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Title: The Relationship of A Problem-Based Calculus Course And Students' Views of Mathematical Thinking

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Margaret L. Niess

It has been held that heuristic training alone is not enough for developing one's mathematical thinking. One missing component is a mathematical point of view. Many educational researchers have proposed problem-based curricula to improve students' views of mathematical thinking. The present study reports findings regarding effects of a problem-based calculus course, using historical problems, to foster Taiwanese college students' views of mathematical thinking.

The present study consisted of three stages. During the initial phase, 44 engineering majors' views on mathematical thinking were tabulated by a six-item, open-ended questionnaire and nine randomly selected students were invited to participate follow-up interviews. Students then received an 18-week problem-based calculus course in which mathematical concepts were problematized in order to challenge their personally expressed empirical beliefs in doing mathematics. Several tasks and instructional approaches served to reach the goal.

Near the end of the semester, all participants answered the same questionnaire and the same students were interviewed to pinpoint their shift in views on mathematical thinking. It was found that participants were more likely to

value logical sense, creativity, and imagination in doing mathematics. Further, students leaned toward a conservative attitude in the certainty of mathematical knowledge. Participants' focus seemingly shifted from mathematics as a product to mathematics as a process.

THE RELATIONSHIP OF A PROBLEM-BASED CALCULUS COURSE AND STUDENTS' VIEWS OF MATHEMATICAL THINKING

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APPROVED:

Redacted for Privacy

Major Professor, representing Mathematics Education

Redacted for Privacy

Head of Department of Science and Mathematics Education

Redacted for Privacy

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

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Po-Hung Liu, Author

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TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (continued)

TABLE OF CONTENTS (continued)

LIST OF TABLES

LIST OF APPENDICES

THE RELATIONSHIP OF A PROBLEM-BASED CALCULUS COURSE AND STUDENTS' VIEWS OF MATHEMATICAL THINKING

CHAPTER 1
THE PROBLEM

Knowing mathematics is doing mathematics and, in its broadest sense, mathematical problem solving is nearly synonymous with doing mathematics (National Council of Teachers of Mathematics [NCTM], 1989). The mathematical problem, according to Halmos (1980), is "the heart of mathematics," which means the desire for solving problems initiates the progress of mathematical knowledge. The central doctrine of mathematical problem solving is thinking about the mathematical problems. As such, the teaching and learning of problem solving to a great extent reflects the value of "thinking" in mathematics. Though the publication of the well-known book *How To Solve It* (Polya, 1945) began to draw public attention to problem solving in mathematics, *An Agenda for Action* (NCTM, 1980), advocating problem solving as *the central focus of school mathematics* [italics added], actually initiated "the decade of problem solving" of the 1980s. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), one of the most influential NCTM publications in the 1980s, further argued that problem solving is "*a primary goal* [italics added] of mathematics instruction and *an integral part* [italics added] of all mathematical activity" (p. 23). While in the recent book *Principles and Standards for School Mathematics* (NCTM, 2000), though still regarded as an integral part of all mathematical learning, the role of problem solving has turned to "*the cornerstone* [italics added] of school mathematics" (p. 182). The central role of problem solving in mathematics education has seemingly shifted to support holistic teaching and learning.

From Problem Solving to Mathematical Thinking

Polya's four-phase theory sketches a blueprint for mathematical problem solving, and heuristic strategies serve as fuel for problem-solving actions. According to Polya, heuristics suggest the methods frequently used by mathematicians in solving mathematical problems, and "the study of heuristics has 'practical' aims; a better understanding of the mental operations typically useful in solving problems could exert some good influence on teaching, especially on the teaching of mathematics" (Polya, 1945, p. 130). In the 1970s and 1980s, heuristics were considered essential for teaching and studying mathematical problem solving. Statistical analysis was the dominant methodology in which control and experimental groups' behavior in employing heuristic strategies were analyzed and compared. With few exceptions (e.g., Post & Brennan, 1976), most studies of heuristics concurrently suggested that teaching heuristics would significantly improve students' ability in employing heuristics to solve non-routine mathematical problems. Though seemingly effective in a laboratory setting, study along this line nevertheless has long been criticized by many scholars for its limited power to prepare students to transfer the ability to new contexts (Lester, 1994; Owen & Sweller, 1989; Sweller, 1990). In addition to knowing heuristics, it has been well-recognized that the student has to learn "when" to use heuristics, a holistic view of monitoring the problem-solving processes.

Given the insufficiency of improving students' problem-solving ability by training them to apply heuristics, mathematics education researchers have revisited the ultimate goals of mathematics instruction and how problem solving fits within the goals. NCTM (1991) contends the goals of teaching mathematics are "to help all students develop mathematical power" and "all students can learn to think mathematically" (p. 21). However, problem solving alone is not enough for developing one's mathematical thinking; metacognitive behavior and a mathematical point of view, a view of thinking mathematically, are two missing parts in the traditional training of problem solving (Schoenfeld, 1992). Defining the meaning of learning to think mathematically or developing mathematical thinking is not easy. Schoenfeld (1992) interpreted the meaning from epistemological, ontological, and pedagogical aspects. He further indicated learning to think mathematically " means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making" (1994b, p. 60). Burton (1984) outlined mathematical thinking in terms of operations, processes, and dynamics in which both inductive and deductive learning are involved. Mason, Burton, and Stacey (1982) sketched the dynamics of mathematical thinking as a helical model constituted of manipulating, getting a sense of pattern, and articulating that pattern symbolically. Yet mathematical thinking should not be mistakenly interpreted as thinking like a mathematician because mathematicians think about mathematics in different ways. Rather, mathematical thinking is a way of seeing the world through the perspectives of mathematicians (Schoenfeld, 1992).

Students acquire their sense of mathematics through the experience of doing mathematics. The major merit of developing students' mathematical thinking is helping them to become mathematical thinkers, not merely doers or solvers. A mathematics thinker, as compared to a doer or solver, is more likely to have a mathematical disposition, an inductive attitude of looking for and exploring patterns to understand mathematical structures. For a mathematical thinker, the joy of solving problems lies in a "STUCK! and AHA!" process as Mason, Burton, and Stacey (1982) indicated rather than obtaining the correct answer. As such, a mathematics thinker shifts knowledge acquisition to knowledge construction, acting as an active member of the mathematics community.

Problem Solvers' Views of Mathematical Thinking

The relationship between students' views or beliefs about doing mathematics and their learning behaviors has increasingly attracted considerable attention in recent years (Carlson, 1999; Franke & Carey, 1997; Higgins, 1997; Kloosterman & Cougan, 1994; Kloosterman and Stage, 1992; Schoenfeld, 1988, 1989). Studying beliefs or views of doing mathematics is based upon the assumption that beliefs or views are potential indicators of the decision individuals make and act as a sort of filter of strategies or approaches they would adopt while engaging in mathematical tasks. As aforementioned, in addition to metacognitive behavior, having an appropriate view of thinking mathematically is another important component for developing the capacity in mathematical thinking. Students develop their world views of doing mathematics from practical experience which in turn may exert an impact on their subsequent learning.

On the basis of empirical investigations, some studies indicated that if students view doing mathematics as a rigid process by following fixed and predetermined procedures, they may be more reluctant to engage in creative mathematical activities. On the contrary, an active view of mathematical thinking and reasoning would potentially promote an individual's desire of being involved in challenging tasks (Bransford, Zech, Schwartz, Barron, and Vye, 1996; Carlson, 1999; Franke & Carey, 1997; Henningsen, & Stein, 1997; Higgins, 1997, Schoenfeld, 1983a, 1989, 1992). In a project-based study, Bransford et al. (1996) found that middle school students' limited views of mathematical thinking were quite similar to their teachers' views, prohibiting their selections of mathematical operations to resolve mathematical tasks. After experiencing a SMART program in which students' views of mathematical thinking were challenged in various ways, students had a much greater appreciation of mathematical thinking and began to understand how to employ mathematical tools more effectively. Moreover, through investigating the relationship between college and graduate students' mathematical behavior and views of doing mathematics, Carlson (1999) reported that mathematics graduate students usually hold more expert views of mathematical thinking (e.g., more persistent and flexible while thinking about mathematics) than their undergraduate peers and are more likely to attempt generic problem-solving approaches. Franke and Carey (1997) and Higgins (1997) both found that, after experiencing problem-solving instruction, students perceived doing mathematics more as a problem-solving endeavor involving communicating mathematical thinking, which exerted an impact on their subsequent learning. While locating factors supporting and inhibiting mathematical thinking, Henningsen and Stein (1997) contended students' mechanical views of thinking prevent them from engaging in high-level cognitive process. By observing two students' behavior in solving geometric construction problems before and after a problem-solving course, Schoenfeld (1983a) suggested that beliefs about doing geometry problems contributed to students' failure or success.

A basic understanding of the intrinsic essence of mathematical knowledge is requisite for mathematical literacy. To reach the goal, students need to comprehend the nature of mathematical thinking (American Association for the Advancement of Science [AAAS], 1990). On the basis of afore-cited empirical evidence, investigating and developing problem solvers' views of mathematical thinking are noteworthy issues to receive further investigation.

Problem-Based Learning

The main idea of problem-based learning is to problematize mathematical concepts—leading students to resolve problematic situations. As defined by Boud and Feletti (1991), problem-based learning is a way of constructing and teaching content using problems as the stimulus for student activity. But they further emphasized that problem-based learning is not "simply the addition of problem-solving activities to otherwise discipline-centered curricula, but a way of conceiving of the curriculum which is centered around key problems in professional practice" (p. 14). Several groups of researchers have suggested that the problem-centered approach or research-based model, similar to the problem-based instruction, may facilitate learners' views and capability of mathematical thinking (Cobb, Wood, Yackel, Nicholls, Wheatly, Trigatti, & Perlwitz, 1991; Cobb, Wood, Yackel, & Perlwitz, 1992; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996; Wood & Sweller, 1996). For instance, Fennema et al. (1996) found that by engaging 18 teachers in a variety of problem-solving situations and encouraging them to talk about their mathematical thinking, fundamental changes were identified in their beliefs about doing and thinking about mathematics, which in turn helped them understand the importance of developing students' mathematical thinking. Henningsen and Stein (1997) and Stein, Grover, and Henningsen (1996) also reported student capacity for high-level mathematical thinking may be achieved by using various mathematical tasks as vehicles to build student capacity for mathematical thinking and reasoning. According to Henningsen and Stein (1997), the nature of problems "can potentially influence and structure the way students think [about mathematics] and can serve to limit or broaden their views of the subject matter with which they are engaged" (p. 525). On the basis of protocol analysis, Schoenfeld (1983a) also attributed his students' progress in problem-solving ability to their belief change rather than gaining more knowledge. The aforementioned findings suggest that in a problem-based learning setting, not only learners' mathematical thinking ability, but their views about thinking mathematically, can be improved.

The influence of problem-based learning on students' views of mathematical thinking may be understood in terms of Piaget's concepts of assimilation and accommodation. Thinking mathematically is experiencing a process of struggling for meaning (Burton, 1984), which is a place for problem-based learning. Problem-based learning is treating mathematics topics as problematic situations, providing students a different vantage point from which to look at doing mathematics (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Oliver, & Wearne, 1996), encouraging students to integrate and connect their existing, but possibly isolated, conceptions about doing mathematics. Namely, problem-based learning provides students a variety of opportunities for assimilation and accommodation. In this manner, a view of mathematical thinking may become an important residue of solving problems and acquiring knowledge.

Mathematics Education in Taiwan Technological College

Though Taiwanese high school students perform well in international mathematics contests (e.g., International Association for the Evaluation of Educational Achievement [IEA], 2000), the teaching of mathematics in Taiwan has long been criticized by its exam-driven approach. Students are able to score high on tests but a deep understanding of mathematics is lacking. The researcher of the present study conducted a pilot study of technological college students in Taiwan and found that they tended to consider mathematics as a matter of calculation and view mathematical thinking as simply an act of solving mathematics problems. It was also found that these rooted misconceptions hindered their learning in mathematics at the college level requiring more sophisticated thinking. Their inappropriate views of doing mathematics and mathematical knowledge proper may have been formed by the precollege discipline in which they were largely trained to solve multiple-choice items. Memorization and rote calculation were the major learning modes. On the basis of the classroom teaching experience of the researcher of the present study, students were able to get answers quickly by following short-cut methods but were totally unaware of the structure of problems. A plausible explanation is this behavior could be largely shaped by the college entrance examination in which they were required to solve 20 mathematics problems in 90 minutes.

The technological college graduates in Taiwan are typically expected to meet technical and practical needs of industrial enterprises; thinking and creativity are not emphasized in their curricula. Whereas, along with the upgrade of Taiwan's enterprises, rote learning of this sort no longer matches the societal demand for improving college students' capability of thinking and creativity. An educational reform call thus has arisen in Taiwan (Chang, 1999; Kuo, 1999).

Statement of the Problem

The chief aims of this research were to investigate whether Taiwanese technological college engineering-major freshmen's initial views of mathematical thinking are consistent with the current view of mathematical thinking and explore the interrelationship between a problem-based calculus course and students' views of mathematical thinking. The major research questions of interest were:

1. What are Taiwanese technological college freshmen engineering-majors' views of mathematical thinking?

2. In what aspects and to what extent, if any, do Taiwanese technological college freshmen engineering majors' views on mathematical thinking change during a problem-based calculus course?

3. What is the relationship, if any, between a problem-based calculus course features and the development of students' views on mathematical thinking?

Significance of the Study

The present study was designed to assess college students' views on mathematical thinking and investigate the development of students' views of mathematical thinking during a problem-based calculus course. Students' views of thinking about mathematics may act as a filter of doing mathematics which in turn exert an influence on their evaluation of ability, on their desire to engage in mathematical activities, and on their ultimate mathematical disposition (NCTM, 1989). Further, teachers' instruction may exert a certain degree of influence on students' views of doing and thinking about mathematics by challenging their existing belief systems (Bransford, et al., 1996; Ford, 1994; Franke & Carey, 1997; Higgins, 1997; Schoenfeld, 1985c, 1988, 1989). In this notion, assessing students' views about mathematical thinking and probing how their views evolve within a problem-based context are important components of the overall assessment of students' mathematical knowledge.

Regardless of the outcome, the present study brings together several significant implications to the study of mathematics education. The relationship between beliefs about abilities in mathematics and doing and learning mathematical problem solving has been well-documented even though there is a lack of a specific working definition of beliefs. Although the global construct makes beliefs difficult to measure, "when specific beliefs are carefully operationalized, appropriate methodology chosen, and design thoughtfully constructed, their study becomes viable and rewarding" (Pajares, 1992, p. 308). Compared to beliefs about holistic ability in mathematics, an individual's views of mathematical thinking is a more specific way of looking at the construct, with more reflection on an individual's spontaneous conceptions; this perspective may be more precise and feasible to investigate. Investigating students' views of mathematical thinking therefore is more likely to sketch the ideal modality rooted in their minds—what mathematical thinking should be or should not be; yet this effort has rarely been studied.

Secondly, a considerable amount of effort has been devoted to improving elementary and high school students' problem-solving ability or mathematical thinking through problem-centered or research-based learning. Few studies, if any, have paid attention to the college level. College mathematics curricula usually require a more advanced level of thinking, yet college teachers often find college freshmen's mathematical thinking ability ill-prepared (Cohen, Knoebel, Kurtz, & Pengelley, 1994). Relevant scholars (Schoenfeld, 1994a; Tall, 1991) have called for improving college students' mathematical thinking and developing a mathematical point of view. However, mathematical problem solving has been conducted in a way of distinction between acquiring background knowledge and applying it to solve non-routine problems, which is inappropriate for mathematics education (Hiebert, et al., 1996). On the contrary, problem-based learning distinguishes it from traditional problem solving by problematizing mathematical concepts which means "allowing students to wonder why things are, to search for solutions, and to resolve incongruities. It means that both curriculum and instruction should begin with problems, dilemmas, and questions for students" (Hiebert, et al., 1996, p. 12). As such, both students' problem-solving ability and their views on thinking about mathematics may be enhanced interactively. Therefore, the present study, exploring college students' views on mathematical thinking in a problem-based learning setting, is a practical action exploring the interactive relationship between curriculum and instruction at the college level.

Thirdly, the mathematics teaching in Taiwan is usually exam-driven, emphasizing memorization and rote learning, at the cost of a moderate view of mathematical thinking. The present study is an attempt to investigate Taiwan college students' views about mathematical thinking and challenge their views on mathematical thinking by problematizing mathematics, a different instructional approach in Taiwan. The findings hopefully can generate productive thought on this issue and draw research interest in Taiwanese mathematics education community.

CHAPTER II
REVIEW OF THE LITERATURE

Problem-based learning is a learning paradigm confronting students with ill-structured, problematic situations in which students assume the role of the stakeholder of these situations and teachers stimulate students' critical thinking via probing, questioning, and challenging student thoughts (Torp & Sage, 1998). Students' conceptual change, in this manner, can be an important residue after engaging problem-solving activities (Heibert, et al, 1996). The purpose of the present study is to explore in what way and to what extent a problem-based calculus course may influence college freshmen's views of mathematical thinking, involving relevant problem-solving aspects of mathematics. In this chapter, recent relevant literature is consulted on these fields in order to reveal the significant role of individuals' views on mathematical thinking.

Heuristic Training Through the Ages

Early studies on mathematical problem solving mainly paid attention to the effect of heuristics in learning problem solving. Heuristics, according to Polya (1945), are means that lead to discovery. The purpose of heuristic training is to explore the rules of discovery or invention. Such attempts can be traced back to Descartes and Leibnitz, who dreamed of establishing universal laws for finding certainty through building up a system of heuristics. Evidence for the validity of Polya's heuristic rules came from early work on computer simulations of human behavior in 1960s. Programmers found that the incorporation of general heuristic rules not only facilitated problem solving by the computer, but closely resembled human behavior while struggling with similar problems (Kilpatrick, 1969). At the time, a good share of the research regarding heuristics was being done by doctoral students (Wills, 1967; Wilson, 1967, cited in Kilpatrick, 1969).

Wilson (1967) explored to what extent subjects taught heuristics were able to successfully implement heuristics on other domains. Subjects were given two tasks and taught to use one of three kinds of heuristics for each: task-specific (applicable to the training task only), means-end (shortening the gap between the given situation and the goal), and planning (omitting details in the given situation and working toward a proposed solution in general terms). Results showed task-specific heuristics did not facilitate performance on the training task as Wilson predicted; on dissimilar tasks, planning heuristics were superior to the other two. Wills (1967) investigated the effect of a two-week lesson in which two groups of eight intermediate algebra classes were given a series of simpler problems and encouraged to search for patterns in the problems, then further generalize the problems. In the experimental group, students were guided by heuristic rules; the control group received no such guidance. On the posttest both groups doubled their pretest performance, except one class in the control group who made little progress. The outcome suggested that the heuristic methods contributed only a minor gain.

Though the findings were not as optimistic as researchers expected, the success of these training programs in specific situations (such as in Wilson's study planning heuristics was effective on dissimilar tasks) encouraged future effort of research on heuristics. Further, the birth of *Journal for Research in Mathematics Education* in 1970 no doubt did provide a more public and convenient forum for discussing relevant educational issues in mathematics. During the 1970s and 1980s, much effort was made to test the power of heuristics by comparing subjects' problem-solving performance between experimental and control groups (Charles & Lester, 1984; Lee, 1982; Lucas, 1974; Post & Brennan, 1976; Vos, 1976).

Lucas (1974) reported an exploratory study attempting to investigate heuristic usage and to analyze the influence of heuristic-oriented teaching on 30 first-year calculus students. Subjects were divided into four groups to correspond with two experimental conditions (exposure to heuristic instruction versus no exposure) and two testing conditions (exposure to both pre- and posttests versus exposure to posttest only). The study was executed in three phases lasting 13 weeks and a comparison between control group (receiving general treatment) and experimental group (receiving heuristics) was made. The statistical outcome showed significant differences favoring the instructional treatment in several respects (e.g., using mnemonic notation more frequently, separating and summarizing data, lower frequencies of reading problem, higher scores on all four aspects). However, data regarding students' pre-instructional status were lacking. Because intact classes were used, whether heuristic students' outperformance can be attributed to treatment is unknown. Further, as a quantitative study, the small sample size also limited the scope of generalization.

