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Abstract Approved Signature redacted for privacy.

Dr. Harry E. Goheen

Let [a,b] be a closed bounded interval of the real line, E^q the q dimensional Euclidean space, and $\gamma : [a,b] \to E^q$ a continuous curve. If $t_0 \in [a,b]$, let $t_0^+ = \sup\{t \in [a,b] : t \geq t_0, \gamma([t_0,t]) = \gamma(t_0)\}$, and $t_0^- = \inf\{t \in [a,b] : t \leq t_0, \gamma([t,t_0]) = \gamma(t_0)\}$.

Let P be a polygon inscribed in γ corresponding to a partition $Q : a = t_0 < t_1 < \ldots < t_n = b$ of [a,b]. Let $\theta(t_{i-1}t_i, t_it_{i+1})$ = the smallest non-negative angle between the directions $\gamma(t_i) - \gamma(t_{i-1})$ and $\gamma(t_{i+1}) - \gamma(t_i)$, with the proviso that $\theta(t_{i-1}t_i, t_it_{i+1}) = 0$ whenever $\gamma(t_{i-1}) = \gamma(t_i)$ or $\gamma(t_i) = \gamma(t_{i+1})$. Let $\kappa(P) = \max\{\theta(t_{i-1}t_i, t_it_{i+1}), i = 1,2,\ldots,n-1\}$, and $\mu(Q) = \max\{t_i^- - t_{i-1}^+, i = 1,2,\ldots n\}$.

A curve γ can be smoothly approximated by inscribed polygons if for every ϵ in the open interval $(0, \frac{\pi}{2})$ there exists a $\delta > 0$ such that $\mu(Q) < \delta$ implies $\kappa(P) < \epsilon$.

The following approximation theorem is the main result in this dissertation.

Approximation Theorem. A curve is regular if and only if it can be smoothly approximated by inscribed polygons.

Besides this approximation theorem, the dissertation also contains research work on integral geometry. The measure of r-planes that intersect a convex set C in E^q can be expressed in the form $\lambda_r W_r(C)$ where λ_r is a constant depending on r, and $W_r(C)$ is the rth quermassintegral of C. It is shown that it is possible to extend this result to a non-convex set T of the type $T = \cup C_i$, where the C_i are convex sets such that $C_i \cap C_j = \emptyset$ for $|i-j| > 1$, provided $W_r(C_i)$ and $W_r(C_i \cap C_{i+1})$ are known. By taking C_i as convex tubes, the calculations for the quermassintegrals are carried out for the case q = 3.

SMOOTH APPROXIMATIONS AND QUERMASSINTEGRALS

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CHAPTER I

Introduction

The fact that a continuous curve $\gamma : [a,b] \to E^q$ can be uniformly approximated by inscribed polygons has played a basic role in mathematics. For example, it underlines the notion of rectifiability and the definition of length. Moreover, in the context of numerical analysis a curve is generally approximated by a finite sequence of points, which is in effect a polygon. Now the following question is a natural one: if γ is not only continuous, but also smooth in some sense, and if one considers a sequence of inscribed polygons with mesh tending to zero, how does the smoothness of γ reflect itself in the behavior of the polygonal sequence? Is there some property of the sequence iteself - which a computer can see - which is characteristic of the fact that the limit curve is smooth? The main result of this dissertation constitutes an answer to this question. If one takes smoothness to mean regularity, the desired property of the approximating polygons turns out to be the following: given $\varepsilon > 0$ there exists a $\delta > 0$ such that, for every partition of [a,b] with mesh less than δ, the angle between successive edges of the corresponding inscribed polygon shall be less than ε. Only one should add that in as much as γ need not be locally one-to-one, the mesh of partitions must be defined in an appropriate way. This, then, is the new concept to which one is led, the notion of "smooth approximation by inscribed polygons." Precise definitions and a proof of the Approximation Theorem will be given in Chapter II.

Though this is undoubtedly our main result, it is not one which we had envisaged from the start. As it turns out, we began our research by investigating two problems pertaining respectively to differential and integral geometry of convex sets. The first dealt with the approximation of q-convex curves by inscribed polygons. In particular, we were interested to find out whether q-convexity was sufficient to guarantee certain good properties of the approximating polygons. Eventually, however, it became clear that q-convexity had little to do with the questions at hand, and this recognition prompted us to consider the problem of which we have spoken before.

As concerns the integral geometry of convex sets, on the other hand, our work has led to a number of results, which will be set forth in chapters III and IV. It is well-known that the measure of r-planes intersecting a convex set C in E^q can be expressed in the form $\lambda_r W_r(C)$ where λ_r is a constant depending on r and $W_r(C)$ is the so-called rth quermassintegral of C. The question arises: can this formula be extended to the case of non-convex sets? We show that this can be done for sets of the form $T = \cup C_i$, where the C_i are convex sets and $C_i \cap C_j = \emptyset$ for $|i-j| > 1$. This in turn leads to the problem of evaluating quermassintegrals for the intersection of convex sets, which in general is a rather difficult matter. We have succeeded in carrying out the calculations for the cases $r = 1, 2$ and $q = 3$ by taking C_i as convex tubes.

CHAPTER II

SMOOTH APPROXIMATION OF SPACE CURVES BY INSCRIBED POLYGONS

Definitions

Let [a,b] be a closed bounded interval of the real line, E^q the q dimensional Euclidean space, and S^{q-1} the q-1 dimensional unit sphere.

(A) A <u>curve</u> is a continuous map $\gamma : [a,b] \to E^q$.

(B) A <u>polygon</u> P (in E^q) is a sequence $a_0, a_1, \ldots, a_n$ of points in E^q such that $a_i \neq a_{i+1}$. The points are called <u>vertices</u> and the line segment joining the points a_{i-1}, a_i is called an <u>edge</u> of P. P is <u>inscribed</u> in γ if there exists a partition $a = t_0 < t_1 < \ldots < t_n = b$ such that $a_i = \gamma(t_i)$.

(C) For every t_0, $t_0' \in [a,b]$, $t_0 \neq t_0'$, let $x_0 = \gamma(t_0)$, $x_0' = \gamma(t_0')$. If $x_0 \neq x_0'$, let $u[t_0, t_0']$ denote the unit vector in the direction from x_0 to x_0'.

Let $t_0 \in [a,b]$, and define

$t_0^+ = \sup\{ t \in [a,b] : t \geq t_0 , \gamma([t_0, t]) = \gamma(t_0) = x_0 \}$,

$t_0^- = \inf\{ t \in [a,b] : t \leq t_0 , \gamma([t, t_0]) = \gamma(t_0) = x_0 \}$.

Then $a \leq t_0^- \leq t_0 \leq t_0^+ \leq b$.

(i) Suppose $t_0^+ < b$. A unit vector u_r is a <u>right tangent</u> to γ at t_0 if, for every sequence $t_n \in [a,b]$ such that $t_n > t_0^+$, $\gamma(t_n) \neq x_0$, and $\lim_{n \to \infty} t_n = t_0^+$, $\lim_{n \to \infty} u[t_0, t_n] = u_r$.

If $t_0^+ = b$, then γ has no right tangent at t_0.

(b) Suppose $t_0^- > a$. A unit vector u_l is a <u>left tangent</u> to γ at t_0 if, for every sequence $t_n \epsilon [a,b]$ such that $t_n < t_0^-$, $\gamma(t_n) \neq x_0$, and $\lim_{n \to \infty} t_n = t_0^-$, $\lim_{n \to \infty} u[t_0, t_n] = - u_l$.

If $t_0^- = a$, then γ has no left tangent at t_0.

Note: If u_r or u_l exists, it is unique.

(iii) Suppose $a < t_0^- \leqslant t_0 \leqslant t_0^+ < b$. A unit vector u is said to be a <u>tangent</u> to γ at t_0 if and only if

(a) the right tangent u_r exists at t_0,

(b) the left tangent u_l exists at t_0, and

(c) $u = u_r = u_l$.

Suppose γ has a unit tangent $u(t)$ at every $t \epsilon [a,b]$. We define a map $\tilde{\gamma} : [a,b] \to S^{q-1}$ by setting $\tilde{\gamma}(t) = u(t)$. We say γ admits <u>continuous tangents</u> if $\tilde{\gamma}$ is continuous.

(D) Let $\theta(t_1 t_2, t_3 t_4)$ = the smallest non-negative angle between $u[t_1, t_2]$ and $u[t_3, t_4]$,

$\theta(t_1 t_2, t_3 t_4) = 0$ if $\gamma(t_1) = \gamma(t_2)$ or $\gamma(t_3) = \gamma(t_4)$,

$\theta(t_1, t_2 t_3) = \theta(t_2 t_3, t_1)$ = the smallest non-negative angle between $u(t_1)$ and $u[t_2, t_3]$,

$\theta(t_1, t_2 t_3) = \theta(t_2 t_3, t_1) = 0$ if $\gamma(t_2) = \gamma(t_3)$,

$\theta(t_1, t_2)$ = the smallest non-negative angle between $u(t_1)$ and $u(t_2)$.

If $Q: a = t_0 < t_1 < \dots < t_n = b$ is a partition of the interval $[a,b]$, and P the corresponding polygon inscribed in γ, let

$$\kappa(P) = \max \{ \theta(t_{i-1}t_i, t_i t_{i+1}), i = 1,2,\dots,n-1 \} ,$$

and $\mu(Q) = \max \{ t_i^- - t_{i-1}^+ , i = 1,2,\dots,n \}$.

A curve γ can be <u>smoothly approximated</u> by inscribed polygons if for every ε in the open interval $(0, \frac{\pi}{2})$ there exists a $\delta > 0$ such that $\mu(Q) < \delta$ implies $\kappa(P) < \varepsilon$.

(E) A rectifiable curve is <u>regular</u> if it is C^1 with respect to arc length parameterization.

Note: Usually a regular curve is defined (for instance, see [1], page 6) as follows: A C^1 curve $\gamma : [a,b] \to E^q$ is regular if $\gamma'(t) \neq 0$ for all $t \in [a,b]$. Our definition concurs with this definition because any C^1 curve $\gamma : [a,b] \to E^q$ with $\gamma'(t) \neq 0$ is rectifiable, and C^1 with respect to arc length. However, unlike the usual definition, our definition is independent of parameterization. We give an example of a curve which is regular, but not regular with respect to a given parameterization.

