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Title RECURSIVE LEAST-SQUARES SMOOTHING WITH CONSTANT OR FACTORIAL WEIGHTS

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This thesis develops recursion formulas for least-squares data smoothing with regard to four classes of functions: linear, quadratic, exponential and trigonometric, which in a linear fashion involve 2, 3, 2 and 2 parameters respectively. The term recursion implies here that the estimates for the parameters of the preceding fit are taken to summarize all past data and are used along with the new observation to yield new estimates of the parameters.

Results for constant weights are given in all cases and in addition factorial weights are considered in the linear and quadratic cases, which yield the simplest results.

In the linear case an effort is made to lead the reader to some sort of intuitive acceptance of the precedents set by Levine [1] with regard to the special forms of linear and quadratic functions adopted for the fitting procedure.

As may be noted in the smoothing equations for each of the four cases, the change in a parameter is always proportional to the difference between the new observation and the corresponding prediction.

Following the trigonometric case an example is given which is intended to illustrate the usefulness of least-squares recursion formulas in terms of the considerable saving of computational labor.

RECURSIVE LEAST-SQUARES SMOOTHING WITH CONSTANT OR FACTORIAL WEIGHTS

by

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RECURSIVE LEAST-SQUARES SMOOTHING WITH CONSTANT OR FACTORIAL WEIGHTS

I. LINEAR SMOOTHING

The statistical foundation for the following development is the Gauss-Markoff Theorem which is now stated.

If a number n of uncorrelated observations $\mathring{x}_j$ are distributed with common variance about means $E(\mathring{x}_j) = \theta_1 z_{j1} + \theta_2 z_{j2} + \cdots + \theta_k z_{jk}$ $(j = 1, 2, \cdots, n)$ where z_{ji} denotes a known constant and θ_i an unknown parameter, then the minimum-variance linear unbiased estimators of the θ_i are the solutions θ_i of a system of linear equations obtained by minimizing the sum of the squared deviations of the $\mathring{x}_j$ from the means $E(\mathring{x}_j)$ with respect to the unknown parameters θ_i. The minimum-variance estimate of a linear combination $\theta_1 z_1 + \cdots + \theta_k z_k$ is $\hat{\theta}_1 z_1 + \cdots + \hat{\theta}_k z_k$.

Suppose it is desirable to fit, by the method of least-squares, the straight line

$$y_n(t) = a_n + b_n t \tag{1.1}$$

to the observations $\mathring{x}_j$ each weighted by

$$w_j = \frac{(p + j - 1)!}{(j-1)!}, \quad \text{where } j = 1, 2, \cdots, n.$$

If the observations occur exactly τ units apart, then a relation giving the ordinate y_n for the corresponding abscissa values at which the observations occur can be stated in terms of j and τ by replacing t by $\tau(j-1)$ in (1.1). Thus $y_n(t) = y_n[\tau(j-1)] = f_n(j)$ or

$$f_n(j) = a_n + b_n\tau(j-1). \tag{1.2}$$

Evaluating $f_n(j)$ at $j = n$ gives

$$f_n(n) = a_n + b_n\tau(n-1). \tag{1.3}$$

Differentiating (1.1) or (1.2) with respect to t and substituting $j = n$ produces

$$\left\{\begin{aligned} b_n &= f'_n(n) \\ a_n &= f_n(n) - f'_n(n)\tau(n-1). \end{aligned}\right. \tag{1.4}$$

Substituting from (1.4) into (1.2) results in

$$f_n(j) = f_n(n) - f'_n(n)\tau(n-1) + f'_n(n)\tau(j-1) \quad \text{or}$$

$$f_n(j) = f_n(n) - f'_n(n)\tau(n-j).$$

For convenience let $f_n(n) = \bar{x}_n$ and

$$f'_n(n)T = \bar{u}_n. \quad \text{Then}$$

$$f_n(j) = \bar{x}_n - \bar{u}_n(n-j) \tag{1.5}$$

which is the form to be fitted in following the precedent of Levine [1].

The weighted sum of the squared deviations of $f_n(j)$ from the observations $\mathring{x}_j$, where $(j = 1, 2, \cdots, n)$ is

$$R_n^2 = \sum_{j=1}^{n} \{ \mathring{x}_j - [\bar{x}_n - (n-j)\bar{u}_n] \}^2 w_j . \tag{1.6}$$

Differentiating R_n^2 with respect first to $\bar{x}_n$ and then to $\bar{u}_n$ and setting these derivatives equal to zero produces the normal equations, the solution of which minimizes R_n^2. The normal equations are thus

$$(1.7) \quad \left\{ \begin{aligned} F_n \bar{x}_n - G_n \bar{u}_n &= \sum_{j=1}^{n} \mathring{x}_j w_j \\ -G_n \bar{x}_n + H_n \bar{u}_n &= -\sum_{j=1}^{n} \mathring{x}_j (n-j) w_j, \quad \text{where} \end{aligned} \right.$$

(1.8) $$F_n = \sum_{j=1}^{n} w_j, \quad G_n = \sum_{j=1}^{n} (n-j)\, w_j \quad \text{and} \quad H_n = \sum_{j=1}^{n} (n-j)^2 w_j \, .$$

Replacing n by $n-1$ in the above equations gives

(1.9) $$\begin{cases} F_{n-1}\overline{x}_{n-1} - G_{n-1}\overline{u}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j w_j \\ \\ -G_{n-1}\overline{x}_{n-1} + H_{n-1}\overline{u}_{n-1} = -\sum_{j=1}^{n-1} \mathring{x}_j (n-j-1)\, w_j \, . \end{cases}$$

Recursion being the objective, explicit presence of the first $n-1$ observations should be eliminated. This is done by subtracting from (1.7) a changed form of (1.9). The next few steps will be concerned with changing (1.9) to a more opportune form.

