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Dale Leroy Harmer for the M. S. in Statistics
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Title A MIXED VARIABLES-ATTRIBUTES DOUBLE SAMPLING PLAN

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This thesis presents a table of dependent, mixed variables and attributes double sampling plans, where the standard deviation of the quality characteristic is assumed to be known, and the acceptance number is zero for the attributes portion of the plans. The table gives sampling plans for ten ranges of lot sizes, each with ten LTPD levels, from one percent to 40 percent. Operating characteristic curves are presented for five representative plans from the table.

In addition, a general discussion of acceptance inspection and lot-by-lot sampling inspection is given. Mixed variables and attributes plans are discussed in some detail for the case of known standard deviation, and somewhat more briefly for the case of unknown standard deviation.

A MIXED VARIABLES-ATTRIBUTES DOUBLE SAMPLING PLAN

by

DALE LEROY HARMER

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In Charge of Major

Chairman of Department of Statistics

Dean of Graduate School

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CHAPTER I

INTRODUCTION

Most of the acceptance sampling plans now in existence are based on either sampling inspection by variables or sampling inspection by attributes. In the present paper, these two types of sampling plans are combined into a double sampling plan where the first sample is inspected by a variables criterion and the second sample by an attributes criterion. A sampling table and five different OC curves are included for the special case where we assume that the standard deviation of the population being sampled is known and the acceptance number on the second sample is zero. The sampling table consists of sampling plans indexed on various LTPD levels from one percent defective to 40 percent defective.

In chapter two, a general discussion of the various possible types of acceptance inspection is presented. The particular kind of acceptance inspection for which sampling plans are used — lot-by-lot sampling inspection - is considered more closely in chapter three. Chapter four includes a discussion of mixed variables and attributes sampling plans in general and a derivation of the operating characteristic equations for independent and dependent mixed variables and attributes sampling plans. The details of the calculation of OC curves are presented for the special case of a dependent plan with known standard deviation and acceptance number zero. In the concluding chapter, the results of this paper are discussed and a sample calculation of the OC function is given. Finally, an example of the use of a mixed variables and attributes sampling plan is given.

CHAPTER 2

ACCEPTANCE INSPECTION

1. Introduction

When a business establishment or a government agency contracts to purchase a number of items of raw materials, parts, or finished products from some supplier the purchase contract will often indicate that the acceptance of the product is subject to having each item meet certain quality specifications. Items failing to meet one or more of these quality specifications are called defective, and each failure to meet a specification is called a defect. Items satisfying the quality specifications are called nondefective. Similarly, in the transfer of goods-in-process or of finished parts among different divisions of the same organization, established quality standards for individual items must be met.

Both the supplier and the receiver of a product realize that, if the product is made by mass production methods, some of the items submitted to the receiver for acceptance will be classified as defective. Furthermore, the number and location of the defectives in any group of items, which we shall call a lot, will seldom be known. To protect the receiver against acceptance of an undue proportion of defective items, to help the supplier to improve the quality of his product, and to assist in quality control and the reduction of production costs when the supplier and the receiver are both in the same organization, some form of acceptance inspection has been found desirable.

2. Types of Acceptance Inspection

Acceptance inspection usually takes one of the three following forms.

2.1. Process Inspection

Under certain circumstances the receiver can run control charts on submitted products or can obtain copies of charts maintained by the supplier at various stages of the production processes. These charts can be of great value to an acceptance program because they supply useful information about the quality level of the product and about the degree of control at various stages in the production process.

2.2. Screening

Using this method of inspection, all the items in a lot are inspected and individually accepted or rejected. This procedure is used when elimination of defective items is essential. Ordinarily, however, the cost of testing each item individually is prohibitive, and if the test damages or destroys the item, screening is impossible. Screening is the only program of sampling inspection that can possibly guarantee the rejection of all the defective items and the acceptance of all the nondefective items that are submitted by a supplier. In practice, however, even screening will usually not eliminate all the defective items from a lot, because inspectors do not work with 100 percent accuracy, especially when large numbers of items are inspected and defective items are not particularly obvious. In some cases sampling inspection with its lower costs may be able to give the same degree of quality assurance as screening.

2.3. Lot-by-lot Sampling Inspection

Under lot-by-lot sampling inspection, the product is divided into appropriate inspection lots (which may be the number of items in a container, the number of items in a shipment, the number of items in a production run, or any other suitable grouping of items); one or several samples of items are drawn from a given lot, and from these samples a decision is made to accept or to reject the lot. For most applications, experience has shown lot-by-lot inspection to be the most satisfactory method of acceptance inspection.

3. Description of Item Quality

In evaluating and describing the quality of an individual item, one of the following general types of inspection is usually used, the choice depending on the quality characteristic being measured and on the device or method that is used in making such measurements.

3.1. Variables Inspection

When inspecting an item on a variable basis, the quality characteristic in question is measured along a continuous scale in terms of centimeters, volts, ohms, pounds, seconds or some such unit.

3.2. Attributes Inspection

Under attribute inspection of an item, the quality characteristic in question is observed or checked with an indicating device and simply classified as defective or nondefective. Use of go and not-go gauges for checking dimensional properties of parts is an example of a widely used method of attribute inspection.

3.3 Counting Defects Per Unit

For certain products, quality characteristics of an item are evaluated by counting the number of a given type of defect in an item.

For example, the quality of the enamel insulation on a copper wire can be measured by enumerating the number of bare spots on a specified length of wire.

4. Description of Lot Quality

In describing the quality of a lot,the following measures are generally used. The choice among these will depend on the nature of the product and on its ultimate use.

4. 1. Percentage Defective

The percentage of defective items in a lot is the most commonly used method of describing lot quality, and it is the basis on which most sampling plans have been constructed. The percentage defective is given by

$$100\,p = 100\,(d/N), \tag{2.1}$$

where p is the proportion of defective items, and d is the number of defective items in a lot containing N items.

4. 2 The Average Number of Defects Per Item of Product

In this case the item quality is usually measured by counting the number of defects per unit for each quality characteristic being inspected. The average number of defects per item, then, is

$$(2.2) \qquad \bar{D} = \sum_{i=1}^{N} (D_i/N),$$

where D_i is the number of defects in the i^{th} item and N is the total number of items in the lot being inspected. Using the number of defects per unit as the measure of item quality, an item will be considered defective if it has more than M defects, where M is some positive integer or zero. Hence, the number of defective items in a lot is given by

$$(2.3) \qquad d = \sum_{i=1}^{N} \delta i, \text{ where } \delta i = 0 \text{ if } D_i \leq M$$
$$= 1 \text{ if } D_i > M.$$

The percentage of defective items, therefore, can be represented in terms of the number of defects per unit by

$$(2.4) \qquad 100\,p = 100 \sum_{i=1}^{N} \delta i/N \ , \text{ where } \delta i = 0 \text{ if } D_i \leq M$$
$$= 1 \text{ if } D_i > M.$$

4.3 The Arithmetic Mean of Item Qualities in a Lot

The use of this statistic assumes that the quality of a lot is adequately represented by the mean values of the quality characteristics being inspected. The mean value of a particular quality characteristic is given by

$$(2.5) \qquad \bar{x} = \sum_{i=1}^{N} (x_i/N),$$

where x_i is a measurement of the quality characteristic of i^{th} item in a lot containing N items. Let us assume that the quality characteristic being measured has a single upper specification limit U, so that the i^{th} item of a lot is acceptable if x_i is less than or equal to U and defective if x_i is greater than U. Then the percentage of defective items in a lot can be represented in terms of the measurements of a quality characteristic by

$$(2.6) \qquad 100p = 100 \sum_{i=1}^{N} (\delta i/N), \text{ where } \delta i \begin{array}{l} = 0 \text{ if } x_i \leq U \\ = 1 \text{ if } x_i > U . \end{array}$$