<u>Example</u> Define $\gamma: [-2, 1] \to E^2$ by

$$\gamma(t) = (- t^2, t^3), -2 \leq t \leq 0,$$
$$= (t^2, t^3), 0 \leq t \leq 1.$$

Here $\gamma'(0) = 0$, and so γ is not regular in the usual sense. Let

$$s(t) = \int_{-2}^{t} ||\gamma'(\tau)|| \, d\tau = \int_{-2}^{t} \sqrt{4\tau^2 + 9\tau^4} \, d\tau .$$

Solving this equation for t in terms of s, and setting

$\gamma(t(s)) = \hat{\gamma}(s)$, we find that

$$\hat{\gamma}(s) = \left(- \frac{((40)^{3/2} - 27s)^{2/3} - 4}{9}, \quad - \left(\frac{((40)^{3/2} - 27s)^{2/3} - 4}{9} \right)^{3/2} \right),$$

$$0 \leq s \leq \frac{(40)^{3/2} - 8}{27},$$

$$= \left(\frac{(27s - (40)^{3/2} + 16)^{2/3} - 4}{9}, \quad \left(\frac{(27s - (40)^{3/2} + 16)^{2/3} - 4}{9} \right)^{3/2} \right),$$

$$\frac{(40)^{3/2} - 8}{27} \leq s \leq \frac{(13)^{3/2} + (40)^{3/2} - 16}{27}.$$

$\hat{\gamma}$ is C^1 with respect to s, and so γ is regular.

Approximation Theorem. A curve is regular if and only if it can be smoothly approximated by inscribed polygons.

I. First we show that if a curve is regular then it can be smoothly approximated by inscribed polygons.

Let $\gamma_r : [a,b] \to E^q$ be a regular curve, and let $\gamma : [0,L] \to E^q$ denote the arc length parameterization of γ_r. It suffices to show that γ can be smoothly approximated by inscribed polygons. For suppose that for every ε in the open interval $(0, \frac{\pi}{2})$ there exists a $\delta > 0$ such that if $\mu(Q) < \delta$ then $\kappa(P) < \varepsilon$, where Q is a partition of [0,L], and P the corresponding polygon inscribed in γ. There exists a monotone continuous function f: $[a,b] \to [0,L]$ such that $\gamma_r = \gamma \circ f$. Moreover, by uniform continuity of f, there exists a $\delta_r > 0$ such that, for every partition Q_r of [a,b] with $\mu(Q_r) < \delta_r$, one has $\mu(Q) < \delta$.

Let P_r be the polygon inscribed in γ_r corresponding to Q_r.

So if $\mu(Q_r) < \delta_r$ then $\kappa(P_r) = \kappa(P) < \varepsilon$.

Let $\gamma : [0,L] \to E^q$ be a regular curve parameterized by arc length s. For every s, s* ε [0,L] , s $\neq$ s* , let o(h) = $\gamma(s) - \gamma(s^*) - h\,\gamma'(s^*)$, where h = s - s*.

<u>Lemma 2.1</u> Given $\varepsilon > 0$ there exists a $\delta > 0$ such that if $0 < |h| < \delta$ then $\dfrac{||o(h)||}{|h|} < \varepsilon$.

<u>Proof of lemma 2.1</u> Let $\varepsilon > 0$ and $\gamma(s) = (\gamma_1(s), \gamma_2(s), \ldots, \gamma_q(s))$. Then

$$\frac{||o(h)||}{|h|} = \left|\left| \frac{\gamma(s) - \gamma(s^*)}{h} - \gamma'(s^*)\right|\right| =$$

$$\sqrt{\left[\frac{\gamma_1(s) - \gamma_1(s^*)}{h} - \gamma_1'(s^*)\right]^2 + \ldots + \left[\frac{\gamma_q(s) - \gamma_q(s^*)}{h} - \gamma_q'(s^*)\right]^2}\ .$$

Since $\gamma \in C^1[0,L]$, $\gamma_i \in C^1[0,L]$ for each i = 1,2,...,q. So given $\varepsilon > 0$ there exists $\delta_i > 0$ such that if $0 < |h| < \delta_i$ then $\left|\dfrac{\gamma_i(s) - \gamma_i(s^*)}{h} - \gamma_i'(s^*)\right| < \dfrac{\varepsilon}{\sqrt{q}}$.

If we choose $\delta = \min(\delta_1, \delta_2, \ldots, \delta_q)$, then for each i = 1,2,...,q, if $0 < |h| < \delta$ then

$\left|\dfrac{\gamma_i(s) - \gamma_i(s^*)}{h} - \gamma_i'(s^*)\right| < \dfrac{\varepsilon}{\sqrt{q}}$. So given $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |h| < \delta$ then $\dfrac{||o(h)||}{|h|} <$

$$\sqrt{q \cdot \frac{\varepsilon^2}{q}} = \varepsilon\ .$$

Now to show that γ can be smoothly approximated by inscribed polygons we need to show that given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that for every s_0, s_1, $s_2 \in [0,L]$ with $s_0 < s_1 < s_2$, if $0 < |s_1 - s_2| < \delta$ then $\theta(s_0 s_1, s_1 s_2) < \varepsilon$. Let $\alpha = \theta(s_1 s_2, s_1)$, and $\beta = \theta(s_1 , s_0 s_1)$.

By the triangle inequality on S^{q-1}, we have

$$\theta(s_0 s_1, s_1 s_2) \leq \alpha + \beta . \tag{2.1}$$

We have

$$\cos \alpha = u(s_1).u[s_1, s_2]$$

$$= \gamma'(s_1). \frac{\gamma(s_2) - \gamma(s_1)}{||\gamma(s_2) - \gamma(s_1)||}$$

$$= \gamma'(s_1). \frac{h\gamma'(s_1) + o(h)}{\sqrt{h^2 + 2h\gamma'(s_1) .o(h) + ||o(h)||^2}} ,$$

where $h = s_2 - s_1$

$$= \frac{h + \gamma'(s_1).o(h)}{h\sqrt{1 + 2 \gamma'(s_1). \frac{o(h)}{h} + (\frac{||o(h)||}{h})^2}}$$

$$= \frac{1 + \gamma'(s_1) . \frac{o(h)}{h}}{\sqrt{1 + 2 \gamma'(s_1). \frac{o(h)}{h} + (\frac{||o(h)||}{h})^2}} \tag{2.2}$$

Let $h = s_2 - s_1$. By lemma 2.1, given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that if $0 < h < \delta$ then $\frac{||o(h)||}{h} < \varepsilon_1$.

Also $\left|\gamma'(s_1) . \frac{o(h)}{h}\right| \leqslant \frac{||o(h)||}{h} < \varepsilon_1$.

Given $\varepsilon_2 > 0$, we can choose $\varepsilon_1 > 0$ so that we have in (2.2)
$1 - \frac{\varepsilon_2^2}{2} < \cos\alpha \leqslant 1$.

Given ε in $(0, \frac{\pi}{2})$, we can choose $\varepsilon_2 > 0$ so that $\alpha < \frac{\varepsilon}{2}$.
That is, given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that

$$\text{if } 0 < |s_1 - s_2| < \delta \text{ then } \alpha < \frac{\varepsilon}{2} . \tag{2.3}$$

Similarly

$$\text{if } 0 < |s_0 - s_1| < \delta \text{ then } \beta < \frac{\varepsilon}{2} . \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1), it follows that given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that if $0 < |s_0 - s_1| < \delta$, $0 < |s_1 - s_2| < \delta$ then $\theta(s_0s_1 , s_1s_2) < \varepsilon$. Hence a regular curve can be smoothly approximated by inscribed polygons.

II. Now assume that γ can be smoothly approximated by inscribed polygons. We will show that γ is regular.

Let $t_0 \in [a,b]$. We first show that γ has a tangent at t_0. It suffices to consider the case where $\gamma([a,t_0]) \neq \gamma(a)$ and $\gamma([t_0, b]) \neq \gamma(b)$.

Consider any sequence $t_n \varepsilon[a,b]$ such that $t_n > t_0^+$, $\gamma(t_n) \neq \gamma(t_0)$ and $\lim_{n \to \infty} t_n = t_0^+$. By compactness of S^{q-1}, the sequence $u[t_o, t_n]$ has a limit point u_r.

Similarly if t'_m is any sequence in $[a,b]$ such that $t'_m < t_0^-$, $\gamma(t'_m) \neq \gamma(t_0)$ and $\lim_{m \to \infty} t'_m = t_0^-$ then the sequence $u[t_0, t'_m]$ has a limit point $- u_1$.

In order to show that γ has a tangent at t_0, it suffices to show that $u_r = u_1$.

Suppose $u_r \neq u_1$. Let ψ be the smallest non-negative angle between u_1 and u_r. Then $\psi \neq 0$. Let $\emptyset = \theta(t'_m t_0, t_0 t_n)$. We may choose ε in $(0, \frac{\pi}{2})$ so that $\varepsilon < \frac{\psi}{2}$. By the definition of smooth approximation, there exists a $\delta > 0$ such that if $0 < |t_n - t_0^+| < \delta$, $0 < |t_m - t_0| < \delta$ then $\emptyset < \varepsilon$. But $\lim \emptyset = \psi$, which is a contradiction.

We now show that γ admits continuous tangents. It suffices to show that given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that if $0 < |t - t_0| < \delta$ then $\theta(t,t_0) < \varepsilon$.

<u>Lemma 2.2</u> Given ε in $(0, \frac{\pi}{2})$ there is a $\delta > 0$ so that for every $t, t_0 \,\varepsilon\, [a,b]$, if $0 < t - t_0 < \delta$ then $\theta(t,t_0 t) < \varepsilon$ and $\theta(t_0, t_0 t) < \varepsilon$.

<u>Proof of Lemma 2.2</u> Let $\frac{\varepsilon}{2} > 0$ and determine the corresponding $\delta > 0$ from the definition of smooth approximation. Let $0 < t - t_0 < \delta$.

If $\gamma(t) = \gamma(t_0)$ then $\theta(t, t_0t) = \theta(t_0, t_0t) = 0$. So assume $\gamma(t_0) \neq \gamma(t)$. Then

$$t^- = \inf\{\tau \varepsilon [a,b] : \tau \leqslant t, \gamma([\tau, t]) = \gamma(t)\} > t_0$$

and there is a sequence t_n such that $t_0 < t_n < t^-$,

$\gamma(t_n) \neq \gamma(t)$, and $\lim_{n \to \infty} t_n = t^-$. By our choice of δ,

$\theta(t_0t_n, t_nt^-) < \frac{\varepsilon}{2}$. Now let $n \to \infty$. Then $\theta(t_0t, t) = \theta(t_0t^-, t^-) \leq \frac{\varepsilon}{2} < \varepsilon$. Likewise, if

$$t_0^+ = \text{Sup}\{\tau \varepsilon [a,b] : \tau \geqslant t_0, \gamma([t_0, \tau]) = \gamma(t_0)\} < t,$$

there is a sequence t_m such that $t_0^+ < t_m < t$, $\gamma(t_m) \neq \gamma(t_0)$, and $\lim_{m \to \infty} t_m = t^+$. As before $\theta(t_0, t_0t) = \theta(t_0^+, t_0^+ t) =$

$$\lim_{m \to \infty} \theta(t_0^+t_m, t_mt) \leq \frac{\varepsilon}{2} < \varepsilon.$$

<u>Lemma 2.3</u> Given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that for every $t, t_0 \varepsilon [a,b]$, if $|t - t_0| < \delta$ then $\theta(t, t_0) < \varepsilon$.

