Effectiveness in Stallings' Proof of Grushko's Theorem

Paul Synhavsky

December 22, 2008

1 Grushko's Theorem

The study of free groups and free products plays an essential role in many aspects of both algebra and topology. One of the most useful theorems on this topic is the following due to I. Grushko [Gru40]:

Theorem 1.1 (Grushko). *Let*

$$\varphi : F \longrightarrow G_1 * G_2$$

be an surjective homomorphism of a free group F onto a free product of groups G_1 and G_2. Then, there exists a decomposition of F as a free product,

$$F = F_1 * F_2$$

such that $\varphi(F_1) \subseteq G_1$ and $\varphi(F_2) \subseteq G_2$.

Algebraically this is useful as it implies every finitely generated group admits a unique decomposition as a free product of indecomposable groups. As an immediate consequence we have additivity of the rank of a free product. Topologically, the connected sum operation, which can be viewed in terms of free products of fundamental groups, plays a critical role in classification of 3-manifolds.

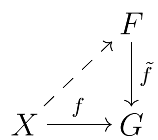
In 1940, Grushko proved Theorem 1.1 in an entirely algebraic context. However twenty years later, early in his career, J. Stallings published several topological proofs and extensions of Grushko's theorem.

In this paper we examine Stallings' most effective proof of the theorem in its original form [Sta65], as well as some of the theorem's consequences. Here we will primarily restrict ourselves to the case where F is of finite rank and the index set of the free product is finite. By induction, it suffices to consider the case of only two factors. This will allow for an effective, algorithmic decomposition of F. It should be noted that both Grushko's theorem and Stallings' proof extend to the general case of arbitrary rank, although there are details which we do not present. See [Sta65] for an outline of this extension. Instead, we focus on the effective nature and algorithmic aspects of Stallings' method.

2 Preliminaries

In this section we state some standard definitions and theorems which will be used throughout the paper.

Definition 2.1. *If X is a subset of a group F, then F is a free group with basis X if, for every group G and every function $f : X \longrightarrow G$, there is a unique homomorphism $\tilde{f} : F \longrightarrow G$ extending f such that the diagram below commutes.*



Definition 2.2. *Let X be a nonempty set and let X^{-1} be a disjoint set in one-to-one correspondence with X via $x \longmapsto x^{-1}$. Define the alphabet on X to be $X \cup X^{-1}$. A word on X of length $n \geqslant 1$ is a function $w : \{1 \ldots, n\} \to X \cup X^{-1}$. Write a word of length n as follows: if $w(i) = x_i^{e_i}$, then*

$$w = x_1^{e_1} \cdots x_n^{e_n}$$

where $x_i \in X$ and $e_i = \pm 1$. The empty word, denoted by 1, is a symbol with length defined to be 0.

Definition 2.3. *A subword of a word $w = x_1^{e_1} \cdots x_n^{e_n}$ is either the empty word or a word of the form $u = x_r^{e_r} \cdots x_s^{e_s}$, where $1 \leqslant r \leqslant s \leqslant n$. The inverse of a word $w = x_1^{e_1} \cdots x_n^{e_n}$ is $w^{-1} = x_n^{-e_n} \cdots x_1^{-e_1}$*

The words of most interest are reduced words.

Definition 2.4. *A word w is reduced if $w = 1$ or if w has no subwords of the form xx^{-1} or $x^{-1}x$, where $x \in X$.*

Proposition 2.5 (**[Rot02]**)**.** *Every word w on a set X is equivalent to a unique reduced word.*

Definition 2.6. *Let $\{G_i : i \in J\}$ be a family of groups. A free product of the G_i is a group H such that*

(i) For each index i there is an injective homomorphism $j_i : G_i \longrightarrow H$.

(ii) For every group G and every family of homomorphisms $\{f_i : G_i \longrightarrow G\}$, there is a unique homomorphism $\psi : H \longrightarrow G$ that extends every f_i.

Example 2.7. *A free group F is a free product of infinite cyclic groups.*

If F is free on a set X, then $\langle x \rangle$ is infinite cyclic for each $x \in X$. A family of homomorphisms, $f_x : \langle x \rangle \longrightarrow G$ determines a function $\tilde{f} : X \longrightarrow G$, in particular $\tilde{f}(x) = f_x(x)$, which extends to a unique homomorphism $\psi : F \longrightarrow G$ with $\psi \mid_{\langle x \rangle} = f_x$.

We have the following normal form statement for free products.

Theorem 2.8 (**Normal Form, [Rot02]**). *If $w \in *G_i$ and $w \neq 1$, then w has a unique factorization*

$$w = g_1 g_2 \cdots g_n$$

where each $g_i \neq 1$ and adjacent factors lie in distinct G_i.

If we regard the elements of $*G_i$ as reduced words, the theorem follows.

The spaces we consider will always be CW-complexes. Given a presentation of any group G, one can build a CW-complex X so that $\pi_1(X) \simeq G$. Recall a CW-complex is a structure on a Hausdorff space X given by an ascending sequence of closed subspaces

$$X^0 \subset X^1 \subset X^2 \cdots$$

which satisfy the following conditions:

1. X^0 has the discrete topology.
2. For $n \geqslant 1$, X^n is obtained from X^{n-1} by adjoining a collection of n-cells via attaching maps.
3. X is the union of the subspaces X^i for $i \geqslant 0$.
4. The space X and the subspaces X^i all have the weak topology.

The proof of the main theorem depends on an application of the Seifert-van Kampen theorem [Mas67]:

Theorem 2.9 (**Seifert-van Kampen**). *Suppose a CW complex X can be expressed as the union of two non-empty, path connected subcomplexes A and B, where $A \cap B$ is simply connected. Then*

$$\pi_1(X) \simeq \pi_1(A) * \pi_1(B).$$

3 Example

To gain a feeling of the flavor of the proof of the theorem, we give an example which demonstrates the effectiveness of Stallings' method.

In this example, we consider the free product of two groups $G_1 * G_2$ where G_1 is a cyclic group of order two with presentation $\langle a : a^2 \rangle$ and G_2 is cyclic of order three with presention $\langle b : b^3 \rangle$. Let F be a free group on two generators x and y. Consider the map

$$\begin{aligned} \varphi : F(x,y) &\longrightarrow \langle a : a^2 \rangle * \langle b : b^3 \rangle \\ \varphi(x) &\longmapsto ab^2 \\ \varphi(y) &\longmapsto aba. \end{aligned}$$

For each $i = 1, 2$, construct a two-dimensinal CW complex B_i with a single vertex v_i that is determined by the presentation of G_i. We then have $\pi_1(B_i, v_i) = G_i$. Let Y be the quotient space of $B_1 \cup B_2$ obtained by identifying the v_i to a single vertex v. Then

$$\pi_1(Y, v) = G_1 * G_2.$$

3.1 Construction of X

Construct a space X composed of two 1-spheres, one for each generator of F, identified at a base point, as follows. Let $g \in \{x, y\}$ be a generator of F, and consider the representation of $\varphi(g)$ in $G_1 * G_2$. This can be expressed as a reduced word in $\{a, b\}$. Let S_g denote the 1-sphere corresponding to g. Divide S_g into n segments by n vertices, one for each letter in $\varphi(g)$.

