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Abstract approved
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L. Schnirelmann's definition of density for sets of positive integers is generalized to density for subsets of certain semigroups called s-sets. Let S be an s-set. For a subset X of S and a finite subset D of S, let X(D) denote the number of elements in the set $X \cap D$. Let $\mathcal{G}$ be any family or non-empty finite subsets of S. Then the density of a subset A of S, with respect to $\mathcal{G}$, is $\alpha = \text{glb}\left\{\frac{A(G)}{S(G)} \mid G \in \mathcal{G}\right\}$.

Axioms are presented which define a family $\mathcal{F}$ of finite subsets of S to be a fundamental family on S. We require in the above definition that $\mathcal{G}$ be either a fundamental family $\mathcal{F}$ or a certain subfamily of a fundamental family called the family of all Cheo sets of the fundamental family. The two densities obtained in this way are called respectively K-density and C-density. The K-density generalizes the density of B. Kvarda and C-density generalizes those

given by F. Kasch and L. Cheo.

The theories of these densities are developed. They include in particular several inequalities for the density of the sum of two subsets of an s-set.

A GENERAL THEORY OF DENSITY IN ADDITIVE NUMBER THEORY

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A GENERAL THEORY OF DENSITY IN ADDITIVE NUMBER THEORY

CHAPTER I

INTRODUCTION

1.1. A General Density

In 1930 L. Schnirelmann [23, 24] introduced the concept of density for a subset A of the set of positive integers: For a positive integer n, let $A(n)$ denote the number of elements in A which do not exceed n. Then the density of A is defined to be

$$(1) \qquad \alpha = \operatorname{glb} \left\{ \frac{A(n)}{n} \,\middle|\, n \geq 1 \right\} .$$

We generalize this definition to subsets of an arbitrary set in the following way.

Definition 1.1. Let S be an arbitrary set. For a subset X of S, and a finite subset D of S, let $X(D)$ denote the number of elements in the set $X \cap D$. Let $\mathfrak{D}$ be any family of non-empty finite subsets of S. Then the density of a subset A of S, with respect to $\mathfrak{D}$, is

$$\alpha = \operatorname{glb} \left\{ \frac{A(G)}{S(G)} \,\middle|\, G \in \mathfrak{D} \right\}$$

This reduces to Schnirelmann's definition if we take S to be the set of positive integers, and $\mathcal{G}$ to be the family of all sets of the form $\{1, 2, \cdots, n\}$ where $n = 1, 2, \cdots$.

The main purpose of this thesis is to develop a general theory of density based upon Definition 1.1. In order to obtain a fruitful theory, it is necessary to place some restrictions on the set S and on the family $\mathcal{G}$. In doing this we have employed an axiomatic approach. Chapter 2 contains two sets of axioms. The first set requires S to be a certain type of abelian semi-group which we will call an s-set. The second set of axioms gives structure to the family $\mathcal{G}$, which is then called a fundamental family on S and is usually denoted by $\mathcal{F}$. Then the basic properties of fundamental families are developed. Some sets in a fundamental family are distinguished in that they are the intersection of all sets of the family which contain a given point of the s-set. Such sets will be called Cheo sets. The family of all Cheo sets is denoted by $\mathcal{H}$.

In Chapter 3 we give some structure theorems for s-sets and fundamental families. We also provide an extensive list of examples.

Chapters 4 and 5 describe special classes of fundamental families.

In Chapter 6 we define and develop the elementary properties of two different densities, K-density and C-density. The K-density is given by Definition 1.1 where S is taken to be an s-set and

$\mathcal{B}$ to be a fundamental family $\mathfrak{F}$ on S. The C-density is given by Definition 1.1 where S is again an s-set and $\mathcal{B}$ is the family $\mathcal{H}$ of all Cheo sets of some fundamental family on S. One of several results is that both K-density and C-density are generalizations of Schnirelmann's original definition.

In Chapters 7 and 8 we develop the theories of K-density and C-density respectively. It is the K-density which has proved most fruitful in extending known results from Schnirelmann density to the more general setting. This is discussed in Section 1.2. The C-density is important as it generalizes other work which is described in Section 1.3. The C-density also presents some interesting problems.

In Chapter 9 we show some cases where the $\alpha + \beta$ theorem holds (see (6) in Section 1.2). In Chapter 10 we discuss further problems.

The general axiomatic approach has provided new insight, and as a result these theories not only extend familiar results to new examples but provide new results for both new and old examples.

1.2. Results for the Positive Integers and Extensions to the General Theory

Most of the density results involve the density of the sum of two sets.

<u>Definition 1.2.</u> Let S be an arbitrary set with a binary operation, denoted by +, defined on it. Let A and B be two subsets of S. The sum of A and B, denoted A+B, is the set

$$A \cup B \cup \{a+b \mid a \in A, \quad b \in B\}.$$

Now let S be the set of positive integers with ordinary addition as the binary operation. Let A and B be subsets of S, and denote by α, β and γ the Schnirelmann densities, given by (1), of A, B and C = A+B respectively. The following is a list of some of the results which have been shown:

(2) If $\alpha + \beta \geq 1$, then $\gamma = 1$ (Schnirelmann[24]).

(3) $\gamma \geq \alpha + \beta - \alpha\beta$ (E. Landau[14] and Schnirelmann[24]).

(4) If $\alpha + \beta < 1$, then $\gamma \geq \beta/(1-\alpha)$ (I. Schur[25]).

(5) $\gamma \geq \min\{1, 2\alpha, 2\beta\}$ (A. Khinchine[9]).

(6) $\gamma \geq \min\{1, \alpha+\beta\}$ (H. Mann[17] and F. Dyson[4]).

We have omitted from this list the famous theorem of A. S. Besicovitch[1] because its statement involves new definitions which would be inconvenient to present now. In Chapter 10 we will discuss this result.

In selecting the axioms, particularly those for fundamental families, we have used as a guide the applicability in the general setting of the methods used in proving (2) through (6). In Chapter 7 using K-density, we show that (2) holds in complete generality and, with an added hypothesis so do (3) and (4). We are unable to obtain extensive generalizations of Results (5) and (6) for K-density. However, Mann's Theorem (6) is shown to hold in some very special cases in Chapter 9.

On the other hand, none of the Results (2)-(6) are known to hold in wide generality when they are stated in terms of C-density, although each does hold for some case. As noted in the following section, (6) has been shown to fail for some cases. In Chapter 8 we obtain some different but no less interesting results for C-density.

1.3. The Generalization of B. Kvarda and F. Kasch and Their Relation to the General Theory

The theory of K-density generalizes the work of B. Kvarda [11], who defined density according to our Definition 1.1 for subsets of I^n, where I^n is the set of all ordered n-tuples $(x_1, \cdots, x_n)$ with each x_i a non-negative integer and $x_1 + \cdots + x_n > 0$. As will be seen, I^n satisfies the axioms for s-sets. She took the family $\mathcal{B}$ to be the set of all non-empty finite subsets F of I^n which have the property that, if $(x_1, \cdots, x_n) \epsilon F$, then all $(y_1, \cdots, y_n) \epsilon I^n$ with $y_i \leq x_i$ $(i = 1, 2, \cdots, n)$ are also in F. As will be seen, this family, which we denote by $\mathcal{K}(I^n)$, satisfies the axioms for fundamental families on I^n. Thus the density of Kvarda is an example of K-density.

With these definitions Kvarda proved that statements (2) and (3) hold. This is the first generalization of the Landau-Schnirelman inequality (3). Kvarda's major work is a generalization of the theorem of Besicovitch and is discussed in Chapter 10.

Preceding Kvarda's work, F. Kasch [7] (and also L. Cheo [2] in the case $n = 2$) defined density for subsets of I^n. Here the family $\mathcal{B}$ was taken to be the family of all sets of the form $\{y \mid y = (y_1, \cdots, y_n) \epsilon I^n, \; y_i \leq x_i \; (i = 1, 2, \cdots, n)\}$ where $(x_1, \cdots, x_n)$ ranges over I^n. It is clear that this family is a subfamily of $\mathcal{K}(I^n)$, and that a set of the above form is the intersection of all $F \epsilon \; (I^n)$ such that $(x_1, \cdots, x_n) \epsilon F$. Thus Kasch's family is the family of all Cheo sets of $\mathcal{K}(I^n)$, and so his density is an example of C-density.

With this density Cheo showed that (2) holds when $n = 2$. In Chapter 8 the same method is used to show that (2) holds for arbitrary positive integral n. The truth of statements (3) or (4) is unknown. Statement (6) is shown by Cheo to fail. The Landau-Schnirelmann inequality (3) can be written $\gamma \geq \alpha + k\beta(1-\alpha)$ where $k = 1$. Kasch proved that for $n \geq 1$ the inequality holds with $k = \alpha/(2n(3(n+1))^n)$. Using a method reminiscent of that employed by Landau in the proof of (3), Kasch improved upon k, for the special case $n = 2$, by showing that the inequality holds for $k = \alpha/2$. He conjectured that for arbitrary positive integral n, the value $k = (\alpha/n)^{n-1}$ may be used. In Chapter 8, as a special case of our main result on C-density, we show that we may take $k = (1-(1-\alpha)^{1/(n-1)})/n$. In Appendix 3 it is shown that this is a better (i. e. larger) k than those mentioned.

We note that in the case $n = 1$ both Kvarda's and Kasch's definition reduces to Schnirelmann's definition. Thus both of these works generalize Schnirelmann's density theory.

1. 4. Other Generalizations and the Literature

Other generalizations of density concepts have been introduced by J. van der Corput and J. Kemperman[3] , E. Harter [6] , B. Muller [20] , and others. In general however, they do not attempt to develop a theory of density as such but restrict themselves to special problems.

There are two excellent bibliographies which cover most of the work on density in additive number theory. They can be found in H. Ostmann [21] , and more recently, in H. Mann [19] .

1. 5. Notation

We adopt the following notation: For two sets A and B, denote by $A \setminus B$ the set of all elements in A which are not in B. In the case where B reduces to a singleton $\{x\}$ we write $A \setminus B$ as $A \setminus x$.

CHAPTER 2

THE AXIOMS AND BASIC RESULTS

In this chapter we present the axioms for s-sets and the axioms for fundamental families. The fundamental properties which follow from these axioms are developed.

2. 1. Axioms for s-sets

Throughout this section, unless otherwise specified, S will denote a non-empty subset of an abelian group Γ. The operation in Γ is denoted by $+$ and the identity element by 0.

Definition 2. 1. For x and y in Γ, we write $x \prec y$ (or $y \succ x$) whenever $y - x \in S$.

Definition 2. 2. For $x \in S$, let $L(x)$ denote the set of all $y \in S$ for which $y \prec x$ or $y = x$. We call $L(x)$ the lower set of x with respect to S.

The set S is called an s-set if the following three axioms are satisfied.

Axiom s. 1. S is closed under $+$.

Axiom s. 2. $0 \notin S$.

Axiom s. 3. $L(x)$ is finite for each $x \in S$.

<u>Theorem 2. 1.</u> Axioms s. 1 and s. 2 are equivalent to the statement that the relation $\prec$ is a partial ordering of Γ, i. e., that $\prec$ is irreflexive and transitive.

Proof. To show irreflexivity, notice that if $x \prec x$ for some $x \in \Gamma$, then $0 = x - x \in S$ contrary to Axiom s. 2. To show transitivity, let x, y and z be elements of Γ with $x \prec y$ and $y \prec z$. Then $y - x$ and $z - y$ are in S, and by Axiom s. 1, $z - x = (z-y) + (y-x) \in S$, and hence $x \prec z$.

Now suppose that $\prec$ is a partial order relation. We prove that S satisfies Axioms s. 1 and s. 2. Since $\prec$ is irreflexive, we have immediately that $0 \notin S$. Now let x and y be in S. We have $x \prec x+y$ and $x+y \prec x+x+y$, and since $\prec$ is transitive, $x \prec x+x+y$. Thus, by the definition of $\prec$, $x+y = (x+x+y) - x \in S$, and we have shown that S is closed under $+$.

Thus the Axioms s. 1-3 can be stated equivalently as the following: $\prec$ is transitive; $\prec$ is irreflexive; $L(x)$ is finite for each $x \in S$.

2. 2. An Equivalent Definition for s-sets

We give another equivalent definition for s-sets which doesn't require the concept of group or order relation. A non-empty set S is called an s-set if the following three conditions hold:

(i) There is a binary operation + defined on S which is associative, commutative, and satisfies the cancellation rules.

(ii) If $x+y$ are in S, then $x+y \neq x$.

(iii) For each $x \in S$, the equation $y+z = x$ has at most finitely many solution pairs y, z in S.

We show that any set S satisfying these conditions also satisfies Axioms s. 1-3. Condition (i) says that S is an abelian cancellation semigroup. By the same method which is used to imbed the set of positive integers in the abelian (additive) group of all integers, so can such a semigroup be imbedded in an abelian group. Thus S is a closed subset of an abelian group, say, Γ. If 0 denotes the identity in Γ, then condition (ii) implies that $0 \notin S$. Now, since S is a subset of an abelian group, we may define $\prec$ and $L(x)$. For each $y \in L(x)$ with $y \prec x$, we have $x - y \in S$. Then $y, x-y$ is a solution pair in S of the equation $y+z=x$. Since, by condition (iii), there are only finitely many such solutions, we obtain that $L(x)$ is finite. Thus S is an s-set.

Conversely, it may be shown that any s-set satisfies the above conditons.

2.3. Some Properties of s-sets

The following theorem lists a few of the simple properties of s-sets.

Theorem 2.2. Let S be an s-set. Let x and y denote arbitrary elements of S and m and n arbitrary positive integers. Then

(i) $nx = \sum_{i=1}^{n} x$ is an element of S;

(ii) $n(x+y) = nx + ny$;

(iii) $(n+m)x = nx + mx$;

(iv) each element of an s-set is of infinite order;

(v) any s-set is infinite.

Proof. Properties (i), (ii) and (iii) are trivial, property (iv) follows from (i) and Axiom s. 2, and property (v) follows from (iv) and the fact that any s-set is non-empty.

Definition 2.3. Let X be a subset of an s-set. A point $x \in X$ is called a minimal point of X if $X \cap L(x) = \{x\}$. The set of all minimal points of X is denoted $Min(X)$.

Theorem 2.3. If a subset X of an s-set is non-empty then $Min(X)$ is also non-empty.

Proof. Assume that X is non-empty and that $Min(X)$ is empty. Let x_1 be an arbitrary point of X. Since $Min(X)$ is empty, there exists an $x_2 \neq x_1$, with $x_2 \in X \cap L(x_1)$. But then there must be an $x_3 \neq x_2$ with $x_3 \in X \cap L(x_2)$. Continuing, we obtain a sequence $x_1 \succ x_2 \succ \cdots$. The members of this sequence are pairwise distinct and they all belong to $L(x_1)$ which implies that $L(x_1)$ is infinite, contrary to s. 3.

The following theorem shows that Definition 2. 3 agrees with the standard definition for minimal point of a partially ordered set.

Theorem 2. 4. If X is a non-empty subset of an s-set and $y \in X$, then there is an $x \in \mathrm{Min}(X)$ such that $x \prec y$ or $x = y$.

Proof. If $y \in \mathrm{Min}(X)$ then take $x = y$. Hence suppose that $y \notin \mathrm{Min}(X)$. Let $x \in \mathrm{Min}(L(y) \cap X)$. Clearly $x \prec y$ or $x = y$ (since $x \in L(y)$), and we show that $x \in \mathrm{Min}(X)$. Now $x \in \mathrm{Min}\,(L(y) \cap X)$ implies $L(x) \cap (L(y) \cap X) = \{x\}$. By the transitivity of $\prec$ we have, since $x \in L(y)$, that $L(x) \subseteq L(y)$. Hence

$$L(x) \cap (L(y) \cap X) = L(x) \cap X = \{x\}$$

and $x \in \mathrm{Min}(X)$. The proof is complete.

As an immediate consequence of Theorem 2. 4 we have

Theorem 2. 5. If $\mathrm{Min}(X) = \{y\}$, then $y \prec x$ for each $x \in X \setminus y$.

Definition 2. 4. Let S be an s-set and let $x \in S$. Denote by $U(x)$ the set of all $y \in S$ with $x \prec y$.

It is important to observe that x is not a member of $U(x)$ whereas $x \in L(x)$.

Definition 2. 5. Let X be a subset of an s-set. A point $x \in X$ is called a maximal point of X if $X \cap U(x) = \phi$. The set of all maximal points of X is denoted $\mathrm{Max}(X)$.

Unlike $\mathrm{Min}(X)$, $\mathrm{Max}(X)$ can be empty for non-empty X, e. g. if $X = S$. However we have the following

Theorem 2. 6. If X is a non-empty and finite subset of an s-set, then $\mathrm{Max}(X)$ is non-empty.

Proof. The proof is analagous to that of Theorem 2. 3. In this case we obtain a sequence $x_1 \prec x_2 \prec \cdots$ in X contrary to the assumption that X is finite.

The following theorem is analagous to Theorem 2. 4.

Theorem 2. 7. If X is a non-empty and finite subset of an s-set, and $x \in X$, then there is a $y \in \mathrm{Max}(X)$ such that $x \prec y$ or $x = y$.

Proof. If $x \in \mathrm{Max}(X)$, then take $y = x$. If $x \notin \mathrm{Max}(X)$, then $X \cap U(x)$ is non-empty and finite. Let $y \in \mathrm{Max}(X \cap U(x))$. Now $y \in U(x)$ and so, by the transitivity of $\prec$, we have

$U(y) \subset U(x)$. We have

$$\begin{aligned} \phi &= (X \cap U(x)) \cap U(y) \\ &= X \cap (U(x) \cap U(y)) \\ &= X \cap U(y), \end{aligned}$$

and so $y \in \mathrm{Max}(X)$. The theorem follows.

Theorem 2.8. If X is finite and $\mathrm{Max}(X) = \{y\}$, then $x \prec y$ for each $x \in X \setminus y$.

Proof. This is an immediate consequence of the preceding theorem.

2.4. Axioms for Fundamental Families

Definition 2.6. For an arbitrary set S let $\mathcal{D} = \mathcal{D}(S)$ denote the family of all non-empty finite subsets of S.

Definition 2.7. Let $\mathfrak{F}$ be an arbitrary subfamily of $\mathcal{D}$ and let F be a set in $\mathfrak{F}$. A point $x \in F$ is called a corner point of F if either $F = \{x\}$ or $F \setminus x \in \mathfrak{F}$. The set of all corner points of F is denoted F^*.

Notice that F^* is dependent upon the family $\mathfrak{F}$. It can happen that a set F may be in two different families and have entirely different sets of corner points relative to the two families.

Now let S be an s-set. A non-empty family $\mathcal{F} \subseteq \mathcal{P}(S)$ will be called a fundamental family on S if the following four axioms are satisfied:

Axiom f. 1. For each $x \in S$ there is an $F \in \mathcal{F}$ with $x \in F$.

Axiom f. 2. The union of any non-empty finite subfamily of $\mathcal{F}$ is a set in $\mathcal{F}$.

Axiom f. 3. The intersection of any non-empty subfamily of $\mathcal{F}$ is a set in $\mathcal{F}$, provided the intersection is non-empty.

Axiom f. 4. If $F \in \mathcal{F}$, then $\mathrm{Max}(F) \subseteq F^*$.

Definition 2. 8. The ordered pair $(S, \mathcal{F})$ is called a density space whenever S is an s-set and $\mathcal{F}$ is a fundamental family on S.

The family $\mathcal{K}(I^n)$ of the Introduction is an example of a fundamental family on the s-set I^n. Thus $(I^n, \mathcal{K}(I^n))$ is an example of a density space. Chapter 3 contains an extensive list of examples of density spaces. In the present chapter we make no further reference to examples but the reader may find it helpful to keep such an example in mind.

2. 5. Remarks on the Axioms

Axioms s. 1-3 are consistent because of the examples of

Section 3.1. The treatment of other logical questions concerning these axioms such as independence, redundancy and categoricalness is not difficult and is omitted.

On the other hand, we consider the following questions concerning Axioms f.1-4: (1) Are they consistent? (2) Are they independent? (3) Are they categorical? (4) Are there any apparent redundancies in the axioms?

The examples of Section 3.2 show that question (1) can be answered in the affirmative. We answer question (2) in Appendix 1 where we show the Axioms f.1-4 are independent.

We do not give a detailed discussion of categoricalness. However, the diversity of the examples of fundamental families given in Section 3.2 is evidence that the axioms are not categorical. It is interesting that, if Axiom f.4 were replaced by

Axiom f.4' If $F \in \mathcal{F}$, then $\mathrm{Max}(F) = F^*$,

then the axioms would be categorical since, in this case, precisely one subfamily of $\mathcal{D}$ would serve as an example. This is part of Theorem 3.8.

The following theorem answers question (4).

Theorem 2.9. (i) Axiom f.2 is redundant in that it is sufficient to state that the union of any two sets of $\mathcal{F}$ is a set of $\mathcal{F}$.
(ii) Axiom f.3 is redundant in that it is sufficient to state that the intersection of any two sets of $\mathcal{F}$, provided it is non-empty, is

a set of $\mathcal{F}$.

Proof. Part (i) follows immediately from the usual induction. To prove (ii), first notice that by induction we can extend this statement to closure of $\mathcal{F}$ under non-empty intersections of non-empty finite subfamilies of $\mathcal{F}$. Now let F_δ be a set in $\mathcal{F}$ for each δ in some non-empty index set Δ, and suppose that $\cap \{F_\delta \mid \delta \in \Delta\}$ is non-empty. Let δ_0 be fixed, $\delta_0 \in \Delta$, and form the family $\{F_\delta \cap F_{\delta_0} \mid \delta \in \Delta\}$. This is a finite class of sets of whose intersection is non-empty. Thus $\cap \{F_\delta \cap F_{\delta_0} \mid \delta \in \Delta\}$ is a set in $\mathcal{F}$, and since $\cap \{F_\delta \mid \delta \in \Delta\} = \cap \{F_\delta \cap F_{\delta_0} \mid \delta \in \Delta\}$, we have the desired result.