<u>Proof of lemma 2.3</u> Let $\frac{\varepsilon}{2} > 0$ and choose $\delta > 0$ as in lemma 2.2. Let $|t - t_0| < \delta$.

If $t > t_0$,

$$\theta(t,t_0) \leqslant \theta(t,t_0t) + \theta(t_0t,t_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

If $t < t_0$,

$$\theta(t,t_0) \leq \theta(t,tt_0) + \theta(tt_0,t_0) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We now show that γ is rectifiable.

Lemma 2.4 Given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that for every t, t_1, t_2, $t' \varepsilon [a,b]$, if $0 < |t - t'| < \delta$ and $t < t_1 < t_2 < t'$ then $\theta(tt',t_1t_2) < \varepsilon$.

Proof of lemma 2.4 Given $\frac{\varepsilon}{3} > 0$ choose $\delta > 0$, the smaller of the delta in the definition of smooth approximation and in lemma 2.3. Then

$$\theta(tt',t_1t_2) \leq \theta(tt',t) + \theta(t,tt_1) + \theta(tt_1,t_1t_2) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Lemma 2.5 Given ε in $(0,\frac{\pi}{2})$ there exists $\delta > 0$ such that for every t_0, $t_n \varepsilon [a,b]$ if

$0 < t_n - t_0 < \delta$ then $\dfrac{\Delta \hat{S}_n}{||\Delta x_n||} < \sec \varepsilon$, where $\Delta x_n = \gamma(t_n) - \gamma(t_0)$, and $\Delta\hat{S}_n$ = length of the inscribed polygon with vertices at $\gamma(t_0)$, $\gamma(t_1),\ldots, \gamma(t_n)$, where $t_0 < t_1 < \ldots < t_n$.

Proof of lemma 2.5 Let H_i be the hyperplane through $\gamma(t_i)$ such that Δx_n is normal to H_i. Let $\alpha_i = ||\gamma(t_i) - \gamma(t_{i-1})||$,

β_i = distance between the hyperplanes H_{i-1} and H_i , and

$\theta_i = \theta(t_0t_n, t_{i-1}t_i)$.

By lemma 2.4, given ε in $(0,\frac{\pi}{2})$ there exists $\delta > 0$ such that if $0 < |t_0 - t_n| < \delta$ then $\theta_i < \varepsilon$, $i = 1,2,\ldots,n$. Hence

$$\frac{\alpha_i}{\beta_i} = \sec \theta_i < \sec \varepsilon . \qquad (2.5)$$

Let $u = \dfrac{\Delta x_n}{||\Delta x_n||}$. Then

$\beta_i = |[\gamma(t_i) - \gamma(t_{i-1})].u| = [\gamma(t_i) - \gamma(t_{i-1})].u$ because $0 \leqslant \theta_i < \varepsilon < \frac{\pi}{2}$. Consequently,

$\Delta x_n = \sum_{i=1}^{n} [\gamma(t_i) - (t_{i-1})]$, and $||\Delta x_n|| = \Delta x_n \cdot u =$

$\sum_{i=1}^{n} [\gamma(t_i) - \gamma(t_{i-1})] .u = \sum_{i=1}^{n} \beta_i$.

Thus,

$$\frac{\Delta \hat{S}_n}{||\Delta x_n||} = \frac{\alpha_1 + \alpha_2 + \ldots + \alpha_n}{\beta_1 + \beta_2 + \ldots + \beta_n} < \sec \varepsilon \text{ , by (2.5).}$$

Lemma 2.6 γ is rectifiable.

Proof of lemma 2.6 Let Q be any partition of [a,b] and P its corresponding polygon. There is a fixed partition Q_1 such that successive partition points of Q_1 differ by less than the δ corresponding to $\varepsilon = 1$ in lemma 2.5. Let $Q^* = Q \cup Q_1$, and let P^* denote the corresponding polygon. Then

$L(P) \leqslant L(P^*) \leqslant L(P_1) \sec 1$

by lemma 2.5, where L(P) is the length of P. Thus, γ is rectifiable.

Finally we show that γ is C^1 with respect to arc length.

Let $a \leqslant t_0 < b$. Consider any sequence $t_n \varepsilon[a,b]$ so that $t_n > t_0$,

$\gamma(t_n) \neq \gamma(t_0)$, and $\lim_{n \to \infty} t_n = t_0$. Let $\Delta x_n = \gamma(t_n) - \gamma(t_0)$, and ΔS_n = arc length measured from $\gamma(t_0)$ to $\gamma(t_n)$. Since γ admits continuous tangents by lemma 2.3, it suffices to show that

$$\lim_{n \to \infty} \frac{\Delta x_n}{\Delta S_n} = u(t_0),$$

where $u(t_0)$ is the unit tangent at t_0, so that γ is C^1 with respect to arc length.

By lemma 2.5, given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that if $0 < |t_0 - t_n| < \delta$ then

$$1 \geq \frac{||\Delta x_n||}{\Delta S_n} > \frac{\Delta \hat{S}_n}{\Delta S_n} \cos \varepsilon .$$

where $\Delta \hat{S}_n$ is the length of any inscribed polygon beginning at $\gamma(t_0)$ and ending at $\gamma(t_n)$.

Since γ is rectifiable by lemma 2.6, there exists a partition Q of $[t_0, t_n]$ so that if $\Delta \hat{S}_n$ is the length of the corresponding inscribed polygon, then

$$\frac{\Delta \hat{S}_n}{\Delta S_n} > 1 - \varepsilon .$$

Hence if $0 < |t_0 - t_n| < \delta$ then

$$1 \geq \frac{||\Delta x_n||}{\Delta S_n} > (1 - \varepsilon) \cos \varepsilon .$$

$$\therefore \quad \lim_{n \to \infty} \frac{||\Delta x_n||}{\Delta S_n} = 1. \tag{2.6}$$

We have

$$\frac{\Delta x_n}{\Delta S_n} = \frac{||\gamma(t_n) - \gamma(t_0)||}{\Delta S_n} \quad u\,[t_0, t_n]$$

$$= \frac{||\Delta x_n||}{\Delta S_n} \quad u[t_0,t_n] \; .$$

$$\therefore \lim_{n \to \infty} \frac{\Delta x_n}{\Delta S_n} = \lim_{n \to \infty} u\,[t_0, t_n] = u(t_0), \text{ by (2.6).}$$

<u>An example:</u> <u>A non-regular C^1 curve.</u> Let $\gamma : [0,1] \to E^2$ be the curve defined by

$$\gamma(t) = (t^3, t^3 \sin \frac{\pi}{t}), \; 0 < t \leqslant 1$$
$$= (0,0) \; , \; t = 0.$$

We show that γ is not regular. Let n be some positive integer, and

$$Q : 0 < \frac{1}{4n + 1} < \frac{2}{4n + 1} < \ldots < \frac{4n + 1}{4n + 1}$$

be a partition of the interval [0,1]. We have

$$\gamma(\frac{1}{4n + 1}) - \gamma(0) = (\frac{1}{(4n + 1)^3} \; , 0) \; ,$$

$$\gamma(\frac{2}{4n + 1}) - \gamma(\frac{1}{4n + 1}) = (\frac{7}{(4n + 1)^3} \; , \; \frac{8}{(4n + 1)^3})$$

If θ is the angle between these two vectors, then $\cos \theta = \frac{7}{\sqrt{113}}$. So $\theta \not\to 0$ as $n \to \infty$. That is, γ cannot be smoothly approximated by inscribed polygons. Hence γ is not regular.

CHAPTER III

Measure of Lines That Intersect a Non-Convex Set

3.1 Measure of Lines that Intersect a Convex Set in E^3.

Let Λ be a line in E^3, and let $m(\Lambda:\Lambda\cap C\neq\phi)$ be the measure of all lines Λ that intersect a convex set C in E^3. Then (see [2], [3], or [4])

$$m(\Lambda:\Lambda\cap C\neq\phi) = \int_{\Lambda\cap C\neq\phi} d\Lambda = \frac{\pi}{2} S, \qquad (3.1)$$

Where S is the surface area of C.

Consider a line segment OA of length σ in E^3. Let $B(x,r)$ be a ball of radius r, centered at x. The union $\bigcup_{x\in OA} B(x,r)$ is a convex tube. Let us denote this tube by $T(r,\sigma)$, or simply by T. The surface area of

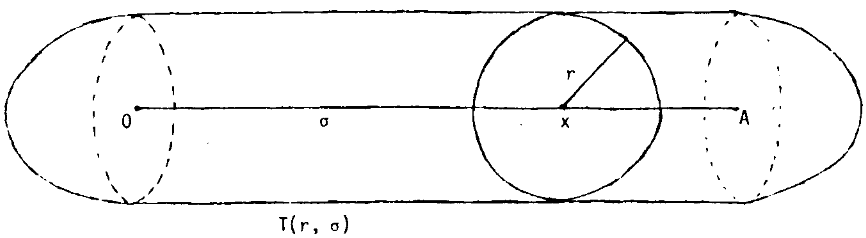


Figure 1

this tube is $2\pi r\sigma + 4\pi r^2$. The measure of all lines Λ that intersect the tube T is given by

$$m(\Lambda:\Lambda\cap T\neq\phi) = \int_{\Lambda\cap T\neq\phi} d\Lambda = \frac{\pi}{2}(2\pi r\sigma + 4\pi r^2) \qquad (3.2)$$

3.2. Tube Around Two Intersecting Segments

Consider two line segments OA, OB of lengths σ_1,σ_2 respectively in E^3. Let the angle between them be φ, $0 < \varphi < \pi$.

Let $T_1(r, \sigma_1) = \bigcup_{x \varepsilon OA} B(x,r)$, and $T_2(r, \sigma_2) = \bigcup_{x \varepsilon OB} B(x,r)$.

Let $T = T_1 \cup T_2$.

Let Λ be a line in E^3, and let $\chi(\Lambda \cap T)$ = number of segments in $\Lambda \cap T$.

Then $\chi(\Lambda \cap T) = 0, 1,$ or 2.