Now, define a map

$$f : S_g \longrightarrow Y$$

so that the restriction of f to each segment is a path in some B_i representing the generator of G_i. In this fashion, f defines a labeling scheme for X so that reading around S_g from the base point is exactly $\varphi(g) \in G_1 * G_2$. It is helpful to think of each segment in X as having a color corresponding to the B_i into which it is mapped. Say for example, the segments mapped into G_1 are red and those mapped into G_2 are blue. We color all the vertices both red and blue, simultaneously.

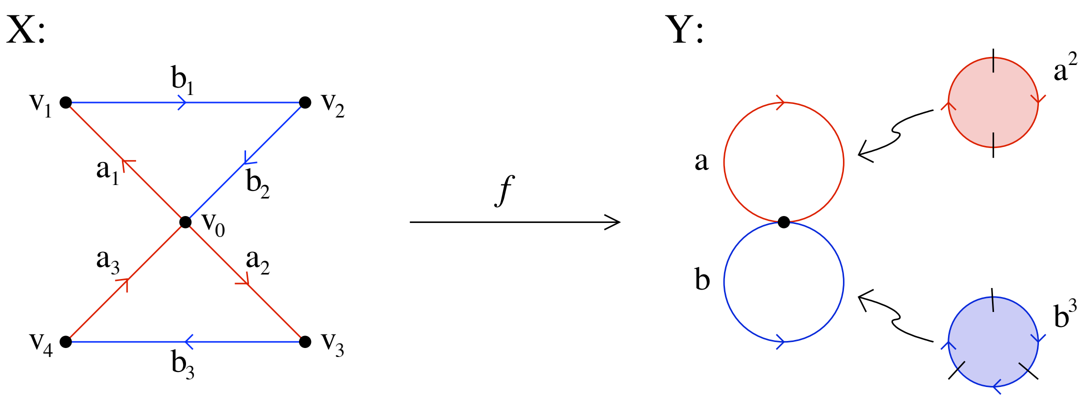


Figure 1: The spaces X and Y.

Now we have that $\pi_1(X) = F(x, y)$ and the union of the maps f defines a unique map $f : X \to Y$ such that the induced homomorphism

$$f_* : \pi_1(X) \longrightarrow \pi_1(Y)$$

is equivalent to our original surjection φ.

Let A_i denote the union of all the vertices of X along with the segments which are mapped into B_i. Then $f(A_i) \subset B_i$ and the intersection of the A_i consists of all the vertices of X.

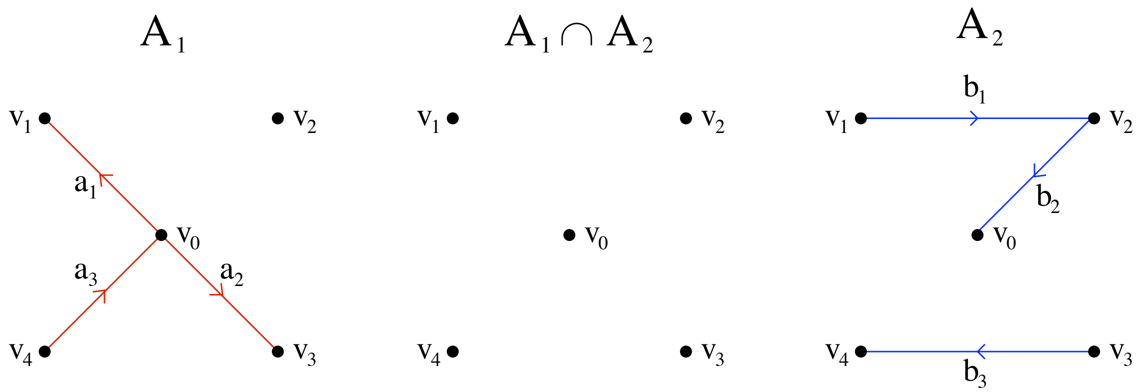


Figure 2: The subcomplexes A_1 and A_2 and their intersection.

In addition to the standard results from the previous section, in the following exposition the notion of a binding tie will be of crucial importance.

Definition 3.1. *Let X and Y be a pair of connected CW-complexes that are each the union of two subcomplexes, $X = A_1 \cup A_2$ and $Y = B_1 \cup B_2$. Let $f : X \to Y$ be a map that sends the n-skeleton of X to the n-skeleton of Y so that $f(A_i) \subseteq B_i$. A path p in X will be called a binding tie if is entirely contained in one A_i while its end points lie in different components of the intersection $A_1 \cap A_2$, and fp is homotopic to a path in $B_1 \cap B_2$ in the B_i to which it was mapped.*

One of the main results in this paper, which allowed Stallings to prove Grushko's theorem in this topological fashion is the following lemma, which we will prove in Section 5.

Lemma 3.2. *Let X and Y be a pair of connected CW-complexes that are each the union of two subcomplexes, $X = A_1 \cup A_2$ and $Y = B_1 \cup B_2$. Let $f : X \to Y$ be a map that sends the n-skeleton of X to the n-skeleton of Y such that $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective. Suppose $B_1 \cap B_2$ consists of a point so that $\pi_1(Y) \simeq \pi_1(B_1) * \pi_1(B_2)$. If $A_1 \cap A_2$ is not connected, there exists a binding tie.*

We can restate Definition 3.1 given our setting now in this example. A path g in X is called a binding tie if $g \subset A_i$ for $i \in \{1, 2\}$, has endpoints in distinct components of $A_1 \cap A_2$ and g has trivial image in $\pi_1(Y)$. Similarly, Lemma 3.2 says that if $A_1 \cap A_2$ is not connected, then there exists a binding tie.

3.2 Decomposition of F

We now outline the method in which to decompose F, and illustrate it with our example. Our strategy is to alter X and the map f so that the domain space reflects the free product structure promised in Grushko's theorem. First we construct a finite, connected, 2-dimensional

CW-complex X' containing X as a deformation retract, and a map $f' : X' \to Y$, which is an extension of f. It follows that $\pi_1(X') \simeq \pi_1(X) = F$, and thus

$$f'_* : \pi_1(X') \to \pi_1(Y)$$

is the given surjection φ.

Next we build X' so that it is the union of connected subcomplexes A'_i so that the following are satisfied

1. $A_i \subset A'_i$
2. $f'(A'_i) \subset B_i$
3. There exits a tree T containing all the vertices of X' such that $T \subset A'_i$ for each i
4. $A'_1 \cap A'_2 = T$.

It follows as a corollary to the Seifert-Van Kampen theorem that $\pi_1(X')$ is the free product of the groups $\pi_1(A'_i)$. Therefore, we let

$$F_i = \pi_1(A'_i).$$

3.3 Construction of X'

We construct X' by adding 2-cells whose boundary is divided into two arcs intersecting only at their endpoints. For each additional 2-cell, we identify one of the arcs with a path in X while leaving the other free. The 2-cells are added by a half-attaching map in an algorithmic manner so that each free edge connects two components of the intersection.

