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Title RECURSIVE LEAST-SQUARES SMOOTHING WITH CONSTANT OR FACTORIAL WEIGHTS

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This thesis develops recursion formulas for least-squares data smoothing with regard to four classes of functions: linear, quadratic, exponential and trigonometric, which in a linear fashion involve 2, 3, 2 and 2 parameters respectively. The term recursion implies here that the estimates for the parameters of the preceding fit are taken to summarize all past data and are used along with the new observation to yield new estimates of the parameters.

Results for constant weights are given in all cases and in addition factorial weights are considered in the linear and quadratic cases, which yield the simplest results.

In the linear case an effort is made to lead the reader to some sort of intuitive acceptance of the precedents set by Levine [1] with regard to the special forms of linear and quadratic functions adopted for the fitting procedure.
As may be noted in the smoothing equations for each of the four cases, the change in a parameter is always proportional to the difference between the new observation and the corresponding prediction.

Following the trigonometric case an example is given which is intended to illustrate the usefulness of least-squares recursion formulas in terms of the considerable saving of computational labor.
RECURSIVE LEAST-SQUARES SMOOTHING WITH CONSTANT OR FACTORIAL WEIGHTS

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I. LINEAR SMOOTHING

The statistical foundation for the following development is the Gauss-Markoff Theorem which is now stated.

If a number of uncorrelated observations \( \hat{x}_j \) are distributed with common variance about means 
\[
E(\hat{x}_j) = \theta_1 z_{j1} + \theta_2 z_{j2} + \cdots + \theta_k z_{jk} \quad (j = 1, 2, \cdots, n)
\]
where \( z_{ji} \) denotes a known constant and \( \theta_i \) an unknown parameter, then the minimum-variance linear unbiased estimators of the \( \theta_i \) are the solutions \( \hat{\theta}_i \) of a system of linear equations obtained by minimizing the sum of the squared deviations of the \( \hat{x}_j \) from the means \( E(\hat{x}_j) \) with respect to the unknown parameters \( \theta_i \). The minimum-variance estimate of a linear combination
\[
\theta_1 z_{1} + \cdots + \theta_k z_{k} \quad \text{is} \quad \hat{\theta}_1 z_{1} + \cdots + \hat{\theta}_k z_{k}.
\]

Suppose it is desirable to fit, by the method of least-squares, the straight line
\[
y_n(t) = a_n + b_n t
\]
(1.1)
to the observations \( \hat{x}_j \) each weighted by
\[
w_j = \frac{(p + j - 1)!}{(j-1)!}, \quad \text{where} \quad j = 1, 2, \cdots, n.
\]
If the observations occur exactly $\tau$ units apart, then a relation giving the ordinate $y_n$ for the corresponding abscissa values at which the observations occur can be stated in terms of $j$ and $\tau$ by replacing $t$ by $\tau(j-1)$ in (1.1). Thus

$$y_n(t) = y_n[\tau(j-1)] = f_n(j) \quad \text{or}$$

(1.2) \hspace{1cm} f_n(j) = a_n + b_n \tau(j-1).

Evaluating $f_n(j)$ at $j = n$ gives

(1.3) \hspace{1cm} f_n(n) = a_n + b_n \tau(n-1).

Differentiating (1.1) or (1.2) with respect to $t$ and substituting $j = n$ produces

$$\begin{cases} 
  b_n = f'_n(n) \\
  a_n = f_n(n) - f'_n(n) \tau(n-1). 
\end{cases}
$$

(1.4)

Substituting from (1.4) into (1.2) results in

$$f_n(j) = f_n(n) - f'_n(n) \tau(n-1) + f'_n(n) \tau(j-1) \quad \text{or}$$

$$f_n(j) = f_n(n) - f'_n(n) \tau(n-j).$$
For convenience let \( f_n(n) = \bar{x}_n \) and

\[
 f'_n(n) \tau = \bar{u}_n. \quad \text{Then}
\]

\[
(1.5) \quad f_n(j) = \frac{\bar{x}_n - \bar{u}_n(n-j)}{\text{which is the form to be fitted in following the precedent of Levine [1].}}
\]

The weighted sum of the squared deviations of \( f_n(j) \) from the observations \( \hat{x}_j \), where \( (j = 1, 2, \ldots, n) \) is

\[
(1.6) \quad R_n^2 = \sum_{j=1}^{n} \left\{ \hat{x}_j - \left[ \frac{\bar{x}_n}{\bar{u}_n(n-j)} \right] \right\}^2 w_j.
\]

Differentiating \( R_n^2 \) with respect first to \( \bar{x}_n \) and then to \( \bar{u}_n \) and setting these derivatives equal to zero produces the normal equations, the solution of which minimizes \( R_n^2 \). The normal equations are thus

\[
\begin{align*}
F \frac{\bar{x}_n}{n} - G \frac{\bar{u}_n}{n} &= \sum_{j=1}^{n} \hat{x}_j w_j, \\
-G \frac{\bar{x}_n}{n} + H \frac{\bar{u}_n}{n} &= \sum_{j=1}^{n} \hat{x}_j(n-j) w_j, \quad \text{where}
\end{align*}
\]

\[
(1.7)
\]
\[(1.8) \quad F_n = \sum_{j=1}^{n} w_j, \quad G_n = \sum_{j=1}^{n} (n-j)w_j \quad \text{and} \quad H_n = \sum_{j=1}^{n} (n-j)^2 w_j.\]

Replacing \(n\) by \(n-1\) in the above equations gives

\[
\begin{align*}
F_{n-1}\bar{x}_{n-1} - G_{n-1}\bar{y}_{n-1} &= \sum_{j=1}^{n-1} \hat{x}_j w_j, \\
-G_{n-1}\bar{x}_{n-1} + H_{n-1}\bar{y}_{n-1} &= -\sum_{j=1}^{n-1} \hat{x}_j (n-j)w_j.
\end{align*}
\]

Recursion being the objective, explicit presence of the first \(n-1\) observations should be eliminated. This is done by subtracting from (1.7) a changed form of (1.9). The next few steps will be concerned with changing (1.9) to a more opportune form.

