figs. 4.3.6.a and 4.3.6.b. Compared to the hydrodynamic force in the horizontal direction, $f_1=F_1/2\rho g a S$, the force in the vertical direction, $f_2=F_2/2\rho g a S$, is much smaller, and the frequency is twice higher.

4.3.c Horizontal Rectangular Cylinder in Roll Motion

The time-domain simulation of the radiation of waves by a horizontal rectangular cylinder in roll motion is
studied. The geometry of the problem and the finite element mesh used in this study are shown in Fig. 4.2.9.
The structure is located with its center of gravity at the still water level. The rolling motion of the structure is periodic in time and is specified as \( -S \sin \omega t \), where \( S = 0.05 \) radian is the amplitude of the motion in radians; \( \omega = 2\pi / T \) is the angular frequency and \( T = 2.0 \text{ sec} \) is the period of the structural motion. The corresponding motion steepness is \( 2aS/L = 0.01 \), \( L = 5.21 \text{ m} \) the wave length corresponding to the period of the structural motion. The fluid is assumed to be initially at rest. The free surface waves generated at selected times are shown in Figs. 4.3.7.a – 4.3.7.j. The evolution and propagation characteristics of the generated free surface waves can be observed. The wave forms grow up to a steady state. Due to the rotation of the structure motion, the generated waves are anti-symmetric about the vertical axis, i.e. the wave forms on two sides of the structure are \( 180^\circ \) out of phase. The development of the unsteady wave forms are seen in Figs. 4.3.7.a – 4.3.7.e. The steady state waves at different times are shown in Figs. 4.3.7.f – 4.3.7.j. The time histories of the free surface elevations at \( x_1 = 0.75 \text{ m} \) and \( 3.1 \text{ m} \) are shown in Figs. 4.3.8.a and 4.3.8.b, respectively. The wave forms develop to steady state in
four wave periods. The comparisons of the steady state wave forms to the results obtained by Black and Mei (17) (dashed line in Figs. 4.3.8) show good agreement. The difference is within 5%. The dimensionless hydrodynamic forces, $f_1 = F_1/2\rho g a^2 S$, $f_2 = F_2/2\rho g a^2 S$ and $f_3 = F_3/2\rho g a^3 S$, acting on the structure are shown in Figs. 4.3.9.a - 4.3.9.c, respectively. The horizontal force and the moment are at the same frequency as the structural motion. However, the vertical force has a frequency two times higher, and the amplitude is much smaller in comparison to the horizontal force and the moment.

4.3.d **Horizontal Rectangular Cylinder in Large Roll Motion**

The time-domain simulation of the radiation of waves by a horizontal rectangular cylinder in large roll motion is studied. The criteria of "large motion" suggested by Ursell (90) is used: motion is considered large if the ratio of twice the amplitude of the motion to the generated wave length is greater than 0.03. The geometry of the problem and the finite element mesh used in this study are shown in Fig. 4.3.10. The roll motion of the structure is specified to be periodic in time as $-S \sin \omega t$, where $S=0.4$ radian and $\omega =$... The corresponding
2aS/L=0.077, \( L \) is the wave length corresponding to the period of the structural motion. To account for large motions of the structure, the position of the structure is changed with time according to the structural motions during each time step. Meanwhile, finite element meshes in the adaptive region are also changed accordingly such that the meshes will not be excessively distorted. General criteria for acceptable geometrical shapes of finite elements can be found in Cook (27). The fluid is assumed to be initially at rest. The generated waves on the free surface at different times are shown in Figs. 4.3.7.a - 4.3.7.j. Similar results are obtained as for small roll motions discussed in 4.3.c. However, distinct differences due to the updates of the structural positions can be observed. Comparisons of Figs. 4.3.11.a - 4.3.11.j to Figs. 4.3.7.a - 4.3.7.j show that in large motions, secondary waves are generated and superposed on the primary waves. The development of the unsteady wave forms are shown in Figs. 4.3.11.a - 4.3.11.e. The steady waves along the free surface at different times are shown in Figs. 4.3.11.h - 4.3.11.j.

Time histories of the free surface elevations at different positions, \( x_1 = -1.0\text{m}, 1.0\text{m}, -3.5\text{m}, \) and \( 3.5\text{m} \) are shown in Figs. 4.3.12.a - 4.3.12.d. The waves develop to
steady state in about six wave periods. Comparisons of Figs. 4.3.8.a - 4.3.8.b to Figs. 4.3.12.a - 4.3.12.d show that the superposing of the secondary effect from large motions have distorted the wave forms; the generated waves are bigger than the results obtained by small motions. The nondimensional hydrodynamic forces, $f_1$ and $f_2$, and moment, $f_3$, acting on the structure are shown in Figs. 4.3.13.a - 4.3.13.c. Comparisons to the results of small motions show interesting differences. The horizontal force and moment are skewed a little, but hardly changed. However, the vertical force increases by about 80%. Effects from large motions of the structure have dominated the hydrodynamic vertical force.

Since only linear wave theory is used for the fluid motion, the question arises as to whether the solution obtained in the above analysis of large structural motions provides an approximation to the nonlinear solution. An attempt was made to provide an answer. A limited nonlinear model of the free surface conditions is incorporated by defining an adaptive mesh along the surface. In addition to updating the structural positions as in the above analysis, the free surface was also updated according to predictions from previous time steps. In Figs. 4.3.14.a - 4.3.14.d the updated free surface
elevations at different times are compared to the results obtained without free surface updates. Two solutions are close to each other before one and half wave periods. After that, the solutions with free surface updates deviate from that without updates, and eventually diverge. Because linear wave theory was used, the updates of the free surface induced errors in the free surface boundary conditions; the accumulations of the errors in the time marching scheme was the reason for divergence. If even smaller time steps were used, the divergence would be delayed. From the above study, no definitive conclusion can be reached as to whether the solutions from linear wave theory with large structural motion could be an approximation to the nonlinear theory. However, the same qualitative result was obtained arising with and without free surface updates. It is necessary to utilize a nonlinear wave theory to study the problems of fluid-structure interactions with large structural motions.