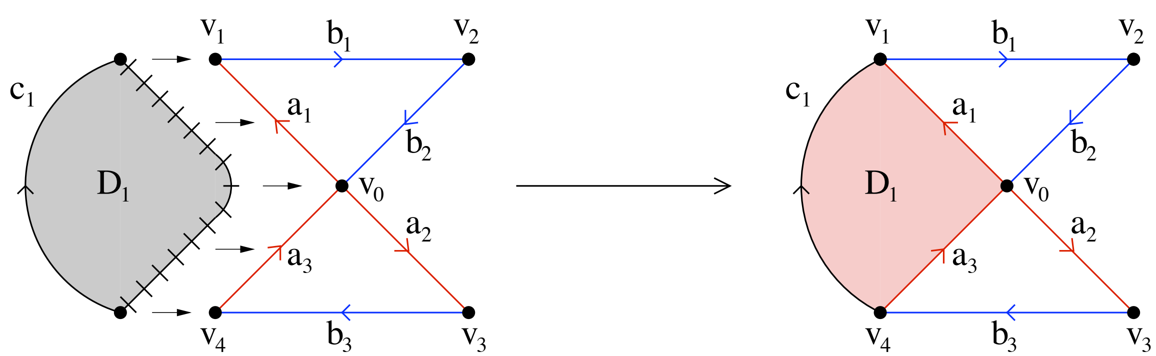


Figure 3: Adding an additional 2-cell via a half-attaching map.

In the figures below, we illustrate the construction of the complex X'. Letting c_i denote the free arc in each new 2-cell, we are able to describe the boundary word of the additional 2-cells as follows.

Noting that as it is extremely difficult to draw all of the added 2-cells, our diagram only depicts the new arcs. Each new 2-cell was added so that its free arc connected two components of the intersection.

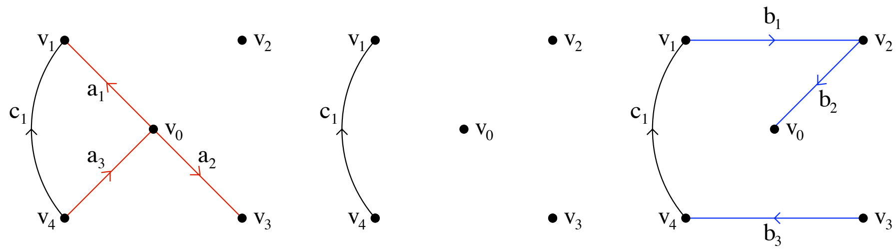


Figure 4: Adding the 2-cell D_1 as in Figure 3 with only the boundary illustrated

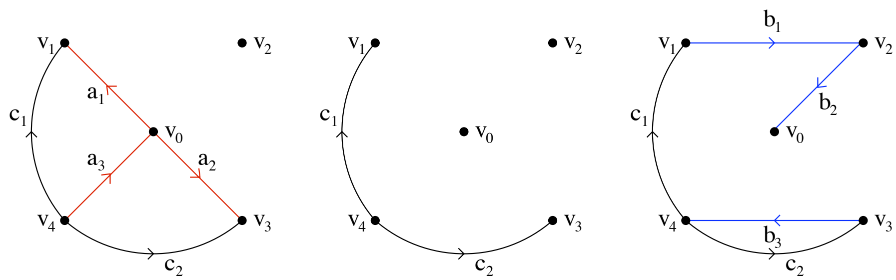


Figure 5: Adding the second 2-cell

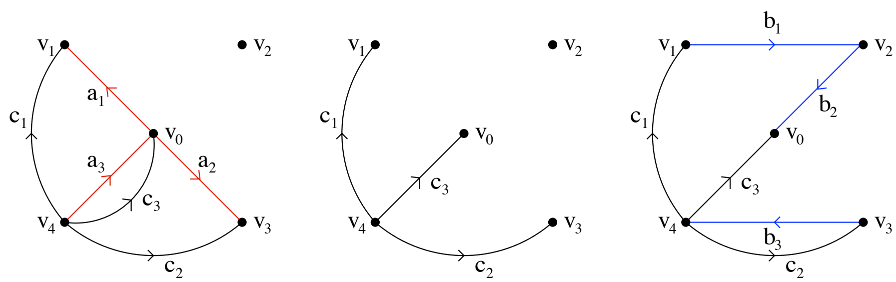


Figure 6: Adding the third 2-cell

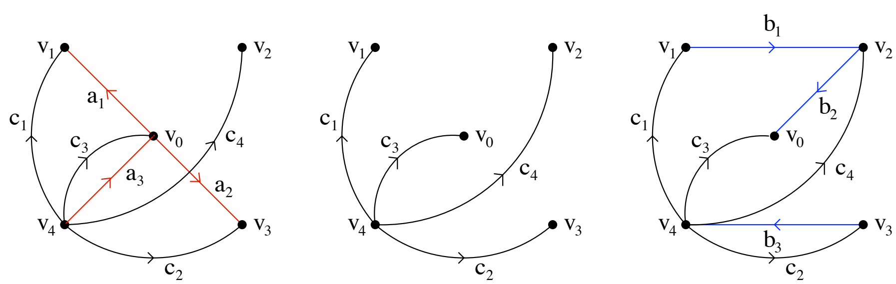


Figure 7: Adding the fourth 2-cell

3.4 Conclusion of Example

We have now built the complex X'. Since each c_i is a free edge in the boundary of a newly added 2-cell, it follows that X' contains X as a deformation retract simply by collapsing the new 2-cells to their boundary edge identified with one of the A_i. Define the subcomplexes A_i' of X' to be the union of A_i along with all the new edges and 2-cells that were attached to segments in A_i. Following our coloring scheme, we could view all the new arcs as both red and blue, the 2-cells corresponding c_1 and c_2 as red, and those corresponding to c_3 and c_4 as blue.

We now extend the map $f : X \to Y$ continuously to a map

$$f' : X' \to Y$$

as follows. Map all the new c_i arcs into the common base point of B_1 and B_2, while mapping the red 2-cells into B_1 and blue 2-cells into B_2. From here we see that conditions required above of X' are satisfied and we have a continuous extension of f. Thus,

$$F(x, y) \simeq \pi_1(X) \simeq \pi_1(X') \simeq \pi_1(A_1') * \pi_1(A_2') \simeq F_1 * F_2.$$

To find a basis for $F_1 * F_2$, view the complex X' as the union of two subcomplexes which are identified along the tree, T, consisting of all the c_i arcs and vertices in X' which are the same as those in X. Each subcomplex will only contain elements that get mapped into one of the B_i or their common basepoint. This is easily visualized by first drawing T as in Figure 8 and then adding the segments from A_i to each diagram.

