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The approximate integration of differential equations can be performed by two methods: step-by-step integration and successive approximations. Unique integrals are obtained by both methods if the Cauchy-Lipschitz hypothesis is satisfied. Numerical processes suitable to calculating machine procedure are almost exclusively limited to those using step-by-step integration. In this thesis are presented two numerical processes for obtaining by the method of successive approximations a segment of the integral curve of the second order differential equation with assigned end points. These processes are called the Short Formula Process and the Long Formula Process.

The Short Formula Process employs Simpson's Rule and Modified Simpson's Rule. The Modified Simpson's Rule determines the second ordinate in terms of the initial ordinate. Except for this ordinate, all other ordinates are found by moving Simpson's Rule over the interval. Since the initial first derivative is not given, a special identity involving a general form of Simpson's Rule is necessary for its determination. The whole process goes forward rapidly in easily remembered steps using simple formulas. In the examples investigated, eight operations usually provide a satisfactory solution.

The Long Formula Process differs from the Short Formula Process in that it applies directly the Three, Five, and Nine Ordinate Formulas. This process shows little promise of replacing the Short Formula Process because of the tediousness of applying formulas using cumbersome coefficients.

Accuracy in both processes is brought about by making the subdivisions of the interval small enough to make nearly zero the neglected terms in the Maclaurin expansions from which all the formulas are derived. To accomplish this, special forms of Lagrange's interpolation formula are applied every alternate time to the second derivatives to bisect each segment of the interval. By this method, fifteen new ordinates are interpolated into the original interval in eight operations.

No rigorous attempt has been made to determine the conditions necessary for the existence of a solution. In practice, if, as the operations continue, the values of the ordinates tend to become stationary, a solution is considered reached.. If, as the operations continue, the values of the ordinates do not tend to become stationary, work is discontinued. No distinction is made between problems having no solution and problems for which the processes fail.

ON THE NUMERICAL INTEGRATION OF THE SECOND ORDER DIFFERENTIAL EQUATION WITH ASSIGNED END POINTS

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TABLE OF CONTENTS

	Page
INTRODUCTION	1
NUMERICAL INTEGRATION.	2
Methods	2
The Cauchy-Lipschitz Hypothesis	3
DEVELOPMENT OF FORMULAS IN TERMS OF ORDINATES.	4
Three Ordinate Formulas	4
Five Ordinate Formulas.	5
Nine Ordinate Formulas.	7
Interpolation Formulas.	8
Modified Simpson's Rule	8
Formula for Determining y_o'	9
APPLICATION OF THE FORMULAS TO A DIFFERENTIAL EQUATION	10
Starting the Solution	10
First Interpolation	11
The Short Formula Process	11
Summary of the Short Formula Process.	14
The Long Formula Process.	16
Comparison of Results	18
A Problem Having No Solution.	18
A Perfect Solution.	19
y" Infinite at One End of Interval.	22
y" Large at One End of Interval	22
PARAMETRIC FORM OF SOLUTION OF A DIFFERENTIAL EQUATION	24
Formation of the Parametric Equation.	24
An Application of the Equations	24
Starting the Solution	25
Continuation of the Process	27
Comparison of Results	27
CONCLUSION	29

ON THE NUMERICAL INTEGRATION OF THE SECOND ORDER DIFFERENTIAL EQUATION WITH ASSIGNED END POINTS

INTRODUCTION

The routine work of the modern scientist and engineer requires the solution of an increasingly greater number of differential equations. In many practical problems, no formal method of solution of the equation is possible. In these cases, approximate methods must be employed. Only a few methods suitable to calculating machine procedure are available. The existing processes are found mainly in the field of step-by-step integration. Among the special types that have been thus integrated is $y'' = f(x,y)$.

In this thesis are presented two methods of approximating the solution to the second order differential equation $y'' = f(x,y,y')$ in which the value of the integral at each end of the interval of integration is given. Both processes are adapted to calculating machine procedure. As far as can be determined, neither of these two processes has been presented before.

NUMERICAL INTEGRATION

Methods

Ince states that "Of all ordinary differential equations ... only certain very special types admit of explicit integration, and when an equation which is not of one or other of these types arises in a practical problem the investigator has to fall back upon purely numerical methods of approximating to the required solution." (2)

Approximate methods may be classified into two groups: (a) successive approximations and (b) step-by-step integration. In the first method, the desired region is assigned in advance or is made as large as possible. In the second method, a form of approximate solution, such as a polynomial of given degree, is chosen arbitrarily in advance and is applied to sub-intervals of not too great extent until the region is covered. The existence of a solution using either method is based on the work of Cauchy (1820 to 1830) and improved by Lipschitz in 1876. Previously in 1768, Euler, in his Institutiones calculi integralis, had suggested this idea as a method of calculation.

Successive approximations may be carried out in several forms, three of which are (a) direct substitution into power series--a method dating back to Newton and Leibnitz; (b) Liouville's method, appearing in the second volume of his journal (1837), in which he writes the solution as an integral equation and proceeds to form a series from this form; and (c) Picard's method of successive substitutions (appearing in Liouville's journal in 1890) in which Cauchy's principle was revived assuming the Cauchy-Lipschitz hypothesis as an initial condition but proceeding in a manner less restricted than Liouville's. That Picard's method actually provides the correct solution was shown by Bendixon and Lindelöf in 1894.(1)

Several step-by-step methods of computation have been devised. The simplest in form and often the most difficult in practice is the Taylor's series. Outstanding objections to this form were overcome by Runge in 1895 and continued by Kutta in 1901. A method especially adapted to the equation $y'' = f(x,y)$ was devised by Milne in 1933.[3]

The Cauchy-Lipschitz Hypothesis

Theoretical considerations in this thesis will be dismissed with a statement of a set of conditions which have been shown to provide a correct solution if they are satisfied.

"In the case of the differential equation of the first order, $dy/dx = f(x,y)$, let f, and $\partial f/\partial y$ be real one-valued continuous functions for all real values of x, y, satisfying the conditions, $|x - x_0| \leqq a$, $|y - y_0| \leqq b$, and let $|f| \leqq M$ be satisfied in this rectangle. Let l denote the [smaller of the values] a, and b/M. Then if the interval, x_0 to x, where $|x - x_0| < l$, be broken up into m equal subintervals of length h, by points, $x_0, x_1, \ldots\ldots, x_{m-1}, x$, one may define $y_1, y_2, \ldots\ldots, y_m$ by the equations $y_{i+1} = y_i + hf(x_i, y_i)$, $i = 0, 1, \ldots\ldots, m - 1$. Then Cauchy showed that under the given hypotheses $y_m(x)$ tended, as m increased, toward a limit function $y(x)$, which is the unique function satisfying the given differential equation with the boundary condition, $y(x_0) = y_0$. The proof was extended to apply to the general system of n such first order equations in n dependent variables, equivalent to a single differential equation of the nth order.

