


AN ABSTRACT OF THE THESIS OF

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Title A DYNAMIC, STEREOSCOPIC DISPLAY OF PHASE-SPACE  
TRAJECTORIES BY PARALLAX SIMULATION

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Phase-space trajectories are often used to describe a system's stability or response characteristics. It is desirable to observe such a phase-space trajectory directly from an analog simulation of a system, but a conventional oscilloscope display does not provide an axis for the third variable.

A third axis could be provided by simulating visual parallax in two planar displays and viewing them stereoscopically. The third axis is then provided by depth perception associated with binocular vision.

The method of producing visual parallax developed in this thesis involves generating a diode equation approximation of the differential parallax equation from optical theory. Once the differential parallax is generated, it is used to modify a planar display so three dimensional vision is simulated with the aid of a stereoscopic viewer.

Photographs of laboratory results are provided as evidence of the success of this approach to the display of phase-space trajectories.

A DYNAMIC, STEREOSCOPIC DISPLAY OF PHASE-SPACE  
TRAJECTORIES BY PARALLAX SIMULATION

by

CHARLES FREDERICK STARNES

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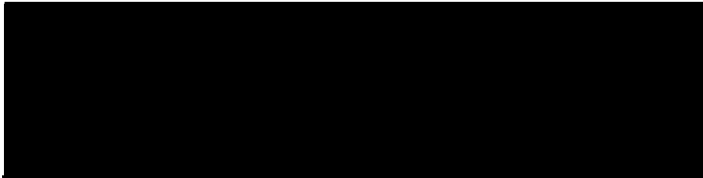
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# A DYNAMIC, STEREOSCOPIC DISPLAY OF PHASE-SPACE TRAJECTORIES BY PARALLAX SIMULATION

## INTRODUCTION

One popular approach to control systems design and analysis employs phase-plane or phase-space criteria as an indication of system stability and general performance. After analytic design methods have been exhausted, the system is usually simulated on a computer so the system response can be further studied and improved. The phase-plane trajectories are easily obtained from an analog computer simulation by displaying the two variables of interest on an oscilloscope. A visual display of a phase-space trajectory is not so easily accomplished.

Systems capable of simultaneously displaying three variables have been developed. The display systems include rotating planes with strobed light beams (3), rotating cathoderay tubes (3), parallax simulation with stereoscopic viewers (8) and repetitive line segment displays that simulate perspective on a plane (6). These systems have the same common drawback of requiring specialized, expensive computer equipment to produce the desired display. Consequently, the objective of this thesis is to develop an inexpensive analog system which will produce a dynamic, stereoscopic display of three variables using parallax simulation.

## THE THEORY OF STEREOSCOPIC VISION

### Depth Perception

Depth perception is dependent on three visual phenomenon. The most obvious of these is perspective. As an object becomes more distant, its size appears to diminish. Seeing a man the size of a house in a photograph forces the observer to conclude that the house is much more distant from the camera.

A second indication of depth is relative motion. As an observer moves, close objects appear to pass by with much greater relative speed than more distant objects.

The third aid to depth perception is visual parallax. Visual parallax is caused by the separation of the eyes resulting in a slight difference between the images viewed. This same effect can be observed by taking two photographs of an object with a camera placed several feet apart for each exposure. The different relative positions of objects observed in the pictures is the same as observed due to the separation of the eyes. When this parallax is caused by binocular vision, the mind interprets it as depth.

With respect to the stereoscopic display of three variables, the relative motion phenomenon is of no value since the observer and coordinate system are stationary. The perspective can be included if a

realistic three dimensional picture is desired. For the purpose of information analysis, however, an isometric representation makes scaling and calibration much more practical. Attention will, therefore, be concentrated on the parallax simulation approach of representing the third variable.

### Derivation of the Differential Parallax Equation

The first problem of concern is finding the relationship between the desired depth and its associated differential parallax. This is a simple geometry problem as shown in Figure 1. For symmetry and convenience, choose to observe the same amount of parallax with each eye in viewing the subject. This does not reduce the generality of the derivation since the similar triangles used remain similar regardless of the observer's position.

The following equations are based on the geometry and notation shown in Figure 1.

By similar triangles:

$$\frac{F + Z}{K} = \frac{Z}{X}$$

Also note that:

$$Z = H - h - F \quad 1$$

This implies:

$$X = \frac{ZK}{F + Z} = \frac{K(H - h - F)}{H - h} \quad 2$$



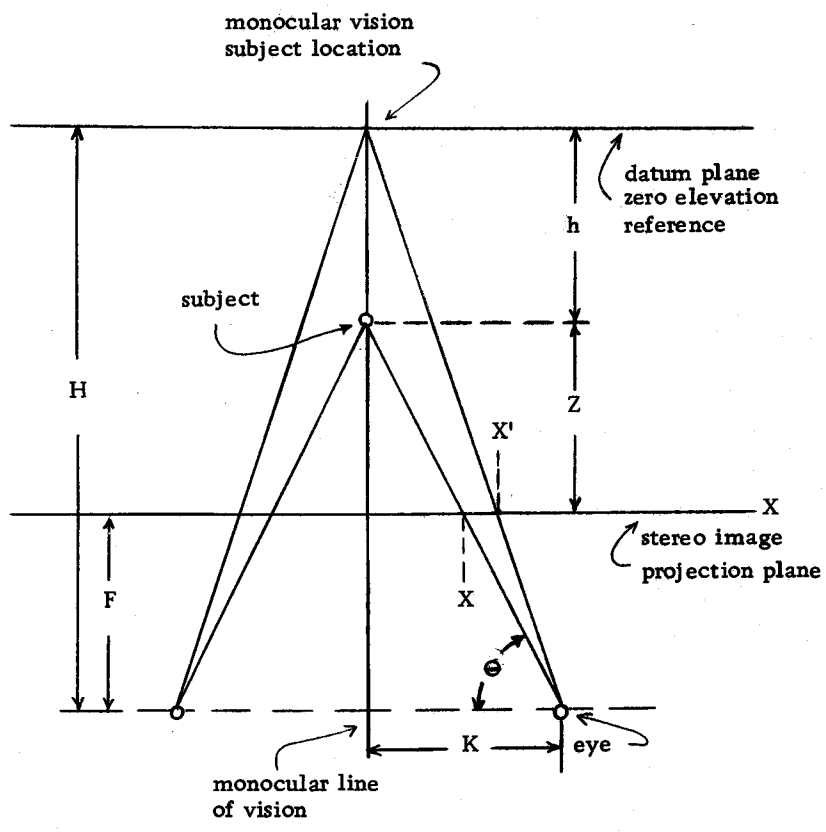


Figure 1. Planar Projection of Visual Parallax

Definitions of Symbols and Terms.