4.4. The Standard Deviation of Item Qualities in a Lot

The standard deviation is a measure of the amount of variation of a quality characteristic from item to item. It is given by

$$(2.7) \qquad s = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}} ,$$

where $\bar{x}$ is the mean of the observations, x_i, from a lot of size N. In this case, an item is considered defective if

$$|x_i - \bar{x}| > L$$

and acceptable otherwise. Here, L is the limit on the variability of the quality characteristic being inspected. The percentage of defective items in a lot is then given by

$$(2.8) \qquad 100\,p = 100 \sum_{i=1}^{N} (\delta i/N)\,, \quad \text{where } \delta i = 0 \text{ if } |x_i - \overline{x}| \leq L$$
$$= 1 \text{ if } |x_i - \overline{x}| > L.$$

CHAPTER 3

LOT-BY-LOT SAMPLING INSPECTION

1. Introduction

The present paper is concerned with a lot-by-lot sampling inspection plan, using both variables inspection and attributes inspection to measure item quality and percent defective to measure lot quality. There are many different lot-by-lot sampling inspection plans available. Some of the more prominent ones are: (a) Dodge and Romig sampling plans (5), which use attributes inspection for item quality and percent defective for lot quality; (b) Military Standard 105C sampling plans (14), which use attributes inspection for item quality and percent defective for lot quality; (c) Bowker and Goode sampling plans (2), which use variables inspection for item quality and percent defective for lot quality; and (d) Military Standard 414 sampling plans, which use variables inspection for item quality and percent defective for lot quality.

2. Attributes Plans Versus Variables Plans

It can be observed from the preceding paragraph that the most commonly used sampling plans use percent defective as the measure of lot quality, but differ in the use of attributes or variables inspection to measure item quality. There are advantages and disadvantages to both variables inspection plans and attributes inspection plans. The principal advantages to the use of attributes sampling plans are:

(1) Inspection by attributes usually requires less skill, less time and less expensive equipment to inspect each item; less record keeping to record the results of inspection; and less arithmetic to determine from the sample whether or not to accept a given lot.

(2) Variables inspection plans are based on the assumption of normality of the quality characteristic being measured, while attributes plans require no such assumption. However, the quality characteristics of most raw materials used by industry and of most manufactured products seem to be distributed in forms that are close enough to the normal distribution for the practical use of variables plans.

(3) At the present time, attributes sampling is more widely known than variables sampling and therefore may require less training of inspectors.

The main advantages to the use of variables sampling plans are:

(1) A smaller sample is required for variables sampling than for attributes sampling to obtain the same discrimination

between good and bad lots. Or, conversely, decisions about a given lot are more reliable if based on a sample inspected by variables than if based on a sample of the same size inspected by attributes. Therefore, variables inspection is preferred if sample items are expensive and inspection is damaging or destructive.

(2) In using variables inspection, the data are accumulated for the most useful forms of control charts — control charts for the mean and control charts for the standard deviation. The receiver is able to give the supplier valuable information that may assist him in improving his manufacturing processes. One of the main purposes of acceptance inspection is to induce the supplier to improve the quality of his product when necessary. The data collected using variables inspection may be very useful in attaining this end.

(3) A minor advantage of variables inspection is that items of borderline quality for a particular characteristic present no problem to the inspector. The inspector's decision as to whether some particular borderline item is defective or not will often decide the disposition of the lot; hence, a decision is difficult to make. Under variables inspection, he has no such decision to make; he simply records some measurement.

It makes little difference in the disposition of the lot if his reading is just under the specification limit, at the limit, or just above it. For this reason, also, the inspector's personal bias through giving borderline items "the benefit of the doubt" is practically eliminated.

3. Mixed Variables and Attributes Sampling Plans

When using mixed variables and attributes sampling plans, a sample is drawn from the lot under consideration and is inspected using a variables criterion. If the lot is acceptable by the variables criterion, sampling is curtailed. If not, a second sample is drawn and is inspected by an attributes criterion.

Mixed variables and attributes plans can be very useful if there is a possibility that the assumption of normality of the distribution of quality characteristics will sometimes be violated. This assumption of normality is commonly violated in one of the following two ways: (1) when the population of quality characteristics is non-homogeneous or (2) when the inspection lot has been screened, resulting in a truncated distribution of quality characteristics.

In the first case, suppose we have a distribution of quality characteristics in which 95 percent of the lot is of acceptable quality, but five percent of the lot is of very bad quality. Let us further assume that a lot containing ten percent defectives is acceptable so that this lot is of acceptable quality. Using a pure variables sampling plan, if any of the extremely bad items is inspected, it is very likely that the lot will be rejected, because the sample mean will be greatly influenced by the extreme values of the quality characteristic. This lot would probably be accepted by a pure attributes sampling plan, but the average sample size would be much larger for the attributes plan than for a mixed variables and attributes plan. By a mixed variables and attributes plan, the lot would probably not be accepted by the variables portion of the plan, but it would have a very good chance of being accepted by the attributes portion of the plan.

Referring to the first case again, let us assume that the distribution of quality characteristics is non-homogeneous in such a way that 95 percent of the lot is of poor enough quality that it should be rejected, but that five percent of the lot is of excellent quality. In this case, it is quite possible that the sample mean will be influenced to such an extent by the extremely good items that the lot would be accepted, although it is of very poor quality. The lot would almost certainly be rejected by a pure attributes plan or by a mixed variables and attributes plan.

In the second case, if a lot has been screened by the producer and all the defective items have been removed, the assumption of normality of the distribution of quality characteristics is a false one. By a pure variables sampling plan, a screened lot will often be rejected if the sample observations are sufficiently close to the limit of the quality characteristics being measured. But a screened lot would never be rejected by an attributes sampling plan, because it contains no defective items. Likewise, it would never be rejected by a mixed variables and attributes sampling plan, because it would always be accepted by the attributes portion of the plan.

When the assumption of normally distributed quality characteristics is justified, a mixed variables and attributes plan requires less sampling on the average than a double sample, pure attributes plan, but more than a double sample, pure variables plan for the same degree of protection. But when the normal assumption is not justified, the mixed plan rejects fewer lots of acceptable quality and accepts fewer lots of bad quality than a pure variable plan does. Hence, a mixed plan is appropriate when the assumption of normality is usually justified, but may sometimes be violated.

4. Operating Characteristics Curves

The behavior of any particular sampling plan is described by its operating characteristic function, $L(p)$. This function gives the probability that the sampling plan will accept a submitted lot which contains a proportion p defective items. The graph of the operating characteristic function is called the operating characteristic curve (OC curve). The OC curve shows, for each possible percentage of defective items in a submitted lot, the probability of accepting a lot of that quality. The ideal sampling plan would have an OC curve like the dotted line in Figure 1. Given that the receiver is willing to accept lots containing a percentage of defective items p' or smaller and wants to reject all other lots, this plan will discriminate perfectly between good and bad lots. That is, it will accept all lots containing a percentage defective of p' or smaller and reject all lots with a percentage defective greater than p'. There is only one possible way to obtain this OC curve, and that is to do 100 percent inspection. However, this OC curve is closely approached when the sample size is close to the lot size. Such a close approach often requires an unjustifiable amount of inspection.