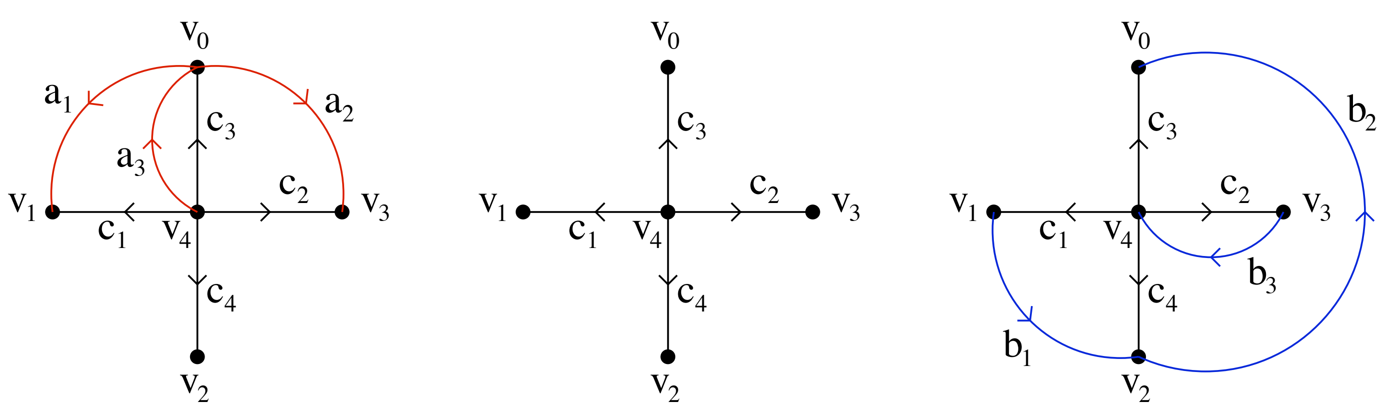


Figure 8: Figure 7 redrawn.

After a homotopy collapse, pick a generator for each side. Let

$$u = c_3^{-1} a_3 \text{ and } v = c_3^{-1} c_2 b_3 c_3.$$

Writing c_3^{-1} in terms of a and b, we have

$$u = c_3^{-1} a_3 = b_2^{-1} b_1^{-1} a_1^{-1} a_3^{-1} b_3^{-1} a_2^{-1} a_3^{-1} a_3$$

After removing the trivial subword $a_3^{-1}a_3$, and observing that

$$x^{-1} = b_2^{-1}b_1^{-1}a_1^{-1} \text{ and } y^{-1} = a_3^{-1}b_3^{-1}a_2^{-1},$$

we have $u = x^{-1}y^{-1}$. A similar expansion and reduction process gives $v = x^{-1}yx$. Let u generate F_1 and v generate F_2. Thus we have a basis for $F_1 * F_2$ such that $\varphi(F_i) \subset B_i$. Since $vu = x^{-1}$ and $u^{-1}v^{-1}u = y^{-1}$, we see that $\{u, v\}$ is indeed a basis for $F(x, y)$. Therefore we have a desired decomposition of F.

4 Terminology

Before we continue to describe Stallings' proof of Grushko's theorem in a general setting, we present some important terminology.

Definition 4.1. *A J-ad $(X; A_\alpha)$ consists of a connected, 2-dimensional complex X together with a set of subcomplexes A_α indexed over $\alpha \in J$, provided*

(i)

$$X = \bigcup_\alpha A_\alpha$$

(ii) For any pair of distinct indices α and β,

$$A_\alpha \bigcap A_\beta = \bigcap_{\lambda \in J} A_\lambda.$$

We will always assume that in a J-ad, X has a base-point which is a vertex in $\cap_{\lambda \in J} A_\lambda$.

Definition 4.2. *A Stallings' system is a pair of J-ads, $(X; A_\alpha)$ and $(Y; B_\alpha)$, together with a map $f : X \to Y$ which sends the n-skeleton of X into the n-skeleton of Y, sends the base-point of X to the base-point of Y, and for every $\alpha \in J$, $f(A_\alpha) \subset B_\alpha$.*

This definition allows us to now define a binding tie in terms of Stallings' systems. We can think of the indices $\alpha \in J$ as different colors.

Definition 4.3. *A path in X is monochromatic if it is lies entirely in one A_α.*

Definition 4.4. *A path in X is called a loop if both end points coincide in a common vertex, and will be called a tie if its end points lie in different components of the intersection $\cap A_\lambda$.*

Definition 4.5. *Given a Stallings' system $f : (X; A_\alpha) \to (Y; B_\alpha)$, a tie $g : I \to X$ is a binding tie if there exists an index α such that $g(I) \subset A_\alpha$ and the composition $fg : I \to Y$ is homotopic in B_α to path in $\cap B_\lambda$.*

Note that a binding tie is always monochromatic. We will be primarily interested in the case where the components of $\cap A_\lambda$ are points or trees. Then the resulting $\cap A'_\lambda$ will be a tree and thus have trivial fundamental group. In this setting, if $\cap B_\lambda$ is simply connected, an equivalent definition for a binding tie is as follows:

Lemma 4.6. *For a given tie $g \subset A_\alpha$, let η represent the path class of g. Suppose $\cap B_\lambda$ is simply connected. If*

$$f_\alpha : A_\alpha \to B_\alpha$$

is the restriction of f to A_α, then g is a binding tie if $f_{\alpha *}(\eta) = 1$ in $\pi_1(B_\alpha)$.

Proof. This follows from Definition 4.5 and the fact that if any path in $\cap B_\lambda$ is homotopic to a constant.

□

5 Some Important Lemmas

We are now in a position to develop the key lemmas that build toward Stallings' proof of Grushko's theorem.

The first step is to connect components of $A_1 \cap A_2$, which we will be able to do when there is a binding tie. For now, consider a Stallings' system $f : (X; A_\alpha) \to (Y; B_\alpha)$.

Let g be any loop or tie in X whose end points lie in $\cap A_\lambda$. It follows that g is homotopic to a product of paths, each of which runs across a single 1-cell. Since each 1-cell is associated with one color, this will be a product of monochromatic paths. By grouping these paths into maximal monochromatic blocks we have $g \sim g_1 \cdots g_n$ where g_i and g_{i+1} have different colors for all i. Then the end points of each g_i lie in the intersection $\cap A_\lambda$.

Choose a base point in each component of $\cap A_\lambda$ such that among these are the left end point of g_1 and right end point of g_n. For $1 \leqslant i \leqslant n$, let h_i be a path in $\cap A_\lambda$ which joins the right end point of g_i to the base point of its component. Then

$$g \sim (g_1 h_1)(h_1^{-1} g_2 h_2) \cdots (h_{n-1}^{-1} g_n).$$

Each term is monochromatic and the end points of each term are among the chosen base points. Therefore each term is either a loop or a tie. This proves that each loop or tie in X is homotopic to a product of monochromatic loops and ties whose end points are among a set of base points, one base point per component of $\cap A_\lambda$.

5.1 Existence of Binding Ties

We now address the question of when a binding tie exists and prove the following key lemma which describes the algorithm which allows us to find a binding tie.