Aside from the above concerns, the most critical issue of Lucas' study is the employment of the system of behavioral analysis. Subjects' problem-solving performance in this study was coded and analyzed according to several visible phenomena; yet this procedure may fail to reflect an individual's inner thinking process as well as distort it. For instance, the frequency of drawing diagrams, which is task-dependent, is not appropriate to be considered as an index of successful implementation of heuristics. The quality of diagrams ought to be far more important than the quantity. Moreover, it is also arguable that a subject's frequency of re-reading problems and time of hesitation were regarded as two indicators of difficulty. Spending less time in re-reading and hesitating was a sign of outperformance in Lucas's study. In this manner, a slow reader might be categorized as a poor solver, which should be inappropriate.

Within the assumption that heuristic training does enhance students' problem solving ability is valid, some researchers asked "Are heuristics good for all?" (Lee, 1982; Vos, 1976). Vos (1976) conducted a study to investigate the effect of heuristic teaching by comparing three instructional modes for promoting five heuristic strategies (drawing a diagram, approximating and verifying, constructing an algebra equation, classifying data, and constructing a chart). The 133 high school students from six mathematics classes across three grade levels were randomly assigned to one of three experimental groups: repetition treatment group (R), list treatment group (L), and behavior instruction group (B). All students were given 20 problems over a 15-week period of time. The R-treatment group was given training problems only without any instruction. The L-treatment group was asked to solve assigned problems, then received extra written instructions. The student then was allowed to return to work on the original tasks after reading these written instructions. The B-treatment group first received individual written instruction in which a specific strategy for solving the training problem was given, then were asked to do the same problem as the R-and L-treatment groups. ANOVA results showed that the B-treatment group though, on the problem-solving test, exhibited the highest proportion of occurrences of the five problem-solving strategies and other strategies in three classes, the score was not significant higher than other others in Algebra II class. The finding suggests that the desire to use heuristics may not guarantee higher performance.

On the other hand, Polya's heuristics were seldom discussed on the basis of Piagetian theory. Lee (1982) hence aimed to answer the following questions: (a) can fourth graders at a concrete operational stage acquire and appropriately use heuristics and (b) are there differences between early and late concrete operational stage children while using heuristics? Through several Piagetian interviews in which students were asked to solve balance and pendulum problems, eight fourth-grade students were classified at early concrete operational stage, denoted by II-A, another eight fourth-grade students at late concrete operational stage, denoted by II-B. All 16 students were then randomly assigned to instruction and no-instruction groups of four students each. The former received extra 20 problem-solving lessons in which four heuristic strategies (understanding the problem, making a plan, carrying out the plan, and looking back) were taught. Nine-week later, differences between instruction and no-instruction students' heuristic behavior were analyzed. It was found that instruction students were more likely to select appropriate heuristics and use them effectively in more cases than their counterparts, yet this did not guarantee their obtaining correct answers. Further, II-B instruction students (late concrete operational stage) were able to employ heuristics more efficiently than II-A instruction students (early concrete operational stage). On the basis of Vos (1976) and Lee (1982), it therefore can be seen that heuristic training may not significantly improve one's problem-solving ability. The issue of idiosyncrasy must be taken into account.

Around 1980, major contributions to the research of heuristics were made by Schoenfeld, who not only devoted himself to exploring effects of heuristics but frankly pointed out the shortage of heuristic training. He argued that, owing to many uncontrolled variables in experimentation, use of artificial instructional environments may be appropriate to test the effect of heuristics (Schoenfeld, 1979b). In his study, seven upper-division science and mathematics majors were recruited as volunteers to participate in the study, four randomly assigned to the experimental group and three to the control group. All subjects were asked to work on five mathematical problems, either in the pre- or posttest, and talk out loud procedure while solving problems. Between the pretest and posttest, five instruction sessions were administered to all students over a two-week period; instruction was mostly carried out through written materials and tape-recorded lectures. In each instruction session, each student was given a tape recorder and booklet containing four practice problems. After solving the problems, students were allowed to turn to the next page in the booklet in which the solutions were shown and turned on the tape recorder to listen to solutions parallel to the written solution. Thus, both groups spent the same amount of time in solving the same problems. The heuristics group, however, received explicit heuristics treatments. At the outset of the practice sessions, heuristics students were given a list of five strategies and told that the experiment was designed try to see how the strategies assisted them in solving the problems. Heuristics students also listened to a ten-minute tape describing the strategies. In each written solution, the strategy used was highlighted for heuristics students only. Further, the heuristics students practiced one specific strategy only in each session; for non-heuristics students, the order of problems was mixed. While solving problems, heuristics students were reminded to refer to the list of strategies at five-minute intervals, non-heuristics students to look back at their work only. It was found that all heuristics students did improve from pre- to posttest, whereas only one nonheuristics student gained similar progress. The present study further looked at protocol data on the posttest; data indicated performance of the heuristics group was substantially different from its counterpart. For instance, on a problem best solved by using the strategy "consider a similar problem with fewer variables," none of the three nonheuristics students solved it, whereas three of four heuristics students successfully solved the problem by using the strategy mentioned above. According to quantitative data and qualitative observation on their solution, the investigator was convinced that "conscious application of some problem-solving strategy does make a difference" (pp.182-83). Nevertheless, an issue of greater concern is whether heuristics students' problem-solving ability really improved after the two-week heuristics training. The present study at most suggested that the experimental group could do better within a similar artificial setting. Regardless of several optimistic findings, noteworthy questions remain: (a) will they do so in unconstrained circumstances; (b) more importantly, can they transfer their training from a laboratory to a regular classroom? For instance, as noted earlier, a student's knowing how to use a strategy is no guarantee that the student will indeed use it. This behavior was frequently observed in heuristics students' problem-solving behavior during the posttest and "the implication of this kind of behavior is serious," Schoenfeld emphasized, "for they point to major difficulties in taking work like this experiment from the laboratory to the classroom" (p. 184).

Rethinking Heuristics Teaching

In another article Schoenfeld (1979a) scrutinized the influence of teaching heuristics. At the outset he confessed the existence of little evidence that general problem-solving skills can be taught, despite the fact that mathematicians do employ heuristics. The background for the above pessimistic statement is not clear. It is possible that, on the basis of his personal experience in teaching heuristics and relevant studies (e.g., Post, 1976), Schoenfeld perceived the equivocal effect of heuristics. However, Schoenfeld after all is an optimist about heuristics. He tried to clarify some issues through exploring positive and negative facets of heuristics teaching. As noted, Polya regards heuristics as a means for discovery. Schoenfeld elaborated more thoroughly: "A *heuristics* [italics added] is a general suggestion or strategy, independent of subject matter, that helps problem solvers approach, understand, and/or efficiently marshal their resources in solving problems" (p. 315).

Schoenfeld added the rationale for studying and teaching of heuristics as follows:

> 1. Through the course of his career, a problem solver develops an idiosyncratic style and method of problem solving. A systematic use of these strategies may take years to develop fully.
> 2. In spite of these idiosyncracies, there is a surprising degree of homogeneity in the approaches of expert problem solvers.
> 3. One could possibly extract a global problem-solving strategy, using first the introspections of experts and then incorporating systematic techniques of artificial intelligence.
> 4. This strategy can serve as a guide to the problem-solving process. Students instructed according to this plan could shorten the long and arduous task of arriving at these general principles themselves. (pp. 315-16)

The last one is the most speculative and merits further concern.

While employing heuristics, a problem solver has to face at least three obstacles: understanding the application of heuristics, sufficient knowledge of subject matter involved, and ability and opportunity to think how to apply the appropriate heuristics. It can be seen that the bulk of instruction in heuristics has largely focused on the first one; insufficient attention to the third barrier may account for the failure of heuristic instruction. Students not only need to know heuristics, but also must learn when to use it. This is what Schoenfeld called "managerial strategy," a means of assessing and allocating an individual's resources. With proper training, Schoenfeld was convinced students are able to learn to apply heuristics in rather sophisticated ways. To elaborate on this idea further, he established a schematic flow chart of problem-solving strategy in which the students were considered as "information processors" and the heuristic strategy as an "executive program" for the information processors. After a problem is given, the first stage for the problem solver is *Analysis*, attempting to understand the statement. After *Analysis*, hopefully one can sense the nature of the problem and proceed to the *Design* stage. *Design*, as defined by Schoenfeld, is a master control, "monitoring the whole process and allocating problem-solving resources efficiently" (p. 325). Once difficulty in making a plan occurs, the problem solver steps into the phase of *Exploration*, which may provide new insights into the problem and send the problem solver back to *Analysis*. In *Exploration*, the majority of heuristics may come into play. *Design* is followed by *Implementation*, a step-by-step execution, and lastly by *Verification*, which may help a problem solver catch silly mistakes or find alternative approaches. It can be readily seen that the above framework is an extended version of Polya's four-stage theory (understanding the problem, making a plan, carrying out the plan, and looking back).

In Schoenfeld's problem-solving class, the aforementioned flow chart was provided for students serving as a means to approach problems; the outcome showed students can learn essential ingredients of a managerial strategy and develop skills in determining appropriate heuristics. Several supportive cases were reported, yet one should bear in mind that the class was experimental (enrollment of eight students) only and relevant data lacking. Hence this perspective proposed by Schoenfeld is suggestive but not conclusive, no more than establishing a rationale and framework for teaching heuristics. The question of whether heuristics can help students transfer their training from a laboratory to a regular classroom remained unanswered. Many mathematics educators hold that heuristics alone can not resolve the "transfer issue" (Lester, 1994; Owen & Sweller, 1989; Sweller, 1990).

Regardless of its inability to resolve the transfer issue, the present theoretical article pointed out several prospective directions for further study on problem solving. First, Schoenfeld postulated that "How successfully one employs a heuristic depends significantly on the way he encodes information and *the perspective he brings to the subject matter* [italics added]" (p. 330). Though what he meant by "perspective" here is not clear, based on his subsequent arguments, it is a reasonable guess that Schoenfeld began to sense the critical role of students' beliefs about mathematics and doing mathematics at the time. Second, Schoenfeld also expressed a concern for the teacher. A successful problem-solving course, he pointed out, "depends on more than the compilation of the strategies which serve as its theoretical foundation. For example, the role of affective considerations, ..." (p. 333). The foresight in some sense initiated the study on teachers' beliefs about teaching mathematical problem solving from the mid-1980s to 1990s. Thirdly, Schoenfeld indicated that the degree of the problem solvers' confidence in solving particular problems may affect performance. Work in this respect has mostly been done by Pajares and Miller (1994, 1995, 1997), on the basis of Bandura's self-efficacy theory (Bandura, 1997). Nevertheless, only quantitative methodology was employed in this area so far. A qualitative approach may play a complementary role in revealing how an individual's confidence relates to his or her problem-solving performance.

Comparison Between Experts and Novices

Given the inability to resolve the "transfer" issue, the rationale of teaching heuristics strategies seems to be descriptive but not prescriptive. The focus of research in this respect then shifted to survey how expert problem solvers successfully executed heuristics. The major question asked was "What makes a good problem solver?" If behaviors of an expert problem solver can be codified, they might be taught to novices. As noted earlier, evidence of the validity of Polya's heuristics came from work on computer simulation of human behavior in the 1960s. Early work in this area was partly done by artificial intelligence programmers (e.g., Newell & Simon, 1972) who tried to duplicate human expertise, but the effort of this kind was announced as a failure (Schoenfeld, 1985).

On the other hand, studies on the issue of expert-novice differences in various fields suggested that mental representation of problems may influence how a problem solver interprets problems (e.g., Silver, 1979). Yet the evidence, though strong, is speculative, since some of these studies paid little attention to subjects' existing differences prior to entering the studies. For example, experts in these studies were older, more trained, more experienced, and more likely possessed a better aptitude for the subject domain. Schoenfeld and Herrmann (1982), therefore, conducted a study seeking to investigate experts' perception of problems in a design avoiding the drawbacks aforementioned. Nineteen college freshmen and sophomores participated; all had similar experiences in mathematics prior to the experiment. Eleven enrolled in a month-long problem-solving course serving as the experimental group, while eight (control group) were paid to attend a computer programming course which also lasted one month. In addition, nine mathematics professors whose perceptions of the problem were treated as a model of expertise were invited to participate in the study.

Before the courses began, both groups performed the card sort and took a mathematics test. One month later, they repeated the card sort and took another mathematics test. In the card sort activities, after reading through 32 problems, all students were asked to decide which problems, if any, were similar mathematically in that they would be solved the same way. Between the two sortings, the experimental group received intensive heuristics training, which stressed a systematic, organized approach to solving problems. Problems studied in this class were similar, but not identical, to those used in the card sort. The experimental students were encouraged to get a feel for the problem, whereas the control group was taught a structured, hierarchical, and orderly way to solve problems using the computer.

According to the scores on posttest mathematical tests, the experimental group showed considerable progress in problem-solving performance, whereas students in the control group did not ($p < .001$). Moreover, the result also indicated a strong change towards perception of problem structure on the part of the experimental group. The experimental group showed a higher degree of homogeneity with regard to deep structure of problems than that of one month before, while the control group showed little change from pre-instruction perception. The investigators therefore were convinced that the dramatic shift was attributed to the instructional treatment focused on understanding and performance. Further, the findings suggested that students' problem perception changed as they acquired problem-solving expertise. After receiving a short period of training, both their performance and perception became more like that of experts.

Note that the use of models of expertise in this study needs more attention. The issue here is what an expert really means. Too often an expert is thought of as a domain-expert who knows the subject matter thoroughly and can solve problems in an automatic way. However, Defranco (1996) challenged this view. While studying problem-solving behavior of eight Ph. D. mathematicians, Defranco found some of them neither solved non-routine mathematical tasks posed nor demonstrated a desired self-regulative behavior. Thus a domain-expert is not necessarily a process-expert. As such, nine mathematics professors' perceptions of the problem in this study being treated as a model of expertise is arguable. The definition of the expert and types of mathematical problems posed then became an important issue for this line of study.

In addition to individual perceptions of problems, Cooper and Sweller (1987) also pointed out another critical factor, rule automation of problem-solving operator, for explaining a problem solver's success or failure. Sweller and his partners (Owen & Sweller, 1985; Cooper and Sweller, 1987; Sweller, 1989) conducted a series of quantitative experiments to examine differences between experts' and novices' mathematical problem solving behavior and performance. In these studies, high-achievement high school students or successful problem solvers were seen as experts. Those studies concurrently suggested that working memory load is an important differentiating factor in solving problems and that this load can be reduced by automating the problem rules. Cooper and Sweller (1987) employed the explanation sheet, in which the significant steps for solving the problem were explicitly demonstrated, to upgrade students' ability in applying problem-solving operators. Results suggested students instructed in this way displayed superior performance on the transfer problem and spent less time solving problems, though automation occurred relatively slowly. So automation of problem-solving operators provided another explanatory mechanism for the issue of transfer. The study proved expert problem solvers more likely to work forward while solving problems, whereas novices appeared to use a means-end strategy, which may have imposed a heavy cognitive load and hence resulted in novices' failure.

Note that the mathematical problems used in Owen and Sweller (1985) were routine algebra problems similar to those in the textbook (e.g., solving the algebra problem $a + b - g = s$, expressing a in terms of the other variables). Use of routine problems may have failed to glean deep insight into students' thinking process, since students probably executed standard algebra operations soon after reading a problem. Non-routine problems, for triggering a problem solver's invisible thought process, therefore were needed. Schoenfeld conducted several studies to test the power of heuristics, yet around the early 1980s, his research agenda turned to probing students' problem-solving processes and locating factors contributing to their failure. What Schoenfeld aimed to do was to get a microscopic view of students' decision-making while attacking a non-routine problem. In doing so, two students in Schoenfeld's problem-solving class (Schoenfeld, 1983b) were asked to solve the problem below in less than 20 minutes:

> Three points are chosen on the circumference of a circle of radius R, and the triangle containing them is drawn. What choice of points results in the triangle with the largest possible area? Justify your answer as well as you can. (p. 371)

During the whole 20-minute session, students mostly embarked on calculation and rarely made any significant plan. Consequently, they were not able to solve the problem directly. Besides, progress and strategies used were not assessed and monitored, frequently resulting in the termination after the exploratory actions became impossible. For instance, several potential approaches did arise throughout, any of which might have led to success, but the nature of their rejection cost them time. Upon encountering an obstacle, they decided to explore another method in a casual way. They could not ask themselves what can be learned from those unsuccessful attempts. Although some correct assumptions were made (e.g., the final answer is symmetric), they were not justified mathematically. Instead, intuition-based empirical investigation was employed. In sum, though the students were aware of a number of clever ideas and had access to a variety of heuristics and algorithmic techniques, their local assessment was working well, whereas their global assessment was poor.

Schoenfeld attributed failure to absence of assessments and strategic decision making. Namely, a poor managerial strategy could be seen in their executive behavior. On the other hand, for building up a model of reasonable problem-solving behavior (the existence of such a model could be debatable), in the same document Schoenfeld investigated a mathematician's problem-solving process in a similar context. The mathematician was a number theorist with a broad mathematical background but had not dealt with geometric problems for years. He was asked to solve a geometric problem below:

> You are given a fixed triangle with base B. Show that it is always possible to construct, with straightedge and compass, a straight line parallel to B and dividing T into two parts of equal area. Can you similarly divide T into five parts of equal area? (pp. 389-90)

Though he began working the problem with less domain-specific knowledge than did Schoenfeld's students, the mathematician marshaled his cognitive resources and ultimately was able to solve the problem. He began by making certain that the problem was fully understood and was soon aware of other information necessary for a solution. The plan was then made and assessed. After entering the implementation phase, two refinements were made showing that the mathematician was still attentive to clarification and simplification. Further, he continued to assess the difficulty of his approaches. According to Schoenfeld, such estimation of problem difficulty could be a major factor in an expert's decisions to pursue or curtail various lines of exploration during the process, with one important element the experts' metacognitive behavior. Subject-matter knowledge was important in the mathematician's success, but metacognitive or control skills provided the keys.

Aside from some potential bias (e.g., (a) students were asked to solve problems in a short 20 minutes, working under time pressure, which may prevent them from elaborating strategies more freely and deeply; (b) students and the mathematician solved different problems), the comparison does shed more light on the difference between experts and novices' problem-solving behavior.

In addition to the control skill, in another study, Schoenfeld (1983a) explored other possible factors affecting individual problem-solving performance. Two protocols were introduced to show two students' behavior change before and after a problem-solving course. In the first, two students, LS and TH, were given two intersecting straight lines and a point P marked on one of them. They were asked to show how to construct, using a straightedge and compass, a circle tangent to both lines and having P as its point of tangency to one of the lines. It appeared that the students had only weak background in the relevant geometric facts and procedures. Nevertheless, the study indicated their domain-specific knowledge was quite adequate to solve the assigned task (for instance, after videotaping, they were given two other proof problems providing information needed to solve the above construction problem and solved both in less than five minutes). After their initial attempts had failed, LS and TH discussed "fitting the circle in," an indication that they did not grasp the essence of the problem. As compared to the previous study (Schoenfeld, 1983b), the students showed better managerial behavior (analyzing a problem before entering the exploration phase); Schoenfeld attributed their failure not to poor control but to intuition-based empirical beliefs with respect to geometry problems. Roughly two-thirds of the allotted time was spent with straightedge and compass in hand and their ideas were largely generated from carefully performed constructions.

After a one-month-long intensive problem-solving course, the same students were asked to solve another geometry problem. They were given three points, A, B, and C, and asked to construct two circles with the same radius, with centers A and B, respectively, such that the common internal tangent to both circles passed through point C. During this second problem session, a significant shift had taken place. The two students recalled more relevant and more accurate information than in the first. Relevant facts were called into play when needed. Even though these students used good sketches for generating ideas and hypotheses, they no longer merely depended on careful constructions. According to Schoenfeld, their intuition-based empirical beliefs had been removed; success in the latter session rested on changes in their beliefs regarding doing geometry construction problems.