A more typical OC curve for sampling inspection plans is represented by the solid curved line in Figure 1. Sampling plans are commonly classified by one or two key points on their OC curves. One commonly used point is (AQL, $1-\alpha$), where AQL is the percentage defective called the "acceptable quality level" and α is

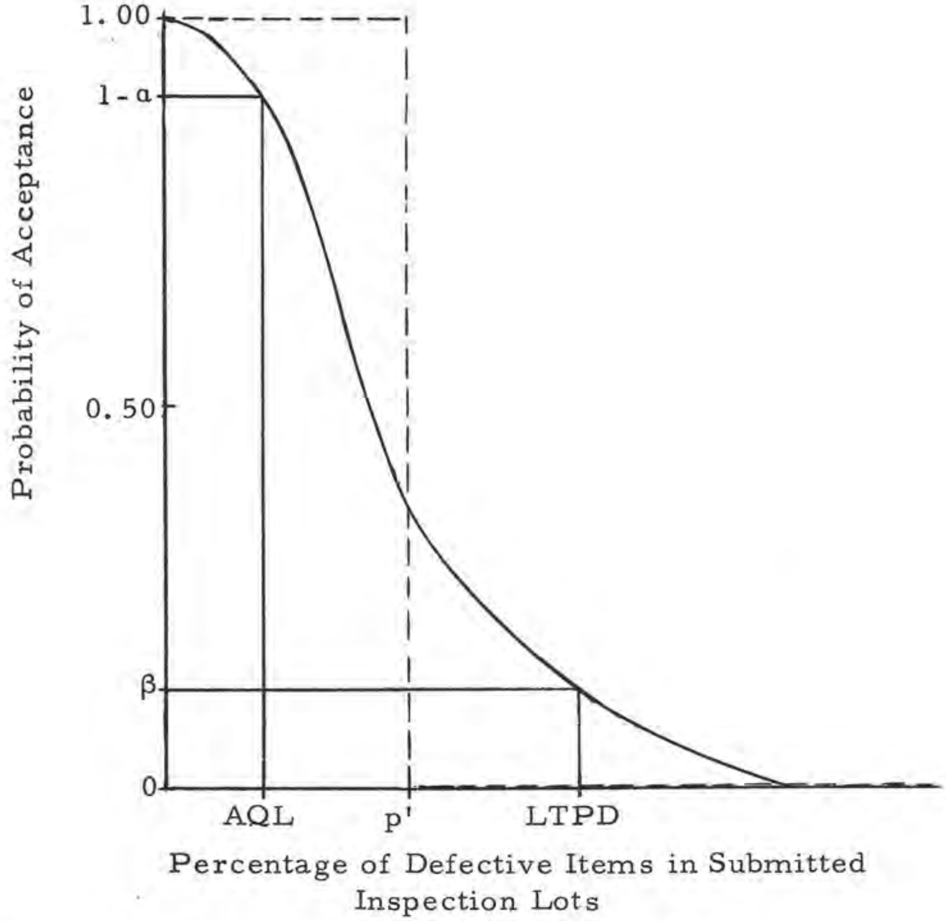

Figure 1. The Operating Characteristic Curve

called the "producer's risk". For an AQL specified by the sampling plan being used, there is a probability α that the sampling plan will reject the lot or a probability $1-\alpha$ that it will accept the lot. A commonly used value of α is 0.05.

Another point on the OC curve that is often used to index sampling plans is the point (LTPD, β). LTPD stands for "lot tolerance percent defective" and is the percentage of defective items in a submitted lot for which the chosen sampling plan will reject a lot of this quality with probability $1-\beta$ or accept it with probability β. A value commonly used for β is 0.10. β is called the "consumer's risk".

A third method of indexing sampling plans, that is commonly used, is by average outgoing quality limit, AOQL. The AOQL is defined as the maximum average proportion of defective items in a product that is finally accepted, if all rejected lots are screened and resubmitted. It is the maximum of the average outgoing quality, AOQ, of a product over all possible values of the proportion defective, p, where

$$\text{(3.1)} \qquad AOQ \approx pL(p) + 0[1-L(p)] = pL(p).$$

Hence, the AOQL is given by

$$\text{(3.2)} \qquad AOQL = \max_p AOQ \approx \max_p pL(p).$$

Using plans indexed on the AOQL, definite assurance can be obtained about the lowest quality of the accepted product, assurance that the average quality of the product finally accepted, regardless of the quality of the product originally submitted, can never be worse than a stated proportion defective, the AOQL.

The sampling plans contained in this paper are indexed on values of the LTPD from one percent through 40 percent.

5. Single, Double and Multiple Sampling Plans

Sampling plans can be based on single, double or multiple sampling. Using a single sampling plan, one sample is drawn from a submitted lot, and a decision is made from this sample either to accept it or reject it. Using a double sampling plan, a sample is drawn and a decision to accept the lot, reject the lot, or take a second sample is made. If a second sample is drawn, the lot is either accepted or rejected on the results of the second sample alone, or on the results of the first and second sample combined. Using a multiple sampling plan, this procedure is repeated until a decision to accept or to reject the lot can be reached.

In general, double sampling plans require a smaller average sample size than single sampling plans, and multiple sampling plans require a smaller average sample size than double sampling plans for the same degree of protection. For example, the average sample sizes for single, double, and multiple attributes sampling plans which have nearly identical OC curves are given in Table 1 for different lot qualities (8, p. 31). For these plans, the first sample was completely inspected.

Table 1. A Comparison of Average Sample Sizes for Single, Double, and Multiple Sampling Plans

Type of Sampling	Sample Size	Percentage of Defectives 1	3	5	8	11
Single	$n = 225$	225	225	225	225	225
Double	$n_1 = 150$	152	162	203	246	214
	$n_2 = 450$					
Multiple	$n_1 = 50$	58	84	151	142	86
	$n_2 = 100$					
	$n_3 = 150$					
	$n_4 = 200$					
	$n_5 = 250$					
	$n_6 = 300$					
	$n_7 = 350$					
	$n_8 = 400$					

It can be seen from the preceding table that, for lots of either high or low quality, double and multiple plans give significant savings in average sample size. For lots of intermediate quality, these savings are considerably reduced and, in the case of the double sampling plan in Table 1, a larger average sample size is actually required than for a single sampling plan, when the lot contains eight percent defective items.

The benefits one derives from using sampling plans requiring a larger and larger number of samples are subject to a law of diminishing returns, however. For, although the average sample size gets smaller the larger the number of samples the plan requires, the plan becomes very difficult to administer. In practice, therefore, single and double sampling plans are most commonly used.

CHAPTER 4

MIXED VARIABLES AND ATTRIBUTES DOUBLE SAMPLING PLANS

1. Introduction

There are two main types of mixed variables and attributes sampling plans: (1) Plans where we assume that the standard deviation of the population of quality characteristics being sampled is known, and (2) plans where this population standard deviation is estimated by an appropriate statistic. The sampling plans presented in this paper are of the former type, and they shall be referred to as "known sigma plans". Unknown standard deviation plans or "unknown sigma plans" will be discussed briefly in section three of this chapter.

2. Known Sigma Plans

2.1. Introduction

There seems to be a real need for known sigma plans. According to Bowker and Goode (2, p. 72):

> For many industrial products, the mean of the measurements of item characteristics changes from lot to lot while the dispersion of the measurements about the means do not change appreciably, so that the standard deviations remain practically constant. This

> occurs because, for many manufacturing processes, the relatively important or assignable causes of item variation that enter to produce noticeable changes affect only the average diameter, average length, the average hardness or the average of other properties. For example, this is commonly the effect of such assignable causes as the slippage of machine adjustments, incorrect tool or machine settings, or pronounced tool wear. The dispersions of measurements about their mean value, on the other hand, will be produced by the constantly present system of minor chance-acting causes inherent in the process — a system that will remain approximately the same regardless of any change in the mean and so will keep the standard deviation almost constant. Under such circumstances, the value for the standard deviation can be accurately estimated from sample data and used in quality control procedures.

There are two significant advantages to using known sigma plans. First, for the same degree of protection, a known sigma sampling plan requires smaller sample sizes than an unknown sigma plan. Secondly, the fact that the sample standard deviation or some other estimate of the population standard deviation does not have to be computed for each sample reduces the necessary clerical work considerably. Once the product standard deviation has been determined, the principal calculations remaining are the relatively simple ones for finding the mean of the measurements for each sample.