First one verifies that

(1.10) $$\begin{cases} F_{n-1} = F_n - w_n, \quad G_{n-1} = G_n - F_n + w_n \quad \text{and} \\ \\ H_{n-1} = H_n - 2G_n + F_n - w_n . \end{cases}$$

Making these substitutions in (1.9) leads to

$$(1.11)\begin{cases} F_n(\bar{x}_{n-1}+\bar{u}_{n-1})-G_n\bar{u}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j w_j + w_n(\bar{x}_{n-1}+\bar{u}_{n-1}) \\ \\ -G_n(\bar{x}_{n-1}+\bar{u}_{n-1})+H_n\bar{u}_{n-1}+F_n(\bar{x}_{n-1}+\bar{u}_{n-1})-G_n\bar{u}_{n-1} = \\ \qquad -\sum_{j=1}^{n-1} \mathring{x}_j(n-j-1)w_j + w_n(\bar{x}_{n-1}+\bar{u}_{n-1}). \end{cases}$$

Subtracting the first equation of (1. 11) from the second and substituting this result for the second changes (1. 11) to

$$(1.12)\begin{cases} F_n(\bar{x}_{n-1}+\bar{u}_{n-1})-G_n\bar{u}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j w_j + w_n(\bar{x}_{n-1}+\bar{u}_{n-1}) \\ \\ -G_n(\bar{x}_{n-1}+\bar{u}_{n-1}) + H_n\bar{u}_{n-1} = -\sum_{j=1}^{n} \mathring{x}_j(n-j)w_j \end{cases}$$

which is therefore the changed form of (1. 9) to be subtracted from (1. 7).

The results of subtracting (1. 12) from (1. 7) follows.

$$(1.13)\begin{cases} F_n[\bar{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1})] - G_n(\bar{u}_n-\bar{u}_{n-1}) = w_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1})] \\ -G_n[\bar{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1})] + H_n(\bar{u}_n-\bar{u}_{n-1}) = 0 . \end{cases}$$

Solving (1.13) for $\bar{x}_n$ and $\bar{u}_n$, the smoothing equations follow.

$$(1.14)\begin{cases} \bar{x}_n = \bar{x}_{n-1} + \bar{u}_{n-1} + \alpha_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})] \\ \bar{u}_n = \bar{u}_{n-1} + \beta_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})], \quad \text{where} \end{cases}$$

$$(1.15)\quad \alpha_n = \frac{w_n H_n}{J_n}, \quad \beta_n = \frac{w_n G_n}{J_n} \quad \text{and} \quad J_n = F_n H_n - G_n^2 .$$

Making formulas (1.14) explicit requires the evaluation of α_n, β_n and hence F_n, G_n and H_n.

$$(1.16)\begin{cases} F_n = \dfrac{(p+n)!}{(n-1)!\,(p+1)}, \quad G_n = \dfrac{(p+n)!}{(n-2)!\,(p+1)(p+2)} \quad \text{and} \\ H_n = \dfrac{(p+n)!\,(p-1+2n)}{(n-2)!\,(p+1)(p+2)(p+3)} \quad \text{by finite integration [2,} \end{cases}$$

p. 20-27].

From (1.16) and the definition of J_n in (1.15),

$$(1.17) \qquad J_n = \frac{(p+n)!^2 (p+1+n)}{(n-1)(n-2)!^2 (p+1)(p+2)^2 (p+3)} .$$

Applying (1.16) and (1.17) to (1.15) yields

$$\alpha_n = \frac{(p-1+2n)(p+2)}{(p+n)(p+1+n)} \quad \text{and} \quad \beta_n = \frac{(p+2)(p+3)}{(p+n)(p+1+n)} .$$

The results of this part on linear smoothing are summarized by the following theorem.

Theorem A: Given the real-valued observations $\mathring{x}_j$, each of which is weighted by $w_j = (p+j-1)!/(j-1)!$ where $j = 1, 2, \cdots, n$ and where successive observations are some constant T units apart, then the least-squares estimates of the parameters $\bar{x}_n$ and $\bar{u}_n$ in the representation $E(\mathring{x}_j) = f_n(j) = \bar{x}_n - (n-j)\bar{u}_n$ are determined recursively as follows:

$$\bar{x}_n = \bar{x}_{n-1} + \bar{u}_{n-1} + \alpha_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})]$$

$$\bar{u}_n = \bar{u}_{n-1} + \beta_n[\mathring{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1})] , \quad \text{where}$$

$$\alpha_n = \frac{(p-1+2n)(p+2)}{(p+n)(p+1+n)} \qquad \text{and} \qquad \beta_n = \frac{(p+2)(p+3)}{(p+n)(p+1+n)} .$$

II. QUADRATIC SMOOTHING

Suppose it is desired to fit by the method of least-squares, the quadratic function

$$(2.1) \qquad f_n(j) = \overline{x}_n - (n-j)\overline{u}_n + (n-j)^2\overline{s}_n$$

to the observations $\overset{\circ}{x}_j$, each of which is weighted by $w_j = (p+j-1)!/(j-1)!$ where $j = 1, 2, \cdots, n$ and where successive observations occur some constant τ units apart. Here $\overline{s}_n = \frac{1}{2}\tau^2$ times the acceleration (or rather its estimate) while $\overline{u}_n = \tau$ times the velocity as before.