One comparative model of expertise in the third protocol, a professional mathematician who had not done any plane geometry for years, was asked to use a straightedge and compass to inscribe a circle in a triangle. Not surprisingly, he solved the problem in a few steps without much struggle; his approach for deriving information used proof-like procedures and was non-empirical in nature.

Schoenfeld (1979a) contended the student's perspective with respect to particular problems may exert a certain degree of impact on problem-solving behavior. This viewpoint was not elaborated further in that article, whereas it became clearer that what he meant by "perspective" is an individual's belief about doing mathematics. Intuition-based, empirical beliefs might put a problem solver in danger of random actions, while metacognition-based, non-empirical beliefs (approaching the problem reasonably, though testing intuitively, and assessing progress frequently) would send the problem solver along an avenue leading to success.

A Framework for the Analysis of Mathematical Behavior

Learning behavior in general, mathematical behavior in particular, is a hard-to-predict consequence of interactions among various cognitive and non-cognitive factors. Exploring individuals' invisible thinking processes to the degree that scientists pursue the laws of the universe can only be achieved from visible and tangible phenomena. Thus, as the lesson learned from the scientific revolution, any attempt to establish a universal rule for interpreting complicated learning behavior may be doomed to failure. Without a generally accepted and workable framework, any engagement in clarifying relevant issues is likely fruitless.

Though mathematical problem solving has been explored for decades, most of the research focused on discrete events rather than holistic, systematic investigation. To gain insight into individual problem-solving behavior and to locate potential variables, a long-term study involving a series of qualitative observations and hypotheses-testing process is indispensable. Most work in this field should be credited to Schoenfeld. As a practicing mathematician producing theorems in topology and measure theory, Schoenfeld was deeply impressed by Polya's ideas, yet his pleasure was lessened by his colleague who coached the team for the Putnam exam and labeled Polya's books worthless. From then on two major questions preoccupied his mind: (a) What does it mean to think mathematically and (b) How can students be helped to think mathematically? Through investigating college students and mathematicians' problem-solving behavior over several years, Schoenfeld recognized that complicated problem-solving is not totally explained by Polya's ideas of heuristics (descriptive but not prescriptive). Knowing heuristics gave no guarantee to successfully employing them. A variety of factors may be in action while doing mathematics.

To establish a theoretical background for future analysis in problem solving that accounts for success or failure of students' problem-solving attempts, Schoenfeld (1985b) mapped out an explanatory framework through summarizing a decade of efforts to understand and teach problem-solving skills. This framework consisted of four major categories: resources, heuristics, control, and belief systems. Performance was first subject to the richness of mathematical knowledge. Resources contained a myriad of facts, non/routine procedures, and skills the individual is capable of bringing to bear on a particular problem. This category served as the database at the individual's disposal, yet the knowledge database was static. No matter how rich the database is, the problem solver can never access it without some knowledge. Heuristics, according to Polya, was considered as means to discovery. To Schoenfeld, heuristics was the tools for resourcefulness and efficiency, strategies for students to make progress. Thus, to be resourceful, students needed to be familiar with a broad range of heuristic techniques. To be efficient, however, the control issue must be carefully dealt with through explicit instruction. Any ideal control behavior included "making plans, selecting goals and subgoals, monitoring and assessing solutions as they evolve, and revising or abandoning plans when assessments indicate such action should be taken" (p. 27). Since "metacognition" was widely used to describe the control issue and its related phenomena like self-regulation, managerial behavior, and decision-making processes, the term "control" was substituted by metacognition while discussing relevant issues thereafter.

Belief systems, the final category, were considered as subtlest. Students failed in solving problems too often not because of a sterile knowledge base, but because of a lack of perception of the usefulness of their mathematical knowledge, and consequently they failed to call upon that knowledge. An invisible factor may prevent students from accessing necessary information available in their mind. Schoenfeld blamed this phenomenon on individual belief systems. "Even the more successful students often held perspectives that were deeply anti-mathematical in fundamental ways and that had clearly negative effects on their problem-solving behavior" (p. 13). Perception of their mathematical knowledge was shaped by their experiences with mathematics. Therefore, "their beliefs about mathematics—consciously held or not—establish the psychological context within which they do mathematics" (p. 14). As such, belief systems are one's mathematical world views shaping the way one does mathematics, which determine how one selects to approach a problem, what strategies are useful, how long and how hard one will work on it, and so on.

Some limitations may be noted in Schoenfeld's framework. Data for building that framework were mostly derived from talented college students in his problem-solving class or from professional mathematicians. How this framework can be extended to general subjects deserves more attention. Also, students' emotional or affective factors occupied little position within the framework (belief can only be partly classified as affective domain). This may be due to younger students, whose performance might be significantly influenced by non-cognitive factors, something not discussed by Schoenfeld. Even so, as noted earlier, his framework does show a holistic, albeit not exhaustive, picture regarding what problem solving is about.

Metacognition

"Knowing mathematics is doing mathematics" has been a slogan proposed by the mathematics community for years, whereas a gap exists between the two extremes. Knowing a large amount of mathematical facts and procedures may not lead students in desired directions, as noted above. In addition to facts and procedure, to reach the destination, students need to, based on their self-perceptions and the problem, ferret out and connect relevant knowledge. The journey should be guided by an invisible mind-map, which monitors one's cognitive progress tightly. This mind-map could be called self-regulation, executive control, or managerial behavior. Or it could be designated by a psychological term, "metacognition," a subspecies of "metamemory," one's awareness of the storage and retrieval of information. Metacognition "refers to one's knowledge concerning one's own cognitive processes and products or anything related to them" (Flavel, 1976, p. 232). Flavell added that metacognition referred to "active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects or data on which they bear, usually in the service of some concrete goal or objective" (p. 232). Over past decades researchers in various fields have been paying attention to individual metacognitive aspects of learning. The research on metacognition impacted mathematics education (Flavel, 1976; Garofalo & Lester, 1985; Schoenfeld, 1985c; Schoenfeld, 1987), but there has been some confusion about the term. One such widespread misconception is about how to distinguish cognition from metacognition. While it may be hard to separate the two constructs, one way to differentiate the two is that cognition is involved in doing, metacognition in planning, choosing, and monitoring what one does (Garofalo & Lester, 1985). Schoenfeld (1987), for sake of simplicity and precision, translated the term into everyday language as "reflections on cognition" or "thinking about your own thinking" (p. 189).

Teaching Metacognition

Good metacognitive skill may prevent students from randomly approaching problems while doing mathematics, but can these skills be taught? The answer to this question is far more important than the previous question: "Can heuristics be taught?" Schoenfeld (1987) proposed a "kitchen sink" approach to developing the students' metacognitive skills, constituted by four techniques: (a) using videotapes; (b) teachers as role models; (c) whole-class discussions of problems with teacher serving as "control"; (d) problem solving in small groups. Since most students are largely unaware of their thinking processes, and self-awareness is a crucial aspect of metacognition, Schoenfeld suggested showing students videotapes of other students working problems in his problem-solving class. Their reactions were intriguing: they first pointed out stupid steps of what students in the videotapes had done, but soon recognized that that could be themselves. According to Schoenfeld, it is easy to analyze someone else's behavior and then to see that the analysis applies to self.

While teaching, teachers usually present the solution in a neat and clean way. However, this produces an unexpected byproduct: students' are likely to think the problem should be solved in the same way if they are able to do the problem. Consequently, the struggling processes are typically hidden from students. In the second techniques, therefore, Schoenfeld suggested teachers work on a problem like a novice doing the problem from scratch, gradually conducting each stage of the solution process. During the whole process, teachers test the hypotheses and assess the approaches frequently until the problem is done, followed by a review of the whole solution. The primary purpose is to focus students' attention on metacognitive behaviors. To increase students' awareness, teachers have to minimize their intervention. When students work as a group on problems, teachers must be a moderator encouraging them to propose suggestions and leading students to verify all possible approaches on their own, rather than directly guiding students to correct solutions. This whole-class activity is aimed at raising students' self-regulation.

The fourth technique Schoenfeld proposed was dividing the whole class into groups of three or four to work on problems. At this stage, the teacher acts like an "intellectual coach," as Schoenfeld called it, answering questions and providing a variety of problem-solving techniques. Teachers move around groups to investigate and inquire about students' work by "What are you doing?" or "Why are you doing it?" or How does it help you?" and so on. These actions force students to defend themselves, hopefully eliciting their self-awareness and self-regulation.

For a course using these methods, Schoenfeld found clear evidence of a marked shift in the students' problem-solving behavior, particularly in the metacognitive aspect. Still, these theoretical ideas were offered without supportive data. Further, the whole methodology has not been adopted by any researcher; true effect of this approach on increasing students' metacognition hence is speculative.

Promoting Metacognitive Skills Through Pair-Work

As Schoenfeld noted, problem solving in small groups is good for strengthening students' metacognitive skills. Although the merits of group learning have been well documented, the effect on metacognition is little known. Goos and Galbraith (1996) conducted a study to explore the nature of secondary students' metacognitive strategy use, plus how these strategies are applied when pairs of students work together on problems. The focus was on the quality of the interaction between two students working collaboratively on applied mathematical problems. Use of pairs of students has its own merit in several aspects, as Schoenfeld (1985b) indicated. First, two students working together produce more verbalization than one because both must explain and defend their own decisions. Second, the double check of mutual ignorance may reduce the pressure of working under observation. Third, the requirement to produce mutually acceptable solutions could provoke considerable reflection on their thinking.

Based on a two-week classroom observation, two senior high school students were selected to participate in the study because they demonstrated a capacity for verbalizing and reflecting on their thinking. According to answers on a questionnaire probing metacognitive awareness, one student, David, perceived himself as a student usually making mechanical errors, though he was clever and good at mathematics. He was aware of his ability and weakness, whereas unable to describe in detail his actions when stuck on a problem. The other student, Rick, was also capable, slightly behind David in his speed in grasping new ideas. He disliked writing down ideas and calculation, but could solve many difficult textbook examples almost entirely in his head. Their metacognitive behaviors were analyzed through two think-aloud videotapes in which they were asked to solve four applied mathematical problems. The protocol was analyzed under a scheme which was an integration of Mason, Burton, and Stacey's (1985) problem solving model with

Schoenfeld's (1985a) episode analysis. Among the four protocols, the "CRICKET" problem was typical and was chosen to be the research sample.

At the outset of the solving process, both boys ignored how given conditions might be used to reach the goal, which resulted in blocks to the solution. Prior to the stage of implementation, they failed to assess the state of their knowledge with respect to the problem. While Rick employed an inappropriate formula, David's monitoring skill was evident as he reminded Rick that this formula was useless. However, Rick ignored his partner's warning and directed him toward an unsuccessful path. During the first implementation stage, David made several local assessments of Rick's flawed procedure and played the role of a skeptic. Nevertheless, while entering the exploration phase, Rick became the major source of new ideas and questioned the rationality of David's approaches. The stage was well-structured, and the pair's monitoring and assessment prevented their straying from the right track, though not thoroughly enough to grasp the key point.

After sometime, David recognized that necessary information was overlooked, prompting him to re-read the problem statement and drew a new diagram. This time the pair carefully checked each other's understanding of the problem. With the discovery of new information, under Rick's investigation, David cautiously made and carried out the plan. After David successfully performed the calculation, Rick assessed the reasonableness of the result.

This protocol showed that Rick and David had differing, yet complementary, metacognitive strengths: Rick as the idea generator and checker of David's calculation, David as a plan maker and executor. Both shared the responsibility for evaluating the accuracy and reasonableness of their result. In sum, the pair effectively played complementary roles in this task, and the resulting control decision led to success. The finding in some sense supported Schoenfeld's suggestion of improving students' metacognition through group work, but one should bear in mind that subjects in this study were carefully selected; whether these findings can be generalized to general subjects continues as an open question. The study also noted that, owing to several personal insistences on wrong procedures, David made poor decisions that guaranteed failure. This result raised a vital issue of the role of the individual's personality in group work, a point merits further investigation.

A Cognitive-Metacognitive Framework

In spite of the importance of recognizing metacognition in mathematical problem solving, no systematic, holistic study has yet been done. Polya initiated the study of heuristics in problem solving; his perception of metacognition can only be considered implicit. In order to devise a tool for analyzing metacognitive aspects of individual problem-solving behavior, Garofalo and Lester (1985) proposed their cognitive-metacognitive framework for studying mathematical performance. The framework, extended from Polya's four-phase description, comprised four categories of activities involved in performing a mathematical task: orientation, organization, execution, and verification. The framework distinguished itself from Polya's four phases (understanding, planning, carrying out the plan, and looking back) by adding assessment activities to each category. For instance, in the category of orientation, assessment of familiarity with the problem and level of difficulty, as well as understanding the problem, were involved. This early estimate of problem difficulty was seen in Schoenfeld's (1983b) study of a mathematician's problem-solving behavior. In the realm of organization, both local and global planning were required to help a problem solver identify the main goal and subgoals. Verification was a holistic evaluation of the previous three steps, checking their accuracy and consistency.

Occurrence of metacognitive behavior is task-dependent; it is therefore not possible to expect these four aspects of metacognition as always involved during solving various kinds of problems. However, as Garofalo and Lester indicated, the framework can be used as a guide in selection of research tasks or design of interview procedures. It can also be employed to organize data and interpret findings.

Even though the role of metacognition in mathematics education has received much attention and evoked extensive discussion, most research is theoretical. Empirical study of this construct is fairly sparse. For instance, Schoenfeld's "kitchen sink" approach for developing metacognitive skills and Garofalo and Lester's framework have rarely if ever been empirically tested. To shed more light on students' invisible thinking processes, sophisticated influence of metacognition cannot be discontinued.

Belief Systems

The last and perhaps the most important category in Schoenfeld's framework is belief systems. These systems contain one's beliefs about self, the environment, the topic, and mathematics. According to Schoenfeld, individual belief systems are world views shaping the way of doing mathematics. Schoenfeld's original concern about one's belief largely is on a practical facet, namely, beliefs about solving problems, not a philosophical facet, the epistemology and ontology issues. He was also aware that the term "belief" is controversial, but "the point of departure, however, should not be controversial" (Schoenfeld, 1985b, p. 44).

Metacognitive behavior has been identified as one of main driving forces throughout the whole problem-solving process. Schoenfeld (1987) even proposed that the study of metacognition is the key for resolving the complicated issue of "transfer." A group of researchers (Garofalo & Lester, 1985; Lester, Garofalo, & Kroll, 1989; Schoenfeld, 1985b, c; Schoenfeld, 1987) suggested that metacognition not only was a force driving cognitive behavior, but also connected to an individual's beliefs. Accordingly, the study on the relation between beliefs and mathematical problem solving soon attracted considerable interest among relevant researchers. On the other hand, it is hard to believe that teaching of problem solving has any chance of success unless the teacher's role in problem solving is clearly and unambiguously defined. Based on studies of teachers' thinking and decision-making, it is now widely recognized that how teachers interpret and implement curriculum is significantly, but not exclusively, influenced by their knowledge and beliefs (Thompson, 1992). Since 1980, many studies have paid attention to teachers' beliefs about mathematics itself and mathematics teaching and learning. The premise underlying this endeavor was that "to understand teaching from teachers' perspectives we have to understand the beliefs with which they define their work" (Nespor, 1987, p.323). Therefore, "awareness of the significant role beliefs play in cognitive behavior and ensuing interest in belief system as a topic of study appear to have developed concurrently, yet independently, among mathematics education researchers interested in teachers' cognitions and those interested in students' cognition. For both groups, the study of beliefs has emerged in recent years as an important, legitimate line of research" (Thompson, 1992, p.131).

Because of the influential role of beliefs in learning and teaching mathematical problem solving, the relevant literature is reviewed in two dimensions: the relationship (a) between students' beliefs about mathematics and learning behaviors in problem solving and (b) between teachers' beliefs about mathematics and their instructional behaviors in problem solving. Since individual views on mathematics are one of the main themes of this thesis, the first dimension is discussed at full length. It should be cautioned again that the term "belief" is controversial. The terms *beliefs*, *views*, *perceptions*, *conceptions* and *values* have been freely used to describe the same concept. Though distinction among those terms has not been clearly identified, for sake of the conventional usage, "belief" is adopted as the general term while discussing relevant phenomena hereafter.

Origins of Students' Beliefs about Mathematics and Their Impact on Learning

While receiving mathematics instruction, some students might show changes in thinking, some might not. Though factors hidden in the phenomenon have not yet been totally coordinated, it is generally held that students' beliefs of mathematics as a discipline and school subject matter are potential keys. How students' beliefs impact problem-solving behavior has received much attention.

It would be naïve to think good teaching necessarily leads to good learning. Any study on the effect of problem solving in mathematics cannot ignore students' reactions to problem-solving environments. Individual behavior may be consciously or unconsciously guided by existing personal beliefs. Students' beliefs of doing and knowing mathematics are acquired through years of experiences at school (Lampert, 1990). Teachers' instructions not only influence students' recognition of subject matter knowledge, but meanwhile convey particular kinds of beliefs of interpreting and implementing that knowledge.

On the basis of previous studies of college students' problem-solving behavior, Schoenfeld (1988) found four common but improper beliefs held by students: (a) the processes of formal mathematics have little or nothing to do with discovery or invention; (b) students who understand the subject matter can solve assigned mathematics problems in five minutes or less; (c) only geniuses are capable of discovering, creating, or really understanding mathematics; (d) one succeeds in school by performing tasks to the letter, as described by the teacher. To pinpoint the origins of such fallacious beliefs, Schoenfeld conducted a whole-year classroom observation of 12 classes. He then selected one typical class, which he observed once a week or more for an entire school year. Two weeks of instruction on geometry were videotaped and analyzed in detail. An 80-item questionnaire (whose validity and reliability were not reported) was filled out by 20 students in the target class and by 210 other students in 11 other classes. The target class scored well on standardized examinations, and the relationship between teacher and students was cordial and respectful. Observation, however, revealed that the primary goal of instruction was largely to have students do well on the statewide exam. Doing proof problems was central to the curriculum. One construction problem appeared on the year-end exam, but approximately 90% of classroom time during the unit on constructions was spent with straightedge and compass, practicing constructions. The objectives of instruction were accuracy and speed with limited emphasis in conceptual understanding. No opportunity was given for discussing why the constructions worked. With the stress on accuracy, students learned to rely on empirical standards to assess the correctness of the construction. In this manner, students were guided to perceive proofs merely as confirmations of known facts; no creativity was considered. Students were requested to follow strictly a two-column protocol format. In both the target class and the other 11 classes, a great deal of time was spent in the consideration of whether the form was acceptable. As a result, students likely believed the form of expression was paramount.

Over the full school year, it was found that students worked exercises that could be done in a short amount of time and thus cannot be called real problems. For example, students were expected to work 60 or more problems during a 54-minute session. Moreover, their teacher advised: "*You'll have to know all your constructions cold so you don't spend a lot of time thinking about them* [italics added]." Students' responses therefore were not convincing. For example, one open-ended item on the questionnaire was: "If you understand the material, how long should it take to answer a typical homework problem? What is a reasonable amount of time to work on a problem before you know it's impossible?" Means for the two parts of the question were only 2.2 minutes ($n = 221$) and 11.7 minutes ($n = 227$), respectively. Students' learning of mathematics would likely be seriously harmed by incorrect beliefs. In addition, an inconsistency was found in students' responses. Students were asked to respond to the statement: "The math that I learn in school is mostly facts and procedures that have to be memorized." With a scale ranging from very true (1 point) to not at all true (4 points), this item received an average score of 1.75—the third strongest "agree rating" of all questions. Yet, the statement "When I do a geometry proof, I get a better understanding of mathematical thinking" received an average score of 1.99, also a very strong agreement. The seemingly conflicting responses, according to Schoenfeld, could be interpreted to mean the former refers to mathematics inside a classroom, the latter outside. Interviewing students' to reveal the contradiction should be an appropriate way, yet it was lacking. Throughout the article, students' interview data were not cited to support the researcher's arguments, weakening the strength of inferences made.