2.2 Independent Plans

Let the measurements $x_1, x_2, \ldots, x_{n_1}$ be a random sample of size n_1 from a normal population with unknown mean μ and known standard deviation σ . An item is considered defective if its measurement exceeds U, the upper limit of the quality characteristic. (We could just as easily have chosen to consider the case where there is a lower limit, L, for the quality characteristic being measured. This would result in only slight changes in the following derivations.) The lot from which the sample is drawn is accepted if $\bar{x} \leq U - k\sigma$, where $\bar{x}$ is the mean of the sample, σ is the known population standard deviation, and k is a constant taken from an appropriate table (2, p. 141). If $\bar{x} > U - k\sigma$, a second sample of maximum size n_2 is drawn. If d_2, the number of defectives in the second sample, is less than or equal to the acceptance number, a, of the sampling plan, the lot is accepted. If d_2 exceeds a, the lot is rejected, sampling being curtailed as soon as a is exceeded. This type of plan is called an independent plan, because only the results of the second sample are considered when applying the second (attributes) portion of the plan.

The probability of acceptance, L(p), of a submitted lot by this plan is given by:

$$
\begin{aligned}
(4.1) \qquad L(p) &= \Pr(\bar{x} \leq U - k\sigma \mid p) + \Pr(\bar{x} > U - k\sigma \mid p)\Pr(d_2 \leq a \mid p) \\
&= \Pr\left[z \leq \sqrt{n_1}(K_p - k)\right] \\
&\quad + \Pr\left[z > \sqrt{n_1}(K_p - k)\right] \sum_{i=0}^{a} \frac{\binom{Np - n_1 p}{i}\binom{Nq - n_1 q}{n_2 - i}}{\binom{N - n_1}{n_2}}
\end{aligned}
$$

where $z = \sqrt{n_1}\bar{x}$, p is the proportion of defective items in the lot, $q = 1 - p$, N is the lot size, and K_p is defined by

$$(4.2) \qquad \int_{K_p}^{\infty} \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \, dx = p.$$

Since L(p) is a function of p with $K_p = \frac{U - \mu}{\sigma}$, we may assume for purposes of computation that the observations are drawn from a normal population with zero mean and unit standard deviation. Hence, the z defined above is a standardized normal deviate, and the probabilities given by the first term and the first factor of the second term in equation (4.1) can be evaluated by looking them up in a table of the cumulative normal probability distribution, such as (13). The remaining factor of the second term can be found in a table of the hypergeometric probability distribution, (12). For large lot sizes, the binomial probability distribution can be used as an approximation to the hypergeometric probability distribution (7, p. 370).

In using an independent mixed variables and attributes sampling plan, information is lost by failing to take into account the first sample for attributes analysis. For this reason a sampling table , Table 2, was calculated for dependent plans rather than for independent plans.

2.3 Dependent Plans

2.3.1. General Procedure. The procedure to be followed in the use of dependent plans is as follows:

(a) A sample of n_1 items is selected at random from a lot of size N.

(b) The quality characteristic being inspected is measured and recorded for each item in the sample.

(c) The sample mean $\bar{x}$ is computed from the n_1 measurements.

(d) $U - k\sigma$ is computed, where U is the upper limit of the quality characteristic being inspected, k is the appropriate value selected from Table 2, and σ is the standard deviation of the quality characteristic of the product, which has been predetermined.

(e) If $\bar{x} \leq U - k\sigma$, the lot is accepted.

(f) If $\bar{x} > U - k\sigma$, the items of the first sample are inspected by an attributes criterion. This involves no reinspection, since those items which are found to have measurements exceeding U are classified as defective. If d_1, the number of defectives in the first sample, is greater than a, the acceptance number, the lot is rejected.

(g) If $\bar{x} > U - k\sigma$ and $d_1 \leq a$, a second sample of up to n_2 items is taken, sampling being curtailed and the lot rejected as soon as the total number of defectives in the first and second samples exceeds a.

(h) If the total number of defectives in the first and second samples, $d = d_1 + d_2$, is smaller than or equal to the acceptance number, a, the lot is accepted.

<u>2.3.2. Derivation of Equation for OC Curves.</u> As in the case of independent plans, since the probability of acceptance, L(p), is a function of p, and $K_p = \frac{U - \mu}{\sigma}$, we may assume for purposes of computation that the observations are drawn from a normal population with zero mean and unit variance. Then the probability of acceptance is given by:

$$(4.3) \qquad L(p) = Pr(\bar{x} \leq K_p - k) + Pr(\bar{x} > K_p - k,\ d \leq a)$$

$$= Pr[\sqrt{n_1}\,\bar{x} \leq \sqrt{n_1}(K_p - k)] + Pr(\bar{x} > K_p - k,\ d \leq a)$$

$$= Pr[z \leq \sqrt{n_1}(K_p - k)] + Pr(\bar{x} > K_p - k,\ d \leq a).$$

Since $z = \sqrt{n_1}\,\bar{x}$ is a standardized normal deviate, the probability given by the first term above may be evaluated by looking in a table of the cumulative normal probability distribution (13). The second term in equation (4.3) may be written as:

$$(4.4) \quad \sum_{i=0}^{a} \Pr(\bar{x} > K_p - k \mid d_1 = i) \Pr(d_1 = i) \Pr(d_2 \le a - i)$$

$$= \sum_{i=0}^{a} \Pr(\bar{x} > K_p - k \mid d_1 = i) \frac{\binom{Np}{i}\binom{Nq}{n_1 - i}}{\binom{N}{n_1}} \sum_{j=0}^{a-i} \frac{\binom{Np-i}{j}\binom{Nq-n_1+i}{n_2-j}}{\binom{N-n_1}{n_2}} .$$

The values of the hypergeometric probabilities above can be found in tables of the hypergeometric probability distribution (12). The conditional probabilities $\Pr(\bar{x} > K_p - k \mid d_1 = i)$ for $i=0, 1, 2, \ldots, a$ are much more difficult to compute. In the present paper we will consider only the case where $a = 0$. A discussion of the problems involved in evaluating these conditional probabilities for larger values of the acceptance number, a, can be found in Gregory and Resnikoff (9, p. 10).

Let us consider the distribution of x, a random variable from a normal population with zero mean and unit variance truncated from above at K_p, where K_p is defined by equation (4.2).

The probability density function of x is:

$$(4.5) \quad f_1(x) = \begin{cases} \frac{1}{q}\phi(x) & \text{for } x \le K_p \\ 0 & \text{for } x > K_p \end{cases}$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$ is the ordinate of the normal distribution. Denoting the k^{th} moment of this density function by a_k, we have

$$(4.6) \qquad a_k = \int_{-\infty}^{\infty} x^k f_1(x)dx = \frac{1}{q}\int_{-\infty}^{K_p} x^k \phi(x)dx.$$

