THE CONVERGENCE OF AITKEN'S δ^2 - PROCESS

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INTRODUCTION

Iteration methods are often useful in approximating the solutions of various types of equations, but it may happen that convergence is impracticably slow. To meet such a difficulty which he encountered while solving an algebraic equation by Bernoulli's method, A. C. Aitken devised his " δ^2 - process " (1, pp.300-303). This technique has proved useful on many examples. The purpose of this thesis is to analyze the method of Aitken, to show that under certain conditions it actually does speed up convergence, and to estimate the degree of improvement.

The remaining part of the introduction will give a brief discussion of Bernoulli's method and the underlying idea of the δ^2 - process. In the second chapter, theorems concerning the δ^2 - process are proved. The third chapter contains a summary of Theorem 3, found in Chapter II, and Chapter IV presents a few practical applications of the δ^2 - process with error estimates.

Bernoulli's method obtains the root, say α_1, of an algebraic equation

$$(1) \qquad b_o x^n + b_1 x^{n-1} + b_2 x^{n-2} + \dots + b_n = 0$$

under the conditions that α_1 is real and the modulus of α_1 is greater than the modulus of any other root of (1.1). If α_1 satisfies these conditions, then Bernoulli's method is briefly as follows. Arbitrary numbers $X_o, X_1, X_2, \dots, X_{n-1}$ are chosen and from (1.1) we form the equation

$$b_o X_n + b_1 X_{n-1} + b_2 X_{n-2} + \dots + b_n X_o = 0 \tag{1.2}$$

from which X_n is determined. We then consider the sequence $X_1, X_2, X_3, \dots, X_n$ and form the equation

$$b_o X_{n+1} + b_1 X_n + b_2 X_{n-1} + \dots + b_n X_1 = 0 \tag{1.3}$$

from which X_{n+1} is determined. The process is contimued in the manner indicated by (1.2) and (1.3) to form an infinite sequence $X_o, X_1, X_2, \dots$, which has the property that

$$\lim_{n \to \infty} X_{n+1} / X_n = \alpha_1 . \tag{1.4}$$

This is Bernoulli's result.

If $\alpha_1, \alpha_2, \dots, \alpha_n$ are the n roots of (1.1) where α_1 and α_2 are real and $|\alpha_1| > |\alpha_2| > |\alpha_k|$, $k = 3, 4, \dots, n$, then Aitken (1, pp.300-303) shows that

$$\lim_{n \to \infty} (S_{n+2} - S_{n+1}) / (S_{n+1} - S_n) = \alpha_2 / \alpha_1 , \tag{1.5}$$

where $S_n = X_{n+1} / X_n$, $n = 0, 1, 2, \dots$.

It should be noted that in this case

(1.6) $$|\alpha_2 / \alpha_1| < 1 .$$

In general, if any infinite sequence $\{S_n\}_0^\infty$ of real numbers satisfies conditions similar to (1.5) and (1.6), the δ^2 - process can be used to accelerate the convergence.

The basic idea of the δ^2 - process is that it gives the exact sum of an infinite geometric series. Thus, when used with an infinite series which is nearly geometric, we can expect to obtain a sum differing little from the true sum of the series. This idea will be made precise by theorems in chapter II .

In 1937 Aitken shows the extended use of the δ^2 - process in finding the latent roots and latent vectors of a matrix (3, pp.291-295). In chapter IV, an example of this type will be given .

THE δ^2 - PROCESS

It will be assumed that an iteration of some type yields a sequence of real numbers $\{S_n\}_1^\infty$ converging to S . The δ^2 - process is then as follows. A first derived sequence $\{S_n^{(1)}\}_1^\infty$ is formed by use of the function $F(S_n, S_{n+1}, S_{n+2})$ where

$$(2.1) \qquad S_n^{(1)} = F(S_n, S_{n+1}, S_{n+2}) = \frac{S_n S_{n+2} - S_{n+1}^2}{S_n - 2S_{n+1} + S_{n+2}} \quad .$$

That is, three successive elements of $\{S_n\}_1^\infty$ are used to form an element of the first derived sequence. This idea can be extended to a second, third, etc., derived sequence where in general

$$(2.2) \qquad S_n^{(r+1)} = F(S_n^{(r)}, S_{n+1}^{(r)}, S_{n+2}^{(r)}), \qquad r = 1, 2, 3, \dots \; .$$

The function $F(x, y, z)$ has the useful property that if $x - 2y + z \neq 0$, then

$$(2.3) \qquad F(x + a, y + a, z + a) = a + F(x, y, z) \; .$$

From (2.1)

$$F(x+a, y+a, z+a) = [(x+a)(z+a) - (y+a)^2] \,/\, [(x+a)-2(y+a)+(z+a)]$$

$$= [xz+az+ax+a^2-y^2-2ay-a^2] \,/\, [x - 2y + z]$$

$$= [(xz - y^2) + a(x - 2y + z)] \,/\, [x - 2y + z]$$

$$= a + F(x, y, z)$$

which proves (2.3).