4.3.e Vibrating Membrane

In this section, the wave radiation by a cable-reinforced, fluid-filled cylindrical membrane vibrating in its floating position is studied. The fluid is assumed to be initially at rest. To specify the motion of the
membrane, Blevins' formulas (18) for the mode shapes of a cylindrical membrane are used to describe the spatial variations of the geometry of the membrane with time; the natural frequencies of the membrane obtained by Lo (58) by using a finite element analysis are used for the frequencies of the motion. The mode shapes of the membrane in accordance with Blevins' formulas (18) can be written as

\[
\begin{align*}
v_n &= n \sin n\theta \sin \left( \frac{2\pi t}{T_n} \right) \\
w_n &= n \sin n\theta \sin \left( \frac{2\pi t}{T_n} \right)
\end{align*}
\]

in which the subscript \( n \) indicates the \( n \)-th mode; \( v_n \) and \( w_n \) are the tangential and normal displacements of a material point on the membrane, respectively; \( \theta \) is the angle in radians between the material point and the horizontal coordinate axis; \( T_n = \frac{2\pi}{\omega_n} \), where \( \omega_n \) is the natural frequency of the \( n \)-th mode. In this study, only the first three mode shapes are used, i.e. \( n=1,2,3 \). The frequencies are specified as \( T_1 = 2\text{sec}, \frac{T_2}{T_1} = 0.25, \frac{T_3}{T_1} = 0.2 \) and the amplitudes of the individual modes are specified as \( S_1 = 0.2a, \frac{S_2}{S_1} = 0.5, \frac{S_3}{S_1} = 0.4 \), where \( a \) is the radius of the cylindrical membrane (\( a = 0.8\text{m} \)). The three mode shapes are calculated and plotted in Figs. 4.3.15.a-4.3.15.c. By superposing the three mode shapes, the resulting motion of the membrane used in this study are as
shown in Fig. 4.3.15.d. The geometrical description of the finite element mesh used in this study is shown in Fig. 4.2.1. The structural motion described in Fig. 4.3.15.d is used for the vibrating motion of the membrane. The relative measure of the structural motion is 

$$\frac{2S}{L} = 0.09$$

where $S$ is the amplitude of the structural motion and $L$ is the wave length corresponding to the first natural frequency and thus, in according to Ursell's criteria (91), is in the range of large motion. To account for large structural motions, an adaptive mesh about the cylinder is used.

The time histories of the vertical hydrostatic force and moment acting on the cylindrical membrane due to large structural motions in the fluid are shown in Figs. 4.3.16.a and 4.3.16.b. In Fig. 4.3.16.a, it can be seen that the vertical hydrostatic force has three different frequency components which correspond to the three natural frequencies of vibration. The time histories of the free surface elevations at positions $x_1 = \pm 1.28\text{m}$ and $\pm 10.88\text{m}$ are shown in Figs. 4.3.17 and 4.3.18. To investigate the effects of the second and third modes of structural motions on the generated wave field, the results in Figs. 4.3.17 and 4.3.18 can be compared to the results shown in Figs. 4.3.5.a and 4.3.5.b which are obtained with a rigid
structure in sway motion only, i.e. the 1st mode of motion for the flexible cylinder. The comparisons indicate that the large first mode motion and the structural motions due to higher modes have a strong effect on the waves in the near-field. Figures 4.3.17.a and 4.3.17.b show the complexities of the near-field waves corresponding to higher-frequency structural motions. The surface waves in the far-field shown in Figs. 4.3.18.a and 4.3.18.b indicate that the higher-frequency waves are riding on the dominant waves due to structural motion in the first mode. The time histories of the nondimensional hydrodynamic forces in horizontal and vertical directions, \( f_1 = \frac{F_1}{2 \rho g a S_1} \) and \( f_2 = \frac{F_2}{2 \rho g a S_1} \), and hydrodynamic moment, \( f_3 = \frac{F_3}{2 \rho g a^2 S_1} \), acting on the structure are shown in Figs. 4.3.19.a - 4.3.19.c. In Fig. 4.3.19.a, it can be seen that the horizontal force has a major frequency corresponding to the first natural frequency of the structure and a secondary frequency corresponding to the third mode. In other words, the horizontal hydrodynamic forces are induced mainly by the first and third modes of the structural motions. In Fig. 4.3.19.b, the vertical force has a major frequency corresponding to the second natural frequency of the membrane, and a secondary frequency which is twice the first natural frequency. Therefore, the
vertical hydrodynamic force is mainly induced by the first and second modes of the structural motion. Compared to the hydrodynamic horizontal and vertical forces, the hydrodynamic moment is much smaller. In Fig. 4.3.19.c, the hydrodynamic moment has frequencies corresponding to the second and third natural frequencies. This is as expected, since the first mode of the structural motion (sway) induces no moment on a circular cylinder.

4.4 Free Vibration of a Circular Cylinder

A rigid body floating on the free surface of the water is given a small vertical displacement from its equilibrium position and is then held fixed. When the fluid has again come to rest, the body is released and a motion of the body and fluid ensues, subject only to the external force of gravity. The ensuing motion consists of a motion of the body, together with a wave motion of the fluid which progressively carries energy away from the body. Ultimately the body and the fluid return to their equilibrium state of rest. This fully transient problem has been studied analytically by Ursell(91). However, only the transient motion of the body was given. The solution was expressed in terms of a kernel function consisting of a force coefficient which was obtained from
an earlier paper(90). Due to the mathematical complexities, Ursell discussed only the asymptotic behavior of the solution. The numerical calculation of the kernel function was studied later by Maskell and Ursell(65). Theoretically, the fluid motion could be calculated by using the known motion of the body. However, due to numerical complications, the fluid motion was left unsolved. To provide for comparisons, in Appendix II, Ursell's analytical solution and Maskell's numerical results of the kernel function are used to calculate the transient motion of the structure.

The finite element method presented in this study is used to solve the above transient problem. The definition sketch of the problem and a typical finite element mesh used in this study are shown in Fig. 4.4.1. Since the problem is symmetric in the horizontal direction, by adding a symmetry boundary at the body, only half of the domain needs to be considered. In this fully transient problem, the waves generated by the motion of the structure are fully unsteady. The use of the nonreflecting boundary condition presented in previous section has some uncertainties. The corresponding wave celerities of the waves passing the nonreflecting boundary are not known. In this study, the wave celerity
corresponding to the natural frequency of the structural motion is used as the limiting wave celerity. Due to this approximation, certain amount of reflections from the boundary is expected. To obtain better numerical results, the nonreflecting boundary is positioned further away from the structure; a distance of six times water depth is used.