"This method was notably improved by Lipschitz, who replaced Cauchy's absolute bound for $\partial f/\partial y$ by the bound for the difference quotient, namely, $|f(x,y) - f(x,Y)| < k\ |y - Y|$, for (x,y) and (x,Y) in the fundamental rectangle, and similarly for the system of the nth order."[1]

DEVELOPMENT OF FORMULAS IN TERMS OF ORDINATES

The set of formulas about to be derived are directly applicable to the differential equation $y'' = f(x,y,y')$ with assigned values of y at the end points. They give values of y and y' directly in terms of y" for $2^n + 1$ ordinates, if the end ordinates are also included. The three-ordinate formulas are exceedingly simple, but the formulas for five and higher ordinates are apparently too complex to be of great practical value. Since the three-ordinate formulas form the basis of the method used in succeeding sections, their derivation will be given in detail.

Three Ordinate Formulas

Let the origin be taken at the midpoint of the interval. Let the distance to either end ordinate be h. If all the derivatives of y with respect to x that are needed exist, Maclaurin's expansion of a function about the origin yields:

$$y_1 = f(h) = y_0 + hy_0' + \frac{h^2}{2}y_0'' + \frac{h^3}{6}y_0''' + \frac{h^4 y_0^{iv}}{24} + \frac{h^5 y_0^{v}}{120} + \frac{h^6 y_0^{vi}}{720} + \cdots$$

$$hy_1' = hf'(h) = hy_0' + h^2 y_0'' + \frac{h^3 y_0'''}{2} + \frac{h^4}{6}y_0^{iv} + \frac{h^5 y_0^{v}}{24} + \frac{h^6 y_0^{vi}}{120} + \cdots$$

$$h^2 y_1'' = h^2 f''(h) = h^2 y_0'' + h^3 y_0''' + \frac{h^4}{2}y_0^{iv} + \frac{h^5}{6}y_0^{v} + \frac{h^6 y_0^{vi}}{24} + \cdots$$

with a corresponding set of three where h is everywhere replaced by - h. This set gives values at the initial ordinate designated as y_{-1}. Since it is assumed that the values of the second derivatives are available from the differential equation $y'' = f(x,y,y')$, they are considered given.

From linear combinations of these expansions result the six formulas:

$$\text{(i)} \quad y_0 = \tfrac{1}{2}(y_1 + y_{-1}) - \frac{h^2}{24}(y_{-1}'' + 10\, y_0'' + y_1'') + \frac{h^6 y_0^{vi}}{480} + \cdots$$

$$\text{(ii)} \quad y_{-1}' = \frac{y_1 - y_{-1}}{2h} - \frac{h}{3}(y_{-1}'' + 2\, y_0'') - \frac{13\, h^4}{360} y_0^{v} + \frac{h^5 y_0^{vi}}{180} + \cdots$$

$$\text{(iii)} \quad y_0' = \frac{y_1 - y_{-1}}{2h} + \frac{h}{12}(y_{-1}'' - y_1'') + \frac{h^4 y_0^{v}}{180} + \cdots$$

$$\text{(iv)} \quad y_1' = \frac{y_1 - y_{-1}}{2h} + \frac{h}{3}(2\, y_0'' + y_1'') - \frac{13\, h^4 y_0^{v}}{360} - \frac{h^5 y_0^{vi}}{180} + \cdots$$

$$\text{(v)} \quad y_1' - y_{-1}' = \frac{h}{3}(y_{-1}'' + 4\, y_0'' + y_1'') - \frac{h^5 y_0^{vi}}{90} + \cdots$$

$$\text{(vi)} \quad y_1 - y_{-1} = \frac{h}{3}(y_{-1}' + 4\, y_0' + y_1') - \frac{h^5 y_0^{v}}{90} + \cdots$$

Formula (vi) is the well known Simpson's Rule, after Thomas Simpson (1743). Its geometric analogue is credited to Cavalieri (1639).

Five Ordinate Formulas

If, in addition to the Maclaurin expansions for $f(\pm h)$ and the derivatives $f'(\pm h)$ and $f''(\pm h)$, are added expansions for the values $f(\pm 2h)$, $f'(\pm 2h)$ and $f''(\pm 2h)$, there result twelve equations in eight unknowns: three ordinates and five first derivatives. In these equations the third, fourth, fifth and sixth derivatives may be eliminated to yield formulas in which the lowest appearing derivative is the seventh.

$$\text{(i')}\quad y_{-1} = y_{-2} + \frac{y_2 - y_{-2}}{4} - \frac{h^2}{480}\left(27\, y_{-2}'' + 332\, y_{-1}'' + 222\, y_o'' + 132\, y_1'' + 7\, y_2''\right) - \frac{h^7 y_o^{vii}}{480} + \cdots$$

$$\text{(ii')}\quad y_o = y_{-2} + \frac{2(y_2 - y_{-2})}{4} - \frac{h^2}{30}\left(y_{-2}'' + 16\, y_{-1}'' + 26\, y_o'' + 16\, y_1'' + y_2''\right) + \frac{h^8 y_o^{viii}}{945} + \cdots$$

$$\text{(iii')}\quad y_1 = y_{-2} + \frac{3(y_2 - y_{-2})}{4} - \frac{h^2}{480}\left(7\, y_{-2}'' + 132\, y_{-1}'' + 222\, y_o'' + 332\, y_1'' + 27\, y_2''\right. + \frac{h^7 y_o^{vii}}{480} + \cdots$$

$$\text{(iv')}\quad y_{-2}' = \frac{y_2 - y_{-2}}{4h} - \frac{2h}{45}\left(7\, y_{-2}'' + 24\, y_{-1}'' + 6\, y_o'' + 8\, y_1'' + \quad\right) - \frac{4h^6 y_o^{vii}}{315} + \cdots$$

$$\text{(v')}\quad y_{-1}' = \frac{y_2 - y_{-2}}{4h} - \frac{h}{720}\left(-27\, y_{-2}'' + 122\, y_{-1}'' + 456\, y_o'' + 150\, y_1'' + 19\, y_2''\right) + \frac{61\, h^6 y_o^{vii}}{10080} + \cdots$$

$$\text{(vi')}\quad y_o' = \frac{y_2 - y_{-2}}{4h} + \frac{h}{90}\left(y_{-2}'' + 28\, y_{-1}'' \qquad - 28\, y_1'' - y_2''\right) - \frac{h^6 v_o^{vii}}{630} + \cdots$$

$$\text{(vii}'\text{)} \quad y_1' = \frac{y_2 - y_{-2}}{4h} + \frac{h}{720}\left(19\, y_{-2}'' + 150\, y_{-1}'' + 456\, y_o'' + 122\, y_1'' - 27\, y_2''\right) + \frac{61\, h^6 y_o^{\,vii}}{10080} + \cdots$$

$$\text{(viii}'\text{)} \quad y_2' = \frac{y_2 - y_{-2}}{4h} + \frac{2h}{45}\left(8\, y_{-1} + 6\, y_o'' + 24\, y_1'' + 7\, y_2''\right) - \frac{4\, h^6 y_o^{\,vii}}{315} + \cdots$$