First one verifies that

\[
\begin{align*}
F_{n-1} &= F_n - w_n, \quad G_{n-1} = G_n - F_n + w_n \quad \text{and} \\
H_{n-1} &= H_n - 2G_n + F_n - w_n.
\end{align*}
\]

Making these substitutions in (1.9) leads to
\[
\begin{align*}
\left\{ \begin{array}{l}
F_n(\overline{x}_{n-1} + \overline{u}_{n-1}) - G_n \overline{u}_{n-1} = \sum_{j=1}^{n-1} \overline{x}_j w_j + w_n(\overline{x}_{n-1} + \overline{u}_{n-1}) \\
-G_n(\overline{x}_{n-1} + \overline{u}_{n-1}) + H_n \overline{u}_{n-1} + F_n(\overline{x}_{n-1} + \overline{u}_{n-1}) - G_n \overline{u}_{n-1} = \\
\sum_{j=1}^{n-1} \overline{x}_j (n-j)w_j + w_n(\overline{x}_{n-1} + \overline{u}_{n-1}).
\end{array} \right.
\end{align*}
\]

Subtracting the first equation of (1.11) from the second and substituting this result for the second changes (1.11) to

\[
\begin{align*}
\left\{ \begin{array}{l}
F_n(\overline{x}_{n-1} + \overline{u}_{n-1}) - G_n \overline{u}_{n-1} = \sum_{j=1}^{n-1} \overline{x}_j w_j + w_n(\overline{x}_{n-1} + \overline{u}_{n-1}) \\
-G_n(\overline{x}_{n-1} + \overline{u}_{n-1}) + H_n \overline{u}_{n-1} = - \sum_{j=1}^{n} \overline{x}_j (n-j)w_j
\end{array} \right.
\end{align*}
\]

which is therefore the changed form of (1.9) to be subtracted from (1.7).

The results of subtracting (1.12) from (1.7) follows.
\[
\begin{align*}
F_n \left[ \bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1}) \right] - G_n \left( \bar{u}_n - \bar{u}_{n-1} \right) &= w_n \left[ \dot{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1}) \right] \\
-G_n \left[ \bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1}) \right] + H_n \left( \bar{u}_n - \bar{u}_{n-1} \right) &= 0.
\end{align*}
\]

Solving (1.13) for \( \bar{x}_n \) and \( \bar{u}_n \), the smoothing equations follow.

\[
\begin{align*}
\bar{x}_n &= \bar{x}_{n-1} + \bar{u}_{n-1} + a_n \left[ \dot{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1}) \right] \\
\bar{u}_n &= \bar{u}_{n-1} + \beta_n \left[ \dot{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1}) \right], \quad \text{where}
\end{align*}
\]

\[
(1.15) \quad a_n = \frac{w_n H_n}{J_n}, \quad \beta_n = \frac{w_n G_n}{J_n} \quad \text{and} \quad J_n = F_n H_n - G_n^2.
\]

Making formulas (1.14) explicit requires the evaluation of \( a_n, \beta_n \) and hence \( F_n, G_n \) and \( H_n \).

\[
\begin{align*}
F_n &= \frac{(p + n)!}{(n-1)! (p+1)} , \quad G_n = \frac{(p+n)!}{(n-2)! (p+1)(p+2)} \quad \text{and} \\
H_n &= \frac{(p+n)! (p-1+2n)}{(n-2)! (p+1)(p+2)(p+3)} \quad \text{by finite integration [2,}
\end{align*}
\]
From (1.16) and the definition of \( J_n \) in (1.15),

\[
J_n = \frac{(p+n)!}{(n-1)(n-2)!} \left( \frac{2(p+1+n)}{2(p+1)(p+2)^2(p+3)} \right)
\]

Applying (1.16) and (1.17) to (1.15) yields

\[
a_n = \frac{(p-1+2n)(p+2)}{(p+n)(p+1+n)} \quad \text{and} \quad \beta_n = \frac{(p+2)(p+3)}{(p+n)(p+1+n)}
\]

The results of this part on linear smoothing are summarized by the following theorem.

**Theorem A:** Given the real-valued observations \( \hat{x}_j \), each of which is weighted by \( w_j = (p+j-1)! / (j-1)! \) where \( j = 1, 2, \ldots, n \) and where successive observations are some constant \( T \) units apart, then the least-squares estimates of the parameters \( \overline{x}_n \) and \( \overline{u}_n \) in the representation

\[
E(\hat{x}_j) = f_n(j) = \overline{x}_n - (n-j)\overline{u}_n
\]

are determined recursively as follows:

\[
\overline{x}_n = \overline{x}_{n-1} + \overline{u}_{n-1} + a_n \left[ \hat{x}_n - (\overline{x}_{n-1} + \overline{u}_{n-1}) \right]
\]

\[
\overline{u}_n = \overline{u}_{n-1} + \beta_n \left[ \hat{x}_n - (\overline{x}_{n-1} + \overline{u}_{n-1}) \right], \quad \text{where}
\]

\[
a_n = \frac{(p-1+2n)(p+2)}{(p+n)(p+1+n)} \quad \text{and} \quad \beta_n = \frac{(p+2)(p+3)}{(p+n)(p+1+n)}
\]
II. QUADRATIC SMOOTHING

Suppose it is desired to fit by the method of least-squares, the quadratic function

\[ (2.1) \quad f_n(j) = \overline{x}_n - (n-j)\overline{u}_n + (n-j)\overline{s}_n \]

to the observations \( \hat{x}_j \), each of which is weighted by

\[ w_j = \frac{(p+j-1)!}{(j-1)!} \]

where \( j = 1, 2, \ldots, n \) and where successive observations occur some constant \( \tau \) units apart. Here

\[ \overline{s}_n = \frac{1}{2} \tau^2 \]

times the acceleration (or rather its estimate) while

\[ \overline{u}_n = \tau \]

times the velocity as before.