From (2.3) we see that any left hand digits common to S_n, S_{n+1} and S_{n+2} can be neglected in the calculation of $S_n^{(1)}$ and then added into the final result. As an example, the numbers $S_n^{(1)} = 15.001418373$, $S_{n+1}^{(1)} = 15.000304169$ and $S_{n+2} = 15.000065221$ are listed in (1, p.302) from which $S_n^{(2)} = 14.999999987$ was calculated. Applying (2.2) and (2.3)

$$S_n^{(2)} = F(S_n^{(1)}, S_{n+1}^{(1)}, S_{n+2}^{(1)})$$

$$= F(15.001418373,\ 15.000304169,\ 15.000065221)$$

$$= 15 + F(.001418373,\ .000304169,\ .000065221)\ .$$

An intuitive reason for defining the function $F(x, y, z)$ can be given by the following theorem.

<u>THEOREM 1</u> : If the sequence $\{S_n\}_0^\infty$ converges geometrically to the limit S, that is to say, for some L, $|L| < 1$, we have

$$(2.4) \qquad S_n - S = L^n(S_o - S), \qquad n = 0, 1, 2, \ldots ,$$

then for $n = 0, 1, 2, \ldots$ we have,

$$(2.5) \qquad F(S_n, S_{n+1}, S_{n+2}) = S\ .$$

<u>PROOF</u>: From (2.4)

$$S_n = S + L^n (S_o - S), \qquad n = 0, 1, 2, \ldots \quad . \tag{2.6}$$

From (2.3) and (2.6), with $a = S$,

$$F(S_n, S_{n+1}, S_{n+2}) = S + F[L^n (S_o - S), L^{n+1}(S_o - S), L^{n+2}(S_o - S)]$$

$$= S + \frac{L^{2n+2}(S_o - S)^2 - L^{2n+2}(S_o - S)^2}{(S_o - S)\, L^n(L - 1)^2}$$

$$= S \quad .$$

This proves Theorem 1. From Theorem 1 we see that, if the sequence $\{S_n\}_1^\infty$ converges geometrically (in the limit), we may expect that the first derived sequence $\{S_n^{(1)}\}_1^\infty$ will converge more rapidly than the original sequence $\{S_n\}_1^\infty$.

Two questions concerning the δ^2 - process which seem immediate are as follows.

1. Does the derived sequence converge to the same limit?
2. If so, is the convergence accelerated?

In answer to the first question, we prove the following theorem.

<u>THEOREM 2</u> : If the sequence $\{S_n\}_1^\infty$ converges to S and $\lim_{n \to \infty} A_{n+1} / A_n = L$, $|L| < 1$, where $A_1 = S_1$ and $A_n = S_n - S_{n-1}$, $n = 2, 3, 4, \ldots$, then

(2.7) $$\lim_{n \to \infty} S_n^{(1)} = S .$$

<u>PROOF</u> : From (2.1) and hypotheses we have for $n = 1, 2, \ldots$,

$$S_n^{(1)} = F(S_n, S_{n+1}, S_{n+2})$$

(2.8) $$= F(S_n, S_n + A_{n+1}, S_n + A_{n+1} + A_{n+2}) .$$

From (2.3)

$$S_n^{(1)} = S_n + F(0, A_{n+1}, A_{n+1} + A_{n+2}) .$$

From (2.1)

$$S_n^{(1)} = S_n + \frac{-A_{n+1}^2}{-2A_{n+1} + A_{n+1} + A_{n+2}}$$

$$= S_n - \frac{A_{n+1}^2}{A_{n+2} - A_{n+1}}$$

(2.9) $$= S_n - \frac{A_{n+1}}{\dfrac{A_{n+2}}{A_{n+1}} - 1} .$$

Also,

(2.10) $$\lim_{n \to \infty} A_n = \lim_{n \to \infty} (S_n - S_{n-1}) = 0 .$$

From (2.9) and (2.10)

$$\lim_{n \to \infty} S_n^{(1)} = \lim_{n \to \infty} \left[S_n - \frac{A_{n+1}}{\frac{A_{n+2}}{A_{n+1}} - 1} \right]$$

$$= S - \frac{0}{L - 1}$$

$$= S \quad .$$

This proves Theorem 2 .