The sum of the squared deviations of the observations $\overset{\circ}{x}_j$ from $f_n(j)$ $(j = 1, 2, \cdots, n)$ is

$$(2.2) \qquad R_n^2 = \sum_{j=1}^{n} \{\overset{\circ}{x}_j - [\overline{x}_n - (n-j)\overline{u}_n + (n-j)^2\overline{s}_n]\}^2 w_j .$$

Differentiating R_n^2 with respect to $\overline{x}_n$, then $\overline{u}_n$ and finally to $\overline{s}_n$ and setting these derivatives equal to zero produces the normal equations, the solution of which minimizes R_n^2. This linear system is thus

$$(2.3)\quad \begin{cases} F_n\bar{x}_n - G_n\bar{u}_n + H_n\bar{s}_n = \sum_{j=1}^{n} \mathring{x}_j w_j \\ -G_n\bar{x}_n + H_n\bar{u}_n - I_n\bar{s}_n = -\sum_{j=1}^{n} \mathring{x}_j (n-j) w_j \\ H_n\bar{x}_n - I_n\bar{u}_n + K_n\bar{s}_n = \sum_{j=1}^{n} \mathring{x}_j (n-j)^2 w_j, \quad \text{where} \end{cases}$$

$$(2.4)\quad \begin{cases} F_n = \sum_{j=1}^{n} w_j, \quad G_n = \sum_{j=1}^{n} (n-j) w_j, \quad H_n = \sum_{j=1}^{n} (n-j)^2 w_j, \\ I_n = \sum_{j=1}^{n} (n-j)^3 w_j \quad \text{and} \quad K_n = \sum_{j=1}^{n} (n-j)^4 w_j \,. \end{cases}$$

Replacing n in (2.3) by $n-1$ gives

$$(2.5)\quad \begin{cases} F_{n-1}\bar{x}_{n-1} - G_{n-1}\bar{u}_{n-1} + H_{n-1}\bar{s}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j w_j \\ -G_{n-1}\bar{x}_{n-1} + H_{n-1}\bar{u}_{n-1} - I_{n-1}\bar{s}_{n-1} = -\sum_{j=1}^{n-1} \mathring{x}_j (n-1-j) w_j \\ H_{n-1}\bar{x}_{n-1} - I_{n-1}\bar{u}_{n-1} + K_{n-1}\bar{s}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j (n-1-j)^2 w_j \,. \end{cases}$$

In order to obtain a system free from explicit mention of the first $n-1$ observations, coefficients $F_{n-1}, G_{n-1}, \cdots, K_{n-1}$ must be expressed in terms of $F_n, \cdots, K_n$ and w_n then substituted into (2.5). The desired system is obtained by subtracting a further altered form of (2.5) from (2.3). The next few steps will be concerned with these alterations.

It may be verified that

$$(2.6)\begin{cases} F_{n-1} = F_n - w_n, \quad G_{n-1} = G_n - F_n + w_n, \quad H_{n-1} = H_n - 2G_n + F_n - w_n \\ \\ I_{n-1} = I_n - 3H_n + 3G_n - F_n + w_n, \text{ and } K_{n-1} = K_n - 4I_n + 6H_n - 4G_n + F_n - w_n. \end{cases}$$

Making substitutions (2.6) into (2.5) yields

$$(2.7)\begin{cases} E_1: F_n\hat{x}_n - G_n\hat{u}_n + H_n\bar{s}_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j w_j + w_n\hat{x}_n \\ E_2: (F_n - G_n)\hat{x}_n + (H_n - G_n)\hat{u}_n + (H_n - I_n)\bar{s}_{n-1} = -\sum_{j=1}^{n-1} \mathring{x}_j (n-j-1) w_j + w_n\hat{x}_n \\ E_3: (H_n - 2G_n + F_n)\hat{x}_n + (I_n - 2H_n + G_n)\hat{u}_n + (K_n - 2I_n + H_n)\bar{s}_{n-1} = \end{cases}$$

$$\sum_{j=1}^{n-1} \mathring{x}_j (n-j-1)^2 w_j + w_n\hat{x}_n, \qquad \text{where}$$

$\hat{x}_n = \overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1}$ and

$\hat{u}_n = \overline{u}_{n-1} + 2\overline{s}_{n-1}$ are the predicted $\overline{x}_n$ and $\overline{u}_n$ respectively.