<u>Nine Ordinate Formulas</u>

These are developed by dividing the interval at the midpoint and employing the Five Ordinate Formulas separately to each half with the requirement that the two applications give the same values for the mid-ordinate and its first derivative. Sixteen formulas are needed. Fourteen are duplicates of (i') to (viii'), excepting (ii') and (vi'), which now take the forms

$$y_o = \frac{y_4 + y_{-4}}{2} - \frac{8h^2}{45}(4\, y_{-3}'' + 3\, y_{-2}'' + 12\, y_{-1}'' + 7\, y_o'' + 12\, y_1'' + 3\, y_2'' + 4\, y_3'')$$

$$y_o' = \frac{y_4 - y_{-4}}{8h} + \frac{2h}{45}(4\, y_{-3}'' + 3\, y_{-2}'' + 12\, y_{-1}'' - 12\, y_1'' - 3\, y_2'' - 4\, y_3'')$$

Interpolation Formulas

The basis for the necessary interpolations is Lagrange's formula:

$$y = \sum_{i=0}^{n} y_i \frac{\prod_{k=0}^{n} (x - x_k)}{\prod_{\substack{k=0 \\ i=0}}^{n} (x_i - x_k)} \quad (k \neq i)$$

If the given ordinates are equally spaced and the interpolated values lie midway between any two consecutive given ordinates, the formula reduces to these simple forms:

Three Ordinates:

$$\text{(I)} \quad \begin{aligned} y_1 &= (1/8)(3\,y_0 + 6\,y_2 - y_4) \\ y_3 &= (1/8)(-\,y_0 + 6\,y_2 + 3\,y_4) \end{aligned}$$

Four Ordinates:

$$\text{(II)} \quad \begin{aligned} y_1 &= (1/16)(5\,y_0 + 15\,y_2 - 5\,y_4 + y_6) \\ y_3 &= (1/16)(-\,y_0 + 9\,y_2 + 9\,y_4 - y_6) \\ y_5 &= (1/16)(\,y_0 - 5\,y_2 + 15\,y_4 + 5y_6) \end{aligned}$$

Five Ordinates:

$$\text{(III)} \quad \begin{aligned} y_1 &= (1/128)(35\,y_0 + 140\,y_2 - 70\,y_4 + 28\,y_6 - 5\,y_8) \\ y_3 &= (1/128)(-5\,y_0 + 60\,y_2 + 90\,y_4 - 20\,y_6 + 3\,y_8) \\ y_5 &= (1/128)(\,3\,y_0 - 20\,y_2 + 90\,y_4 + 60\,y_6 - 5\,y_8) \\ y_7 &= (1/128)(-5\,y_0 + 28\,y_2 - 70\,y_4 + 140\,y_6 + 35\,y_8) \end{aligned}$$

Modified Simpson's Rule

By combining Simpson's Rule for ordinates located at $x = h$ and $3h$ and Newton's Three-Eighths Rule for ordinates located at $x = 0$ and $3h$ a formula for determining the ordinate at $x = h$ is obtained. Newton's Three Eighths Rule gives

$$\text{(vii)} \quad y_3 - y_0 = \frac{3h}{8}(y_0' + 3\,y_1' + 3\,y_2' + y_3') + \frac{3\,h^5}{80} y_{+3/2}^{\text{v}} + \ldots$$

Simpson's Rule (vi) gives

$$\text{(viii)} \quad y_3 - y_1 = \frac{h}{3}(y_1' + 4\,y_2' + y_3') - \frac{h^5 y_2^{v}}{90} + \ldots$$

The subtraction of (viii) from (vii) furnishes the formula

$$\text{(ix)} \quad y_1 - y_o = \frac{h}{24}(9\,y_o' + 19\,y_1' - 5\,y_2' + y_3') + \frac{h^5}{720}(27\,y^{v}_{3/2} + 8\,y_2^{v} \,..$$

By a similar process there is obtained

$$\text{(x)} \quad y_1' - y_o' = \frac{h}{24}(9\,y_o'' + 19y_1'' - 5y_2'' + y_3'') + \frac{h^5}{720}(27y^{vi}_{3/2} + 8y_2^{vi} \,..$$

Formula for Determining y_o'

In the Short Formula process, the first integral of the differential equation is known except for an additive constant, y_o'. This constant can be determined from the assigned end points as follows:

Let $y' - y_o'$ be integrated over the interval x_o to a, subject to the conditions $y = y_o$ at $x = x_o$ and $y = y_a$ at $x = a$. Then it follows directly that $\int_{x_o}^{a}(y' - y_o')\,dx = y_a - y_o - (a - x_o)\,y_o'$. If the integral is replaced by a general form of Simpson's Rule (vi), the formula to determine y_o' is

$$\begin{aligned}\text{(xi)} \quad (a - x_o)\,y_o' = (y_a - y_o) \; - \; \frac{h}{3}\Big\{4\,\big[(y_1' - y_o') + \ldots \\ + (y_i' - y_o') + \ldots\big] + 2\,\big[(y_2' - y_o') + \ldots \\ + (y_k' - y_o') + \ldots\big] \quad + (y_a' - y_o')\Big\}\end{aligned}$$

where $i = 1, 3, 5, \ldots, a - 1$, and $k = 2, 4, 6, \ldots, a - 2$.

APPLICATION OF THE FORMULAS TO A DIFFERENTIAL EQUATION

The differential equation is taken in the form $y'' = (f(x,y,y')$ with the assigned end points (x_o,y_o) and (a,y_a). To illustrate the procedure, the processes are applied to the differential equation $(1 + x^2)\, y'' = x - 2y - 4xy'$, for which is desired the solution passing through (0,1) and (2,2).