The example considered here is that the free vibrations of a floating circular cylinder with a radius \( a = 0.8 \text{m} \), and an initial vertical displacement \( \delta U_2 = -0.05 \text{m} \) in a water of depth 10 m. The heave response of the structure calculated using the methods of present study and from Ursell's analytical solution are shown in Fig. 4.4.2. The comparisons show very good agreement. The plot in Fig. 4.4.2 is an obvious damped vibration. The fluid has a significant damping effect on the structural motion. The structural vibration is damped out in four motion periods.

The dimensionless vertical hydrodynamic force,

\[ f_2 = \frac{F_2}{\rho ga |\delta U_2|/l} \]

\( f_2 \) is plotted in Fig. 4.4.3. As is expected, a maximum force occurs immediately after the motion starts. In Fig. 4.4.4, the wave celerities used in the numerical calculations for the nonreflecting boundary condition are shown. Since an extrapolating scheme is
used, it is to be understood that this curve provides only an approximation to the actual wave celerity.

The free surface elevations at different times are shown in Figs. 4.4.5.a - 4.4.5.1. The generated surface waves are shown to develop due to the continuous addition of energy from the motions of the structure. Meanwhile, due to the decaying motions of the structure, the waves do not reach a steady state, instead they remain in a fully transient situation and propagate dispersively away from the structure. Figures 4.4.6.a - 4.4.6.f show histories of the free surface elevations at different positions. The wave groups are shown to be moving away from the structure. Eventually, the waves will move out of the domain.

A parametric study of the effects of water depth on the motions of the structure was considered. Different water depths were used: 2.0m, 1.6m, and 1.2m, namely, 2.5a, 2.0a, and 1.5a. The heave motions of the structure are compared in Figs. 4.4.7 to that obtained for deep water. In Fig. 4.4.7.a, we see that the shallower the water depth is, the stronger is the damping effect from the water. The structural motions are damped out faster, and the frequency of structural motion is lower in shallower water. Fig. 4.4.7.b offers a more detailed look
Fig. 4.3.1.a. Free surface elevations at t=0.5s for a horizontal circular cylinder in heave motion

Fig. 4.3.1.b. Free surface elevations at t=1.0s for a horizontal circular cylinder in heave motion
Fig. 4.3.1.c. Free surface elevations at $t=1.5s$ for a horizontal circular cylinder in heave motion

Fig. 4.3.1.d. Free surface elevations at $t=2.0s$ for a horizontal circular cylinder in heave motion
Fig. 4.3.1.e. Free surface elevations at $t=4.0\text{s}$ for a horizontal circular cylinder in heave motion

Fig. 4.3.1.f. Free surface elevations at $t=14.0\text{s}$ for a horizontal circular cylinder in heave motion
Fig. 4.3.1.g. Free surface elevations at $t=14.5s$ for a horizontal circular cylinder in heave motion

Fig. 4.3.1.h. Free surface elevations at $t=15.0s$ for a horizontal circular cylinder in heave motion
Fig. 4.3.1.i. Free surface elevations at t=15.5s for a horizontal circular cylinder in heave motion

Fig. 4.3.1.j. Free surface elevations at t=16.0s for a horizontal circular cylinder in heave motion
Fig. 4.3.2.a. Free surface elevations at $x_1=1.28\text{m}$ for a horizontal circular cylinder in heave motion

Fig. 4.3.2.b. Free surface elevations at $x_1=10.88\text{m}$ for a horizontal circular cylinder in heave motion
Fig. 4.3.3. Dimensionless Hydrodynamic force $f_2$ on a horizontal circular cylinder in heave motion.
Fig. 4.3.4.a. Free surface elevations at \( t=0.5\,\text{s} \) for a horizontal circular cylinder in sway motion.

Fig. 4.3.4.b. Free surface elevations at \( t=1.0\,\text{s} \) for a horizontal circular cylinder in sway motion.
Fig. 4.3.4.c. Free surface elevations at t=1.5s for a horizontal circular cylinder in sway motion

Fig. 4.3.4.d. Free surface elevations at t=2.0s for a horizontal circular cylinder in sway motion
Fig. 4.3.4.e. Free surface elevations at $t=4.0\text{s}$ for a horizontal circular cylinder in sway motion

Fig. 4.3.4.f. Free surface elevations at $t=14.0\text{s}$ for a horizontal circular cylinder in sway motion
Fig. 4.3.4.g. Free surface elevations at $t=14.5\text{s}$ for a horizontal circular cylinder in sway motion

Fig. 4.3.4.h. Free surface elevations at $t=15.0\text{s}$ for a horizontal circular cylinder in sway motion
Fig. 4.3.4.i. Free surface elevations at $t=15.5\,\text{s}$ for a horizontal circular cylinder in sway motion

Fig. 4.3.4.j. Free surface elevations at $t=16.0\,\text{s}$ for a horizontal circular cylinder in sway motion
Fig. 4.3.5.a. Free surface elevations at $x_1=1.28\text{m}$ for a horizontal circular cylinder in sway motion.

Fig. 4.3.5.b. Free surface elevations at $x_1=10.88\text{m}$ for a horizontal circular cylinder in sway motion.
Fig. 4.3.6.a. Dimensionless hydrodynamic horizontal force on a horizontal circular cylinder in sway motion.

Fig. 4.3.6.b. Dimensionless hydrodynamic vertical force on a horizontal circular cylinder in sway motion.
Fig. 4.3.7.a. Free surface elevations at $t=0.5s$ on a horizontal rectangular cylinder in roll motion

Fig. 4.3.7.b. Free surface elevations at $t=1.0s$ for a horizontal rectangular cylinder in roll motion
Fig. 4.3.7.c. Free surface elevations at t=1.5s for a horizontal rectangular cylinder in roll motion

Fig. 4.3.7.d. Free surface elevations at t=2.0s for a horizontal rectangular cylinder in roll motion
Fig. 4.3.7.e. Free surface elevations at t=4.0s for a horizontal rectangular cylinder in roll motion

Fig. 4.3.7.f. Free surface elevations at t=14.0s for a horizontal rectangular cylinder in roll motion
Fig. 4.3.7.g. Free surface elevations at t=14.5s for a horizontal rectangular cylinder in roll motion

Fig. 4.3.7.h. Free surface elevations at t=15.0s for a horizontal rectangular cylinder in roll motion
Fig. 4.3.7.i. Free surface elevations at $t=15.5 \text{s}$ for a horizontal rectangular cylinder in roll motion

Fig. 4.3.7.j. Free surface elevations at $t=16.0 \text{s}$ for a horizontal rectangular cylinder in roll motion
Fig. 4.3.8.a. Free surface elevations at $x_1 = 0.75\text{m}$ for a horizontal rectangular cylinder in roll motion.