When $E_2 - E_1$ is substituted for E_2 and $E_3 - 2E_2 + E_1$ is substituted for E_3 the following system results.

$$(2.8)\quad \left\{ \begin{aligned} F_n\hat{x}_n - G_n\hat{u}_n + H_n\overline{s}_{n-1} &= \sum_{j=1}^{n-1} \mathring{x}_j w_j + w_n\hat{x}_n \\ -G_n\hat{x}_n + H_n\hat{u}_n - I_n\overline{s}_{n-1} &= -\sum_{j=1}^{n} \mathring{x}_j (n-j) w_j \\ H_n\hat{x}_n - I_n\hat{u}_n + K_n\overline{s}_{n-1} &= \sum_{j=1}^{n} \mathring{x}_j (n-j)^2 w_j . \end{aligned} \right.$$

System (2.8) is equivalent to (2.5) and when (2.8) is subtracted from (2.3), the following system results which is free from explicit mention of the first $n-1$ observations.

$$(2.9)\quad \left\{ \begin{aligned} F_n(\overline{x}_n - \hat{x}_n) - G_n(\overline{u}_n - \hat{u}_n) + H_n(\overline{s}_n - \overline{s}_{n-1}) &= w_n(\mathring{x}_n - \hat{x}_n) \\ -G_n(\overline{x}_n - \hat{x}_n) + H_n(\overline{u}_n - \hat{u}_n) - I_n(\overline{s}_n - \overline{s}_{n-1}) &= 0 \\ H_n(\overline{x}_n - \hat{x}_n) - I_n(\overline{u}_n - \hat{u}_n) + K_n(\overline{s}_n - \overline{s}_{n-1}) &= 0 . \end{aligned} \right.$$

The solution of (2.9) produces the smoothing equations:

$$(2.10)\quad \begin{cases} \bar{x}_n = \bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1}+\alpha_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})] \\ \bar{u}_n = \bar{u}_{n-1}+2\bar{s}_{n-1}+\beta_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})] \\ \bar{s}_n = \bar{s}_{n-1}+\gamma_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})]\ , \quad \text{where} \end{cases}$$

$$(2.11)\quad \begin{cases} \alpha_n = \dfrac{w_n(H_nK_n-I_n^2)}{D_n}\ , \quad \beta_n = \dfrac{w_n(G_nK_n-H_nI_n)}{D_n} \\ \gamma_n = \dfrac{w_n(G_nI_n-H_n^2)}{D_n} \quad \text{and where} \\ D_n = F_n(H_nK_n-I_n^2)-G_n(G_nK_n-H_nI_n)+H_n(G_nI_n-H_n^2)\ . \end{cases}$$

Obtaining explicit expressions for the smoothing equations (2.10) requires that α_n, β_n, and γ_n be evaluated in terms of the constants p and n. This further requires the evaluation of the sums $F_n, G_n, \cdots, K_n$ in terms of p and n.

The obvious approach would be to calculate α_n, β_n and γ_n directly from their definitions in (2.11). However this leads to serious computational obstructions. Therefore formulas (conjectured by Dr. E. L. Kaplan) for α_n, β_n and γ_n are proved demonstratively instead.

It is nevertheless still necessary to express F_n, G_n, H_n, I_n and K_n in closed form as functions of p and n, $(p \geq 0;$ integer $n \geq 3)$. This is done next.

$$(2.12)\begin{cases} F_n = \dfrac{(p+n)!}{(n-1)!(p+1)}, \quad G_n = \dfrac{(p+n)!}{(n-2)!(p+2)^{(2)}}, \quad H_n = \dfrac{(p+n)!(p-1+2n)}{(n-2)!(p+3)^{(3)}} \\[2ex] I_n = \dfrac{(p+n)![6(p+1+n)(n-2)+(p+3)(p+4)]}{(n-2)!(p+4)^{(4)}} \qquad \text{and} \\[2ex] K_n = \dfrac{(p+n)![(p+4)(p+5)(13p+11+14n)-12(p+2+n)(p+1+n)(p+9-2n)]}{(n-2)!(p+5)^{(5)}} \end{cases}$$

by finite integration [2, p. 20-27]. Here $(p+j)^{(k)} \equiv (p+j)(p+j-1)(p+j-2)\ldots(p+j-k+1)$ has k factors.

Crucial elements in the following argument are two well known properties from elementary matrix theory which are stated next.

<u>Property 1:</u> If each element in a row of a square matrix is multiplied by its own cofactor, the sum of the resulting products is the determinant of the matrix.

<u>Property 2:</u> Consider two rows of a square matrix. If each element in one row is multiplied by the cofactor of the corresponding element of the other row, the sum of the

resulting products are zero.

Consider the matrix of coefficients of system (2.3)

$$C = \begin{bmatrix} F_n & -G_n & H_n \\ -G_n & H_n & -I_n \\ H_n & -I_n & K_n \end{bmatrix}$$

and the cofactors of the first row. Applying these properties to the cofactors of the first row gives

$$(2.13)\quad \left\{\begin{aligned} F_n(H_nK_n-I_n^2)-G_n(G_nK_n-H_nI_n)+H_n(G_nI_n-H_n^2) &= D_n \\ -G_n(H_nK_n-I_n^2)+H_n(G_nK_n-H_nI_n)-I_n(G_nI_n-H_n^2) &= 0 \\ H_n(H_nK_n-I_n^2)-I_n(G_nK_n-H_nI_n)+K_n(G_nI_n-H_n^2) &= 0 \quad . \end{aligned}\right.$$

When (2.13) is multiplied by $\frac{w_n}{D_n}$ the result is

$$(2.14)\quad \left\{\begin{aligned} F_n\alpha_n - G_n\beta_n + H_n\gamma_n &= w_n \\ -G_n\alpha_n + H_n\beta_n - I_n\gamma_n &= 0 \\ H_n\alpha_n - I_n\beta_n + K_n\gamma_n &= 0 , \end{aligned}\right.$$

which follows from (2.11) and amounts to a simple change of variables in (2.9).