In particular, $a_0 = 1$, $a_1 = -\frac{1}{q}\phi(K_p)$, and $a_2 = 1 + a_1 K_p$. Now let us consider the distribution of the random variable

$$(4.7) \qquad z = \frac{\bar{x} - a_1}{\sigma / \sqrt{n_1}}$$

where $\bar{x}$ is the mean of a sample of n_1 observations from the density function (4.5). According to the central limit theorem (7, p. 121), the distribution of z approaches the normal distribution with zero mean and unit standard deviation as $n_1 \longrightarrow \infty$. That is, z is asymptotically normally distributed with zero mean and unit standard deviation. Hence, the probability density function of z, $f_{n_1}(z)$, can be represented by the following asymptotic expansion, called an Edgeworth series (3, p. 228):

$$(4.8) \quad f_{n_1}(z) = \phi(z) - \frac{\gamma_1}{3!\sqrt{n_1}}\phi^{(3)}(z) + \frac{1}{n_1}\left[\frac{\gamma_2}{4!}\phi^{(4)}(z) + \frac{10\gamma_1^2}{6!}\phi^{(6)}(z)\right] + O\left(\frac{1}{n_1^{3/2}}\right)$$

where $O\left(\frac{1}{n_1^{3/2}}\right)$ means that the neglected terms of this expansion are of order $n_1^{-3/2}$. The symbols γ_1 and γ_2 above are the coefficients of skewness and excess, respectively, given by

$$(4.9) \qquad \gamma_1 = \frac{\mu_3}{(\mu_2)^{3/2}} \quad \text{and} \quad \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3,$$

where the central moments, μ_k, of $f_1(x)$ are given by

$$(4.10) \qquad \mu_k = \int_{-\infty}^{K_p} (x-a_1)^k f_1(x)dx.$$

The symbol $\phi^{(k)}(z)$ represents the k^{th} derivative of the normal density function. Therefore, letting

$$(4.11) \qquad z_0 = \frac{K_p - k - a_1}{\sigma_1/\sqrt{n_1}} \quad ,$$

where $\sigma_1 = \sqrt{\mu_2}$ is the standard deviation of $f_1(x)$, we have

$$(4.12) \qquad \Pr(\bar{x} > K_p - k \mid d_1 = 0) = \int_{\bar{x} = K_p - k}^{\infty} f_{n_1}(z)dz = \int_{z_0}^{\infty} f_{n_1}(z)dz$$

$$= \int_{z_0}^{\infty} \phi(x)dx + \frac{\gamma_1}{6\sqrt{n_1}} \phi^{(2)}(z_0) - \frac{\gamma_2}{24n_1} \phi^{(3)}(z_0)$$

$$- \frac{\gamma_1^2}{72n_1} \phi^{(5)}(z_0) + O\left(\frac{1}{n_1^{3/2}}\right) \quad .$$

Hence we obtain the approximation,

$$(4.13) \quad \Pr(\bar{x} > K_p - k \mid d_1 = 0) \approx 1 - F(z_0) + \frac{\gamma_1}{6\sqrt{n_1}} \phi^{(2)}(z_0)$$

$$- \frac{\gamma_2}{24n_1} \phi^{(3)}(z_0) - \frac{\gamma_1^2}{72n_1} \phi^{(5)}(z_0) \ ,$$

where $F(z_0) = \int_{-\infty}^{z_0} \phi(x)dx$ is the cumulative normal distribution function.

2.3.3. Calculation of k-Values and OC Curves. For the case under consideration in this paper — where the acceptance number, a, for the attributes portion of the sampling plan is zero— equations (4.3) and (4.4) are somewhat simplified, and we have

$$(4.14) \qquad L(p) = F[\sqrt{n_1}(K_p - k)] + Pr(d_1 = 0,\ d_2 = 0) Pr(\bar{x} > K_p - k \mid d_1 = 0).$$

For computation purposes, let

$$(4.15) \qquad p_1 = F[\sqrt{n_1}(K_p - k)],$$

where $F(x)$ is the cumulative normal distribution function defined above,

$$(4.16) \qquad p_2 = Pr(d_1 = 0,\ d_2 = 0) = \frac{\binom{Nq}{n_1}}{\binom{N}{n_1}} \; \frac{\binom{Nq-n_1}{n_2}}{\binom{N-n_1}{n_2}} ,$$

and

$$(4.17) \qquad p_3 = Pr(\bar{x} > K_p - k \mid d_1 = 0),$$

which is approximated by (4.13) above. Then the equation for the OC function is given by

$$(4.18) \qquad L(p) = p_1 + p_2 p_3 .$$

The value of p_1 can be determined by looking in a table of the cumulative normal distribution function (13), p_2 can be found in tables of the hypergeometric probability distribution (12), and p_3 can be evaluated using equation (4.13). The constants K_p, γ_1, γ_2, a_1 and σ_1 can be found in Tables IV and V of Gregory and Resnikoff (9). The derivatives of the normal density function can be found in (11).

For large lot sizes, N, the hypergeometric distribution can be approximated by the binomial distribution (7, p. 370). This approximation is necessary because the hypergeometric distribution is not well tabulated for large values of N. When the binomial approximation is used, equation (3.16) becomes

$$(4.19) \quad p_2 = \Pr(d_1=0,\ d_2=0) \cong q^{n_1} q^{n_2} = q^{n_1+n_2} .$$

To calculate the k-values of Table 2, indexed on the LTPD levels indicated, equation (4.18) was solved by an iterative procedure for the k-value which gave an $L(p) = 0.100$. This calculation was done on an IBM 7090 computer, while the author was with the Boeing Company in the summer of 1962. The program for this calculation was written by Mr. John Elliott. For a sample calculation of $L(p)$, see section 2 of chapter 5. Chapter 5 also contains the OC curves for five representative sampling plans from Table 2.

3. Unknown Sigma Plans

Gregory and Resnikoff (9, p. 20-24) discuss the cases where the standard deviation of the quality characteristic being inspected is estimated by the sample standard deviation,

$$(4.20) \quad s = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (x_i - \overline{x})^2} \ ,$$

and by the extreme deviation from the mean,

$$(4.21) \quad v = x_{(n)} - \overline{x} \ .$$

In the above two equations, $\overline{x}$ is the mean of a sample of size n and $x_{(n)}$ is the largest observation in the sample.

In the former case, no practical way has yet been devised to calculate the OC curves for such plans. In the latter case, ease of computation is limited to the case where the acceptance number, a, is zero for the attributes portion of the plan.

CHAPTER 5

RESULTS

1. Sampling Plans and OC Curves

A table of mixed variables and attributes sampling plans is presented in Table 2, and five representative OC curves are given in Figures 2, 3, 4, and 5. The table is indexed horizontally by LTPD level and vertically by inspection lot size. For a specified LTPD and lot size, the table gives the first sample size, second sample size, and k-value. For example, if we have a lot containing 300 items, and we want a LTPD of five percent, our first sample should contain eight items, and our second sample should contain 34 items. The k-value for the required plan in 2.523.

The OC curves presented are for sampling plans (8, 1), (5, 5), (2, 7), (1, 10), and (10, 10), where sampling plan (i, j) is the plan given by the i^{th} row and j^{th} column of Table 2. The OC curves for plans (1, 10) and (10, 10) are plotted on the same graph to show the similarity of OC curves within the same LTPD level. The AOQL is easily calculated for each of five plans for which OC curves are given, using equations (3.1) and (3.2). The AOQL's are given on their respective OC curves.