2.6. Properties of Fundamental Families

The following theorems describe the structure of a fundamental family. Throughout this section $(S, \mathcal{F})$ is an arbitrary density space (Definition 2.8).

Theorem 2.10. For each $F \in \mathcal{F}$, the set of corner points F^* of F is non-empty.

Proof. The theorem follows immediately from Axiom f. 4 and Theorem 2.6.

Definition 2.9. Let $x \in S$. Denote by $[x]$ the intersection of all F such that $x \in F \in \mathcal{F}$. We call $[x]$ the Cheo set of $\mathcal{F}$ determined by x.

<u>Theorem 2.11.</u> For each $x \in S$ we have $[x] \in \mathcal{F}$.

Proof. By Axiom f.1 there is at least one F with $x \in F \in \mathcal{F}$. Since $\cap \{F \mid x \in F \in \mathcal{F}\}$ is non-empty, we have $[x] \in \mathcal{F}$ by Axiom f.3.

<u>Theorem 2.12.</u> If $x \in F \in \mathcal{F}$, then $[x] \subseteq F$.

Proof. This is an immediate consequence of Definition 2.9.

<u>Theorem 2.13.</u> For each $x \in S$ we have $[x]^* = \{x\}$.

Proof. Assume y is a corner point of $[x]$ and $y \neq x$. Then $F = [x] \setminus y \in \mathcal{F}$. Hence $x \in F$, and by Theorem 2.12 we have $[x] \subseteq F$, a contradiction. Thus, there is no corner point of $[x]$ different from x. Applying Theorem 2.10 we obtain the desired result.

In view of Theorem 2.13, we can restate Theorem 2.12 thus: If $[x]^* \subseteq F \in \mathcal{F}$, then $[x] \subseteq F$. The following theorem is a generalization.

<u>Theorem 2.14.</u> If F and G are in $\mathcal{F}$, and $F^* \subseteq G$, then $F \subseteq G$.

Proof. Let $H_0 = \cup\{[x] \mid x \in F^*\}$. By Axiom f. 2, $H_0 \in \mathcal{F}$. By Theorem 2. 12, we have $H_0 \subseteq F$. Since $F^* \subseteq G$, we have $H_0 \subseteq G$ also. If $H_0 = F$, then the theorem is proved. Suppose, on the other hand, that there is an $x_1 \in F \setminus H_0$ and let $H_1 = H_0 \cup [x_1]$. For the same reasons as before $H_1 \in \mathcal{F}$ and $H_1 \subseteq F$. If $H_1 \neq F$, then let $x_2 \in F \setminus H_1$ and $H_2 = H_1 \cup [x_2]$. Again $H_2 \in \mathcal{F}$ and $H_2 \subseteq F$. Continuing in this manner we must eventually arrive at an $x_k \in F \setminus H_{k-1}$ such that $H_k = H_{k-1} \cup [x_k] = F$. We show that x_k is a corner point of F, i. e., $F \setminus x_k \in \mathcal{F}$. For

$$
\begin{aligned}
F \setminus x_k = H_k \setminus x_k &= (H_{k-1} \cup [x_k]) \setminus x_k \\
&= H_{k-1} \cup ([x_k] \setminus x_k)
\end{aligned}
$$

which is in $\mathcal{F}$ by Theorem 2. 13 and Axiom f. 2. Thus $x_k \in F^*$, but this is a contradiction since x_k is also in $F \setminus H_{k-1}$ which is disjoint from F^* since $F^* \subseteq H_0 \subseteq H_{k-1}$. Thus the assumption that H_0 is different from F leads to a contradiction and the theorem is proved.

As an immediate consequence of Theorem 2. 14 we have

Theorem 2. 15. If F and G are in $\mathcal{F}$, then $F = G$ if and only if $F^* = G^*$.

Thus a fundamental set is uniquely determined by its set of corner points. In particular, if a set $F \in \mathcal{F}$ has the single corner point x, then F is identical with $[x]$.

The next theorem offers a useful decomposition of a set in $\mathcal{F}$.

<u>Theorem 2.16.</u> For each $F \in \mathcal{F}$, we have

$$F = \cup \{[x] \mid x \in F^*\}.$$

Proof. If $x \in F^*$, then $x \in F$, and so by Theorem 2.12, we have $[x] \subseteq F$. Thus $\cup \{[x] \mid x \in F^*\} \subseteq F$. On the other hand, by Axiom f.2, $\cup \{[x] \mid x \in F^*\} \in \mathcal{F}$. Clearly $F^* \subseteq \cup \{[x] \mid x \in F^*\}$ so that, by Theorem 2.14, we have $F \subseteq \cup \{[x] \mid x \in F^*\}$. The proof is complete.

<u>Theorem 2.17.</u> Let X be a non-empty finite subset of S. Then the set $F = \cup \{[x] \mid x \in X\}$ is in $\mathcal{F}$ and furthermore $F^* \subseteq X$.

Proof. The first part is an immediate consequence of Axiom f.2. For the second part, suppose there is a point $y \in F^*$ such that $y \notin X$. By the definition of F there must be an $x \in X$ such that $y \in [x]$. Let $F_1 = F \setminus y$ which is in $\mathcal{F}$. Since $x \in F_1$ we have, by Theorem 2.12, that $[x] \subseteq F_1$ which implies that $y \in F_1$, a contradiction.

<u>Theorem 2.18.</u> For each $x \in S$, we have $[x] \subseteq L(x)$.

Proof. From Theorem 2.6, Axiom f.4 and Theorem 2.13 we obtain

$$\phi \neq \mathrm{Max}([x]) \subseteq [x]^* = \{x\} .$$

Thus $\mathrm{Max}([x]) = \{x\}$. By Theorem 2.8, we have $y \prec x$ for each $y \in [x] \setminus x$. Thus for each $y \in [x]$, we have $y \in L(x)$ and the proof is complete.

Theorem 2.19. Let $F \in \mathcal{F}$. If x and y are distinct points of F^*, then $x \notin [y]$.

Proof. By Theorem 2.16 we have $F = \cup \{[z] \mid z \in F^*\}$. If $x \in [y]$, then by Theorem 2.12, $[x] \subseteq [y]$ and so $F = \cup \{[z] \mid z \in F^* \setminus x\}$. But Theorem 2.17 implies that $F^* \subseteq F^* \setminus x$, a contradiction. Thus $x \notin [y]$.

Definition 2.10. A point $x \in S$ is an essential point of the density space $(S, \mathcal{F})$ if $[x] = \{x\}$.

The essential points of $(S, \mathcal{F})$ are of importance in the proofs of some of the density theorems of Chapters 6, 7 and 8. The following theorem characterizes the set of essential points of $(S, \mathcal{F})$.

Theorem 2.20. Let $x \in S$. Then x is an essential point if and only if there is an $F \in \mathcal{F}$ such that $x \in \mathrm{Min}(F)$.

Proof. If x is essential, then take $F = [x]$. Clearly

$x \in \mathrm{Min}(\{x\}) = \mathrm{Min}([x]) = \mathrm{Min}(F)$.

Now suppose there is an $F \in \mathcal{F}$ such that $x \in \mathrm{Min}(F)$. By Theorem 2.12, we have $[x] \subseteq F$, and by Theorem 2.18, $[x] \subseteq L(x)$. Thus

$$\phi \neq [x] \subseteq L(x) \cap F = \{x\}$$

since x is a minimal point of F (Definition 2.3). Hence $[x] = \{x\}$ and x is essential.

Moreover, we have the following theorem.

<u>Theorem 2.21.</u> If $x \in \mathrm{Min}(S)$, then x is an essential point.

Proof. If $x \in \mathrm{Min} S$, then $\{x\} = L(x) \cap S = L(x)$. Hence as before $\phi \neq [x] \subseteq L(x) = \{x\}$ and the theorem follows.

<u>Definition 2.11.</u> Let $\mathcal{K} = \mathcal{K}(S)$ denote the family of all sets $F \in \mathcal{D}(S)$ with the property that if $x \in F$, then $L(x) \subseteq F$.

In Section 3.2 it is shown that $\mathcal{K}$ and $\mathcal{D}$ are fundamental families on S. We mention this now because of the following theorem which shows that $\mathcal{K}$ and $\mathcal{D}$ occupy special places in the class of all fundamental families on S.

<u>Theorem 2.22.</u> For any fundamental family $\mathcal{F}$ on S we have $\mathcal{K} \subseteq \mathcal{F} \subseteq \mathcal{D}$.

Proof. By definition $\mathcal{F} \subseteq \mathcal{D}$. Let $K \in \mathcal{K}$. For $x \in S$, let $[x]$ denote the Cheo sets of $\mathcal{F}$ determined by x (as opposed to the Cheo set of any other family). Define $F = \cup \{[x] \mid x \in K\}$. We have by Axiom f. 2, that $F \in \mathcal{F}$, and we show that $F = K$. By Theorem 2. 18, we have $[x] \subseteq L(x)$. Thus, for each $x \in K$, we have $[x] \subseteq L(x) \subseteq K$, and so $F \subseteq K$. On the other hand, we have for each $x \in K$, that $x \in [x] \subseteq F$, and so $K \subseteq F$. Thus $K = F$, $K \in \mathcal{F}$ and $\mathcal{H} \subseteq \mathcal{F}$.

Theorem 2. 23. For each $x \in S$, we have $L(x) \in \mathcal{F}$.

Proof. In view of Theorem 2. 22 it is sufficient to prove that $L(x) \in \mathcal{H}$. To prove this we need to show for each $y \in L(x)$, that $L(y) \subseteq L(x)$. This follows immediately from the transitivity of the relation $\prec$.

Theorem 2. 24. Let $\{\mathcal{F}_\delta \mid \delta \in \Delta\}$ be an arbitrary non-empty class of fundamental families on S. Then $\cap \{\mathcal{F}_\delta \mid \delta \in \Delta\}$ is again a fundamental family on S.

Proof. Let $\mathcal{F} = \cap \{\mathcal{F}_\delta \mid \delta \in \Delta\}$. By Theorem 2. 23, we have $L(x) \in \mathcal{F}_\delta$ for all $x \in S$ and $\delta \in \Delta$. Hence, for all $x \in S$, we have $L(x) \in \mathcal{F}$. Thus $\mathcal{F}$ satisfies Axiom f. 1 since for each $x \in S$, we have $x \in L(x) \in \mathcal{F}$. Axioms f. 2 and f. 3 are clearly satisfied by $\mathcal{F}$. To show that Axiom f. 4 holds, let $x \in \mathrm{Max}(F)$ where $F \in \mathcal{F}$. Since $F \in \mathcal{F}_\delta$ for each δ, it follows from Axiom f. 4 that x is in the set of corner points of F relative to $\mathcal{F}_\delta$ for each $\delta \in \Delta$. Thus $F \setminus x \in \mathcal{F}_\delta$ (or $F = \{x\}$) for each $\delta \in \Delta$, and so $F \setminus x \in \mathcal{F}$ (or $F = \{x\}$). Thus $x \in F^*$ and $\mathrm{Max}(F) \subseteq F^*$.

Theorems 2.22 and 2.24 imply the following theorem.

Theorem 2.25. The class of all fundamental families on S forms a complete lattice with respect to the partial ordering by set inclusion " $\subseteq$ " .

Proof. The theorem follows from the well known fact that any class of sets closed under arbitrary intersections and containing a largest set is a complete lattice with respect to the partial ordering " $\subseteq$ " .

We conclude this chapter with the following definition which extends Definitions 2.2, 2.4 and 2.9.

Definition 2.12. Let

$$L(0) = [0] = \phi \ , \quad \text{and} \quad U(0) = S$$

where 0 denotes the identity of the containing group of S.

CHAPTER 3

STRUCTURE THEOREMS AND EXAMPLES

In this chapter we prove several theorems on the structure of s-sets and fundamental families. Included also is an extensive list of examples.

3.1. Examples of s-sets

We begin with some theorems which enable us to construct new s-sets from given ones.

Theorem 3.1. If T is a closed subset of an s-set S, then T is an s-set.

Proof. Denote the containing group by Γ. Clearly, T satisfies Axioms s.1 and s.2. Let $\prec_S$ and $\prec_T$ denote the partial orderings of Γ with respect to S and T respectively (Definition 2.1). Similarly, let $L_S(x)$ and $L_T(x)$ be the lower sets of x with respect to S and T respectively (Definition 2.2). If $y \prec_T x$, then $x - y \in T \subseteq S$, and so $y \prec_S x$. Thus $L_T(x) \subseteq L_S(x)$ for each $x \in T$, and so T satisfies Axiom s.3. This completes the proof.

We can apply Theorem 3.1 to obtain many new examples. For instance, the following theorem shows that a translation of an s-set by an element of the s-set forms a new s-set.

Theorem 3.2. Let S be an s-set and x a fixed element of S. Then $T = \{y + x \mid y \in S\}$ and $T' = \{y + x \mid y \in S \text{ or } y = 0\}$ are s-sets.

Proof. Let $y_1 + x$ and $y_2 + x$ be two elements in T or T'. Then $(y_1 + x) + (y_2 + x) = (y_1 + x + y_2) + x$. Clearly, $(y_1 + x + y_2) \in S$ so that T and T' are closed subsets of S. By Theorem 3.1, T and T' are s-sets.

Definition 3.1. For a set X contained in a group Γ, we denote by X^0 the set $X \cup \{0\}$ where 0 is the identity of Γ. We will use the symbol 0 indifferently for the identity element of any (additive) group.

The following definition shows us how to take a special product of a class of sets, each a subset of a group.

Definition 3.2. Let be given, for each δ in a non-empty index set Δ, a set X_δ contained in an abelian group Γ_δ. Consider the set X of all functions f defined on Δ which satisfy the following two properties:

(i) $f(\delta) \in X_\delta^0$ for each $\delta \in \Delta$,

(ii) the set of δ for which $f(\delta) \neq 0$ is non-empty and finite.

We denote the set X by $\overline{\Pi}\{X_\delta \mid \delta \in \Delta\}$ and we call X the product of the X_δ. A function $f \in X$ is sometimes denoted by the indexed array $(x_\delta \mid \delta \in \Delta)$ where $x_\delta = f(\delta)$. The elements of X will be referred to as either points or functions.

<u>Theorem 3.3.</u> Let S_δ be an s-set for each δ in some non-empty index set Δ. Let $S = \overline{\Pi}\{S_\delta \mid \delta \in \Delta\}$. Then

(i) S is a subset of an abelian group;

(ii) for $f \in S'$ $L(f) = \overline{\Pi}\{L_\delta(f(\delta)) \mid \delta \in \Delta\}$

where $L_\delta(x)$ denotes the lower set of $x \in S_\delta$ with respect to S_δ.

Proof. Let Γ_δ denote a group containing S_δ. Then $\Gamma = \overline{\Pi}\{\Gamma_\delta \mid \delta \in \Delta\}$ is the (outer) direct sum of the Γ_δ without the identity element. Thus $\Gamma' = \Gamma \cup \{0\}$ is an abelian group, and clearly $S \subset \Gamma'$. This proves (i). We denote by $\prec$ and $\prec_\delta$ the partial orderings of Γ' and Γ_δ with respect to S and S_δ. If f_1 and f_2 are in S, then a necessary and sufficient condition that $f_1 \prec f_2$ or $f_1 = f_2$ is that, for each $\delta \in \Delta$, we have $f_1(\delta) \prec_\delta f_2(\delta)$ or $f_1(\delta) = f_2(\delta)$. Formula (ii) follows immediately.

<u>Theorem 3.4.</u> If S_δ is an s-set for each $\delta \in \Delta$, then $S = \overline{\Pi}\{S_\delta \mid \delta \in \Delta\}$ is an s-set. Here, addition is defined on S by

the formula

$$(f_1 + f_2)(\delta) = f_1(\delta) + f_2(\delta).$$

Proof. We omit most of the details. From Theorem 3. 3(i), S is a subset of an abelian group and it is clear that S is closed under addition. Conditions (ii) of Definition 3. 2 assures us that 0 (i. e. the zero function) is not in S. Thus S satisfies Axioms s. 1 and s. 2. From Theorem 3. 3(ii) we have $L(f) = \overline{\Pi}\, \{L_\delta(f(\delta)) | \delta \in \Delta\}$ which is finite in view of condition (ii) of Definition 3. 2. Thus S satisfies Axiom s. 3.

<u>Definition 3. 3.</u> In the case where $S_\delta = S$ for each $\overset{?\ \delta}{S}$ in Δ we denote $\overline{\Pi}\, \{S | \delta \in \Delta\}$ by S^Δ. In the case where Δ is finite of order n we denote S^Δ by S^n.

Notice that S^n is isomorphic to the set of all ordered n-tuples $(s_1, \cdots, s_n)$ where $s_i \in S^0$ and at least one $s_i \neq 0$. (For the definition of isomorphism between s-sets see Chapter 5).

In defining a particular example of an s-set we will often omit reference to the containing group Γ. In view of Section 2. 2 this can do no harm.

<u>Example ss-1.</u> The set I of positive integers is an s-set. Here $L(x) = \{1, 2, \cdots, x\}$ and $\mathrm{Min}(I) = \{1\}$.

<u>Example ss-2.</u> For an arbitrary positive integer n, I^n

is an s-set. Notice that $I' = I$ and I^2 is isomorphic to the set of positive Gaussian integers, i.e., the set of all $x + yi$ with x and y non-negative integers and $x + y > 0$. For I^n,

$L((x_1, x_2, \cdots, x_n)) = \{y \mid y = y_1, \cdots, y_n) \in I^n, \quad y_i \leq x_i \quad \text{for } i = 1, 2, \cdots, n\}$.

$\mathrm{Min}(I^n) = \{e_1, \cdots, e_n\}$ where $e_i = (\delta_{1i}, \delta_{2i}, \cdots, \delta_{ni})$ where $\delta_{ji} = 0$ if $j \neq i$ and $\delta_{ji} = 1$ if $j = i$.

<u>Example ss-3.</u> I^I is an s-set. This can be identified with the set of all sequences of non-negative integers with at most finitely many non-zero entries and at least one non-zero entry. We have $L((x_1, x_2, \cdots)) = \{y \mid y = (y_1, \cdots) \in I^I, \quad y_i \leq x_i \quad \text{for } i = 1, 2, \cdots\}$ and $\mathrm{Min}(I^I) = \{e_1, e_2, \cdots\}$ where $e_i = (\delta_{1i}, \delta_{2i}, \cdots)$.

<u>Example ss-4.</u> Let Δ be an uncountable index set. Then I^Δ is an example of an s-set which is uncountable, since $\mathrm{Min}(I^\Delta)$ is the set of all functions $e_\delta (\delta \in \Delta)$, where e_δ is defined by the formula

$$e_\delta(\lambda) \begin{cases} 0 & \text{if } \lambda \neq \delta, \\ 1 & \text{if } \lambda = \delta, \end{cases}$$

and, clearly, this set has the same cardinality as Δ.

In Appendix 2 we will show that the following example is not isomorphic to any s-set which is a closed subset of I^Δ for any Δ, even though it is contained in the additive group of rational numbers.

Example ss-5. Let

$$S = \{n + (i/2^n) \mid n \geq 1, \quad 0 \leq i < 2^n\}.$$

We show that S is an s-set.

(i) S is closed. Let $x = n + (i/2^n)$ and $y = m + (j/2^m)$ be arbitrary points of S. Then

$$z = x + y = (n+m) + \frac{i2^m + j2^n}{2^{n+m}}.$$

Write

$$\frac{i2^m + j2^n}{2^{n+m}} = k + \frac{P}{2^{n+m}},$$

where k is an integer ≥ 0 and $0 \leq P < 2^{n+m}$. Then we have

$$z = (n+m+k) + \frac{P}{2^{n+m}} = (n+m+k) + \frac{2^k P}{2^{n+m+k}}$$

where $0 \leq 2^k P < 2^{n+m+k}$ and so $z \in S$.

(ii) 0 is not in S by definition.

(iii) For each $x \in S$, L(x) is finite. Clearly, if $y \prec x$ then $y < x$. Hence,

$$L(x) \subseteq \{y \mid y \in S, \quad y \leq x\}.$$

This set is finite, for if $y = n(i/2^n) \leq x$, then $1 \leq n \leq x$ and $0 \leq i < 2^n \leq 2^x$.

All of the examples so far have the property that the containing group Γ can be taken to be torsion free, i.e., with no non-zero element of finite order. Our next theorem shows how to construct s-sets for which this does not hold.

Theorem 3.5. Let T be an s-set and G a finite abelian group. Then the set

$$S = G \times T = \{(x, y) \mid x \in G, \ y \in T\}$$

is an s-set where addition on S is defined by the equation,

$$(x, y) + (x', y') = (x + x', \ y + y').$$

Proof. If Γ is a group containing T then

$$G \times \Gamma = \{(x, y) \mid x \in G, \ y \in \Gamma\}$$

is an abelian group containing S. It is clear that S is closed and that $0 \, (=(0, 0))$ is not in S. Now, $L((x, y) = \{(x', y') \mid x' \in G, \ y' \in L_T(y)\}$, and this set is finite. This completes the proof.

Notice that if $G \neq \{0\}$, then any group Γ' containing $G \times T$ must contain a subgroup isomorphic to G, and so Γ' cannot be torsion free.

Our next example is just a special case of Theorem 3.5.

Example ss-6. Let $S = G \times I$ where $G = \{0, 1\}$ is the abelian group of order 2.

3.2. Examples of Density Spaces

We begin with a theorem that characterizes all examples of fundamental families on a given s-set. It could thus be taken for the definition of fundamental families.

Theorem 3.6. Let S be an arbitrary s-set. Corresponding to each $x \in S$, let $B(x)$ be a subset of S satisfying the following three conditions:

(b. 1) $x \in B(x)$,

(b. 2) $B(x) \subseteq L(x)$,

(b. 3) if $y \in B(x)$, then $B(y) \subseteq B(x)$.