The question concerning the accelerated convergence by use of the δ^2 - process, i.e., forming a derived sequence $\{S_n^{(1)}\}_1^{\infty}$ from the convergent sequence $\{S_n\}_1^{\infty}$, will be answered in the following theorem which imposes certain conditions on $\{S_n\}_1^{\infty}$. Also, an estimate of the error of $S_n^{(1)}$ will be found, but in most cases will be difficult to apply. Nevertheless, the accelerated convergence will be shown.

<u>THEOREM 3</u>: If $\{S_n\}_1^{\infty}$ is a convergent sequence converging to S and

(2.11) $$\lim_{n \to \infty} A_{n+1} / A_n = L \; , \qquad |L| < 1 \; ,$$

where $A_1 = S_1$ and $A_n = S_n - S_{n-1}$, $n = 2, 3, 4, \ldots$, then given any $\epsilon > 0$, there exists a positive integer N such that for $n \geq N$

$$|S - S_n^{(1)}| \leqq |A_{n+2}| \quad \epsilon \quad .$$

PROOF : As in (2.8) and (2.9)

$$S_n^{(1)} = F(S_n, S_{n+1}, S_{n+2}) = S_n - \frac{A_{n+1}}{\frac{A_{n+2}}{A_{n+1}} - 1} \quad . \tag{2.12}$$

Also,

$$S_n = S_1 + (S_2 - S_1) + (S_3 - S_2) + \dots + (S_n - S_{n-1})$$

$$= \sum_{k=1}^{n} A_k \quad . \tag{2.13}$$

Then ,

$$\sum_{k=1}^{\infty} A_k = \lim_{n \to \infty} \sum_{k=1}^{n} A_k = \lim_{n \to \infty} S_n = S \; . \tag{2.14}$$

From (2.12), (2.13) and (2.14)

$$S - S_n^{(1)} = S - S_n + \frac{A_{n+1}}{\frac{A_{n+2}}{A_{n+1}} - 1}$$

$$= \sum_{k=n+1}^{\infty} A_k + \frac{A_{n+1}}{\frac{A_{n+2}}{A_{n+1}} - 1}$$

$$S - S_n^{(1)} = \sum_{k=n+3}^{\infty} A_k + A_{n+1} + A_{n+2} + \frac{A_{n+1}^2}{A_{n+2} - A_{n+1}}$$

$$= \sum_{k=n+3}^{\infty} A_k + \frac{A_{n+2}^2 - A_{n+1}^2 + A_{n+1}^2}{A_{n+2} - A_{n+1}}$$

$$= \sum_{k=n+3}^{\infty} A_k + \frac{A_{n+2}^2}{A_{n+2} - A_{n+1}}$$

$$(2.15) \qquad = A_{n+2} \left\{ \sum_{k=n+3}^{\infty} \frac{A_k}{A_{n+2}} + \frac{A_{n+2}}{A_{n+2} - A_{n+1}} \right\} .$$

From (2.15)

$$(2.16) \qquad |S - S_n^{(1)}| = |A_{n+2}| \left| \sum_{k=n+3}^{\infty} \frac{A_k}{A_{n+2}} - \frac{A_{n+2}}{A_{n+1} - A_{n+2}} \right| .$$

The proof will now be separated into three cases, namely,

(2.17) case 1, $0 < L < 1$

(2.18) case 2, $-1 < L < 0$

(2.19) case 3, $L = 0$.

<u>CASE 1</u> : From (2.11) and (2.17), given any $\sigma > 0$, where $0 < L - \sigma$ and $L + \sigma < 1$, there exists a positive integer N_1 such

that for $m \geqq N_1$

(2.20) $$0 < L - \sigma < A_{n+1} / A_n < L + \sigma < 1 \quad n = m+1, m+2, \ldots .$$

For simplicity let

(2.21) $$p = L - \sigma \quad \text{and} \quad q = L + \sigma \ .$$

Then (2.20) becomes

(2.22) $$0 < p < A_{n+1} / A_n < q < 1 \qquad n = m+1, m+2, \ldots .$$

From (2.22)

(2.23) $$0 < p < A_{m+2} / A_{m+1} < q < 1 \qquad m \geqq N_1 \ .$$

From (2.23)

(2.24) $$0 < \frac{p}{1 - p} < \frac{\frac{A_{m+2}}{A_{m+1}}}{1 - \frac{A_{m+2}}{A_{m+1}}} < \frac{q}{1 - q} \qquad m \geqq N_1 \ .$$

Also,

(2.25) $$\frac{A_{m+2}}{A_{m+1} - A_{m+2}} = \frac{\frac{A_{m+2}}{A_{m+1}}}{1 - \frac{A_{m+2}}{A_{m+1}}} \ .$$