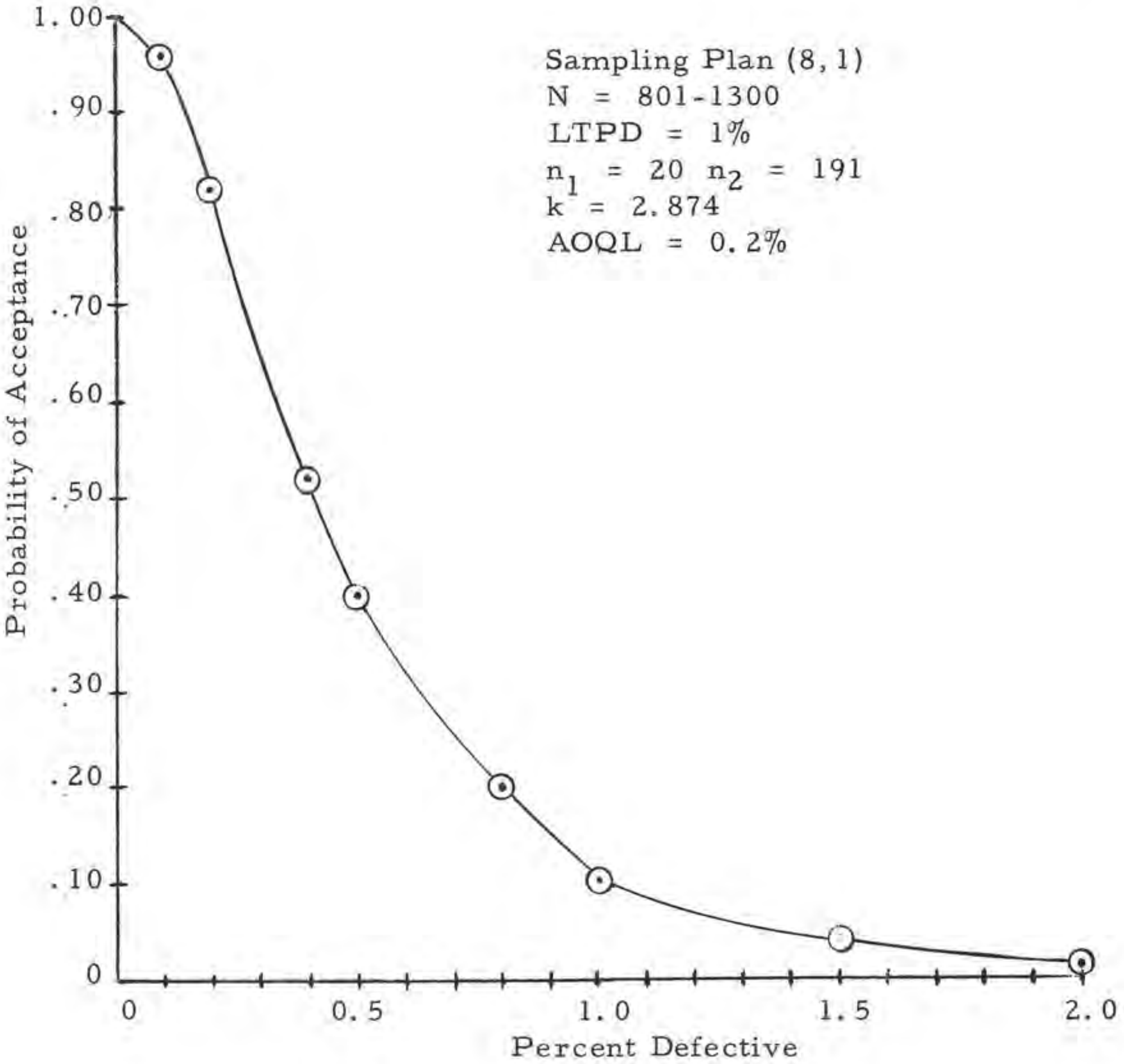

Figure 2. Operating Characteristic Curve

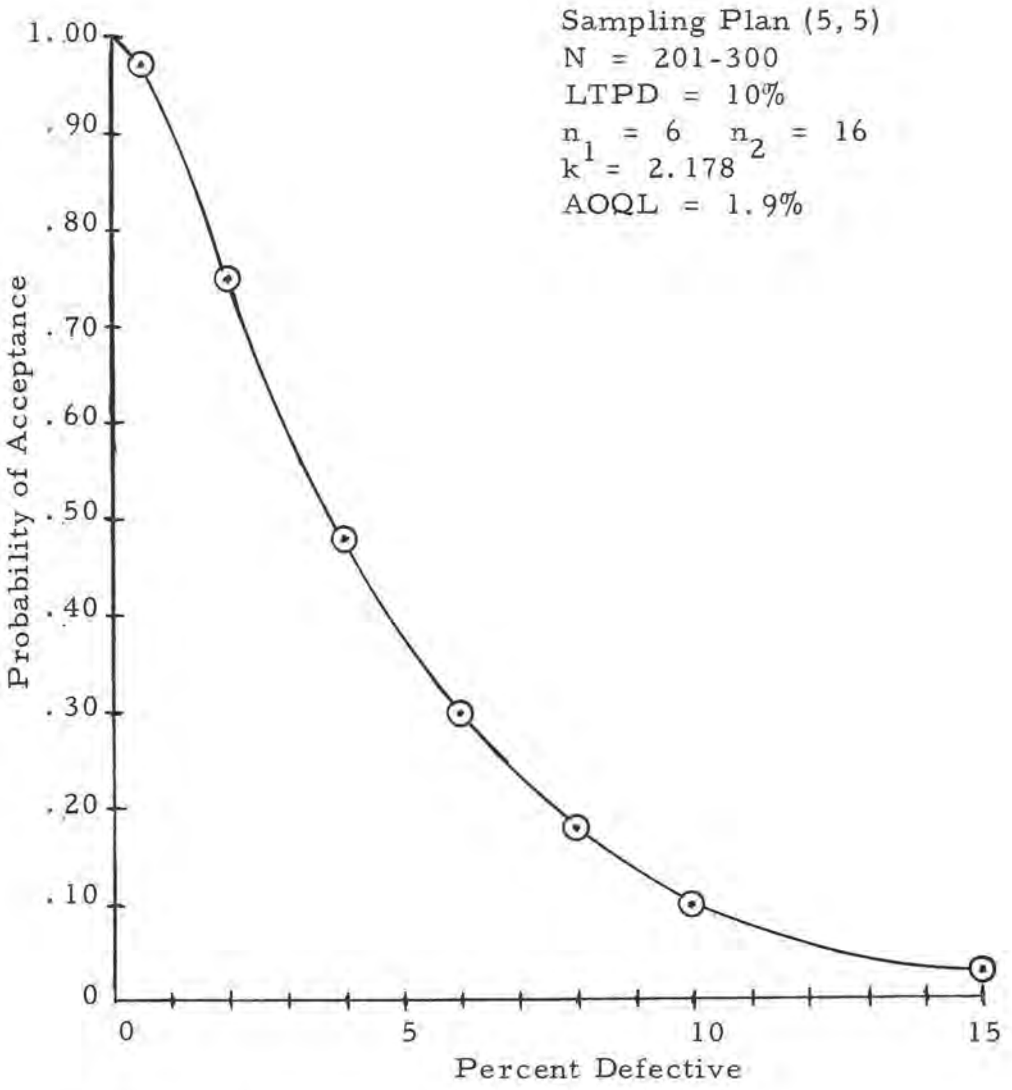

Figure 3. Operating Characteristic Curve

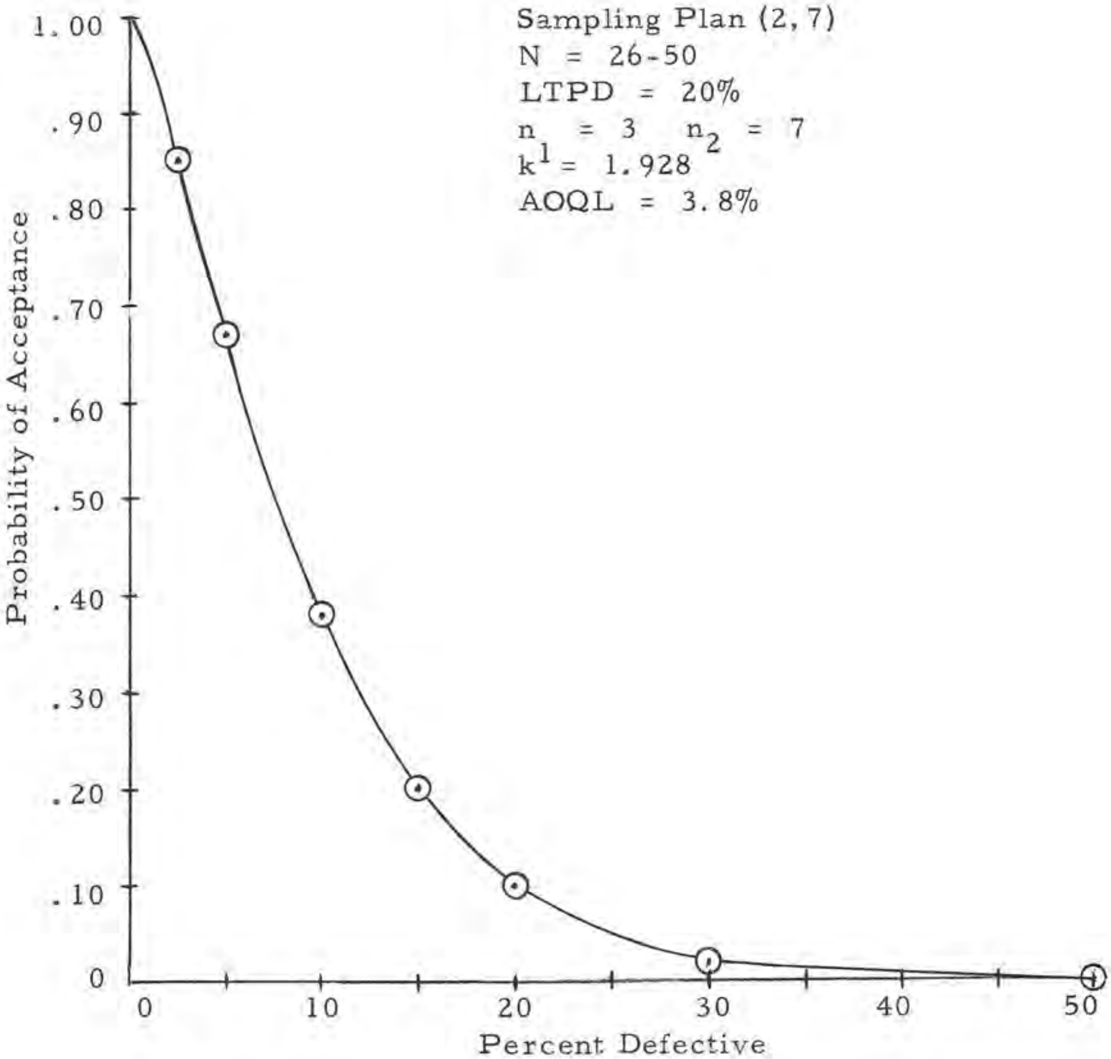

Figure 4. Operating Characteristic Curve