Substitution from (2.12) and

$$(2.15)\quad \left\{\begin{aligned} \alpha_n &= \frac{(p+3)[3(n-1)(p+n)+(p+1)(p+2)]}{(p+2+n)^{(3)}} \\ \beta_n &= \frac{3(p+3)(p+4)(p-1+2n)}{2(p+2+n)^{(3)}} \\ \gamma_n &= \frac{(p+3)(p+4)(p+5)}{2(p+2+n)^{(3)}} \end{aligned}\right\} \quad \left(\begin{array}{c}\text{Kaplan's}\\ \text{conjectures}\end{array}\right)$$

into (2.14) makes (2.14) a system of identities in p and n. Therefore formulas (2.15) are correct expressions for α_n, β_n, and γ_n. The question of whether they constitute the unique solution will be considered next. However this question is the same as whether the least-squares solution for the parameters $\overline{x}_n$, $\overline{u}_n$ and $\overline{s}_n$ is unique, since the coefficient matrix is the same in both cases. Therefore the question will be considered from the latter point of view.

Consider the matrix

$$M = \begin{bmatrix} \sqrt{w_1} & \sqrt{w_2} & \cdots & \sqrt{w_j} & \cdots & \sqrt{w_{n-2}} & \sqrt{w_{n-1}} & \sqrt{w_n} \\ -(n-1)\sqrt{w_1} & -(n-2)\sqrt{w_2} & \cdots & -(n-j)\sqrt{w_j} & \cdots & -2\sqrt{w_{n-2}} & -\sqrt{w_{n-1}} & 0 \\ (n-1)^2\sqrt{w_1} & (n-2)^2\sqrt{w_2} & \cdots & (n-j)^2\sqrt{w_j} & \cdots & 2^2\sqrt{w_{n-2}} & \sqrt{w_{n-1}} & 0 \end{bmatrix}$$

each of whose columns is composed of the coefficients of the expression for the estimated ordinate at one of the n abscissa values. The product of M and its transpose is the coefficient matrix C whose determinant is hence the sum of the squares of all the 3×3 minors of M. Obviously if any one of these minors is non-zero, then the determinant of C is non-zero, which implies that any system having C as its coefficient matrix has a unique solution. It is an easy matter to show that the minor consisting of the last three columns of M is non-zero for $n \geq 3$.

A formal statement of this part on quadratic smoothing is given by the following theorem.

<u>Theorem B:</u> Given the real valued observations $\mathring{x}_j$ each of which is weighted by $w_j = (p+j-1)!/(j-1)!$, where $j = 1, 2, \cdots, n$ and where successive observations are some constant T units apart, also given that $E(\mathring{x}_j)$ has the form $f_n(j)=\bar{x}_n-(n-j)\bar{u}_n+(n-j)^2\bar{s}_n$, then the least-squares estimates of the parameters $\bar{x}_n$, $\bar{u}_n$ and $\bar{s}_n$ are determined recursively as follows:

$$\bar{x}_n = \bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1} + \alpha_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})],$$

$$\bar{u}_n = \bar{u}_{n-1}+2\bar{s}_{n-1}+\beta_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})] \quad \text{and}$$

$$\bar{s}_n = \bar{s}_{n-1}+\gamma_n[\mathring{x}_n-(\bar{x}_{n-1}+\bar{u}_{n-1}+\bar{s}_{n-1})], \quad \text{where}$$

$$\alpha_n = \frac{(p+3)[3(n-1)(p+n)+(p+1)(p+2)]}{(p+2+n)^{(3)}} \quad ,$$

$$\beta_n = \frac{3(p+3)(p+4)(p-1+2n)}{2(p+2+n)^{(3)}} \qquad \text{and}$$

$$\gamma_n = \frac{(p+3)(p+4)(p+5)}{2(p+2+n)^{(3)}} \quad .$$

III. EXPONENTIAL SMOOTHING

Consider the observations $\mathring{x}_1, \mathring{x}_2, \cdots, \mathring{x}_j, \cdots, \mathring{x}_n$ which occur sequentially and exactly τ units apart. Suppose it is required to fit

(3. 1) $$y_n(t) = a_n + b_n e^{\zeta t}$$

to these data by the method of least-squares. Consequently if $\zeta\tau = q$, then $y_n(t) = y_n[\tau(j-1)] = f_n(j)$ since the time of the j^{th} observation is $\tau(j-1)$. i. e.,

(3. 2) $$f_n(j) = a_n + b_n e^{q(j-1)} .$$

The sum of the squared deviations of the observations $\mathring{x}_j$ from the $f_n(j)$ is

$$R_n^2 = \sum_{j=1}^{n} \{\mathring{x}_j - [a_n + b_n e^{q(j-1)}]\}^2,$$

constant (unit) weights being assumed in this case.

Differentiating R_n^2 first with respect to a_n and then b_n and setting these derivatives equal to zero produces

$$(3.3)\quad \begin{cases} na_n + b_n \sum_{j=1}^{n} e^{q(j-1)} = \sum_{j=1}^{n} \mathring{x}_j \\ a_n \sum_{j=1}^{n} e^{q(j-1)} + b_n \sum_{j=1}^{n} e^{2q(j-1)} = \sum_{j=1}^{n} \mathring{x}_j e^{q(j-1)} \end{cases}$$

which are the normal equations.