From (2.24) and (2.25)

$$(2.26) \qquad 0 < \frac{p}{1-p} < \frac{A_{m+2}}{A_{m+1} - A_{m+2}} < \frac{q}{1-q} .$$

From (2.22)

$$(2.27) \qquad 0 < p^{r-2} < \frac{A_{m+r}}{A_{m+2}} = \frac{A_{m+r}}{A_{m+r-1}} \; \frac{A_{m+r-1}}{A_{m+r-2}} \cdots \frac{A_{m+3}}{A_{m+2}} < q^{r-2}$$

for $r = 3, 4, \ldots$.

From (2.27)

$$(2.28) \qquad 0 < \sum_{k=1}^{r-2} p^k < \sum_{k=3}^{r} \frac{A_{m+k}}{A_{m+2}} < \sum_{k=1}^{r-2} q^k$$

for $r = 3, 4, \ldots$. Letting r become infinite in (3.28) we have,

$$(2.29) \qquad 0 < \sum_{k=1}^{\infty} p^k \leqq \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} \leqq \sum_{k=1}^{\infty} q^k .$$

Also,

$$\sum_{k=1}^{\infty} p^k = \frac{p}{1-p}, \qquad \sum_{k=1}^{\infty} q^k = \frac{q}{1-q}$$

so that from (2.29)

$$(2.30) \qquad 0 < \frac{p}{1-p} \leqq \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} \leqq \frac{q}{1-q} \qquad m \geqq N_1 .$$

From (2.16), (2.26) and (2.30)

$$|S - S_m^{(1)}| = |A_{m+2}| \ \left| \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} - \frac{A_{m+2}}{A_{m+1} - A_{m+2}} \right|$$

$$\leqq |A_{m+2}| \ \left|\frac{q}{1-q} - \frac{p}{1-p}\right|$$

$$(2.31) \qquad = |A_{m+2}| \ \left|\frac{q-p}{(1-q)\ (1-p)}\right| \qquad m \geqq N_1 \ .$$

From (2.21) and (2.31)

$$|S - S_m^{(1)}| \leqq |A_{m+2}| \ \left| \frac{(L+\sigma) - (L-\sigma)}{(1-L-\sigma)(1-L+\sigma)} \right|$$

$$(2.32) \qquad = |A_{m+2}| \ \left| \frac{2\sigma}{(1-L)^2 - \sigma^2} \right| \qquad m \geqq N_1 \ .$$

Since σ was arbitrarily small, given any $\epsilon > 0$, there exists a positive integer N_2 such that for $m \geqq N_2$

$$(2.33) \qquad \left| \frac{2\sigma}{(1-L)^2 - \sigma^2} \right| \leqq \epsilon \ .$$

Let $N = \max. \{N_1, N_2\}$. Then, for $m \geqq N$, we have from (2.32) and (2.33)

$$|S - S_m^{(1)}| \leqq |A_{m+2}| \ \epsilon \ .$$

<u>CASE 2</u> : From (2.11) and (2.18), given any $\sigma > 0$ where $-1 < L - \sigma$ and $L + \sigma < 0$, there exists a positive integer N_1 such

that for $m \geqq N_1$

$$(2.34) \qquad -1 < L - \sigma < A_{n+1} / A_n < L + \sigma < 0$$

$$n = m+1, m+2, \ldots \ .$$

For simplicity let

$$(2.35) \qquad p = L - \sigma \quad \text{and} \quad q = L + \sigma \ .$$

From (2.34) and (2.35)

$$(2.36) \qquad -1 < p < A_{n+1} / A_n < q < 0 \qquad n = m+1, m+2, \ldots \ .$$

Since

$$\frac{A_{m+k}}{A_{m+2}} = \frac{A_{m+k}}{A_{m+k-1}} \ \frac{A_{m+k-1}}{A_{m+k-2}} \ \cdots \ \frac{A_{m+3}}{A_{m+2}}$$

for $k = 3, 4, 5, \ldots$, from (2.36) we have the following inequalities:

$$-1 < p < A_{m+3} / A_{m+2} < q < 0$$

$$0 < q^2 < A_{m+4} / A_{m+2} < p^2$$

$$\vdots$$

$$p^{2r-1} < A_{m+2r+1} / A_{m+2} < q^{2r-1}$$

$$q^{2r} < A_{m+2r+2} / A_{m+2} < q^{2r}$$

$$\vdots \ .$$

From the preceding inequalities we have

$$(2.37) \qquad p + q^2 + \dots + q^{2r-2} + p^{2r-1} < \sum_{j=3}^{2r+1} \frac{A_{m+j}}{A_{m+2}} < q + p^2 + \dots + p^{2r-2} + q^{2r-1}$$