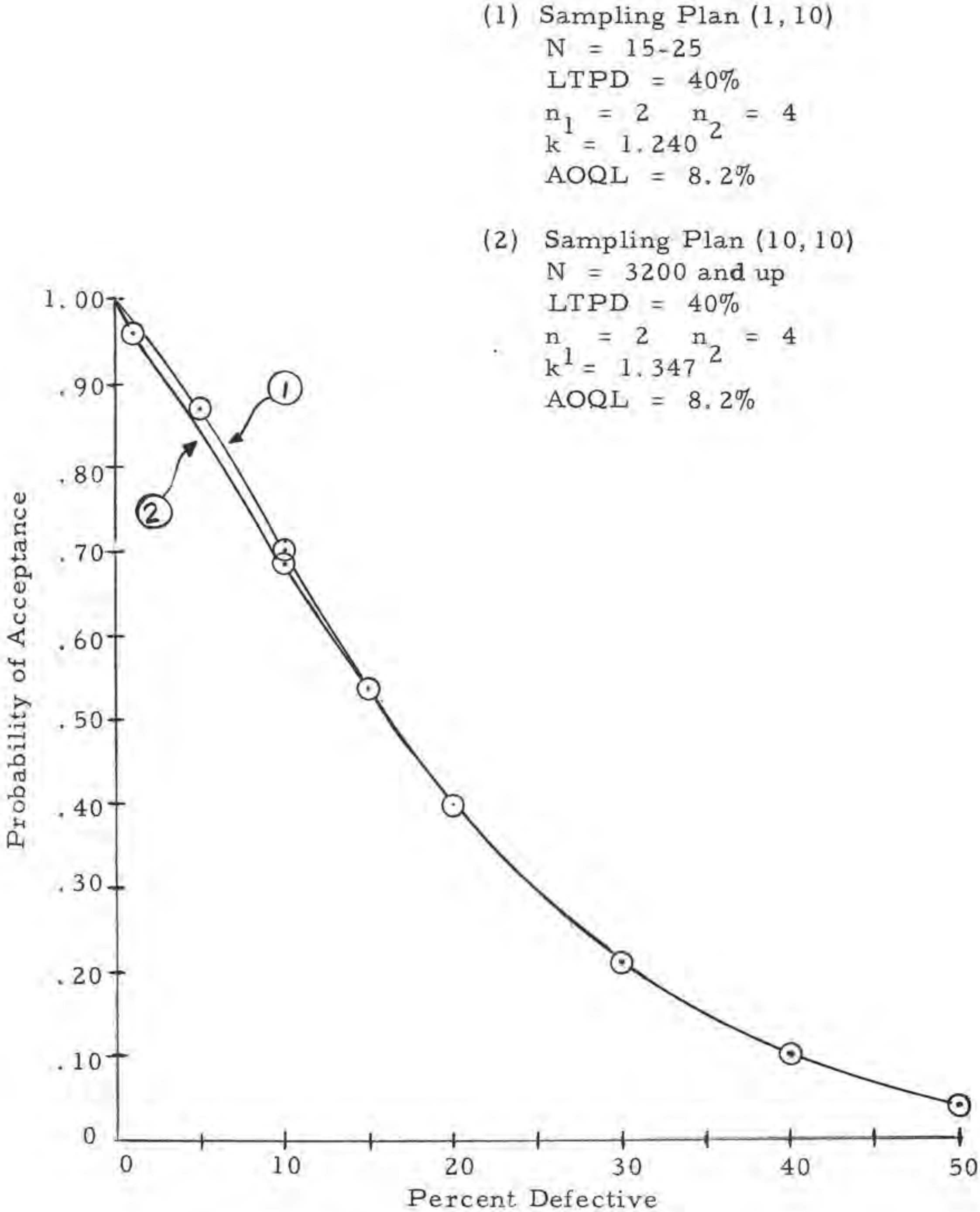

Figure 5. Operating Characteristic Curves

Table 2. Variables - Attributes Double Sampling Plan (Known Sigma)

KEY: (Factors for a Probability of Acceptance of 10% at the Quality Listed)