Starting the Solution

The first approximation to the solution is assumed to be a straight line connecting the assigned end points. The midpoint of the line has the coordinates $x_m = \frac{1}{2}(x_o + a)$ and $y_m = \frac{1}{2}(y_o + y_a)$. The slope of the line is $y_m' = (y_a - y_o)/(a - x_o)$. The coordinates of the two end points and of the midpoint together with the slope are substituted into the differential equation to give the three second derivatives y_o'', y_m'', y_a''. For example, in the chosen equation, the end points are as (0,1) and (2,2). Then $x_m = 1$, $y_m = 1.5$, and $y_m' = 0.5$. The differential equation gives $y_o'' = -2$, $y_m'' = -2$, and $y_a'' = -1.2$. These values of x_m, y_m, and y_m' are now refined by substituting the y'' values into equations (i) to (iv) inclusive, of the Three Ordinate Formulas. The value of h is given by $(a - x_o)/(N - 1)$ where N is the number of ordinates in the interval. The equations become, since $h = 1$,

$$y_m = \tfrac{1}{2}(2 + 1) - \frac{1}{24}(-2 + 10(-2) - 1.2) = 2.47$$

$$\text{(A)}\quad y_o' = \frac{2 - 1}{2(1)} - \frac{1}{3}(-2 + 2(-2) = 2.50$$

$$y_m' = 0.43, \text{ and } y_a' = -1.22.$$

The work up to this point may be conveniently shown in tabular form:

x	y"	y' - y_o'	y'	y
0			0.5	1.0
1			0.5	1.5
2			0.5	2.0
0	- 2		2.50	1.00
1	- 2		0.43	2.47
2	- 1.2		- 1.22	2.00

First Interpolation

Another set of values for y" is found from the differential equation using the values (A) and the end points. These are $y_o" = -2$, $y_m" = -2.608$, $y_a" = 0.992$. At this stage, a second derivative is interpolated midway between $y_o"$ and $y_m"$ and another midway between $y_m"$ and $y_a"$. The interpolation is carried out using Lagrange's three ordinate formulas (I). Five second derivatives are now available. The interpolated values are $y_1" = -2.830$ and $y_3" = -1.334$. Let $y_m" = y_2"$ and $y_a" = y_4"$. At this point, either of the two processes: The Short Formula Process or The Long Formula Process, may be employed. The solution by The Long Formula Process is given in a later section.

The Short Formula Process

The Short Formula Process uses the Simpson and the Modified Simpson Rules together with the Formula for Determining y_o'.

Formula (x) gives the value of $y_1' - y_o'$ and formula (v) gives $y_2' - y_o'$. The values for $y_3' - y_1'$ and $y_4' - y_2'$ are found by moving formula (v) forward one ordinate at a time. By adding $y_3' - y_1'$ to $y_1' - y_o'$, $y_3' - y_o'$ is obtained. By adding $y_4' - y_2'$ to $y_2' - y_o'$, $y_4' - y_o'$ is obtained. This gives a set of values $y' - y_o'$. When the column headed $y' - y_o'$ is completed, y_o' is determined from formula (xi). This value of y_o' is entered opposite x_o in the column headed y'. The algebraic addition of y_o' to each value in the $y' - y_o'$ column gives a complete table of first integrals of the y'' column. Since y_o is given, the integration of the y' column follows directly. The process this time goes forward in only one column, headed y, with the use of formulas (vi) and (ix). The complete process thus requires only a five column table.

The procedure applied to the example is as follows: $h = \frac{1}{2}$,

From (x), $y_1' - y_o' = \frac{1}{48}\,(9(-2) + 19(-2.830) + \ldots) = -\,1.251.$

From (v), $y_2' - y_o' = \frac{1}{6}\,(-2 + 4(-2.830) - 2.608) = -\,2.655,$

$$y_3' - y_1' = \frac{1}{6}\,(-2.830 + \ldots.) = -\,2.433,$$

and $y_4' - y_2' = -\,1.159.$

In terms of y_o', these are $y_3' - y_o' = -\,3.684$ and $y_4' - y_o' = -\,3.814$. These four values are now substituted into formula (xi) to give $y_o' = 2.905$. This value is entered in the y' column as indicated in the preceding paragraph. The y' column is completed by adding this value of y_o' algebraically to each value in the $y' - y_o'$ column.

The integration of the y' column is exactly as before, beginning with formula (ix) and following with formula (vi). The work of this section forms a table like this:

x	y"	y' - y$_o$'	y'	y
0	-2.		2.905	1.000
0.5	-2.830	-1.251	1.654	2.517
1.0	-2.608	-2.655	0.251	2.629
1.5	-1.334	-3.684	- .779	2.470
2.0	.992	-3.814	- .909	2.000

The Short Formula Process, from this point on, is merely a repetition of the procedure given in the last two paragraphs with the subdivision of each segment of the interval every alternate time after a set of ordinates has been calculated. After each interpolation, a refinement of the values is made on the assumption that variations due to interpolation will be practically eliminated.

The rest of the work necessary to complete the illustrative example is shown in the table.

x	y''	$y' - y_0'$	y'	y
0	-2.000		4.089	1.000
0.5	-5.698	-2.348	1.741	2.445
1.0	-2.631	-4.571	-0.482	2.762
1.5	0.380	-4.988	-0.899	2.264
2.0	0.053	-4.580	-0.491	2.000
0	-2.000		4.208	1.000
0.25	-6.581	-1.203	3.005	1.921
0.50	-6.298	-2.885	1.323	2.463
0.75	-3.891	-4.175	0.033	2.615
1.00	-1.298	-4.815	-0.607	2.534
1.25	0.351	-4.903	-0.695	2.364
1.50	0.728	-4.745	-0.537	2.207
1.75	0.312	-4.605	-0.397	2.094
2.00	0.386	-4.548	-0.340	2.001
0.00	-2.000		3.938	1.000
0.25	-6.209	-1.152	2.786	1.857
0.50	-5.658	-2.708	1.230	2.359
0.75	-2.931	-3.800	0.138	2.511
1.00	-1.320	-4.267	-0.329	2.480
1.25	-0.001	-4.484	-0.546	2.367
1.50	0.095	-4.369	-0.431	2.235
1.75	0.109	-4.443	-0.505	2.136
2.00	0.145	-4.313	-0.375	2.000