Lemma 5.1. *Let $f : (X; A_\alpha) \to (Y; B_\alpha)$ be a Stallings' system. Suppose $\cap B_\lambda$ consists of a single point, so that $\pi_1(Y)$ is naturally the free product $*_\lambda \pi_1(B_\lambda)$. Suppose the induced map $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective. If the intersection $\cap A_\lambda$ is not connected, then there exists a binding tie.*

Proof. Suppose $\cap A_\lambda$ is not connected. Let t be a tie in X whose end points lie in different components of $\cap A_\lambda$. Let τ be the path class of t. Since $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective, there exists a loop l in X based at the base point with path class $\lambda \in \pi_1(X)$ such that $f_*(\lambda) = f_*(\tau)$. Then since $f_*(\lambda^{-1}\tau) = 1$, we have that $l^{-1}t = g$ is the desired tie.

From above, we can assume that g is a product of monochromatic loops and ties with $g \sim g_1 \cdots g_n$, where g_i and g_{i+1} have different colors for all i. If we let η represent the equivalence class of g, and η_i denote the equivalence class for each g_i, then $f_*(\eta) = f_*(\lambda^{-1}\tau) = 1$.

If g_i is a loop and $f_*(\eta_i) = 1$, then its end points coincide and $g' = g_1 \cdots g_{i-1} g_{i+1} \cdots g_n$ is a product of fewer terms. Thus, if g_i is a loop, then $f_*(\eta_i) \neq 1$. If η' denotes the class g', then

$$f_*(\eta') = f_*(\eta_1) \cdots f_*(\eta_{i-1}) \cdot 1 \cdot f_*(\eta_{i+1}) \cdots f_*(\eta_n) = f_*(\eta) = 1.$$

After a finite number of such reductions, grouping terms into monochromatic blocks when necessary, we obtain the desired g. The number of factors n will remain positive since the endpoints of g remain unchanged throughout the reductions.

Now the terms $f_*(\eta_i)$ and $f_*(\eta_i + 1)$ lie in different free factors $\pi_1(B_\lambda)$ of Y, for all i. Therefore, there must exist an index i such that $f_*(\eta_i) = 1$. Otherwise, by Theorem 2.8, we would have a reduced word of length greater than 1 in the free product representing the identity. Furthermore, g_i is not a loop, and hence a tie which is monochromatic. It is a binding tie since $f_*(\eta_i) = 1$. Since $\pi_1(Y)$ is a free product, we must have $f_*(\eta_i) = 1$ in the factor $\pi_1(B_\lambda)$ to which it is mapped.

□

5.2 Construction in the Case of Binding Ties.

Let

$$f : (X; A_\alpha) \to (Y; B_\alpha)$$

be a Stallings' system and $g : I \to A_\alpha$ be a binding tie of color α. Let D be a closed 2-cell whose boundary is divided into two one 1-cells c_1 and c_2 which intersect only at their end points. Identify c_1 with the unit interval so that g is a map of c_1 into A_α. Let X' denote the quotient space of X and D obtained by identifying each point x in c_1 with its image $g(x) \in X$.

Then X' is a CW-complex containing X as a subcomplex and two additional cells, the free edge c_2 and the 2-cell D. Clearly X is a deformation retract of X'.

Let A'_α denote the union of A_α and D along with its boundary arcs by the given identification. For any index $\beta \neq \alpha$, let A'_β denote the union of A_β and c_2, identifying the end points of c_2 with the end points of g.

Then $(X'; A'_\alpha)$ is a J-ad containing $(X; A_\alpha)$. Moreover, $\cap A'_\lambda$ consists of $\cap A_\lambda$ and the 1-cell c_2 which joins two distinct components of $\cap A_\lambda$.

We now extend the map f to a map

$$f' : (X'; A'_\alpha) \to (Y; B_\alpha).$$

Let f' map the 1-cell c_2 into the single vertex v of Y and let $f'(c_1)=fg$. Then f' maps D into B_α continuously. This is because g is a binding tie, and thus fg is homotopic to a path in $\cap B_\lambda$.

This construction may be used to connect any two components of $\cap A_\lambda$ whenever we have a binding tie. When the number of components of $\cap A_\lambda$ is finite, we may repeat this construction inductively until $\cap A_\lambda$ is connected. Thus we have the following lemma.

Lemma 5.2. *Let $f : (X; A_\alpha) \to (Y; B_\alpha)$ be a Stallings' system. Then there is a J-ad $(X'; A'_\alpha)$ containing $(X; A_\alpha)$, such that X is a deformation retract of X', and such that each component of $\cap A'_\lambda$ consists of components of $\cap A_\lambda$ joined by arcs. Additionally, there is a Stallings' system $f' : (X'; A'_\alpha) \to (Y; B_\alpha)$ which extends f, such that with respect to f' there are no binding ties.*

We are most interested in the case where the components of $\cap A_\lambda$ are points. This will allow our construction of $\cap A'_\lambda$ to be a tree which will have trivial fundamental group, leading to an application of the Seifert-van Kampen theorem, and thus the proof of Grushko's theorem.

5.3 Note on a remark of Stallings

In Stallings' paper [Sta65], he notes the following:

> In the case where F is of finite rank, and if we can effectively determine whether any element of G_i is trivial, and if for each element of $*G_i$ we can effectively find a preimage in F, then this proof (of Grushko's theorem) and the free product structure of F is effective. For, by Lemma 5.1, we can discover binding ties by algorithm whenever they exist.

However, here we show that if F has finite rank, it is possible to effectively determine when a map φ is surjective. Moreover, if φ is surjective, this method provides explicit preimages as generators for the decomposition of F.

To do this, we expand upon Lemma 5.1, illustrating exactly how to discover binding ties. Consider any map from a free group F of finite rank to an arbitrary free product of groups. Build a space X as in the example. Suppose the intersection of the A_i is not connected. Consider any two vertices v_i and v_j such that each is contained in a different component of $\cap A_i$. Now consider any one A_i based at v_i, and denote the image $f_*(\pi_1(A_i, v_i)) = S \leqslant \pi_1(B_i)$.

Suppose p is any path in A_i joining v_i to v_j. Let τ be the path class of p. We would like to know if there exists a binding tie joining v_i and v_j. Consider $f_*(p)$. If $f_*(p) \in f_*(\pi_1(A_i, v_i))$, then there exists a loop $l \in A_i$ based at an end point of p such that $f_*(l) = f_*(p)$. Thus $f_*(pl^{-1}) = 1 \in f_*(\pi_1(A_i, v_i))$, so $t = pl^{-1}$ is a binding tie joining v_i to v_j.

This Illustrates the importance of requiring that f_* be surjective. Suppose that f_* is not surjective and let q be any loop in A_i based at an end point of p. Let λ be the path class of q. Then pq^{-1} is a loop in A_i based at v_i. Thus $\tau\lambda^{-1} \in \pi_1(A_i, v_i)$.

Now $f_*(\tau\lambda^{-1}) = f_*(\tau)f_*(\lambda^{-1}) = f_*(\tau)\lambda^{-1}$, and hence $f_*(\tau) \in S\lambda^{-1}$. If λ^{-1} is not an element of S, then $S \neq S\lambda^{-1}$ and thus $f_*(\tau)$ is not contained in S. Therefore p cannot be a binding tie connecting v_i and v_j.