1st Sample	2nd Sample	
k value		Note: Acceptance Number for 2nd Sample is zero

LTPD LEVELS

LOT SIZE	.010	.025	.050	.075	.100	.150	.200	.250	.300	.400
15-25			8 | 11	7 | 10	6 | 10	4 | 7	3 | 6	2 | 5	2 | 4	2 | 4
			2,210	2.050	1,877	1,973	1,875	2,128	1,890	1,240
26-50		8 | 29	8 | 25	7 | 14	6 | 12	4 | 9	3 | 7	2 | 6	2 | 5	2 | 4
		2,481	2,132	2,328	2,063	1,981	1,928	2,015	1,740	1,286
51-100	10 | 64	8 | 60	8 | 29	7 | 19	6 | 14	4 | 10	3 | 7	2 | 6	2 | 5	2 | 4
	2,819	2,430	2,308	2,215	2,150	2,032	2,163	2,212	1,831	1,313
101-200	10 | 127	8 | 66	8 | 32	7 | 21	6 | 15	4 | 10	3 | 8	2 | 7	2 | 5	2 | 4
	2,770	2,641	2,474	2,282	2,237	2,196	1,964	1,953	1,889	1,330
201-300	10 | 151	10 | 69	8 | 34	7 | 22	6 | 16	4 | 11	3 | 8	2 | 7	2 | 5	2 | 4
	2,932	2,684	2,523	2,299	2,178	2,037	2,000	1,978	1,916	1,337
301-500	10 | 174	10 | 74	8 | 35	7 | 22	6 | 16	4 | 11	3 | 8	2 | 7	2 | 5	2 | 4
	2.991	2,704	2,591	2,404	2,238	2,065	2,023	1,992	1,932	1,340
501-800	10 | 190	10 | 76	8 | 36	7 | 23	6 | 16	4 | 12	3 | 8	2 | 7	2 | 5	2 | 4
	3,054	2,803	2,580	2,312	2,292	1,960	2,039	2,002	1,942	1,342
801-1300	20 | 191	10 | 80	8 | 37	7 | 23	6 | 17	4 | 12	3 | 8	2 | 7	2 | 5	2 | 4
	2,874	2,711	2,523	2,347	2,153	1,965	2,049	2,008	1,949	1,345
1301-3200	20 | 200	10 | 85	8 | 38	7 | 24	6 | 17	4 | 12	3 | 8	2 | 7	2 | 5	2 | 4
	2,945	2,636	2,479	2,265	2,168	1,972	2,058	2,013	1,955	1,346
3200 & Up	20 | 210	10 | 90	8 | 39	7 | 24	6 | 18	4 | 12	3 | 8	2 | 7	2 | 5	2 | 4
	2,882	2,581	2,432	2,276	2,087	1,975	2,063	2,016	1,958	1,347

2. Sample Calculation of L(p)

Let us consider the calculation of L(p) for the case of an inspection lot of size 20, containing five percent defective items with a LTPD of 40 percent. This corresponds to sampling plan (1, 10) and the L(p) is given by equation (4. 18) as

$$L(p) = p_1 + p_2 p_3$$

where $p_1 = F[\sqrt{n_1}(K_p - k)]$,

$$p_2 = \frac{\binom{Nq}{n_1}\binom{Nq-n_1}{n_2}}{\binom{N}{n_1}\binom{N-n_1}{n_2}},$$

and
$$p_3 \approx 1 - F(z_0) + \frac{\gamma_1}{6\sqrt{n_1}}\phi^{(2)}(z_0) - \frac{\gamma_2}{24\,n_1}\phi^{(3)}(z_0) - \frac{\gamma_1^2}{72\,n_1}\phi^{(5)}(z_0).$$

The value of z_0 is calculated by

$$z_0 = \sqrt{n_1}\left(\frac{K_p - k - a_1}{\sigma_1}\right).$$

For sampling plan (1, 10),

$N = 20$ $\qquad k = 1.240$

$n_1 = 2$ $\qquad \sqrt{n_1} = 1.414214$

$n_2 = 4$

In addition, for the case we are considering,

$$p = 0.05 \qquad \begin{aligned} K_p &= 1.644854 \\ \gamma_1 &= -0.337588 \\ \gamma_2 &= -0.227324 \\ \gamma_1^2 &= 0.113966 \end{aligned} \qquad \begin{aligned} a_1 &= -0.108564 \\ \sigma_1 &= 0.899801 \end{aligned}$$

where K_p, γ_1, γ_2, a_1, and σ_1 were taken from Table IV and Table V of Gregory and Resnikoff (9). Then,

$$\sqrt{n_1}\,(K_p - k) = 0.572550$$

$$p_1 = F(.5726) = 0.716542$$

$$p_2 = \frac{\binom{19}{2}}{\binom{20}{2}} \cdot \frac{\binom{17}{4}}{\binom{18}{4}} = 0.700000$$

$$z_0 = 0.806938$$

$$p_3 \approx 1 - F(.8069) + \frac{-0.337588}{6(1.414214)}\,\phi^{(2)}(.807) - \frac{-0.227324}{24(2)}\,\phi^{(3)}(.807)$$

$$- \frac{.113966}{72(2)}\,\phi^{(5)}(.807) \approx 0.218059$$

Finally,

$$L(p) = 0.716542 + (0.700000)(.218059) = 0.869183$$

According to Cramér (3, p. 229), the error in the above Edgeworth series approximation to p_3 is of the same order of magnitude as the first neglected term. The first neglected term is

$$(5.1)\quad E = \frac{1}{n_1^{3/2}}\left[\frac{\gamma_3}{120}\,\phi^{(4)}(z_0) + \frac{\gamma_1\gamma_2}{144}\,\phi^{(6)}(z_0) + \frac{\gamma_1^3}{1296}\,\phi^{(8)}(z_0)\right],$$

where $\gamma_3 = \frac{\mu_5}{(\mu_2)^{5/2}} - 10\gamma_1$ (9, p. 17).

For the preceding approximation to p_3, $E = 0.000407$. Since p_3 is multiplied by p_2 in the calculation of the $L(p)$, this means the error in $L(p)$ is of the order of $0.7(0.000407) = .000285$. This error is of no consequence in the calculation of OC curves and k-values. For most calculations of $L(p)$, the error would be smaller than this, since n_1 is always greater than or equal to two and p_2 is usually smaller than 0.7.

To construct the OC curves in Figures 2, 3, 4, and 5, $L(p)$ was calculated for seven or eight values of p, for each of the five sampling plans represented, and a smooth curve was drawn through the points thus obtained.

To calculate the k-value for a given LTPD level and lot size and somewhat arbitrarily chosen first and second sample sizes, different k-values were tried in equation (4.18) by an iterative procedure until one was found which gave a $L(p)$ of 0.100. This computation was done on an IBM 7090 computer.

3. Example of the Use of the Mixed Variables and Attributes Sampling Plans

Let us consider an example illustrating how a mixed variables and attributes plan from Table 2 on page 40 might be used. Suppose we have received a lot containing 75 steel castings which have a specified minimum yield point of 53, 000 p. s. i. The standard deviation σ is known to be 2000 p. s. i. We want to use a mixed variables and attributes plan to determine whether or not we should accept this lot if we specify a LTPD of 25 percent.

From Table 2 we look for the entries corresponding to a lot size of 75 items and a LTPD level of 25 percent. We find that if we take a first sample of two items and a second sample of six items (if necessary), the k-value is 2. 212. This example is a case of a single lower limit L, rather than a single upper limit for which the OC equation was derived. However, the derivation goes through the same for this case, except for slight modifications, and the results are identical for both cases. In this case, we will accept the lot on the first sample if $\bar{x} \geq L + k\sigma$ and take a second sample if $\bar{x} < L + k\sigma$ and there are no defective items in the first sample. We calculate

$$L + k\sigma = 53,000 + 2.212\ (2,000) = 57,424.$$

Let us suppose that we draw a sample of two items from the lot and find

$$x_1 = 55,496 \ , \quad x_2 = 53,052$$

We calculate $\overline{x} = 54,274$ and, since neither of these items is defective, we take a second sample of six items. Suppose we find that

$$\begin{array}{lll} x_3 = 56,491 & x_5 = 53,789 & x_7 = 54,032 \\ x_4 = 59,907 & x_6 = 54,476 & x_8 = 55,091 . \end{array}$$

Since none of these items is defective, we will accept the lot.

Let us suppose that the preceeding lot had been screened, unknown to us, and that we had decided to use a single-sample variables acceptance plan. We can find such a plan in Bowker and Goode (2, p. 199). Using this plan, we find that the k-value is 1.177 and the sample size is seven. We will accept the lot if $\overline{x} \geq L + k\sigma$, and reject it if $\overline{x} < L + k\sigma$. We find that

$$L + k\sigma = 53,000 + 1.177(2,000) = 55,354,$$

and from the first seven sampled items above, $\overline{x} = 55,320$. Therefore, we would erroneously reject the lot using a pure variables plan, although we found no defective items in the sample.

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