In a companion study, Schoenfeld (1989) further explored aspects of the relationship between students' beliefs about mathematics and their performance with the intention of supplying quantitative data to supplement qualitative observations reported earlier. Subjects in this study were 230 high school students (112 females, 118 males), all enrolled in the classes of teachers who agreed to participate and all on the academic, college-bound track. Yet selection of the participating class was not discussed.

A questionnaire with 70 Likert-type, 4-point items and 11 open-ended questions was developed for the study. Multiple-choice questions asked about: (a) attributions of success or failure; (b) students' perceptions of mathematics and school practice; (c) their views of school mathematics, English, and social studies; (d) nature of geometric proof, reasoning, and constructions; (e) personal and scholastic performance and motivation. Open questions were designed to give students an opportunity to present more extended answers to issues of interest. Questionnaires were administered to subjects at their teachers' convenience during the last two weeks.

Though the questionnaire covers various areas, only items related to belief are discussed below. Most students considered mathematics an objective discipline that can be mastered. They believed work, not luck, accounted for good grades and placed much more emphasis on work than on inherent talent. The study also found students were expected to memorize subject matter. When asked how important memorizing was in learning mathematics, some typical answers were:

> You must know certain rules, which are a part of all mathematics; Without knowing these rules, you cannot successfully solve a problem; Memorizing is very important, and in geometry, especially for the final exam, because I am required to write proofs from memory (p.344).

It is worth noting that an inconsistency between students' views of doing geometry problems and problem-solving behavior was found. Students disagreed with the statement: "Constructions have little to do with other things in geometry like proofs and theorems," indicating they were aware of connections between deductive and constructive mathematics. Still, in interviews conducted concurrently with this study, each of the 21 students in the target class attacked a geometric construction problem using trial-and-error to verify an intuition-based conjecture, regardless of proof-related knowledge.

Students' responses to items concerning classroom practice were consistent with classroom observations, revealing a byproduct of drill exercises. Average time expressed by the 206 respondents to the question: "How long should it take to solve a typical homework problem?" was just under two minutes. The longest of the 215 responses to: "What is a reasonable amount of time to work on a problem before you know it's impossible?" was 20 minutes and the average of 12 minutes manifested students' shortage of patience in doing mathematics.

The results of statistical analysis also showed students' perceptions of mathematics differed significantly from English and social studies. Students believed that solving problems depended on knowing "rules"—mathematics is presumed more rule-bound than English and social studies. Yet, when asked questions about "certainty" of the three subject matters, students gave unexpected results. Mathematics received the strongest disagreement: students held that mathematics is the least certain subject among the three. The most likely reason for this result is that, as Schoenfeld indicated, in school partial credit is frequently given; hence, mathematics work is not simply right or wrong. If this result is the case, it supports the afore-cited conjecture that students might hold two distinct images of mathematics—one in school, concerned with rules and format only; another (real mathematics) involving reasoning and creativity. The former could be driven much more by students' long-term school experience, whereas the latter is merely an image verbally conveyed by teachers or public, but which they have never experienced. These incompatible images seem rooted in students' minds without connection and reflection. As a result, students may fail to comprehend the value of mathematics in daily life and human culture, one goal of education (NCTM, 1989). A good deal of work is necessary to convert those rhetorical advances into substantive ones.

Effort to Change Students' Views of Mathematics

Views on the nature of mathematics are shifting. For over two thousand years, it has been seen as a body of infallible and objective knowledge, far removed from the values of humanity. Currently, this view is being challenged (Ernest, 1991; Hersh, 1986; Tymoczko, 1986) a shift that calls for corresponding action in classrooms. As seen earlier, children normally perceive mathematics as a set of rules and procedures in which problems are solved by applying computational algorithms explicitly taught by the teacher. Students expect these algorithms to be fairly routine tasks without elaborate thinking, an experience that shapes children's perception of mathematics as a static body of knowledge not created but replicated in particular ways. The vision of mathematics portrayed in reform documents (e.g., NCTM, 1989, 2000) requires students to think differently from the way they currently do about the nature of mathematical knowledge. However, the above traditional view of mathematics may influence their eagerness and willingness to participate in active and meaningful leaning, essentially inhibiting them from engaging in problem-solving tasks envisioned by current reform movements (Garofalo, 1989; Lampert, 1990; Schoenfeld, 1983a, 1992).

Students' views of mathematics are typically described as shaped or affected, at least in part, by their classroom teacher (Cobb, 1987; Ford, 1994; Schoenfeld, 1983a, 1988). The nature of the classroom environment strongly influences how students view a subject, the way they believe mathematics would be done, and what they consider appropriate responses to mathematical questions (Garofalo, 1989). Students' beliefs also may differ among classrooms and school systems; classrooms with variant demographic characteristics may be influenced by similar principles about teaching and learning. Thus Franke and Carey (1997) investigated how CGI (Cognition Guided Instruction) affected elementary students' perceptions of mathematics in problem-solving environments. The study summarized major positive effects of CGI teaching as follows:

1. Children recognized and accepted a variety of solutions, as well as assuming a shared responsibility with the teacher for learning;

2. Children's perceptions of what it meant to succeed in mathematics were not determined for these children by speed and accuracy;

3. The children's focus on strategy and answers provided a way to determine success, communicate mathematically, and resolve a problem-solving conflict.

In their study, Franke and Carey defined a problem-solving classroom as one where students have opportunities to engage consistently in problem solving, discuss their solution strategies and build their own informal strategies for solving problems. Problem solving has been interpreted in many different ways. Compared with other relevant research, Franke and Carey showed a clearer picture of what problem solving was about in their study. Yet their study heavily emphasized the use of strategies in problem solving rather than developing students' thinking ability to resolve new, unfamiliar problems.

Similarly, Higgins (1997) conducted a one-year investigation of the effects of a systematic, heuristic instruction on middle-school students' attitudes and beliefs about problem solving and on problem-solving ability. Two 6th-grade teachers, four 7th-grade teachers and their students (n = 137) participated. One 6th-grade teacher and two 7th-grade teachers had received training in the teaching of mathematical problem solving at least five years before the study. They received a three-week problem-solving training one more time and were provided specific coaching on the nature of problem-solving instruction. The (heuristic) students of these (heuristic) teachers received problem-solving instruction, thus serving as an experimental group.

Heuristic students were taught skills through explicit problem-solving teaching. This phase of instruction was completed in five weeks. After the initial five-week instructional phase, teachers engaged students in weekly challenge problems involving a situation or task for which there was no immediately obvious solution. This explicit teaching is akin to Schoenfeld's (1979b, 1980, 1982) proposed method, whose purpose was to keep reminding students to employ problem-solving skills. One major drawback is that while solving a problem, students were all aware of the possible tools. Consequently, the opportunity for developing personal, informal approaches, a quite important goal for teaching problem solving, was precluded.

Further, a 39-item Likert-type questionnaire, adapted from Schoenfeld (1989), was administered to all students near the end of the school year and used to explore relationships between students' beliefs about mathematics and their problem-solving instruction. Three students per teacher were chosen by randomly selecting one questionnaire. Thus a group of nine heuristic and nine non-heuristic students were selected for follow-up individual interviewing. During these interviews, students were asked to solve four non-routine problems by using a "think aloud" strategy and also requested not to erase answers on the papers. Among the 39 questionnaire items, the study found some statistically significant differences between heuristic and non-heuristic students. The former typically saw mathematics as more than facts and procedures to be memorized and considered it more important to do well in mathematics than non-heuristic students did. Besides, non-heuristic students were more dependent on textbooks or teachers to justify their answers. While some optimistic signs were found from heuristic students' responses, since pre-measurement of students' initial views was lacking, it can not be ascertained that different views between groups were solely created by the treatment.

In interviews, while asked for the definition of mathematical problem solving, heuristic students usually associated it with skills learned in class, but none had clear ideas of what was meant by problem solving, likely resulting from explicit teaching of problem-solving skills. On the other hand, not surprisingly, "solving a problem" or "finding an answer" were identified as problem solving by non-heuristic students.

Students' views of school mathematics were also investigated. Every heuristic student claimed mathematics was useful and gave examples of daily applications, but perception of mathematics' utility declined with the proficiency level of non-heuristic students. The three high-ability non-heuristic students gave answers similar to those given by heuristic students; low-ability non-heuristic students indicated they did not find mathematics useful. Students' responses to the role of memorization in learning mathematics were quite different between groups. Six of nine heuristic students claimed memorization was important only in some areas; three did not believe it at all important. On the other hand, eight non-heuristic students found memorization very important in learning, similar to the finding in the afore-cited relevant research.

Assessment of students' views of understanding in mathematics revealed one difference between groups. When asked: "How can you tell if you understand something in mathematics?", heuristic students tended to focus on different solutions to problems, along with ability to make generalizations and convey one's thoughts. Six non-heuristic students usually associated understanding with speed. Further, it is interesting to note that the two groups did not differ in the responses on beliefs about mathematical discovery. All but two perceived mathematics as a discipline in which one could be creative and make personal discoveries. Five from each group claimed they had actually made discoveries on their own, but supportive evidence was lacking and no further attempt was made to validate students' professed statements.

The two groups differed in the length of time they said they would work on a problem before they would believe it was impossible. The responses of heuristic students ranged from four hours to one week; the non-heuristic students ranged from five to 80 minutes. Heuristic students recalled the longest time they had worked on a problem, mostly ranging from one hour to two weeks. By contrast, the majority of non-heuristic students answered in terms of minutes, which echoes Schoenfeld's (1989) finding.

In regard to students' ability to solve four non-routine problems, average scores of heuristic students were generally higher than those of non-heuristic ones on all dimensions of the four problems. Despite the optimistic outcomes, as noted earlier, validity of problems and extent to which they related to students' curricula is unknown; heuristic students' performance might be boosted by familiarity with these types of problems. As such, the conclusion that heuristic students' problem-solving ability had been improved should be carefully made.

In addition to numerical ratings, the study also employed verbal protocols for insight into students' thought processes and showed qualitative differences in favor of heuristics students in four aspects: they tended to (a) make generalized statements about solutions, (b) use estimation to verify reasonableness of results, (c) verbalize solution strategies using vocabulary had learned in problem-solving instruction, and (d) approach problems through reasoning and logic. These qualitative differences to some extent can be regarded as one optimistic effect of problem-solving instruction; further supportive data were unreported.

Regardless of several positive findings, it should be cautioned that the narrow perspective held by the study may be responsible for failure of heuristic students to recognizing the real nature of problem solving. As noted earlier, heuristic students were unable to characterize problem solving but associated it with skills. Such beliefs about problem solving were cast into students' minds, as seen in Franke and Carey (1997). This misconception may engender a belief that all problems can and should be resolved by following extant skills or strategies. Accordingly, students' creativity was sacrificed and developing students' mathematical thinking through problem solving became impossible.

Teachers' Beliefs and Mathematical Problem Solving

As noted previously, students' views of mathematical thinking play a fundamental role in involving problem-solving activities, to a great extent shaped by teachers' instruction. As mediators of mathematical concepts, teachers must convey contemporary views toward mathematics in order to help students comprehend the nature of mathematical thinking. Whereas the goal cannot be achieved without teachers' enriched understanding and appropriate belief about this subject, teachers' beliefs about mathematics and their potential effect on instruction consequently have received much study. Though the main theme of this thesis focuses on the student side, as the dual role of the researcher/instructor played in this study, it would be a sad mistake to omit the teacher's part.

Mathematics Teachers' Beliefs about Mathematics

Teachers from time to time have to face a variety of dilemmas while making decisions in myriad situations. Even when a decision is made, their unconscious beliefs can affect the final implementation. To explore the issue, Raymond (1997) studied beginning elementary teachers' beliefs about the nature, learning, and teaching of mathematics through interviewing teachers and classroom observations. The results showed that although their beliefs were diverse, from traditional to non-traditional, all six participating teachers expressed "primarily non-traditional" or "non-traditional" beliefs about learning and teaching mathematics. More precisely, they concurrently held that mathematics is best learned and taught through problem solving, despite some of their views of it as a rigid and static body of knowledge.

Consistent with Raymond's finding, the informant in Cooney (1985), a beginning high school teacher, professed that mathematics is useful, logical, axiomatic and hard, which is more like a mixture of Platonist and instrumentalist views. Conversely, his views about teaching and learning it were quite active, repeatedly emphasizing mathematics as essentially problem solving. Raymond and Cooney's findings seemingly suggested that beginning teachers are likely to espouse non-parallel, if not inconsistent, beliefs between the nature of mathematics and teaching and learning of this subject.

On the other hand, Thompson's (1984) study of experienced teachers tells a different story. Three participating teachers expressed consistent beliefs about the nature, teaching, and learning of mathematics. For example, two junior high school mathematics teachers viewed it as an exact discipline and professed that the teacher's role is to present the content in a clear, logical, and precise manner. On the contrary, another participant regarded it as a challenging and rigorous subject whose study provides the opportunity for a wide spectrum of high-level mental activities and expressed a dynamic view that a teacher should create an open classroom atmosphere and encourage students to guess, conjecture, and reason on their own. From these findings it could be said that, as compared to beginning teachers, experienced teachers' beliefs about the nature, teaching, and learning of mathematics are more consistent.

The Relationship Between Beliefs and Instructional Practice

In addition to investigating teachers' beliefs, researchers were also concerned with the impact of these beliefs on instructional practice, especially focusing on the issue of inconsistency. The three case studies in Thompson (1984) concurrently showed that teachers' instructional behaviors were intimately related to their professed beliefs about mathematics. The two who viewed mathematics as an organized and exact subject tended to convey mathematical ideas in a formal, prescriptive approach; the teacher claiming mathematics is a challenging discipline did employ a variety of problem-solving approaches to arouse students' interest. It appears teachers' beliefs about mathematics, along with those about learning and teaching, were the best indicators of instructional behavior. However, this is not the case in Cooney (1985) and Raymond (1997). Fred, a beginning teacher in Cooney (1985), while professing that problem solving was central to teaching mathematics, expressed frustration after ten weeks of teaching and wondered whether a teacher could actually teach problem solving. He tended to see problem solving as added-on activities in which problems used were recreational, less tied to curricula. His failure in implementing problem-solving instruction may be attributed to misunderstanding one key feature of problem solving: it should be incorporated into ordinary academics rather than treated as extra.

Among the original six participating beginning teachers, Raymond (1997) selected Joanna as the study target because she was a dramatic case. Her beliefs about the nature of mathematics were fairly traditional, a predictable, certain, absolute, and fixed subject matter having no aesthetic value; whereas beliefs about the learning and teaching of mathematics were primarily non-traditional, professing that problem solving is a big part of mathematics and that teachers should demonstrate a variety of ways to look at the same question. Yet in class she maintained a controlled, disciplined atmosphere where students were quiet and on task. Joanna conducted a considerable amount of teacher-directed, teacher-student dialogue during her lessons, in sharp contrast to her professed beliefs about learning and teaching mathematics, but consistent with beliefs about its nature.

What lesson is learned from the above diverse findings? The three teachers in Thompson (1984), whose teaching behavior were consistent with their professed beliefs, were all mathematics teachers who had at least three years' experience; participants in Cooney (1985) and Raymond (1997), whose practice may not always reflect their professed beliefs, were beginning mathematics teachers. Do these results suggest that the experienced teachers' behavior is more consistent with their beliefs, whereas the beginning teachers' behavior is not? Thompson (1992) pointed out that "A search of the literature in mathematics education revealed no single study specific to the topic of beliefs involving both pre- and in-service teachers, or a mix of teachers from elementary, middle, and high school levels" (p.131). Similarly, it is also evident that few, if any, studies have simultaneously investigated the issue of beginning and experienced teachers' consistency between their professed beliefs and teaching behavior. A single study involving the two groups may help to verify the above conjecture.

In addition, two methodological issues in the research of this type are the procedure for conducting interviews and/or classroom observations. To avoid unexpected bias, interviews and observations need to be conducted by different persons, or observations need to be followed by interviews. Among the afore-cited studies, only Thompson (1984) conducted the study in this way. Also, to reveal the actual relationship between teachers' beliefs and instructional practice, adequate observation time is necessary. Classroom observations in those studies ranged from nine consecutive classes to a whole school year. Once methodological issues are clarified, it no doubt would provide more clues to reconcile diverse findings.

Furthermore, differing from the afore-cited studies aiming to compare teachers' professed beliefs to instructional practice, Chapman (1997) investigated three teachers' metaphors in teaching problem solving and hoped to capture how teachers experienced the teaching of problem solving in a holistic context representative of their perspectives (i.e., their personal meanings, their personal contexts of making sense of what they actually did in the classroom, what they considered meaningful and important in characterizing their teaching). It was found that participants interpreted problem solving in different ways and showed a variety of approaches. Through interviewing and classroom observation over two-year period, the three participants' problem-solving teaching was categorized as "community" (developing ability to resolve conflicts through communication), "adventure"(involving struggle, perseverance, and risk taking), and "game" (involving fun, gratification, personal skills, and challenge), respectively. The position held by Chapman was that a metaphor is a possible interpretation of how teachers may make sense of their teaching of problem solving. However, in what way and to what extent the study of metaphor can be supplemented to the study of teachers' beliefs and applied to practical mathematical teaching were not clearly addressed.

The Correlation Between Teachers' and Students' Beliefs about Mathematics

As noted, it is suggested that teachers' beliefs play a profound and sophisticated role in their instructional behavior and subsequently exert a certain degree of impact on students. Schoenfeld (1988, 1989) explored the issue, but without direct attention to the correlation between teachers and students' beliefs. Ford (1994) aimed to probe what beliefs teachers hold about problem solving in mathematics and to what extent the beliefs of teachers are reflected in the beliefs of students in their classroom. The study first found that participating teachers held inaccurate understandings of problem solving, believing it is primarily the application of computational skills in everyday life. Through administration of questionnaire and interviews, it was also found that their students' beliefs about the nature of problem solving were for the most part consistent with the views held by their teachers. Yet a methodological drawback that may weaken the inference made by the study is that students' pre-instruction views were not investigated at the outset; hence to what degree teachers' inappropriate views had been conveyed to students remains unknown.

The Effect of Problem-Solving Training on Teachers' Beliefs

According to NCTM (1989, 2000), mathematics teachers are expected to involve students in problem-solving activities. However, as seen in Ford (1994), many teachers lack appropriate understanding of problem solving. Prior to conducting these kinds of learning activities, teachers have to experience problem-solving training as well. How teachers perceive this kind of training is a vital issue. If problem-solving training does not affect school teachers too much, then it is a naive assumption that these teachers would likely view problem solving as a central role in school mathematics. Raymond and Santos (1995) conducted a study to investigate how pre-service elementary teachers struggled and changed their beliefs about mathematics in a problem-solving class. After receiving one semester of problem-solving training in which a variety of approaches were designed to shaking existing beliefs of doing mathematics and eliciting intrinsic reflection, the study found that participating teachers' views of learning and teaching mathematics had been challenged, if not changed. Raymond and Santos consequently suggested this kind of problem-solving course may help pre-service teachers to question their belief systems. Yet, beginning teachers more often demonstrate inconsistency between professed beliefs and practical behaviors. As such, a more important issue is in what way and to what extent their belief shifts may contribute to their beginning years of teaching practice. A subsequent tracing investigation therefore is necessary.

Though not totally consistent and perhaps even controversial, the afore-cited findings to some extent support the philosophical argument proposed by mathematician Rene Thom: "whether one wishes it or not, all mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (cited in Steiner, 1987, p.7). A teacher's sense of mathematical enterprise may determine the nature of classroom environment a teacher creates (Schoenfeld, 1992), which subsequently exerts certain impact on students' belief about knowing and doing mathematics. Hersh (1986) also contended that the teacher's view of how classroom teaching should take place is strongly based on a teacher's understanding of the nature of mathematics, not on what he or she sees as the best way to teach. Accordingly, integrating relevant findings regarding both learning and teaching of problem solving, by minor paraphrasing Hersh's argument (Hersh, 1986, p. 13), it may be said the issue is not "What is the best way to teach and learn?" but "What is mathematics really all about?". It seems that mathematical problem solving is more complicated than people once thought.