Repeating this process for all the components in $\cap A_i$, it is possible to determine if there are any binding ties connecting the component containing v_i to another component. This process is algorithmic in nature in that it is only necessary to check one path connecting any two components of $\cap A_i$ in each A_i. If the process terminates before the intersection of A_i is connected, by Lemma 5.1 the given map cannot be surjective. Additionally, it is effective in the sense that when a map is surjective, it produces explicit generators as we saw in Section 3.

5.4 Illustration of Surjectivity

To illustrate the effectiveness, we consider a simple example. Define a homomorphism

$$\begin{array}{rcl} \varphi : F(x,y) & \longrightarrow & F(a) * F(b) \\ \varphi(x) & \longmapsto & ab \\ \varphi(y) & \longmapsto & aba^{-1}. \end{array}$$

Build the complex X as a union of 1-spheres, S_x and S_y, as in Section 3. Now A_a consists of three three 1-cells associated with a, and four vertices, while A_b consists of two 1-cells associated with b and the same four vertices. Thus the intersection of A_a and A_b is four points.

In order to determine if φ is surjective, and if so, a decomposition of $F(x,y)$, we attempt to build the complex X'. Viewing the diagram below, we attempt join two components of $A_a \cap A_b$, say v_1 and v_3, by attaching a disc D_1 with two 1-cells as its boundary, and one boundary component identified with a tie in one A_i while leaving the other free. The free edge will lie in the intersection of A_a and A_b, which would then have one fewer component.

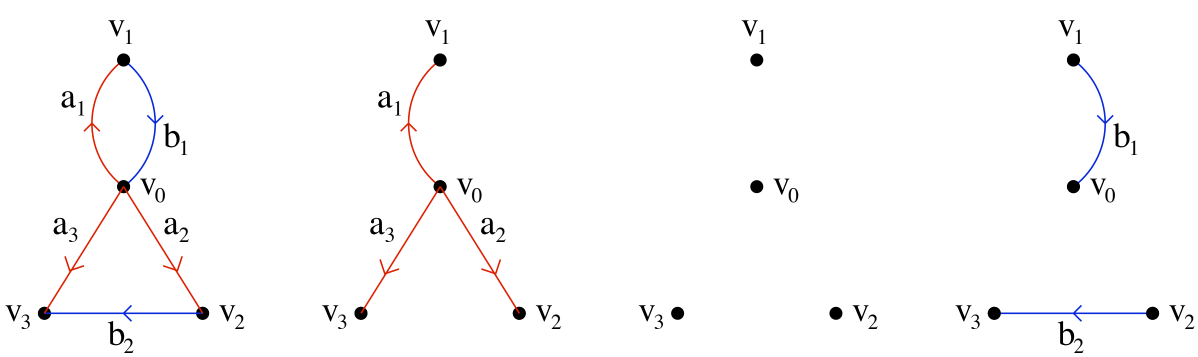


Figure 9: The complex X along with the subcomplexes A_a and A_b and their intersection.

Since only A_a has a path connecting v_1 to v_3, specifically $a_1^{-1}a_3$, we only need consider $f_*(\pi_1(A_a, v_1)) = 1$. In other words, if c_1 is any path connecting v_1 and v_3, then $c_1 a_3^{-1} a_1$ is a loop in A_a based at a_1. Thus $f_*(c_1 a_3^{-1} a_1) \in f_*(\pi_1(A_a)) = 1$. Since

$$f_*(c_1 a_3^{-1} a_1) = f_*(c_1) f_*(a^{-1}a) = f_*(c_1) a^{-1} a = f_*(c_1),$$

we have that $f_*(c_1) \in f_*(\pi_1(A_a, v_1))$ and thus $a_3^{-1}a_1$ is a binding tie.

We attempt to connect v_1 to another component in $A_a \cap A_b$. Suppose c_2 is a free edge of a disc D_2 attached to $X' = X \cup D_1$, and c_2 connects v_1 to v_0 or v_2. First consider $A'_a = A_a \cup D_1$. If c_2 connects v_1 and v_0, then $c_2 a_1$ is a loop in A'_a. Thus,

$$f_*(c_2 a_1) \in f_*(\pi_1(A'_a)) = f_*(\pi_1(A_a)) = 1.$$

Since $f_*(c_2 a_1) = f_*(c_2) f_*(a) = f_*(c_1) a$, and since $a \neq 1$, we have $f_*(c_2) \in \langle a \rangle \neq f_*(\pi_1(A_a, v_1))$. However if we check any path connecting v_1 and v_2, say $a_1^{-1}a_2$, then $c_2 a_2^{-1} a_1$ is a loop in A'_a, and as with c_1, we have that $f_*(c_2 a_2^{-1} a_1) \in f_*(\pi_1(A_a)) = 1$. Since

$$f_*(c_2 a_2^{-1} a_1) = f_*(c_2) f_*(a^{-1}a) = f_*(c_2) a^{-1} a = f_*(c_2),$$

we have that $f_*(c_2) \in f_*(\pi_1(A_a, v_1))$, and thus $a_2^{-1}a_1$ is a binding tie.

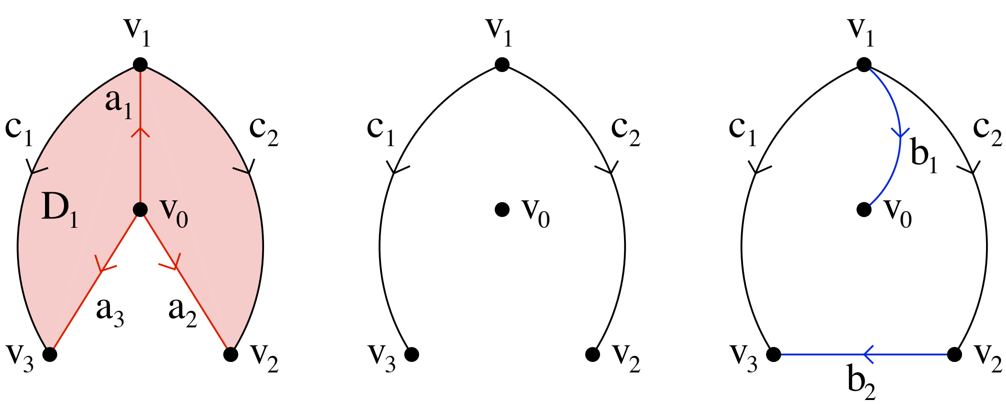


Figure 10: The subcomplexes A'_a and A'_b and their intersection after adding two 2-cells.

The complex X' now consists of the union of X along with the 2-cells D_1 and D_2. The intersection $A'_a \cap A'_b$ has only two components, v_0 and the complex consisting of three vertices and the 1-cells c_1 and c_2. It remains to find a binding tie connecting v_0 to this complex. From above, we have seen that there cannot be any such path in A'_a, so we consider A'_b.