and

$$(2.38) \qquad p + q^2 + \dots + p^{2r-1} + q^{2r} < \sum_{j=3}^{2r+2} \frac{A_{m+j}}{A_{m+2}} < q + p^2 + \dots + q^{2r-1} + p^{2r}$$

for $r = 1, 2, 3, \dots$. Letting r become infinite in (2.37) and (2.38) we have

$$(2.39) \qquad \sum_{j=1}^{\infty} p^{2j-1} + \sum_{j=1}^{\infty} q^{2j} \leqq \sum_{k=3}^{\infty} \frac{A_{m+k}}{A_{m+2}} \leqq \sum_{j=1}^{\infty} q^{2j-1} + \sum_{j=1}^{\infty} p^{2j} .$$

From (2.39)

$$(2.40) \qquad \frac{p}{1-p^2} + \frac{q^2}{1-q^2} \leqq \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} \leqq \frac{q}{1-q^2} + \frac{p^2}{1-p^2}$$

for $m \geqq N_1$.

From (2.36)

$$(2.41) \qquad -1 < p < A_{m+2} / A_{m+1} < q < 0 \qquad\qquad m \geqq N_1 .$$

For simplicity let

(2.42) $$t = A_{m+2} / A_{m+1} \ .$$

From (2.41) and (2.42)

(2.43) $$-1 < p < t < q < 0 \ .$$

From (2.43)

(2.44) $$p / (1 - p^2) < t / (1 - t^2) < q / (1 - q^2)$$

and

(2.45) $$q^2 / (1 - q^2) < t^2 / (1 - t^2) < p^2 / (1 - p^2) \ .$$

From (2.44) and (2.45)

(2.46) $$\frac{p}{1 - p^2} + \frac{q^2}{1 - q^2} < \frac{t}{1 - t^2} + \frac{t^2}{1 - t^2} < \frac{q}{1 - q^2} + \frac{p^2}{1 - p^2} \ .$$

Also,

(2.47) $$\frac{t}{1 - t} = \frac{t}{1 - t^2} + \frac{t^2}{1 - t^2}$$

so that from (2.46)

(2.48) $$\frac{p}{1 - p^2} + \frac{q^2}{1 - q^2} < \frac{t}{1 - t} < \frac{q}{1 - q^2} + \frac{p^2}{1 - p^2} \ .$$

From (2.42)

$$(2.49) \qquad \frac{t}{1-t} = \frac{\dfrac{A_{m+2}}{A_{m+1}}}{1 - \dfrac{A_{m+2}}{A_{m+1}}} = \frac{A_{m+2}}{A_{m+1} - A_{m+2}} \quad .$$

From (2.48) and (2.49)

$$(2.50) \qquad \frac{p}{1-p^2} + \frac{q^2}{1-q^2} < \frac{A_{m+2}}{A_{m+1} - A_{m+2}} < \frac{q}{1-q^2} + \frac{p^2}{1-p^2}$$

for $m \geqq N_1$. From (2.16), (2.40) and (2.50)

$$|s - s_m^{(1)}| = |A_{m+2}| \; \left| \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} - \frac{A_{m+2}}{A_{m+1} - A_{m+2}} \right|$$

$$\leqq |A_{m+2}| \; \left| \frac{q}{1-q^2} + \frac{p^2}{1-p^2} - \frac{p}{1-p^2} - \frac{q^2}{1-q^2} \right|$$

$$= |A_{m+2}| \; \left| \frac{q-q^2}{1-q^2} - \frac{p-p^2}{1-p^2} \right|$$

$$= |A_{m+2}| \; \left| \frac{q}{1+q} - \frac{p}{1+p} \right|$$

$$(2.51) \qquad = |A_{m+2}| \; \left| \frac{q-p}{(1+q)\,(1+p)} \right| \qquad m \geqq N_1 \; .$$

From (2.35) and (2.51)

$$|S - S_m^{(1)}| \leq |A_{m+2}| \; \left|\frac{L + \sigma - L + \sigma}{(1 + L + \sigma)(1 + L - \sigma)}\right|$$

$$(2.52) \qquad = |A_{m+2}| \; \left|\frac{2\sigma}{(1 + L)^2 - \sigma^2}\right| \qquad m \geq N_1 .$$

Since σ was arbitrarily small, given any $\epsilon > 0$, there exists a positive integer N_2 such that for $m \geq N_2$.

$$(2.53) \qquad \left|\frac{2\sigma}{(1 + L)^2 - \sigma^2}\right| \leq \epsilon .$$