Investigation of Mathematical Thinking

The aforementioned discussion explores relationships between mathematical problem solving and individual beliefs about mathematics. This section specifically focuses on the issue of mathematical thinking. For all students to learn how to think mathematically is one of major goals of teaching mathematics (NCTM, 1991), yet the goal has not been achieved in ordinary school mathematics instruction. Schoenfeld (1992) even indicated problem solving alone is not enough for developing learners' mathematical thinking because two critical parts, metacognition and a mathematical point of view, are missing. In the 1990s, several studies thus paid additional attention to probing one's mathematical thinking, not just to performance.

Groups of researchers (Clark & Peterson, 1986; Fennema & Franke, 1992; Putnam, Lampert, & Peterson, 1990) suggested as teachers' knowledge of students' thinking grew, their beliefs about instruction were modified and instructional change occurred. It is therefore noteworthy to identify what knowledge enables teachers to modify instruction so that students learn mathematics with conceptual understanding. Fennema, Carpenter, Franke, Levi, Jacob, and Empson (1996) conducted a four-year study of changes in beliefs and instruction of 21 elementary teachers. These teachers' baseline data were gathered in the first year by classroom observation, semi-structured interviews, and field notes. The CGI Belief Scale, a paper-and-pencil Likert-type instrument, investigated teachers' beliefs. On the other hand, students' baseline data (problem-solving ability, conceptual understanding, and computational skill) were collected via project-constructed tests.

For the next three years, the 21 teachers were invited to participate in a teacher development program (Cognitively Guided Instruction, CGI) focused on (a) boosting their knowledge of children's mathematical thinking and (b) how students' thinking could form a basis for development of more advanced mathematical ideas.

In the late spring of Year 0, teachers with little or no previous exposure to CGI participated in a $2^1/2$-day workshop. Over the following three years, all teachers participated in several CGI workshops with various time spans, from $2^1/2$ hours to two days. All content was covered by the end of Year 1; during succeeding years, workshops focused on helping teachers review and reflect on content and students' thinking, as well as encouraging teachers to reorganize lesson plan by referring to students' thinking.

To survey the teachers' change during CGI, their instruction and beliefs about mathematics teaching were both categorized into four levels and contrasted. Despite a relation between levels of instruction and beliefs appearing obvious, no overall pattern emerged because teachers' beliefs and instruction were not always categorized at the same levels. Also, there was no consistency of change in belief preceding a change in instruction or vice versa. Among the 17 teachers whose final ratings were higher than their initial rating on both beliefs and instruction, six teachers' beliefs changed before their instruction changed; five teachers' instruction changed before beliefs, and six changed simultaneously. Consequently, it seems the relation between alteration of instruction and beliefs is a complex issue and left unexplored in this study.

Another important concern was whether changes in teachers' instruction were reflected in changes in their students' learning. This study examined the issue from two aspects. First, in the case of a teacher gaining an instructional level, it was then investigated whether the change was followed by increase in students' achievement. Next, for the instances in which there were increases in students' achievement, the study subsequently considered whether those increases were preceded by changes in the given teachers' level of instruction. Overall, among the 11 teachers whose data were completely collected over the four years, it was reported that change in teacher's level of instruction was reflected in students' achievement. Nevertheless, the reason why only 11 of the 21 participating teachers data were completely collected and the significant level of students' changes in achievement were not addressed.

The present longitudinal study suggested the CGI program helped teachers develop alternative beliefs in which they tended to understand and appreciate the variety of students' thinking and strategies. Shift in belief, meanwhile, changed their concept of their role as teachers; they came to believe their job is not to tell students how to think, but to create an environment in which students' knowledge and thinking grows as they engage in problem-solving activities. While correlation between gain of beliefs and instruction was not linear, increases of both generally contributed to their students' academic progress. One of the main purposes of this study was to examine what outcome may occur when teachers came to understand and appreciate students' thinking. Nonetheless, though quantitative data demonstrated progress in achievement, they failed to reveal in what aspects students' thinking evolved along with teachers' change in instruction and beliefs. Follow-up qualitative investigation may shed more light on this concern.

More researchers (Cobb, Wood, Yackel, Nicholls, Wheatly, Trigatti, & Perwitz, 1991; Wood & Sellers, 1996) conducted longitudinal investigations to gauge effect of reform-based curriculum carried out on 2nd- and 3rd-grade students, respectively. To echo with current mathematics education reform movement, they developed problem-centered instruction built on a social constructivist point of view wherein learning was seen as both acculturation and individual construction; teacher and students mutually constructed taken-to-be-shared mathematical interpretation and understanding. Cobb et al. (1991) had 10 second-grade classes participating in a yearlong project situating students in a problem-centered instruction environment. Ten volunteering teachers were invited to participate in a one-week summer institute designed to help teachers understand aspects of their current practice as problematic. At the end of the institute, all participants received a complete set of instructional activities together with notes to guide their use of activities in classroom, along with extensive support throughout that school year. On the other hand, eight non-project teachers participated, bypassing the summer institute. A 33-item instrument (Cronbach alpha reliability was .96) was utilized to assess the 18 teachers' beliefs about teaching arithmetic in 2nd grade and found that beliefs of project teachers were more consistent with socioconstructivism than were those of non-project teachers (t [17]=36.66, p <.0001).

Two standardized arithmetic tests (ISTEP and Project Arithmetic Test), composed by several subscales, were administered in the 18 teachers' classrooms to evaluate students' computational ability and conceptual understanding. According to ANOVA analysis, project students significantly outdid non-project ones on the ISTEP Concepts & Applications scale (F[1,332]=6.83, p <.01) and on the Relational Scale of Project Arithmetic Test (F[1,332]=67.42, p <.0001). As for the Computation Scale of ITTEP and Instrumental Scale of Project Arithmetic Test, no significant difference was found. In addition to students' achievement, a five-point-scale questionnaire also assessed students' own goals and beliefs about the reasons for success in mathematics, indicating project students (a) were less motivated to be superior to peers (F[1,332]= 11.07, p <.001), (b) valued attempting to understand and collaborate more than non- project students (F[1,332]=11.55, p <.001), and (c) saw less value in conforming to others' solution methods (F[1,332]=59.32, p <.0001).

Within the identical theoretical framework and carried out in similar manner, Wood and Sellers (1996) compared 2nd and 3rd-grade students in problem-centered classes for two years (project students) with those in problem-centered classrooms for one year only (non-project students), and with students in conventional classrooms for two years (textbook students). Result indicated project students achieved significantly higher on the Computational Scale (F=4.48, p <.05) than non-project and textbook students; no significant difference was found between any of the groups on the Concepts and Applications Scale; project students significantly outperformed both non-project and textbook students on total test (F=8.11, p <.05). No significant differences were found between non-project and textbook classes. Further, project and non-project students were compared by arithmetic test, showing insignificant disparity on the Instrumental Scale but project classes scored significantly higher than non-project classes on the Relational Scale (t=8.86, p <.001). On the Belief Scale, project students more strongly believed success in mathematics relates to finding their own or different ways to solve problems rather than conforming to methods shown by the teacher (t=8.20, p <.0001).

Findings reported by Cobb et al. (1991), along with Wood and Sellers (1996), concurrently suggested a reform curriculum centered on the investigation of problem resolution via social interaction between teacher and students may create a microcosm of mathematical culture in which students' conceptual understanding and thinking can be enhanced without sacrificing computational ability. Meanwhile, project students' beliefs about doing mathematics also improved. Regardless of optimistic outcomes, similar to a study by Fennema et al. (1996), use of standardized achievement test and reported quantitative data failed to reveal in what particular aspects students' thinking strategies were upgraded, a major goal of problem-centered curriculum. Furthermore, teachers' previous experience is one critical factor affecting the degree of success in implementing curriculum. In both studies, however, whether the project teachers had received similar training in problem-centered curriculum was unknown. Particularly, Cobb et al. (1991) provided no students' baseline data, perhaps weakening credibility.

Contrary to above-cited quantitative studies probing conceptual understanding or thinking, Henningsen and Stein (1997) employed qualitative investigation to identify classroom-based factors shaping students' engagement in mathematical tasks that were set up to encourage high-level mathematical thinking and reasoning. Data in the present study were drawn from four primary school teachers' classrooms over three years. Trained observers took detailed field notes on mathematics instruction and students' reactions to it. All classroom episodes were videotaped and, following their observations, those observers created Classroom Observation Instrument (COI) including 144 samples by referring to field notes and videotaped lessons. The COI sketched the physical setting of classrooms, instructional events, nature of tasks used in classrooms, behavior of the students as they engaged in these tasks. Among the 144 samples, 58 were identified as being set up to encourage doing mathematics and promote thinking. Based on these COI samples, an effort was made to examine factors associated with maintenance or decline of doing-mathematics tasks. Analysis indicated five influential factors in maintaining student engagement at a high level of thinking and reasoning in mathematics: (a) tasks built on prior knowledge, (b) scaffolding, (c) appropriate time frame, (d) modeling of high level performance, and (e) sustained press for explaining and meaning. The findings show teachers' dominant role in proactively supporting students' high-level engagement.

Factors leading to the decline of students' engagement were also identified. It was found that the removal of challenging aspects of the tasks, shifts in focus from understanding to correctness or completeness of answers, and inappropriate amounts of time allotted to the tasks were three major factors resulting in students' thinking processes declining into the use of procedures without connection to meaning and understanding. Among the three, the first one is particularly noteworthy. Teachers and students perceived demanding tasks as ambiguous and risky; hence it was often seen that teachers reduced complexity of tasks so as to manage the accompanying anxiety. All the same, when this was done, cognitive demands were weakened and students' cognitive processes fell into predictable and mechanical thinking. Inappropriateness of tasks likewise caused both a decline from doing mathematics into unsystematic exploration and to almost no mathematical activity. Plus, lack of motivation, prior knowledge, or suitably specific expectations were categorized as reasons for inappropriate tasks.

Despite their detailed analysis and description, Henningsen and Stein (1996) failed to provide a satisfactory account in several respects. First, the four teachers in their study were selected from an initial twelve teachers at four sites. Criteria for the choice, however, were not addressed, nor was the background of these teachers, such that insight into correlation between their training and instruction fails to materialize. Second, interpretation demonstrated in the present study solely based upon the COI samples and no interviews were conducted with teachers. In this manner, prejudiced analysis might occur for lack of a two-way communication. Third, students' responses were totally lacking in this report, which may lead to a biased situation inclined toward the teachers' side. As seen above, the present study suggested teachers' dominated role in influencing students' engagement in higher-level thinking (either in teaching approach or selecting of the problems). It is generally held that students are also responsible for their learning, and learners can never really learn without active participation. The implication made in this study to some degree ignored the role of students play in classroom. Though the chief purpose of this study was to provide a detailed qualitative portrait to illustrate factors associated with tasks set up to engage students in cognitive processes and high-level thinking, it at most provided a qualitative description rather than interpretation. For achieving insightful qualitative understanding, both out- and insiders' responses are indispensable.

The Use of Historical Problems to Develop Students' Mathematical Thinking

The merit of incorporating history in mathematics education has received considerable attention and discussion for decades. Whenever the place of the history of mathematics in the teaching of mathematics is debated, two questions are always asked (Heiede, 1996). First, why should history of mathematics have a place in mathematics teaching and learning? Second, what could be done to gain this place for integrating history into school mathematics? Among the benefits advocated by relevant scholars, the idea of using historical problems in mathematics teaching has received increasing attention (Avital, 1995; Barbin, 1996; Furinghetti, 1997; Katz, 1997; Rickey, 1995; Siu, 1995a, 1995b; Swetz, 1995a, 1995b). In contrast to telling mathematical stories to draw interest and improve attitudes (both are merely related to affective domain), using historical problems in classroom is an advance of not only being able to benefit students' affective domain but also to cognitive domain. The noted Norwegian mathematician Niels Henrick Abel once said that if one wants to make progress in mathematics, one should study the masters. Mathematical concepts have continually evolved and been revised through ages. The wisdom behind these great endeavors may provide deep insight into mathematical thinking. As Ernest (1998) put it, "Mathematicians in history struggled to create mathematical processes and strategies which are still valuable in learning and doing mathematics" (p. 25).

Mathematical thinking is a combination of complicated processes involving guessing, induction, deduction, specification, generalization, analogy, formal and informal reasoning, and verification. Yet modern mathematicians and most mathematical texts, influenced by contemporary deductivist doctrine, present the final product in a neat and polished format, which "hides the struggle, hides the adventure. The whole story vanishes" (Lakatos, 1976, p.142). Unlike others, the great mathematician Leonhard Euler in the 18th century was not reluctant to demonstrate his process of discovery. As Polya (1954) applauded:

> Yet Euler seems to me almost unique in one respect: he takes pains to present the relevant inductive evidence carefully, in detail, in good order. He presents it convincingly but honestly, as a genuine scientist should do. His presentation is 'the candid exposition of the ideas that led him to those discoveries' and has a distinctive charm. Naturally enough, as any other author, he tries to impress his readers, but as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself. (vol. 1, p. 90)

Thus, through analyzing Euler's problem-solving process, one may shed more light on the nature of mathematical thinking. Siu (1995a, 1995b) discussed numerous examples of Euler's approaches to solving mathematical problems to explicate how Euler's mathematical mind worked. For instance, in the solution of the problem of Seven Bridges of Konigsberg, Euler illustrated how generalization and specialization complement each other, introduced good notation, broke down the problem into subproblems and assembled these to give a solution to the problem. These procedures are quite typical in the work of mathematicians and worth pointing out to students.

In addition to presenting single typical solutions, demonstrating multiple methods for a particular problem provides an effective way to teach problem solving and develop mathematical insights (Swetz, 1995b). Alternative solutions for a particular historical problem from different persons, time periods, and cultures can be assembled and assigned as exercise for students to compare. Students can be advanced in this manner from *knowing* to *appreciating* the solutions. For instance, prior to Newton and Leibniz, several mathematicians were devoted to studying the tangent to a curve. Reńe Descartes and Pierre De Fermat, two contemporaries, developed distinct techniques in this regard, the former geometrically, the latter in an analytic fashion, neater and more concise. Despite the considerable criticism his method aroused, Fermat's idea of infinitesimal and analytic style not only brought about the birth of calculus but also ushered in the analytic era of mathematics.

Calculation of the sum of harmonic series $1+ 1/2 + 1/3 + 1/4 +\ldots$, a common sight in modern textbooks, may also serve as a good example along this line. The above sum is equal to infinity, usually surprising to students. Prior to revealing the secret, students had better be encouraged to explore and discover the fact on their own, followed by demonstration of approaches employed by ancient mathematicians such as Johann Bernoulli, Nicole Oresme, and Pietro Mengoli, (Dunham, 1990). As Dunham indicated, Bernoulli's approach is trickier; Oresme's idea clear and concise; the beauty of Mengoli's method is its self-replicating nature. When witnessing three different methods given by three great mathematicians, students may learn to appreciate the intrinsic nature of each approach. In this fashion, stereotyped thinking that mathematical problems always have only one rigid and strict method can be eliminated and students will be convinced that mathematics may involve creative work.

The solutions from different cultures may also serve the purpose very well. In the Chinese mathematics classic *Jiu Zhang Suan Shu* (*Nine Chapters on the Mathematical Art*), the area of a circle equals half the perimeter times half the diameter without any proof or interpretation; in Archimedes's *Measurement of a Circle*, the area of any circle is equal to a right-angle triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle, proved by double *reductio ad adsurdum*. The two different styles feature the chief distinction between ancient Chinese and Greek mathematical ways of thinking. Ancient problems were typically empirically based and task oriented; some early mathematics texts, Eastern or Western, only provided a solution for a specific problem without any general approach or rationale. Students may be encouraged to resolve the puzzle and led on a journey of discovery (Swetz, 1995b). More important, multiple approaches and proofs collected from history serve not merely to convince students but to "enlighten" them and further broaden their perspectives of mathematical thinking (Siu, 1993). Nonetheless, the aforementioned theoretical arguments have yet been empirically investigated.

Summary and Conclusion

Early teaching and study of mathematical problem solving were preoccupied with the effect of Polya's heuristic strategies—means leading to discovery. Yet an overemphasis on heuristics resulted in another kind of rote learning, having only short-term effects. It was found that students tended to have difficulty transferring these strategies to a new context. On the basis of his systematic investigation on college and senior students, and comparison between expert and novice problem solvers, Schoenfeld (1985a) mapped out an explanatory framework for analyzing students' problem-solving behavior, comprising four major categories: resources, heuristics, control, and belief systems. Among the four, beliefs systems—belief about self, doing mathematics, and context—are an individual's mathematical world view shaping the way one does mathematics, permeating through the whole process of doing mathematics. Students typically view mathematics as a set of rules and procedures in which each problem is supposed to be resolved by following specific algorithms. This misconception, largely shaped by ordinary school mathematics, contributed to students' perceptions of mathematics as a static body of knowledge with no place for creativity, that in turn may profoundly influence students' participation in meaningful problem-solving activities (Franke & Carey, 1997; Lampert, 1990; Schoenfeld, 1983a, 1992).

Some basic understanding of the nature of mathematical thinking is requisite for scientific literacy (AAAS, 1990), and one of the main goals of teaching mathematics is to help all students learn to think mathematically (NCTM, 1991). Schoenfeld (1992,1994) indicated that problem solving alone is not enough for developing an individual's mathematical thinking. Several problem-based reform curricula (e.g., CGI and problem-centered instruction) thus were developed to promote one's conceptual understanding and thinking strategies. Theses studies usually adopted a quantitative approach and measured students' achievement by standardized tests. Students' intrinsic thinking processes, in this manner, were undetectable. In contrast, Henningsen and Stein (1997) conducted qualitative survey to identify classroom-based factors that support and inhibit students' high-level thinking and demonstrated the influential role a teacher plays in the classroom. Nevertheless, most studies of this line focused on elementary level; relevant information regarding high school and college students are left unexplored. A qualitative investigation beyond primary level paying attention to how students' mathematical thinking and views on mathematical thinking evolve in a course should merit further examination. Moreover, despite the numerous discussions on the issue of using historical problems to foster students' ability and conceptions of mathematical thinking, empirical research designed to examine the effect is rare. An investigation of how these ideas can be integrated into the problem-based curriculum may generate precious information for future study on the area of mathematical thinking.

CHAPTER III
DESIGN AND METHOD

The main purposes of the present thesis were, through an exploratory study, to investigate Taiwanese technological college engineering-major freshmen's initial views about mathematical thinking and explore the interrelationship between the problem-based calculus course (that used historical problems) and these students' views on mathematical thinking. The major research questions were:

1. What are Taiwanese technological college freshmen engineering majors' views of mathematical thinking?
2. In what aspects and to what extent, if any, do Taiwanese technological college freshmen engineering majors' views on mathematical thinking change during a problem-based calculus course?
3. What is the relationship, if any, between a problem-based calculus course features and students' views on mathematical thinking?

Participants

Of the fifty-three students in the class, forty-four engineering-major freshmen, enrolled at a mid-size four-year technological college in central Taiwan during the fall semester of 2001, participated in the present study. The calculus course is required, serving as a basic course of their professional training. The goal of technological and vocational education in Taiwan is to inculcate students with practical skills that fulfill industrial and business demand, rather than focus on the typical educational goals of research colleges. The Taiwanese student in technological and vocational institutes traditionally receives more professional training with less general education. In this manner, mathematics, as an academic subject in technological and vocational institutes, is usually regarded as a knowledge source for a profession, rather than a discipline for training one's mathematical thinking. For example, prior to entering college, all vocational high school students must pass a multiple-choice entrance examination in mathematics; on the basis of colleagues' observation, these students are usually trained to pick the correct answer but fail to demonstrate the process correctly and precisely. Memorization then is a dominant way of learning mathematics.

The Course Curriculum and Instruction

Calculus is required for all engineering-majors, serving as a basic discipline for taking advanced mathematics courses such as engineering mathematics. Typically, the course is processed in a rather traditional way in which teachers are expected to cover a considerable amount of content and students are required to do many rote exercises in textbooks. The calculus course in this study was reorganized in order to match the purpose of the present thesis.

The Curriculum

The textbook used was *Calculus* (Bradly & Smith, 1999). Course goals and schedule, along with scope and sequences, are listed in Appendices A, B, and C respectively. To enrich students' understanding of the importance of the topic, several handouts collected from various sources were assigned to students as supplemental materials. All handouts, written in Chinese, (see Appendix C, only titles are shown) related to instructional topics or classroom activities. Topics without adequate Chinese materials were developed by the researcher/instructor.