Let

$$(3.4)\qquad G_n = \sum_{j=1}^{n} e^{q(j-1)} \quad \text{and} \quad H_n = \sum_{j=1}^{n} e^{2q(j-1)}.$$

Then

$$(3.5)\quad \begin{cases} na_n + G_n b_n = \sum_{j=1}^{n} \mathring{x}_j \quad \text{and} \\ G_n a_n + H_n b_n = \sum_{j=1}^{n} \mathring{x}_j e^{q(j-1)} \quad . \end{cases}$$

Replacing n by $n-1$ gives

$$(3.6)\quad \begin{cases} (n-1)a_{n-1} + G_{n-1}b_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j \quad \text{and} \\ \\ G_{n-1}a_{n-1} + H_{n-1}b_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j e^{q(j-1)} \quad . \end{cases}$$

Subtracting (3. 6) from (3. 5) and making the substitution

$$(3.7)\qquad G_{n-1} = G_n - e^{q(n-1)} \qquad \text{and} \qquad H_{n-1} = H_n - e^{2q(n-1)}$$

results in the following system which no longer explicitly involves the first $n-1$ observations.

$$(3.8)\quad \begin{cases} n(a_n - a_{n-1}) + G_n(b_n - b_{n-1}) = \mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)} \\ \\ G_n(a_n - a_{n-1}) + H_n(b_n - b_{n-1}) = (\mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)})e^{q(n-1)} \end{cases}$$

Hence

$$(3.9)\quad \begin{cases} D_n(a_n - a_{n-1}) = (\mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)})(H_n - G_n e^{q(n-1)}) \\ \\ D_n(b_n - b_{n-1}) = (\mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)})(ne^{q(n-1)} - G_n), \quad \text{where} \end{cases}$$

$$D_n = nH_n - G_n^2 .$$

Solving (3.9) for a_n and b_n gives the following recursion relations (the smoothing equations).

$$(3.10)\quad \begin{cases} a_n = a_{n-1} + \alpha_n(\overset{\circ}{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)}) \\ b_n = b_{n-1} + \beta_n(\overset{\circ}{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)})e^{q(n-1)}, \end{cases}$$

where $\alpha_n = \dfrac{H_n - G_n e^{q(n-1)}}{D_n}$ and $\beta_n = \dfrac{ne^{q(n-1)} - G_n}{D_n}$.

Finally inserting the following explicit evaluations

$$(3.11)\qquad G_n = \sum_{j=1}^{n} e^{q(j-1)} = \frac{e^{qn}-1}{e^q - 1} \quad \text{and} \quad H_n = \sum_{j=1}^{n} e^{2q(j-1)} = \frac{e^{2qn}-1}{e^{2q}-1}$$

yields the following theorem on the smoothing of exponential trends:

<u>Theorem C:</u> Consider the real observations $\overset{\circ}{x}_j$ $(j = 1, 2, \cdots, n)$ which successively occur a constant τ units apart with $E(\overset{\circ}{x}_j) = f_n(j) = a_n + b_n e^{q(j-1)}$. Then the least-squares estimates of the parameters a_n and b_n are determined recursively as follows:

$$a_n = a_{n-1} + \alpha_n(\mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)})$$

$$b_n = b_{n-1} + \beta_n(\mathring{x}_n - a_{n-1} - b_{n-1}e^{q(n-1)}),$$ where

$$\alpha_n = \frac{(e^q - 1)(e^{q(n-1)} - 1)}{(e^q + 1)(e^{qn} - 1) - n(e^q - 1)(e^{qn} + 1)}$$ and

$$\beta_n = \frac{(e^{2q} - 1)[(e^{qn} - 1) - ne^{q(n-1)}(e^q - 1)]}{(e^{qn} - 1)[(e^q + 1)(e^{qn} - 1) - n(e^q - 1)(e^{qn} + 1)]}.$$

IV. TRIGONOMETRIC SMOOTHING

Suppose it is desired to fit by least-squares

$$y_n(t) = b_n \sin \omega t + c_n \cos \omega t \tag{4.1}$$

to the observations $\mathring{x}_1, \mathring{x}_2, \cdots, \mathring{x}_j, \cdots, \mathring{x}_n$ which are made successively T units of abscissa apart. Then the t value of the j^{th} observation can be given by $t = T(j-1)$. Further, if $\omega T=q$, then $y_n(t) = y_n[T(j-1)] = f_n(j)$. That is

$$f_n(j) = b_n \sin q(j-1) + c_n \cos q(j-1). \tag{4.2}$$

The sum of the squared deviations from the least-squares curve $f_n(j)$ is

$$R_n^2 = \sum_{j=1}^{n} \{ \mathring{x}_j - [b_n \sin q(j-1) + c_n \cos q(j-1)] \}^2 .$$