- Datum Plane = The zero reference elevation (depth).
- $H$  = The distance from the observer's eye to the datum plane
- $F$  = The focal distance of the stereoscope
- $h$  = The elevation of the subject from the datum plane
- $Z$  = The distance of the subject from the focal plane of the stereoscope
- $X'$  = The parallax necessary to locate the datum plane with respect to the focal plane
- $X$  = The parallax necessary to locate the subject with respect to the focal plane
- $Dp = X' - X$  = The differential parallax which locates the subject with respect to the datum plane
- $K$  = Half of the distance between the eyes
- $\theta$  = The angle of the line of sight from the eye to the subject

At the datum plane  $h = 0$ ;  $Z = H - F$

$$X' = \frac{K(H - F)}{F + (H - F)} = \frac{K(H - F)}{H} \quad 3$$

Now the Differential Parallax is defined by:

$$D_p = X' - X = \frac{K(H - F)}{H} - \frac{K(H - h - F)}{H - h} \quad 4$$

$$D_p = \frac{K(H^2 - HF - Hh + hF - H^2 + Hh + HF)}{H(H - h)} = \frac{FK(h)}{(H - h)H}$$

$$D_p = \frac{h}{(H - h)} \left( \frac{FK}{H} \right) \quad 5$$

The differential parallax is the deviation in the position of the subject observed with binocular vision as compared with the position seen with monocular vision. This must be added to the horizontal planar coordinate of the phase-plane trajectory to produce the desired stereoscopic image.

#### Practical Limits of Binocular Vision

The differential parallax is defined by equation 5 for subject elevations ranging from an infinite distance from the observer ( $+\infty$ ) to a position directly between the observer's eyes ( $-H$ ). These limits are completely unrealistic. As soon as the differential parallax causes less than twenty seconds arc between the converging lines of sight, no change in elevation can be detected (2). As the subject

approaches within approximately six centimeters of the observers eyes, it breaks into a double image because the line of sight of the eyes can no longer converge on the subject.

The actual limits of elevation to be generated and observed are dependent on the optical constants (K, H and F) of the stereoscopic viewer. The simulation of visual parallax should not be limited to one set of optical constants if the simulation technique is to be of any practical value. For preliminary studies, therefore, elevations of plus and minus one half the distance from the datum plane to the observer are considered. Specific optical constants will not be chosen until the general differential parallax function is generated.

## PRELIMINARY STUDIES

### Diode Equation Approximation

Of the approaches investigated for the simulation of differential parallax (see Appendix), the most promising is a diode equation approximation. The plot of differential parallax as a function of subject elevation on cartesian coordinates (Figure 2) is observed to resemble the diode characteristic curve (Figure 4) (1, 4).

Consider using the resistance characteristics of a diode as the input impedance of an operational amplifier. A function of the form  $E_o = R_f I_s (e^{av} - 1)$  is generated (see Appendix). Putting the differential parallax equation into the same form yields  $Dp = R_f K (e^{ah} - 1)$ .

The approach now is to solve for the constants,  $R_f$  and  $a$ , corresponding with the values for differential parallax calculated by the exact differential parallax equation. Then plotting  $\ln(0.5K + Dp)$  as a function of elevation yields a straight line plot which demonstrates the correspondence between the differential parallax equation and the diode equation (see Figure 15).

### Diode Circuit Considerations

With evidence that the diode equation approximation approach is sound, consideration to the practical circuitry is in order. Using