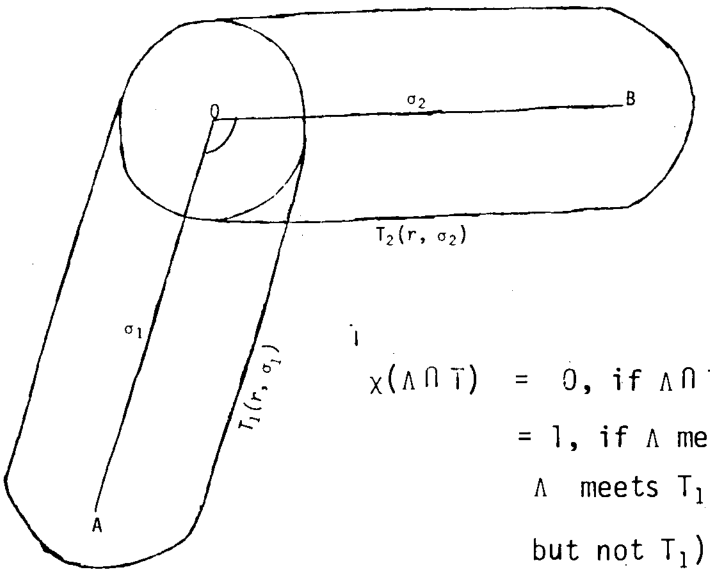


Figure 2

$$\chi(\Lambda \cap T) = 0, \text{ if } \Lambda \cap T = \phi \tag{3.3}$$

= 1, if Λ meets T in one segment (that is, if Λ meets $T_1 \cap T_2$, or, T_1 but not T_2, or, T_2 but not T_1)

= 2, if Λ meets T in two segments (that is, if Λ meets both T_1 and T_2, but not $T_1 \cap T_2$).

Figures (a), (b) , and (c) respectively show the cases where $\chi(\Lambda \cap T) = 0$, $\chi(\Lambda \cap T) = 1$, and $\chi(\Lambda \cap T) = 2$. (See following page for figures).

Theorem 3.1.

$$\int_{\Lambda \cap T} \chi(\Lambda \cap T)\, d\Lambda = \int_{\Lambda \cap T_1} d\Lambda + \int_{\Lambda \cap T_2} d\Lambda - \int_{\Lambda \cap (T_1 \cap T_2)} d\Lambda \tag{3.4}$$

Proof:

Case (i): $\chi(\Lambda \cap T) = 0$

If $\Lambda \cap T = \phi$, then $\Lambda \cap T_1$, $\Lambda \cap T_2$, and $\Lambda \cap (T_1 \cap T_2)$ are all empty. Hence both sides of the equation (3.4) are zero, and the equation is true.

Figure 3

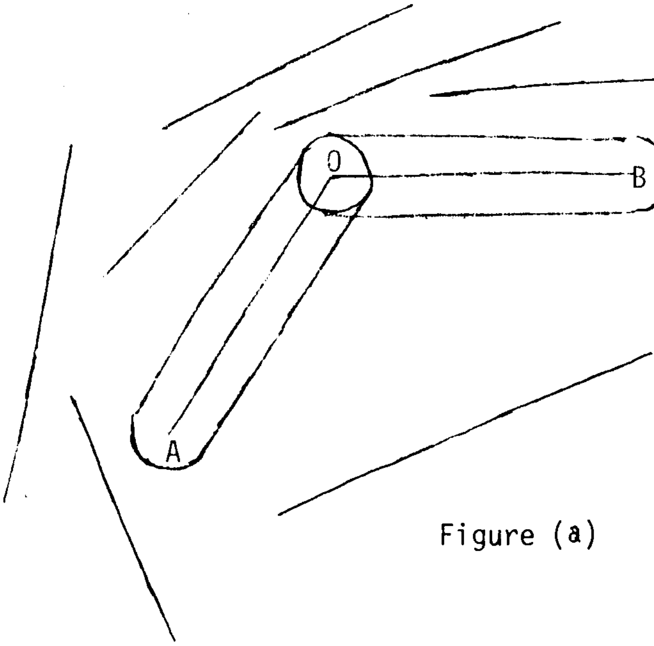


Figure (a)

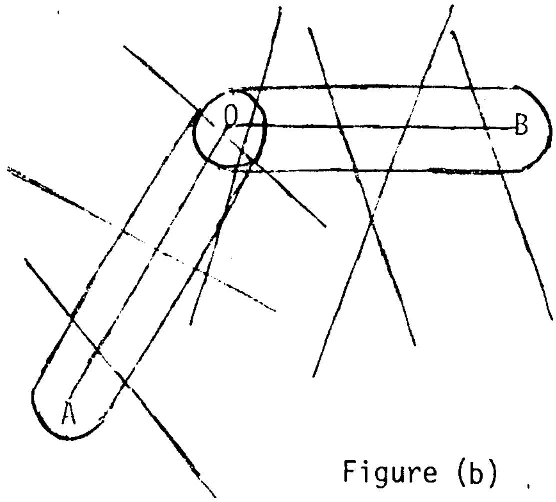


Figure (b)

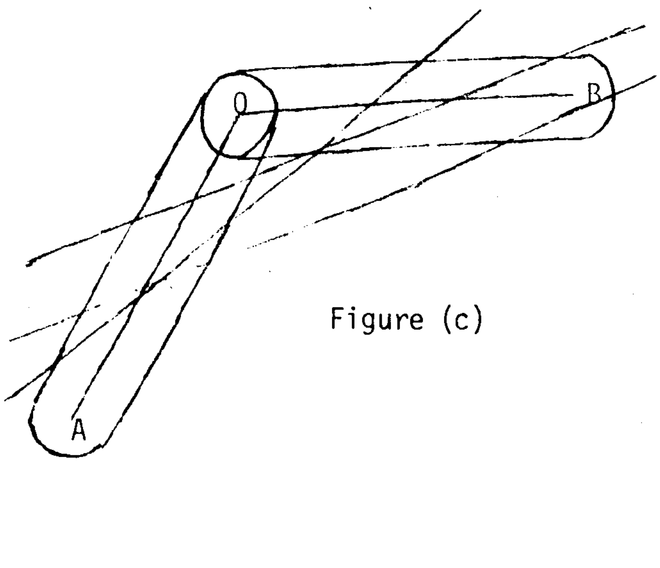


Figure (c)

<u>Case (ii):</u> $\chi(\Lambda \cap T) = 1$

(a) Let $\Lambda \cap (T_1 \cap T_2) \neq \phi$. Then neither of the sets $\Lambda \cap T_1$, $\Lambda \cap T_2$ is empty. So, the lines that intersect T are counted once on the right hand side of equation (3.4). (Because the lines that intersect T are counted in each of the integrals on the right hand side of (3.4), and hence are counted 1 + 1 - 1 = 1 time). So, the equation (3.4) holds.

(b) Let $\Lambda \cap (T_1 \cap T_2) = \phi$. Then one of the sets $\Lambda \cap T_1$,$\Lambda \cap T_2$ is empty and the other is not empty and equals $\Lambda \cap T$. So equation (3.4) holds in this case.

<u>Case (iii):</u> $\chi(\Lambda \cap T) = 2$.

Here $\Lambda \cap (T_1 \cap T_2) = \phi$, $\Lambda \cap T_1 \neq \phi$, and $\Lambda \cap T_2 \neq \phi$

The lines Λ that intersect T are counted 1 + 1 - 0 = 2 times on the right hand side of equation (3.4). So the equation holds in this case too.

Hence, we have:

$$\int \chi(\Lambda \cap T)\, d\Lambda = \frac{\pi}{2}(4\pi r^2 + 2\pi r \sigma_1) + \frac{\pi}{2}(4\pi r^2 + 2\pi r \sigma_2) - \frac{\pi}{2} S(T_1 \cap T_2),$$

where $S(T_1 \cap T_2)$ is the surface area of $T_1 \cap T_2$. Therefore,

$$\int \chi(\Lambda \cap T)\, d\Lambda = \pi^2 r(\sigma_1 + \sigma_2) + 4\pi^2 r^2 - \frac{\pi}{2} S(T_1 \cap T_2). \quad (3.5)$$

<u>3.3. Tube Around a Polygon.</u>

Consider a polygon P in E^3, consisting of the edges e_k of lengths σ_k, k = 1,2,....,n. Let $T(r, \sigma_k) = \bigcup_{x \varepsilon e_k} B(x,r)$. Assume that

$e_i \cap e_j = \phi$ for $|i-j| > 1$. Let r be such that $T(r, \sigma_i) \cap T(r, \sigma_j) = \phi$ for $|i-j| > 1$. To see that this can be done with r small enough, we choose r as follows. Let $\min_{\substack{x_i \varepsilon e_i \\ x_j \varepsilon e_j}} |x_i - x_j| = \delta_{ij} > 0$. Let $\min_{|i-j|>1} \delta_{ij} = \delta$.

Choose r so that $2r < \delta$. Then

$$T(r, \sigma_i) \cap T(r, \sigma_j) = \phi \quad \text{for } |i-j| > 1. \tag{3.6}$$

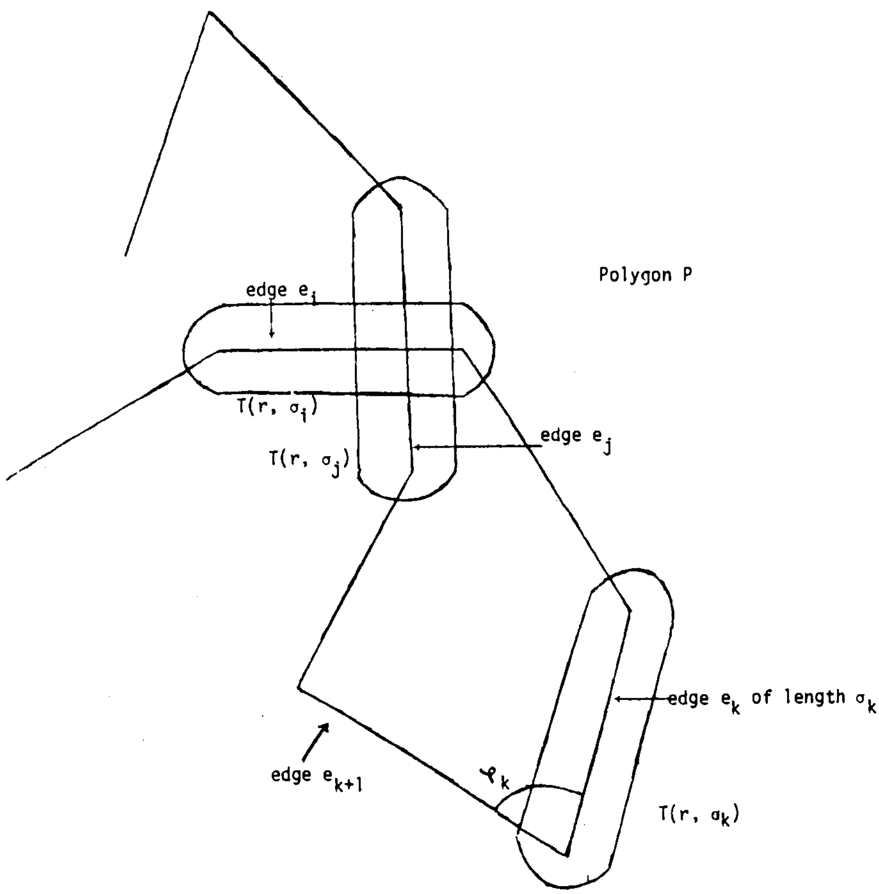


Figure 4

Call P closed if $e_1 \cap e_n$ is a point such that the last vertex of P in e_n is also the first vertex of P in e_0, and $e_i \cap e_j = \emptyset$ for $1 < |i - j| < n - 1$. If P is closed we can choose r similarly. For $1 < |i - j| \neq n - 1$, let $\min_{\substack{x_i \in e_i \\ x_j \in e_j}} |x_i - x_j| = \delta_{ij} > 0$. Let $\min_{1 < |i-j| < n-1} \delta_{ij} = \delta$. Now choose r so that $2r < \delta$. Let

$$T(r, P) = T(r,\sigma_1) \cup \ldots\ldots\ldots\ldots \cup T(r, \sigma_n), \tag{3.7}$$

and $\chi(\Lambda \cap T(r, P))$ = number of segments in $\Lambda \cap T(r, P)$. (3.8)