The sum of the squared deviations of the observations \( \hat{x}_j \) from \( f_n(j) \) (\( j = 1, 2, \ldots, n \)) is

\[ (2.2) \quad R_n^2 = \sum_{j=1}^{n} \left\{ \hat{x}_j - \left[ \overline{x}_n - (n-j)\overline{u}_n + (n-j)\overline{s}_n \right] \right\}^2 w_j \]

Differentiating \( R_n^2 \) with respect to \( \overline{x}_n \), then \( \overline{u}_n \) and finally to \( \overline{s}_n \) and setting these derivatives equal to zero produces the normal equations, the solution of which minimizes \( R_n^2 \). This linear system is thus
\[
\begin{aligned}
F_{n} & \quad \overline{x}_{n} - G_{n} \overline{u}_{n} + H_{n} \overline{s}_{n} = \sum_{j=1}^{n} \hat{x}_j w_j \\
\text{(2.3)}

-G_{n} \overline{x}_{n} + H_{n} \overline{u}_{n} - I_{n} \overline{s}_{n} & = -\sum_{j=1}^{n} \hat{x}_j (n-j) w_j \\
H_{n} \overline{x}_{n} - I_{n} \overline{u}_{n} + K_{n} \overline{s}_{n} & = \sum_{j=1}^{n} \hat{x}_j (n-j)^2 w_j, \quad \text{where}
\end{aligned}
\]

\[
\begin{aligned}
F_{n} & = \sum_{j=1}^{n} w_j, \quad G_{n} = \sum_{j=1}^{n} (n-j) w_j, \quad H_{n} = \sum_{j=1}^{n} (n-j)^2 w_j, \\
\text{(2.4)}

I_{n} & = \sum_{j=1}^{n} (n-j)^3 w_j \quad \text{and} \quad K_{n} = \sum_{j=1}^{n} (n-j)^4 w_j.
\end{aligned}
\]

Replacing \( n \) in (2.3) by \( n-1 \) gives

\[
\begin{aligned}
F_{n-1} & \quad \overline{x}_{n-1} - G_{n-1} \overline{u}_{n-1} + H_{n-1} \overline{s}_{n-1} = \sum_{j=1}^{n-1} \hat{x}_j w_j \\
\text{(2.5)}

-G_{n-1} \overline{x}_{n-1} + H_{n-1} \overline{u}_{n-1} - I_{n-1} \overline{s}_{n-1} & = -\sum_{j=1}^{n-1} \hat{x}_j (n-1-j) w_j \\
H_{n-1} \overline{x}_{n-1} - I_{n-1} \overline{u}_{n-1} + K_{n-1} \overline{s}_{n-1} & = \sum_{j=1}^{n-1} \hat{x}_j (n-1-j)^2 w_j.
\end{aligned}
\]
In order to obtain a system free from explicit mention of the first \( n-1 \) observations, coefficients \( F_{n-1}, G_{n-1}, \ldots, K_{n-1} \) must be expressed in terms of \( F_n, \ldots, K_n \) and \( w_n \) then substituted into (2.5). The desired system is obtained by subtracting a further altered form of (2.5) from (2.3). The next few steps will be concerned with these alterations.

It may be verified that

\[
\begin{align*}
F_{n-1} &= F_n - w_n, \\
G_{n-1} &= F_n - F_n + w_n, \\
H_{n-1} &= H_n - 2G_n + F_n - w_n \\
I_{n-1} &= I_n - 3H_n + 3G_n - F_n + w_n, \text{ and } K_{n-1} &= K_n - 4I_n + 6H_n - 4G_n + F_n - w_n.
\end{align*}
\]

(2.6)

Making substitutions (2.6) into (2.5) yields

\[
\begin{align*}
E_1: F_n G_n H_n + S_n &= \sum_{j=1}^{n-1} x_j w_j + w_n, \\
E_2: (F_n G_n) A_n + (H_n - G_n) B_n + (H_n - I_n) \bar{S}_n &= \sum_{j=1}^{n-1} x_j (n-j-1) w_j + w_n, \\
E_3: (H_n - 2G_n + F_n) A_n + (I_n - 2H_n + G_n) B_n + (K_n - 2I_n + H_n) \bar{S}_n &= \sum_{j=1}^{n-1} x_j (n-j-1) w_j + w_n,
\end{align*}
\]

(2.7)

where
\[ \hat{x}_n = \overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1} \] and
\[ \hat{u}_n = \overline{u}_{n-1} + 2\overline{s}_{n-1} \] are the predicted \( \overline{x}_n \) and \( \overline{u}_n \) respectively.