Let

$$\mathcal{F}_B = \{F \mid F \in \mathcal{D}(S),\ x \in F \text{ implies } B(x) \subseteq F\} .$$

Then $\mathcal{F}_B$ is a fundamental family on S. Conversely, given any fundamental family $\mathcal{F}$ on S, there exists a function $B(x)$ satisfying conditions (b. 1-3) such that $\mathcal{F}_B = \mathcal{F}$.

Proof. Condition (b. 1) implies that $B(x)$ is non-empty, and (b. 2) implies that $B(x)$ is finite so that each $B(x) \in \mathcal{D}(S)$. Now, (b. 3) implies that $B(x) \in \mathcal{F}_B$. In view of (b. 1), Axiom f. 1 holds. Axioms f. 2 and f. 3 follow immediately from the definition of $\mathcal{F}_B$. Let $F \in \mathcal{F}_B$ and let $y \in F \setminus x$, where $x \in Max(F)$. By condition (b. 2) and the assumption that $x \in Max(F)$, we have that

$x \notin B(y)$. Thus $B(y) = B(y) \setminus x \subseteq F \setminus x$, and so $F \setminus x$ is in $\mathcal{F}_B$. Hence $x \in F^*$ for each $x \in \mathrm{Max}(F)$, and we obtain Axiom f. 4. We conclude that $\mathcal{F}_B$ is a fundamental family on S.

Now, let an arbitrary fundamental family $\mathcal{F}$ on S be given. We take $B(x) = [x]$. Condition (b. 1) is satisfied because of the definition of $[x]$ (Definition 2. 9). Condition (b. 2) follows from Theorem 2. 18, and Condition (b. 3) follows immediately from Theorems 2. 11 and 2. 12. Thus, by the first part of this theorem, $\mathcal{F}_B$ is a fundamental family on S. Clearly, by Theorem 2. 12, each $F \in \mathcal{F}$ satisfies the condition, $x \in F$ implies $B(x) \subseteq F$, and hence $\mathcal{F} \subseteq \mathcal{F}_B$. Now, let $F \in \mathcal{F}_B$. By Condition (i) and the definition of $\mathcal{F}_B$ we have that $F = \cup \{B(x) | x \in F\}$. This set is in $\mathcal{F}$ since it is a finite union of Cheo sets in $\mathcal{F}$. Thus $\mathcal{F}_B \subseteq \mathcal{F}$ and the proof of Theorem 3.6 is complete.

<u>Theorem 3. 7.</u> Let S be an s-set and let B(x) satisfy Conditions (b. 1-3). Then the Cheo set of $\mathcal{F}_B$ determined by x is equal to B(x).

Proof. Since $x \in B(x) \in \mathcal{F}_B$ we have by Theorem 2. 12 that $[x] \subseteq B(x)$. On the other hand, since $x \in [x] \in \mathcal{F}_B$ we have by the definition of $\mathcal{F}_B$ that $B(x) \subseteq [x]$.

Along with the description of an example of a density space $(S, \mathcal{F})$, we will characterize the set of corner points of a set of $\mathcal{F}$, indicate the essential points of $(S, \mathcal{F})$, and show anything else of interest concerning the density space.

Example ff-1. The pair $(S, \mathcal{K})$ where S is an arbitrary s-set and where $\mathcal{K} = \mathcal{F}_B$ with $B(x) = L(x)$ is a density space. It is clear that $L(x)$ satisfies Conditions (b. 1) and (b. 2). Condition (b. 3) follows from the transivity of $\prec$. Notice that this definition of $\mathcal{K}$ is equivalent to Definition 2. 11.

As $\mathcal{K} = \mathcal{K}(S)$ is a very important fundamental family, we list its properties in the following theorem.

Theorem 3. 8. The fundamental family $\mathcal{K}$ has the following properties:

(i) For each $F \in \mathcal{K}$, we have $\mathrm{Max}(F) = F^*$.

(ii) $\mathcal{K}$ is the only fundamental family on S satisfying property (i).

(iii) The Axioms f. 1, f. 2, f. 3 and f. 4' (see Section 2. 5) are categorical.

(iv) The Cheo set of $\mathcal{K}$ determined by $x \in S$ is $L(x)$.

(v) The essential points of $(S, \mathcal{K})$ are just the minimal points of S.

Proof. (i) From Axiom f. 4 we obtain $\mathrm{Max}(F) \subseteq F^*$. On the other hand, let $x \in F^*$ and suppose there exists an element y

such that $y \in F$ with $x \prec y$. Then $x \in L(y)$ and, since $y \in F \setminus x \in \mathcal{K}$, $L(y) \subsetneq F \setminus x$. This is a contradiction. Hence $x \in Max(F)$, and we obtain $Max(F) = F^*$.

(ii) Let $\mathcal{F}$ be any fundamental family on S with the property that $Max(F) = F^*$ for each $F \in \mathcal{F}$. By Theorem 2.22 we have $\mathcal{K} \subseteq \mathcal{F}$. Let F be any set in $\mathcal{F}$ and suppose that F is not in $\mathcal{K}$. Then there is an x such that both $x \in F$ and $L(x) \not\subseteq F$. Let $y \in L(x) \setminus F$ and let $G = F \cup [y]$ where $[y]$ denote the Cheo set of $\mathcal{F}$ determined by y. Then $G \in \mathcal{F}$ and so $Max(G) = G^*$ Since $G \setminus y = F \cup ([y] \setminus y)$, we have that $y \in G^*$. But $y \notin Max(G)$ since $x \in G$ and $y \prec x$. This is a contradiction and we conclude that $\mathcal{F} = \mathcal{K}$.

(iii) From (ii) it is clear that the only family satisfying the four Axioms f. 1, f. 2, f. 3 and f. 4' is $\mathcal{K}$.

(iv) For $x \in S$ we have, by Theorem 3.7, that $[x] = B(x)$ where, by definition, $B(x) = L(x)$.

(v) By (iv), an element $x \in S$ is an essential point of $(S, \mathcal{K})$ if and only if $L(x) = \{x\}$. Now $L(x) = \{x\}$ if and only if $L(x) \cap S = \{x\}$, that is, if and only if $x \in Min(S)$.

<u>Example ff-2.</u> As a special case of Example ff-1 we have that $(I^n, \mathcal{K})$ is a density space.

This density space is the one used by Kvarda and Kasch in defining K-density and C-density respectively for subsets of I^n according to Definition 1.1. In the special case $n = 1$ we have Schnirelmann's original definition. Notice that when we replace Axiom f.4 by Axiom f.4' and specify the s-set to be I^n we get an axiomatic characterization of these densities.

Example ff-3. The pair $(S, \mathcal{D})$ where S is an arbitrary s-set and where $\mathcal{D} = \mathcal{F}_B$ with $B(x) = \{x\}$ is a density space. Clearly $B(x)$ satisfies conditions (b.1-3). Notice that this definition of $\mathcal{D}$ is equivalent to Definition 2.6. Note that for each $x \in S$ we have $[x] = \{x\}$ so that every point of S is an essential point of $(S, \mathcal{D})$. Note also that $F^* = F$ for all $F \in \mathcal{D}$.

Example ff.4. The pair $(I^n, \mathcal{F}_R)$ is a density space where R is defined as follows. For $x = (x_1, x_2, \cdots, x_n) \in I^n$ define $R(x)$ to be the set of points

$$\left(\frac{ix_1}{d}, \frac{ix_2}{d}, \cdots, \frac{ix_n}{d}\right)$$

where $d = \gcd\{x_1, \cdots, x_n\}$ ($\gcd$ = greatest common divisor) and $i = 1, 2, \cdots, d$ (we show $\mathcal{F}_R$ is a fundamental family). Thus $R(x)$ satisfies (b.1) since $x = (dx_1/d, \cdots, dx_n/d) \in R(x)$. Condition (b.2) follows from the fact that $ix_j/d \leq x_j$ for $i = 1, \cdots, d$ and $j = 1, \cdots, n$. To prove Condition (b.3) notice that

$\gcd\{ix_1/d, \cdots, ix_n/d\} = i$ and $(i'(ix_j)/d)/i = i'x_j/d$ for $1 \leq i' \leq i$ and $1 \leq j \leq n$. Thus

$$R((\frac{ix_1}{d}, \cdots, \frac{ix_n}{d})) \subseteq R(x) .$$

The essential points of $(I^n, \mathcal{F}_R)$ are just those points $x = (x_1, \cdots, x_n)$ for which $d = \gcd\{x_1, \cdots, x_n\} = 1$.

Independent of the work on this thesis, Kvarda and R. Killgrove [13] have given a definition of density for subsets I^n which amounts to the C-density with respect to $\mathcal{F}_R$. That is, the density of a subset A of I^n is given by Definition 1. 1 where the family $\mathcal{B}$ is taken to be the family of all Cheo sets of $\mathcal{F}_R$.

Example ff-5. The pair $(I^n, \mathcal{F}_{H_j})$ is a density space where j is an integer, $1 \leq j \leq n$, and H_j is defined as follows. For each $x = (x_1, \cdots, x_n) \in I^n$, define $H_j(x)$ to be the set of points

$$\{(x_1, \cdots, x_{j-1}, i, x_{j+1}, \cdots, x_n) \mid \eta(x) \leq i \leq x_j\} .$$

Here $\eta(x)$ is defined to be 1 if $x_k = 0$ for all k where $k \neq j$, and $\eta(x) = 0$ otherwise. Now $H_j(x)$ satisfies the Conditions (b. 1-3) so that $\mathcal{F}_{H_j}$ is a fundamental family on I^n.

The essential points of $(I^n, \mathcal{F}_{H_j})$ are all points $x = (x_1, \cdots x_n) \in I^n$ such that $x_j = 0$ together with the point e_j which has unit j^{th} coordinate and all other coordinates zero.

The preceding two examples can be extended to the s-sets I^{Δ} for arbitrary Δ as is shown in Example ff-6 and ff-7.

Example ff-6. The pair $(I^{\Delta}, \mathcal{F}_R)$ is a density space where R is defined as follows. Recall that an element of I^{Δ} is a function f on Δ into I^0 such that $f(\delta) = 0$ for all but finitely many $\delta \in \Delta$, and $f(\delta) \neq 0$ for at least one δ. For a function $f \in I^{\Delta}$, let $d = \gcd\{f(\delta) \mid \delta \in \Delta\}$, and define $R(f)$ to be the set of functions g_i $(1 \leq i \leq d)$ defined by the formula

$$g_i(\delta) = \frac{if(\delta)}{d} \quad .$$

Then, as before, $R(f)$ satisfies Conditions (b. 1-3).

We remark that Theorem 2. 24 provides a method for obtaining new fundamental families on a given s-set from old ones. Moreover, if $\mathcal{D}'$ is any subfamily of $\mathcal{D}$, then the family which is the intersection of all fundamental families $\mathcal{F}$ on the s-set which contain $\mathcal{D}'$ is a fundamental family, and is called the fundamental family generated by $\mathcal{D}'$.

Example ff-7. The pair $(I^{\Delta}, \mathcal{F}_{H_\lambda})$ is a density space where λ is a fixed element in Δ, and H_λ is defined as follows. For $f \in I^{\Delta}$, let $H_\lambda(f)$ be the set of all functions $g \in I^{\Delta}$ defined by the formula

$$\begin{cases} g(\delta) = f(\delta) & \text{if} \quad \delta \neq \lambda , \\ \eta(f) \leq g(\delta) \leq f(\lambda) & \text{if} \quad \delta = \lambda , \end{cases}$$

where

$$\eta(f) = \begin{cases} 1 & \text{if} \quad f(\delta) = 0 \quad \text{for} \quad \delta \neq \lambda , \\ 0 & \text{otherwise} . \end{cases}$$

By Theorem 2. 24 we have that $\cap \{\mathcal{F}_{H_\lambda} | \lambda \in \Delta\}$ is a fundamental family on I^Δ. We note that this family is $\mathcal{K}(I^\Delta)$.

Example ff-8. Ths pair $(I^2, \mathcal{F}_B)$ is a density space where B is defined below. As usual we represent I^2 as the set of all ordered pairs (x, y) where x and y are non-negative integers and $x + y > 0$. Define $B((x, y))$ as follows:

$$B((x, y)) = \begin{cases} \{(x, y)\} & \text{if } x > 0 \text{ and } y \text{ is odd,} \\ \{(x, i) | i = 0, 2, 4, \cdots, y\} & \text{if } x > 0 \text{ and } y \text{ is even,} \\ \{(x, i) | i = 1, 2, \cdots, y\} & \text{if } x = 0 . \end{cases}$$

The abundance of examples of density spaces is indicated by how easy one can obtain spaces from the space $(I, \mathcal{K})$ by varying I and $\mathcal{K}$ slightly. For example, we can take closed subsets of I. The following is an example where $\mathcal{K}$ is varied.

Example ff-9. The pair $(I, \mathcal{F}_B)$ is a density space where

B is defined as follows:

$$B(x) = \begin{cases} \{x\} & \text{if } x \text{ is odd,} \\ \{2, 4, \cdots, x\} & \text{if } x \text{ is even.} \end{cases}$$

The following theorem provides a method by which we can construct a fundamental family on an s-set S, where S is the product of a class $\{S_\delta \mid \delta \in \Delta\}$ of s-sets.

Theorem 3. 9. For each δ in an index set Δ, let $(S_\delta, \mathcal{F}_\delta)$ be a density space, and let $S = \overline{\Pi} \{S_\delta \mid \delta \in \Delta\}$. Then $\mathcal{F}_B$ is a fundamental family on S where B is defined as follows: Denote by $[x]_\delta$ the Cheo set of $\mathcal{F}_\delta$ determined by $x \in S_\delta$. For a function $f \in S$ define $B(f)$ to be the set

$$\overline{\Pi} \{[f(\delta)]_\delta \mid \delta \in \Delta\}.$$

Proof. Condition (b. 1) follows since $f \in \overline{\Pi} \{[f(\delta)]_\delta \mid \delta \in \Delta\}$. Letting $L_\delta(x)$ denote the lower set of $x \in S_\delta$ with respect to S_δ, we know, by Theorem 2. 18, that $[f(\delta)]_\delta \subseteq L_\delta(f(\delta))$ for each $\delta \in \Delta$. Hence

$$B(f) = \overline{\Pi} \{[f(\delta)]_\delta \mid \delta \in \Delta\} \subseteq \overline{\Pi} \{L_\delta(f(\delta)) \mid \delta \in \Delta\} = L(f)$$

where this last equality holds by Theorem 3. 3(ii). Thus B satisfies Condition (b. 2). Now, if $g \in B(f)$, then $g(\delta) \in [f(\delta)]_\delta^0$ for each $\delta \in \Delta$. Hence, by Theorem 2.12, we have $[g(\delta)]_\delta \subseteq [f(\delta)]_\delta$, and so

$$B(g) = \overline{\Pi}\,\{[g(\delta)]_\delta \mid \delta \in \Delta\} \subseteq \overline{\Pi}\,\{[f(\delta)]_\delta \mid \delta \in \Delta\} = B(f)\ .$$

Thus Condition (b. 3) is satisfied and the proof is complete.

Fundamental families on product s-sets are studied in Section 8. 2.

Definition 3. 4. The fundamental family $\mathcal{F}_B$ of Theorem 3.9 is denoted by

$$\mathbf{P}\{\mathcal{F}_\delta \mid \delta \in \Delta\}\ ,$$

and the density space $(S, \mathcal{F}_B)$ by

$$\mathbf{P}\{S_\delta, \mathcal{F}_\delta) \mid \delta \in \Delta\}\ .$$

Theorem 3. 10. For any non-empty class $\{S_\delta \mid \delta \in \Delta\}$, we have

(i) $\mathcal{H}(\overline{\Pi}\,\{S_\delta \mid \delta \in \Delta\}) = \mathbf{P}\{\mathcal{H}(S_\delta) \mid \delta \in \Delta\}$.

Also,

(ii) $\mathcal{H}(I^n) = \mathbf{P}\{\mathcal{H}(I) \mid \delta = 1, 2, \cdots, n\}$.

Proof. We have that $\mathbf{P}\{\mathcal{H}(S_\delta) \mid \delta \in \Delta\} = \mathcal{F}_B$ where

$$\begin{aligned} B(f) &= \overline{\Pi}\{[f(\delta)]_{\delta} \mid \delta \in \Delta\} \\ &= \overline{\Pi}\{L_{\delta}(f(\delta)) \mid \delta \in \Delta\} \\ &= L(f) \, . \end{aligned}$$

Thus from Example ff-1, we have $\mathcal{F}_B = \mathcal{K}$. Now (iii) follows from (i) if we let $\Delta = \{1, 2, \cdots, n\}$, and $S_{\delta} = I$ for each $\delta \in \Delta$.

CHAPTER 4

TRANSFORMATION PROPERTIES

In the proofs of some of the theorems of Chapters 7 and 8 it will be necessary to assume that the fundamental family has one of two transformation properties which are defined and discussed in this chapter. Even though they are not used until Chapter 7 we present them now in order not to disturb the continuity of the theory of density developed in Chapters 6 through 9.

4. 1. Definitions

There are two transformation properties which we define now.

Definition 4. 1. Let $(S, \mathcal{F})$ be a density space. Let F be an arbitrary set of $\mathcal{F}$ and x an element of F^0. Let $D = F \cap U(x)$ ($U(x)$ is defined in Definitions 2. 4 and 2. 12). Define $T_1[D]$ to be the set $\{y-x | y \in D\}$. If $T_1[D]$ is either in $\mathcal{F}$ or is empty for every $D = F \cap U(x)$ where $F \in \mathcal{F}$ and $x \in F^0$, then $\mathcal{F}$ satisfies the first transformation property or $\mathcal{F}$ is transformation -1.

Definition 4. 2. Again let $(S, \mathcal{F})$ be a density space. Let $x \in S$ and let F be any set of $\mathcal{F}$ or let F be empty. Let

$D = [x] \setminus F$. Define $T_2[D]$ to be the set $\{x-y \mid y \in D \setminus x\}$. If $T_2[D]$ is either in $\mathcal{F}$ or is empty for each set $D = [x] \setminus F$ where $x \in S$ and $F \in \mathcal{F} \cup \{\phi\}$, then $\mathcal{F}$ satisfies the second transformation property or $\mathcal{F}$ is transformation -2.

Just how these properties are used will be seen in Chapters 7 and 8, particularly in the proofs of the theorems which generalize the inequalities of Landau and Schnirelmann and of Schur.

4. 2. Existense of Fundamental Families with the Transformation Properties

There are fundamental families which are transformation -1 and fundamental families which are transformation -2. In particular, we have

Theorem 4. 1. For an arbitrary s-set S, $\mathcal{K}(S)$ is both transformation -1 and transformation -2.

Proof of transformation -1. Let $D = F \cap U(x)$ where $F \in \mathcal{K}$ and $x \in F^0$. Suppose that $T_1[D]$ is non-empty. To show that $T_1[D] \in \mathcal{K}$ is to show that for any $y \in T_1[D]$ we have $L(y) \subseteq T_1[D]$. For this it is sufficient to show that, if $y \in T_1[D]$ and $z \prec y$ $(z \in S)$, then $z \in T_1[D]$. Thus let $y \in T_1[D]$ and $z \prec y$ $(z \in S)$. We have $x \prec x+z$, and so $x+z \in U(x)$. Next $y = d-x$ for some $d \in D$ so that $x+y \in D \subseteq F$. Thus, since $x+z \prec x+y$, we have $x+z \in F$. Hence $x+z \in D$, and so $z = (x+z) - x \in T_1[D]$. This completes the proof.

Proof of transformation -2. Recall that in the present case we have $[x] = L(x)$ for $x \in S$. Let $D = L(x) \setminus F$ where $x \in S$ and $F \in \mathcal{K} \cup \{\phi\}$. Suppose that $D \setminus x$ is non-empty so that $T_2[D]$ is non-empty. Let $y \in T_2[D]$ and let $z \in S$ with $z \prec y$. As before we prove that $z \in T_2[D]$. We have $y = x-d$ for some $d \in D \setminus x$ so that $x - y \in D$. Now $x-y \prec x-z$ so that, if $x-z \in F$, then also $x-y \in F$, a contradiction. Thus $x-z \notin F$. Furthermore, $x-z \prec x$ so that $x-z \in L(x)$. Thus $x-z \in D$ and since $x-z \neq x$, we have $x-z \in D \setminus x$. Hence $z = x-(x-z) \in T_2]D]$.

The only examples of Section 3.2 where the fundamental family fails to be transformation - 1 are Example ff-4 and the more general Example ff-6. The only example where the fundamental family fails to be transformation - 2 is Example ff-8.

CHAPTER 5

SEPARATED FUNDAMENTAL FAMILIES

In this chapter we define and treat the concept of separated fundamental families. These families have important applications to the theory of C-density (Definition 6.2), and to the proofs of the $\alpha + \beta$ theorem (Chapter 9).

5.1. Definition of a Separated Fundamental Family

Let $(S, \mathcal{F})$ be an arbitrary density space.

Definition 5.1. The family $\mathcal{F}$ is separated if, whenever x and y are points of S such that $x \notin [y]$ and $y \notin [x]$, then $[x] \cap [y]$ is empty.

Thus from Theorem 2.12, given any two Cheo sets of $\mathcal{F}$, either one is contained in the other or they are disjoint.

5.2. Characterization of Separated Fundamental Families of the Form $\mathcal{K}(S)$

In this section we prove that $\mathcal{K}(S)$ is separated if and only if S is isomorphic to I. For convenience we define isomorphism between s-sets.

Definition 5.2. An s-set S_1 is isomorphic to an s-set S_2 if there exists a one-to-one function τ on S_1 onto S_2 such that

$$\tau(x+y) = \tau(x) + \tau(y)$$

for each x and y in S_1. If S_1 is isomorphic to S_2, then S_2 is isomorphic to S_1, and S_1 and S_2 are isomorphic. The function τ is an isomorphism.

The following two theorems lead up to Theorem 5.3 which is the main result of this section. Theorem 5.1 is somewhat stronger than what is needed, but it is presented here since it is of interest by itself.