Theorem 3.2

$$\int \chi(\Lambda \cap T(r, P))\, d\Lambda = \sum_{k=1}^{n} \int_{\Lambda \cap T(r, \sigma_k)} d\Lambda - \sum_{k=1}^{n} \int_{\Lambda \cap [(T(r, \sigma_k) \cap T(r, \sigma_{k+1})]} d\Lambda \tag{3.9}$$

where in the second integral, e_{n+1} stands for e_1 if P is closed, and $e_{n+1} = \phi$ if P is not closed.

Proof:

Let $\chi(\Lambda \cap T(r, P)) = \ell \ (\leqslant n)$.

The lines Λ that intersect $T(r, P)$ are counted ℓ times on the left hand side of equation (3.9). By reasoning similar to that in Theorem 3.1, the lines Λ that intersect $T(r, P)$ are counted ℓ times on the right hand side of equation (3.9).

Hence we have

$$\int \chi(\Lambda \cap T(r, P))\, d\Lambda = A_1 - A_2, \text{ where} \tag{3.10}$$

$$A_1 = \sum_{k=1}^{n} \frac{\pi}{2} (2\pi r\sigma_k + 4\pi r^2) = \pi^2 r L(P) + 2\pi^2 r^2 n, \tag{3.11}$$

where $L(P) = \sum_{k=1}^{n} \sigma_k$ is the length of P, and

$$A_2 = \frac{\pi}{2} \sum_{k=1}^{n} S[T(r, \sigma_k) \cap T(r, \sigma_{k+1})] = \frac{\pi}{2} \sum_{k=1}^{n} S(D_k),$$

Where

$$D_k = T(r, \sigma_k) \cap T(r, \sigma_{k+1}), \tag{3.13}$$

and $S(D_k)$ is the surface area of D_k.

First let P be closed. Then

$$A_2 = \frac{\pi}{2} \sum_{k=1}^{n} S(D_k) \tag{3.14}$$

$$= \frac{\pi}{2} \{ \sum_{k=1}^{n} S(B_r) + \sum_{k=1}^{n} [S(D_k) - S(B_r)] \},$$

Where $S(B_r)$ is the surface area of the ball of radius r, in E^3.

Therefore,

$$A_2 = \frac{\pi}{2} . \; 4 \pi r^2 n + \frac{\pi}{2} \sum_{k=1}^{n} [S(D_k) - S(B_r)] \tag{3.15}$$

and so,

$$\int \chi(\Lambda \cap T(r,P)) d\Lambda = \pi^2 r \, L(P) - \frac{\pi}{2} \sum_{k=1}^{n} [S(D_k) - S(B_r)] \tag{3.16}$$

Next, suppose P is not closed. Then

$$A_2 = \frac{\pi}{2} \underbrace{(2\pi r^2 + 2\pi r^2)}_{''} + \frac{\pi}{2} \sum_{k=1}^{n-1} S[T(r, \sigma_k) \cap T(r, \sigma_{k+1})]$$

(the surface area of the two half balls at the ends of P)

$$= \frac{\pi}{2} . \; 4\pi r^2 + \frac{\pi}{2} \sum_{k=1}^{n-1} S(D_k)$$

$$= \frac{\pi}{2} . \; 4\pi r^2 + \frac{\pi}{2} \{ \sum_{k=1}^{n-1} S(B_r) + \sum_{k=1}^{n-1} [S(D_k) - S(B_r)] \}$$

$$= \frac{\pi}{2} \cdot 4\pi r^2 + \frac{\pi}{2} 4\pi r^2 (n-1) + \frac{\pi}{2} \sum_{k=1}^{n-1} S(D_k) - S(B_r)]$$

$$= 2\pi r^2 n + \frac{\pi}{2} \sum_{k=1}^{n-1} [S(D_k) - S(B_r)], \text{ and so}$$

$$\int \chi(\Lambda \cap T(r, P)) d\Lambda = \pi^2 r L(P) - \frac{\pi}{2} \sum_{k=1}^{n-1} [S(D_k) - S(B_r)] \qquad (3.18)$$

we therefore have,

$$\frac{1}{\pi^2 r} \int \chi (\Lambda \cap T(r, P) d\Lambda = L(P) - \frac{1}{2\pi} \sum_k \frac{S(D_k) - S(B_r)}{r},$$

where $\sum_k = \sum_{k=1}^{n}$ if P is closed, and $\sum_k = \sum_{k=1}^{n-1}$ if P is not closed.

Theorem 3.3.

$\frac{S(D_k) - S(B_r)}{r} \to 0$ as $\varphi_k \to \pi$, where φ_k is the angle between the edges e_k and e_{k+1} of P.

Proof.

To show this, let us consider two line segments OA, OB of lengths σ_1 and σ_2 respectively. Let φ be the angle AOB. With centers on these segments draw balls of radius r. Then we get two tubes $T_1(r, \sigma_1)$ and $T_2(r, \sigma_2)$.

Let C_1 and C_2 be the cylindrical surfaces of the tubes T_1 and T_2 respectively. (That is, C_1 is the surface of the tube T_1 with half balls removed on the two ends of the segment OA). The intersection of the surfaces C_1 and C_2 is a plane curve, which is in fact an ellipse.

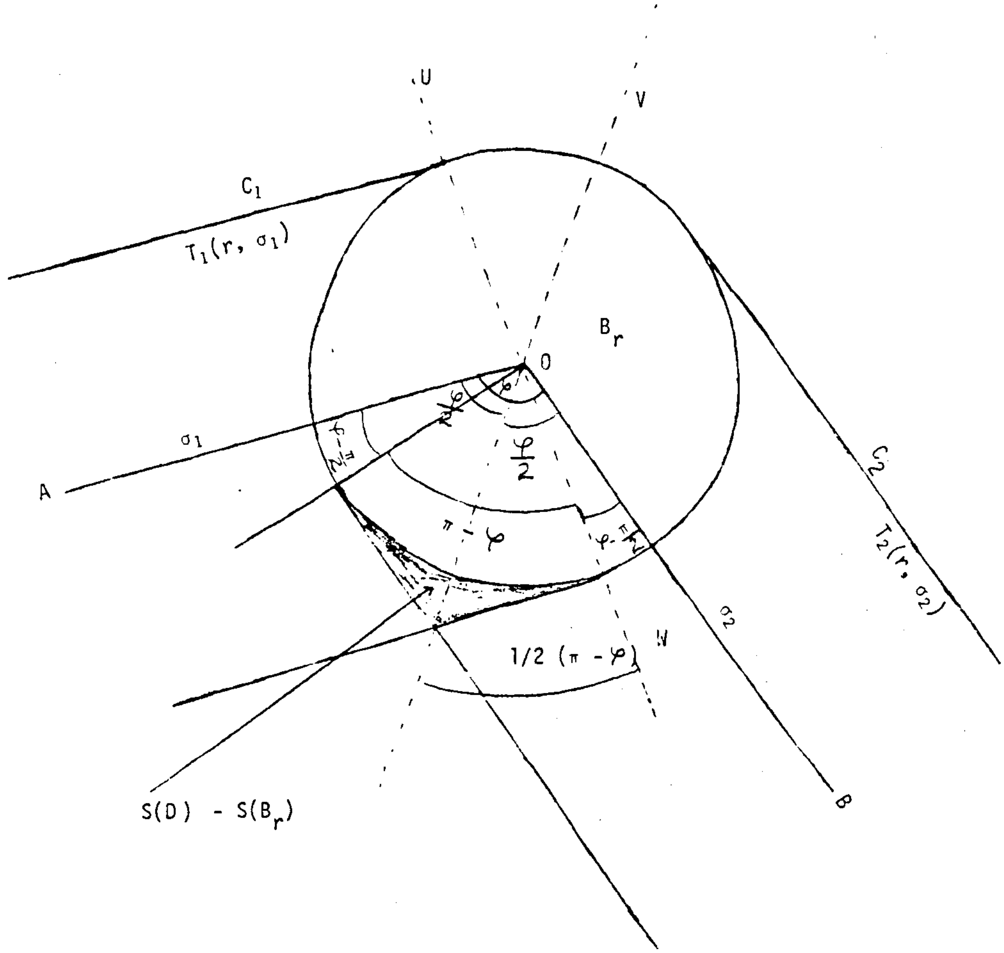

U
V
C_1
$T_1(r, \sigma_1)$
B_r
O
σ_1
A
$\frac{\varphi}{2}$
$\pi - \varphi$
C_2
$T_2(r, \sigma_2)$
σ_2
W
1/2 (π - φ)
B
S(D) - S(B_r)

Figure 5

Let V be the plane of intersection $C_1 \cap C_2$. The plane V bisects the angle between OA and OB. In other words, if W is the plane containing the segments OA and OB, the line of intersection of the planes V and W, makes an angle $\frac{\varphi}{2}$ with OA, and an angle $\frac{\varphi}{2}$ with OB.

Let O be the origin, and the axis of C_1 (that is, the direction along OA) be the x_3-axis, and the plane U through O containing end cross section of C_1 be the (x_1, x_2)-plane.

Let the line of intersection of the plane W (containing the line segments OA, OB) and the plane U be the x_2-axis. Then the line of intersection of the planes U and V will be the x_1-axis.