x	y"	y' - y_o'	y'	y
0.000	-2.000		3.824	1.000
0.125	-4.663	-0.433	3.394	1.455
0.250	-5.882	-1.106	2.718	1.838
0.375	-5.998	-1.858	1.966	2.131
0.500	-5.342	-2.573	1.251	2.331
0.625	-4.238	-3.175	0.649	2.448
0.750	-2.999	-3.627	0.197	2.500
0.875	-1.929	-3.932	-0.108	2.503
1.000	-1.322	-4.129	-0.305	2.478
1.125	-0.572	-4.257	-0.433	2.430
1.250	-0.294	-4.292	-0.468	2.374
1.375	-0.203	-4.338	-0.514	2.313
1.500	-0.118	-4.343	-0.519	2.247
1.625	0.041	-4.364	-0.540	2.182
1.750	0.249	-4.331	-0.507	2.115
1.875	0.379	-4.305	-0.481	2.055
2.000	0.200	-4.081	-0.257	2.003
0.000	-2.000		3.809	1.000
0.125	-4.393	-0.409	3.400	1.454
0.250	-5.783	-1.056	2.753	1.840
0.375	-5.993	-1.806	2.003	2.138
0.500	-5.331	-2.518	1.291	2.342
0.625	-4.238	-3.121	0.688	2.465
0.750	-3.098	-3.575	0.234	2.520
0.875	-2.126	-3.903	-0.094	2.529
1.000	-1.368	-4.115	-0.306	2.501
1.125	-0.788	-4.252	-0.443	2.456
1.250	-0.452	-4.322	-0.513	2.393
1.375	-0.147	-4.366	-0.557	2.329
1.500	0.037	-4.364	-0.555	2.256
1.625	0.212	-4.357	-0.548	2.190
1.750	0.263	-4.316	-0.507	2.120
1.875	0.304	-4.292	-0.483	2.063
2.000	0.011	-4.254	-0.445	2.000

Summary of the Short Formula Process

By this process, the complete integration of a differential equation may be carried out in thirteen steps:

1. Assume a straight line as the first guess.
2. Correct the guess using the Three Ordinate Formulas.
3. Interpolate with Lagrange's formulas (I).
4. Apply the Modified Simpson Rule to the y" column to get $y_1'-y_o'$.

5. Apply Simpson's Rule to the y" column to get remaining $y' - y_0'$.
6. Use formula (xi) to get y_0'.
7. Fill in y' column.
8. Apply the Modified Simpson Rule to y' column to get y_1.
9. Apply Simpson's Rule to the y' column to get remaining ordinates.
10. Check the calculated value of end ordinate with true value.
11. Repeat steps 4 to 10 inclusive.
12. Interpolate, using either (II) or (III) of Lagrange's formulas.
13. Repeat steps 4 to 12 inclusive until solution is reached.

In tabular form steps 4 to 10 inclusive can be performed as rapidly as the calculating machine can be operated and the figures recorded. The actual number of repetitions of step 13 needed for a solution depends almost entirely on three uncontrollable factors: (a) the length of the interval $a - x_0$, (b) the given differential equation, and (c) the number of significant figures desired in the result. Ordinarily, when a short interval in which h will become 0.1 in a few operations and when not more than four significant figures are desired, eight operations suffice. This number of operations gives fifteen new ordinates in the interval. Except for the calculation of the values of the y" column, the table can be filled in at the rate of from 60 to 100 figures of four digits each in an hour. In eight operations, about 200 figures are entered. The calculation of the y" values depends upon the given equation.

Although in the earlier examples, Lagrange's formulas (III) were used for interpolation, later examples showed that the set (II) can be used when changes in the y" column are uniform. The set (II) is operated by applying the formulas for y_1 and y_3 to the first four values and then moving the formula for y_3 ahead one row in the table at each step until the end is reached. The final interpolated value is found from the formula for y_5. Since the coefficients in the formula for y_3 are -1 and 9, this process is very rapid with a calculator.

<u>The procedure as outlined in steps 1 to 13 using Lagrange's four ordinate formulas (II) for interpolation in step 12 is, in the author's</u>

<u>opinion, the simplest that has been devised for the numerical integration of a second order differential equation with assigned end points.</u>

It is possible that the Long Formula Process may prove simpler if the first derivative is missing. When the ordinate is missing, the Short Formula Process is greatly shortened.

The Short Formula Process is especially adapted to an electric calculating machine. It is simple enough so that all steps can proceed from memory. The actual work is simple enough so that a machine operator can solve the differential equation without recourse to methods beyond the level of High School Algebra.

The Long Formula Process

After a solution has been started, the Long Formula Process may be substituted for The Short Formula Process.

The Long Formula Process uses the Three, Five, and Nine Ordinate Formulas exclusively. The new ordinates are interpolated in the same manner as in the Short Formula Process. In principle, the mid-ordinate and slope found from the first guess are refined by repeated use of the formulas. In practice, the formulas (i) to (iv) and (i') to (viii') together with the two new formulas derived under the "Nine Ordinate Formulas" are applied directly to the table of y" values.

The process does not show promise of replacing the Short Formula Process. Except for the formulas for y_0 and y'_0, the formulas contain such large coefficients that the process is tedious and cumbersome. These formulas were applied to the same differential equation as was used in the Short Formula Process. The results were not as good after nine operations as had been obtained in eight operations by the simpler process.

The complete solution of the differential equation by the Long Formula Process follows. The table takes a different form. Two applications of the Three Ordinate Formulas, three applications of the Five Ordinate Formulas and three applications of the Nine Ordinate Formulas are shown in the table.

x=	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
y	1.0				1.5				2.0
y'	0.5				0.5				0.5
y"									
y	1.00				2.47				2.00
y'	2.50				0.43				-1.22
y"	-2.00				-2.00				-1.20
y	1.00		2.22		2.70		2.50		2.00
y'	3.06		1.73		0.20		-0.88		-0.87
y"	-2.00		-3.07		-2.83		-1.29		1.55
y	1.00		2.57		2.81		2.40		2.00
y'	4.25		1.73		-0.50		-0.95		-0.54
y"	-2.00		-5.92		-2.60		0.38		0.99
y	1.00		2.51		2.59		2.24		2.00
y'	4.33		1.39		-0.64		-0.61		-0.38
y"	-2.00		-6.48		-1.46		0.74		0.46
y	1.000	1.872	2.366	2.506	2.441	2.319	2.213	2.118	2.000
y'	3.996	2.795	1.416	0.035	-0.378	-0.475	-0.505	-0.474	-0.502
y"	-2.000	-6.454	-5.840	-3.257	-0.810	0.386	0.209	-0.471	0.208
y	1.000	1.841	2.333	2.487	2.459	2.361	2.236	2.112	2.000
y'	3.821	2.740	1.213	0.136	-0.367	-0.472	-0.510	-0.447	-0.406
y"	-2.000	-5.928	-5.651	-2.795	-1.185	-0.395	0.032	0.205	0.403
y	1.000	1.831	2.321	2.491	2.472	2.372	2.224	2.116	2.000
y'	3.792	2.697	1.249	0.207	-0.377	-0.482	-0.526	-0.492	-0.433
y"	-2.000	-5.809	-5.273	-2.964	-1.225	-0.434	0.027	0.213	0.250
y	1.000	1.835	2.333	2.507	2.486	2.382	2.248	2.117	2.000
y'									
y"	-2.000	-5.750	-5.312	-3.106	-1.218	-0.423	0.052	0.237	0.293