When \( E_2 - E_1 \) is substituted for \( E_2 \) and \( E_3 - 2E_2 + E_1 \) is substituted for \( E_3 \) the following system results.

\[
\begin{align*}
F_{n n} \hat{x}_n - G_{n n} \hat{u}_n + H_{n n} \overline{s}_{n-1} &= \sum_{j=1}^{n-1} \hat{x}_j w_j + w_n \hat{x}_n \\
-G_{n n} \hat{x}_n + H_{n n} \hat{u}_n - I_{n n} \overline{s}_{n-1} &= -\sum_{j=1}^{n} \hat{x}_j (n-j)w_j \\
H_{n n} \hat{x}_n - I_{n n} \hat{u}_n + K_{n n} \overline{s}_{n-1} &= \sum_{j=1}^{n} \hat{x}_j (n-j)^2 w_j.
\end{align*}
\]

System (2.8) is equivalent to (2.5) and when (2.8) is subtracted from (2.3), the following system results which is free from explicit mention of the first \( n-1 \) observations.

\[
\begin{align*}
F_n (\overline{x}_n - \hat{x}_n) - G_n (\overline{u}_n - \hat{u}_n) + H_n (\overline{s}_n - \overline{s}_{n-1}) &= w_n (\overline{x}_n - \hat{x}_n) \\
-G_n (\overline{x}_n - \hat{x}_n) + H_n (\overline{u}_n - \hat{u}_n) - I_n (\overline{s}_n - \overline{s}_{n-1}) &= 0 \\
H_n (\overline{x}_n - \hat{x}_n) - I_n (\overline{u}_n - \hat{u}_n) + K_n (\overline{s}_n - \overline{s}_{n-1}) &= 0.
\end{align*}
\]
The solution of (2.9) produces the smoothing equations:

\[
\begin{align*}
\bar{x}_n &= \bar{x}_{n-1} + \bar{u}_{n-1} + \bar{s}_{n-1} + a_n \left[ \bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1} + \bar{s}_{n-1}) \right] \\
\bar{u}_n &= \bar{u}_{n-1} + 2\bar{s}_{n-1} + \beta_n \left[ \bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1} + \bar{s}_{n-1}) \right] \\
\bar{s}_n &= \bar{s}_{n-1} + \gamma_n \left[ \bar{x}_n - (\bar{x}_{n-1} + \bar{u}_{n-1} + \bar{s}_{n-1}) \right], \text{ where}
\end{align*}
\]

(2.10)

\[
\begin{align*}
a_n &= \frac{w_n (H_n K_n - I_n^2)}{D_n}, \quad \beta_n = \frac{w_n (G_n K_n - H_n I_n)}{D_n} \\
\gamma_n &= \frac{w_n (G_n I_n - H_n^2)}{D_n}
\end{align*}
\]

(2.11)

and where

\[
D_n = F_n (H_n K_n - I_n^2) - G_n (G_n K_n - H_n I_n) + H_n (G_n I_n - H_n^2).
\]

Obtaining explicit expressions for the smoothing equations

(2.10) requires that \(a_n\), \(\beta_n\), and \(\gamma_n\) be evaluated in terms of the constants \(p\) and \(n\). This further requires the evaluation of the sums \(F_n, G_n, \ldots, K_n\) in terms of \(p\) and \(n\).

The obvious approach would be to calculate \(a_n\), \(\beta_n\), and \(\gamma_n\) directly from their definitions in (2.11). However this leads to serious computational obstructions. Therefore formulas (conjectured by Dr. E. L. Kaplan) for \(a_n\), \(\beta_n\) and \(\gamma_n\) are proved demonstratively instead.
It is nevertheless still necessary to express \( F_n, G_n, H_n, I_n \) and \( K_n \) in closed form as functions of \( p \) and \( n \), \((p \geq 0;\text{integer } n \geq 3)\). This is done next.

\[
\begin{align*}
F_n &= \frac{(p+n)!}{(n-1)!n+1}, \quad G_n = \frac{(p+n)!}{(n-2)!(p+2)^2}, \quad H_n = \frac{(p+n)!(p-1+2n)}{(n-2)!(p+3)^3} \\
I_n &= \frac{(p+n)! \left[ 6(p+1+n)(n-2)+(p+3)(p+4) \right]}{(n-2)!(p+4)^4} \\
K_n &= \frac{(p+n)! \left[ (p+4)(p+5)(13p+11+14n)-12(p+2+n)(p+1+n)(p+9-2n) \right]}{(n-2)!(p+5)^5}
\end{align*}
\]

by finite integration \([2, \text{p. 20-27}]\). Here \((p+j)^{(k)} = (p+j)(p+j-1)(p+j-2)\ldots(p+j-k+1)\) has \(k\) factors.

Crucial elements in the following argument are two well known properties from elementary matrix theory which are stated next.

**Property 1:** If each element in a row of a square matrix is multiplied by its own cofactor, the sum of the resulting products is the determinant of the matrix.

**Property 2:** Consider two rows of a square matrix. If each element in one row is multiplied by the cofactor of the corresponding element of the other row, the sum of the
resulting products are zero.