Since x_3 is perpendicular to the plane U, and OA makes an angle $\frac{\varphi}{2}$ with the line of intersection of the planes V and W, it is clear that the angle between the planes U and V is $\frac{\pi}{2} - \frac{\varphi}{2}$.

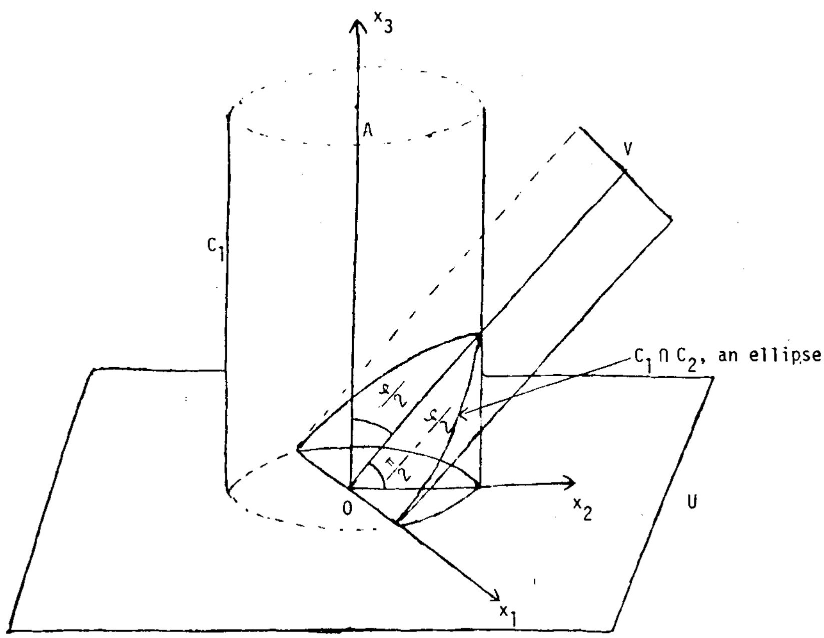


Figure 6

Let us now compute the surface area of the cylinder C_1 below the plane V. If x_1 is the distance from $C_1 \cap C_2$ to the plane (x_1, x_2), then

$$x_3 = \tan\left(\frac{\pi}{2} - \frac{\varphi}{2}\right) x_2 = \cot\frac{\varphi}{2}\; x_2 = r \cot\frac{\varphi}{2} \sin\psi \;, \text{ where}$$

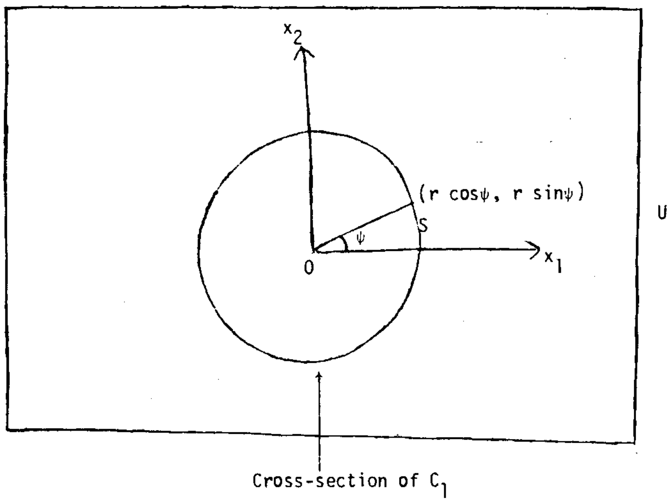


Figure 7

(r, ψ) is the polar coordinate corresponding to the point (x_1, x_2) in the plane U. Therefore, the surface area of C_1 below the plane V

$$= 2 \int_0^{\frac{\pi}{2}} x_3(\psi)\, ds(\psi)$$

Since the arc length $s(\psi) = r\psi$, the surface area of the cylinder below the plane V

$$= 2 \int_0^{\frac{\pi}{2}} r^2 \cot\frac{\varphi}{2} \sin\psi \, d\psi = 2r^2 \cot\frac{\varphi}{2} \quad .$$

Similarly, the surface area of the cylinder C_2 above the plane V = $2r^2 \cot\frac{\varphi}{2}$. Hence S(D), the total surface area of $T_1 \cap T_2$,

$$= 4r^2\cot\frac{\varphi}{2} + 4\pi r^2\ \frac{2\pi - (\pi - \varphi)}{2\pi}$$

$$= 4r^2\cot\frac{\varphi}{2} + 2r^2(\pi + \varphi).$$

$$\therefore\quad S(D) - S(B_r) = 4r^2\cot\frac{\varphi}{2} + 2r^2(\pi + \varphi) - 4\pi r^2$$

$$= 4r^2[\cot\frac{\varphi}{2} + \frac{\varphi}{2} - \frac{\pi}{2}]$$

$$\therefore\quad \frac{S(D) - S(B_r)}{r} = 4r[\cot\frac{\varphi}{2} + \frac{\varphi}{2} - \frac{\pi}{2}]$$

Let $F(\varphi) = \cot\frac{\varphi}{2} + \frac{\varphi}{2} - \frac{\pi}{2}$

$$\therefore\quad F'(\varphi) = \frac{1 - \operatorname{cosec}^2\frac{\varphi}{2}}{2} < 0$$

$\therefore\quad F(\varphi) \downarrow$ as $\varphi \uparrow$ from 0 to π

But $F(\pi) = 0 + \frac{\pi}{2} - \frac{\pi}{2} = 0$

$\therefore\quad F(\varphi) \to 0$ as $\varphi \to \pi$.

Let us now consider a regular space curve $\gamma : [0,L] \to E^3$, parameterized by arc length s. Let $Q : 0 = s_0 < s_1 < \ldots < s_n = L$ be a partition of $[0,L]$ and P be the corresponding polygon inscribed in γ. Let e_k be the kth edge of P obtained by joining the vertices s_{k-1} and s_k. By the Approximation Theorem in Chapter II, given ε in $(0,\frac{\pi}{2})$ there exists $\delta > 0$ such that if $\mu(Q) < \delta$ then $|\varphi_k - \pi| < \varepsilon$, where φ_k is the angle between the edges e_k and e_{k+1}. Let $r < \frac{1}{2}\min\{\sigma_k,\ k = 1,2,\ldots,n\}$, where σ_k is the length of e_k.

Theorem 3.4.

$$\sum_k \frac{S(D_k) - S(B_r)}{r} \to 0 \text{ as } \mu(Q) \to 0.$$

Proof.

We have, for $\theta^2 < \frac{\pi^2}{4}$,

$$\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\theta^5}{15} + \frac{17\theta^7}{315} + \ldots\ldots + \frac{2^{2n}(2^{2n}-1)\,B_n}{(2n)!}\,\theta^{2n-1} + \ldots.$$

Where B_n represents the nth Bernoulli number.

$$\therefore\ \cot\frac{\varphi}{2} = \tan\left(\frac{\pi}{2} - \frac{\varphi}{2}\right) = \left(\frac{\pi}{2} - \frac{\varphi}{2}\right) + \frac{\left(\frac{\pi}{2} - \frac{\varphi}{2}\right)^3}{3} + \ldots\ldots.$$

$$\therefore\ F(\varphi) = \cot\frac{\varphi}{2} + \frac{\varphi}{2} - \frac{\pi}{2} = \frac{(\pi - \varphi)^3}{24} + \frac{(\pi - \varphi)^5}{240} + \ldots\ldots.$$

$$< \frac{(\pi - \varphi)^3}{4} \text{ for sufficiently small } \pi - \varphi.$$

Since γ is regular, given ε in $(0, \frac{\pi}{2})$ there exists $\delta > 0$ such that if $\mu(Q) < \delta$ then $|\varphi_k - \pi| < \varepsilon$.

$$\therefore\ S(D_k) - S(B_r) = 4r^2\,F(\varphi) < r^2\,|\pi - \varphi|^3 < r^2\varepsilon^3$$

$$\therefore\ \sum_k [S(D_k) - S(B_r)] < nr^2\varepsilon^3.$$

Choose r so that $r < \min \frac{\sigma_k}{2} < \min \frac{s_k - s_{k-1}}{2}$, $k = 1, 2, \ldots\ldots, n$.

Then

$$nr < n \min \frac{s_k - s_{k-1}}{2} < \frac{L}{2} .$$

$$\sum_k \frac{S(D_k) - S(B_r)}{r} < \frac{L}{2} \varepsilon^3 \to 0 \text{ as } \mu(Q) \to 0.$$

CHAPTER IV

Quermassintegrals of the Intersection of Two Convex Tubes

Quermassintegrals of a Convex Set: Definition. Let C be a convex set, and B be the unit ball, in E^q. For $\rho \geqslant 0$, let $C + \rho B = \bigcup_{x \in C} B(x,\rho)$. The set $C + \rho B$ is called "the parallel set in the distance ρ of the convex set C". If $V(C + \rho B)$ is the volume of $C + \rho B$, then we have the formula (see [3] pp. 220-221):

$$V(C + \rho B) = \sum_{p=0}^{q} \binom{q}{p} W_p(C) \rho^p.$$

This is called Steiner's formula for parallel convex sets. $W_p(C)$, $p = 0, 1, 2, \ldots, q$, is called the pth quermassintegral of C. The constant term $W_0(C)$ gives the volume of C.

The intersection of the convex tubes $T_1(r,\sigma_1)$ and $T_2(r,\sigma_2)$ is a convex set D. In this chapter we show how to compute the quermassintegrals of D.