Comparison of Results

For the assigned end points, direct integration of $(1 + x^2)\ y'' = x - 2\ y - 4\ xy'$ by formal methods gives the equation $6(1 + x^2)\ y = x^3 + 23\ x + 6$. In the accompanying table, the correct values of the integral for seventeen ordinates are given in the second column. In the third column are the results for the same ordinates using the Short Formula Process. In the last column are the values for nine ordinates found by the Long Formula Process.

x	True Values	Short Formula	Long Formula
0.000	1.000	1.000	1.000
0.125	1.457	1.454	
0.250	1.846	1.840	1.835
0.375	2.145	2.138	
0.500	2.350	2.342	2.333
0.625	2.471	2.465	
0.750	2.525	2.520	2.507
0.875	2.529	2.529	
1.000	2.500	2.501	2.486
1.125	2.450	2.456	
1.250	2.387	2.393	2.382
1.375	2.319	2.329	
1.500	2.244	2.256	2.248
1.625	2.182	2.190	
1.750	2.117	2.120	2.117
1.875	2.056	2.063	
2.000	2.000	2.000	2.000

A Problem Having No Solution

If the end points (0,0) and (1.6, 1) are assigned to the differential equation $y'' = -\left(\frac{\pi}{1.6}\right)^2 y$, no solution exists. The family of solutions vanishing at $x = 0$ is $y = A \sin (\pi x/1.6)$, where A is an arbitrary constant. At $x = 1.6$, $y = 0$ independently of A so that no solution can be found. The Short Formula Process is used on the problem. The values of the ordinates found in each of the eight operations are given in the table. The non-existence of a solution is indicated by the steadily increasing values of the ordinates as the operations are continued.

x	1st	2nd	3rd	4th	5th	6th	7th	8th
0.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.1							0.770	0.901
0.2					1.045	1.289	1.514	1.772
0.3							2.210	2.586
0.4			1.087	1.522	1.967	2.417	2.832	3.314
0.5							3.364	3.934
0.6					2.656	3.247	3.794	4.424
0.7							4.109	4.768
0.8	0.500	1.117	1.751	2.389	3.029	3.669	4.274	4.955
0.9							4.485	4.976
1.0					3.037	3.631	4.194	4.827
1.1							4.150	4.512
1.2			1.741	2.218	2.672	3.127	3.560	4.047
1.3							3.270	3.432
1.4					1.968	2.218	2.449	2.718
1.5							1.976	1.869
1.6	1.000	1.000	1.000	0.999	1.001	1.001	1.000	1.000

The Perfect Solution

A perfect solution for four significant figures is reached when the calculated and true values do not vary by more than one unit in the last significant figure. In this example, the results are within the required limits. The equation integrated is $y'' = (1 + y'^2)^{3/2}$ with the assigned end points (0,0) and (0.8,0.2). Since they are not needed in the calculations, the y column values are omitted from the table until the final operation is reached. The behavior of the y' column indicates that results correct to at least three significant figures can be obtained at the end of the sixth operation. Lagrange's formulas (II) are used in the interpolations. A check with the formal solution of the differential equation shows that the seventeen ordinates lie on the circle having the equation $(x - 0.1790)^2 + (y - 0.9838)^2 = 1$. The work for this example is given in full.

x	y"	y' - y_0'	y'	y
0			.25	
.4			.25	
.8			.25	
0	1.0952		-.1881	
.4	1.0952		+.2500	
.8	1.0952		.6881	
0	1.0535		-.1826	
.2	0.9936	.2020	+.0194	
.4	1.0952	.4082	.2256	
.6	1.3604	.6510	.4684	
.8	1.7886	.9632	.7806	
0	1.0504		-.1830	
.2	1.0006	.2035	+.0205	
.4	1.0773	.4087	.2257	
.6	1.3465	.6473	.4643	
.8	2.0416	.9757	.7927	
0	1.0507		-.1822	
.1	1.0135	.1030	-.0792	
.2	1.0006	.2035	+.0213	
.3	1.0194	.3042	.1220	
.4	1.0774	.4087	.2265	
.5	1.1675	.5208	.3386	
.6	1.3402	.6449	.4627	
.7	1.6316	.7928	.6106	
.8	2.0779	.9764	.7942	
0	1.0502		-.1820	
.1	1.0094	.1027	-.0793	
.2	1.0007	.2030	+.0210	
.3	1.0224	.3039	.1219	
.4	1.0779	.4086	.2266	
.5	1.1768	.5209	.3389	
.6	1.3378	.6460	.4640	
.7	1.6085	.7921	.6101	
.8	2.0825	.9745	.7925	

x	y"	y' - y_o'	y'	y
0	1.0501		-.1820	
.05	1.0257	.0519	-.1301	
.10	1.0094	.1027	-.0793	
.15	1.0012	.1530	-.0290	
.20	1.0007	.2029	+.0209	
.25	1.0075	.2532	.0712	
.30	1.0224	.3038	.1218	
.35	1.0454	.3556	.1736	
.40	1.0780	.4085	.2265	
.45	1.1209	.4636	.2816	
.50	1.1771	.5208	.3388	
.55	1.2479	.5816	.3996	
.60	1.3397	.6459	.4639	
.65	1.4543	.7160	.5340	
.70	1.6074	.7920	.6100	
.75	1.8110	.8776	.6956	
.80	2.0773	.9741	.7921	

0	1.0501		-.1821	0
.05	1.0255	.0519	-.1302	-.0078
.10	1.0094	.1027	-.0794	-.0130
.15	1.0013	.1530	-.0291	-.0157
.20	1.0007	.2030	+.0209	-.0159
.25	1.0076	.2532	.0711	-.0136
.30	1.0223	.3039	.1218	-.0088
.35	1.0455	.3556	.1735	-.0014
.40	1.0779	.4086	.2265	+.0086
.45	1.1213	.4636	.2815	.0213
.50	1.1770	.5209	.3388	.0368
.55	1.2488	.5816	.3995	.0552
.60	1.3396	.6461	.4640	.0768
.65	1.4569	.7160	.5339	.1017
.70	1.6072	.7923	.6102	.1303
.75	1.8075	.8776	.6955	.1629
.80	2.0761	.9742	.7921	.2000

y" Infinite at One End of Interval

When y" becomes infinite at one end of the interval, the Short Formula Process fails to give a solution. In this case, the result reached by stepping the process out from the initial point does not check with the value assigned at the end of the interval.