We have

$$f_*(\pi_1((A'_b), v_1)) = f_*\langle c_2 b_2 c_1^{-1}\rangle = \langle b\rangle.$$

If c_3 is a path joining v_1 to v_0, then $c_3 b_1^{-1}$ is a loop in A_b. Since $f_*(b) \in f_*(\pi_1((A_b), v_1))$, there exists a path p with the same endpoints as b_1 and $f_*(p) = 1$. We select a loop l based at the base point in A'_b such that $f_*(l) = f_*(b_1)$. Pick $l = c_1 b_2^{-1} c_2^{-1}$. It follows that $f_*(b_1 * l^{-1}) = 1$. So we let $p = b_1 * l^{-1}$ be the path to connect. Then

$$f_*(c_3 p) = f_*(c_3) f_*(p) = f_*(c_3) \cdot 1 = f_*(c_3).$$

Hence p is a binding tie, and we have built a J-ad $(X'; A'_i)$ so that the intersection of the A_i is connected. We can now follow the method illustrated in Section 3 to find explicit preimages for each B_i.

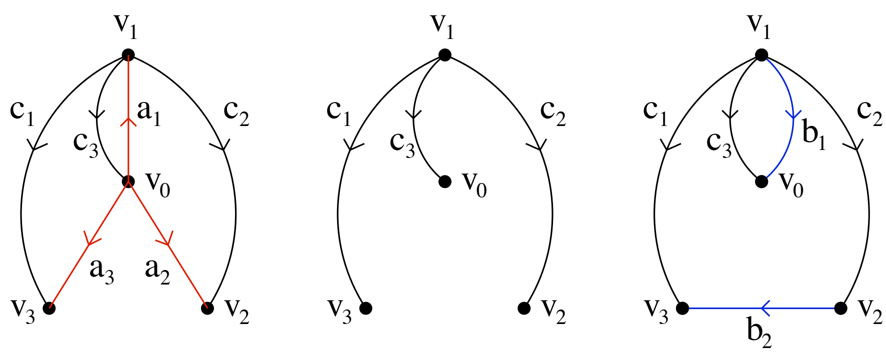


Figure 11: The connected intersection.

5.5 Illustration of Non-surjectivity

Consider a homomorphism of the groups above with $\varphi(x) \longmapsto ab$. Suppose now that $\varphi : y \longmapsto ba^{-1}$. We ask if this map is surjective. As above, we begin by attaching D_1 in the same fashion, and have a binding tie connecting two of the three vertices in the intersection of the A_i. It remains to attach a disc with free edge connecting v_1 and v_0 as there are only three vertices in the intersection $A_a \cap A_b$ under this map.

As before, $f_*(\pi_1(A_a, v_1)) = 1$. Suppose c_2 is a free edge of a disc D_2 attached to $X' = X \cup D_1$, and c_2 connects v_1 to v_0. Then $c_2 a_1$ is a loop in A'_a. So,

$$f_*(c_2 a_1) \in f_*(\pi_1(A'_a)) = f_*(\pi_1(A_a)) = 1.$$

Since $f_*(c_2 a_1) = f_*(c_2) f_*(a_1) = f_*(c_1) a$, and since $a \neq 1$, we have that

$$f_*(c_2) \in \langle a\rangle \neq f_*(\pi_1(A_a, v_1)).$$

Similarly, $f_*(\pi_1(A_b, v_1)) = 1$. Suppose c_2 is a free edge of a disc D_2 attached to $X' = X \cup D_1$, and c_2 connects v_1 to v_0. Then $c_2 b_1^{-1}$ is a loop in A'_b. So

$$f_*(c_2 b_1^{-1}) \in f_*(\pi_1(A'_b)) = f_*(\pi_1(A_b)) = 1.$$

Since $f_*(c_2 b_1^{-1}) = f_*(c_2) f_*(b_1^{-1}) = f_*(c_1) b^{-1}$, and since $b \neq 1$, we have that

$$f_*(c_2) \in \langle b \rangle \neq f_*(\pi_1(A_b, v_1)).$$

Hence there is no binding tie connecting v_1 and v_0 in either A_a or A_b. Therefore the intersection of the A'_i is not connected and there is not binding tie, so φ cannot be surjective.

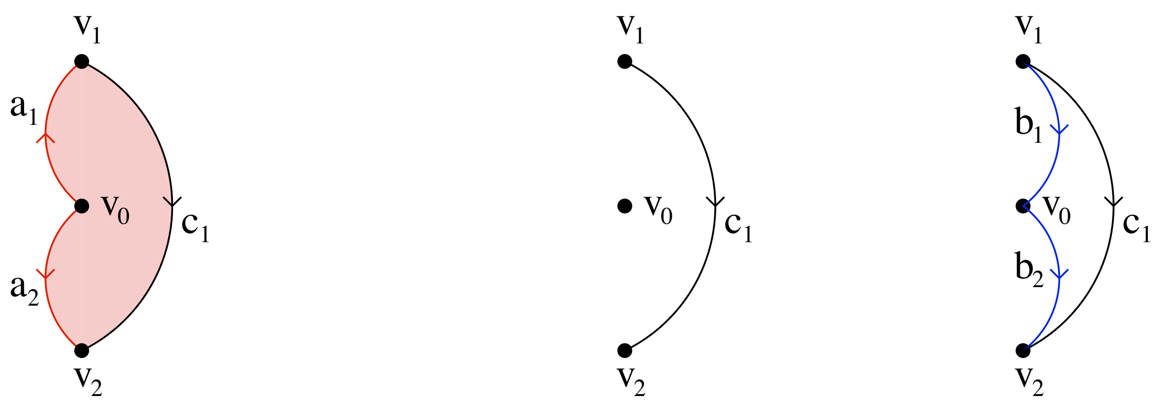


Figure 12: No additional binding tie can exist.

6 Proof of Grushko's Theorem

We are now able to prove Grushko's Theorem in the case of finite rank.

Proof of Theorem 1.1. Let B_i be a 2-dimensional complex equipped with a base point determined by the presentation of G_i. Then we have $\pi_1(B_i) \simeq G_i$. Let Y denote the quotient space obtained by identifying all the base points of G_1 and G_2. Then $\pi_1(Y) \simeq G_1 * G_2$.

A basis for F is given by $\{x, y\}$. Each generator $g \in \{x, y\}$ has a unique representation under φ as a reduced word in $G_1 * G_2$, say for example $\varphi(g) = a_1 a_1 \cdots a_n$, where each a_i belongs to one G_i. For each index generator g, let S_g denote a 1-sphere divided into n segments by n vertices starting at a base point. Label these segments in order $W_1, W_2, \ldots, W_n$. Define a continuous map $f_g : S_\gamma \to Y$ so that the restriction of f to W_i is a path in B_i representing a_i.

Repeat this for each generator, and let X denote the union of all the S_x and S_y, identifying the base points to a single vertex. Then X is a bouquet of two 1-spheres, one for each generator of F, so $\pi_1(X) \simeq F$. The union of the f_g defines a unique map $f : X \to Y$

so it follows that the induced homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is equivalent to the homomorphism φ.