Fig. 4.3.8.b. Free surface elevations at $x_1 = 3.1\text{m}$ for a horizontal rectangular cylinder in roll motion.
Fig. 4.3.9.a. Dimensionless hydrodynamic horizontal force on a horizontal rectangular cylinder in roll motion

Fig. 4.3.9.b. Dimensionless hydrodynamic vertical force on a horizontal rectangular cylinder in roll motion
Fig. 4.3.9.c. Dimensionless hydrodynamic moment on a horizontal rectangular cylinder in roll motion
Fig. 4.3.10. Geometry and finite element mesh of a horizontal rectangular cylinder in large roll motion.
Fig. 4.3.11.a. Free surface elevations at \( t=0.5\)s for a horizontal rectangular cylinder in large roll motion

Fig. 4.3.11.b. Free surface elevations at \( t=1.0\)s for a horizontal rectangular cylinder in large roll motion
Fig. 4.3.11.c. Free surface elevations at $t=1.5s$ for a horizontal rectangular cylinder in large roll motion

Fig. 4.3.11.d. Free surface elevations at $t=2.0s$ for a horizontal rectangular cylinder in large roll motion
Fig. 4.3.11.e. Free surface elevations at t=4.0s for a horizontal rectangular cylinder in large roll motion

Fig. 4.3.11.f. Free surface elevations at t=14.0s for a horizontal rectangular cylinder in large roll motion
Fig. 4.3.11.g. Free surface elevations at $t=14.5s$ for a horizontal rectangular cylinder in large roll motion

Fig. 4.3.11.h. Free surface elevations at $t=15.0s$ for a horizontal rectangular cylinder in large roll motion
Fig. 4.3.11.i. Free surface elevations at \( t = 15.5 \)\( s \) for a horizontal rectangular cylinder in large roll motion.

Fig. 4.3.11.j. Free surface elevations at \( t = 16.0 \)\( s \) for a horizontal rectangular cylinder in large roll motion.
Fig. 4.3.12.a. Free surface elevations at $x_1 = -1.0\text{m}$ for a horizontal rectangular cylinder in large roll motion.

Fig. 4.3.12.b. Free surface elevations at $x_1 = +1.0\text{m}$ for a horizontal rectangular cylinder in large roll motion.
Fig. 4.3.12.c. Free surface elevations at $x_1=-3.5\text{m}$ for a horizontal rectangular cylinder in large roll motion.

Fig. 4.3.12.d. Free surface elevations at $x_1=+3.5\text{m}$ for a horizontal rectangular cylinder in large roll motion.
Fig. 4.3.13.a. Dimensionless hydrodynamic horizontal force on a horizontal rectangular cylinder in large roll motion

Fig. 4.3.13.b. Dimensionless vertical hydrodynamic force on a horizontal rectangular cylinder in large roll motion
Fig. 4.3.13.c. Dimensionless hydrodynamic moment on a horizontal rectangular cylinder in large roll motion
Fig. 4.3.14.a. Free surface elevations at $t=1.0\text{s}$ for a horizontal rectangular cylinder in large roll motion with and without free surface updates.

Fig. 4.3.14.b. Free surface elevations at $t=2.0\text{s}$ for a horizontal rectangular cylinder in large roll motion with and without free surface updates.
Fig. 4.3.14.c. Free surface elevations at t=3.0s for a horizontal rectangular cylinder in large roll motion with and without free surface updates.

Fig. 4.3.14.d. Free surface elevations at t=4.0s for a horizontal rectangular cylinder in large roll motion with and without free surface updates.
Fig. 4.3.15.a. Mode shape of a cylindrical membrane in the 1st mode

Fig. 4.3.15.b. Mode shape of a cylindrical membrane in the 2nd mode
Fig. 4.3.15.c. Mode shape of a cylindrical membrane in the 3rd mode

Fig. 4.3.15.d. Motions of a cylindrical membrane with first three modes
Fig. 4.3.16.a. Dimensionless hydrostatic vertical force on a vibrating cylindrical membrane

Fig. 4.3.16.b. Dimensionless hydrostatic moment on a vibrating cylindrical membrane
Fig. 4.3.17.a. Free surface elevations at $x_1 = -1.28$ m for a vibrating cylindrical membrane

Fig. 4.3.17.b. Free surface elevations at $x_1 = +1.28$ m for a vibrating cylindrical membrane
Fig. 4.3.18.a. Free surface elevations at $x_1 = -10.88\,\text{m}$ for a vibrating cylindrical membrane.

Fig. 4.3.18.b. Free surface elevations at $x_1 = +10.88\,\text{m}$ for a vibrating cylindrical membrane.
Fig. 4.3.19.a. Dimensionless hydrodynamic horizontal force on a vibrating cylindrical membrane

Fig. 4.3.19.b. Dimensionless hydrodynamic vertical force on a vibrating cylindrical membrane
Fig. 4.3.19.c. Dimensionless hydrodynamic moment on a vibrating cylindrical membrane
Fig. 4.4.1. Definition sketch for free vibrations of a circular cylinder
Fig. 4.4.2. Heave motions for free vibrations of a circular cylinder
Fig. 4.4.3. Dimensionless hydrodynamic vertical force due to free vibrations of a circular cylinder.
Fig. 4.4.4. Wave celerity at the artificial boundary for free vibrations of a circular cylinder.
Fig. 4.4.5.a. Free surface elevations at $t=0.5s$ for free vibrations of a circular cylinder

Fig. 4.4.5.b. Free surface elevations at $t=1.0s$ for free vibrations of a circular cylinder
Fig. 4.4.5.c. Free surface elevations at $t=1.4s$ for free vibrations of a circular cylinder

Fig. 4.4.5.d. Free surface elevations at $t=2.0s$ for free vibrations of a circular cylinder
Fig. 4.4.5.e. Free surface elevations at $t=3.0s$ for free vibrations of a circular cylinder.