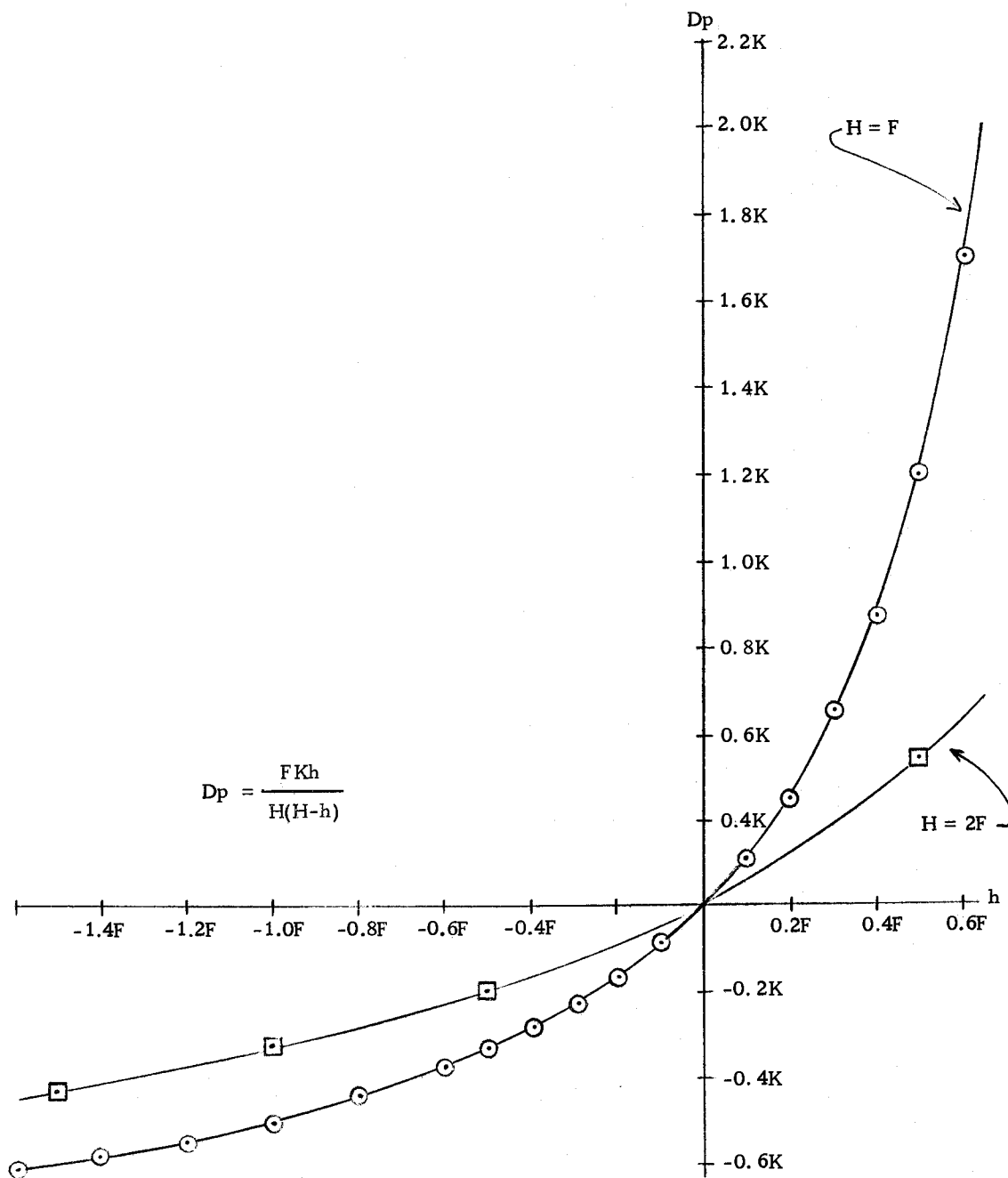


Figure 2. Differential Parallax as a Function of Elevation.

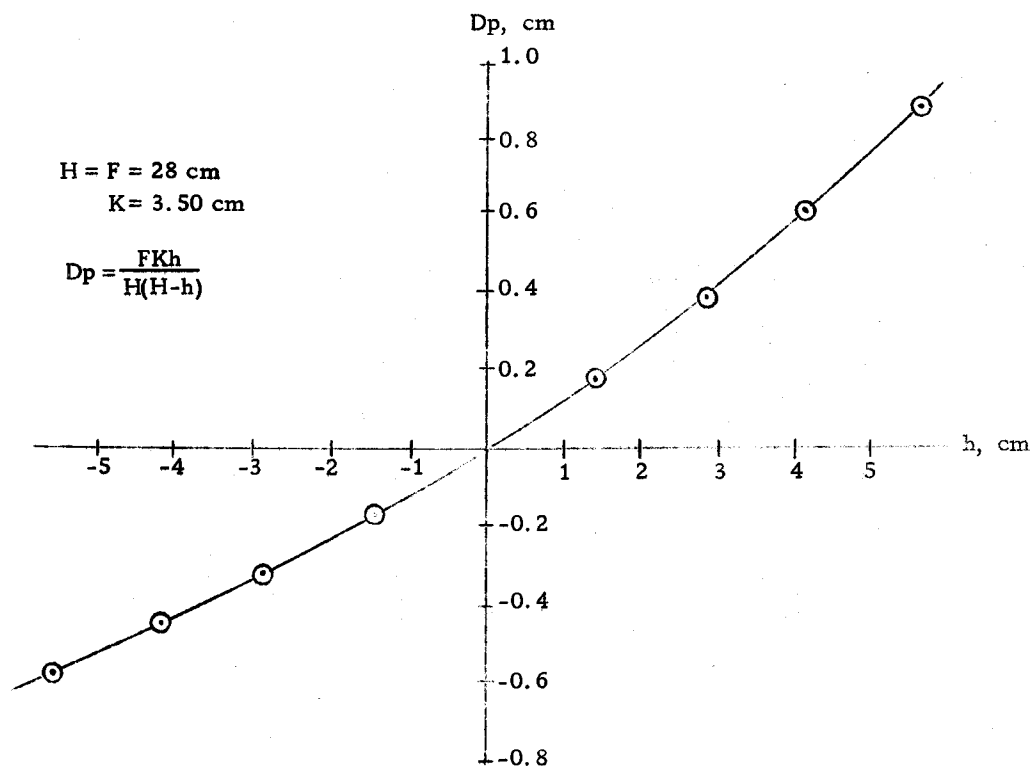


Figure 3. Differential Parallax Function for Known Constants

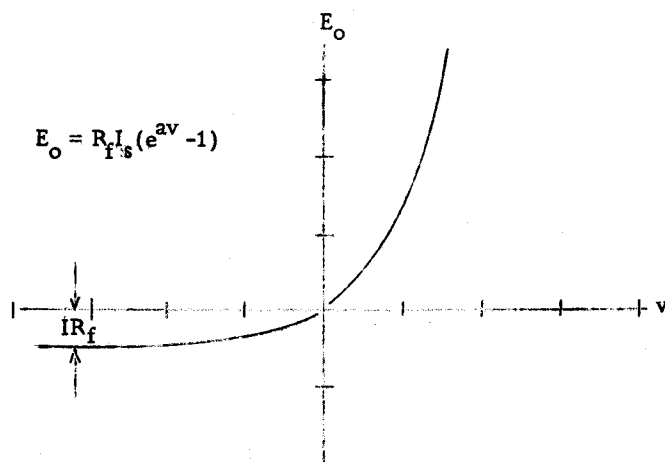


Figure 4. Operational Amplifier Response with Diode Input Impedance

a diode to generate the differential parallax, immediately presents two problems. The first is locating a diode with the desired constants. The second is the necessity of having a different diode for every new set of optical constants used. For these reasons, the PN junction of a transistor is selected as the function generating (diode equation) element. Curve shape control can then be achieved by biasing the collector junction, thus eliminating the need for more than one component for changes in optical constants.

Additional function control is available from scaling and axis translation provided by conventional operational amplifier techniques. Now the function generation is not dependent on matching diode characteristics exactly with the differential parallax function to be generated.

### Circuit Performance Specifications

Even though some degree of function control is available, the circuit design should be based on one desirable set of optical constants. The most important of these is the location of the datum plane with respect to the focal distance of the viewer. The other optical constants are not of concern yet since they don't determine the exponential curve shape as does the location of the datum plane.