When the end points (0,0) and (1.6, 0.9798) are assigned for the equation $y'' = 1.1\,(1 + y'^2)^{3/2}$, y" becomes infinite at x = 1.6. Three applications of the process are enough to show that the solution cannot be reached by this method.

x	y"	$y' - y_0'$	y'	y
0.0			0.6124	0.0000
0.8			0.6124	0.4899
1.6			0.6124	0.9798
0.0	1.7737		-0.8066	0.0000
0.8	1.7737		+0.6124	-0.0777
1.6	1.7737		2.0314	0.9798
0.0	2.3325		-0.8829	0.0000
0.4	0.6903	0.4921	-0.3908	-0.2318
0.8	1.7737	0.8725	-0.0104	-0.3275
1.2	5.8256	2.2961	1.4132	0.1308
1.6	12.7650	5.0455	4.1626	1.3073

y" Large at One End of Interval

If at a few points in the interval, y" becomes very large in proportion to the other values of y", repeated application of the Short Formula Process makes the fit to the solution increasingly poor.

When the end points (0,0) and (1.6, 0.5) are assigned to the equation $y'' = (1 + y'^2)^{3/2}$, y" becomes infinite at x = 1.6373. At the sixth operation, the value of y" at x = 1.6 is beginning to control the table. At the eighth operation, y" at x = 1.6 is eight times the next largest y" value in the table. This value has now become large enough to affect the values of all fifteen interpolated ordinates. The agreement with the true values becomes worse at each operation beyond this stage. The process for the sixth and eighth operations is shown. Correct values are shown in the column headed Y.

Sixth Operation

x	y"	y' - y_0'	y'	y	Y
0.0	2.0807		-0.8399	0.0000	0.0000
0.2	1.3767	0.3361	-0.5038	-0.1323	-0.1287
0.4	1.1108	0.5799	-0.2600	-0.2077	-0.2033
0.6	1.0053	0.7911	-0.0488	-0.2385	-0.2287
0.8	1.0577	0.9926	+0.1527	-0.2279	-0.2161
1.0	1.2182	1.2214	0.3815	-0.1756	-0.1613
1.2	1.7349	1.5036	0.6637	-0.0717	-0.0561
1.4	3.9759	2.0303	1.1904	+0.1062	+0.1239
1.6	23.8969	3.9904	3.1505	0.5000	0.5000

Eighth Operation

x	y"	y' - y_0'	y'	y	Y
0.0	2.2134		-0.8392	0.0000	0.0000
0.1	1.6738	0.1919	-0.6473	-0.0739	-0.0738
0.2	1.3732	0.3427	-0.4965	-0.1308	-0.1287
0.3	1.1950	0.4706	-0.3686	-0.1740	-0.1708
0.4	1.0875	0.5841	-0.2551	-0.2050	-0.2033
0.5	1.0266	0.6896	-0.1496	-0.2253	-0.2199
0.6	1.0014	0.7906	-0.0486	-0.2351	-0.2287
0.7	1.0071	0.8909	+0.0517	-0.2350	-0.2274
0.8	1.0436	0.9930	0.1538	-0.2247	-0.2161
0.9	1.0899	1.0999	0.2607	-0.2041	-0.1943
1.0	1.1009	1.2098	0.3706	-0.1725	-0.1613
1.1	1.3594	1.3283	0.4891	-0.1297	-0.1159
1.2	1.6370	1.4823	0.6431	-0.0735	-0.0561
1.3	2.1650	1.6640	0.8248	-0.0002	+0.0217
1.4	2.7993	1.9188	1.0796	+0.0939	0.1239
1.5	8.8492	2.4044	1.5652	0.2234	0.2649
1.6	74.7006	5.6820	4.8428	0.5000	0.5000

PARAMETRIC FORM OF SOLUTION OF A DIFFERENTIAL EQUATION

Formation of the Parametric Equations

From the identity $dy/dx = (dy/ds)/(dx/ds)$, the identity $d^2y/dx^2 = (d^2y/ds^2)/(dx/ds)^4$ follows by ordinary differentiation. Further, if $s = kt$, it follows that $d^2y/dx^2 = k^2(d^2y/dt^2)/(dx/dt)^4$ and $dy/dx = (dy/dt)/(dx/dt)$. If accents denote differentiation with respect to t, the original differential equation

(1) $\qquad d^2y/dx^2 = f(x,y,dy/dx)$ $\qquad$ becomes

(2) $\qquad k^2\, y''/x'^4 = f(x,y,y'/x')$.

If s denotes the length of the arc of the curve, an element of arc, ds, is given by $ds^2 = dx^2 + dy^2$. Since $ds = k\,dt$, x' and y' are connected by the relation

(3) $\qquad x'^2 + y'^2 = k^2$.

The constant k can be eliminated from (2) and (3) by substituting (3) into (2) and differentiating (3). The equations replacing the single differential equation (1) take the forms

(4) $\qquad (x'^2 + y'^2)\, y''/x'^4 = f(x,y,y'/x')$

(5) $\qquad x'\, x'' + y'\, y'' = 0$.

The substitution of (2) into (5) gives

(6) $\qquad k^2\, x''/x'^3y' = -f(x,y,y'/x')$.

Equations (2) and (6) form a pair of parametric equations replacing the original equation (1).

An Application of the Equations

If equations (2) and (6) are applied to the equation $d^2y/dx^2 = [1 + (dy/dx)^2]^{3/2}$, there is obtained the simple pair $y'' = kx'$ and $x'' = - ky'$. If the end points assigned are (0,0) and (1.6, 0.5), the problem becomes the same one which in the last chapter failed to yield a solution. The integral is found by the simultaneous solution of this pair of equations using a double application of the Short Formula Process at each operation. The particular problem studied is carried ten operations. This many operations interpolates a total of 31 new pairs of values of x and y between (0,0) and (1.6,0.5).

Starting the Solution

For convenience, the equations are integrated from $t = 0$ to $t = 3.2$ The table is set up in nine columns headed t; x'', $x' - x_0'$, x' and x; y'', $y' - y_0'$, y' and y. The connection between the block relating to x and the block relating to y is made through the parametric differential equations. Otherwise, the solution is just the same as though two separate equations were being solved in parallel columns. Each block is started separately. For the x-block a straight line connecting $t = 0$, $x = 0$ and $t = 3.2$, $x = 1.6$ is assumed. This gives $x_m' = 0.5000$. For the y-block, a straight line connecting $t = 0$, $y = 0$ and $t = 3.2$ $y = 0.5000$ is assumed. This gives $y'_m = 0.1562$. From the relationship (3), $k = 0.5238$. Substituting for k, x' and y' in the pair of differential equations gives $y''_m = 0.2619$ and $x''_m = -0.0647$. The Three Ordinate Formulas are applied directly to each block in turn to give the x' and y' values needed. New values of k are found again from (3). The table on the next page shows the first five operations.