Fig. 4.4.5.f. Free surface elevations at $t=4.0s$ for free vibrations of a circular cylinder.
Fig. 4.4.5.g. Free surface elevations at $t=6.0\,s$ for free vibrations of a circular cylinder

Fig. 4.4.5.h. Free surface elevations at $t=8.0\,s$ for free vibrations of a circular cylinder
Fig. 4.4.5.i. Free surface elevations at $t=10.0s$ for free vibrations of a circular cylinder

Fig. 4.4.5.j. Free surface elevations at $t=12.0s$ for free vibrations of a circular cylinder
Fig. 4.4.5.k. Free surface elevations at $t=14.0\text{s}$ for free vibrations of a circular cylinder

Fig. 4.4.5.1. Free surface elevations at $t=16.0\text{s}$ for free vibrations of a circular cylinder
Fig. 4.4.6.a. Free surface elevations at $x_1=0.8\text{m}$ for free vibrations of a circular cylinder

Fig. 4.4.6.b. Free surface elevations at $x_1=4.64\text{m}$ for free vibrations of a circular cylinder
Fig. 4.4.6.c. Free surface elevations at $x_1=8.00\text{m}$ for free vibrations of a circular cylinder.

Fig. 4.4.6.d. Free surface elevations at $x_1=12.32\text{m}$ for free vibrations of a circular cylinder.
Fig. 4.4.6.e. Free surface elevations at $x_1=17.92$m for free vibrations of a circular cylinder

Fig. 4.4.6.f. Free surface elevations at $x_1=23.68$m for free vibrations of a circular cylinder
Fig. 4.4.7.a. Structural motions for free vibrations of a circular cylinder with water depth 2.0m, 1.6m and 1.2m
Fig. 4.4.7.b. Structural motions for free vibrations of a circular cylinder with water depth 2.0m, 1.6m and 1.2m
in the first three periods at the shifts of the heave
motions of the structure due to changes in water depth.

4.5 Hydrodynamics of a Moored, Floating Structure

Various kinds of moored floating structures, such as
floating bridges and breakwaters, have been employed in
coastal and harbor waters. Their structural motions, wave
attenuation characteristics and moorings have been
investigated by Adee, et al. (1), by Yamamoto, et al. (99)
and by Ijima, et al. (50).

Finite element solutions were obtained in the present
study for responses to a rightwardly travelling
monochromatic incident wave field of a semi-submerged
rectangular cylinder cross-spring moored to the sea floor
in the case of finite water depth. The geometry of the
structure, arrangement of the moorings and the meshes used
in the finite element solutions are shown in Fig. 4.5.1.
Incident wave conditions are specified as wave amplitude
A=0.05m, wave period T=2.0sec ( \( \omega^2 h \gamma = 2.0 \), wave length
L=12.16m ). The fluid is assumed to be initially at rest.
Ijima's solution (50) is used for comparisons. The
solution technique used by Ijima (50) was an analytical,
series expansion method implemented by a numerical method
to solve for the coefficients of the series expression.
The sway, heave and roll motions of the structure are shown in Figs. 4.5.2 - 4.5.4. The structure is seen to have large unsteady motions at the beginning and then approach as steady state periodic motions. The comparisons of the steady structural motions to Ijima's results (dashed lines) show very good agreement. Figures 4.5.5 - 4.5.6 show the reflected and transmitted waves on the structural surfaces and in the far-field. A check on the energy conservation shows a 3% difference between Ijima's (50) and the present results. The mooring forces in the springs are shown in Figs 4.5.7.a and 4.5.7.b. Due to the unsteady motions of the structure, the mooring forces are also unsteady. Comparison of Figs. 4.5.7.a and 4.5.7.b shows that the sea-side spring is subjected to larger forces than the lee-side spring.

4.6 Hydrodynamics of Multiple Floating Structures

In this section, time-domain simulations of the hydrodynamic response of two closely spaced moored horizontal floating structures are performed to investigate the wave interference phenomena and the sheltering effects of the structures. The same incident wave train as in Sec. 4.5 is assumed. The fluid is assumed to be initially at rest. The geometry and data of
the problem and the finite element mesh are shown in Fig. 4.6.1. Since effects due to a second placed on the lee-side structure of the first structure are to be considered, the structural conditions of each individual structure were specified the same as the case of the single structure shown in Fig. 4.5.1. The spacing between two structures is the width of the structure = 2a.

For the first example, two floating structures without inter-structural constraints are considered. The nondimensional horizontal and vertical displacements and rotations for the wave-side structure are shown in Figs. 4.6.2.a - 4.6.2.c, and for the lee-side structure are shown in Figs. 4.6.3.a - 4.6.3.c. The dimensionless mooring forces for the first and second structures are shown in Figs. 4.6.4 and 4.6.5, respectively. Upon comparisons to the results of a single structure, as given in Sec. 4.5, it can be observed that because of the interference effects from the second structure, the time of unsteady structural motions has significantly increased. The horizontal and rotational structural motions are increased, but vertical motions have decreased. Due to the sheltering effects from the first structure, the responses of the second structure can be seen to be smaller than those of the first structure. The
wave conditions in the far-field, near-field and between two structures are shown in Figs. 4.6.6.a - 4.6.6.f.

In the second example, the effects of inter-structural constraints on the structural motions are studied. The same geometrical specifications as in the previous case, Fig. 4.6.1, are considered. A single horizontal spring connects the two structures. The structural motions of two floating structures with a weak inter-structural constraint in $x_1$-direction, $Q_{1111}/pga = Q_{1212}/pga = Q_{1112}/pga = Q_{1211}/pga = 1.0$ are shown in Figs. 4.6.7.a - 4.6.7.c. The structural motions of the structures with a strong inter-structural constraint in $x_1$-direction, $Q_{1111}/pga = Q_{1212}/pga = Q_{1112}/pga = Q_{1211}/pga = 8.0$, are shown in Fig. 4.6.8.a - 4.6.8.c. From these results, it can be observed that with weak inter-structural constraint in the $x_1$-direction, the horizontal structural motions of the two structures are related with a phase difference; whereas with a strong inter-structural constraint, the horizontal structural motions tend to be in phase and the relative structural motions are small.