For the first approximation, specify that the datum plane be located at the focal distance ( $H = F$ ) of the viewer. The general

function to be generated in terms of eye span and focal distance, is then defined in Figure 2. A circuit designed to generate this function will provide the desired differential parallax for any eye span and focal distance with a simple scale change.

The degree of accuracy required of the function generating circuit is defined by the optical constants and limits of binocular vision. Assume the stereoscope has an eye span of seven centimeters and a focal distance of 28 centimeters (Figure 3). The angle error of the line of sight must be less than 20 seconds or 0.00582 radians to be negligible. Simple trigonometry then defines a maximum permissible differential parallax error of 0.163 centimeters. It is reasonable and sufficient to impose a maximum error limit of plus or minus 0.08 centimeters on the circuit specifications.



## PARALLAX SIMULATION CIRCUIT DESIGN

### Control

The final circuit design satisfying the desired performance specifications is shown in Figure 5. The control features with respect to the function generated are as follows.

The curve shape is controlled by the D. C. bias voltage on the transistor collector junction. The operating characteristics are shown in Figure 7 for several different bias levels.

Axis translation is achieved by varying the gain of the D. C. input to the summation junction of the differential parallax generating operational amplifier.

Curve segment selection is accomplished by attenuation of the input function (to limit segment size) and by addition of a D. C. level to the input function to bias the base junction of the transistor at the center of the desired operating characteristic. This mode of control is shown in Figure 8.

Scaling is accomplished by varying the feedback resistor of the differential parallax generating operational amplifier.

Using the controls described, the function being generated can easily be made to fit any of the differential parallax curves desired. Figures 13 and 14 show two curves that were generated for differing