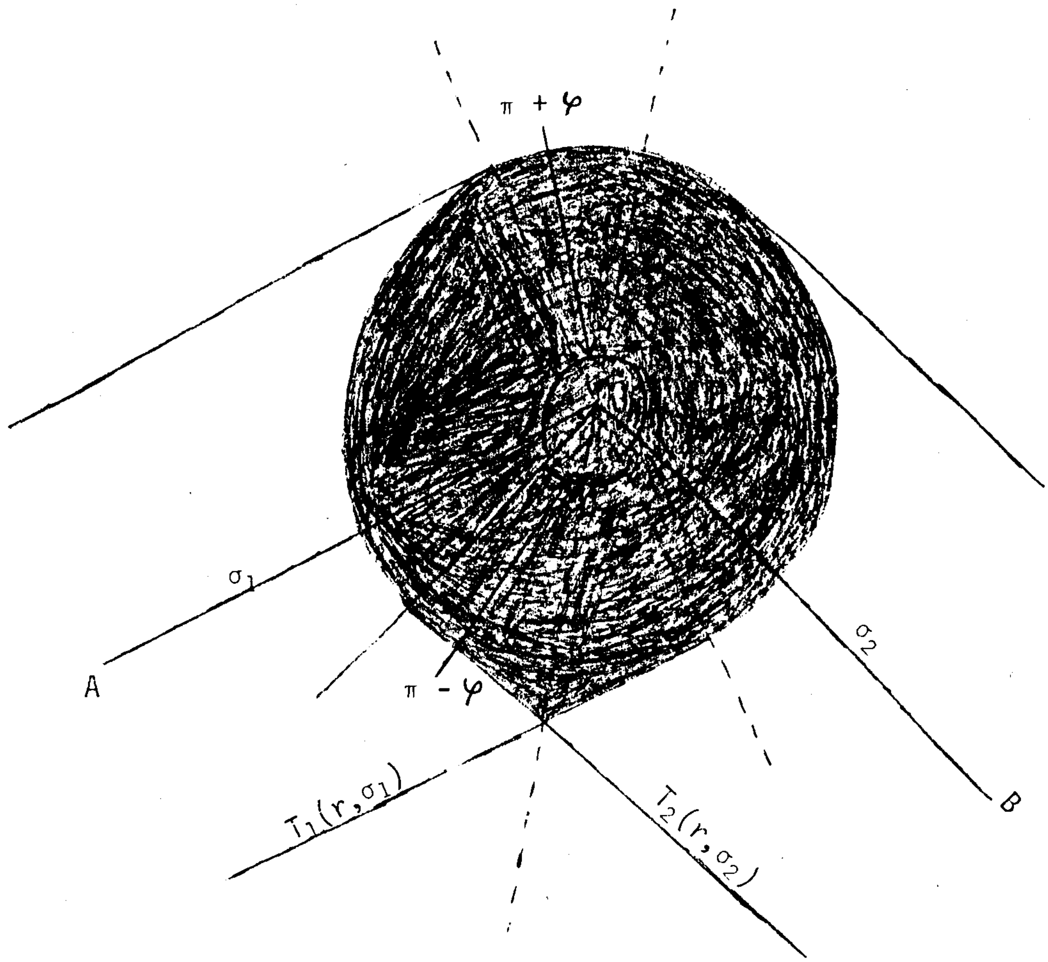


Figure 8

To find the quermassintegrals of D, we compute the volume $V(D + \rho B)$

We have $D + \rho B = D \cup \bigcup_{x \in \partial D} B(x,\rho)$.

Let us first compute the volume of D. Let λ be a ray on the (x_1,x_2) - plane with x_1- axis as the polar axis, 0 as the pole, and let λ make an angle ψ with the x_1-axis. Then

$$V(D) = \frac{4\pi r^3}{3} \cdot \frac{\pi + \varphi}{2\pi} + \iiint \lambda d\lambda d\psi dx_3$$

We have

$$x_3 = x_2 \cot \frac{\varphi}{2} = \lambda \sin \psi \cot \frac{\varphi}{2}$$

$$\therefore V(D) = \frac{4\pi r^3}{3} \quad \frac{\pi + \varphi}{2\pi} + 4 \int_{\psi=0}^{\frac{\pi}{2}} \int_{\lambda=0}^{r} \int_{x_3=0}^{\lambda \sin\psi \cot\frac{\varphi}{2}} \lambda d\lambda d\psi dx_3$$

$$= \frac{4r^3}{3}\left(\frac{\pi}{2} + \frac{\varphi}{2}\right) + \frac{4r^3}{3} \cot \frac{\varphi}{2}$$

To compute the volume $V(D + \rho B)$, we divide the region $D + \rho B$ into four regions: D, I, II, III so that

$$V(D + \rho B) = V(D) + V_1 + V_2 + V_3,$$

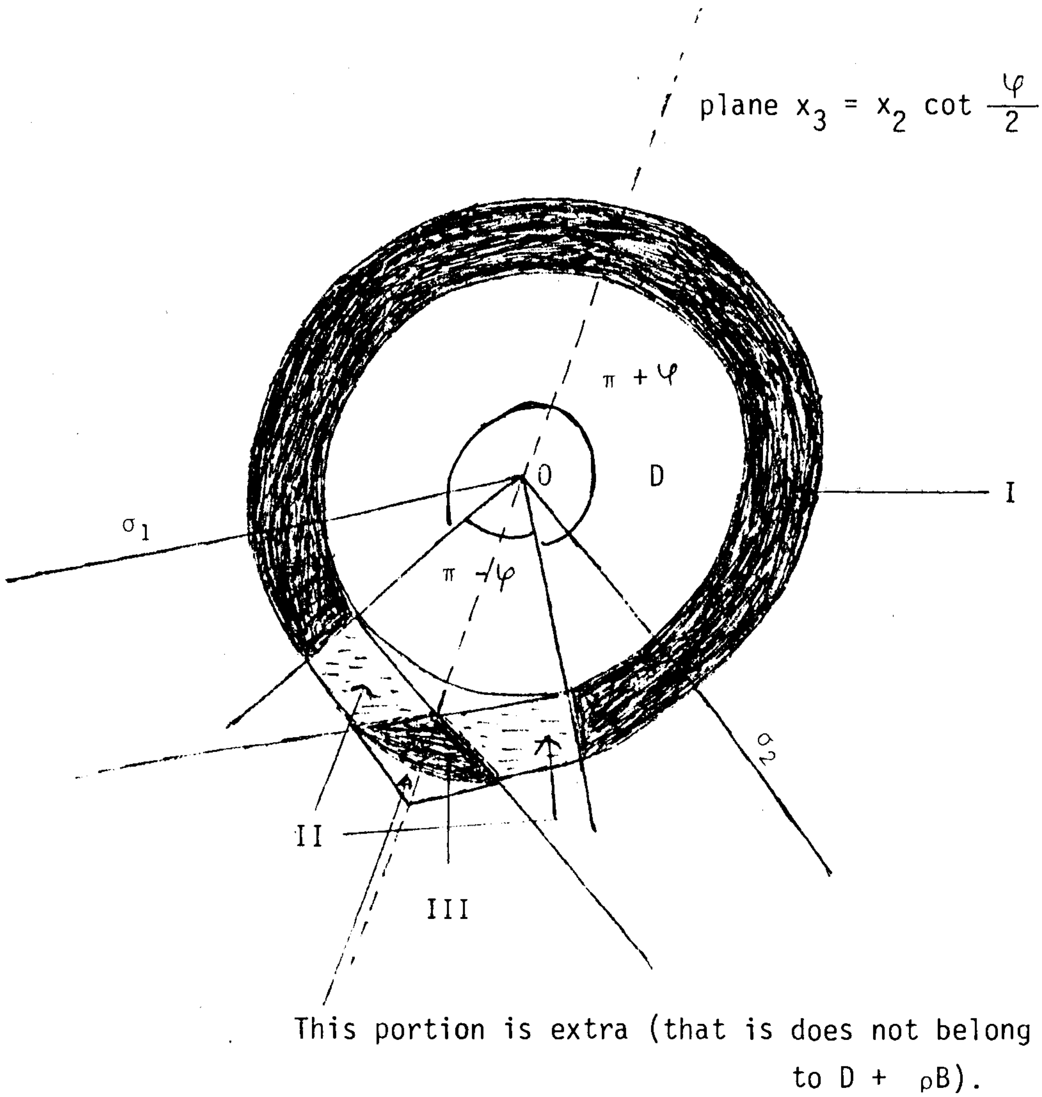
where V_1, V_2, V_3 are the volumes of the regions I, II, and III, respectively.

Region I = The region between the balls of radii r and $r + \rho$, as shown,

Region II = The region between the cylindrical surface C_1 of $T_1(r, \sigma_1)$ and the cylindrical surface of $T_1(r + \rho, \sigma_1)$ up to the height $C_1 \cap C_2$, and the similar region between the cylindrical surface C_2 of $T_2(r, \sigma_2)$ and the cylindrical surface of $T_2(r + \rho, \sigma_2)$, and,

Region III = the crescent-like region, above the region II, obtained by adding the balls on $C_1 \cap C_2$.

Figure 9: $D + \rho B$



The portion of II and III below the plane $x_3 = x_2 \cot \frac{\varphi}{2}$

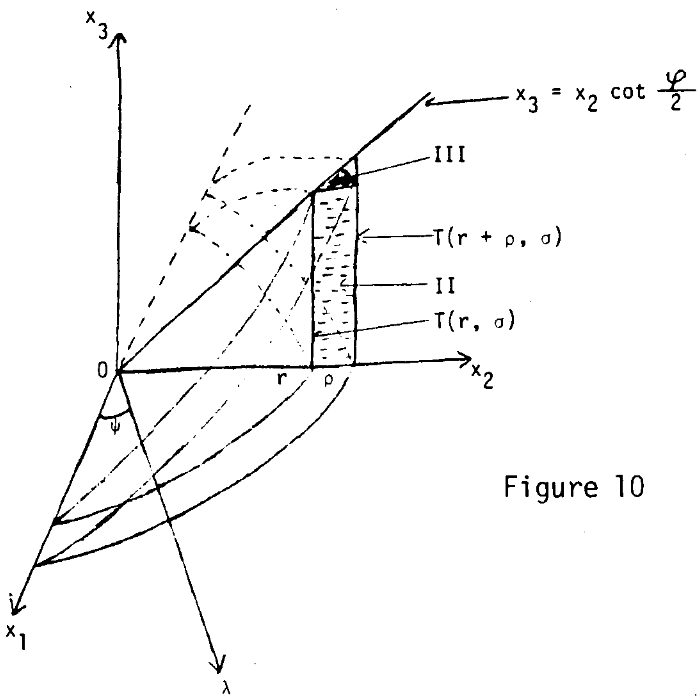
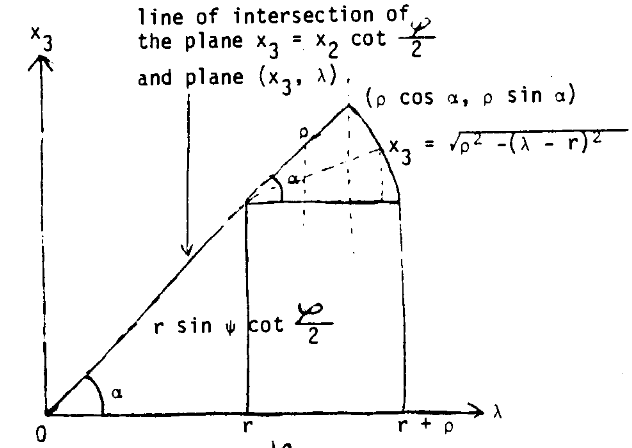