As a final example, a model of two floating bridges under construction at Mercer Island, Washington is considered. A second bridge is being constructed on lee
side existing Murrow bridge. Since a two-dimensional finite element model is used to analyze the problem, typical cross-sections for each individual bridges are used in the present study. The geometrical dimensions (60), wave characteristics and the finite element mesh used in this study are shown in Fig. 4.6.9. Design wave conditions, H=2.15' and T=2.70sec, are considered in the analysis. Linear springs are used to model the mooring cables, and an equivalent spring constant is used. The fluid is assumed to be initially at rest. The structural motions of the two bridges are given in Figs. 4.6.10.a - 4.6.10.c. The maximum amplitude of horizontal displacement for the Murrow bridge is 0.26' and for the new bridge is 0.13'; the maximum vertical amplitude for the Murrow bridge is 0.13' and for the new bridge is 0.22'; the maximum amplitude of rotation for the Murrow bridge is 0.12° and for the new bridge is 0.06°. These are well within acceptable levels for the specified design waves.
Fig. 4.5.1. Definition sketch of a cross-moored, floating rectangular structure.
Fig. 4.5.2. Sway motion of a cross-moored, floating rectangular structure

Fig. 4.5.3. Heave motion of a cross-moored, floating rectangular structure
Fig. 4.5.4. Roll motion of a cross-moored, floating rectangular structure
Fig. 4.5.5.a. Reflected waves at $x_1=-0.5h$ by a cross-moored, floating rectangular structure

Fig. 4.5.5.b. Reflected waves at $x_1=-3.5h$ by a cross-moored, floating rectangular structure
Fig. 4.5.6.a. Transmitted waves at $x_1=0.5h$ by a cross-moored, floating rectangular structure.

Fig. 4.5.6.b. Transmitted waves at $x_1=3.5h$ by a cross-moored, floating rectangular structure.
Fig. 4.5.7.a. Mooring forces of no. 1 spring on a cross-moored, floating rectangular structure

Fig. 4.5.7.b. Mooring forces of no. 2 spring on a cross-moored, floating rectangular structure
Fig. 4.6.1. Geometry and finite element mesh of two floating structures
Fig. 4.6.2.a. Sway motions of the structure at wave-side of two floating structures

Fig. 4.6.2.b. Heave motions of the structure at wave-side of two floating structures
Fig. 4.6.2.c. Roll motions of the structure at wave-side of two floating structures
Fig. 4.6.3.a. Sway motions of the structure at lee-side of two floating structures

Fig. 4.6.3.b. Heave motions of the structure at lee-side of two floating structures
Fig. 4.6.3.c. Roll motions of the structure at lee-side of two floating structures
Fig. 4.6.4.a. Mooring forces of no. 1 spring on the structure at wave-side of two floating structures

Fig. 4.6.4.b. Mooring forces of no. 2 spring on the structure at wave-side of two floating structures
Fig. 4.6.5.a. Mooring forces of no. 1 spring on the structure at lee-side of two floating structures

Fig. 4.6.5.b. Mooring forces of no. 2 spring on the structure at lee-side of two floating structures
Fig. 4.6.6.a. Free surface elevations at $x_1 = -10a$ for two floating structures

Fig. 4.6.6.b. Free surface elevations at $x_1 = -3a$ for two floating structures
Fig. 4.6.6.c. Free surface elevations at $x_1 = -a$ for two floating structures

Fig. 4.6.6.d. Free surface elevations at $x_1 = a$ for two floating structures
Fig. 4.6.6.e. Free surface elevations at $x_1=3a$ for two floating structures.

Fig. 4.6.6.f. Free surface elevations at $x_1=10a$ for two floating structures.
Fig. 4.6.7.a. Sway motions for two floating structures with weak inter-structural constraint in x₁-direction.
Fig. 4.6.7.b. Heave motions for two floating structures with weak inter-structural constraint in x-direction.

\[ t/T, T=2.8365 \]

\[ U_e/A \]

\[ U_e/A \]
Fig. 4.6.7.c. Roll motions for two floating structures with weak inter-structural constraint in $x_1$-direction
Fig. 4.6.8.a. Sway motions for two floating structures with strong inter-structural constraint in x1-direction
Fig. 4.6.8.b. Heave motions for two floating structures with strong inter-structural constraint in $x_1$-direction
Fig. 4.6.8.c. Roll motions for two floating structures with strong inter-structural constraint in $x_1$-direction
Fig. 4.6.9. Geometry and finite element mesh of two floating bridges
Fig. 4.6.10.a. Sway motions for two floating bridges
Fig. 4.6.10.b. Heave motions for two floating bridges
Fig. 4.6.10.c. Roll motions for two floating bridges
5.0 CONCLUSIONS

5.1 Summary

A finite element model of fluid-structure interactions in the time domain has been presented. A numerical method for the simulation of the time dependent interactions between transient surface waves and unsteady structural motions was developed and validated by several series of numerical examples.

Galerkin's method of weighted residuals was used to formulate an integral expression corresponding to the combined initial and boundary value problem of linear wave theory. That integral was discretized with isoparametric volume and surface elements. Quadratic shape functions were used.

Structural motions were considered to be large relative to the wave length. An Eulerian description of the fluid motions was adopted, but a Lagrangian description was used for the structural motions and for fluid-structure interfaces that deform with time. Adaptive finite element meshes were used in regions surrounding the moving structures to accommodate structural motions, and to avoid excessive element distortions.
In considering unsteady characteristics of surface waves, Orlanski's condition was incorporated into the finite element fluid formulation at transmitting boundaries which artificially truncated the domain but still simulate an infinite or semi-infinite fluid domain. Orlanski's condition was modified: the phase velocity was assumed to be a slowly varying function in time but invariant along the artificial boundaries. An extrapolation equation was proposed for the calculation of the wave celerity.