Let $N = \max. \{N_1, N_2\}$. Then, for $m \geq N$ we have from (2.52) and (2.53)

$$(2.54) \qquad |S - S_m^{(1)}| \leq |A_{m+2}| \; \epsilon .$$

<u>CASE 3</u> : From (2.11) and (2.19), given any σ , where $0 < \sigma < 1$, there exists a positive integer N_1 such that for $m \geq N_1$

$$(2.55) \qquad |A_{n+1} / A_n| < \sigma \qquad n = m+1, m+2, \ldots .$$

From (2.55) for $r = 3, 4, \ldots$

$$(2.56) \qquad \left|\frac{A_{m+r}}{A_{m+2}}\right| = \left|\frac{A_{m+r}}{A_{m+r-1}} \frac{A_{m+r-1}}{A_{m+r-2}} \cdots \frac{A_{m+3}}{A_{m+2}}\right| < \sigma^{r-2} .$$

From (2.56) for $r = 3, 4, 5, \ldots$

$$(2.57) \qquad \left| \sum_{j=3}^{r} \frac{A_{m+j}}{A_{m+2}} \right| \leq \sum_{j=3}^{r} \left| \frac{A_{m+j}}{A_{m+2}} \right| < \sum_{j=3}^{r} \sigma^{j-2} .$$

Letting r become infinite in (2.57)

$$(2.58) \qquad \left| \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} \right| \leq \sum_{j=3}^{\infty} \sigma^{j-2} = \frac{\sigma}{1-\sigma} .$$

From (2.55)

$$(2.59) \qquad -\sigma < A_{m+2} / A_{m+1} < \sigma \qquad m \geq N_1 .$$

Also,

$$(2.60) \qquad \frac{A_{m+2}}{A_{m+1} - A_{m+2}} = \frac{\dfrac{A_{m+2}}{A_{m+1}}}{1 - \dfrac{A_{m+2}}{A_{m+1}}} .$$

From (2.59)

$$(2.61) \qquad -\frac{\sigma}{1-\sigma} < \frac{\dfrac{A_{m+2}}{A_{m+1}}}{1 - \dfrac{A_{m+2}}{A_{m+1}}} < \frac{\sigma}{1-\sigma} \qquad m \geq N_1 .$$

From (2.60) and (2.61)

$$(2.62) \qquad -\frac{\sigma}{1-\sigma} < \frac{A_{m+2}}{A_{m+1} - A_{m+2}} < \frac{\sigma}{1-\sigma} \qquad m \geq N_1 .$$

From (2.16), (2.58), and (2.62)

$$|S - S_m^{(1)}| = |A_{m+2}| \ \left| \sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} - \frac{A_{m+2}}{A_{m+1} - A_{m+2}} \right|$$

$$\leqq |A_{m+2}| \ \left| \frac{2\sigma}{1 - \sigma} \right| \qquad m \geqq N_1 . \tag{2.63}$$

Since σ was arbitrarily small, given any $\epsilon > 0$, there exists a positive integer N_2 such that for $m \geqq N_2$

$$| 2\sigma / (1 - \sigma) | \leqq \epsilon . \tag{2.64}$$

Let $N = \max. \{N_1, N_2\}$. Then for $m \geqq N$ we have from (2.63) and (2.64)

$$|S - S_m^{(1)}| \leqq |A_{m+2}| \ \epsilon .$$

This completes the proof of Theorem 3 .

SUMMARY OF THE δ^2 - PROCESS

A summary of Theorem 3 will be given in order to show if and when the δ^2 - process should be used. Of course, it will be assumed that the sequence $\{S_n\}_1^\infty$ satisfies the hypotheses of Theorem 3, that is,

$$(3.1) \qquad \lim_{n\to\infty} \frac{S_{n+1} - S_n}{S_n - S_{n-1}} = \lim_{n\to\infty} \frac{A_{n+1}}{A_n} = L \qquad |L| < 1 .$$

It is easily seen that (2.21) and (2.35) are not necessary for (2.31) and (2.51) respectively. We need only (2.22) and (2.36) in order to assure the inequalities (2.31) and (2.51). With this in mind we separate the summary into the three cases corresponding to those found in Theorem 3.