t	x"	x' - x_o'	x'	x	y"	y' - y_o'	y'	y
0			.5000	0			.1562	0
1.6			.5000	0.8			.1562	.25
3.2			.5000	1.6			.1562	.50
0	-.0647		.6035	0	.2619		-.2628	0
1.6	-.0647		.5000	0.8828	.2619		.1562	.0852
3.2	-.0647		.3965	1.6000	.2619		.5752	.5000
0	.1730	0	.4767		.3972	0	-.3350	
.8	.0666	.0975	.5742		.3108	.2807	-.0543	
1.6	-.0647	.0999	.5766		.2619	.5073	.1723	
2.4	-.2208	-.0126	.4641		.2507	.7098	.3748	
3.2	-.4018	-.2600	.2167		.2770	.9184	.5834	
0	.1918	0	.5119	0	.2730	0	-.3482	0
.8	.0313	.0872	.5991	0.4528	.3312	.2431	-.1051	-.1852
1.6	-.1037	.0569	.5688	0.9272	.3470	.5186	.1704	-.1595
2.4	-.2236	-.0747	.4372	1.3359	.2768	.7754	.4272	.0825
3.2	-.3630	-.3061	.2058	1.6001	.1349	.9424	.5942	.5001
0	.2156	0	.4863		.3169	0	-.3902	
.4	.1429	.0720	.5583		.3750	.1419	-.2483	
.8	.0639	.1135	.5998		.3644	.2908	-.0994	
1.2	-.0184	.1227	.6090		.3588	.4341	.0439	
1.6	-.1012	.0987	.5850		.3378	.5758	.1856	
2.0	-.1820	.0420	.5283		.3070	.7030	.3128	
2.4	-.2569	-.0461	.4402		.2629	.8196	.4294	
2.8	-.3220	-.1622	.3241		.2041	.9114	.5212	
3.2	-.3736	-.3019	.1844		.1294	.9808	.5906	

Continuation of the Process

The first interpolation is made with Lagrange formulas (I). All later interpolations are made with the formulas (II). The process is continued as in the preceding examples, except that now k must also be determined for each value of t. k varies with each ordinate and from one operation to the other. As the solution is approached, k becomes more nearly constant. If the solution is perfect, k has the same value for all ordinates in the interval. This property of k offers it much promise of serving as a valuable guide in this type of solution.

Comparison of Results

Formal solution gives the equations

$$x = 0.6373 + \sin (0.6211\ t - 0.6909)$$

$$y = 0.7706 - \cos (0.6211\ t - 0.6909)$$

For 33 ordinates, the true values of x and y for a given value of t are shown in the columns headed X and Y in the table. The third and fifth columns give the calculated values. Much better results are obtained here than were obtained in the case where the Short Formula Process was applied directly.

t	X	x	Y	y
0.0	0.0000	0.0000	0.0000	0.0000
0.1	0.0491	0.0491	-0.0381	-0.0380
0.2	0.1004	0.1005	-0.0731	-0.0728
0.3	0.1538	0.1539	-0.1048	-0.1044
0.4	0.2091	0.2092	-0.1335	-0.1326
0.5	0.2660	0.2661	-0.1579	-0.1573
0.6	0.3243	0.3245	-0.1792	-0.1784
0.7	0.3840	0.3840	-0.1968	-0.1959
0.8	0.4445	0.4446	-0.2106	-0.2096
0.9	0.5058	0.5058	-0.2207	-0.2196
1.0	0.5676	0.5676	-0.2270	-0.2258
1.1	0.6295	0.6296	-0.2294	-0.2282
1.2	0.6917	0.6916	-0.2279	-0.2267
1.3	0.7535	0.7534	-0.2226	-0.2215
1.4	0.8150	0.8148	-0.2135	-0.2123
1.5	0.8757	0.8755	-0.2006	-0.1995
1.6	0.9356	0.9353	-0.1838	-0.1828
1.7	0.9942	0.9938	-0.1635	-0.1626
1.8	1.0515	1.0512	-0.1396	-0.1386
1.9	1.1072	1.1067	-0.1121	-0.1113
2.0	1.1611	1.1607	-0.0816	-0.0804
2.1	1.2129	1.2124	-0.0471	-0.0465
2.2	1.2626	1.2622	-0.0098	-0.0091
2.3	1.3098	1.3092	+0.0305	+0.0309
2.4	1.3544	1.3541	0.0736	0.0742
2.5	1.3963	1.3957	0.1197	0.1197
2.6	1.4353	1.4350	0.1683	0.1683
2.7	1.4712	1.4706	0.2187	0.2186
2.8	1.5038	1.5036	0.2715	0.2717
2.9	1.5331	1.5327	0.3262	0.3260
3.0	1.5590	1.5589	0.3827	0.3828
3.1	1.5813	1.5810	0.4406	0.4402
3.2	1.6000	1.6000	0.5000	0.4999

CONCLUSION

A comparison of the two processes developed in this thesis shows that in ordinary practice the Short Formula Process using Simpson's Rule and the Modified Simpson's Rule proceeds so rapidly and simply that it is greatly superior to the Long Formula Process using the Three, Five, and Nine Ordinate Formulas. The tabular form taken by the Short Formula Process is short, compact, and easily duplicated.

Accuracy in both processes is brought about by reducing h to a value small enough to make nearly zero the terms neglected in the Maclaurin expansions of the formulas used. In ordinary cases where h becomes at least one-tenth by the seventh operation, a satisfactory solution is reached in the eighth.

No rigorous attempt has been made to determine the requirements for the existence of a solution. Each of the examples presented displays special characteristics which may serve as predictors of a successful solution. Problems to be solved may be classified as (a) those for which the Short Formula Process gives a perfect solution, (b) those for which no solution is possible with the assigned end points, and (c) those capable of solution but to which the Short Formula Process is inapplicable. A perfect solution would appear to be indicated by (a) repetition of values for several operations, (b) steady approach of a value to a definite limit as operations proceed, and (c) in the parametric form, the tendency of k to remain constant throughout the interval and from one operation to the next. That these conditions will insure a correct solution has not been examined. From the examples no distinction can be made between the failure of the process and non-existence of a solution. In both instances, the indication that the ordinates are not going to become stationary in value after a few operations is taken as sufficient evidence that the continuance of the process will be unprofitable.

BIBLIOGRAPHY

1. Committee on Numerical Integration, "Numerical Integration of Differential Equations." _Bulletin of the National Research Council_, No. 92, 1933, pp. 51-70.

2. Ince, E. L. _Ordinary Differential Equations_. London: Longmans, Green and Co. Ltd. 1927, p. 540.

3. Milne, W. E. "On the Numerical Integration of a Certain Differential Equation of the Second Order." _American Mathematical Monthly_. 40: 322-327, 1933.