Consider the matrix of coefficients of system (2.3)

\[
C = \begin{bmatrix}
F_n & -G_n & H_n \\
-G_n & H_n & -I_n \\
H_n & -I_n & K_n
\end{bmatrix}
\]

and the cofactors of the first row. Applying these properties to the cofactors of the first row gives

\[
\begin{align*}
\frac{F_n (H_n K_n - I_n^2)}{D_n} - \frac{G_n (G_n K_n - H_n I_n)}{D_n} + \frac{H_n (G_n I_n - H_n^2)}{D_n} &= D_n \\
\frac{-G_n (H_n K_n - I_n^2)}{D_n} + \frac{H_n (G_n K_n - H_n I_n)}{D_n} - \frac{I_n (G_n I_n - H_n^2)}{D_n} &= 0 \\
\frac{H_n (H_n K_n - I_n^2)}{D_n} - \frac{I_n (G_n K_n - H_n I_n)}{D_n} + \frac{K_n (G_n I_n - H_n^2)}{D_n} &= 0
\end{align*}
\]

When (2.13) is multiplied by \( \frac{w_n}{D_n} \) the result is

\[
\begin{align*}
\frac{F_n a_n - G_n \beta_n + H_n \gamma_n}{D_n} &= w_n \\
\frac{-G_n a_n + H_n \beta_n - I_n \gamma_n}{D_n} &= 0 \\
\frac{H_n a_n - I_n \beta_n + K_n \gamma_n}{D_n} &= 0
\end{align*}
\]

which follows from (2.11) and amounts to a simple change of variables in (2.9).
Substitution from (2.12) and

\[
\begin{align*}
\alpha_n &= \frac{(p+3)[3(n-1)(p+n)+(p+1)(p+2)]}{(p+2+n)^3} \\
\beta_n &= \frac{3(p+3)(p+4)(p-1+2n)}{2(p+2+n)^3} \\
\gamma_n &= \frac{(p+3)(p+4)(p+5)}{2(p+2+n)^3}
\end{align*}
\]

(2.15) (Kaplan's conjectures)

into (2.14) makes (2.14) a system of identities in \(p\) and \(n\).

Therefore formulas (2.15) are correct expressions for \(\alpha_n\), \(\beta_n\), and \(\gamma_n\). The question of whether they constitute the unique solution will be considered next. However this question is the same as whether the least-squares solution for the parameters \(\overline{x}_n\), \(\overline{v}_n\), and \(\overline{s}_n\) is unique, since the coefficient matrix is the same in both cases. Therefore the question will be considered from the latter point of view.

Consider the matrix

\[
M = \begin{bmatrix}
\sqrt{w}_1 & \sqrt{w}_2 & \ldots & \sqrt{w}_j & \ldots & \sqrt{w}_{n-2} & \sqrt{w}_{n-1} & \sqrt{w}_n \\
-(n-1)\sqrt{w}_1 & -(n-2)\sqrt{w}_2 & \ldots & -(n-j)\sqrt{w}_j & \ldots & -2\sqrt{w}_{n-2} & -\sqrt{w}_{n-1} & 0 \\
(n-1)^2\sqrt{w}_1 & (n-2)^2\sqrt{w}_2 & \ldots & (n-j)^2\sqrt{w}_j & \ldots & 2^2\sqrt{w}_{n-2} & \sqrt{w}_{n-1} & 0
\end{bmatrix}
\]
each of whose columns is composed of the coefficients of the expression for the estimated ordinate at one of the \( n \) abscissa values.

The product of \( M \) and its transpose is the coefficient matrix \( C \) whose determinant is hence the sum of the squares of all the \( 3 \times 3 \) minors of \( M \). Obviously if any one of these minors is non-zero, then the determinant of \( C \) is non-zero, which implies that any system having \( C \) as its coefficient matrix has a unique solution.

It is an easy matter to show that the minor consisting of the last three columns of \( M \) is non-zero for \( n \geq 3 \).

A formal statement of this part on quadratic smoothing is given by the following theorem.

**Theorem B:** Given the real valued observations \( \hat{x}_j \) each of which is weighted by \( w_j = (p+j-1)! / (j-1)! \), where \( j = 1, 2, \ldots, n \) and where successive observations are some constant \( T \) units apart, also given that \( E(x_j) \) has the form \( f_n(j) = x_n - (n-j)u_n + (n-j)^2 s_n \), then the least-squares estimates of the parameters \( \overline{x}_n, \overline{u}_n \) and \( \overline{s}_n \) are determined recursively as follows:

\[
\overline{x}_n = \overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1} + a_n \left[ \overline{x}_n - (\overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1}) \right],
\]

\[
\overline{u}_n = \overline{u}_{n-1} + 2\overline{s}_{n-1} + \beta_n \left[ \overline{x}_n - (\overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1}) \right] \quad \text{and}
\]

\[
\overline{s}_n = \overline{s}_{n-1} + \gamma_n \left[ \overline{x}_n - (\overline{x}_{n-1} + \overline{u}_{n-1} + \overline{s}_{n-1}) \right],
\]

where
\[ a_n = \frac{(p+3)[3(n-1)(p+n)+(p+1)(p+2)]}{(p+2+n)^3}, \]

\[ \beta_n = \frac{3(p+3)(p+4)(p-1+2n)}{2(p+2+n)^3} \]

and

\[ \gamma_n = \frac{(p+3)(p+4)(p+5)}{2(p+2+n)^3}. \]
III. EXPONENTIAL SMOOTHING

Consider the observations \( \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_j, \ldots, \dot{x}_n \) which occur sequentially and exactly \( \tau \) units apart. Suppose it is required to fit

\[
y_n(t) = a_n + b_n e^{\tau t}
\]

(3.1)

to these data by the method of least-squares. Consequently if \( \tau = q \), then

\[
y_n(t) = y_n[\tau(j-1)] = f_n(j) \text{ since the time of the } j^{th} \text{ observation is } \tau(j-1). \quad \text{i.e.,}
\]

\[
f_n(j) = a_n + b_n e^{q(j-1)}.
\]

(3.2)

The sum of the squared deviations of the observations \( \dot{x}_j \)
from the \( f_n(j) \) is

\[
R_n^2 = \sum_{j=1}^{n} \left\{ \dot{x}_j - [a_n + b_n e^{q(j-1)}] \right\}^2,
\]

constant (unit) weights being assumed in this case.