Dynamic analysis procedures for the problem of transient fluid-structure interactions were described. A partitioned integration scheme for the fluid-structure system was used, and the solution state was advanced over each of the subsystems in a staggered iterative fashion. An iterative approach was adopted for the fluid-structure coupling analysis to account for the nonconservative interaction terms. The implicit Newmark's method was used to integrate the structural equations. A nodal partitioned explicit-implicit scheme was developed for the fluid where an explicit central difference method was used for the explicit nodal group; and an implicit backward difference method was adopted for the implicit nodal group.
5.2 Observations and Discussions

The first series of time-domain simulations were of the flap wavemaker problem was intended to demonstrate the validity of the partitioned explicit-implicit solution scheme for the fluid and to study effects of the scheme adopted to artificially truncated fluid regions. First, an extremely long channel was modelled with a total reflection boundary assumed at the end of the channel. The free surface elevations calculated at times before the waves were reflected compared extremely well with Kennard's analytic solution (55) for an infinite channel. The flap wavemaker initially generated a long stretched wave followed by transition waves, and finally by steady state waves.

Next, a short channel with the modified Orlanski's boundary was modelled to simulate an infinite long channel. The free surface elevations, when compared to those for the long channel without reflected waves, showed excellent agreements. This demonstrated the ability of the modified Orlanski's condition in unsteady wave problems.

Orlanski's boundary was also applied to multiple-frequency wavemaker problems to show the applicability of the Orlanski's condition for a multiple-frequency wave
field and the results were generally successful. The principal frequency of the multiple-frequency waves is suggested for use to calculate the steady state phase velocity in the modified Orlanski's condition, but after long transient periods reflections from the artificial boundary due to higher frequency waves should be expected to contaminate numerical results.

In the second series of time-domain simulations, problems of wave diffractions by fixed horizontal structures were considered. Both circular and rectangular cylinders were studied. A hyperbolic tangent ramping function was used to describe the suddenly applied incident waves: they were ramped up to full amplitude in a quarter of a wave period. The numerical results show the scattered waves behind and in front of the structures to gradually change from unsteady to steady state. The wave forms were fully developed in about four wave periods. Comparisons of the steady state results from the present time-domain solutions and the existing frequency-domain results showed very good agreements.

In the third series of time-domain simulations, problems of wave radiation by floating horizontal structures were considered. Cases of a circular cylinder
in periodic heave motions, sway motions, and a rectangular in roll motions were considered. Unsteady waves were generated initially and steady wave forms were fully developed in about four wave periods. Comparisons to frequency-domain solutions showed very good agreements.

Large, periodic roll motions of the rectangular cylinder were also studied to investigate the effects of adaptive structural motions on generated waves. Adaptive meshes were used to account for deformations of the structural boundaries and to avoid excessive element distortions. It was found that secondary waves were generated and the hydrodynamic vertical force was significantly increased.

The case of a rectangular cylinder with large periodic roll motions was reconsidered, but with the free surface boundary updated. For shorter transient intervals similar results were obtained as in the analysis without updates. No definitive conclusion was reached as to whether the solutions from linear wave theory with large structural motion could be an approximation to the nonlinear wave theory. It is recommended to utilize a nonlinear wave theory to study the problems of fluid-structure interactions with large
structural motions.

The problem of a vibration membrane in the fluid was considered to investigate the interactions of a deformable body with the fluid. Adaptive meshes were used to accommodate large motions of the vibrating membrane. The first mode of the natural frequencies of the membrane was used to calculate the propagating wave celerity in the modified Orlanski's condition.

A fully transient problem of the free vibration of a floating circular cylinder with an initial displacement was studied to validate the procedures for the dynamic coupling between the structure and fluid. Results for transient structural motions were compared to analytic solutions by Ursell (91) and showed excellent agreements. The structure showed damped vibration motions, and were damped out in four structural natural periods. The fluid provided damping effects to structural motions by radiating surface waves. The generated waves were developing due to continuous structural motions, but because of decaying structural amplitudes, the waves did not reach steady state, instead remaining in a fully transient situations. A parametric study of the effects of the water depth on the structural motions was conducted. It was found that the shallower the water,
the stronger the damping effects from the fluid and the higher the frequency of the structural motion.

In the last series of time-domain simulations, the interactions of moored floating structures and incident water waves with or without inter-structural constraints were studied. The structural motions, wave attenuations and mooring forces in the time domain were considered. First, the problem of a moored, floating rectangular structure subjected to the action of incident waves was considered. Results were compared to a frequency-domain analytic solution by Ijima (50), and showed very good agreement. The time-dependent structural motions were seen to have large unsteady motions initially and to then approach steady periodic motions. From this study, the value of time-domain simulations of fluid-structure interactions can be seen: unsteady structural responses may be two to three times results predicted by steady state analysis.

Next, the interference phenomena of two moored, floating structures were studied. Because of the second structure, the unsteady structural motions lasted for a longer time than in the case of a single structure. Due to sheltering effects from the wave-side structure, the responses of the lee-side structure were smaller than
those of wave-side structure. With a strong inter-structural constraint in the horizontal direction, the horizontal motions of two structures tended to be in phase, i.e., relative motions were small, whereas with a weak inter-structural constraint, the horizontal motions of two structures were related with a phase difference.

In the final example, an actual model of two floating bridges subjected to design wave conditions was considered. The mooring cables were simulated by elastic springs and equivalent spring constants were used. The structural responses and mooring forces obtained in the present analysis appear realistic and reasonable. The finite element method thus proved to be a useful tool in the time-domain analysis of wave-structure interactions.

### 5.3 Possible Extensions

In the analysis of interactions between nonlinear waves and highly deformable large membranes in the time domain, three phases of research are necessary. In the first phase, which is the subject of the present study, time-domain simulation of the interactions between linear waves and rigid structures are studied. A dynamic fluid model with suitable time dependent radiation condition for the artificial boundary and dynamic analysis
procedures for coupled fluid-structure interactions would be developed and validated. An adaptive mesh would be used to account for changes of the structural positions with time. Additional research is necessary to achieve a model of the nonlinear interactions of waves with highly deformable bodies. In the next phase, time domain simulations of the interactions between nonlinear waves and rigid structures should be developed. The dynamic coupled fluid-structure model developed in the present study could be utilized with a nonlinear fluid model and an adaptive mesh implemented for the free surface boundary. In the third phase of the research, time-domain simulations of the interactions between nonlinear waves and highly deformable membranes should be investigated. The dynamic coupled fluid-structure model developed in previous phases would be used. A dynamic model for the deformable membrane as well as dynamic analysis procedures for nonlinear coupling systems would be developed and verified.