Theorem 5.1. Let S be an arbitrary s-set. Each element $x \in S$ can be written as a finite linear combination

$$(1) \qquad x = n_1 e_1 + n_2 e_2 + \cdots + n_k e_k$$

where the n_i are positive integers and each $e_i \in \mathrm{Min}(S)$.

Proof. Suppose there are points of S not represented in the form (1). By Theorem 2.3, we may take x to be a minimal point of the set of those not representable. If $L(x) = \{x\}$, then $x \in \mathrm{Min}(S)$, and so x is representable in the form (1) (take $k = 1$, $n_1 = 1$ and $e_1 = x$). If there is a $y \in L(x)$ such that $y \neq x$, then

$y \prec x$ and $x-y \prec x$, and so both y and $x-y$ are representable. Thus

$$y = n_1 e_i + \cdots + n_k e_k$$

$$x-y = m_1 e'_1 + \cdots + m_{k'} e'_{k'} \quad ,$$

and so $x = y + (x-y)$ is representable in the form (1). This is a contradiction and the proof is complete.

Theorem 5.2. An s-set S is isomorphic to I if and only if $\mathrm{Min}(S)$ reduces to a singleton $\{x\}$.

Proof. If $\mathrm{Min}(S) = \{x\}$, then by Theorem 2.2(i), $nx \in S$ for each positive integer n, and by Theorem 5.1, each $y \in S$ has the form $y = nx$ for some positive integer n. Thus $S = \{nx \mid n = 1, 2, \cdots\}$. The function τ on S onto I, defined by $\tau(nx) = n$, clearly satisfies the requirements of Definition 5.2 so that S is isomorphic to I.

Now suppose that S is isomorphic to I and let τ be the isomorphism. Because of the manner in which the order relation $\prec$ is defined in terms of the operation $+$ we have that τ preserves the order relation, and hence that $\tau(L_S(x)) = L_I(\tau(x))$ for each $x \in S$. Thus, if $x \in \mathrm{Min}(S)$, then $L_I(\tau(x)) = \tau(L_S(x)) = \tau(\{x\}) = \{\tau(x)\}$ which implies that $\tau(x) \in \mathrm{Min}(I)$. Thus $\tau(x) = 1$ for each $x \in \mathrm{Min}(S)$, and so S has one and only one minimal point.

Theorem 5. 3. The fundamental family $\mathcal{K}(S)$ is separated if and only if S is isomorphic to I.

Proof. In view of Theorem 5. 2 it is sufficient to prove that $\mathcal{K}(S)$ is separated if $\mathrm{Min}(S)$ is a singleton, and that $\mathcal{K}(S)$ is not separated if $\mathrm{Min}(S)$ has at least two distinct points.

If $\mathrm{Min}(S) = \{x\}$, then $S = \{nx \mid n = 1, 2, \cdots\}$. Clearly $L(nx) = \{ix \mid i = 1, 2, \cdots, n\}$. If $F \in \mathcal{K}(S)$ then $F = \cup\{L(y) \mid y \in F\}$. Clearly then $F = L(mx)$ where $m = \max\{i \mid ix \in F\}$. Hence every set in $\mathcal{K}(S)$ is of the form $L(y)$ for some $y \in S$. If nx and mx are two distinct points of S then either $n < m$ or $m < n$, i. e., $nx \in L(mx)$ or $mx \in L(nx)$. Thus the condition of Definition 5. 1 is satisfied (vacuously), and so $\mathcal{K}(S)$ is separated.

Now suppose x and y are two distinct elements of $\mathrm{Min}(S)$. Consider the two points $x+y$ and $y+y$. If $x+y = y+y$, then $x = y$, a contradiction. Also, if $x+y \prec y+y$, then $x \prec y$ contradicting the fact that $y \in \mathrm{Min}(S)$. Hence $x+y \notin L(y+y)$. Similarly, we obtain $y+y \notin L(x+y)$. But $y \in L(x+y) \cap L(y+y)$ and so the condition of Definition 5. 1 is not satisfied. Thus $\mathcal{K}(S)$ is not separated. This completes the proof.

Along with $\mathcal{K}(I)$ (and $\mathcal{K}(S)$ where S is isomorphic to I), all the families from Example ff-3 through ff-9 are separated.

5.3. Decomposition of a Set in a Separated Fundamental Family

Let $\mathcal{F}$ be a separated fundamental family on an s-set S. By Theorem 2.16 we may write for any $F \in \mathcal{F}$ that $F = \cup\{[x] | x \in F^*\}$. By Theorem 2.19 and Definition 5.1 we have that the Cheo sets $[x]$ for $x \in F^*$ are pairwise disjoint. Thus we have shown

Theorem 5.4. If F is a set of a separated fundamental family $\mathcal{F}$ on an s-set, then F is the union of disjoint Cheo sets:

$$F = \cup\{[x] \mid x \in F^*\} .$$

CHAPTER 6

INTRODUCTION TO DENSITY

In this chapter we define the two densities which will be studied in this thesis. Also the elementary properties of these densities are developed.

6.1. Definitions

Throughout this section $(S, \mathcal{J})$ is an arbitrary density space.

Definition 6.1. Let X be a subset of S. For any finite (possibly empty) subset D of S we let $X(D)$ be the number of elements in the set $X \cap D$. Furthermore, if D is non-empty, let $q(X, D)$ be the quotient $X(D)/S(D)$.

A few of the important properties of the counting function $X(D)$ are listed in the following theorem, the proof of which is immediate.

Theorem 6.1. The function $X(D)$ has the following properties:

(i) If $D_1, D_2, \cdots, D_n$ is a finite collection of finite subsets of S, then

$$X(D_1 \cup \cdots \cup D_n) \leq X(D_1) + \cdots + X(D_n)$$

with equality holding if the D_i are pairwise disjoint.

(ii) $S(D)$ is just the number of elements in D.

(iii) If $D_1 \subseteq D_2$, then $X(D_2 \setminus D_1) = X(D_2) - X(D_1)$.

(iv) If $X \subseteq Y$, then $(Y \setminus X)(D) = Y(D) - X(D)$.

Definition 6. 2. Let A be an arbitrary subset of S. The K-density of A with respect to $\mathcal{F}$ is

$$d(A, \mathcal{F}) = \text{glb}\,\{q(A, F) \mid F \in \mathcal{F}\} .$$

Definition 6. 3. Let $A \subseteq S$. The C-density of A with respect to $\mathcal{F}$ is

$$d_c(A, \mathcal{F}) = \text{glb}\,\{q(A, [x]) \mid x \in S\}$$

where $[x]$ denotes the Cheo set of $\mathcal{F}$ determined by x (Definition 2. 9).

As already noted in Example ff-2, for the density space $(I^n, \mathcal{K})$ we have that $d(A, \mathcal{K})$ is the density defined by Kvarda [11] and $d_c(A, \mathcal{K})$ is the density used by Cheo [2] and Kasch [7].

The following theorem provides alternate definitions for K- and C- density, and follows immediately from Definitions 6.2 and 6. 3.

Theorem 6. 2. The density $d(A, \mathcal{F})$ is the largest real number α such that $A(F) \geq \alpha\, S(F)$ for each set $F \in \mathcal{F} \cup \{\phi\}$. Similarly $d_c(A, \mathcal{F})$ is the largest real number α_c such that

$A([x]) \geq \alpha_c S([x])$ for each $x \in S^0$.

6.2. Elementary Properties of K- and C- density

In this section $(S, \mathcal{F})$ is an arbitrary density space. We adopt a shorter notation and write $d(A, \mathcal{F}) = \alpha$ and $d_c(A, \mathcal{F}) = \alpha_c$ where A is an arbitrary subset of S.

Theorem 6.3. We have $0 \leq \alpha \leq \alpha_c \leq 1$. Furthermore, $\alpha = \alpha_c = 1$ if and only if $A = S$.

Proof. The fact that $\alpha \leq \alpha_c$ follows from the relation

$$\{q(A, F) \mid F \in \mathcal{F}\} \supseteq \{q(A, [x]) \mid x \in S\}.$$

Now, for each $F \in \mathcal{F}$, we have $0 \leq A(F) \leq S(F)$, and so $0 \leq q(A, F) \leq 1$ from which it follows that $0 \leq \alpha \leq 1$ and $0 \leq \alpha_c \leq 1$. For the second part, if $A = S$, then clearly $\alpha = \alpha_c = 1$, since $q(S, F) = 1$ for each $F \in \mathcal{F}$. Now assume there is an $x \in S \setminus A$. Then $A([x]) < S([x])$, and so $\alpha \leq \alpha_c \leq q(A, [x]) < 1$.

Theorem 6.4. If $D_1, \cdots, D_n$ is a finite collection of pair-wise disjoint non-empty finite subsets of S, and $D = \cup \{D_i \mid i = 1, \cdots, n\}$, then for any set $A \subseteq S$ we have

$$q(A, D) \geq \min \{q(A, D_i) \mid i = i, \cdots, n\}.$$

Proof. If $n = 1$ the theorem is immediate. For $n = 2$ we have, using Theorem 6.1(i), that

$$q(A, D) = \frac{A(D_1) + A(D_2)}{S(D_1) + S(D_2)} .$$

Letting $a_i = A(D_i)$ and $s_i = S(D_i)$ $(i = 1, 2)$ we have that, if

$$\frac{a_1 + a_2}{s_1 + s_2} < \frac{a_i}{s_i} , \quad (i = 1, 2)$$

then both $a_1 s_2 < a_2 s_1$ and $a_2 s_1 < a_1 s_2$, a contradiction. Thus

$$q(A, D) = \frac{a_1 + a_2}{s_1 + s_2} \geq \min \{\frac{a_i}{s_i} \mid i = 1, 2\}$$

$$= \min \{q(A, D_i) \quad (i = 1, 2\} .$$

Let $n > 2$ and suppose the theorem is true if n is replaced by any smaller integer. Let $D' = \cup \{D_i \mid i = 1, \cdots, n-1\}$. Then

$$q(A, D) \geq \min\{q(A, D'), \ q(A, D_n)\}$$

$$\geq \min \{\min \{q(A, D_i) \mid i = 1, \cdots, n-1\}, \ q(A, D_n)\}$$

$$= \min \{q(A, D_i) \mid i = 1, \cdots, n\} .$$

The following theorem shows that there is no difference between K- and C- density when the fundamental family is assumed to be separated.

<u>Theorem 6.5.</u> If $\mathcal{F}$ is a separate fundamental family, then the density given in Definition 6.2 and 6.3 are identical. That is, $\alpha = \alpha_c$ for each $A \subseteq S$.

Proof. By Theorem 6.3 we have $\alpha \leq \alpha_c$, Let F be an arbitrary set in $\mathcal{F}$. By Theorem 5.4 we have $F = \cup\{[x] \mid x \in F^*\}$ where this is a union of pairwise disjoint sets. Hence, we have by Theorem 6.4 that

$$\begin{aligned} q(A, F) &\geq \min\{q(A, [x]) \mid x \in F^*\} \\ &\geq \alpha_c \; . \end{aligned}$$

Since F is arbitrary, we conclude that $\alpha \geq \alpha_c$, and the theorem is proved.

<u>Theorem 6.6.</u> If $\alpha > 0$ or $\alpha_c > 0$, then A contains all the essential points of $(S, \mathcal{F})$ (Definition 2.10).

Proof. Suppose x is an essential point not in A. Then

$$\begin{aligned} 0 = A(\{x\}) = A([x]) &\geq \alpha_c S([x]) = \alpha_c S(\{x\}) \\ &= \alpha_c \geq \alpha \geq 0 \; . \end{aligned}$$

Thus $\alpha = \alpha_c = 0$.

The notation introduced in the following definition will be used extensively throughout the remainder of this thesis.

Definition 6.4. For a set $X \subseteq S$, the set $S \setminus X$ is denoted by $\overline{X}$.

The following theorem offers an equivalent definition for the K-density of a set $A \nsupseteq S$.

Theorem 6.7. If $\overline{A}$ is non-empty, then

$$\alpha = \text{glb}\,\{q(A,F) \mid F \in \mathcal{F} \text{ and } F^* \subseteq \overline{A}\}.$$

Proof. Let $\alpha' = \text{glb}\,\{q(A,F) \mid F \in \mathcal{F},\ F^* \subseteq \overline{A}\}$. Clearly, $\alpha' \geq \alpha$. Now let F be an arbitrary fundamental set. If $A(F) = S(F)$ then $q(A,F) = 1 > \alpha'$. If $A(F) < S(F)$, then let $G = \cup\{[x] \mid x \in F \cap \overline{A}\}$. By Theorem 2.17, we have that $G \in \mathcal{F}$ and $G^* \subseteq F \cap \overline{A}$. Thus $G^* \subseteq \overline{A}$. Also $G^* \subseteq F$ which implies, by Theorem 2.14, that $G \subseteq F$. Thus $F = G \cup (F \setminus G)$, and G and $F \setminus G$ are disjoint. Finally, by the construction of G we have $F \cap \overline{A} \subseteq G$, and so $F \setminus G \subseteq A$, so that $A(F \setminus G) = S(F \setminus G)$. Hence, we have

$$q(A,F) = \frac{A(F)}{S(F)} = \frac{A(G) + A(F\setminus G)}{S(G) + S(F\setminus G)}$$

$$= \frac{A(G) + S(F\setminus G)}{S(G) + S(F\setminus G)} \geq \frac{A(G)}{S(G)} = q(A,G) \geq \alpha' .$$

Thus $\alpha \geq \alpha'$, and the theorem is proved.

The natural question to ask is whether or not $\alpha_c = \text{glb}\,\{q(A,[x])\,|\,x \in \bar{A}\}$. In general this is not true as the following example shows. Consider the density space $(I^2, \mathcal{K})$. Take A to be the set of all ordered pairs $(x, y) \in I^2$ which satisfy at least one of the following conditions :

(i) (x, y) is equal to one of $(0,1)$, $(0, 2)$, $(1, 0)$, $(2, 0)$ or $(3, 3)$;

(ii) $x \geq 4$;

(iii) $y \geq 4$.

Then $\alpha_c = 1/3$ while, recalling that $[x] = L(x)$, $\text{glb}\,\{q(A,[x]\,|\,x \in \bar{A}\} = 4/11$.

However, if $\mathcal{F}$ is assumed to be separated, then we can prove the following theorem.

<u>Theorem 6.8.</u> If $\bar{A}$ is non-empty and the fundamental family $\mathcal{F}$ is separated, then

$$\alpha_c = \text{glb}\,\{q(A,[x])\,|\,x \in \bar{A}\}.$$

Proof. Let $\alpha' = \text{glb}\{q(A,[x])\,|\,x \in \bar{A}\}$. It is clear that $\alpha' \geq \alpha_c$. Now let $F \in \mathcal{F}$ such that $F^* \subseteq \bar{A}$. From Theorems 5.4 and 6.4 we obtain

$$q(A, F) \geq \min\{q(A,[y])\,|\,y \in F^*\} \geq \alpha' .$$

The last inequality follows since each $y \in \bar{A}$. Hence, by Theorem

6.7 we have $\alpha \geq \alpha'$, and so, since $\alpha = \alpha_c$ (Theorem 6.5), we have $\alpha_c \overset{>}{\leq} \alpha'$ and the theorem is proved.

The last theorem of this chapter expresses some simple but useful density relations.

<u>Theorem 6.9.</u> Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two fundamental families on S, and let A_1 and A_2 be two subsets of S.

(i) If $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $d(A, \mathcal{F}_1) \geq d(A, \mathcal{F}_2)$ for each $A \subseteq S$.

(ii) If $A_1 \subseteq A_2$, then $d(A_1, \mathcal{F}) \leq d(A_2, \mathcal{F})$ for each fundamental family $\mathcal{F}$ on S.

(iii) If $A_1 \subseteq A_2$, then $d_c(A_1, \mathcal{F}) \leq d_c(A_2, \mathcal{F})$ for each fundamental family $\mathcal{F}$ on S.

Proof. Part (i) follows from the fact that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies that

$$\{q(A, F) \mid F \in \mathcal{F}_1\} \subseteq \{q(A, F) \mid F \in \mathcal{F}_2\}.$$

Parts (ii) and (iii) follow since $A_1 \subseteq A_2$ implies that $A_1(F) \leq A_2(F)$ and thus $q(A_1, F) \leq q(A_2, F)$ for each $F \in \mathcal{F}$.

CHAPTER 7

K-DENSITY

In this chapter we present some theorems which involve K-density. Most of them require the concept of sum of sets which we define in the first section.

7.1. The Sum of Subsets of an s-set

Definition 7.1. Let S be an s-set and let A and B be subsets of S. The sum $A + B$ of A and B is the set

$$A \cup B \cup \{a+b \mid a \in A,\ b \in B\}.$$

The sum of a finite number of subsets of S, say $A_1, A_2, \cdots A_n$ $(n \geq 2)$, is defined recursively by

$$\sum_{i=1}^{n} A_i = \sum_{i=1}^{n-1} A_i + A_n$$

where $\sum_{i=1}^{1} A_i$ is defined to be A_1. In the case where $A_i = A$ for $i = 1, 2, \cdots, n$ $(n \geq 1)$, we denote $\sum_{i=1}^{n} A_i$ by nA.

We note that an equivalent definition of the sum of n sets $A_1, \cdots, A_n$ is given by

$$\sum_{i=1}^{n} A_i = S \cap \{a_1 + \cdots + a_n \mid a_i \in A_i^0, \ 1 \le i \le n\} .$$

7.2. K-density Results Involving the Sum of Sets

In all that follows $(S, \mathcal{J})$ is a density space, and A and B are arbitrary subsets of S. We write $C = A+B$ and furthermore, we write $d(A, \mathcal{J}) = \alpha$, $d(B, \mathcal{J}) = \beta$, and $d(C, \mathcal{J}) = \gamma$.

Theorem 7.1. $\gamma \geq \max\{\alpha, \beta\}$.

Proof. From Definition 7.1 we have $A, B \subseteq C$ so that, by Theorem 6.9(ii), we have $\alpha \leq \gamma$ and $\beta \leq \gamma$.

Theorem 7.2. If $x \in \overline{C}$, then

$$A(L(x)) + B(L(x)) \leq S(L(x)) - 1 .$$

Proof. Let $A \cap L(x) = \{a_1, a_2, \cdots, a_n\}$. Since $x \in \overline{C}$ it follows from Definition 7.1 that $x \in \overline{B}$ also. Furthermore, for each i, $1 \leq i \leq n$, we have $x - a_i \in L(x)$. Also $x - a_i \in \overline{B}$ since, if $x - a_i \in B$, then $a_i + (x - a_i) = x$ would be in C contrary to hypothesis. Thus, letting X denote the set $\{x, x-a_1, \cdots, x-a_n\}$, we have $X \subseteq \overline{B} \cap L(x)$. Hence

$$A(L(x)) + 1 = S(X) \leq \overline{B}(L(x)) = S(L(x)) - B(L(x)) ,$$

and the theorem follows.

Theorem 7.3. If $\alpha + \beta \geq 1$, then $\gamma = 1$.

Proof. By Theorem 2.23 we have $L(x) \in \mathcal{F}$ for each $x \in S$. If $\gamma < 1$, then by Theorem 6.3, there is an $x \in \overline{C}$. By Theorem 7.2

$$A(L(x)) + B(L(x)) \leq S(L(x)) - 1 .$$

Dividing this inequality by $S(L(x))$, and applying the definition of α and β we obtain

$$\alpha + \beta \leq q(A, L(x)) + q(B, L(x)) \leq 1 - 1/S(L(x)) < 1 .$$

This contradicts our assumption that $\alpha + \beta \geq 1$, and the theorem follows.

The following theorem offers a useful property of finite subsets of S.

Theorem 7.4. If X is a non-empty finite subset of S, then we may index the elements of X, $X = \{x_1, x_2, \cdots, x_n\}$ in such a way that the following condition is satisfied:

$$\text{If } x_i \prec x_j, \text{ then } i < j .$$

Proof. Let $x_1 \in \mathrm{Min}(X)$. Take $x_2 \in \mathrm{Min}(X \setminus x_1)$, $x_3 \in \mathrm{Min}(X \setminus \{x_1, x_2\})$, and so on. Theorem 2.3 and the finiteness of

X, $X\setminus x_1$, $X\setminus\{x_1, x_2\}, \cdots$ assure us that this process exhausts X. We show that the resulting indexing of X satisfies the stated condition. Let $x_i \prec x_j$. Since $x_i \neq x_j$ we have either $j < i$ or $i < j$. If $j < i$, then $x_i \in X\setminus\{x_1, \cdots x_{j-1}\} = Y$. But $x_j \in \mathrm{Min}(Y)$ so that $x_i \prec x_j$ is impossible. Thus we must have $i < j$.

The following theorem is used to prove Theorems 7.6 and 7.7.

<u>Theorem 7.5.</u> Suppose that $\mathcal{F}$ is transformation -2 (Definition 4.2), and that $\overline{C}$ is non-empty. Let $F \in \mathcal{F}$ such that $F^* \subseteq \overline{C}$. If H is any set such that $F^* \subseteq H \subseteq \overline{C} \cap F$, then

$$\overline{B}(F) \geq \alpha(S(F) - S(H)) + S(H).$$

Proof. Let $g_1, g_2, \cdots, g_n$ be the points of H, indexed according to Theorem 7.4, so that

(1) $g_i \prec g_j$ implies $i < j$.

Let $D_1 = [g_1]$ and for $1 \leq i < n$, let $D_{i+1} = [g_{i+1}]\setminus \cup\{[g_j] \mid 1 \leq j \leq i\}$. Then we have

(2) $D_i \cap D_j$ is empty if $i \neq j$,

(3) $\cup\{D_i \mid 1 \leq i \leq n\} = F$,

and

(4) $g_i \in D_i$ for $1 \leq i \leq n$.