Differentiating R_n^2 with respect to first b_n and then c_n and setting these derivatives equal to zero produces the normal equations,

$$(4.3)\quad \begin{cases} b_n \sum_{j=1}^{n} \sin^2 q(j-1) + c_n \sum_{j=1}^{n} \sin q(j-1)\cos q(j-1) = \sum_{j=1}^{n} \mathring{x}_j \sin q(j-1) \\ b_n \sum_{j=1}^{n} \sin q(j-1)\cos q(j-1) + c_n \sum_{j=1}^{n} \cos^2 q(j-1) = \sum_{j=1}^{n} \mathring{x}_j \cos q(j-1). \end{cases}$$

(4.4) Defining $J_n = \sum_{j=1}^{n} \sin^2 q(j-1)$, $I_n = \sum_{j=1}^{n} \sin q(j-1)\cos q(j-1)$ and

$K_n = \sum_{j=1}^{n} \cos^2 q(j-1)$ changes (4.3) to

$$(4.5)\quad \begin{cases} J_n b_n + I_n c_n = \sum_{j=1}^{n} \mathring{x}_j \sin q(j-1) \\ I_n b_n + K_n c_n = \sum_{j=1}^{n} \mathring{x}_j \cos q(j-1) \;. \end{cases}$$

Replacing n by $n-1$ in (4.5) gives

$$(4.6)\quad \begin{cases} J_{n-1} b_{n-1} + I_{n-1} c_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j \sin q(j-1) \\ I_{n-1} b_{n-1} + K_{n-1} c_{n-1} = \sum_{j=1}^{n-1} \mathring{x}_j \cos q(j-1) \;. \end{cases}$$

Subtracting (4.6) from (4.5) yields

$$(4.7)\quad \begin{cases} J_n b_n - J_{n-1} b_{n-1} + I_n c_n - I_{n-1} c_{n-1} = \mathring{x}_n \sin q(n-1) \\ \\ I_n b_n - I_{n-1} b_{n-1} + K_n c_n - K_{n-1} c_{n-1} = \mathring{x}_n \cos q(n-1) \quad . \end{cases}$$

From the definitions of J_n, I_n and K_n in (4.4) it may be observed that

$$(4.8)\quad \begin{cases} J_{n-1} = J_n - \sin^2 q(n-1), \quad I_{n-1} = I_n - \cos q(n-1) \sin q(n-1) \text{ and} \\ \\ K_{n-1} = K_n - \cos^2 q(n-1). \end{cases}$$

Making these substitutions in (4.7) gives

$$(4.9)\quad \begin{cases} J_n(b_n - b_{n-1}) + I_n(c_n - c_{n-1}) = [\mathring{x}_n - b_{n-1} \sin q(n-1) - c_{n-1} \cos q(n-1)] \sin q(n-1) \\ \\ I_n(b_n - b_{n-1}) + K_n(c_n - c_{n-1}) = [\mathring{x}_n - b_{n-1} \sin q(n-1) - c_{n-1} \cos q(n-1)] \cos q(n-1) \end{cases}$$

wherein $b_{n-1}\sin q(n-1) + c_{n-1}\cos q(n-1)$ may be denoted by $\hat{x}_n$, the expected value of the n^{th} observation as predicted from the first $n-1$ observations. If also $D_n = J_n K_n - I_n^2$, the solution of (4.9) takes the following form.

$$(4.10)\quad \begin{cases} b_n = b_{n-1} + \beta_n(\mathring{x} - \hat{x}_n) \\ c_n = c_{n-1} + \gamma_n(\mathring{x} - \hat{x}_n), \quad \text{where} \end{cases}$$

$$(4.11)\quad \begin{cases} \beta_n = [K_n \sin q(n-1) - I_n \cos q(n-1)] \div D_n \quad \text{and} \\ \gamma_n = [-I_n \sin q(n-1) + J_n \cos q(n-1)] \div D_n \quad . \end{cases}$$

By the methods of finite integration it becomes apparent that

$$(4.12)\quad \begin{cases} J_n = \sum_{j=1}^{n} \sin^2 q(j-1) = [(2n-1)\sin q - \sin q(2n-1)] \div 4 \sin q \ , \\ I_n = \sum_{j=1}^{n} \sin q(j-1) \cos q(j-1) = [\cos q - \cos q(2n-1) \div 4 \sin q \text{ and} \\ K_n = \sum_{j=1}^{n} \cos^2 q(j-1) = [(2n+1)\sin q + \sin q(2n-1) \div 4 \sin q \ . \end{cases}$$

Hence

$$D_n = J_n K_n - I_n^2 = (n^2 \sin^2 q - \sin^2 qn) \div 4 \sin^2 q \ ,$$

$$\beta_n = \frac{2n\sin^2 q \sin q(n-1)}{n^2\sin^2 q - \sin^2 qn}$$

$$\gamma_n = \frac{2\sin q[n\sin q\cos q(n-1) - \sin qn]}{n^2\sin^2 q - \sin^2 qn} .$$

The results of this section on trigonometric smoothing are summarized by the following theorem.

Theorem D: Consider the real observations $\mathring{x}_j$ $(j = 1, 2, \cdots, n)$ which occur successively some constant τ units apart. Suppose $E(\mathring{x}_j)$ has the form $f_n(j) = b_n \sin q(j-1) + c_n \cos q(j-1)$. Then the least-squares estimates of the parameters b_n and c_n are determined recursively as follows:

$$(4.13)\begin{cases} b_n = b_{n-1} + T\, n\sin q \sin q(n-1) \quad \text{and} \\ c_n = c_{n-1} + T[n\sin q\cos q(n-1) - \sin qn] \end{cases}$$

where

$$T = \frac{2\sin q[\mathring{x}_n - b_{n-1}\sin q(n-1) - c_{n-1}\cos q(n-1)]}{n^2\sin^2 q - \sin^2 qn} .$$

V. AN EXAMPLE

The following example is given to illustrate, in a small way, the labor saving advantages of recursive smoothing. It will be done by two methods.