Weekly problems (in Appendix D), differing from ordinary exercises in nature, were adopted from various sources (including the textbook) and served as challenging tasks to motivate mathematical thinking and put students in stressful situations. Schoenfeld (1983a) indicated that students are more prone to think a mathematical problem should be solved in minutes and otherwise cannot be solved. The weekly problems were designed to challenge this misconception as well as highlight the importance of the topic. Although the problems were considered challenging, they were not problems beyond the students' capability. With few exceptions, all problems were assigned closely related to the mathematics taught at the time the problem was presented.

Problem-Based Learning

The central doctrine of the problem-based learning is to problematize the topics. The meaning of "problem-based" in the present study thus was twofold. First, at the outset of each topic, the researcher/instructor proposed to elicit students' interest and curiosity by questioning them why they thought the topic was important, what the key concepts were, and how to resolve the problem, followed by a background introduction aimed at showing students the origins and importance of the mathematics concept. Second, students were assigned several problems as weekly homework. With few exceptions, the problems were closely related to the current class topic; students then applied known mathematics to solve the assigned problems; all non-routine tasks were of various difficulty levels gleaned from various sources. Students solved the problems through either individual work or group cooperation, depending on the difficulty level and characteristics of the problem assigned.

Collaborative Learning

In order to focus on problem-based learning and elicit higher-order thinking, some weekly problems required group collaboration. The design of the group work was to help students experience the social interaction of solving mathematical problems and enhance their communication ability while verbalizing mathematical ideas. Prior to group discussion, each student was asked to work the assigned problem alone and submit the draft solution, either complete or incomplete. The purpose was to emphasize the importance of individual learning in group tasks, to ensure that each student thoroughly understood the problem, and to constructively cooperate in the group. Students were asked to work in the groups outside classroom sharing their own ideas with others. When the group work was done, each student was required to hand in the revised solution. The two versions of the solution occupied equal weight of the final grade for the problem. In addition, students from each group were randomly selected to present their solution before the whole class.

Interaction in the Classroom

Given the traditional authority of the teacher's role in Taiwan, the typical classroom climate in Taiwan was reserved. Teachers are expected to impart material in a clear and concise way; students have learned to be as passive knowledge receivers rather than active constructors, severely hampering learning and opportunities for developing mathematical thinking. The researcher/instructor established a classroom environment that encouraged students to think about mathematics and make plausible guesses. For instance, as answers for weekly problems were collected, students demonstrating elaborative thinking were invited to share their ideas on the board, followed by a whole-class discussion. All students were encouraged to question, even challenge, other students' presentations; this activity was designed to engaging student involvement, eliciting higher-order thinking, and creating an open interactive environment.

Metacognitive Teaching

An important key feature of this course was that, the researcher/instructor served as a role model for metacognitive behavior. Teachers in a mathematics class commonly tend to present the solution to a problem in a clear format. Yet the byproduct of this approach is that the struggle of doing mathematics is diminished, giving students an inaccurate impression that every problem should be done in this way. "In consequence, the give-and-take of real problem-solving—the false starts, the recoveries from them, the interesting insights, and the ways we capitalize on them, and so on—are all hidden from students" (Schoenfeld, 1987, p. 200). Therefore, the researcher/instructor in the present study acted as a novice working problems from scratch, demonstrating various plausible approaches, and asking students to evaluate the possibility and difficulty of each approach. The students had opportunities to witness and experience the cost of an incorrect approach.

The Researcher/Instructor

The researcher was the instructor of this problem-based calculus course. The benefit for the study conducted by the researcher/instructor was that it allows the researcher/instructor a role in creating the setting to be investigated, to examine phenomenon from the inside, to learn that which is less visible by outsiders (Ball, 2000). With central issues in mind, the researcher/instructor was able to design the context and methods, probe the issues, try new approaches or materials, and examine the findings systematically. As Lampert (1990) indicated, the aim of this line of research is not to determine whether general propositions about learning or teaching are true or false. Rather, it is to further the understanding of the characteristics of particular kinds of human thought and to build an empirical link between theoretical analysis and practice. Nevertheless, this kind of work faces_ challenges in convincing others that findings are unbiased and worthwhile. Some concerns should be noted; one of which is whether the researcher/instructor is well equipped to be a designer, developer, and enactor of the practice. Therefore, the following sections regarding the researcher/instructor's background, views on mathematical thinking, and dilemmas are provided to clarify the role the researcher/instructor plays in the present study.

Academic Background

The researcher/instructor earned B.S. and M.S. degrees in Mathematics at the same university in Taiwan. Most college- and graduate-level courses taken by the researcher were within the pure mathematics domain. As such, the researcher/instructor received traditional and formal training in which axiomatic and deductive approaches constituted the chief mode of learning, with all subject matter treated hierarchically. The researcher/instructor was not satisfied with the course design of the program then, since in this manner, the intrinsic thinking process and developmental background of mathematical knowledge were all hidden from learners and the mathematics as a discipline was distorted.

After years of teaching mathematics at a technological college in Taiwan, the researcher/instructor further pursued a doctoral degree in mathematics education at a mid-size public university in the United States. His main interest was mathematical problem solving. Through studying relevant theoretical and empirical literature, the researcher/instructor gradually comprehended the importance of eliciting students' mathematical thinking and interest in learning mathematics. Since then, providing students opportunities to look at mathematics in a different way through various problem-solving activities has been the researcher/instructor's main instructional goal.

Views About Mathematical Thinking

The researcher/instructor developed his mathematical thinking mostly from reading history of mathematics. His interest in the history of mathematics began around the second year as a master graduate student. Only when immersing himself in the study of history did the researcher/instructor view the whole body of mathematical knowledge as an enriched organism, such as why certain mathematical concepts are important and how a mathematical idea evolves over the course of time. In this manner, the research/instructor's views on mathematical thinking have been influenced. For instance, from the mathematician's struggle and patience while doing mathematics, the research/instructor learned the importance of persistence in mathematical thinking. Considering the illogical development of some mathematical concepts, the researcher/instructor recognized the potentially fallible aspects of mathematical thinking and the necessity of rigor. More important, from reading history, the researcher/instructor was convinced that an inductive attitude toward mathematical thinking is imperative in mathematics; deduction and logic do not describe the whole domain of doing mathematics.

The researcher/instructor's overall views depicted mathematics as more than a school subject, a tool used by laypersons and scientists, or a thought product of mathematicians. Rather mathematics is a treasure encompassing the value of thinking and a precious heritage intimately tied to human culture. As a whole, the researcher/instructor's mathematical point of view delineated doing mathematics as a multi-dimensional interactive process between inductive thinking and deductive reasoning, concrete objects and abstract concepts, specification and generalization.

Dilemmas

The researcher/instructor regarded his role in the present thesis more as a researcher, which is theory-laden, than as an instructor, which is practice-laden. As a researcher/instructor, it was expected that several dilemmas would challenge his decision-making ability on various occasions.

The first dilemma the researcher/instructor faced prior to the study was the adoption of a textbook for the course. The researcher/instructor hoped to pick a Chinese edition of a calculus textbook, serving the goal of this study as well as being easily understood by participants. However, after an extensive search, no texts met both conditions. The researcher/instructor then searched for English versions and considered several editions. Among them, *Calculus* (Bradley & Smith, 1999) attracted the researcher/instructor's interest for its problem-based approach to guide the students' development of the mathematical concepts.

One of the major dilemmas for the researcher/instructor was to balance the extensive textbook content and create an open classroom climate to promote students' higher-order thinking. Calculus in a technological institute is usually viewed as a knowledge resource to support students in taking advanced mathematical or professional courses like engineering mathematics. Subsequently, related courses generally use a traditional teaching approach, heavily relying on memorization and rote calculation. Consequently, the researcher/instructor, a practicing teacher working within such an environment, was required to cover a certain amount of content while at the same time creating a problem-based learning environment.

Another difficulty laid in the validity of the present study. It was possible the researcher/instructor, either consciously or unconsciously, allotted a great amount of time to augment students' grasp of views on mathematical thinking and deviate from the mathematics content to be taught. An external observer or videotape recording of classroom episodes may serve to monitor but at the same time make the environment too artificial, defeating the researcher/instructor's purpose. Hence audiotapes provided a partial check of the researcher/instructor's classroom action. In doing so, the researcher/instructor created a checklist (Appendix E) including aforementioned important features of the course such as problematizing mathematical concepts, metacognitive teaching, increasing student involvement through questioning, whole-class discussion and so on. For each week, an invited reviewer, one of researcher/instructor's colleagues, then randomly listened to one-hour classroom episode and commented on the researcher/instructor's instruction on the basis of the checklist. The reviewer's comment, to some degree, was designed to help the researcher/instructor focus on the planned instructional approach.

Data Sources

The purpose of the study was to investigate the influence of a problem-based calculus course on students' mathematical thinking. Four instruments were designed to gather relevant data for addressing the research questions: the mathematics biography for collecting information about students' past learning experience in mathematics, open-ended questionnaire for investigating students' pre- and post- instruction views of mathematical thinking, follow-up interviews for validating students' written responses, and students' in-class reflection for collecting students' spontaneous thinking toward the classroom activity. Further, since the researcher was the instructor in the present study, unexpected bias might occur and result in misinterpreted findings. To minimize any potential bias, a researcher's diary functioned as a guide keeping the researcher/instructor on track.

The Mathematics Biography

All participants were asked to complete their mathematics biographies (Appendix F) at the outset of the semester describing their experiences in learning mathematics, any significant events or people influencing their disposition or attitude toward learning mathematics, how important mathematics is in their minds, and how they evaluated their capability and performance in such learning. Students' mathematics biographies served as auxiliary data in interpreting students' initial views of doing mathematics.

Open-Ended Questionnaire

An open-ended questionnaire was used to investigate participating students' views about mathematical thinking, before and after the instructional session. Use of open-ended items allowed respondents to express more freely their opinions on issues of interest, as compared to the closed format, Likert-scale questionnaire. Nevertheless, it is also possible that the respondent might miss the point if the scope of the question asked was too broad. For this reason, a brief background introduction of certain items was appropriate. In addition to encouraging respondents to elaborate more on their intrinsic thinking, asking them to justify the answer by giving examples may give the researcher a better position to interpret their responses.

On the basis of the goals of the present study and taking into account the above concerns, the researcher developed a questionnaire in a four-stage process. A draft questionnaire consisting of 16 open-ended items (Appendix G) was created and administered to around 100 college students with the background similar to the participants in the present study. This stage was aimed to test the researcher's personal ideas and collect students' opinions in a broad sense. In the second stage, by considering the main purpose of the present study and the vagueness or semantic overlap of the statements, nine items in the draft version were removed; four items were merged to two items; two items were revised; three new items were added. The second version, consisting of eight items, was then sent to a panel consisting of a mathematician, a mathematics education researcher, and a psychologist, to check its validity. Each panel member was told the purpose of the study and the goal of each item. The mathematician suggested a minor modification of the third question ("Is there any difference between a mathematician's way of thinking and a layperson's?" rather than "What's the difference between a mathematician's way of thinking and layperson's?"). The mathematics educator reminded the researcher/instructor that some questions might be hard to answer and whether the designed curriculum may help students to rethink the questions. The psychologist found no ambiguous wording. The third version was then formed according to the panel's comments and administered to 40 students with a background similar to the future participants in the present study. Five respondents selected at random were invited to interview to check the reliability of their responses. Though sometimes students exhibited difficulty in explaining their views or held conflicting conceptions with respect to certain items, generally speaking, their oral and written responses were satisfactorily consistent. Lastly, upon further consideration of the purpose of the present study, a revised version was developed by removing two additional items. The resulting questionnaire is presented in Appendix H.

In-Class Reflection Reports

The present study lasted 18 weeks. It was not possible for students to memorize all major events occurring in the classroom and their ideas while involved in the problem-based activities. In-class reflection reports (Appendix I), therefore, were designed to motivate critical thinking and help students reflect on classroom events. The activities were administered monthly (Week 2, Week 5, Week 10, and Week 16). If any significant difference occurred between participants' pre- and post-instruction responses, data collected by this instrument served as supplementary evidence for pursuing potential factors.

Follow-Up Interviews

To avoid misinterpreting students' written responses and to elaborate on their intrinsic thinking, a 20% representative random sample (nine students) was selected and invited to participate in the follow-up, one-on-one interview soon after the pre-instruction questionnaire was administered. In the semi-structured interviews, students were asked to read and explain their written responses to each item on the open-ended questionnaire. The researcher asked probing questions to elicit the interviewee's conscious or unconscious conceptions regarding issues of interest, such as identifying the most important components in mathematical thinking. The same procedure was conducted on the same random sample after the post-instruction questionnaire was administered.

The Researcher's Diary

The researcher's diary functioned not only as the researcher's self-reflection document but also as a guide throughout the class, an opportunity to keep a narrative account of the researcher's perspectives of actions in the classroom. Audiotapes recording classroom events constituted the main source for writing the diary. In this manner, the researcher's diary assisted the researcher in capturing the essence of classroom activities and interpreted their meaning for future teaching episodes. Major items in the researcher's diary were the researcher's self-reflection on classroom episodes, evaluation of teaching strategies, reassessment of subsequent lesson plans, and reactions to students' responses, both on questionnaires and in the follow-up interviews. As noted, several dilemmas could challenge the researcher's decision-making in various occasions. Resolution of such a dilemma was mainly on the basis of the purpose of the study and classroom episodes of the day. Episodes recorded on audiotape helped the researcher recall how he employed instruments and conducted relevant activities so as to prevent the teaching deviating from the goal of the present study or being distorted by some personal bias. All these considerations were included in the researcher's diary, in an ongoing attempt to portray his practice systematically and secure the descriptive, interpretive, and theoretical validity (Maxwell, 1992) of future findings.

Data Collection

The entire data collection procedure proceeded in three stages: (a) initial instruction stage, (b) instruction stage, and (c) late instruction stage, in which the instruction stage was throughout the 18-week-long session and overlapped with the initial and late instruction stages. A timetable of data collection is shown in Appendix K.

Initial Instruction Stage

Three types of data (pre-instruction questionnaire, mathematics biography, and pre-instruction interview transcripts) were collected during this stage, the first three weeks of the semester. The questionnaire investigating pre-instruction views was administered to all participants at the first class meeting soon after the process of human subjects was completed. To obtain the students' responses, students were asked to answer the questionnaire in class. The researcher also reminded students that their responses would not affect their class standing and that no correct or definite answers were sought. The questionnaire, comprising six open-ended items, was finished in fifty minutes. In addition to responding to the questionnaire, all students were requested to hand in their mathematics biographies at the next class meeting. Students' mathematics biographies served as auxiliary data for interpreting pre-instruction views. A random sample of 20% of participating students (9 students) were generated and invited to participate in the follow-up one-on-one interviews. Their written responses on the questionnaire and mathematics biography were read to generate questions to be asked in the follow-up interview. All interviews were audio-taped and transcribed.

Instruction Stage

In addition to questionnaires and interview transcripts, three sorts of data were gathered during this stage: (a) students' in-class reflection reports, (b) students' solutions to weekly problems, and (c) the researcher's diary.

As noted, for investigating their instant reactions and to encourage them to reflect their own thinking while involved in classroom activities, students were asked to submit their reflection reports. These data served as supplementary evidence for interpreting post-instruction responses in contrast to pre-instruction responses on the questionnaire.

The instructor assigned students several problems as weekly homework. The problems were aimed at problematizing mathematical topics as well as developing students' mathematical thinking. Students, therefore, were reminded not only that the correct answer was required but also that all approaches used, successful or not, should be included. Further, writing a research dairy was an ongoing process throughout the session. Major events occurring in the classroom, highlights of students' reflection reports and resolutions of weekly assignment, and the researchers' reactions constituted the content of the research dairy. This document was self-dialectic in nature; the researcher kept asking himself whether the instructional objectives had been achieved. If the answer was yes, he considered the next step. If the answer was no, he considered what caused the failure, how the situation could be remedied. By referring to the audiotape recording of daily classroom episodes and students' in-class reflection reports, a search was made for the solution.

Late Instruction Stage

Post-instruction questionnaire and follow-up interview transcripts were collected during the last three weeks of the semester. The post-instruction questionnaires, comprising those same items as the pre-instruction questionnaire, were administered to all participating students in class during the 16th week (two weeks before the end of the semester). As previously indicated, students were reminded that responses had no bearing on their class standing and that no correct or definite answers were sought. The researcher stressed, in class, that the point was to explain their current thinking; therefore a recall of previous responses was not necessary. More importantly, to minimize potential bias, the researcher cautiously avoided mentioning that difference in written responses was his main concern.

During the 16th and 17th weeks of the semester, several follow-up interviews were conducted with the same nine randomly selected students participating in pre-instruction interviews. Prior to the post-instruction interviews, these students' in-class reflection reports and responses to post-instruction questionnaires were analyzed to pinpoint special features or inconsistencies and thereby generate questions for post-instruction interviews. The context of the post-instruction interview was identical to that of the pre-instruction interview. All interviews were audio-taped and transcribed.

Data Analysis

Though the research problem was clear at the outset of the present study and the context was defined, to prevent the researcher from unfocused data, an ongoing analysis was necessary. The entire data analysis process consisted of three major phases: (a) validating and interpreting participants' pre-instruction views, (b) validating and interpreting participants' post-instruction views, and (c) answering questions of interest.

Phase I: Validating and Interpreting Pre-Instruction Views

This phase aimed to interpret all 44 participating students' pre-instruction views about mathematical thinking. After the first day of administration of pre-instruction questionnaires, each participating student's responses were read, analyzed, and treated as independent objects to identify their initial views on mathematical thinking. For the nine students in the random sample, a triangulation was adopted to reach this goal. Three documents (pre-instruction questionnaires, mathematics biography, and pre-instruction interview transcripts) were used to provide more detailed insight into their conceptions. These students' responses to items on the questionnaire were read and analyzed, then their mathematics biographies were compared and contrasted to seek corroborating or conflicting evidence. By meticulously analyzing the two documents, the researcher generated questions to be asked in the pre-instruction interviews, during which an effort was made to elicit the interviewees' conscious or unconscious, consistent or inconsistent views by asking probing questions. If any discrepancies appeared, the researcher consulted with the interviewees to determine which profiles more appropriately represented their views.

Phase II: Validating and Interpreting Post-Instruction Views

Responses to post-instruction questionnaires were validated on a basis similar to that of Phase I. For the nine students in the random sample, three documents (post-instruction questionnaires, in-class reflection reports, and post-instruction interview transcripts) provided insight into their views on mathematical thinking. One difference from Phase I was the use of students' in-class reflection reports. These documents were alternative information to shed light on students' thinking and feeling throughout, hopefully to help interpret the post-instruction views in many ways. If students' professed views shifted near the end of the semester, the reflection reports perhaps could provide clues. For instance, if a student's view on mathematical thinking shifted from deductive to inductive, the researcher asked the student to defend his or her answer by giving examples, and further consult to in-class reflection reports to identify the possible factors. If supportive evidence was not found, either in interview transcripts or reflection reports, then the assertion that their views had been upgraded would not be made.

Phase III: Answering Questions of Interest

The ultimate aim of analyzing various data was to answer questions guiding the present study via looking for special patterns or features of participants' views about mathematical thinking before and after instruction. Since the present research is an exploratory study through practical action, a certain degree of subjectivity is unavoidable. To minimize potential bias stemming from dual roles of researcher/instructor, answering questions of interest was on a conservative basis. With the lack of a control group, the influence, if any, of the problem-based course on students' views of mathematical thinking was not expected to be interpreted in a causal way.

By exploring and analyzing participating students' mathematics biographies, pre-instruction questionnaires, and follow-up interview transcripts, a holistic profile was mapped for interpreting the participants' initial views. An investigation was made of whether students' views reflected current views of mathematical thinking. In either case, on the basis of mathematics biographies and interview transcripts, underlying causes were sought to reveal the relationship between the formation of students' current views and their past learning experiences.