Differentiating \( R_n^2 \) first with respect to \( a_n \) and then \( b_n \) and setting these derivatives equal to zero produces
which are the normal equations.

Let

\[
G_n = \sum_{j=1}^{n} e^{q(j-1)} \quad \text{and} \quad H_n = \sum_{j=1}^{n} e^{2q(j-1)}.
\]

Then

\[
\left\{
\begin{aligned}
na_n + b_n \sum_{j=1}^{n} e^{q(j-1)} &= \sum_{j=1}^{n} \hat{x}_j \\
G_n a_n + H_n b_n &= \sum_{j=1}^{n} \hat{x}_j e^{q(j-1)}
\end{aligned}
\]

Replacing \( n \) by \( n-1 \) gives
\[
\begin{align*}
(n-1)a_{n-1} + G_{n-1}b_{n-1} &= \sum_{j=1}^{n-1} \hat{x}_j \quad \text{and} \\
G_{n-1}a_{n-1} + H_{n-1}b_{n-1} &= \sum_{j=1}^{n-1} \hat{x}_j e^{q(j-1)}
\end{align*}
\]  
(3.6)

Subtracting (3.6) from (3.5) and making the substitution

\[
G_{n-1} = G_n e^{q(n-1)} \quad \text{and} \quad H_{n-1} = H_n e^{2q(n-1)}
\]

results in the following system which no longer explicitly involves the first \(n-1\) observations.

\[
\begin{align*}
(na_{n-1} + G_n(b_n - b_{n-1})) &= \hat{x}_n a_{n-1} - b_{n-1} e^{q(n-1)} \\
G_n(a_n - a_{n-1}) + H_n(b_n - b_{n-1}) &= (\hat{x}_n a_{n-1} - b_{n-1} e^{q(n-1)}) e^{q(n-1)}
\end{align*}
\]  
(3.8)

Hence

\[
\begin{align*}
D_n(a_n - a_{n-1}) &= (\hat{x}_n a_{n-1} - b_{n-1} e^{q(n-1)})(H_n - G_n e^{q(n-1)}) \\
D_n(b_n - b_{n-1}) &= (\hat{x}_n a_{n-1} - b_{n-1} e^{q(n-1)}) (n e^{q(n-1)} - G_n)
\end{align*}
\]  
(3.9)
\[ D_n = nH_n - G_n^2. \]

Solving (3.9) for \( a_n \) and \( b_n \) gives the following recursion relations (the smoothing equations).

\[
\begin{align*}
\frac{a_n}{a_{n-1}} &= a_n + a_n (\dot{x}_n - a_{n-1} - b_{n-1} e^{q(n-1)}) \\
\frac{b_n}{b_{n-1}} &= b_n + \beta_n (\dot{x}_n - a_{n-1} - b_{n-1} e^{q(n-1)}) e^{q(n-1)},
\end{align*}
\]

where

\[
\frac{H_n - G_n e^{q(n-1)}}{D_n} \quad \text{and} \quad \beta_n = \frac{ne^{q(n-1)} - G_n}{D_n}.
\]

Finally inserting the following explicit evaluations

\[ G_n = \sum_{j=1}^{n} e^{q(j-1)} = \frac{e^{qn} - 1}{e^{q} - 1} \quad \text{and} \quad H_n = \sum_{j=1}^{n} e^{2q(j-1)} = \frac{e^{2qn} - 1}{e^{2q} - 1} \]

yields the following theorem on the smoothing of exponential trends:

**Theorem C:** Consider the real observations \( \dot{x}_j \) \((j = 1, 2, \ldots, n)\) which successively occur a constant \( T \) units apart with \( E(\dot{x}_j) = f_n(j) = a_n + b_n e^{q(j-1)} \). Then the least-squares estimates of the parameters \( a_n \) and \( b_n \) are determined recursively as follows:
\[ a_n = a_{n-1} + a_n (\hat{x}_n - a_{n-1} - b_{n-1} e^{q(n-1)}) \]

\[ b_n = b_{n-1} + \beta (\hat{x}_n - a_{n-1} - b_{n-1} e^{q(n-1)}), \text{ where} \]

\[ a_n = \frac{(e^q - 1)(e^{q(n-1)} - 1)}{(e^q + 1)(e^{qn - 1}) - n(e^q - 1)(e^{qn + 1})} \]

and

\[ \beta_n = \frac{(e^{2q} - 1)[(e^{qn - 1}) - ne^{q(n-1)}(e^q - 1)]}{(e^{qn - 1})[(e^q + 1)(e^{qn - 1}) - n(e^q - 1)(e^{qn + 1})]} . \]
IV. TRIGONOMETRIC SMOOTHING

Suppose it is desired to fit by least-squares

\[ y_n(t) = b_n \sin \omega t + c_n \cos \omega t \]  

(4.1)

to the observations \( x_1, x_2, \ldots, x_j, \ldots, x_n \) which are made successively \( T \) units of abscissa apart. Then the \( t \) value of the \( j^{\text{th}} \) observation can be given by \( t = \tau(j-1) \). Further, if \( \omega \tau = q \), then

\[ y_n(t) = y_n[\tau(j-1)] = f_n(j) \]

That is

(4.2)

\[ f_n(j) = b_n \sin q(j-1) + c_n \cos q(j-1) \]

The sum of the squared deviations from the least-squares curve \( f_n(j) \) is

\[ R_n^2 = \sum_{j=1}^{n} \{ x_j - [b_n \sin q(j-1) + c_n \cos q(j-1)] \}^2 \]