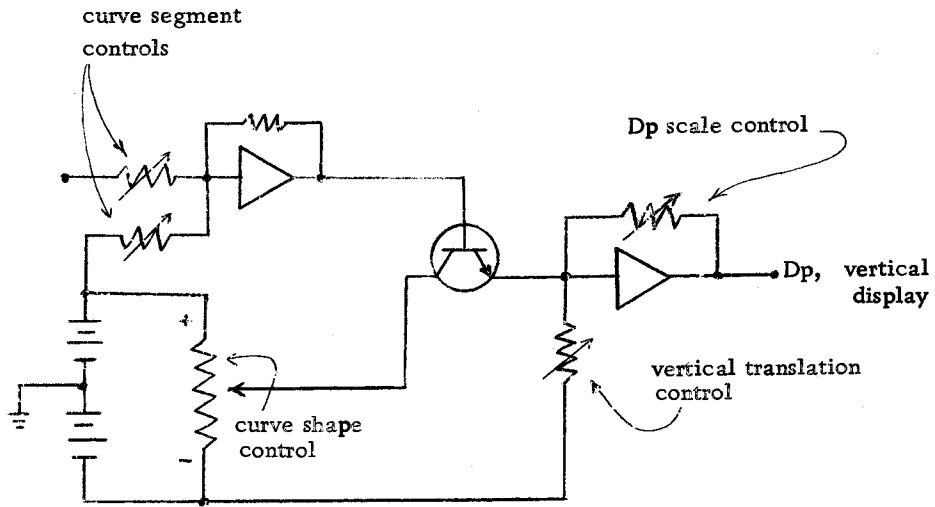


Figure 5. Differential Parallax Simulation Circuit

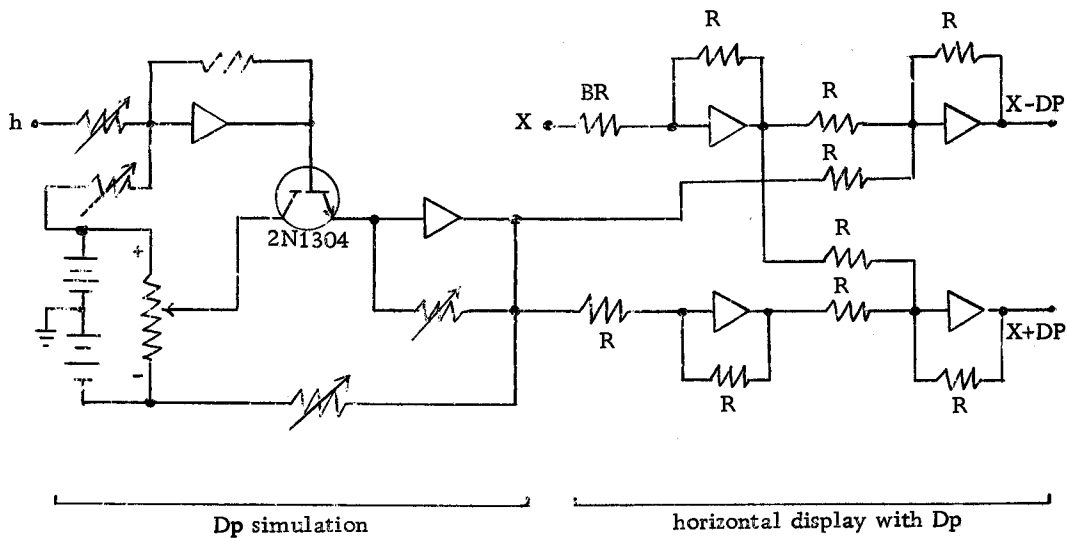


Figure 6. Stereoscopic Trajectory Simulation Circuit

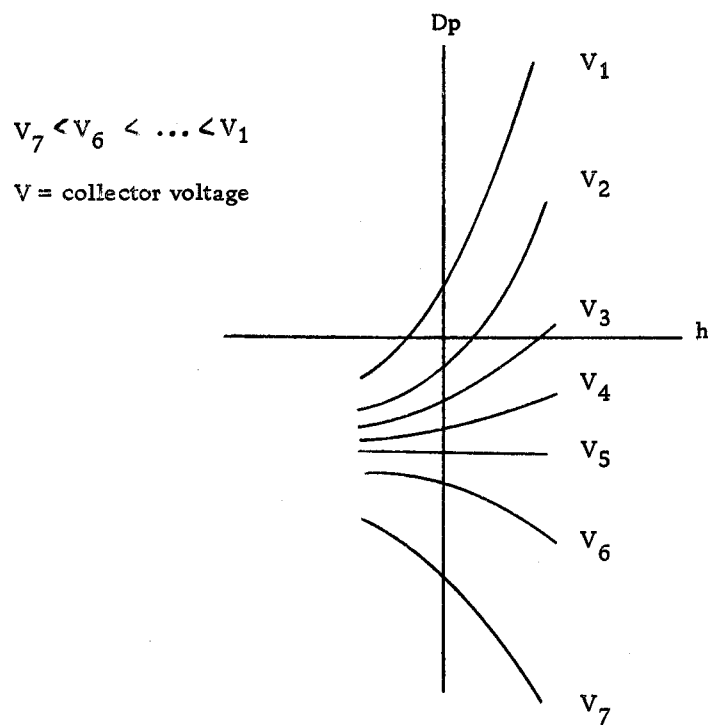


Figure 7. Function Shape Control

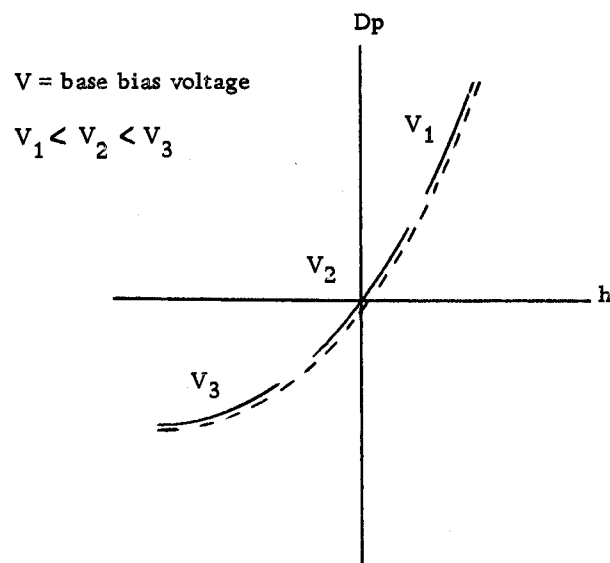


Figure 8. Curve Segment Control

maximum elevation limits when used with a stereoscope and datum plane satisfying the specifications described on page 11.

### Stability

There has been no attempt made to define specific bias voltages for the functions generated. This information would be useless because the circuit is not long-time stable. Once the circuit is operating and drift has stabilized, the function generated is stable. After putting the circuit to rest and reactivating it, however, new control settings are required to duplicate the desired function. Replacing the transistor also necessitates adjustment. This is no real problem because the adjustments are not difficult to make.

### Calibration

Calibration of the elevation axis is a relatively simple matter. A sinusoidal input voltage is adjusted until the desired maximum elevation deflection is achieved on the differential parallax function. The scale calibration is then the input voltage divided by the deflection. Elevation measurements can then be made easily with a parallax bar viewed through a stereoscope.

### Phase-Space Generation

Now that the differential parallax has been successfully

simulated, it must be used to modify the phase-plane trajectory so the third variable can be seen stereoscopically. This is a simple summation process shown in Figure 6. If a dynamic, visual output is desired, the right and left eye images can be generated simultaneously on two oscilloscopes placed facing each other and viewed through two 45 degree mirrors. Another useful display method is to take photographs of each image and view them with a standard stereoscope.

This latter method was the one used to verify that the techniques described will produce the desired phase-space. By taking photographs it is necessary to have only one stereoscopic image displayed at a time. This allows a reduction of the number of operational amplifiers required. The experimental circuit used for demonstration also employed some switching logic which reduced the number of operational amplifiers needed from six to two (see Figure 16).

## LABORATORY RESULTS

### Sources of Error

Electronically produced stereoscopic displays suffer from three types of error if caution is not observed. One is elevation exaggeration. This occurs if the optical constants of the stereoscopic viewer do not correspond with the differential parallax function generated. An example of this distortion is shown in Figure 9. The lissajous should appear to be in one plane slicing at an angle through the datum plane due to a triangle function at the elevation input. Instead, the lissajous appears to plunge out of the plane at one extreme because of exaggerated elevation.

Another source of error is horizontal drift. This shows up as an additional differential parallax and will displace the entire subject some constant error distance from the datum plane.

A third source of error is vertical drift of the image. This can not be tolerated. Any vertical drift that is detectable will appear to be vertical parallax and will prevent the eyes from being able to focus on the subject. Only the horizontal parallax can be compensated for by the eyes to bring an image into focus.

Care should be taken to match up the images with their corresponding eyes. If this is not done, pseudoscopic illusion occurs

causing everything on the elevation axis to be seen in reverse.

### Planning Phase-Space Displays

An elevation deflection of plus and minus two or three centimeters from the datum plane seems most satisfactory for display purposes with the stereoscope specified (p. 11). Observers with sub-standard binocular vision have difficulty focusing on larger deflections while less deflection makes an inadequate display of the third variable. For persons with strong binocular vision, plus and minus five centimeters deflection from the datum plane is even more effective, but does cause more eye strain.

One consideration that should be made in planning a stereoscopic display is the relative datum plane projection positions of the subject's elevation extremes. For example, when the close portion of the subject is viewed in focus, more distant portions appear as a double image. If the split image is not in the direct line of sight, it will not be disturbing. If, however, the split image is in the direct line of sight, the eyes find it difficult to decide on which objective to focus. This implies that large extremes in elevation can be viewed with less strain when large contrasts in elevation are widely separated on the datum plane. If the extremes in elevation are close on the datum plane, a reduction in elevation deflection aids clear viewing.

### Sample Phase-Space Photographs

To support the claims and conclusions of this study, photographs demonstrating the stereoscopic display of three variables are shown in Figures 10, 11 and 12. All three have a  $\sin wt$  horizontal input and a  $\cos wt$  vertical input, but differ in the elevation input. Figure 10 has a square wave elevation so the lissajous is split in half between two planes parallel with the datum plane. Figure 11 has a ramp elevation input which makes the lissajous spiral into the datum plane. Figure 12 has a triangle input which makes the lissajous look like a planar ellipse slicing the datum plane at an angle.