Consider the problem of fitting the function

$$(5.1)\quad \begin{cases} f_4(j)=b_4\sin\frac{\pi}{6}(j-1)+c_4\cos\frac{\pi}{6}(j-1) \quad \text{to the observations} \\ \overset{\circ}{x}_1=2,\ \overset{\circ}{x}_2=\frac{3+2\sqrt{3}}{2},\ \overset{\circ}{x}_3=\frac{2+3\sqrt{3}}{2},\ \overset{\circ}{x}_4=2 \quad \text{after the} \end{cases}$$

first three have already been fitted with $f_3(j)=3\sin\frac{\pi}{6}(j-1)+2\cos\frac{\pi}{6}(j-1)$ and where j is the number of the observation, starting with 1.

First the problem will be done using the traditional least-squares method, then it will be done recursively.

The normal equations are

$$(5.2)\quad \begin{cases} b_4\sum_{j=1}^{4}\sin^2\frac{\pi}{6}(j-1)+c_4\sum_{j=1}^{4}\sin\frac{\pi}{6}(j-1)\cos\frac{\pi}{6}(j-1)=\sum_{j=1}^{4}\overset{\circ}{x}_j\sin\frac{\pi}{6}(j-1) \quad \text{and} \\ b_4\sum_{j=1}^{4}\sin\frac{\pi}{6}(j-1)\cos\frac{\pi}{6}(j-1)+c_4\sum_{j=1}^{4}\cos^2\frac{\pi}{6}(j-1)=\sum_{j=1}^{4}\overset{\circ}{x}_j\cos\frac{\pi}{6}(j-1)\,. \end{cases}$$

There are five sums of four terms each to be evaluated before the system can be solved.

$$\sum_{j=1}^{4} \sin^2 \frac{\pi}{6}(j-1) = 0 + \frac{1}{4} + \frac{3}{4} + 1 = 2$$

$$\sum_{j=1}^{4} \cos^2 \frac{\pi}{6}(j-1) = 1 + \frac{3}{4} + \frac{1}{4} + 0 = 2$$

$$\sum_{j=1}^{4} \sin \frac{\pi}{6}(j-1) \cos \frac{\pi}{6}(j-1) = 0 + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + 0 = \frac{\sqrt{3}}{2}$$

$$\sum_{j=1}^{4} \mathring{x}_j \sin \frac{\pi}{6}(j-1) = 0 + \frac{3+2\sqrt{3}}{4} + \frac{9+2\sqrt{3}}{4} + 2 = 5 + \sqrt{3}$$

$$\sum_{j=1}^{4} \mathring{x}_j \cos \frac{\pi}{6}(j-1) = 2 + \frac{6+3\sqrt{3}}{4} + \frac{2+3\sqrt{3}}{4} + 0 = \frac{8+3\sqrt{3}}{2} .$$

Making these substitutions in (5.2), the system to be solved becomes

$$2b_4 + \frac{\sqrt{3}}{2} c_4 = 5 + \sqrt{3}$$

$$\frac{\sqrt{3}}{2} b_4 + 2c_4 = \frac{8+3\sqrt{3}}{2} .$$ Hence,

$$b_4 = \frac{31}{13} \quad \text{and} \quad c_4 = \frac{26 + 2\sqrt{3}}{13} .$$

Nexy by way of contrast, these same coefficients (of f_4) will be calculated using recursion formulas (4. 13) on page 27.

Substituting $n = 4$, $q = \frac{\pi}{6}$, $b_3 = 3$, $c_3 = 2$, and $\overset{\circ}{x}_4 = 2$,

$$b_4 = 3 + \left[\frac{2(\frac{1}{2})(2-3-0)}{16(\frac{1}{4}) - \frac{3}{4}}\right] (4)(\frac{1}{2})(1) \quad = \frac{31}{13} \quad \text{and}$$

$$c_4 = 2 + \left[\frac{2(\frac{1}{2})(2-3-0)}{16(\frac{1}{4}) - \frac{3}{4}}\right] [4(\frac{1}{2})(0) - \frac{\sqrt{3}}{2}] \; = \frac{26+2\sqrt{3}}{13} \; .$$

The saving of computational energy is even more impressive if one considers the additional computation encountered in fitting to many observations instead of only four as in this example; each of the five sums would have as many terms as there are observations.

<u>Comment</u>: Formulas (4. 13) are useful for fitting purposes even though the axis of oscillation is not zero, provided the mean value μ is known <u>a priori</u>. Instead of fitting to the observations $\overset{\circ}{x}_j$ themselves, formulas (4. 13) could be used to fit $f_n(j)$ to $\overset{\circ}{x}_j - \mu$, $(j = 1, 2, \cdots, n)$. Then μ would be added back to $f_n(j)$ in order to obtain $g_n(j) = b_n \sin q(j-1) + c_n \cos q(j-1) + \mu$ which would be a fitting for the untransformed observations $\overset{\circ}{x}_j$.

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