Now (2) follows immediately from the definition of the D_i. To see (3), notice that since $F^* \subseteq H$, we have by Theorem 2.16, for each $x \in F$ that $x \in [g_i]$ for some i. Since

$$\cup \{D_i \mid 1 \leq i \leq n\} = \cup \{[g_i] \mid 1 \leq i \leq n\},$$

then (3) follows. To prove (4) suppose that $g_i \notin D_i$ for some i. Clearly $i > 1$. Then $g_i \in \cup \{[g_j] \mid 1 \leq j \leq i-1\}$ so that $g_i \in [g_{j_0}]$ for some $j_0 < i$. Hence, by Theorem 2.18, we have $g_i \prec g_{j_0}$, which by property (1) implies $i < j_0$, a contradiction.

Now since $\mathcal{F}$ is transformation -2 and because of the way each D_i is defined, we have

(5) $T_2[D_i] \in \mathcal{F}$ or is empty,

(6) $S(T_2[D_i]) = S(D_i)-1$.

Now, for each $a \in A \cap T_2[D_i]$, there exists a unique $x \in D_i \setminus g_i$ such that $a = g_i - x$. It follows that $x \in \overline{B}$ for, otherwise $a + x = g_i$ would be in C, contrary to hypothesis. Also from (4), we know $g_i \in \overline{B} \cap D_i$ and so

$$\begin{aligned} \overline{B}(D_i) &\geq A(T_2[D_i]) + 1 \\ &\geq \alpha S(T_2[D_i]) + 1. \end{aligned}$$

The second inequality follows from (5) and the definition of α. Thus, from (6), we have

(7) $\overline{B}(D_i) \geq \alpha(S(D_i)-1) + 1$.

Summing (7) over i, applying Theorem 6. 1(i) in view of (2) and (3), we obtain,

(8) $\overline{B}(F) \geq \alpha(S(F)-n) + n$.

Now, $n = S(H)$ and so (8) becomes

$$\overline{B}(F) \geq \alpha(S(F)-S(H))+S(H)$$

which is the desired result.

Note that the property (1) is used only in the proof of (4). In the case $H = F^*$, the proof can be simplified somewhat by using instead, Theorem 2. 19 to prove (4).

The following theorem is similar to Theorem 7. 2. It can be used in much the same way to prove Theorem 7. 3 with the additional hypothesis that the family $\mathcal{F}$ is transformation -2, although we omit this proof. More importantly it is used to prove Theorem 8. 4.

Theorem 7. 6. Suppose $\mathcal{F}$ is transformation -2. Let $\overline{C}$ be non-empty and let $F \in \mathcal{F}$ such that $F^* \subseteq \overline{C}$. Then

$$\alpha S(F) + B(F) < S(F) .$$

Proof. By the preceeding theorem we have

$$\overline{B}(F) \geq \alpha S(F) + (1-\alpha)S(H)$$

for any H with $F^* \subseteq H \subseteq \overline{C} \cap F$. By Theorem 7.1, since $\gamma < 1$, we have $\alpha < 1$ so that $(1-\alpha)S(H) > 0$. Hence

$$\overline{B}\,(F) > \alpha S(F) \quad .$$

Since $\overline{B}(F) = S(F) - B(F)$ we have the desired result.

Theorem 7.7. Suppose $\mathcal{F}$ is transformation -2. If $\overline{C}$ is non-empty and $F \epsilon \mathcal{F}$ such that $F^* \subseteq \overline{C}$, then

$$C(F) \geq \alpha C(F) + B(F).$$

Proof. In Theorem 7.5 let $H = \overline{C} \cap F$. Then $S(H) = \overline{C}(F)$ and

$$\overline{B}(F) \geq \alpha(S(F) - \overline{C}(F)) + \overline{C}(F) \, .$$

Since $\overline{B}(F) = S(F) - B(F)$ and $\overline{C}(F) = S(F) - C(F)$, we have

$$C(F) \geq \alpha C(F) + B(F)$$

and the theorem is proved.

The following theorem generalizes the inequality of Schur mentioned in the Introduction. This accounts for the only generalization known to this writer of this inequality.

Theorem 7.8. If the fundamental family $\mathcal{F}$ is transformation -2 and $\alpha + \beta < 1$, then

(9) $\gamma \geq \beta/(1-\alpha)$.

Proof. If $\gamma = 1$, then, since $\beta/(1-\alpha) < 1$, the theorem follows. If $\gamma < 1$, then $\overline{C}$ is non-empty, and by Theorem 7.7, if $F \in \mathcal{F}$ and $F^* \subseteq \overline{C}$, then

$$C(F) \geq \alpha C(F) + B(F) .$$

Dividing by $S(F)$ we obtain

$$q(C,F) \geq \alpha q(C,F) + q(B,F)$$

$$\geq \alpha\gamma + \beta .$$

We apply Theorem 6.7 and obtain that

$$\gamma \geq \alpha\gamma + \beta ,$$

which, because $\alpha < 1-\beta \leq 1$, is equivalent to

$$\gamma \geq \beta/(1-\alpha) ,$$

and the theorem follows.

The next theorem leads to the generalization of the Landau-Schnirelmann inequality. The proof is a refinement of the proof given by Kvarda [11] for the specialization of this theorem to the density space $(I^n, \mathcal{H})$.

Theorem 7.9. Suppose $\mathcal{F}$ satisfies the first transformation property (Definition 4.1). Then for each $F \in \mathcal{F}$, with $\mathrm{Min}(F) \subseteq A$ we have

(10) $C(F) \geq A(F) + \beta \bar{A}(F)$.

Proof. Let F be an arbitrary set of $\mathcal{F}$ such that $\mathrm{Min}(F) \subseteq A$. By Theorem 2.3, we then have $\phi \neq \mathrm{Min}(F) \subseteq A \cap F$ and so $A \cap F \neq \phi$. Let the set $A \cap F$ be indexed $\{a_1, a_2, \cdots, a_n\}$ where, according to Theorem 7.4, we may assume that

(11) if $a_i \prec a_j$, then $i < j$.

We define inductively $D_i = U(a_i) \cap G_i \cap F$ where $G_i = \cup \{L(x) | x$ satisfies conditions $N(i)\}$.

$$\text{Conditions } N(i) \begin{cases} \text{(a)} & a_i \in L(x) \\ \text{(b)} & a_j \notin L(x) \quad \text{if} \quad j > i \ . \end{cases}$$

Now $G_i \cap F = \cup \{L(x) \cap F | x$ satisfies $N(i)\}$ which is a finite union of sets in $\mathcal{F}$ (since each $L(x) \cap F$ is a set in $\mathcal{F}$, and $L(x) \cap F \subseteq F$ so that only finitely many such sets are possible). Thus $G_i \cap F \in \mathcal{F}$, and clearly $a_i \in G_i \cap F$. Hence, since $\mathcal{F}$ is transformation -1, and because of the way D_i is defined, we have for each i, $1 \leq i \leq n$, that

(12) $T_1[D_i] \in \mathcal{F}$ or $T_1[D_i]$ is empty.

Also, from the definition of $T_1[D_i]$, it is clear that

(13) $S(T_1[D_i]) = S(D_i)$.

We prove next that the sets D_i, $1 \leq i \leq n$, satisfies the following two properties:

(14) If $i \neq j$, then $D_i \cap D_j$ is empty,

and

(15) $\cup \{D_i \mid 1 \leq i \leq n\} = \overline{A} \cap F$.

To prove (14) let $i < j$ and let $y \in D_j$. If $y \in D_i$, then there exists an x satisfying Conditions $N(i)$ such that $y \in U(a_i) \cap L(x) \cap F$. But then, since $j > i$ and $a_j \in L(y) \subseteq L(x)$, Condition $N(i)(b)$ is violated. We conclude that $y \notin D_i$ and property (14) follows.

To show (15) we first prove that $D_i \subseteq \overline{A} \cap F$ for each i, $1 \leq i \leq n$. Let $y \in D_i$. There exists an x satisfying Conditions $N(i)$ such that $y \in U(a_i) \cap L(x) \cap F$. Thus $a_i \prec y$ so that if $y = a_j$, then we would have by (11) that $i < j$. But then Condition $N(i)(b)$ is violated. Thus $y \neq a_j$ for all j and so $y \notin A \cap F$. Clearly $y \in F$ so that $y \in \overline{A} \cap F$. It follows that

$$\cup \{D_i \mid 1 \leq i \leq n\} \subseteq \overline{A} \cap F.$$

We prove now the reverse inclusion. Let $x \in \overline{A} \cap F$. Choose i_0 to be the largest index such that $a_{i_0} \prec x$. Since $\mathrm{Min}(F) \subseteq A$, such an index must exist by Theorem 2. 4. We show that x satisfies the Conditions $N(i_0)$. We have $a_{i_0} \in L(x)$ since $a_{i_0} \prec x$. By the maximality of i_0, $a_j \notin L(x)$ for $j > i_0$. Thus

Conditions $N(i_0)(a,b)$ are satisfied and so $x \in U(a_{i_0}) \cap L(x) \cap F \subseteq D_{i_0}$. We conclude that $\overline{A} \cap F \subseteq \cup \{D_i \mid 1 \leq i \leq n\}$ and the proof of (15) is complete.

Now for each $b \in B \cap T_1[D_i]$ we have that $b = x - a_i$ where $x \in D_i$ is uniquely determined. It follows that $x = a_i + b \in C$ so we have that

$$(16) \quad C(D_i) \geq B(T_1[D_i]) .$$

Now (14) and (15) imply

$$(17) \quad \overline{A}(F) = \sum_{i=1}^{n} S(D_i).$$

Thus, we have (giving justifications to the right) that

$$
\begin{aligned}
C(F) &= A(F) + C(\overline{A} \cap F) && \\
&= A(F) + C(\cup \{D_i \mid 1 \leq i \leq n\}) && (15) \\
&= A(F) + \sum_{i=1}^{n} C(D_i) && \text{Theorem 6. 1(i)} \\
&\geq A(F) + \sum_{i=1}^{n} B(T_1[D_i]) && (16) \\
&\geq A(F) + \beta \sum_{i=1}^{n} S(T_1[D_i]) && \text{(12) and Def. of } \beta \\
&\geq A(F) + \beta \sum_{i=1}^{n} S(D_i) && (13) \\
&= A(F) + \beta\, \overline{A}(F) . && (17)
\end{aligned}
$$

This completes the proof.

Theorem 7.10. If $\mathcal{F}$ is transformation -1, then

$$\gamma \geq \alpha + \beta - \alpha\beta .$$

Proof. If $\alpha = 0$, then the inequality reduces to $\gamma \geq \beta$ which is true by Theorem 7.1. Thus we assume that $\alpha > 0$. By Theorem 6.6, we have that A contains all the essential points of $(S, \mathcal{F})$ so that, by Theorem 2.20, we have that $\mathrm{Min}(F) \subseteq A$ for any $F \in \mathcal{F}$. Thus, for each $F \in \mathcal{F}$ we have, by Theorem 7.9, that

$$\begin{aligned} C(F) &\geq A(F) + \beta\,\overline{A}(F) \\ &= A(F)\,(1-\beta) + \beta\, S(F) \\ &\geq \alpha(1-\beta)\, S(F) + \beta\, S(F) \\ &= (\alpha + \beta - \alpha\beta)\, S(F) . \end{aligned}$$

Dividing by $S(F)$ we obtain

$$q(C, F) \geq \alpha + \beta - \alpha\beta$$

for each $F \in \mathcal{F}$ and the theorem follows.

7.3. A Theorem on Bases

In this section we prove a famous result on bases. Let $(S, \mathcal{F})$ be a density space. We begin with the definition of a basis.

Definition 7.2. A set $A \subseteq S$ is a basis for S if $nA = S$ for some positive integer n (see Definition 7.1).

Theorem 7.11. If $d(A, \mathcal{F}) > 0$, then A is a basis for S.

Proof. By Theorem 2.22 we have $\mathcal{K}(S) \subseteq \mathcal{F}$, and by Theorem 6.9(i) we have $d(A, \mathcal{K}) \geq d(A, \mathcal{F})$, so that $d(A, \mathcal{K}) > 0$. Let $\alpha = d(A, \mathcal{K})$. It suffices to show that, if $\alpha > 0$, then A is a basis for S. We give two proofs, one based upon Theorem 7.8 and the other upon Theorem 7.10.

First, by Theorem 4.1, $\mathcal{K}$ is transformation -2 and so we may apply Theorem 7.8. For $n \geq 1$ denote the number $d(nA, \mathcal{K})$ by γ_n. By Theorem 6.3 it is sufficient to show that $\gamma_n = 1$ for some integer n.

If $\alpha = 1$, then $\gamma_1 = \alpha = 1$ and we are done. Hence assume $\alpha < 1$. Since by definition $kA = A+(k-1)A$ for $k \geq 2$, then if $\alpha + \gamma_{k-1} \geq 1$, we have by Theorem 7.3 that $\gamma_k = 1$. If $\alpha + \gamma_{k-1} < 1$, then by Theorem 7.8, we have

$$\gamma_k \geq \gamma_{k-1}/(1-\alpha) .$$

Hence

(18) $\gamma_k \geq \min\{1, \gamma_{k-1}/(1-\alpha)\}$. $(k \geq 2)$

Now we prove by induction that

(19) $\min\{1, \gamma_m/(1-\alpha)\} \geq \min\{1, \alpha/(1-\alpha)^m\}$.

For $m = 1$ we have equality. Now, for $m \geq 2$, we have by (18) that

$$\min \{1, \gamma_m/(1-\alpha)\}$$

$$\geq \min \{1, (\min \{1, \gamma_{m-1}/(1-\alpha)\})/(1-\alpha)\}$$

$$\geq \min \{1, (\min \{1, \alpha/(1-\alpha)^{m-1}\})/(1-\alpha)\}$$

$$= \min \{1, \min \{1/(1-\alpha), \alpha/(1-\alpha)^m\}\}$$

$$= \min \{1, 1/(1-\alpha), \alpha/(1-\alpha)^m\}$$

$$= \min \{1, \alpha/(1-\alpha)^m\} .$$

Hence, from (18) and (19) we have

$$\gamma_k \geq \min \{1, \alpha/(1-\alpha)^{k-1}\} . \qquad (k \geq 2)$$

Since $\alpha > 0$, then for sufficiently large n we have $\alpha/(1-\alpha)^{n-1} \geq 1$, and so $\gamma_n = 1$. This completes the proof.

Alternatively, by Theorem 4. 1, $\mathcal{H}$ is transformation -1 and we may apply Theorem 7. 10. Using the same notation as above we have by Theorem 7. 10 that

$$\gamma_k \geq \alpha + \gamma_{k-1} - \alpha\gamma_{k-1}, \qquad (k \geq 2)$$

which is equivalent to

$$(1-\gamma_k) \leq (1-\alpha)(1-\gamma_{k-1}). \qquad (k \geq 2)$$

Thus it follows by induction on k that

$$(1-\gamma_k) \leq (1-\alpha)^k .$$

Since $\alpha > 0$, we have for sufficiently large k, that

$$(1-\alpha)^k < \alpha .$$

Hence for n sufficiently large

$$(1-\gamma_{n-1}) < \alpha .$$

Thus $\alpha + \gamma_{n-1} > 1$ and so, by Theorem 7.3,

$$A + (n-1)A = nA = S,$$

and the proof is complete.

CHAPTER 8

C - DENSITY

In this chapter we develop some of the properties of C-density (Definition 6.3). Generally, little is known about C-density, and the results obtained are sometimes weaker than those obtained for K-density. On the other hand, the simple nature of the C-density seems to permit a wider range of possibilities for future research.

8.1. C-density Results Involving the Sum of Sets

Let $(S, \mathfrak{F})$ be an arbitrary density space. Let A and B be subsets of S. We write $C = A+B$, $d_c(A, \mathfrak{F}) = \alpha_c$, $d_c(B, \mathfrak{F}) = \beta_c$ and $d_c(C, \mathfrak{F}) = \gamma_c$. Our first theorem is an immediate consequence of Theorem 6.5.

Theorem 8.1. If the family $\mathfrak{F}$ is separated (Definition 5.1), then all the density theorems of Chapter 7, for K-density, are true for C-density. Thus

(i) $\alpha_c + \beta_c \geq 1$ implies $\gamma_c = 1$,

(ii) $\gamma_c \geq \beta_c/(1-\alpha_c)$ if $\alpha_c + \beta_c < 1$ and $\mathfrak{F}$ is transformation -2,

(iii) $\gamma_c \geq \alpha_c + \beta_c - \alpha_c\beta_c$, if $\mathfrak{F}$ is transformation -1,

(iv) $\alpha_c > 0$ implies A is a basis for S.

Accordingly, for the remainder of this section we will make no assumption as to whether the family $\mathcal{F}$ is separated or not.

Theorem 8.2. $\gamma_c \geq \max\{\alpha_c, \beta_c\}$.

Proof. From Definition 7.1 we have that $A, B \subseteq C$ so that, by Theorem 6.9 (iii), $\gamma_c \geq \alpha_c$ and $\gamma_c \geq \beta_c$.

None of the parts of Theorem 8.1 are known to hold in general without the separated hypothesis. However, we obtain other special theorems corresponding to the parts of Theorem 8.1 by applying the methods of Chapter 7. Corresponding to Theorem 8.1(i) we have the following two theorems.

Theorem 8.3. If $\mathcal{F} = \mathcal{K}(S)$ and $\alpha_c + \beta_c \geq 1$, then $\gamma_c = 1$.

Proof. Since $\mathcal{F} = \mathcal{K}$ we have by Theorem 3.8(iv) that $[x] = L(x)$ for each $x \in S$. If $\gamma_c < 1$ then there exists an $x \in \overline{C}$ and so, by Theorem 7.2, we have

$$A(L(x)) + B(L(x)) \leq S(L(x)) - 1 .$$

Thus, dividing by $S(L(x))$, we have

$$q(A, L(x)) + q(B, L(x)) < 1 ,$$

and so $\alpha_c + \beta_c < 1$. The theorem is proved.

Theorem 8.4. Let $\mathcal{F}$ be transformation -2 and let α

denote the K-density of A with respect to $\mathcal{F}$. If $\alpha + \beta_c \geq 1$, then $\gamma_c = 1$.

Proof. Suppose $\gamma_c < 1$ and let x be an arbitrary point in $\bar{C}$. We recall that $[x]$ is a set in $\mathcal{F}$ and $[x]^* = \{x\}$ so that $[x]^* \subseteq \bar{C}$. By Theorem 7.6 we have

$$\alpha S([x]) + B([x]) < S([x]) .$$

Dividing this inequality by $S([x])$ we obtain

$$\alpha + q(B, [x]) < 1 .$$

Thus $\alpha + \beta_c < 1$, contrary to hypothesis, and the proof is complete.

Since $\beta_c \geq \beta = d(B, \mathcal{F})$ (Theorem 6.3) we see that $\alpha + \beta \geq 1$ implies $\alpha + \beta_c \geq 1$, but not conversely. Thus we conclude that, when applicable, Theorem 8.4 is a stronger result than Theorem 7.3. However, Theorem 7.3 is always applicable while Theorem 8.4 requires that $\mathcal{F}$ be transformation -2.

The methods of Chapter 7 fail to provide a satisfying result corresponding to Theorem 8.1(ii). Using a similar argument as in the proof of Theorem 8.4 we see that, if $\mathcal{F}$ is transformation -2 and $x \epsilon \bar{C}$, then by Theorem 7.7, we have

$$C([x]) \geq \alpha C([x]) + B([x]) ,$$

and so

$$q(C, [x]) \geq \alpha\, q(C, [x]) + q(B, [x])$$

$$\geq \alpha \gamma_c + \beta_c \quad .$$

We cannot, however, replace the left hand side of this inequality by γ_c since we do not have in general that $\gamma_c = \text{glb}\,\{q(C, [x]) \mid x \in \overline{C}\}$ as is shown by the example which follows Theorem 6.7.

On the other hand Theorem 7.9 provides the following result which corresponds to Theorem 8.1(iii).

Theorem 8.5. If $\mathcal{F}$ is transformation -1, then $\gamma_c \geq \alpha_c + \beta - \alpha_c \beta$.

Proof. We can assume $\alpha_c > 0$ since, if $\alpha_c = 0$, then the inequality reduces to $\gamma_c \geq \beta$, which clearly holds, since $\gamma_c \geq \beta_c$ (Theorem 8.2), and $\beta_c \geq \beta$ (Theorem 6.3). Let x be an arbitrary point of S. Then $[x] \in \mathcal{F}$, and since $\alpha_c > 0$, we have by Theorems 2.20 and 6.6 that $\text{Min}([x]) \subseteq A$. Thus by Theorem 7.9 we obtain

$$\begin{aligned} C([x]) &\geq A([x]) + \beta \overline{A}([x]) \\ &= A([x])(1-\beta) + \beta S([x]) \\ &\geq \alpha_c (1-\beta) S([x]) + \beta S([x]) \\ &= (\alpha_c + \beta - \alpha_c \beta)\, S([x]) \quad , \end{aligned}$$

and the theorem follows upon division by $S([x])$.

The methods of Chapter 7 do not seem to provide us with a C-density result corresponding to Theorem 8.1(iv).

8.2. The C-density on a Density Space which is a Product of Density Spaces

There is a method available for working with C-density which requires that the density space be a finite product of density spaces. The following discussion will introduce the concepts and notation which we will need.

Let $(S_\delta, \mathcal{J}_\delta)$ be a density space for each δ in a non-empty index set Δ. Let $(S, \mathcal{J}) = \mathbb{P}\{(S_\delta, \mathcal{J}_\delta) \mid \delta \in \Delta\}$. Thus $S = \overline{\Pi}\{S_\delta \mid \delta \in \Delta\}$ and $\mathcal{J} = \mathbb{P}\{\mathcal{J}_\delta \mid \delta \in \Delta\}$ (see Theorems 3.4 and 3.9). We consider the two different projections on S given in the following defintion.