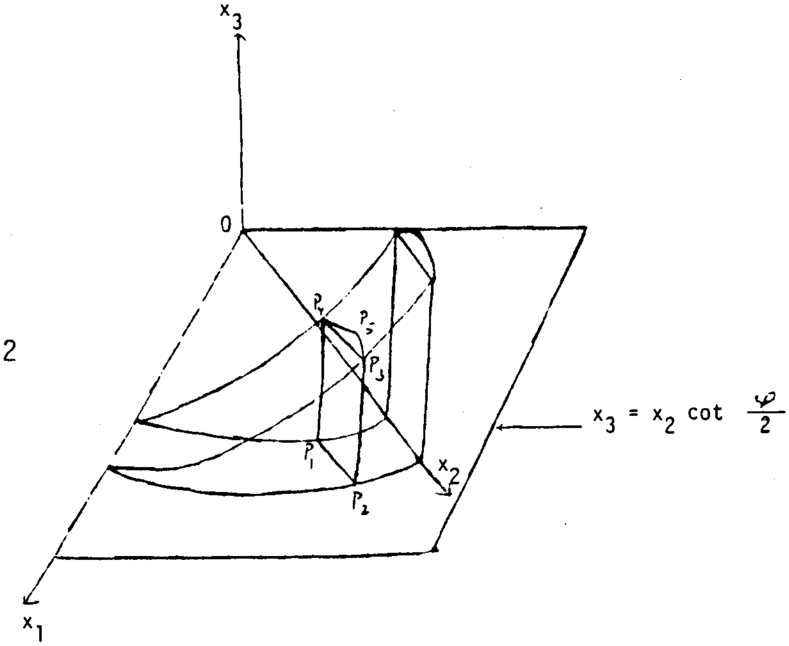
Figure 10



$$\tan \alpha = \frac{r \sin \psi \cot \frac{\varphi}{2}}{r} = \sin \psi \cot \frac{\varphi}{2}$$

Figure 11

Figure 12



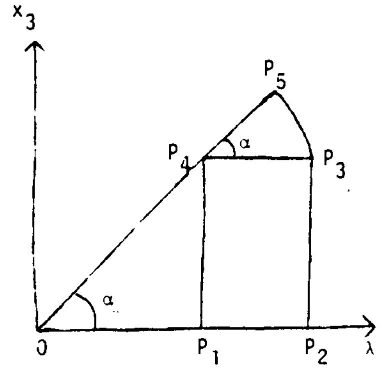


Figure 13

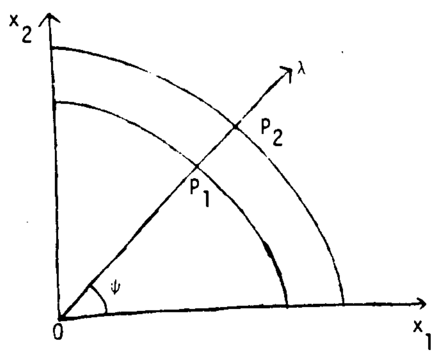


Figure 14

$$V_2 = 4\int_0^{\frac{\pi}{2}} \Big(\int_{P_1P_2P_3P_4} \lambda d\lambda dx_3\Big)\, d\psi \qquad V_3 = 4\int_0^{\frac{\pi}{2}} \Big(\int_{P_3P_4P_5} \lambda d\lambda dx_3\Big)\, d\psi$$

$P_1 = (r \cos \psi, r \sin \psi)$

$P_2 = ((r + \rho) \cos \psi, (r + \rho) \sin \psi)$

$P_3 = ((r + \rho) \cos \psi, (r + \rho) \sin \psi, r \sin \psi \cot \frac{\varphi}{2})$

$P_4 = (r \cos \psi, r \sin \psi, r \sin \psi \cot \frac{\varphi}{2})$

$P_5 = ((r + \rho \cos \alpha) \cos \psi, (r + \rho \cos \alpha) \sin \psi, r \sin \psi \cot \frac{\varphi}{2} + \rho \sin \alpha)$

$$\therefore V_1 = \frac{4\pi(r+\rho)^3}{3} \cdot \frac{\pi+\varphi}{2\pi} - \frac{4\pi r^3}{3} \cdot \frac{\pi+\varphi}{2\pi}$$

$$V_2 = 4\int_0^{\frac{\pi}{2}} \left(\int_r^{r+\rho} \left(\int_0^{r\sin\psi\cot\frac{\varphi}{2}} dx_3\right) \lambda d\lambda\right) d\psi$$

$$V_3 = 4\int_0^{\frac{\pi}{2}} \left[\int_r^{r+\rho\cos\alpha} \left(\int_{r\sin\psi\cot\frac{\varphi}{2}}^{r\sin\psi\cot\frac{\varphi}{2}+(\lambda-r)\tan\alpha} dx_3\right)\lambda d\lambda + \int_{r+\rho\cos\alpha}^{r+\rho} \left(\int_{r\sin\psi\cot\frac{\varphi}{2}}^{r\sin\psi\cot\frac{\varphi}{2}+\sqrt{\rho^2-(\lambda-r)^2}} dx_3\right)\lambda d\lambda\right] d\psi,$$

where $\tan\alpha = \sin\psi \cot\frac{\varphi}{2}$

$$V_2 = 4\int_0^{\frac{\pi}{2}}\int_r^{r+\rho} r\sin\psi\cot\frac{\varphi}{2}\,\lambda d\lambda d\psi$$

$$= 4\int_0^{\frac{\pi}{2}} r\sin\psi\cot\frac{\varphi}{2}\left[\frac{(r+\rho)^2}{2} - \frac{r^2}{2}\right] d\psi$$

$$= 2r\cot\frac{\varphi}{2}(2r\rho+\rho^2)$$

Now

$$\int_r^{r+\rho\cos\alpha} \left(\int_{r\sin\psi\cot\frac{\varphi}{2}}^{r\sin\psi\cot\frac{\varphi}{2}+(\lambda-r)\tan\alpha} dx_3\right)\lambda d\lambda = \int_r^{r+\rho\cos\alpha} (\lambda^2-\lambda r)\tan\alpha\, d\lambda$$

$$= \frac{\rho^3\cos^2\alpha\sin\alpha}{3} + \frac{r\rho^2\cos\alpha\sin\alpha}{2}, \text{ and}$$

$$\int_{r+\rho\cos\alpha}^{r+\rho} \left(\int_{r\sin\psi\cot\frac{\varphi}{2}}^{r\sin\psi\cot\frac{\varphi}{2}+\sqrt{\rho^2-(\lambda-r)^2}} dx_3\right)\lambda d\lambda = \int_{r+\rho\cos\alpha}^{r+\rho} \sqrt{\rho^2-(\lambda-r)^2}\,\lambda d\lambda$$

$$= \int_{\rho \cos \alpha}^{\rho} \sqrt{\rho^2 - u^2}\,(r + u)\, du$$

$$= r\rho^2 \left(1/2\ \alpha - 1/4 \sin 2\alpha \right) + 1/3\ \rho^3 \sin^3\alpha\ )\ d\psi$$

$$V_3 = 4 \int_0^{\frac{\pi}{2}} \left(\frac{\rho^3 \cos^2\alpha \sin \alpha}{3} + \frac{r\rho^2 \alpha}{2} + \frac{\rho^3 \sin^3\alpha}{3} \right) d\psi$$

$$= \frac{4\rho^3}{3} \int_0^{\frac{\pi}{2}} \sin \alpha\, d\psi + 2r\rho^2 \int_0^{\frac{\pi}{2}} \alpha\, d\psi$$

$$= \frac{4\rho^3}{3} \int_0^{\frac{\pi}{2}} \frac{\sin \psi \cot \frac{\varphi}{2}}{\sqrt{1 + \cot^2 \frac{\varphi}{2} \sin^2 \psi}}\, d\psi + 2r\rho^2 \int_0^{\frac{\pi}{2}} \arctan(\sin\psi \cot \frac{\varphi}{2})\, d\psi$$

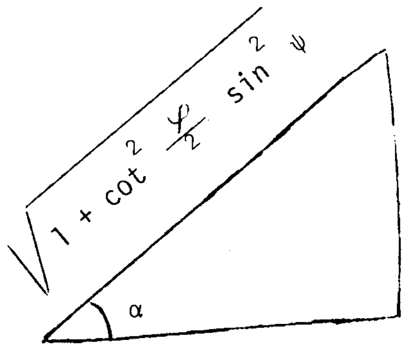


Figure 15

$$= \frac{4\rho^3}{3} \left(\frac{\pi}{2} - \frac{\varphi}{2} \right) + 2r\rho^2\, G(\varphi), \text{ where}$$

$$G(\varphi) = \int_0^{\frac{\pi}{2}} \arctan\left(\sin \psi \cot \frac{\varphi}{2}\right) d\psi$$

$$\therefore \quad V_1 = 2/3\ (\pi + \varphi)\ (3r^2\rho + 3r\rho^2 + \rho^3)$$

$$V_2 = 2r \cot \frac{\varphi}{2}\ (2r\rho + \rho^2), \text{ and}$$

$$V_3 = \frac{4\rho^3}{3}\left(\frac{\pi}{2} - \frac{\varphi}{2}\right) + 2r\rho^2\, G(\varphi), \text{ where}$$

$$G(\varphi) = \int_0^{\frac{\pi}{2}} \arctan\left(\sin\psi \cot\frac{\varphi}{2}\right) d\psi$$

$$\therefore \quad V(D + \rho B) = V(D) + \left[2r^2(\pi + \varphi) + 4r^2 \cot\frac{\varphi}{2}\right]\rho$$

$$+ \left[2r(\pi + \varphi) + 2r\cot\frac{\varphi}{2} + 2r\, G(\varphi)\right]\rho^2$$

$$+ 4/3\left[\frac{\pi}{2} + \frac{\varphi}{2} + \frac{\pi}{2} - \frac{\varphi}{2}\right]\rho^3$$

$$= W_0(D) + 3\, W_1(D) + 3\, W_2(D) + W_3(D).$$

$$\therefore \quad W_0(D) = V(D) = \frac{4r^3}{3}\left(\frac{\pi}{2} + \frac{\varphi}{2}\right) + \frac{4r^3}{3}\cot\frac{\varphi}{2}$$

$$W_1(D) = 1/3\left[2r^2(\pi + \varphi) + 4r^2\cot\frac{\varphi}{2}\right] = 1/3\ S(D),$$

where $S(D)$ is the surface area of D

$$W_2(D) = 1/3\left[2r(\pi + \varphi) + 2r\cot\frac{\varphi}{2} + 2r\, G(\varphi)\right], \text{ where}$$

$$G(\varphi) = \int_0^{\frac{\pi}{2}} \arctan\left(\sin\psi\cot\frac{\varphi}{2}\right) d\psi$$

$$W_3(D) = \frac{4\pi}{3}.$$

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