The present study explored the relationship, if any, between a problem-based course and students' views of mathematical thinking. It was not expected that students' views were enriched in all aspects. Therefore, data analysis was on an aspect-by-aspect basis. Namely, if a student's views of mathematical thinking shifted for certain aspects, it was not boldly reported that the student's holistic understanding of mathematical thinking had been enriched. For instance, after taking a one-semester course of this kind, students might develop an appreciation of the importance of persistence in doing mathematics yet still show poor thought on the complementary relationship between induction and deduction in mathematical thinking. In this case, factors contributing to or hampering progress in their professed claims were located.

The problem-based calculus course contained various features such as problematizing mathematical topics, cooperative group activity, using weekly problems, and metacognitive teaching. If the course did exert a certain degree of influence on students' views of mathematical thinking, a further analysis was made to investigate what aspects of the course were related to the participating students' conception on certain facets. For instance, if a student's post-instruction responses showed a more enriched view on mathematicians' thinking processes, the analysis then aimed to establish any potential link between the course feature and students' view shift. The link though can not be interpreted as a causal-effect inference, further investigation focusing on clarifying the interactive relationship is imperative.

The present thesis is an exploratory study investigating the relationship of a problem-based calculus course, using historical problems, on Taiwanese college students' views of mathematical thinking. As cited above, through gathering and analyzing independent but complementary data sources, an attempt was made to secure the findings and identify any potential links between any students' shift in their views of mathematical thinking and the designed course features.

CHAPTER IV
RESULTS

The chief aim of the present study was to explore the interrelationship between a problem-based calculus course, using historical problems, and Taiwanese technological college engineering-major freshmen's views of mathematical thinking. To achieve this goal, the study explored students' initial views of mathematical thinking at the outset of the course and probed their post-instruction views on mathematical thinking near the end of the semester. The ultimate goals were (a) to depict how students' conceptions regarding thinking mathematically evolved during the investigated course, and (b) to identify any potential course features which may have played a role in fostering the learners' views of mathematical thinking.

The chapter consists of three sections: demonstrating pre-instruction views, characterizing post-instruction views, and answering questions of interest. The first section analyzes past learning experiences and initial views of mathematical thinking on the basis of various data sources: mathematics biographies, open-ended pre-instruction questionnaires, and semi-structured follow-up interviews. The second section plots changes in participants' views of mathematical thinking after an 18-week-long problem-based course. The third section explores potential interrelationships between the course features and participants' view-shifting. Special heed is paid to which aspects and to what extent this problem-based curriculum builds learners' conceptual frameworks associated with thinking on mathematics.

Pre-Instruction Views

This two-part section focus on the relationship of participants' perspectives to their past learning experiences in mathematics and current views of mathematics.

Students' mathematics biographies were analyzed to describe past experiences in learning mathematics, including any significant events or people influencing their dispositions or attitudes toward this field, importance of the subject, and self-evaluation of capability and performance in doing mathematics. Their pre-instruction views regarding various aspects of mathematical thinking (persistence and creativity, for instance) were then investigated to set the stage for the discussion, comparing and contrasting learning experiences and initial views.

Participants' Learning Experiences

Mathematics biographies provided the chief source for interpreting past learning experiences. Follow-up individual interviews conducted with the nine randomly selected students, nonetheless, served to establish the validity and reliability of the self-reports. Written responses were found congruent with oral answers expressed during interviews. A coding system was developed to identify individual participants. Each code began with one or two letters followed by a two-digit number. Students not interviewed were coded from S01 to S35; nine students in the random sample were coded from RS01 to RS09. Although 44 students consented to participate, 10 did not return their mathematics biographies. Thus the report in this section was based on 34 students' written responses and the nine random sample students' oral responses. The list of students who did not return mathematics biography is provided in Appendix L.

The most prominent finding gleaned from the mathematics biographies was the teachers' dominant role in the participants' learning of mathematics. With regard to the most important people or events influencing past learning in mathematics, 21 of the 34 respondents (62%) mentioned style of teaching as exerting significant impact on their dispositions or attitudes toward mathematics. S13's response was a typical one:

> But I formed a passive habit of learning [math]. The class tutor during the 5th and 6th grade was a math teacher; I began to understand math from that time on. In the eighth grade, the math teacher's teaching was so vivid and vigorous that my math record progressed rapidly...such a pity that the 12th grade math teacher did not do his duty well, such that my record fell off a little bit. Hence the teacher's instruction is extremely important to me. (S13, mathematics biography)

Among them, several had been affected by the teacher's encouragement or enlightenment while learning, as shown by the following quotes:

> I did poorly in elementary school math. I remember I was unable to memorize timetables until 5th grade. Just after entering senior high, I did not think that my math can be remedied...whereas, unexpectedly, the math teacher told us in the first class that despite how bad you guys' math is, you are going to have a new beginning as long as you study hard from now on. I was thinking this could be my turning point...it turned out that the more I learn [math], the more I got fulfillment. (S04, mathematics biography)

> Math is fun for me. I can always score high as long as I learn diligently. I had a cool math teacher in the 11th grade. His homework cudgeled my brain. Assigned problems were so flexible that deeper comprehension was required. No matter how difficult a problem, he said, nothing is too hard if you understand basic definitions. Math makes me quick-witted. (S16, mathematics biography)

On the other hand, two students described unhappy memories of junior high school teachers:

> My performance at the elementary level was none too good, whereas a junior high math teacher was quite strict. You got paddled if you failed to pass exams. To avoid punishment, the best thing to do is to study math obediently. (S34, mathematics biography)

> My math record was about average and not extremely outstanding. In the 9th grade, the teacher kept a close watch, such that learning was partly forced, more or less dampening my interest in math. (S35, mathematics biography)

Five participants, instead of referring to the teacher's influence, exhibited a tendency of self-recognition while studying mathematics. They appeared to acknowledge the problems of mechanical learning:

> I used to focus on applying formulas while learning math, instead of the rationale for those formulas. Not until the 12^{th} grade did I recognize that memorizing is not enough for employing knowledge learned. (S06, mathematics biography)

> My parents sent me to learn mental calculation in kindergarten. Calculation was easy as pie for me then, but my thinking changed soon after attending elementary school. Good in mental calculation may not guarantee good math. What is important is the thought of concepts. (S09, mathematics biography)

Four of these five students were confident of their capability in doing mathematics. On the other hand, those who reported past learning of mathematics as significantly affected by teachers were less likely to demonstrate this sort of self-recognition, while considering themselves good at mathematics. The finding suggests that, on the basis of mathematics biographies, a majority of these students manifested a passive habit of studying the subject. An initiative spirit of learning was hardly seen in their professed claims, even for students reporting high self-evaluation.

The effect of passive learning habits also was reflected by their views. The random sample students were asked to propose the best way to learn mathematics. Several respondents, including their enthusiasm about doing mathematics, mentioning more practice as the key:

> In my case, I'll read first and go to solve problems. I feel that examples are used to applied theorems. If merely paying attention to definition, some stuff would be hard to understand. Just doing problems. *Doing more, understanding will follow* [italics added]. (RS03, pre-instruction interview)

These responses endorsed a common view that practice makes perfect in mathematics.

In addition, participants were asked about the importance of mathematics. There were only 15 students responding to the question and all acknowledged that mathematics was an influential subject, with their reasons falling into three main categories. First, five students regarded mathematics as a significant subject because of its importance in school records or exams.

> In junior high, my math record was the top ten in class. This was still the case in senior high. Mathematics is rather important because it is a subject making me outperform others. (S05, mathematics biography)

> Mathematics, as a subject, occupies much weight at all school levels. Therefore it is very important to me. (S08, mathematics biography)

> Mathematics is quite important for me. My English is so poor that I gave it up before taking the college entrance examination and busied myself preparing for mathematics. It should be credited to mathematics that I can attend this college. (S34, mathematics biography)

These quotes mirror a shared conception regarding mathematics held by students and the reality of Taiwanese mathematics education. In Taiwan, mathematical achievement has long been viewed as an index for assessing an individual's capability of learning in general, especially for engineering majors, and exam scores were equated with individual achievement. As such, pursuing high scores was the students' ultimate goal of learning, sacrificing practical and intellectual values of mathematics. As S09 professed:

> Though I always earn high scores in mathematics after attending high school, I can never realize what mathematics is for. (S09, mathematics biography)

Four students concurred with the role of significance of mathematics in professional training; three mentioned the importance of mathematics in daily life. They were more likely to relate the practical use of mathematics in daily life:

> I understood the importance of mathematics while learning electricity. Mathematics sustains various theories and experiments…I am quite interested in mathematics. I myself feel that advanced mathematics is only useful for advanced theories. Just like when going shopping on street, it is impossible to say to the store keeper: "Hi, boss, give me the square root of 2 pounds of stuff!" (S27, mathematics biography)

The aforementioned quote conveyed a pragmatist view in which mathematics is treated less as a discipline and more as a tool; the value of mathematics relied on its utility of application rather than intellectual function. Of all the respondents, only one associated mathematics with thinking in the mathematics biography:

> I study and think on my own while learning mathematics. Not until 12^{th} grade, because of its difficulty, did I begin to listen to teachers' explanation. I feel that learning mathematics is a good way for training one's thinking ability. The more difficult problems I solved, the more sense of fulfillment I got. (S20, mathematics biography)

The statement also reveals S20's active tendency in doing mathematics.

On the basis of the mathematics biographies, participants generally showed some degree of confidence in their capability of doing mathematics. Among 25 students reporting their self-evaluation of mathematical ability, only six considered themselves bad at mathematics; four demonstrated a positive attitude toward the subject:

> My evaluation for current mathematical ability is that, despite those requiring memorizing formulas, I can use logical reasoning to look at a problem from various angles. I am not a high achiever in mathematics, but I am convinced that I can get what I want as long as I study hard. (S18, mathematics biography)

> I have learned mathematics since the 1st grade and been quite interested in it. Mathematics attracts me via processes of discovering and solving problems. I don't see myself as smart, yet I am willing to give it my best try because I am always interested in calculus, though also feel that it is hard. (S25, mathematics biography)

Based on the mathematics biographies, three common features of participants emerged.

1. The teacher played an influential role in learning of mathematics at various levels. Appropriate or suitable styles of instruction led them to gain better understanding of mathematical concepts and helped develop good habits for doing mathematics. Contrarily, the exam-driven, cramming method of teaching placed a negative impact on their disposition for doing mathematics.

2. With few exceptions, students tended to hold that more practice is the best way to learn mathematics, though a majority of them considered themselves to be good at mathematics. This view may be related to the previous characteristic. Moreover, some identified the importance of basic definitions and principles,

whereas the significance of a holistic conceptual understanding of and a persistence in doing mathematics were not emphasized.

3. In spite of exhibiting a positive disposition and enthusiasm toward the subject, an appreciation for the value of mathematics was lacking in the respondents' statements. They either associated the importance of mathematics with professional training or with utility in daily life, a pragmatist point of view. The merit of mathematics as an intellectual discipline was scarcely addressed.

Pre-Instruction Views of Mathematical Thinking

The students responded to a six-item, open-ended questionnaire (Appendix H) at the first class meeting of this calculus course in which they were asked to describe their thinking about the questions of interest in a few sentences. The purpose of this pre-instruction questionnaire was to generate a profile of the participants' pre-instruction views regarding mathematical thinking in particular and mathematics in general in order to further validate and elaborate on the individual interviews conducted with nine students in the random sample. Questions posed during these semi-structured interviews served not only to elicit the interviewees' intrinsic thought but also help them reflect on extant concepts, consistent or inconsistent. An attempt was made to pursue the interviewee's line of thought to a satisfactory level of clearness and explicitness. Some conflicting ideas or misunderstandings of problems surfaced while they further explicated their thinking; yet generally speaking, the participants' written and oral responses showed acceptable consistency.

The six-item questionnaire investigated the participants' macro- and micro-views about mathematical thinking prior to entering the course. Its scope encompassed various dimensions of mathematical thinking. The main foci were (a) the essence of mathematical thinking, (b) distinguished features of mathematics, (c) development of mathematical knowledge, (d) persistence of doing mathematics, (e) creativity and flexibility of deriving solutions, and (f) a mathematician's way of thinking. During analysis, each item was treated as an independent entity and followed by contrasting relevant data sources to seek patterns or interrelationships among items, if any.

The first item asked students to define mathematical thinking on the basis of their understanding, with the intent of profiling the essence of the construct in their minds. As shown in the Table 1, 45% (20 of 44 respondents) associated mathematical thinking with ways of solving problems or deriving answers. For all tables, the number of responses identifies the numbers of students who pronounced the specific comment. Some students pronounced more than one comment.

Table 1

Major Pre-Instruction Responses Regarding Mathematical Thinking

What is mathematical thinking	Number of responses	Percentage
Ways of solving problems or achieving answers	20	45%
A process of logical thinking or reasoning	12	27%
Calculation and operation	4	9%
Recall of formulas	4	9%
No responses	4	9%

The outcome revealed that the participants were more likely to perceive thinking of mathematics as a solution-oriented process, in which executing mathematical operations is the means and reaching final answers is the end:

> [Mathematical thinking] should be the thinking of solving mathematical problems. (S18, pre-instruction questionnaire)

> [Mathematical thinking is] figuring out how to calculate on the basis of demands of problems and identify rules, or pondering the methods to find answers. (RS06, pre-instruction questionnaire)

> My understanding to "mathematical thinking" is "multiple solutions." For instance, when a teacher poses a question like $2x + y = 6$, students may give a variety of answers such as "equations," "x = a certain number, y = a certain number," or someone else would answer "a line" (all quotation marks original). Therefore, the question is fixed but perspectives may vary. (S05, pre-instruction questionnaire)

Further, participants tended to relate solving problems to deriving answers by following predetermined routes. Thinking processes were referred to as recalling fixed formulas:

> [Mathematical thinking is when] you are given a problem to figure out how to do and then to search for a rule to work it out...[that is] *to identify a preset method to solve it* [italics added]. (RS06, pre-instruction interview)

> Mathematical thinking should be so-called recalling mathematical formulas in order to apply them on problem solving...[I was] always *pondering* [italics added] when taking math tests. While pondering, I often can get solutions by means of applying a series of formulas. (S06, pre-instruction questionnaire)

It seems students interpreted thinking of mathematics as ways of recalling and applying formulas. Student S24 confessed in the pre-instruction questionnaire that his way of mathematical thinking was doing problems and using formulas rather than anything to do with thinking. This finding highlights a widespread phenomenon in Taiwan: students are usually trained to do routine mathematical tasks, and the use of formulas or rules becomes the dominant way.

Twelve respondents (27%) referred to mathematical thinking as a process of logical thinking or reasoning; doing mathematics trains a person's logical reasoning:

> Mathematical thinking is pondering one thing by means of integrating logical concepts. We cannot merely rely on memorizing while solving mathematical problems. For bringing [skills] into full play, considering fundamental concepts is required. (S28, pre-instruction questionnaire)

> [Mathematical thinking is] thinking on problems via learned mathematical principles and logic. Just like if you want to find the volume of an object, you have to evaluate it by using various methods for deriving volumes. (RS05, pre-instruction questionnaire)

As compared to previous responses, associating mathematical thinking with deriving answers by means of formulas and following routine procedures, the statements of this sort no doubt demonstrated a thoughtful insight; nevertheless, a concern soon emerged after carefully investigating their personally expressed thinking. Asked to elaborate further the meaning of mathematical thinking, student RS05 gave an unexpected answer:

> [Mathematical thinking] means that you are given a direction and you have to do it by following the direction…[logic] is a mathematical concept, a fixed procedure frequently employed. (RS05, pre-instruction interview)

RS05's statements seemingly did not differ much from the previous ones in which following predetermined routes was demanding. Logic, in RS05's mind, was merely the rule on which all operational steps underlie. Plus, participant RS02 endorsed the view that mathematics is for fostering one's concept of logical thinking, whereas confessed that he had never experienced the merit. This fact also manifested itself in the following quotes professed by RS08:

> I have never felt that so far [mathematical thinking is one way of training one's logical reasoning]. Teachers told me so…I was told that mathematics is for training my brain and thinking, yet I have never had a feeling of this kind. (RS08, pre-instruction interview)

Thus, the participants' association of mathematical thinking with logic could be merely rhetorical and superficial.

Four participants did not respond to the item, perhaps demonstrating that some had rarely reflected upon their own thinking while doing mathematics. An individual's views about something are generally shaped by years of working in this field. This inability to answer the question could be attributed to the way they were taught to launch into deriving a solution by executing a calculation soon after reading problems. In their minds, the position for mathematical thinking had been taken by mathematical operations. This case was not limited to passive learners. Participant RS03, an active student exhibiting an extremely positive disposition toward doing mathematics, also confessed that:

> [Mathematical thinking] is the thinking on solving mathematical problems…[I] never paid attention to and heard of the issue. Therefore, I never thought of it. (RS03, pre-instruction interview)

Though mathematical thinking is hard to define as a set of features that are necessary and sufficient (Sternberg & Ben-Zeev, 1996), the finding clearly reveals that, as a rule, these students focused primarily on answers rather than the processes of obtaining the answers. A holistic perspective of what it means to do mathematics was lacking in participants' views of problem solving.

Problem solving is a process of applying knowledge to unknown fields. The second questionnaire item aimed at exploring how students reacted to difficult situations. The primary strategies adopted by the participants are shown in Table 2.

Table2

Major Pre-Instruction Responses Regarding Strategies to Unfamiliar Situations

Instant strategies to unfamiliar situations	Number of responses	Percentage
Seeking assistance (looking for materials, asking for help)	12	27%
Thinking for a while then asking for help	8	18%
Going back to basic concepts	5	11%
Testing alternative approaches	4	9%
Recall similar problems or formulas	4	9%
Skipping it	2	5%
Other	9	20%

When asked to point out the quick strategies they would use to solve a difficult position, 12 (27%) respondents reported that the first thing they would do is seek external assistance, such as asking for someone else's help or referring to relevant material:

> [My] instant reaction is looking for books. I'll skip it first while taking the exam and, when the rest of the problems are done, go back to see if I can think of related problems in order to associate it with methods of resolving. (S06, pre- instruction questionnaire)

> The first thing to do is searching for similar type of problems. If it does not work, [I would] ask for classmates' assistance or, if they cannot do anything helpful, ask the teacher to solve. (S24, pre-instruction questionnaire)

> [I would] think on it for a while, but that would not endure too long. Probably I would try similar methods two or three times: otherwise I'll go ask someone. It would not last too long. (RS05, pre-instruction interview)

Moreover, two respondents professed that they would skip it directly. The responses demonstrated that almost one third of the participants lacked personal perseverance while doing mathematics. One reason for this shortage could be that they were generally solving problems under time or exam pressure. As the student RS07 confessed in the questionnaire and interview:

> I would follow the feeling and think of it randomly. If still no clue at all, skip it. But for record and responsibility, I guess I will ask for help from people whose math better than me. (RS07, pre-instruction questionnaire)

> It would be exciting if I got the problems solved. If I failed, I would ask someone or give up…It was probably short of time [of thinking or referring to relevant material]. There were a lot of mathematics problems…for responsibility, seeking for help from people quite strong on math. I don't have confidence in insisting to solve a problem. (RS07, pre-instruction interview)

With the participants' concern for solving a large number of problems hurriedly, some of them tended to quickly quit thinking with a challenging task. Several respondents, though not deciding to ask for outside assistance immediately, adopted conservative strategies to deal with difficult positions, such as recalling formulas, similar problems, or what teachers taught.