<u>Definition 8.1.</u> Let f be a function in S and δ a point in Δ. Then define

$$P_\delta(f) = f(\delta) ,$$

and

$$p_\delta(f) = g$$

where g is the function in S defined by the formula

$$g(\lambda) = \begin{cases} f(\lambda) & \text{if } \lambda \neq \delta , \\ 0 & \text{if } \lambda = \delta . \end{cases}$$

Sometimes we allow f to be the zero function (in S^0). In this case we define $P_\delta(0) = 0$ and $p_\delta(0) = 0$.

<u>Definition 8. 2.</u> We adopt the following notation. As usual, the Cheo set of $\mathcal{F}$ determined by a point $f \in S$ is denoted by $[f]$, but in order to avoid any possible confusion, the Cheo set of $\mathcal{F}_\delta$ determined by a point $s \in S_\delta$ is denoted by $H_\delta(x)$.

Recall from Theorems 3. 9 and 3. 7, that $[f] = \overline{\Pi}\{H_\delta(f(\delta)) \mid \delta \in \Delta\}$.

<u>Definition 8. 3.</u> Denote by $S_{(\delta)}$ the set of all $f \in S$ with the property that $f(\lambda) = 0$ for all $\lambda \neq \delta$. Let $A \subseteq S$ and denote the set $A \cap S_{(\delta)}$ by $A_{(\delta)}$. Finally, let $A_\delta = \{P_\delta(f) \mid f \in A_{(\delta)}\} = P_\delta(A_{(\delta)})$.

Notice that $S_{(\delta)}$ and S_δ are different since $S_{(\delta)}$ is a subset of S and S_δ is not. However, the correspondence

$$f \longleftrightarrow P_\delta(f)$$

establishes an isomorphism between $S_{(\delta)}$ and S_δ.

<u>Theorem 8. 6.</u> If $f \in S_{(\delta)}$, then

$$A([f]) = A_\delta(H_\delta(f(\delta))) .$$

Proof. We establish a one-to-one correspondence between the sets $A \cap [f]$ and $A_\delta \cap H_\delta(f(\delta))$. Let $g \in A \cap [f]$. Since $g \in [f]$ we have $g(\lambda) \in H_\lambda^0(f(\lambda))$ for each $\lambda \in \Delta$. Here $H_\lambda^0(x) = H_\lambda(x) \cup \{0\}$. Since $f(\lambda) = 0$ for all λ such that $\lambda \neq \delta$ we have $g(\lambda) \in H_\lambda^0(0) = \{0\}$, i.e. $g(\lambda) = 0$, for $\lambda \neq \delta$. Thus $g \in S_{(\delta)}$, and since $g \in A$, we have $g \in A_{(\delta)}$. Hence $P_\delta(g) \in A_\delta$. Since $g \in [f]$ we have $P_\delta(g) = g(\delta) \in H_\delta^0(f(\delta))$, and since $g(\delta) \neq 0$ we have $P_\delta(g) \in H_\delta(f(\delta))$. Thus $P_\delta(g) \in A_\delta \cap H_\delta(f(\delta))$.

Now if g_1 and g_2 are distinct points in $A \cap [f]$, then, since $g_1(\lambda) = 0 = g_2(\lambda)$ for $\lambda \neq \delta$, we must have $g_1(\delta) \neq g_2(\delta)$. Hence $P_\delta(g_1) \neq P_\delta(g_2)$.

So far we have shown that P_δ takes points of $A \cap [f]$, in a one-to-one manner, into points of $A_\delta \cap H_\delta(f(\delta))$. It remains only to show that each $x \in A_\delta \cap H_\delta(f(\delta))$ is the image under P_δ of some $g \in A \cap [f]$. Naturally, we take g to be the junction defined by

$$g(\lambda) = \begin{cases} 0 & \text{if } \lambda \neq \delta, \\ x & \text{if } \lambda = \delta. \end{cases}$$

From the definition of A_δ it follows that $g \in A$, and since $g(\lambda) \in H_\lambda^0(f(\lambda))$ for each $\lambda \in \Delta$, we have $g \in [f]$. The proof is complete.

Theorem 8.7. For each $\delta \in \Delta$ we have $d_c(A, \mathcal{J}) \leq d_c(A_\delta, \mathcal{J}_\delta)$.

Proof. We have for fixed δ,

$$\begin{aligned} d_c(A, \mathcal{J}) &\leq \text{glb}\left\{\frac{A([f])}{S([f])} \mid f \in S_{(\delta)}\right\} \\ &= \text{glb}\left\{\frac{A_\delta(H_\delta(f(\delta))}{S_\delta(H_\delta(f(\delta))} \mid f \in S_{(\delta)}\right\} \\ &= \text{glb}\left\{\frac{A_\delta(H_\delta(x))}{S_\delta(H_\delta(x))} \mid x \in S_\delta\right\} \\ &= d_c(A_\delta, \mathcal{J}_\delta) . \end{aligned}$$

The second step follows from Theorem 8.6.

For the remainder of this chapter we assume that the index set Δ is finite, and moreover, that $\Delta = \{1, 2, \cdots, n\}$ for some positive integer n. We will use the following theorem, which has been proved in less abstract form by Loomis and Whitney [16] and Kemperman [8].

Theorem 8.8. Let X be a finite non-empty subset of S^0. Then we have

$$(S^0(X))^{n-1} \leq \prod_{\delta=1}^{n} S^0(p_\delta(X)) ,$$

where $S^0(Z)$ is the number of elements in $S^0 \cap Z$.

Proof. The proof is by induction on n. In the case $n = 1$

we have $p_1(X) = \{0\}$ so that

$$(S^0(X))^{n-1} = 1 = S^0(p_1(X)) = \prod_{\delta=1}^{n} S^0(p_\delta(X)) .$$

In the case $n = 2$ we have that X is contained in the ordinary cartesian product of the two sets $(p_2(X))_1$ and $(p_1(X))_2$. Thus

$$\begin{aligned}(S^0(X))^{n-1} &= S^0(X) \le S^0((p_2(X))_1 \times (p_1(X))_2) \\ &= S^0(p_2(X))\cdot S^0(p_1(X)) = \prod_{\delta=1}^{n} S^0(p_\delta(X)) .\end{aligned}$$

Now let $n \geq 3$ and assume the theorem is true if n is replaced by $n-1$. For each $x \epsilon S_n^0$ let

$$Y_x = \{f \mid f \epsilon X, \quad f(n) = x \} .$$

Note that Y_x is non-empty for at most finitely many x. For $\delta = 1, 2, \cdots, n-1$, let

$$B_{\delta, x} = p_\delta(Y_x) .$$

Since the Y_x have fixed n^{th} coordinate we may apply our induction hypothesis and obtain, for $x \epsilon S_n^0$,

$$(1) \quad (S^0(Y_x))^{n-2} \le \prod_{\delta=1}^{n-1} S^0(B_{\delta, x}) .$$

This inequality holds even if Y_x is empty since, in that case, both sides are zero.

Now X is equal to the disjoint union of the Y_x where x ranges over S_n^0, so we have

$$(2)\quad S^0(X) = \sum_{x\in S_n^0} S^0(Y_x)\ ,$$

and, for $\delta = 1, 2, \cdots, n-1$,

$$(3)\quad S^0(p_\delta(X)) = \sum_{x\in S_n^0} S^0(B_{\delta,x})\ .$$

Furthermore, since $p_n(Y_x) \subseteq p_n(X)$, we have

$$(4)\quad S^0(Y_x) = S^0(p_n(Y_x)) \le S^0(p_n(X))\ .$$

Now, since $n > 2$, expression (1) can be seen to be equivalent to

$$(5)\quad (S^0(Y_x))^{1/2}\ \Big(\prod_{\delta=1}^{n-1} S^0(B_{\delta,x})\Big)^{1/2(n-2)}$$

$$\le (S^0(Y_x))^{1/(n-1)}\ \Big(\prod_{\delta=1}^{n-1} S^0(B_{\delta,x})\Big)^{1/(n-1)}\ .$$

In the following chain of inequalities we assume that all summations are over $x\in S_n^0$ and all products are from $\delta = 1$ to $\delta = n-1$ unless otherwise noted. We have

$$(6) \quad (S^0(X))^{n-1} = \{\Sigma S^0(Y_x)\}^{n-1}$$

$$= \{\Sigma [S^0(Y_x) \cdot S^0(Y_x)]^{1/2}\}^{n-1}$$

$$(7) \quad \leq \{\Sigma [S^0(Y_x) \cdot (\Pi S^0(B_{\delta,x}))^{1/(n-2)}]^{1/2}\}^{n-1}$$

$$(8) \quad \leq \{\Sigma [S^0(Y_x))^{1/(n-1)} \cdot (\Pi S^0(B_{\delta,x}))^{1/(n-1)}]\}^{n-1}$$

$$= \{\Sigma [\Pi ((S^0(Y_x))^{1/(n-1)} \cdot S^0(B_{\delta,x}))^{1/(n-1)}]\}^{n-1}$$

$$(9) \quad \leq \Pi (\Sigma [(S^0(Y_x))^{1/(n-1)} \cdot S^0(B_{\delta,x})])$$

$$(10) \quad \leq \Pi [(S^0(p_n(X))^{1/(n-1)} \cdot (\Sigma S^0(B_{\delta,x}))]$$

$$(11) \quad = \Pi [(S^0(p_n(X))^{1/(n-1)} \cdot S^0(p_\delta(X))]$$

$$= \prod_{\delta=1}^{n} S^0(p_\delta(X)) \quad .$$

Here, (6) follows from (2); (7) from (1); (8) from (5); (9) from the standard generalization of the Cauchy inequality which can be found, e.g., in [5]; (10) from (4); and (11) from (3). This completes the proof.

Our main result on C-density follows.

<u>Theorem 8.9.</u> Assume for each $\delta \in \Delta$, that $\mathcal{J}_\delta$ is transformation -1. Let A and B be arbitrary subsets of S. We denote by α_c and γ_c the numbers $d_c(A, \mathcal{J})$ and $d_c(C, \mathcal{J})$

where $C = A + B$. Furthermore, we write, for $\delta = 1, 2, \cdots, n$, β_δ to be the K-density $d(B_\delta, \mathcal{J}_\delta)$ and $\beta' = \min\{\beta_1, \cdots, \beta_n\}$. Then we have

$$\gamma_c \geq \alpha_c + \frac{1 - (1-\alpha_c)^{1/(n-1)}}{n} \beta'(1-\alpha_c) .$$

Proof. Let z be an arbitrary point in S. Let δ be a fixed element of Δ. Let $E_\delta = \{f \mid f \in [z]^0, f(\delta) = 0\}$. For each $f \in E_\delta$, let

$$Z_f = \{f + g \mid g \in G(f)\}$$

where

$$G(f) = \{g \mid g \in S^0_{(\delta)} \text{ such that } g(\delta) \in H^0_\delta(z(\delta))\}$$

if $f \neq 0$, and

$$G(0) = \{g \mid g \in S_{(\delta)} \text{ such that } g(\delta) \in H_\delta(z(\delta))\} .$$

We show that

$$(12) \quad [z] = \cup \{Z_f \mid f \in E_\delta\} ,$$

and that this union is disjoint. If $x \in [z]$, then $x = f + g$ where $f = p_\delta(x)$ and g is defined by

$$g(\lambda) = \begin{cases} 0 & \text{if } \lambda \neq \delta , \\ x(\delta) & \text{if } \lambda = \delta . \end{cases}$$

Now, $f \in [z]^0$ since

$$f(\lambda) = \begin{cases} x(\lambda) \in H_\lambda^0(z(\lambda)) & \text{if} \quad \lambda \neq \delta\,, \\ 0 \in H_\delta^0(z(\delta)) & \text{if} \quad \lambda = \delta. \end{cases}$$

Note, in particular, that $f(\delta) = 0$ so that $f \in E_\delta$. We have to show that $g \in G(f)$. Since $g(\lambda) = 0$ for all $\lambda \neq \delta$, we have $g \in S^0_{(\delta)}$. Thus it is sufficient to show that $g(\delta) \in H_\delta^0(z(\delta))$, if $f \neq 0$, and $g(\delta) \in H_\delta(z(\delta))$, if $f = 0$. Now $g(\delta) = x(\delta)$, and since $x \in [z]$ we know $x(\delta) \in H_\delta^0(z(\delta))$. In the case $f = 0$ we must have $x(\delta) \neq 0$, and so $x(\delta) \in H_\delta(z(\delta))$. Thus $x \in Z_f$, and so $[z] \subseteq \cup \{Z_f | f \in E_\delta\}$.

To prove the reverse inclusion notice that, if $x = f+g \in Z_f$, then $x(\lambda) = f(\lambda) \in H_\lambda^0(z(\lambda))$, for $\lambda \neq \delta$. Furthermore $x(\delta) = g(\delta) \in H_\delta^0(z(\delta))$. Hence $x \in [z]^0$. Since $f \neq 0$ implies $x \neq 0$, and $f = 0$ implies $g \neq 0$ implies $x \neq 0$, we have $x \neq 0$, and so $x \in [z]$. Thus $[z] \supseteq Z_f$, and so $[z] \supseteq \cup \{Z_f | f \in E_\delta\}$. Thus
(12) is proved, but it remains to show the disjointness of the union.

Let $f_1, f_2 \in E_\delta$ and suppose that $f_1 \neq f_2$. Let $x_1 = f_1 + g_1$ with $g_1 \in G(f_1)$ and $x_2 = f_2 + g_2$ with $g_2 \in G(f_2)$. Since $f_1 \neq f_2$ we must have $f_1(\lambda) \neq f_2(\lambda)$ for some $\lambda \neq \delta$ (since $f_1(\delta) = f_2(\delta) = 0$). But then $x_1(\lambda) = f_1(\lambda) \neq f_2(\lambda) = x_2(\lambda)$ so that $x_1 \neq x_2$. We conclude that $Z_{f_1} \cap Z_{f_2} = \phi$.

Now let f be a fixed element of E_δ. Let $a_{1,f}, a_{2,f}, \cdots, a_{k(f),f}$ be all the points of $G(f)$ such that $f + a_{i,f} \in A$. Let $a_i = P_\delta(a_{i,f})$ so that the set $\{a_1, \cdots, a_{k(f)}\}$ is a subset of S_δ. Moreover, by the definition of $G(f)$, each $a_i \in H^0_\delta(z(\delta))$. Let us assume, according to Theorem 7.4, that the indices are so arranged that $a_i \prec_\delta a_j$ implies $i < j$. Now, we define inductively $D_i = U_\delta(a_i) \cap G_i \cap H_\delta(z(\delta))$ where $G_i = \cup \{L_\delta(x) \mid x$ satisfies Conditions $N(i)\}$, where the Conditions $N(i)$ are given in the proof of Theroem 7.9. Here $L_\delta(x)$ is the lower set of x with respect to S_δ and similarly for $U_\delta(x)$. We have that the D_i satisfy the following properties .

(13) $T_1[D_i] \in \mathcal{J}_\delta$ or $T_1[D_i] = \phi$;

(14) $S_\delta(T_1[D_i]) = S_\delta(D_i)$;

(15) if $i \neq j$, then $D_i \cap D_j = \phi$;

(16) $\cup \{D_i \mid 1 \leq i \leq k(f)\}$

$$= (\cup \{U_\delta(a_i) \mid 1 \leq i \leq k(f)\}) \cap (H_\delta(z(\delta)) \setminus \{a_i \mid 1 \leq i \leq k(f)\}).$$

Now (13) follows since $\mathcal{J}_\delta$ is transformation -1, and (14) is immediate from the definition of $T_1[D_i]$.

To prove (15) let $i < j$ and let $y \in D_j$. If $y \in D_i$, then there is an x satisfying Conditions $N(i)$ such that $y \in U(a_i) \cap L(x) \cap H_\delta(z(\delta))$. But then, since $j > i$ and $a_j \in L(y) \subseteq L(x)$, Condition $N(i)(b)$ is violated. Hence $D_i \cap D_j = \phi$.

Notice the difference here in statement (16) as compared to

(15) of Chapter 7. This is necessary since we have no guarantee, as we had in Theorem 7.9, that $\operatorname{Min}(H_\delta(z(\delta))) \subseteq \{a_i \mid 1 \leq i \leq k(f)\}$. To prove (16) first let $x \in D_i$. Then $x \in U_\delta(a_i)$ so that $x \in \cup \{U_\delta(a_j) \mid 1 \leq j \leq k(f)\}$. Also $x \in H_\delta(z(\delta))$. We need only show that $x \neq a_j$ for $1 \leq j \leq k(f)$. Since $x \in U_\delta(a_i)$ we have $x \neq a_i$, and since $a_i \prec_\delta x$, if $x = a_j$ then by our special indexing $i < j$. Now $x \in D_i$ implies there exists y satisfying Conditions $N(i)$ such that $x \in U_\delta(a_i) \cap L(y) \cap H_\delta(z(\delta))$. In particular, $x \in L(y)$ and so $x \neq a_j$ for all $j > i$ by Condition $N(i)(b)$. Hence we have proved that the left hand side of (16) is included in the right hand side. To prove the reverse inclusion, first let $x \in (\cup \{U_\delta(a_i) \mid 1 \leq i \leq k(f)\}) \cap (H_\delta(z(\delta)) \setminus \{a_i \mid 1 \leq i \leq k(f)\})$. Let i_0 be the largest index such that $x \in U_\delta(a_{i_0})$. We show that x satisfies the Conditions $N(i_0)$. We have $a_{i_0} \in L_\delta(x)$ since $a_{i_0} \prec_\delta x$. By the maximality of i_0 we have $a_j \notin L(x)$ for $j > i_0$. Thus Conditions $N(i_0)$ are satisfied and so $x \in U_\delta(a_{i_0}) \cap H_\delta(z(\delta)) \cap L(x) \subseteq D_{i_0}$. This completes the proof of (16).

Now, for each i, let D'_i and D''_i be the subsets of S defined by

$$D'_i = \{x \mid x \in S_{(\delta)},\quad x(\delta) \in D_i\},$$

$$D''_i = \{x \mid x \in S_{(\delta)},\quad x(\delta) \in T_1[D_i]\}.$$

We have $D'_i \subseteq G(f)$, for if $g \in D'_i$, then $g \in S_{(\delta)}$ and

$g(\delta) \epsilon D_i \subseteq H_\delta(z(\delta))$, this last inclusion following immediately from (16). Now any point $x \epsilon [z]$ of the form $f + g$ $(g \epsilon G(f))$ with $x(\delta) \epsilon D_i$ is not in A, for we have $f(\delta) = 0$ and so $g \epsilon D_i'$. Since $g(\delta) \epsilon D_i$ we have $g(\delta) \neq a_j$ (by(16)) and so $g \neq a_{j,f}$ for all j, $1 \leq j \leq k(f)$. Thus $f + g \notin A$.

For each $b \epsilon B \cap D_i''$ we have $a_{i,f} + b \epsilon D_i' \subseteq G(f)$, and so $f + a_{i,f} + b \epsilon Z_f$. Also $f + a_{i,f} + b \epsilon \bar{A}$ since $(f + a_{i,f} + b)(\delta) \epsilon D_i$. Furthermore, since $f + a_{i,f} \epsilon A$ and $b \epsilon B$ we have $f + a_{i,f} + b \epsilon C$. Thus

$$f + a_{i,f} + b \epsilon (C \setminus A) \cap Z_f ,$$

and so, since (15) implies that the D_i' are disjoint,

$$C(Z_f) - A(Z_f) \geq \sum_{i=1}^{k(f)} B(D_i'') .$$

Hence,

$$
\begin{aligned}
C(Z_f) &\geq A(Z_f) + \sum_{i=1}^{k(f)} B(D_i'') \\
&= A(Z_f) + \sum_{i=1}^{k(f)} B_\delta(T_1[D_i]) \\
&\geq A(Z_f) + \beta_\delta \sum_{i=1}^{k(f)} S_\delta(T_1[D_i]) \\
&= A(Z_f) + \beta_\delta \sum_{i=1}^{k(f)} S_\delta(D_i) \\
&= A(Z_f) + \beta_\delta \sum_{i=1}^{k(f)} S(D_i') \ .
\end{aligned}
$$

The second step follows from the definition of D_i'' and B_δ; the third step from (13) and the definition of β_δ; the fourth from (14); and the last step from the definition of D_i'.

Denoting $\sum_{i=1}^{k(f)} S(D_i')$ by $M(f)$, we sum the last inequality over the set E_δ and obtain, using (12) and the disjointness of the Z_f,

$$
C([z]) \geq A([z]) + \beta_\delta \sum_{f \in E_\delta} M(f) \ .
$$

Letting $\sigma_\delta = \sum_{f \in E_\delta} M(f)$ we have

$$C([z]) \geq A([z]) + \beta_\delta \sigma_\delta \geq A([z]) + \beta' \sigma_\delta \quad .$$

Summing this inequality over the n values of δ and dividing by n, we obtain

$$(17) \quad C([z]) \geq A([z]) + \frac{\beta'}{n} \sum_{\delta=1}^{n} \sigma_\delta \quad .$$

We prove next that σ_δ is the number of elements $x \in [z]$ for which

(i) $x \in \overline{A}$,

(ii) there exists $a \in A \cap [z]$ such that $a(\lambda) = x(\lambda)$ for $\lambda \neq \delta$ and $x(\delta) \in U_\delta(a(\delta))$.

Suppose x satisfies Conditions (i) and (ii). Write $x = f + g$ where $f = p_\delta(x)$. Then, we can write the element a, whose existence is assured by (ii), as $a = f + a_{i,f}$. In view of (16), since $x(\delta) \in U_\delta(a(\delta))$ and $x(\delta) \neq a_{j,f}$ (since $x \in A$) we have that $x(\delta) \in D_j$ for some j. Thus $g \in \cup \{D'_j \mid 1 \leq j \leq k(f)\}$ and so x is counted by $M(f)$.

On the other hand, any $x = f + g \in Z_f$ with $g \in \cup \{D'_j \mid 1 \leq j \leq k(f)\}$ (i.e. any point counted by $M(f)$) clearly satisfies Conditions (i) and (ii). Thus we see that $\sigma_\delta = \sum_{f \in E_\delta} M(f)$ counts those and only those points which satisfy (i) and (ii).