Extensions of the present finite element method are suggested for the following areas:

(1) Experimental verification of the time-domain wavemaker problem, and quantitative determination of time dependency of the wave celerity;
(2) Incorporation of nonlinear free surface boundary conditions into the finite element fluid model to treat nonlinear wave problems;

(3) Incorporation of deformable structural models into present computer program to treat problems of interactions between waves and deformable structures;

(4) Development of a internal fluid/structure/external fluid, three-phase interaction model wherein the internal fluid may also have free surface effects, and nonlinear boundaries are considered; and

(5) Development of a wavemaker/fluid/structure/beach model to provide prototype numerical model testings, where actual beach conditions can be incorporated directly into the present finite element program.
BIBLIOGRAPHY


APPENDIX
APPENDIX I - KENNARD'S ANALYTICAL SOLUTION

In the following, Kennard's analytical solution is summarized and applied to a flap wavemaker. For detailed derivations see Ref. (55).

The generation of surface waves by a moving partition in a semi-infinite channel with a finite water depth, h, was studied in Ref. (55). Linear wave theory and a source method were used to solve the corresponding linearized combined initial and boundary value problem. The solutions of the velocity potential, \( \phi \), and the free surface elevation, \( \eta \), can be expressed as

\[
\phi = \phi_0 + \frac{2gS}{\pi} \int_0^\infty \int_0^\infty \delta_1 \sin \delta \frac{\cosh k(h-x_2)}{\cosh kh} \cdot \cos kx_1 Q(k,t-\tau) \, dk \, d\tau \tag{A.1}
\]

\[
\eta = \frac{S}{2\pi} \int_0^\infty \int_0^\infty \cos \delta \tau \cos kx_1 Q(k,t-\tau) \, dk \, d\tau \tag{A.2}
\]
in which

\[ \phi' = \frac{S}{2\pi} \int_0^h \frac{\partial u(\beta,t)}{\partial t} \, d\beta \]

\[ \ln \left[ \frac{\cosh \frac{\partial x_1}{2h} - \cos \frac{\partial (\beta + x_2)}{2h}}{\cosh \frac{\partial x_1}{2h} + \cos \frac{\partial (\beta + x_2)}{2h}} \frac{\cosh \frac{\partial x_1}{2h} - \cos \frac{\partial (\beta + x_2)}{2h}}{\cosh \frac{\partial x_1}{2h} + \cos \frac{\partial (\beta + x_2)}{2h}} \right] d\beta \]

\[ Q(k,t) = \int_0^h \frac{\partial u(\beta,t)}{\partial t} \left( \exp(-k\beta) + \frac{2 \sinh k\beta}{\exp(2kh)+1} \right) d\beta \]

\[ (A.3) \]

\[ \delta^2 = gk \tanh kh \]

\[ (A.4) \]

where \( u(\beta,t) \) is the function describing the motion of the moving object, and \( S \) is the amplitude of the motion.

Note that in Eqs. (A.1)-(A.5), the coordinate system is located at the still water level with positive \( x_2 \)-axis pointed downward.

For a full-depth hinged flap wavemaker initially at rest in its extreme left position, \( u(x_2,t) \) can be expressed as

\[ u(x_2,t) = -\frac{h-x_2}{h} \cos \omega t \]

\[ (A.5) \]

Upon substitution of Eq. (A.6), Eq. (A.4) becomes
\[ Q(k, t - \tau) = \]
\[
\begin{cases}
- \omega \cos \omega (t - \tau) \left[ \frac{1}{k} + \frac{1}{k^2 h} \left( e^{kh} - 1 \right) + \frac{2}{e^{2kh} + 1} \left( \frac{\sinh kh}{k^2} - \frac{1}{k} \right) \right], & k \neq 0 \\
- \omega \cos \omega (t - \tau) \left[ \frac{h}{2} \right], & k = 0
\end{cases}
\]

(A.7)

Any scheme can be used to numerically integrate Eqs. (A.1) and (A.2). However, it is necessary to approximate the infinite integrals shown in those equations. From an analysis of orders of magnitude, it has been shown that if the error of the integration is m\% of S, then the infinite integration limit can be replaced by

\[ k, \quad \text{where} \]
\[ k = \frac{m}{S} \]  \quad (A.8)

Using Eqs. (A.3), (A.5) and (A.7)-(A.8), one can integrate Eqs. (A.1) and (A.2).
APPENDIX II - URSELL'S ANALYTICAL SOLUTION

The transient motion corresponding to the free vibration of a floating circular cylinder initially displaced in the vertical direction from its equilibrium position was given by Ursell (91) as

\[ x_2(t) = x_2(0) \cdot f \left[ t \left( \frac{g}{a} \right)^{1/2} \right] \]  \hspace{1cm} (B-1)

where \( x_2(t) \) is the vertical displacement of the structure at time \( t \); \( x_2(0) \) is the small initial displacement; \( g \) is the gravitational constant; \( a \) is the radius of the circular cylinder. The kernel function \( f \) in Eq. (B-1) can be expressed as

\[
\int_{\infty}^{\infty} \frac{u (1+\Gamma) \exp(-iu\tau)}{\sqrt{1 - \frac{g}{a} u^2 (1+\Gamma)/4}} \, du
\]  \hspace{1cm} (B-2)

in which \( i = \sqrt{-1} \) and the force coefficient \( \Gamma \) is given by Ursell (90).

The numerical calculation of the kernel function \( f \) has been studied by Maskell and Ursell (65). Equation (B-2) was expressed as the sum of a polar component, \( \bar{f} \), and an integral component, \( \hat{f} \):

\[ f(\tau) = \bar{f}(\tau) + \hat{f}(\tau) \]  \hspace{1cm} (B-3)

in which

\[ \bar{f}(\tau) = 0.9664 \exp(-0.1309\tau) \cos(0.9117\tau - 0.4805) \]  \hspace{1cm} (B-4)

and \( \hat{f}(\tau) \) is plotted in Fig. B-1. For \( \tau \geq 8 \), \( \hat{f}(\tau) \) can be approximated by \( -(4/\tau^2) \, \tau^{-2} \).
Fig. B.1. The integral component of the kernel function in Ursell's analytical solution.