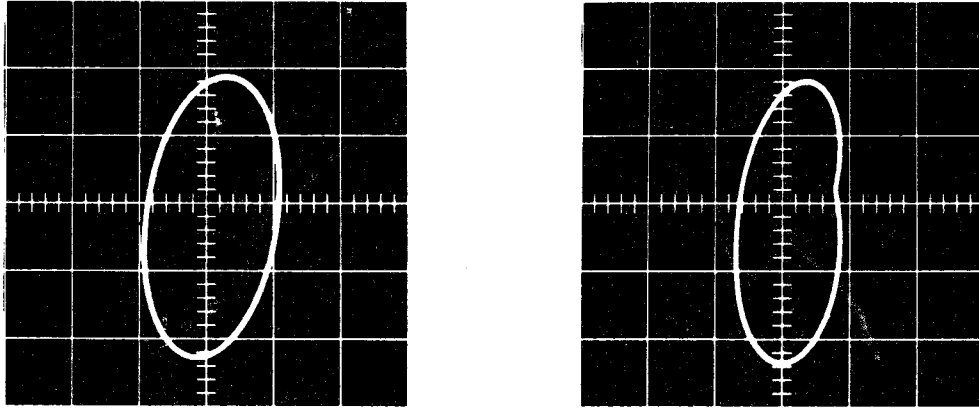


Figure 9. An Example of Exaggerated Elevation

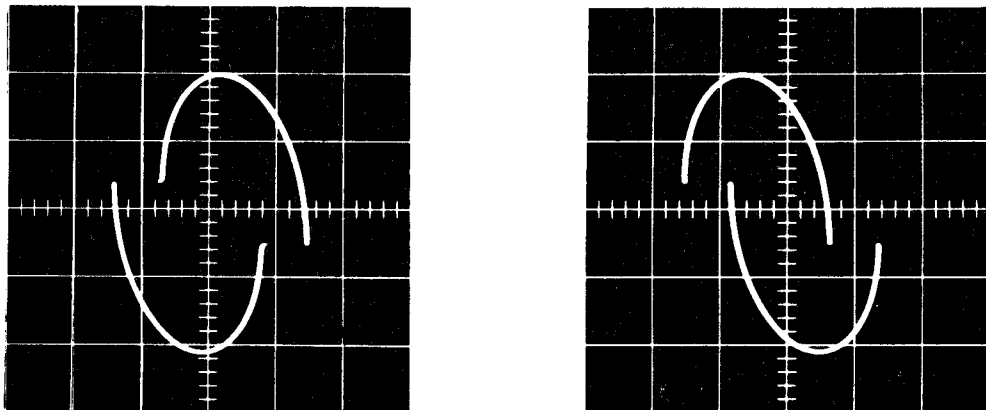


Figure 10. A Trajectory with Square Wave Elevation

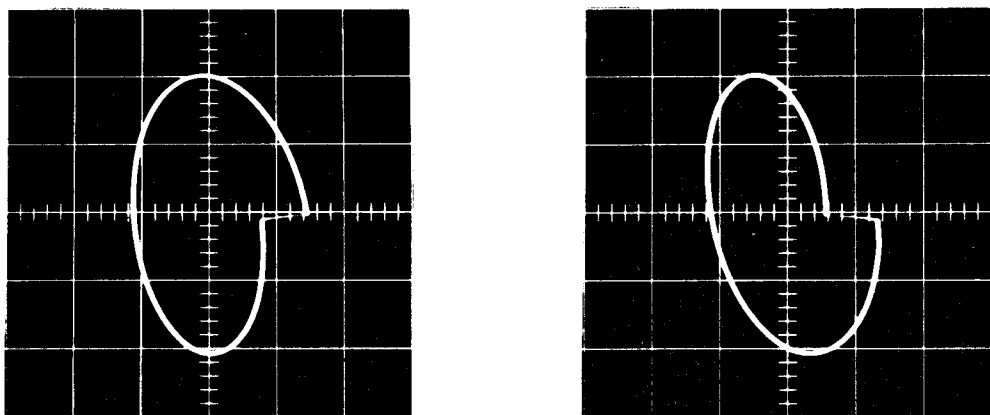


Figure 11. A Trajectory with a Ramp Elevation

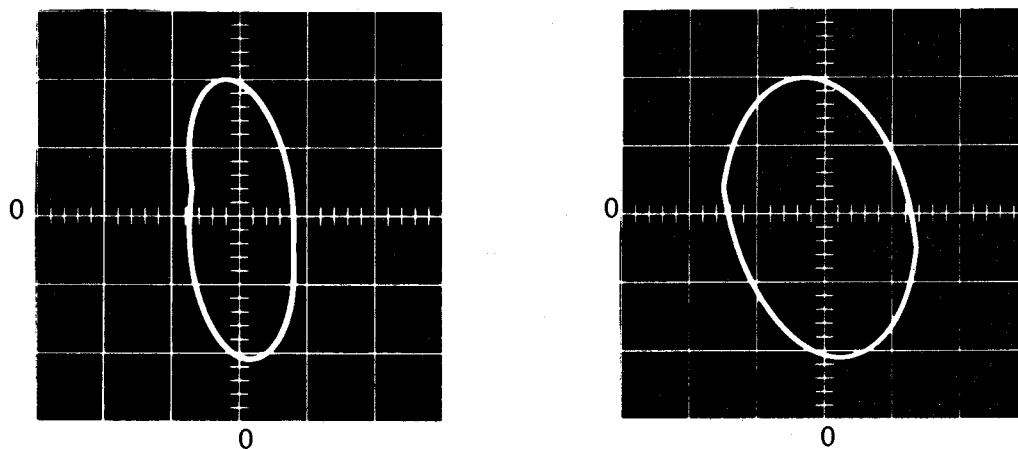


Figure 12. A Trajectory with a Triangle Elevation

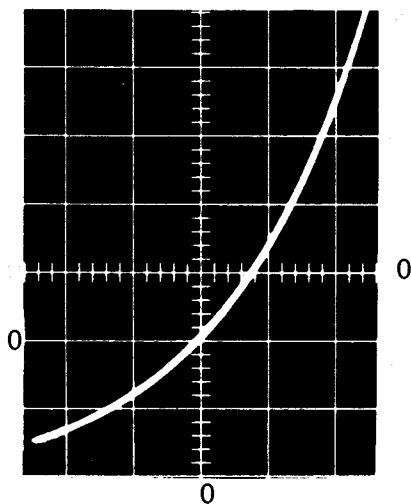


Figure 13. Diode Equation  
Approximation 1

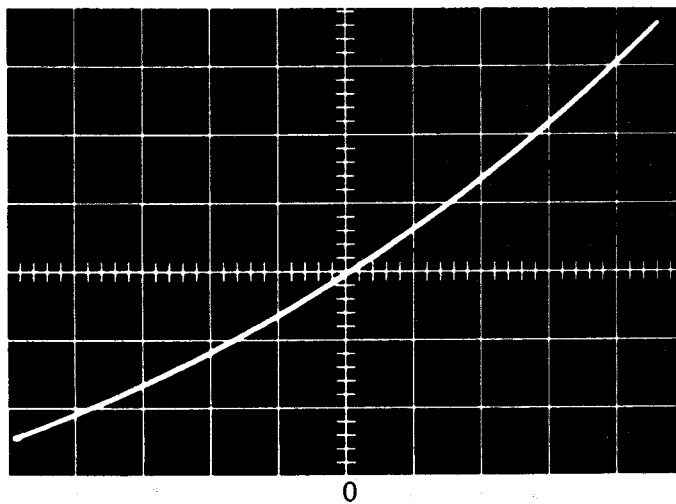


Figure 14. Diode Equation  
Approximation 2

## CONCLUSION

The purpose of this thesis was to develop an inexpensive analog system which could produce a dynamic, stereoscopic display of three variables using parallax simulation. This has been shown to be feasible by generating a diode equation approximation of the differential parallax equation which is used to modify a planar trajectory. The success of this approach is demonstrated with stereoscopic photographs which can be viewed and calibrated in three dimensions with a stereoscope.

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## PRELIMINARY STUDIES

The first idea that occurred with respect to generating the differential parallax function was to derive an analog computer transfer function that would duplicate or at least approximate the optical equations. An attempt to take the Laplace transform of the differential parallax function proved to be futile as the successive terms diverged.

The next approach was a McLaren's series approximation which could be generated term by term and summed on the analog computer. The difficulty here was at least the first four terms would be needed to produce a worst case error of ten percent or less. Fourteen integrations and a summation would be required to generate this approximation. The McLaren's series was ruled out as requiring an excessive amount of hardware and introducing too many sources of error.

A Fourier series approximation was considered next, but was discarded because voltage controlled, variable frequency generators would be required to generate each term.

Non-linear devices were investigated next. Non-linear resistors could not be located with the needed characteristics and frequency response. The diode equation was then investigated as a possible approximation of the differential parallax function. The merit of this approach became the subject of this thesis.