Now, let Y be the set of all $x \in [z]$ which satisfy Condition (i), and for some δ, Condition (ii). Then by the preceding remarks we have

$$(18) \quad \sum_{\delta=1}^{n} \sigma_\delta \geq S(Y) = \bar{A}([z]) - S(X) \ ,$$

where $X = (\bar{A} \cap [z]) \setminus Y$. Thus X is the set of all $x \in \bar{A} \cap [z]$ for which Condition (ii) is satisfied for no δ.

We assume that, for each $x \in S_{(\delta)}$, $1 \leq \delta \leq n$, with $x(\delta) \in \mathrm{Min}(H_\delta(z(\delta)))$, that $x \in A$. Otherwise, applying Theorem 2.20 to obtain that $[x] = \{x\}$, we have $a_c = 0$ and the theorem clearly holds. It follows that, if $y \in S_{(\delta)} \cap [z]$, then $y \in Y$ or $y \in A$. Thus, we have shown that for each $x \in X$, $x(\delta) \neq 0$ for at least two distinct δ. Hence $p_\delta(x) \neq 0$ for each $x \in X$ and $1 \leq \delta \leq n$. Let $x \in X$. We have $(p_\delta(x))(\lambda) = x(\lambda)$ for $\lambda \neq \delta$. Now if $x(\delta) \neq 0$, then $x(\delta) \in S_\delta = U_\delta(0) = U_\delta((p_\delta(x))(\delta))$ so that, since Condition (ii) is satisfied for no δ, we have $p_\delta(x) \in \bar{A}$. If $x(\delta) = 0$, then $x = p_\delta(x)$, and so $p_\delta(x) \in \bar{A}$. Thus, for any δ, $1 \leq \delta \leq n$, we have

$$p_\delta(X) = \bar{A} \cap p_\delta(X) \subseteq \bar{A} \cap p_\delta([z]) = \bar{A} \cap [p_\delta(z)] \ .$$

The second relation follows from the fact that $X \subseteq [z]$. The last equality can be seen to hold by observing the expression for a Cheo set of $\mathcal{J}$. Using Theorem 8.8 we obtain

$$(19)\qquad (S(X))^{n-1} \leq \prod_{\delta=1}^{n} S(p_\delta(X))$$

$$\leq \prod_{\delta=1}^{n} \bar{A}([p_\delta(z)])$$

$$\leq (1-a_c)^n \prod_{\delta=1}^{n} S([p_\delta(Z)]) \ .$$

The first inequality follows from Theorem 8.8 since $S^0(X) = S(X)$ and (as we have $p_\delta(x) \neq 0$, $x \in X$, $1 \leq \delta \leq n$) $S^0(p_\delta(X)) = S(p_\delta(X))$. The last step follows from the relations

$$A([f]) \geq a_c S([f]) \ ,$$

and

$$\bar{A}([f]) = S([f]) - A([f]) \ ,$$

so that

$$(1-a_c)\, S([f]) \geq \bar{A}([f]) \ .$$

Now, making use of the formula

$$S([f]) = \Big(\prod_{\delta=1}^{n} \{S_\delta(H_\delta(f(\delta))) + 1\}\Big) - 1 \ ,$$

we have

$$\prod_{\delta=1}^{n} S([p_\delta(z)]) = \prod_{\delta=1}^{n} \Big(\frac{S([z]) + 1}{S_\delta(H_\delta(z(\delta))) + 1} - 1\Big)$$

$$\leq \prod_{\delta=1}^{n} \Big(\frac{S([z])}{S_\delta(H_\delta(z(\delta))) + 1}\Big)$$

$$= \frac{(S([z]))^n}{S([z]) + 1} \leq (S([z]))^{n-1} \ .$$

Hence (19) yields $S(X) \leq (1-a_c)^{n/(n-1)} S([z])$. Substituting this into (18) we obtain

$$(20) \quad \sum_{\delta=1}^{n} \sigma_\delta \geq \bar{A}([z]) - (1-a_c)^{n/(n-1)} S([z]),$$

and finally (17) and (20) give us

$$C([z]) \geq A([z]) + \frac{\beta'}{n}[\bar{A}([z]) - (1-a_c)^{n/(n-1)} S([z])]$$

$$= A([z])\left(1-\frac{\beta'}{n}\right) + \frac{\beta'}{n}\left(1-(1-a_c)^{n/(n-1)}\right)S([z]) .$$

Dividing by $S([z])$ we obtain

$$q(C,[z]) \geq q(A,[z])\left(1-\frac{\beta'}{n}\right) + \frac{\beta'}{n}\left(1-(1-a_c)^{n/(n-1)}\right)$$

$$\geq a_c\left(1-\frac{\beta'}{n}\right) + \frac{\beta'}{n}\left(1-(1-a_c)^{n/(n-1)}\right)$$

$$= a_c + \frac{(1-(1-a_c)^{1/(n-1)}}{n} \beta' (1-a_c) .$$

Since z is an arbitrary element of S, the proof is complete.

Under an additional assumption we may replace the β' by $\beta_c = d_c(B, \mathfrak{F})$ as shown in the following theorem.

Theorem 8.10. Under the hypotheses of Theorem 8.9, if we further assume that each $\mathfrak{F}_\delta$ is separated, then

$$\gamma_c \geq a_c + \frac{(1-(1-a_c)^{1/(n-1)})}{n} \beta_c(1-a_c) .$$

Proof. We have by Theorem 6.5 that $d_c(B_\delta, \mathfrak{F}_\delta) = d(B_\delta, \mathfrak{F}_\delta) = \beta_\delta$ for each δ, and by Theorem 8.7, $d_c(B, \mathfrak{F}) \leq d_c(B_\delta, \mathfrak{F}_\delta)$. Hence

$\beta_c = d_c(B, \mathcal{F}) \leq \min\{\beta_1, \cdots \beta_n\} = \beta'$. The theorem follows immediately.

The following special case of Theorem 8.10 improves the work of Kasch mentioned in the Introduction: Let $S_\delta = I$ and $\mathcal{F}_\delta = \mathcal{K}(I)$. Then $S = I^n$ and $\mathcal{F} = \mathbf{P}\{\mathcal{K}(I) \mid 1 \leq \delta \leq n\} = \mathcal{K}(I^n)$. Since $\mathcal{K}(I)$ is transformation -1 (Theorem 4.1) and separated (Theorem 5.3), it follows that the inequality of Theorem 8.10 holds. In Appendix 3 we show that this inequality betters those proved and conjectured by Kasch.

We conclude this chapter with simple applications of Theorems 8.9 and 8.10. Let $\Delta = \{1, 2\}$, $S_1 = I$, $S_2 = I^2$, $\mathcal{F}_1 = \mathcal{K}(I)$, and $\mathcal{F}_2 = \mathcal{K}(I^2)$. Then S is isomorphic with I^3, and $\mathcal{F} = \mathcal{K}(I^3)$. We have $\beta_1 = d(B_1, \mathcal{K}(I)) = d_c(B_1, \mathcal{K}(I)) \geq d_c(B, \mathcal{K}(S)) = \beta_c$. Hence $\beta' = \min\{\beta_1, \beta_2\} \geq \min\{\beta_c, \beta_2\} = \beta''$. By Theorem 8.9 we have

$$\gamma_c \geq \alpha_c + \frac{1-(1-\alpha_c)}{2} \beta'(1-\alpha_c)$$

$$= \alpha_c + \frac{\alpha_c}{2} \beta'(1-\alpha_c).$$

Also, since $\beta' \geq \beta''$, we have

$$\gamma_c \geq \alpha_c + \frac{\alpha_c}{2} \beta''(1-\alpha_c) .$$

Clearly, if $\beta_2 \geq \beta_c$, then $\beta'' \geq \beta_c$, and so

$$\gamma_c \geq \alpha_c + \frac{\alpha_c}{2} \beta_c(1-\alpha_c) \;.$$

We summarize this example by saying that if A and B are subsets of I^3, and if the K-density of the restriction of B to one of the 2-dimensional hyperplanes with respect to $\mathcal{K}(I^2)$ is greater than or equal to the C-density of B with respect to $\mathcal{K}(I^3)$, then the inequality

$$\gamma_c \geq \alpha_c + \frac{\alpha_c}{2} \beta_c(1-\alpha_c)$$

holds.

Now let $(I^4, \mathcal{F}) = \mathbf{P}\{(S_i, \mathcal{F}_i) \mid i = 1, 2\}$

where $S_1 = S_2 = I^2$ and $\mathcal{F}_1 = \mathcal{F}_2$ is the fundamental family on I^2 given in Example ff-4. Since $\mathcal{F}_i$ $(i = 1, 2)$ is separated we apply Theorem 8.10 to the density space $(I^4, \mathcal{F})$ and obtain the inequality

$$\gamma_c \geq \alpha_c + \frac{\alpha_c}{2} \beta_c (1-\alpha_c).$$

Many more applications like the preceding can be made.

CHAPTER 9

THE $\alpha + \beta$ THEOREM

In this chapter we discuss extensions of Mann's famous $\alpha + \beta$ theorem.

9.1. The $\alpha + \beta$ Conjecture

Let $(S, \mathcal{F})$ be an arbitrary density space. We make the following conjecture.

$\alpha + \beta$ Conjecture. If A and B are subsets of S, and $C = A + B$, and as usual $\alpha = d(A, \mathcal{F})$, $\beta = d(B, \mathcal{F})$ and $\gamma = d(C, \mathcal{F})$, then

$$\gamma \geq \min\{1, \alpha + \beta\} .$$

In this section and the next we treat some cases where this conjecture can be seen to hold.

The conjecture holds for the density space $(I, \mathcal{H})$. This is the famous theorem of H. Mann. We do not give a proof of this theorem but refer the reader to one of the following papers: Mann [17], Dyson [4], Khinchine [10], van der Corput-Kemperman [3].

We can use Mann's theorem to prove the $\alpha + \beta$ Conjecture for a new and general class of density spaces.

Theorem 9.1. Let S be an arbitrary s-set, and X a subset of S with the property that if x and y are distinct elements of X, then the equation $mx = ny$ is unsolvable in positive integers m and n. For each $z \in S$ define $R(z)$ by the formula

$$R(z) = \begin{cases} \{ix \mid 1 \leq i \leq n\} & \text{if } z = nx \ (x \in X, \ n \geq 1), \\ \{z\} & \text{otherwise} . \end{cases}$$

Then $R(z)$ satisfies Condition (b.1-3) of Theorem 3.6, and so $\mathcal{F}_R = \{F \mid F \in \mathcal{D}(S), \ z \in F \text{ implies } R(z) \subseteq F\}$ is a fundamental family on S. Finally, the $\alpha + \beta$ Conjecture holds for the density space $(S, \mathcal{F}_R)$.

Proof. We prove first that $R(z)$ satisfies Conditions (b.1-3) of Theorem 3.6. Clearly, $z \in R(z)$, and $R(z) \subseteq L(z)$. Now let $z' \in R(z)$. If $z' = z$, then clearly $R(z') \subseteq R(z)$. If $z' \neq z$, then we must have $z = nx$ and $z' = ix$ for some $x \in X$ and $1 \leq i \leq n$. But then

$$R(z') = \{jx \mid 1 \leq j \leq i\} \subset \{jx \mid 1 \leq j \leq n\} = R(z) .$$

Thus $\mathcal{F}_R$ is a fundamental family on S.

Now, by Theorem 3.7, we have that the Cheo set of $\mathcal{F}_R$ determined by $z \in S$ is just $R(z)$, i.e., $[z] = R(z)$. Let $z_1, z_2 \in S$ such that $z_1 \notin R(z_2)$ and $z_2 \notin R(z_1)$. If $R(z_1) = \{z_1\}$

or $R(z_2) = \{z_2\}$, then clearly $R(z_1) \cap R(z_2) = \phi$. Hence suppose that $R(z_1) = \{ix \mid 1 \leq i \leq n\}$ where $z_1 = nx$, $x \in X$ and $n \geq 1$, and similarly that $R(z_2) = \{jy \mid 1 \leq j \leq m\}$ where $z_2 = my$, $y \in X$ and $m \geq 1$. If $x = y$, then

$$(\min\{n, m\})\, x \in R((\max\{n, m\})x)$$

contrary to our assumption. Hence $x \neq y$, and if $z \in R(z) \cap R(z_2)$, then for some i and j we would have $ix = z = jy$ contradicting the stated property of the set X. Hence $R(z_1) \cap R(z_2) = \phi$ and $\mathcal{J}_R$ is separated (Definition 5. 1), and so, by Theorem 6. 5, we have for any $Y \subseteq S$,

$$\begin{aligned} d(Y, \mathcal{J}_R) &= d_c(Y, \mathcal{J}_R) \\ &= \operatorname{glb}\{q(Y, [z]) \mid z \in S\} \\ &= \operatorname{glb}\{q(Y, R(z)) \mid z \in S\} . \end{aligned}$$

Now let A, B and $C = A + B$ be subsets of S. Let $\alpha = d(A, \mathcal{J}_R)$, $\beta = d(B, \mathcal{J}_R)$ and $\gamma = d(C, \mathcal{J}_R)$. For fixed $x \in X$ consider the set $\{ix \mid i \geq 1\}$. This set is isomorphic to I, and we may apply Mann's theorem to obtain for each $i \geq 1$, that

$$q(C, R(ix)) \geq \min\{1, \alpha_x + \beta_x\}$$

where $\alpha_x = \operatorname{glb}\{q(A, R(ix)) \mid i \geq 1\}$ and similarly for β_x. Clearly $\alpha_x \geq \alpha$ and $\beta_x \geq \beta$ so that, for all $x \in X$ and $i \geq 1$, we have

$$q(C, R(ix)) \geq \min\{1, \alpha + \beta\} .$$

Now, suppose $z \neq ix$ for all $i \geq 1$ and all $x \in X$. If $z \in C$, then $q(C, R(z)) = 1 \geq \mathrm{Min}\{1, \alpha+\beta\}$. If $z \in C$, then $q(C, R(z)) = 0$ and so $\gamma = 0$ which implies, by Theorem 7.1, that $\alpha = \beta = 0$. Thus in all cases

$$q(C, R(z)) \geq \min\{1, \alpha+\beta\}$$

and the $\alpha+\beta$ Conjecture holds.

Note that the density space of Example ff-4 is a special case of that defined in Theorem 9.1. Here we take $S = \Gamma^n$ and X to be the set of all $(x_1, \cdots, x_n) \in \Gamma^n$ for which $\gcd\{x_1, \cdots, x_n\} = 1$. The $\alpha+\beta$ Conjecture holds for Example ff-4.

A method similar to the one used in the proof of the preceding theorem can be used to prove the following theorem. We state it without proof.

Theorem 9.2. Let S be an s-set, and u a fixed element of S, and X a subset of S with the following properties:

(i) $u \in X$,

(ii) if x and y are distinct elements of X then the equation $x+mu = y+nu$ is unsolvable in non-negative integers m and n. For each $z \in S$ define $H(z)$ by the formula

$$H(z) = \begin{cases} \{x+iu \mid 0 \leq i \leq n\} & \text{if } z = x+nu \ (x \in X, \quad n \geq 0), \\ \{z\} & \text{otherwise.} \end{cases}$$

Then $H(z)$ satisfies Conditions (b. 1-3) of Theorem 3.6, and so $\mathscr{F}_H$ is a fundamental family on S. Finally, the $\alpha+\beta$ Conjecture holds for the density space $(S, \mathscr{F}_H)$.

Note that the $\alpha+\beta$ Conjecture holds for Example ff-5 by this theorem. Here we take $S = I^n$ and, for fixed j, $1 \leq j \leq n$, take $u = e_j = (0, \cdots, 0, 1, 0, \cdots, 0)$ where the 1 appears in the j^{th} place, and take X to be the set consisting of u and all points $(x_1, \cdots, x_n) \in I^n$ with $x_j = 0$.

9. 2. Discrete Cases

We will call a fundamental family $\mathscr{F}$ on an s-set S discrete of order n if $\mathscr{F}$ satisfies the following two conditions: (i) $\mathscr{F}$ is separated; (ii) for each $x \in S$, $S([x]) \leq n$ with equality holding for some x.

Theorem 9. 3. For any s-set S, and for any positive integer n, there is a fundamental family on S which is discrete of order n.

Proof. Let x be a fixed element of S and let n be a fixed positive integer. For $y \in S$, define $B(y)$ by the formula

$$B(y) = \begin{cases} \{ix \mid 1 \leq i \leq n\} & \text{if } y = nx, \\ \{y\} & \text{otherwise.} \end{cases}$$

Then Conditions (b. 1-3) of Theorem 3.6 hold and thus $\mathcal{F}_B$ is a fundamental family on S. Clearly, if $y_1 \notin B(y_2)$ and $y_2 \notin B(y_1)$, then $B(y_1) \cap B(y_2) = \phi$ so that $\mathcal{F}_B$ is separated. Also $S(B(y)) \leq n$ for all y and $S(B(nx) = n$ so that $\mathcal{F}_B$ is discrete of order n.

Evidently, if $n > 1$, there will be many different discrete fundamental families of order n on S. For $n = 1$, $\mathcal{D}(S)$ is the only example of a discrete fundamental family of order 1 on S. The $\alpha + \beta$ Conjecture is known to hold in the discrete case only for $n = 1$ and $n = 2$ as shown in the following theorem.

Theorem 9.4. Let $(S, \mathcal{F})$ be a density space where $\mathcal{F}$ is discrete of order 1 or 2. Then the $\alpha + \beta$ Conjecture holds.

Proof. Let A, B and $C = A+B$ be subsets of S and define α, β and γ as usual.

1. The order of $\mathcal{F}$ is 1. Since $\mathcal{F}$ is separated we have that

$$\gamma = \operatorname{glb}\{q(C, [x]) \mid x \in S\}$$
$$= \operatorname{glb}\{q(C, \{x\}) \mid x \in S\} .$$

Hence the only possible values for γ are 0 or 1. If $\gamma = 1$ then $\gamma \geq \min\{1, \alpha + \beta\}$. If $\gamma = 0$, then, by Theorem 7.1, $0 \geq \max\{\alpha, \beta\}$, and so $\alpha + \beta = 0$, and the $\alpha + \beta$ -Conjecture holds in all cases.

2. The order of $\mathcal{F}$ is 2. As before

$$\gamma = \text{glb}\{q(C, [x])\,|\, x \in S\}.$$

Now, for each $x \in S$, $q(C,[x])$ has one of the values $0, \frac{1}{2}$ or 1. Thus $\gamma = 0, \frac{1}{2}$ or 1. Similar statements hold for α and β. If $\gamma = 0$ or 1 we argue as above to obtain $\gamma \geq \min\{1, \alpha+\beta\}$. Now, if $\gamma = \frac{1}{2}$ and $\alpha+\beta \leq \frac{1}{2}$ we are done. The case $\gamma = \frac{1}{2}$ and $\alpha+\beta > \frac{1}{2}$ is impossible since, if $\alpha+\beta > \frac{1}{2}$, then $\alpha+\beta \geq 1$ and so, by Theorem 7.3, $\gamma = 1$. The proof is complete.

The method used in proving the preceding theorem fails for $n \geq 3$.

9.3. Other Methods

In all of the cases where we have shown that the $\alpha+\beta$ Conjecture holds the fundamental family has been separated. Evidently much stronger methods will be required to prove the $\alpha+\beta$ Conjecture for many cases. There seems to be some hope in applying the methods of Mann and Dyson, although the author cannot report any particular successes along these lines. Both of these methods involve slight transformations of one or both of the sets A and B. It is true that both methods use the linearity properties of the integers and that these properties all but disappear in the general s-set. But both of these methods have been used in proving results, related to the $\alpha+\beta$ Theorem, concerning subsets of finite abelian groups where the linearity properties vanish.

On the other hand, the method used by Khinchine in proving the weaker result

$$\gamma \geq \min\{1, 2\alpha, 2\beta\}$$

seems more remote in its possible application to s-sets in general. A good account of Khinchine's proof can be found in Landau [15]. It is seen that a double induction is employed and the linearity properties of the integers are used over and over again. Such observations leave little hope in applying it in a more general setting.

CHAPTER 10

FURTHER PROBLEMS

10.1. Research Problems

In this section we give brief discussions of several general problems.

(i) Let $(S, \mathcal{F})$ be a density space. For a set $A \subseteq S$ let us define the modified Besicovitch (or Erdos) density of A, with respect to $\mathcal{F}$, to be

$$\alpha_1 = \text{glb}\left\{\frac{A(F)}{S(F)+1} \;\middle|\; F \in \mathcal{F},\quad A(F) < S(F)\right\} .$$

For the density space $(I^n, \mathcal{H})$, Kvarda [12] has shown that

(1) $C(F) \geq \alpha_1(S(F)+1) + B(F)$

for each $F \in \mathcal{H}(I^n)$ such that $F^* \subseteq \bar{C}$. Here A and B are arbitrary subsets of I^n and $C = A + B$.

Dividing (1) by $S(F)$ we obtain that

$$q(C, F) \geq \alpha_1 + \beta ,$$

and so, by Theorem 6.7, we have

(2) $\gamma \geq \alpha_1 + \beta$.

Moreover, if we divide (1) by $S(F) + 1$ we get

(3) $\dfrac{C(F)}{S(F)+1} \geq \alpha_1 + \dfrac{B(F)}{S(F)+1} \geq \alpha_1 + \beta_1$

for each $F \in \mathcal{F}$ with $F^* \subseteq \bar{C}$. We prove that

$$\gamma_1 = \text{glb}\{\frac{C(F)}{S(F)+1} \mid F \in \mathcal{F},\ \ F^* \subseteq \bar{C}\}.$$

Let γ' denote this glb. Since $F^* \subseteq \bar{C}$ implies that $C(F)<S(F)$, we have $\gamma_1 \leq \gamma'$. Now let $F \in \mathcal{F}$ such that $C(F) < S(F)$. Define, as in the proof of Theorem 6.7, the set

$$G = \cup \{[x] \mid x \in \bar{C} \cap F\}.$$

Then $G \in \mathcal{F}$, $G^* \subseteq \bar{C}$, $G \subseteq F$ and $F \setminus G \subseteq C$ so that

$$\frac{C(F)}{S(F)+1} = \frac{C(G)+C(F\setminus G)}{S(G)+1+S(F\setminus G)} = \frac{C(G)+S(F\setminus G)}{S(G)+1+S(F\setminus G)}$$

$$\geq \frac{C(G)}{S(G)+1} \geq \gamma' .$$

Thus $\gamma_1 \geq \gamma'$ and the proof is complete. Hence we obtain from (3) that

(4) $\gamma_1 \geq \alpha_1 + \beta_1$.