Differentiating \( R_n^2 \) with respect to first \( b_n \) and then \( c_n \) and setting these derivatives equal to zero produces the normal equations,
\[
\begin{align*}
\left\{ \begin{array}{c}
b_n \sum_{j=1}^{n} \sin^2 q(j-1) + c_n \sum_{j=1}^{n} \sin q(j-1) \cos q(j-1) = \sum_{j=1}^{n} x_j \sin q(j-1) \\
b_n \sum_{j=1}^{n} \sin q(j-1) \cos q(j-1) + c_n \sum_{j=1}^{n} \cos^2 q(j-1) = \sum_{j=1}^{n} x_j \cos q(j-1) \\
\end{array} \right. \\
(4.3)
\end{align*}
\]

(4.4) Defining \( J_n = \sum_{j=1}^{n} \sin^2 q(j-1), \ I_n = \sum_{j=1}^{n} \sin q(j-1) \cos q(j-1) \) and 

\[
K_n = \sum_{j=1}^{n} \cos^2 q(j-1)
\]
changes (4.3) to

\[
\begin{align*}
J_n b_n + I_n c_n &= \sum_{j=1}^{n} x_j \sin q(j-1) \\
(4.5)
I_n b_n + K_n c_n &= \sum_{j=1}^{n} x_j \cos q(j-1) \\
\end{align*}
\]

Replacing \( n \) by \( n-1 \) in (4.5) gives

\[
\begin{align*}
J_{n-1} b_{n-1} + I_{n-1} c_{n-1} &= \sum_{j=1}^{n-1} x_j \sin q(j-1) \\
(4.6)
I_{n-1} b_{n-1} + K_{n-1} c_{n-1} &= \sum_{j=1}^{n-1} x_j \cos q(j-1) \\
\end{align*}
\]
Subtracting (4.6) from (4.5) yields

\[
\begin{align*}
J_n b_n - J_{n-1} b_{n-1} + I_n c_n - I_{n-1} c_{n-1} &= \hat{x}_n \sin q(n-1) \\
I_n b_n - I_{n-1} b_{n-1} + K_n c_n - K_{n-1} c_{n-1} &= \hat{x}_n \cos q(n-1)
\end{align*}
\]

From the definitions of \( J_n, I_n \) and \( K_n \) in (4.4) it may be observed that

\[
\begin{align*}
J_{n-1} &= J_n \sin^2 q(n-1), \quad I_{n-1} = I_n \cos q(n-1) \sin q(n-1) \\
K_{n-1} &= K_n - \cos^2 q(n-1).
\end{align*}
\]

Making these substitutions in (4.7) gives

\[
\begin{align*}
J_n (b_n - b_{n-1}) + I_n (c_n - c_{n-1}) &= [\hat{x}_n - b_n - I_n \sin q(n-1) - c_{n-1} \cos q(n-1)] \sin q(n-1) \\
I_n (b_n - b_{n-1}) + K_n (c_n - c_{n-1}) &= [\hat{x}_n - b_n - I_n \sin q(n-1) - c_{n-1} \cos q(n-1)] \cos q(n-1)
\end{align*}
\]

wherein \( b_{n-1} \sin q(n-1) + c_{n-1} \cos q(n-1) \) may be denoted by \( \hat{x}_n \), the expected value of the \( n \)th observation as predicted from the first \( n-1 \) observations. If also \( D_n = J_n K_n - I_n^2 \), the solution of (4.9) takes the following form.
\[
\begin{align*}
\begin{cases}
  b_n = b_{n-1} + \beta_n \left( \ell - \ell_n \right) \\
  c_n = c_{n-1} + \gamma_n \left( \ell - \ell_n \right), \text{ where}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
  \beta_n = \left[ K \sin q(n-1) - I_n \cos q(n-1) \right] \div D_n \text{ and} \\
  \gamma_n = \left[ -I_n \sin q(n-1) + J_n \cos q(n-1) \right] \div D_n .
\end{cases}
\end{align*}
\]

By the methods of finite integration it becomes apparent that

\[
\begin{align*}
\begin{cases}
  J_n = \sum_{j=1}^{n} \sin^2 q(j-1) = \left[ (2n-1)\sin q - \sin q(2n-1) \right] \div 4 \sin q , \\
  I_n = \sum_{j=1}^{n} \sin q(j-1)\cos q(j-1) = \left[ \cos q - \cos q(2n-1) \right] \div 4 \sin q \text{ and} \\
  K_n = \sum_{j=1}^{n} \cos^2 q(j-1) = \left[ (2n+1)\sin q + \sin q(2n-1) \right] \div 4 \sin q .
\end{cases}
\end{align*}
\]

Hence

\[
D_n = J \ n \ K - I_n^2 = \left( n^2 \sin^2 q - \sin^2 qn \right) \div 4 \sin^2 q ,
\]
\[ \beta_n = \frac{2n \sin^2 q \sin (n-1)}{n \sin^2 q - \sin^2 qn} \]

\[ \gamma_n = \frac{2 \sin q [n \sin q \cos (n-1) - \sin qn]}{n^2 \sin^2 q - \sin^2 qn} \]

The results of this section on trigonometric smoothing are summarized by the following theorem.