DERIVATION OF A DIODE EQUATION APPROXIMATION  
OF THE DIFFERENTIAL PARALLAX EQUATION

The diode equation is (1, 4):

$$I = I_s (e^{av} - 1)$$

Where  $v$  differs from the external source voltage by the  $IR$  drop across the doped crystal. The value of  $v$  is, therefore, always less than the actual voltage across the diode's terminals (1).

Solving for resistance yields:

$$R = \frac{E}{I} = \frac{E}{I_s (e^{av} - 1)}$$

Now let  $R$  be the input impedance of an operational amplifier.

$$E_o = \frac{ER_f}{R_i} = \frac{ER_f I_s (e^{av} - 1)}{E} \quad \text{(a non-inverting amplifier is assumed here)}$$

$$E_o = R_f I_s (e^{av} - 1)$$

Put the differential parallax equation into exponential form.

$$K = I_s, \quad h = v, \quad Dp = E_o$$

$$Dp = R_f K (e^{ah} - 1)$$

$$(Dp + R_f K) = R_f K e^{ah}$$

$$\ln(Dp + R_f K) = \ln(R_f K) + ah$$

Using values of  $h$  and  $D_p$  calculated from the exact  $D_p$  equation, evaluate the corresponding constants  $R_f$  and  $a$  using:

$$a = \ln\left(\frac{D_p}{KR_f} + 1\right) \quad \text{and} \quad a = a_1 = a_2 \quad \text{evaluated for:}$$

$$h_1 = 0.2F$$

$$D_{p1} = 0.25K$$

$$h_2 = -0.2F$$

$$D_{p2} = -0.1667K$$

(Figure 2)

Solving the two resulting equations simultaneously for  $R_f$  and  $a$  yields:

$$R_f = 0.5, \quad a = 2.03$$

The resulting equations are:

$$D_p = 0.5K(e^{2.03h} - 1)$$

$$\ln(D_p + 0.5K) = \ln(0.5K) + (2.03)h$$

This exponential approximation of the differential parallax function can be shown to be valid for elevations between plus and minus half the focal distance ( $\pm 0.5F$ ) by plotting  $\ln(D_p + 0.5K)$  as a function of elevation using  $* D_p = \frac{Kh}{(H-h)}$ . The result is:

$$D_p = \frac{Kh}{(H-h)} \doteq 0.5K(e^{2.03h} - 1)$$

$$\ln(D_p + 0.5K) \doteq \ln\left(\frac{Kh}{H-h} + 0.5K\right) \doteq \ln(0.5K) + (2.03)h$$

---

\* This is the exact differential parallax equation for the case where the datum plane is located at the focal distance of the viewer ( $H=F$ ).

The exponential form is derived from Figure 15 as a result of plotting  $\ln\left(\frac{Kh}{H-h} + 0.5K\right)$  for exact values of differential parallax.