A problem is to what extent can the inequality (1) be extended. It is clear upon study of Kvarda's proof that inequality (1) may be extended by her method to the density space $(I^\Delta, \mathcal{K})$ for arbitrary non-empty Δ. If we are to use Kvarda's method in other density spaces, it can be seen that the family $\mathcal{F}$ necessarily satisfies the following properties: the family $\mathcal{F}$ is translation -1 ; if $D = U(x) \cap F\ (x \in F^0)$, and $y_1, y_2 \in D$ with $y_1 \prec y_2$, then $y_2 - y_1 \in T_1[D]$; if $x_i \in F^0\ (i = 1, 2, \cdots, k)$, and $D_i = U(x_i) \cap F$, then $S(\cup T_1[D_i]) + 1 \leq S(\cup D_i)$. We do not claim that these properties are sufficient.

We remark that for the density space $(I, \mathfrak{K})$ Kvarda's inequality (1) was proved earlier by the methods of Mann [18] and Besicovitch. This latter method was first used for this purpose by P. Scherk [22]. Consequently inequalities (2) and (4) were also obtained for this special case at an earlier date although inequality (4) does not appear explicitly in the literature. Kvarda's method of proof for this special case is essentially different than the two earlier methods which have not yet been applied successfully to other density spaces.

(ii) Our second problem concerns the so called essential component theorem. Given a density space $(S, \mathfrak{F})$, we call a set $B \subseteq S$ a K essential component (C essential component) if, for each $A \subseteq S$ with $0 < d(A, \mathfrak{F}) < 1$ $(0 < d_c(A, \mathfrak{F}) < 1)$ we have $d(C, \mathfrak{F}) > d(A, \mathfrak{F})$ $(d_c(C, \mathfrak{F}) > d_c(A, \mathfrak{F}))$ where $C = A + B$. The essential component theorem states that, if B is a basis (Definition 7.2), then B is a (K or C) essential component. Kasch [7] has shown this theorem for the density space $(I^n, \mathfrak{K})$ where the theorem is stated for C-density. Nothing else is known for any other density space.

(iii) R. Stalley [26] has defined a modified Schnirelmann density for infinite sets of positive integers. We generalize his definition. For the density space $(S, \mathcal{F})$ we define the K^*-density of an infinite subset A of S, to be

$$\alpha^* = d^*(A, \mathcal{F}) = \text{g l b}\,\{q(A, F) \mid F \in \mathcal{F},\ F^* \subseteq A\}\ .$$

Correspondingly, we define the C^*-density of A to be

$$\alpha_c^* = d_c^*(A, \mathcal{F}) = \text{g l b}\,\{q(A, [x]) \mid x \in A\}\ .$$

Both of these definitions reduce to Stalley's for the density space $(I, \mathcal{K})$. For this density space he has shown, among several other results, that if $\alpha^* + \beta^* > 1$, then $\gamma^* = 1$ where α^*, β^*, and γ^* are the K^* (or C^*) densities of the infinite sets $A, B \subseteq S$ and $C = A + B$ respectively. This result does not necessarily hold for other density spaces as the following example shows. Consider the density space $(I^2, \mathcal{K})$. Let A be the set of all pairs $(x, y) \in I^2$ such that either $x \geq \overset{3}{\cancel{4}}$, $y \geq \overset{3}{\cancel{4}}$ or (x, y) equals one of $(0, 1)$, $(1, 0)$, $(2, 0)$, ~~(3, 0)~~. Let $B = A$. Then $\alpha^* + \beta^* = \alpha_c^* + \beta_c^* = 12/11 > 1$, but $\gamma^* = \gamma_c^* = 9/11 < 1$.

If the above result doesn't hold, are there constants $k < 2$ such that $\alpha^* + \beta^* > k$ implies $\gamma^* = 1$? If so what is their greatest lower bound ?

(iv) Another problem, barely touched in this thesis, is the purely algebraic problem of characterizing the class of all s-sets. The examples of Section 3. 1 and the proof in Appendix 2 indicate that this is no trivial problem. To the best of the author's knowledge, the study of s-sets as algebraic entities in themselves has not yet been carried out. It would be useful to know, for instance, when an s-set S is imbeddable in an s-set of the form $G \times I^{\Delta}$. The work in Appendix 2 suggests that a necessary and sufficient condition might be that, for arbitrary $x \in S$, there exists positive constants r and t, depending on x, such that $S(L(nx)) \leq tn^r$, for $n \geq 1$.

10.2. Concluding Remarks

The problems of the preceding section by no means exhaust the possible areas for research. For instance, we have not mentioned the generalization of asymptotic density. There is also the problem of continuing to improve upon the results of Chapters 7 and 8. To embark upon research on any of these problems is to tacitly agree that the foundation for the theory which we have set forth is one worth keeping. This brings up the important question of whether or not the axiomatic foundation can or should be changed.

One possibility is to remove from the axioms for s-sets Axiom s.3. This would then allow sets like the positive rationals or the positive reals to be considered. Retaining the same axioms for fundamental families, much of the theory would go through unaltered. The most important exception is that we would not be sure that $\mathcal{H}$ is a fundamental family since we would not be sure that

$L(x)$ is finite. Looking back, we find many important results depending on the fact that $\mathcal{K}$ is a fundamental family.

Another possibility is to leave the axioms for s-sets as they are and replace Axiom f. 4 with the weaker statement of Theorem 2.10, i.e., that for each $F \in \mathcal{F}$ we have $F^* \neq \phi$. This would have the effect of enlarging the class of fundamental families on an s-set. Theorem 2.22 would no longer hold and hence neither would several results which depend on it. The definition of the transformation properties would have to be revised.

Other changes in the axiomatic formulation can be considered. The question as to whether any of these changes are worth while can be answered only through long and serious research. It is hoped that this thesis helps to induce that research.

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APPENDICES

APPENDIX 1

In this appendix we will prove that, given an arbitrary s-set S, the axioms for a fundamental family $\mathcal{F}$ on S are independent. The proof results from the construction of four subfamilies of $\mathcal{D}(S)$, namely $\mathcal{F}_i$ $(i=1,2,3,4)$, where $\mathcal{F}_i$ satisfies Axiom f. j $(j \neq i)$, and fails to satisfy Axiom f. i. In each of the following constructions let x denote an arbitrary element of S.

(i) Let $\mathcal{F}_1 = \{\{x\}\}$, i.e., the family which consists of the single set $\{x\}$. Then $\mathcal{F}_1$ satisfies Axioms f. 2-4, but clearly fails to satisfy Axiom f. 1.

(ii) For a positive integer n let

$$R(n) = \{nx, (n-2)x, \cdots, 2x\}$$

if n is even, and

$$R(n) = \{nx, (n-2)x, \cdots, x\}$$

if n is odd. For $n \leq 0$, define $R(n)$ to be the empty set.

Let $\mathcal{F}_2$ be the family of all non-empty sets of the form $R(n) \cup X$ where $n = 0, 1, 2, \cdots$ and X is any (possibly empty) finite subset of S which contains no integer multiple of x. We prove that $\mathcal{F}_2$ satisfies all the axioms except Axiom f. 2.

Let $y \in S$. If $y = nx$, then take $F = R(n) \cup \phi$ (where, as usual, ϕ denotes the empty set). If y is not a multiple of x, then take $F = R(0) \cup \{x\}$, and we have proved Axiom f. 1.

To show Axiom f. 3, it is sufficient, after Theorem 2. 9(ii), to prove that the non-empty intersection of two sets of $\mathcal{F}_2$ is a set of $\mathcal{F}_2$. Thus, let $R(n) \cup X_1$ and $R(m) \cup X_2$ be sets of $\mathcal{F}_2$ such that

$$(R(n) \cup X_1) \cap (R(m) \cup X_2) = (R(n) \cap R(m)) \cup (X_1 \cap X_2)$$

is non-empty. If n and m are both even or both odd, then $R(n) \cap R(m) = R(\min\{n, m\})$. Otherwise $R(n) \cap R(m) = \phi = R(0)$. Also $X_1 \cap X_2$ has the desired properties. Thus, $(R(n) \cup X_1) \cap (R(m) \cup X_2)$ is of the form $R(k) \cup X$ and, since it is non-empty, it is in $\mathcal{F}_2$.

To prove Axiom f. 4 holds, let $R(n) \cap X$ be a set of $\mathcal{F}_2$ with more than one point in it, and let y be a maximal point of $R(n) \cup X$. If $y \in R(n)$, then $y = nx$ and $(R(n) \cup X)\setminus y = (R(n)\setminus y) \cup X$. Since $R(n)\setminus nx = R(n-2)$, we obtain that y is a corner point of $R(n) \cup X$. Now, if $y \in X$, then

$$(R(n) \cup X)\setminus y = R(n) \cup (X\setminus y)$$

which, clearly, is in $\mathcal{F}_2$. Thus y is a corner point of $R(n) \cup X$.

Finally, $\mathcal{F}_2$ does not satisfy Axiom f. 2, since $R(1)$ and $R(2)$ are sets of $\mathcal{F}_2$ but $R(1) \cup R(2)$ is not.

(iii) With $R(n)$ defined as above, let $\mathcal{F}_3$ be the family of all sets $F \in \mathcal{P}(S)$ with the property, if $nx \in F$, then $R(n-1) \subseteq F$ or $R(n-2) \subseteq F$ (possibly both). Axiom f. 1 is shown to hold similarly as for $\mathcal{F}_2$ above. Axiom f. 2 follows immediately from the definition. To show that Axiom f. 4 holds, let $F \in \mathcal{F}_3$ such that F has more than one point in it. Let y be a maximal point of F. If y is not a multiple of x then $F \setminus y$ still has the required property. If $y = nx$, then, by the maximality of y, n is the largest integer such that $nx \in F$. Thus, $F \setminus y$ still satisfies the requirements of a set of $\mathcal{F}_3$.

$\mathcal{F}_3$ is shown to fail Axiom f. 3 by the following example: $\{3x, 2x\}$ and $\{3x, x\}$ are sets of $\mathcal{F}_3$ but their intersection, $\{3x\}$, is not.

(iv) Finally, define $\mathcal{F}_4$ to be all sets of the form $R(2n) \cup R(2n-1) \cup X$ where $n = 1, 2, \cdots$, and X is restricted as in (ii).

Axiom f. 1 is shown similarly as for $\mathcal{F}_2$. Axioms f. 2 and f. 3 follow immediately from the equations

$$(R(2n) \cup R(2n-1)) \cup (R(2m) \cup R(2m-1))$$
$$= R(2(\max\{m,n\})) \cup R(2(\max\{m,n\})-1)\ ;$$

$$(R(2n) \cup R(2n-1)) \cap (R(2m) \cup R(2m-1))$$
$$= R(2(\min\{m,n\})) \cup R(2(\min\{m,n\})-1)\ .$$

Finally, the set $\{2x, x\} \in \mathcal{F}_4$ but $\{x\} \in \mathcal{F}_4$. Thus $2x$ is not a corner point of $\{2x, x\}$ which shows that $\mathcal{F}_4$ fails to satisfy Axiom f. 4. This completes the proof of the independence of the Axiom f. 1-4.

APPENDIX 2

We prove that the s-set of Example ss-5 is not isomorphic to any closed subset of an s-set of the form I^{Δ}. For any integer $j \geq 1$, we have $j + 0/2^j = j \in S$. We estimate the number $S(L(j))$ for $j = 1, 2, \cdots$. First we prove

(1) $x \in S$, $x \leq j/2$ imply $x \in L(j)$.

If $x = j/2$, then $j-x = x \in S$ and so $x \prec j$. If x is equal to an integer, then $j-x$ is an integer ≥ 1 and so $j-x \in S$, whence $x \prec j$. Thus, we assume that $x = n+i/2^n < j/2$ where $i \neq 0$. We have $n < j/2$ which implies $j - 2n - 1 \geq 0$. Hence

$$j-x = (j-n-1) + \frac{2^n - i}{2^n} = (j-n-1) + \frac{2^{j-2n-1}(2^n - i)}{2^{j-n-1}}$$

which is in S by definition. Hence, again, $x \prec j$ and so (1) is proved.

Next we prove

(2) $S(\{x | x \in S, \ x \leq j/2\}) \geq 2^k - 1$

where k denotes the greatest integer $\leq j/2$. First note that (2) is true if $j = 1$. Hence, assume that $j > 1$. Clearly

$$S(\{x | x \in S, \ x \leq j/2\}) \geq S(\{x | x \in S, \ x \leq k\})$$
$$= 1 + \sum_{i=1}^{k-1} S(\{x | x \in S, \ i \leq x < i + 1 \}) .$$

As is easily seen from the definition of S, we have

$$S(\{x \mid x \in S,\ \ i \leq x < i+1\}) = 2^i$$

and so

$$S(\{x \mid x \in S,\ \ x \leq j/2\}) \geq 1 + \sum_{i=1}^{k-1} 2^i = 2^k - 1 .$$

This completes the proof of (2).

Now, if $x \in L(j)$ and $x \leq j/2$, then $j - x \prec j$ and so $j - x \in L(j)$ and $j - x \geq j/2$. Thus, (1) and (2) yield

$$\begin{aligned} S(L(j)) &\geq 2(S(\{x \mid x \in S,\ \ x \leq j/2\})) - 1 \\ &\geq 2^{k+1} - 2 - 1 = 2^{k+1} - 3 \geq 2^{j/2} - 3 . \end{aligned}$$

Now let us suppose that S is isomorphic to $S' \subseteq I^{\Delta}$. Under this isomorphism 1 corresponds to some function $f \in S'$. Then j corresponds to the function jf. As can be seen

$$S'(L_{S'}(g)) \leq \prod_{\delta \in \Delta} (g(\delta) + 1)$$

for any $g \in S'$. If we let N denote the (finite) number of δ for which $f(\delta) \neq 0$, then

$$S(L(j)) = S'(L_{S'}(jf)) \leq \prod_{\delta \in \Delta} (jf(\delta) + 1) \leq j^N \prod_{\delta \in \Delta} (f(\delta) + 1).$$

Thus for all $j \geq 1$ we have

$$j^N \prod_{\delta \in \Delta} (f(\delta) + 1) \geq S(L(j)) \geq (\sqrt{2})^j - 3 .$$

This is a contradiction since, for large enough j, the right hand side will exceed the left hand side. Thus, it is impossible that S is isomorphic to any subset of any I^{Δ}.

APPENDIX 3

For convenience we define

$$K(z,n) = \frac{1-(1-z)^{1/(n-1)}}{n}$$

$$K_1(z,n) = \left(\frac{z}{n}\right)^{n-1} ,$$

$$K_2(z,n) = \frac{z}{2n(3(n+1))^n} .$$

In this appendix we prove that

$$\alpha_c + K(\alpha_c, n)\beta_c(1-\alpha_c)$$

$$\geq \max\{\alpha_c + K_1(\alpha_c, n)\beta_c(1-\alpha_c),\ \alpha_c + K_2(\alpha_c, n)\beta_c(1-\alpha_c)\},$$

where n is a positive integer and $0 \leq \alpha_c \leq 1$, and that strict inequality holds except in the case $n \leq 2$, $\alpha_c = 0, 1$ or $\beta_c = 0$. This shows that, for the density space $(I^n, \mathcal{H})$, the inequality of Theorem 8.10 improves upon those proved and conjectured by Kasch [7] (see Section 1.3). Clearly it suffices to show that

$$K(z,n) \geq K_i(z,n) \qquad (i=1,2)$$

for $n \geq 1$ and $0 \leq z \leq 1$, and that strict inequality holds for $n \geq 3$ and $0 < z$.

We will use the following lemma: if $n \geq 3$, then

$$(1) \quad n^{(n+1)(n-2)} > (n-1)^{n(n-1)} .$$

Let $d(x) = (x+1)(x-2)\log x - x(x-1)\log(x-1)$. We show that $d(x) > 0$ for $x \geq 4$. Now

$$d'(x) = \log\left(\frac{x-1}{x}\right)^{-2x+1} - 1 - \frac{2}{x} .$$

From the relation

$$\left(\frac{x-1}{x}\right)^{-x} \geq e$$

it follows that

$$(2) \quad d'(x) \geq \log e^2 + \log\left(\frac{x-1}{x}\right) - 1 - \frac{2}{x}$$

$$= \frac{x-2}{x} + \log\left(\frac{x-1}{x}\right) .$$

Now $e^y > 1+y$, and so $e^{1-(2/x)} \geq 2 - (2/x)$ so that

$$e^{1-(2/x)}\left(1 - \frac{1}{x}\right) \geq \left(2 - \frac{2}{x}\right)\left(1 - \frac{1}{x}\right)$$

$$= 2 - \frac{4}{x} + \frac{2}{x^2} .$$

Thus $e^{1-(2/x)}\left(1 - \frac{1}{x}\right) > 1$ if $x \geq 4$. Taking log of both sides we have

$$\left(1 - \frac{2}{x}\right) + \log\left(1 - \frac{1}{x}\right) > 0 \qquad (x \geq 4) .$$

Thus, from (2), we obtain that $d'(x) > 0$ if $x \geq 4$. Hence $d(x)$ increases as x increases through values ≥ 4, and so $d(x) \geq d(4) > 0$ for $x \geq 4$. Thus (1) follows for $n \geq 4$. If

$n = 3$, then

$$n^{(n+1)(n-2)} = 3^4 = 81 > 64 = 2^6 = (n-1)^{n(n-1)}$$

and so (1) is proved for $n \geq 3$.

We now show that $K(z,n) \geq K_1(z,n)$ for $n \geq 1$, $0 \leq z \leq 1$ with strict inequality if $n \geq 3$ and $0 < z$. In the cases $n = 1$ and $n = 2$ it is clear that we have equality for all z. Hence suppose $n \geq 3$ and let

$$f(x) = n^{\frac{n-2}{n-1}} \left(1 - x^{\frac{1}{n-1}}\right)^{\frac{1}{n-1}} - 1 + x \, .$$

We show that $f(x) \geq 0$ for $0 \leq x \leq 1$. Now

$$f'(x) = 1 - n^{\frac{n-2}{n-1}} \left(\frac{1}{(n-1)^2}\right)\left(\left(1 - x^{\frac{1}{n-1}}\right) x\right)^{\frac{2-n}{n-1}} \, .$$

Hence $f'(x) = 0$ if and only if $x - x^{n/(n-1)} = 1/A$ where $A = (1/n)(n-1)^{(2(n-1))/(n-2)}$. Define $g(x) = x - x^{n/(n-1)}$. Then $g'(x) = 1-(n/(n-1))x^{1/(n-1)}$, and so $g'(y) = 0$ only when $y = ((n-1)/n)^{n-1}$. Hence $g(x) \geq g(y) = (1/n)((n-1)/n)^{n-1}$. From (1) we obtain

$$n^{n+1} > (n-1)^{\frac{n(n-1)}{n-2}} = (n-1)^{\frac{2(n-1)}{n-2}} (n-1)^{n-1} \, ,$$

and so

$$\frac{n}{(n-1)^{\left(\frac{2(n-1)}{n-2}\right)}} > \frac{(n-1)^{n-1}}{n^n} \, ,$$

which is

$$\frac{1}{A} > g(y) \ .$$

Thus $g(x) = 1/A$ is impossible and hence so is $f'(x) = 0$ impossible for $0 \leq x \leq 1$. Hence $f'(x)$ is of constant sign. For x sufficiently close to zero it can be seen that $f'(x) < 0$ so that $f'(x) < 0$ for all x, $0 \leq x \leq 1$. Hence $f(x)$ is a strictly decreasing function, and so for all x, $0 \leq x \leq 1$,

$$f(x) \geq f(1) = 0 \ .$$

Furthermore, if $x \neq 1$, then $f(x) > 0$.

Now, if we let $x = 1-z$, then we have

$$f(x) = (K(z,n))^{1/(n-1)} - (K_1(z,n))^{1/(n-1)}$$

and so $K(z,n) \geq K_1(z,n)$ for $n \geq 3$, $0 \leq z \leq 1$, and strict inequality holds if $0 \neq z$.

Finally, we show that $K(z,n) \geq K_2(z,n)$. For the cases $n+1$ and $n+2$ it can be seen by direct computation that strict inequality holds if $z > 0$. Assume $n \geq 3$ and let $B = 2n(2(n+1))^n$. Let

$$h(x) = B(K(1-x,n) - K_2(1-x,n))$$

$$= \frac{B}{n}(1 - x^{1/(n-1)}) - 1 + x.$$

Now

$$h'(x) = 1 - \frac{B}{n(n-1)} x^{\frac{2-n}{n-1}} ,$$

and so $h'(y) = 0$ only when $y = \left(\frac{n(n-1)}{B}\right)^{\frac{n-1}{2-n}} > 1$,

and so $h'(x)$ is of constant sign in the interval $0 \leq x \leq 1$. This sign must be negative so that $h(x)$ is strictly decreasing for these values of x. Hence we have

$$h(x) \geq h(1) = 0 \qquad (0 \leq x \leq 1)$$

and $h(x) > 0$ if $x \neq 1$. Substituting $z = 1-x$ we obtain the desired result.

Let $n = 3$ and $z = 15/16$. Then $K(z,n) = 1/4$, $K_1(z,n) = 25/256$ and $K_2(z,n) = 15/2^{11} \cdot 3^4$. The preceding example gives an idea of how much better K can be than K_1 or K_2.