**Theorem D:** Consider the real observations \( \hat{x}_j \) (\( j = 1, 2, \cdots, n \)) which occur successively some constant \( T \) units apart. Suppose \( E(\hat{x}_j) \) has the form \( f_n(j) = b_n \sin (j-1) + c_n \cos (j-1) \). Then the least-squares estimates of the parameters \( b_n \) and \( c_n \) are determined recursively as follows:

\[
\begin{align*}
\beta_n &= \beta_{n-1} + T n \sin q \sin (n-1) \quad \text{and} \\
\gamma_n &= \gamma_{n-1} + T [n \sin q \cos (n-1) - \sin qn] \\
\end{align*}
\]

where

\[
T = \frac{2 \sin q [\hat{x}_n - b_{n-1} \sin (n-1) - c_{n-1} \cos (n-1)]}{n^2 \sin^2 q - \sin^2 qn}
\]
V. AN EXAMPLE

The following example is given to illustrate, in a small way, the labor saving advantages of recursive smoothing. It will be done by two methods.

Consider the problem of fitting the function

\[
\begin{align*}
  f_4(j) &= b_4 \sin \frac{\pi}{6}(j-1) + c_4 \cos \frac{\pi}{6}(j-1) \\
\end{align*}
\]

(5.1)

\[
\begin{align*}
  \mathcal{A}_1 &= 2, \quad \mathcal{A}_2 = \frac{3+2\sqrt{3}}{2}, \quad \mathcal{A}_3 = \frac{2+3\sqrt{3}}{2}, \quad \mathcal{A}_4 = 2
\end{align*}
\]

after the first three have already been fitted with \( f_3(j) = 3\sin \frac{\pi}{6}(j-1)+2\cos \frac{\pi}{6}(j-1) \) and where \( j \) is the number of the observation, starting with \( 1 \).

First the problem will be done using the traditional least-squares method, then it will be done recursively.

The normal equations are

\[
\begin{align*}
  b_4 \sum_{j=1}^{4} \sin \frac{\pi}{6}(j-1) + c_4 \sum_{j=1}^{4} \sin \frac{\pi}{6}(j-1) \cos \frac{\pi}{6}(j-1) &= \sum_{j=1}^{4} \mathcal{A}_j \sin \frac{\pi}{6}(j-1) \\
  b_4 \sum_{j=1}^{4} \sin \frac{\pi}{6}(j-1) \cos \frac{\pi}{6}(j-1) + c_4 \sum_{j=1}^{4} \cos \frac{\pi}{6}(j-1) &= \sum_{j=1}^{4} \mathcal{A}_j \cos \frac{\pi}{6}(j-1) \\
\end{align*}
\]

(5.2)
There are five sums of four terms each to be evaluated before the system can be solved.

\[
\sum_{j=1}^{4} \sin^{2} \frac{\pi}{6}(j-1) = 0 + \frac{1}{4} + \frac{3}{4} + 1 = 2
\]

\[
\sum_{j=1}^{4} \cos^{2} \frac{\pi}{6}(j-1) = 1 + \frac{3}{4} + \frac{1}{4} + 0 = 2
\]

\[
\sum_{j=1}^{4} \sin \frac{\pi}{6}(j-1) \cos \frac{\pi}{6}(j-1) = 0 + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{4} + 0 = \frac{\sqrt{3}}{2}
\]

\[
\sum_{j=1}^{4} x_{j} \sin \frac{\pi}{6}(j-1) = 0 + \frac{3 + 2\sqrt{3}}{4} + \frac{9 + 2\sqrt{3}}{4} + 2 = 5 + \sqrt{3}
\]

\[
\sum_{j=1}^{4} x_{j} \cos \frac{\pi}{6}(j-1) = 2 + \frac{6 + 3\sqrt{3}}{4} + \frac{2 + 3\sqrt{3}}{4} + 0 = \frac{8 + 3\sqrt{3}}{2}
\]

Making these substitutions in (5.2), the system to be solved becomes

\[2b_{4} + \frac{\sqrt{3}}{2} \cdot c_{4} = 5 + \sqrt{3}\]

\[\frac{\sqrt{3}}{2} b_{4} + 2c_{4} = \frac{8 + 3\sqrt{3}}{2}\]

Hence,

\[b_{4} = \frac{3}{13} \quad \text{and} \quad c_{4} = \frac{26 + 2\sqrt{3}}{13}\]
Nexy by way of contrast, these same coefficients (of \( f_4 \)) will be calculated using recursion formulas (4.13) on page 27.

Substituting \( n = 4, \ q = \frac{\pi}{6}, \ b_3 = 3, \ c_3 = 2, \) and \( \hat{x}_4 = 2, \)

\[
\begin{align*}
  b_4 &= 3 + \left[ \frac{2(\frac{1}{2})(2-3-0)}{16(\frac{1}{4}) - \frac{3}{4}} \right] (4)(\frac{1}{2})(1) = \frac{31}{13} \\
  c_4 &= 2 + \left[ \frac{2(\frac{1}{2})(2-3-0)}{16(\frac{1}{4}) - \frac{3}{4}} \right] \left[ 4(\frac{1}{2})(0) - \frac{\sqrt{3}}{2} \right] = \frac{26+2\sqrt{3}}{13}.
\end{align*}
\]

The saving of computational energy is even more impressive if one considers the additional computation encountered in fitting to many observations instead of only four as in this example; each of the five sums would have as many terms as there are observations.

**Comment:** Formulas (4.13) are useful for fitting purposes even though the axis of oscillation is not zero, provided the mean value \( \mu \) is known *a priori*. Instead of fitting to the observations \( \hat{x}_j \) themselves, formulas (4.13) could be used to fit \( f_n(j) \) to \( \hat{x}_j - \mu, \) \((j = 1, 2, \cdots, n)\). Then \( \mu \) would be added back to \( f_n(j) \) in order to obtain \( g_n(j) = b_n \sin q(j-1) + c_n \cos q(j-1) + \mu \) which would be a fitting for the untransformed observations \( \hat{x}_j. \)
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