To each segment W_i of S_g, associate an index such that $f(W_i) \subset B_i$. Define A_i to be the union of all vertices in X along with all the segments associated to the chosen index. Then $(X; \{A_i\})$ and $(Y; \{B_i\})$ are J-ads over the index set $\{1, 2\}$ of the free product $G_1 * G_2$, and together with f give a Stallings system. We note that the intersection $A_1 \cap A_2$ is the set of all vertices in X, and in general a given A_i is not connected.

By Lemma 5.2, there is a Stallings system $f' : (X'; A'_i) \to (Y; B_i)$ such that X is a deformation retract of X', and f' is an extension of f such that in X' there are no binding ties. It is still possible to identify $\varphi : F \to G_1 * G_2$ with the induced map $f_* : \pi_1(X') \to \pi_1(Y)$. By Lemma 5.1, the intersection $A_1 \cap A_2$ must be connected, and since the components of $A_1 \cap A_2$ are points then $A_1 \cap A'_2$ is a tree. It then follows from the Theorem 2.9 that $\pi_1(X') \simeq *\pi_1(A'_i)$. Setting $F_i = \pi_1(A'_i)$, we have

$$F \simeq \pi_1(X) \simeq \pi_1(X') \simeq \pi_1(A'_1) * \pi_1(A'_2) \simeq \pi_1(A_1) * \pi_1(A_2) \simeq F_1 * F_2.$$

This completes the proof. □

7 Consequences

The importance of Grushko's theorem is immediately realized in several corollaries.

Definition 7.1. *The rank of a free group F, denoted by rank(F), is the number of elements in a basis for F.*

Definition 7.2. *The rank of a group G, denoted by rank(G), is the minimum number of elements in a generating set for G.*

Corollary 7.3. *The rank of a free product of groups is additive. That is, for groups A and B, rank$(A * B)$ =rank(A) + rank(B).*

Proof. Consider the free product of groups $A * B$. Suppose rank $A = n$ and rank $B = m$ where n and m are finite. Then $A * B$ is finitely generated by $n + m$. Thus rank$(A * B) \leqslant$ rank(A) + rank(B).

Suppose rank $A * B = k$. Let F be a free group of order k mapping onto $A * B$.

$$\varphi : F \longrightarrow A * B$$

Then by Grushko's Theorem, we can decompose $F = F_a * F_b$ such that $\varphi(F_a) \subseteq A$ and $\varphi(F_b) \subseteq B$. Let $X = \{g_1, \ldots, g_k\}$ denote the set of basis elements for $F_n * F_m$ such that $\{\varphi(g_1), \ldots, \varphi(g_i)\}$ is a generating set for A and $\{\varphi(g_{i+1}, \ldots, \varphi(g_k)\}$ is a generating set for B. Therefore we have $i \geqslant n$ and $k - i \geqslant m$. Thus,

$$n + m \geqslant rank(A * B) = k \geqslant n + m.$$

Hence $rank(A * B) = rank$(A)+rank(B).

□

Definition 7.4. *A group G is freely indecomposable if whenever $G \neq \{1\}$ and if $G \simeq H * K$, then either H or $K = 1$.*

Example 7.5. *Some familiar classes of groups which are freely indecomposable are all finite groups and all abelian groups.*

Another important consequence is that every finitely generated group admits a unique decomposition as a free product of indecomposables. This is sometimes known as the Grushko decomposition theorem [Sta77].

Theorem 7.6. *Let G be a finitely generated group. Then it can be decomposed as free product*

$$G = G_1 * G_2 \cdots * G_r * F_s$$

where F_s is a free group of rank s, and each G_i is neither infinite cyclic nor decomposable.

The existence of such a decomposition follows directly from Grushko's theorem, while the Kurosh subgroup theorem [Kur56] implies the G_i are unique up to conjugation in G, and that the rank of F_s is determined by G.

The Kneser conjucture [Pap58] is a topological analogue of Grushko's theorem in the context of 3-manifolds.

Theorem 7.7 (Kneser). *Let M be an orientable 3-manifold, such that there exist non-trivial groups A and B so that,*

$$\pi_1(M) = A * B.$$

Then there exists an embedding of a 2-sphere without self-intersections S in M separating M into two components M' and M'', with $\pi_1(M') \simeq A$ and $\pi_1(M'') \simeq B$.

Stallings' proved Kneser's conjecture in his thesis as well as an alternate topological proof of Grushko's theorem than the one we have seen here. Although it is beyond the scope of this paper, it is easy to imagine the value of Grushko's theorem when considering the classification of 3-manifolds. For instance, if we can find a decomposition of $\pi_1(M)$ into indecomposables, then we have a decomposition of M. For an excellent introduction to this theory, see [Pap58].

8 Acknowledgments

I would like to thank Bill Bogley, my advisor, without whom this paper would not have been possible. Thanks are also due to Aaron Wood for help with the graphics.

References

[Gru40] I. A. Grushko. On the bases of a free product of groups. *Matematicheskii Sbornik*, 8:169–182, 1940.

[Kne29] Hellmuth Kneser. Geschlossene flchen in dreidimensionalen mannigfaltigkeiten. *Jahresber. Deutsch. Math. Verein.*, 38:248–260, 1929.

[Kur56] A. G. Kurosh. *The theory of groups. Vol. II.* Chelsea Publishing Company, New York, N.Y., 1956. Translated from the Russian and edited by K. A. Hirsch.

[Mas67] William S. Massey. *Algebraic topology: An introduction.* Harcourt, Brace & World, Inc., New York, 1967.

[MKS04] Wilhelm Magnus, Abraham Karrass, and Donald Solitar. *Combinatorial group theory.* Dover Publications Inc., Mineola, NY, second edition, 2004. Presentations of groups in terms of generators and relations.

[Pap58] C. D. Papakyriakopoulos. Some problems on 3-dimensional manifolds. *Bull. Amer. Math. Soc.*, 64:317–335, 1958.

[Rot95] Joseph J. Rotman. *An introduction to the theory of groups*, volume 148 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, fourth edition, 1995.

[Rot02] Joseph J. Rotman. *Advanced modern algebra.* Prentice Hall Inc., Upper Saddle River, NJ, 2002.

[Sta65] John R. Stallings. A topological proof of Grushko's theorem on free products. *Math. Z.*, 90:1–8, 1965.

[Sta77] John Stallings. Coherence of 3-manifold fundamental groups. In *Séminaire Bourbaki, Vol. 1975/76, 28 ème année, Exp. No. 481*, pages 167–173. Lecture Notes in Math., Vol. 567. Springer, Berlin, 1977.

[Sti93] John Stillwell. *Classical topology and combinatorial group theory*, volume 72 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, second edition, 1993.

[SW79] Peter Scott and Terry Wall. Topological methods in group theory. In *Homological group theory (Proc. Sympos., Durham, 1977)*, volume 36 of *London Math. Soc. Lecture Note Ser.*, pages 137–203. Cambridge Univ. Press, Cambridge, 1979.