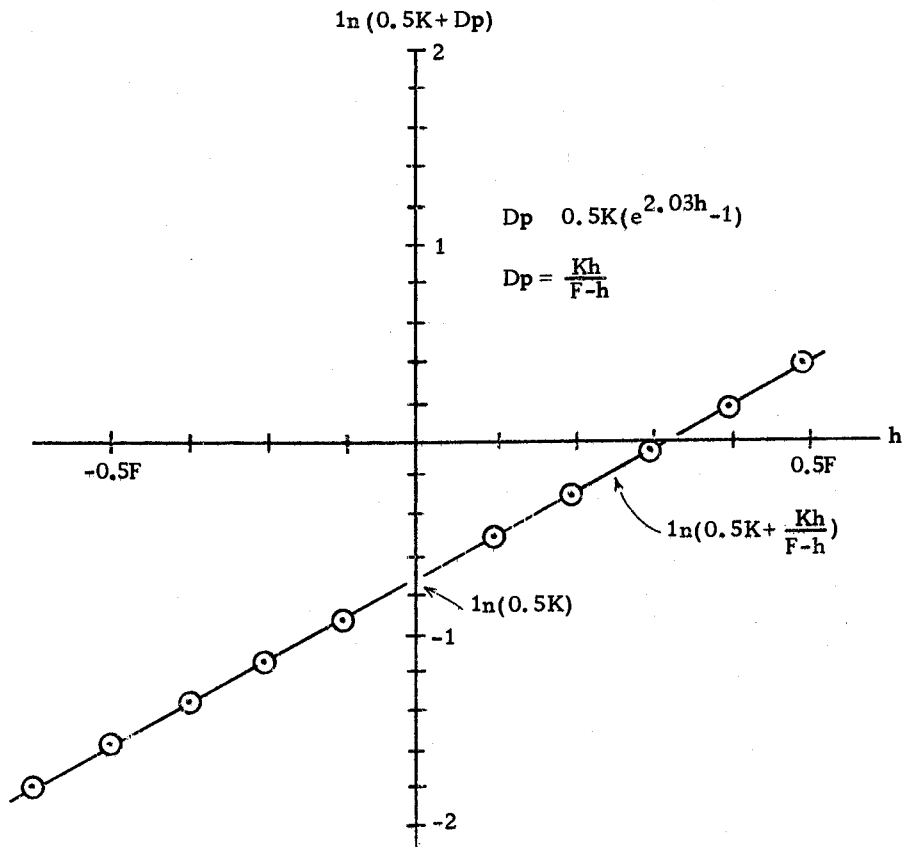


Figure 15. Demonstration of a Diode Equation Approximation for Differential Parallax

Switching Logic						
Sw. no.	1	2	3	4	5	6
	U	U	U	U	U	D -Dp
	U	U	U	D	D	U X - Dp
	D	D	D	D	D	U X + Dp output

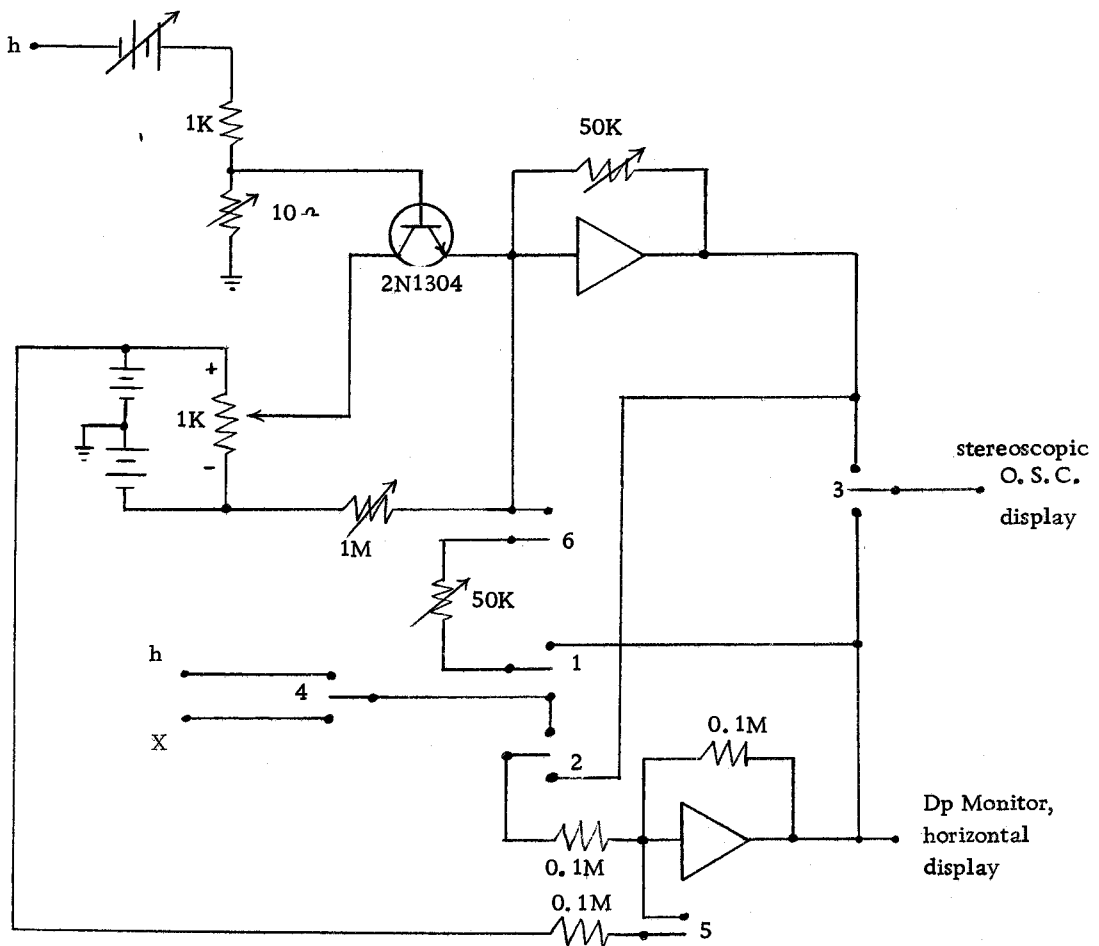


Figure 16. Laboratory Circuit Used to Simulate Stereoscopic Phase-Space Trajectories

## ADDITIONAL USES

There is evidence that the differential parallax generating circuit may have some potential beyond its use in this study. The operating characteristics can be made almost linear with proper adjustment. In this operating state, the slope is dependent on the collector voltage. If this property is used in an operational amplifier circuit, the result would be a voltage-controlled, variable gain.

An additional application for the stereoscopic display would be to combine it with a repetitive line display to generate a three-dimensional surface. This at the present is one of the most difficult functions to display effectively.