<u>CASE 1</u> : If p and q can be determined as in (2.22), then

$$(3.2) \qquad |A_{m+2}| > p\,|A_{m+1}| \qquad m \geq N_1 .$$

From (2.30) and (3.2)

$$\begin{aligned} |S - S_{m+3}| &= \left| \sum_{k=1}^{\infty} A_k - \sum_{k=1}^{m+3} A_k \right| \\ &= \left| \sum_{k=m+4}^{\infty} A_k \right| \\ &\geq |A_{m+3}| \; |p / (1 - p)| . \end{aligned}$$

(3.3) $$|S - S_{m+3}| \geq |A_{m+2}| \; | p^2 / (1 - p) | \qquad m \geq N_1 .$$

From (2.31)

(3.4) $$|S - S_m^{(1)}| \leq |A_{m+2}| \; \left|\frac{q - p}{(1 - q)(1 - p)}\right| \qquad m \geq N_1 .$$

with $|q - p|$ sufficinetly small, the improvement of $S_m^{(1)}$ over S_{m+3} is easily seen from (3.3) and (3.4). It may only be possible to estimate p and q in which case (3.4) gives only an estimate of the error of $S_m^{(1)}$ and not an exact error bound.

<u>CASE 2</u>: If p and q can be determined as in (2.36), then

(3.5) $$|A_{m+2}| \geq |q| \; |A_{m+1}| \qquad m \geq N_1 .$$

Also, if p and q are sufficiently near to L, from (2.40) we have

(3.6) $$\sum_{k=m+3}^{\infty} \frac{A_k}{A_{m+2}} \leq \frac{q}{1 - q^2} + \frac{p^2}{1 - p^2} < 0 \qquad m \geq N_1 .$$

From (3.5) and (3.6)

$$|S - S_{m+3}| = | \sum_{k=m+4}^{\infty} A_k |$$

(3.7) $$\geq |A_{m+2}| \; \left|\frac{q^2}{1 - q^2} + \frac{p^2 q}{1 - p^2}\right| \quad m \geq N_1 .$$

From (2.51)

(3.8) $$|S - S_m^{(1)}| \leq |A_{m+2}| \; \left|\frac{q - p}{(1 + q)(1 + p)}\right| \qquad m \geq N_1 .$$

With $|q - p|$ sufficiently small, the improvement of $S_m^{(1)}$ over S_{m+3} is easily seen from (3.7) and (3.8). As in the preceding case, it may only be possible to estimate p and q, in which case, (3.8) gives only an estimate of the error of $S_m^{(1)}$ and not an exact error bound.

<u>CASE 3</u> : In this case the original sequence $\{S_n\}_1^{\infty}$ will converge rapidly and the δ^2 - process need not be used. Also, from (2.58) and (2.63)

$$(3.9) \qquad |S - S_{m+2}| \leqq |A_{m+2}| \; |\sigma / (1 - \sigma)|$$

and

$$(3.10) \qquad |S - S_m^{(1)}| \leqq |A_{m+2}| \; |2\sigma / (1 - \sigma)| \, .$$

From (3.9) and (3.10), we cannot draw any conclusions concerning acceleration.

NUMERICAL EXAMPLES

Two examples will be given with estimated error bounds. In the first example, the root of largest modulus of an algebraic is approximated using Bernoulli's method. In the second example, the latent vector, associated with the latent root of largest modulus of a matrix, is approximated by the sequence $AX_1, A^2X_1, A^3X_1, \ldots$ (3, pp.269-304) .

EXAMPLE 1 : The equation to be considered is,

$$x^3 - 2x^2 - 5x + 6 = 0 , \tag{4.1}$$

whose roots are 1, -2, and 3 . Following the method in chapter 1, we have

$$X_{m+3} = 2X_{m+2} + 5X_{m+1} - 6X_m$$

for $m = 1, 2, 3, \ldots$. The calculations are given in tables one and two.

The values $X_1 = -6$, $X_2 = 5$, and $X_3 = 2$ were arbitrarily chosen. The values of p_m and q_m were taken from the corresponding sequences A_8 / A_7, A_{10} / A_9, A_{12} / A_{11}, ... and A_9 / A_8, A_{11} / A_{10}, A_{13} / A_{12}, ... , since the first sequence is monotone increasing and the second sequence is monotone decreasing. The sequence $\{S_m\}$ converging to the root $S = 3$ of (4.1), is listed in Table 1, and in Table 2, the derived sequence $\{S^{(1)}_{m-2}\}$ is listed. The estimated error bound of $S_{m-2}^{(1)}$ is denoted by E_{m-2} in Table 2, where

$$E_{m-2} = |A_m| \ \left| \frac{q_m}{1 + q_m} - \frac{p_m}{1 + p_m} \right|$$

from (2.51). Since the true value of the root is 3, we see that the estimated error bound is approximately twenty times as large as the actual error of $S_{m-2}^{(1)}$.

m	X_m	$S_m = X_m / X_{m-1}$	$A_m = S_m - S_{m-1}$	A_m / A_{m-1}
1	-6			
2	5			
3	2			
4	65			
5	110			
6	553	4.845454545		
7	1,226	2.300187617	-2.545266928	
8	4,457	3.635399673	1.335212056	-0.524586259
9	11,846	2.657841597	-0.977558076	-0.732136945
10	38,621	3.260256626	0.602415029	-0.616244746
11	109,730	2.841200383	-0.419056243	-0.695627139
12	341,489	3.112084206	0.270883823	-0.646414002
13	999,902	2.928065032	-0.184019174	-0.679328769
14	3,048,869	3.049167818	0.121102786	-0.658098737
15	9,048,314	2.967760831	-0.081406987	-0.672213990
16	27,341,561	3.021729904	0.053969073	-0.662953820
17	81,631,478	2.985618780	-0.036111124	-0.669107731

TABLE 1

m	p_m	q_m	E_{m-2}	$S_{m-2}^{(1)}$
9	-0.732136945	-0.524586259	1.593243640	3.071034321
10	-0.732136945	-0.616244746	0.679176811	3.030566724
11	-0.695627139	-0.616244746	0.284797545	3.013117244
12	-0.695627139	-0.646414002	0.123869143	3.005729974
13	-0.679328769	-0.646414002	0.053419375	3.002505201
14	-0.679328769	-0.658098737	0.023450067	3.3001102169
15	-0.672213990	-0.658098737	0.010253195	3.000485659
16	-0.672213990	-0.662953820	0.004523599	3.000214572
17	-0.669107731	-0.662953820	0.001992582	3.000094917

TABLE 2

EXAMPLE 2: The equation to be considered is,

$$AX = \lambda X \tag{4.2}$$

where A is a second order matrix with elements

$$a_{11} = 1, \quad a_{12} = 9, \quad a_{21} = 1, \quad a_{22} = 1 .$$

The symbol $\{c, d\}$ will be used to denote a column vector, where c is the first and d the second component. The latent roots of the matrix A are $\lambda = -2$ and $\lambda = 4$. The latent vector associated with $\lambda = 4$ is $\{3, 1\}$. We arbitrarily chose the initial value of X as

$$X_1 = \{x_1, y_1\} = \{1, 1\}$$

and denoted $A^{m-1} X_1$ as

$$A^{m-1} X_1 = \{x_m, y_m\} \qquad m = 2, 3, 4, \ldots . \tag{4.3}$$

In the calculations to follow, the vector $\{x_m, y_m\}$ in (4.3) will be transformed into the vector

$$\{x_m / y_m, 1\} \tag{4.4}$$

and for uniformity we let $S_m = x_m / y_m$.

Since the second component of (4.4) is 1, it will not be listed. The quantities S_m, A_m, A_m / A_{m-1}, p_m, q_m, E_{m-2} and $S_{m-2}^{(1)}$ in tables three and four are similar to the corresponding quantities in tables one and two, so little discussion will be devoted to them. Comparing E_{m-2} and $S_{m-2}^{(1)}$ in table 4, we see that the estimated error bound, E_{m-2}, is approximately ten times as large the actual error of $S_{m-2}^{(1)}$, since the true value of first component is 3 .

m	x_m	y_m	$S_m=x_m/y_m$	$A_m=S_m-S_{m-1}$	A_m/A_{m-1}
1	1	1	1.000000000		
2	10	2	5.000000000		
3	28	12	2.333333333	-2.666666667	
4	136	40	3.400000000	1.066666667	-0.400000000
5	496	176	2.818181818	-0.581818182	-0.545454545
6	2,080	672	3.095238095	0.277056277	-0.476190475
7	8,128	2,752	2.953488372	-0.141749723	-0.511627906
8	32,896	10,880	3.023529411	0.070041039	-0.494117642
9	130,816	43,776	2.988304093	-0.035225318	-0.502923978
10	524,800	174,592	3.005865102	0.017561009	-0.498533725
11	2,096,128	699,392	2.997071742	-0.008793360	-0.500732047

TABLE 3

m	p_m	q_m	E_{m-2}	$s_{m-2}^{(1)}$
5	-0.545454545	-0.400000000	0.310303029	3.023529411
6	-0.545454545	-0.476190475	0.080598190	3.005865102
7	-0.511627906	-0.476190475	0.019636325	3.001465201
8	-0.511627906	-0.494117642	0.004964150	3.000366232
9	-0.502923978	-0.494117642	0.001233610	3.000091553
10	-0.502923978	-0.498533725	0.000309296	3.000022887
11	-0.500732047	-0.498533725	0.000077209	3.000005721

TABLE 4

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