Conversely, some others expressed alternative views while facing predicaments. Eight students claimed that they would think on their own for a while before asking for help; four expressed a more active disposition—testing all approaches:

> [My] instant reactions are 1. thinking on the content first; 2. pondering why I feel it difficult; 3. searching for key concepts; 4. discussing with classmates; 5. inquiring of teachers. (S09, pre-instruction questionnaire)

> I usually try to think on it for a while. If there are some clues, I would make a try to solve; otherwise I would go back to fundamental concepts instead…and seek assistance from able people if I cannot understand. (RS09, pre-instruction questionnaire)

A student, RS03, exhibited more persistence toward doing mathematics:

> Just pick up the problem and do it. Then keep thinking…because any problems more or less can be classified as certain kinds…then go figure it out and derive a reasonable explanation. If it is still not working, look at it again in different ways…I am normally quite persistent. As a rule, I would not give up until exhausted. (RS03, pre-instruction interview)

It is interesting to know what creates RS03's persistence. One plausible explanation is that he was rather confident of his mathematical ability. However, when asked to explain, he responded:

> There is little to do with confidence. *This is what mathematics is all about* [italics added]: you have to think. Mathematics is not so hard as it appears. You would feel it easy when you achieve a breakthrough in your thinking. (RS03, pre-instruction interview)

RS03 not only gave a thoughtful account of mathematical thinking, but also showed extreme persistence in doing mathematics. He mentioned that he had thought about problems for weeks.

Previous results indicated several participants generally showed a traditional disposition while stuck. Nonetheless, S03 responded on the pre-instruction questionnaire that he would imagine what he could do if he were a high achiever in mathematics, suggesting that a model of mathematical behavior could exist in his mind.

The mathematician is typically regarded as the outstanding mathematical problem solver. Though this may or may not be the case (Defranco, 1996), laypersons usually conceptualize mathematicians' ways of thinking as an archetype. On the basis of this notion, students were asked to propose, in their imagination, how mathematicians think of a mathematical problem and any differences in their thinking from laypersons. Respondents' thought on models of mathematical thinking was elicited.

Table 3

Major Pre-Instruction Responses Regarding How Mathematicians Think

How Mathematicians Think	Number of responses	Percentage
Thinking from diverse angles/alternative approaches	8	18%
Being able to find a quickest (best) way	8	18%
Starting from basic concepts/fundamental principles	5	11%
Thinking hard	4	9%
No answer	6	14%

As demonstrated in Table 3, eight of the forty-four respondents (18%) considered mathematicians as usually able to attack problems from diverse angles or apply alternative approaches, flexible thinkers less likely tied down by fixed procedures:

> I guess they are able to think of the most basic ideas and then solve a problem by integrating various basic ideas. (S09, pre-instruction questionnaire)

> They mull over problems from diverse aspects and try to derive solutions by using all kinds of approaches. (S14, pre-instruction questionnaire)

On the questionnaire, the participant RS01 indicated that a mathematician's thinking should be multi-faceted. He elaborated further in the follow-up interview:

> Just like me...I can only use extant methods invented by someone else to solve a problem. Real thinkers probably would think of varied aspects and compare different items to attain more convenient solutions belonging to themselves. They are more flexible. (RS01, pre-instruction interview)

An intriguing question following this finding concerns what enables mathematicians to do so. Two students perceived mathematicians as owning solid knowledge background:

> Mathematicians probably know more content than laypersons. Therefore they can solve a problem in multiple ways. Laypersons can only apply known knowledge. (S08, pre-instruction questionnaire)

> They [mathematicians] own a solid foundation and thus are abler to work things out. Especially when a problem can be solved in multiple ways, they are capable of deriving different solutions. (RS06, pre-instruction interview)

In addition to characterizing mathematicians as more knowledgeable, another two participants attributed mathematicians' flexibility to more practice:

> I guess mathematicians can be mathematicians because *they definitely do more problems than laypersons* [italics added]. Their thoughts and ways of thinking form a nerve network able to access anywhere, whereas laypersons' thinking is like a water pipe: only one route. (S18, pre-instruction questionnaire)

The result endorses a view that content knowledge and practice are keys to performing well in mathematics, as S04 and S16 indicated:

> *A bunch of mathematical equations will surface in* [*mathematicians'*] *brains* [italics added], and then they pick one according to the problem. (S04, pre-instruction questionnaire)

> Mathematicians' brains must be filled with various kinds of definitions and solutions for solving problems. In this manner, they are able to solve problems *by using very simple, quick and precise approaches* [italics added]. (S16, pre-instruction questionnaire)

Similar to the above vision, eight participants claimed that mathematicians usually can solve problems in the quickest ways:

> They derive the solution to a problem by using the most basic concept, then extend it to attain the quickest method. (RS07, pre-instruction questionnaire)

This impression could be shaped by past teachers' instructional approaches, as evident in RS07's responses:

> Teachers used to solve problems via some basic but trifling methods, then introduced a rather fast approach. It could be very simple to employ fundamental ways, but we all felt tired; it remained unsolved after 20 more minutes. Mathematicians are more familiar with methods. (RS07, pre-instruction interview)

The view of mathematicians as effective solvers may lead students to believe there is always a shortcut for deriving answers and only expert solvers recognize. Students as such would be reluctant to devote themselves to thinking hard about problems if they do not consider themselves capable.

Although mathematicians' persistence in doing mathematics was not considered by most participants, four respondents cited hard thinking as critical to a mathematician's vocation. S07 professed that mathematicians devote their whole lives to mathematics and so are able to draw an analogy while tackling problems; S24 claimed, "mathematicians' way of thinking is beating their brains." Ideas of this kind can be characterized by S28's statement:

> I guess mathematicians normally think hard day and night while solving a mathematical problem. They will try to derive solutions by any means, no matter how long, how difficult. (S28, pre-instruction questionnaire)

It is noteworthy that S25 and S27 proposed specific descriptions regarding mathematicians' work. S27 submitted four stages of their thinking process:

> 1. Locating where [he] gets stuck. 2. searching for applicable equations. 3. testing by experimental ways. 4. discussing with friends. (S27, pre-instruction questionnaire)

While S27 provided a comprehensive picture of mathematicians' ways of thinking, S25 plotted a more clear understanding of the process for solving problems:

> Mathematicians usually carry out a survey with respect to problems, boldly make a conjecture, testing it repeatedly by using known and unknown, then constantly verify result when all done. (S25, pre-instruction questionnaire)

The statement to some extent revealed the empirical aspect of mathematics. S25's perspective regarding mathematical thinking also appeared in his math biography, where he professed that the process of discovering and solving problems is what attracts him to mathematics.

In addition, participants were asked to outline, based on their understanding, differences between mathematicians' and laypersons' modes of thinking. This item attempted to elicit the archetype of thinking in mathematics, which may be implicit in their minds, by contrasting distinctions of both sides. Further, students' responses to this question function as auxiliary evidence of the previous question.

Seven of the forty-four (16%) respondents held that the major difference between mathematicians and laypersons is mathematicians' greater ability to attack problems from various angles and employ alternative approaches, whereas the laypersons' thinking is more rigid:

> Mathematicians are likely to think and prove from a variety of facets, but laypersons usually limit themselves within the scope of what they learned. (S23, pre-instruction questionnaire)

> Laypersons normally focus all their attention to the same point, such that few methods can be thought out. Mathematicians tend to approach [the problem] from multiple facets, less likely to confine [themselves] to a fixed area. (RS04, pre-instruction questionnaire)

One student clearly accounted for mathematicians' flexibility:

> Mathematicians: intuitional reaction but solutions and approaches may not be the easiest and quickest. [They] also attack from alternative angles and are less likely to use particular methods too often to solve problems.
> Laypersons: directly associate to formulas or rapid ways and constantly use one fixed method to derive answers. (RS09, pre-instruction questionnaire)

Still, four respondents also asserted that laypeople tend to think using formulas and mathematicians using theories. Note S10's alternative perspective regarding mathematicians' thinking:

> *Mathematicians are inclined to look for problems* [italics added] and take a further step to think on the basis of theories developed by themselves. Laypersons typically focus on the solution to a problem and consider types of problems. (S10, pre-instruction questionnaire)

The view that the mathematician is a problem discoverer as well as a problem solver was not voiced by others.

Another chief distinction proposed by the respondents was logical reasoning. Five of the forty-four (11%) participants claimed mathematicians, as a rule, think of the problem based on logical reasoning and hence are more methodical and rigorous:

> Because mathematicians own a solid base of concepts, [they] must be thinking by logic on the basis of concepts; laypersons normally think by experiences or intuition. (RS05, pre-instruction questionnaire)

This response seemingly conveyed a belief that experiences and intuition are more unreliable. Nevertheless, some other participants (S13, S31, RS03) regarded experiences and intuition as keys that help mathematicians to think about problems. It is therefore noteworthy to further investigate participants' conceptions of experiences and intuition in mathematical thinking.

Mathematics is generally seen as a discipline requiring creativity. Schoenfeld (1989) reported a conflicting finding that high schoolers regarded mathematics as a subject involving creativity yet at the same time they viewed memorization as the best way to learn mathematics. In this study, while responding to the previous question (distinctions between mathematicians' and laypersons' thinking), only one participant associated mathematics with creativity. Hence it is noteworthy to scrutinize participating college freshmen's view on the issue. Respondents were asked to express their opinions on two contrasting viewpoints:

problem solving in mathematics as a thinking process involving creativity or as one that requires following predetermined procedures for deriving correct answers.

Table 4

Major Pre-Instruction Responses Regarding Creativity in Mathematics

Creativity/Following Preset Procedures	Number of responses	Percentage
Involving creativity	26	59%
Following preset procedures	3	7%
Both	5	11%
Depends (on doers or occasions)	8	18%
No answer	2	5%

As in Table 4, over half of the participants (26 of the 44, 59%) thought problem solving in mathematics was much like a creative activity, but their reasons varied. Twelve claimed that solving problems involves personal creativity because there are always various ways to do mathematics:

> I think that many problems may not be solved in only one way. Sometimes the point is not the answer but the thinking process. If thinking processes go awry, it will be worthless, even if correct answers are obtained. (S08, pre-instruction questionnaire)

> Individuals' ways of thinking differ, and methods or principles used are also distinct. I consider it unnecessary to follow rigid procedures while answering as long as problems can be worked out. Many mathematics problems could be solved not only in one way; processes and speed are not the same, but answers can be obtained in the long run. (S10, pre-instruction questionnaire)

> Some people are able to work out effortlessly problems which could be solved in several steps. Formulas are created by people too. Thus [solving problems] is more related to personal creativity. We probably can create our own formulas. (RS02, pre-instruction interview)

Participants' concept of this issue (associating creativity in mathematics with multiple approaches to solving problems) was largely congruent with the aforementioned finding that mathematicians are likely to attack problems from diverse approaches. Some interpreted mathematical creativity from other aspects:

> I consider solving problems as a thinking process involving personal creativity, not for deriving answers. *Mathematics is the mother of science. Mathematics would no longer be mathematics if obtaining correct answers were the main purpose* [italics added]. In that human progress relies on innovation in science, it would be in vain if [we] only seek correct answers without creative thinking. (S09, pre-instruction questionnaire)

S09 exhibited an understanding of mathematics as a creative discipline. His thoughtful statement about mathematical thinking was also manifested by his claim regarding the definition of mathematical thinking:

> Mathematical thinking can be applied to many things, such as (1) counting money...(2) architecture particularly requires it...(3) writing computer programs is also mathematical thinking, thinking of logic, thinking of the application of calculus. (S09, pre-instruction questionnaire)

As compared to other participants' interpretation of mathematical thinking, restricted to the field of doing mathematics, S09 demonstrated a wider vision pertinent to the construct.

Five (11%) respondents held a neutral position: both creativity and preset procedures are required for doing mathematics. Some arguments presented a two-level concept of following a preset procedure at the bottom, creativity at the top:

> For superficial learning, obtaining answers is paramount, so following known predetermined procedures is necessary. Going deeper, [you] would feel the process of creative thinking is more likely to achieve fulfillment than merely pursuing answers. (S06, pre-instruction questionnaire)

> I consider it okay to follow known predetermined procedures at first, yet after getting more comprehension, personal creativity is needed. We can learn how to make a cake at the outset, for instance, but how to make your product attractive hinges on your own ideas (S28, pre-instruction questionnaire)

In addition, eight (18%) respondents perceived the issue as doer- or occasion-dependent:

> If you are more stupid, you will think step by step. Smart guys are more likely to use formulas and then find a common pattern. That will come sooner if one solves it in this way. (RS05, pre-instruction interview)

> The former [creativity] is for people good at thinking; they are able to locate ways for realizing problems more easily...the latter [following known preset procedure] is for people good at calculating; they are able to derive correct answers without further breakthrough. (S15, pre-instruction questionnaire)

This individual-dependent outlook mentioned suggested these students typically were limited by self-recognition of capability. S06, who previously claimed following a preset procedure as essential for superficial learning, further confessed in the questionnaire that he is a superficial learner and hence merely able to go after fixed steps. Similar to the doer-dependent view, an occasion-dependent perspective was evidenced in S29's and S31's statements:

> On exams, [following preset procedure] can lead to correct answers and gain high score; on ordinary occasions, personal creative [mathematical] thinking is not limited to predetermined processes. (S29, pre-instruction questionnaire)

> When time is limited, following predetermined procedure should be the fastest way to derive answers. Conversely, unexpected outcomes could occur through personal thinking. (S31, pre-instruction questionnaire)

These statements represented a split view in which mathematics was divided into two distinct subjects rather than a whole one, a view that could be shaped by the exam-driven educational climate in Taiwan. As RS07 claimed, students were normally trained to stick to known prearranged routes to obtain accurate answers, though she seemingly felt that mathematics should contain a lively component.

On the basis of written responses, though few participants thought mathematics ought to proceed in a lockstep fashion, the case may not be as clear as it appeared. When asked to elaborate further, RS01 held a rigid position:

> Why does 1 + 1 = 2? Can't it be 3? This is defined by ancestors...Our current thought could see it as 0 if 1 + 1 was previously defined as 0. I think [doing mathematics] is following known procedure. (RS01, pre-instruction interview)

It seems that RS01 associated doing mathematics with arithmetic. RS06 took a neutral stance in the questionnaire but confessed in the follow-up interview she tended to regard solving problems more as predetermined activities. An image of mathematics as inflexible seemingly looms large in many participants' minds.

Participants in the present study have formally studied mathematics for at least 12 years. It is therefore presumed that they have impressions regarding the discipline. The findings revealed the participating students, in general, demonstrated a wide range of diversity regarding doing and thinking about mathematics. The fourth questionnaire item was designed to investigate participants' conceptual frameworks about holistic mathematical knowledge.

As shown in Table 5, eight participants (18%) were silent on this concern. RS08 made no comment to the issue, but explained further in the interview:

> Feeling about mathematics is just like that ...it has little to do with life...the abstraction of math is quite abstract. (RS08, pre-instruction interview)

RS08 still had difficulty explicitly illustrating his mental image regarding mathematics during the interview, suggesting a certain portion of participants lacked an understanding of the subject, even a superficial one.

Table 5

Major Pre-Instruction Responses Regarding The Essence of Mathematics

What Mathematics Is	Number of responses	Percentage
A subject of studying numbers	9	20%
No answer	8	18%
A tool of daily life	7	16%
A subject possessing infallible knowledge	5	11%
A subject for studying science and nature	5	11%

Nine participants (20%), the largest portion, associated mathematics with numbers and symbols in which the main subjects in mathematics are numbers and mathematics is the subject dealing with the relationships of numbers:

> [Mathematics is] the philosophy of numbers. (S01, pre-instruction questionnaire)

> Mathematics is the game constituted by a pile of numbers and symbols. (S24, pre-instruction questionnaire)

> Mathematics is the study of numbers and their changes, looking for something wonderful, like Pascal's triangle. It's so cool. (RS06, pre-instruction interview)

Connection between numbers and mathematics made by these respondents could be attributed to their past learning in which they were mostly trained to do calculation.

Another account given by participants was that mathematical outcomes are absolute and unchangeable. Five respondents (11%) professed that results in mathematics must be the same, regardless of time and procedure:

> Mathematics always has a fixed outcome in spite of evolving over complicated and lengthy time. (S17, pre-instruction questionnaire)

> [Mathematics is] the most natural study and is an absolute certainty. (S32, pre-instruction questionnaire)

> Mathematics is a result-oriented study. It is correct as long as answers can be acquired. (RS05, pre-instruction questionnaire)

RS05's description is particularly noteworthy because he expressed an outcome-determinate point of view. Asked to make a interpretation, he claimed:

> Teachers reminded us that, unless asked to use a particular method, you would attain scores if you can write down the answers...From childhood, you always get scores as long as answers are correct. (RS05, pre-instruction interview)

RS05's version fully illustrated a significant impact of school mathematics on individual concepts about this discipline. Students were generally taught to believe there is always a fixed, unchangeable answer, and what students need to learn is identifying a route leading to the preset outcome.

Seven participants (16%) interpreted mathematics in a pragmatic way—i.e., that it should function as a practical tool in daily life—as manifested by S37's claim:

> It is enough [for me] as long as I can do addition and subtraction and make no mistake while shopping. Do we really need calculus for shopping? (S37, pre- instruction questionnaire)

Another participant complained:

> Mathematics is felt like a virtual stuff...[I] don't know how to apply it in daily life, even after long-term learning. (S13, pre-instruction questionnaire)

Still others distinguished mathematics from arithmetic:

> Mathematics is [only] for doing research. Day-to-day needs do not go beyond operations like addition, subtraction, multiplication and division. (S04, pre- instruction questionnaire)

Several interviewees, though not addressing this opinion on the questionnaire, also endorsed the aforementioned pragmatic view of mathematics, showing this sort of vision could be prevalent.

Contrary to the concepts mentioned above, some respondents saw mathematics from an alternative window. Five (11%) professed mathematics as fundamental to science and inextricably related to the study of reality:

> Mathematics is the mother of science. For embodying all science, being or have been studied, scientists should fit scientific data into mathematics. (S07, pre-instruction questionnaire)

> All phenomena in daily life, such as nature, physics, astronomy and geography, really involve mathematical components. (S35, pre-instruction questionnaire)

Moreover, participant S09, showing thoughtful views on several previous items, portrayed mathematics as the heart of human beings and even drew a picture putting it in the middle, central to various scientific disciplines.

For eliciting more information, participants were further asked to compare essential differences between mathematics and other subjects like science and art, proposed to evince respondents' intrinsic thought by comparing and contrasting mathematics with these two human endeavors. Twelve participants (37%) did not explicitly delineate mathematics from other disciplines, including five above-cited respondents unable to illustrate the essence of mathematics. Furthermore, participants' responses were divergent; the most distinguishing feature of mathematics voiced by the participants was its abstraction. Seven indicated, in contrast with science and art, the objects studied by mathematics are invisible and thus more abstract:

> Mathematics is a subject exploring principles, distinct from others in that its theories are quite abstract. (RS03, pre-instruction questionnaire)

> Mathematics is a more metaphysical discipline. With some exceptions (such as geometry, numbers) in which observable objects are available, others are not; art and science have concrete objects. Most require mental calculation, more likely belonging to thought aspect. (S25, pre-instruction questionnaire)

RS03 further elaborated his vision during interview:

> Some stuff is visible. You may refer to data. As for mathematics, assume you are given a principle like formulas of Fibonacci sequence. You have to figure out what makes this so. Someone is able to appreciate art; it [art] is supposed to be more concrete. If you are given a mathematical result without background introduction, you would wonder why. Memorizing is necessary if you don't get it. (RS03, pre-instruction interview)

RS03 was also asked to compare abstractions in mathematics and in paintings:

> It takes a genius to appreciate abstract painting…Abstract painting is understandable for those who know principles. (RS03, pre-instruction interview)

Since RS03 previously professed that mathematics is a subject exploring principles, he was questioned again to clarify his position. He defended:

> Scope of art is quite broad. For instance, a well-made cup is an art too. Scope of art is rather extensive, which cannot be limited to painting. Scope of [the abstraction of] mathematics is smaller. (RS03, pre-instruction interview)

Consequently, RS03's perspective about the abstraction in mathematics remained unclear. A reasonable interpretation could be that, in RS03's mind, abstraction in art is visible, in mathematics invisible; artistic abstractions can be appreciated through eyes, yet mathematical ones cannot. Actually the scope of mathematical abstraction is also wide-ranging. Another reasonable explanation might be that he had more experiences with art than he did with mathematics in this respect.

Though most students held that mathematics is more akin to science in nature, some pointed out distinctions:

> Science has to rely on observation; results alone are not sufficient. During the processes, you normally can find a certain phenomenon, a kind of regular law, benefiting future science. [As compared to mathematics] science requires more observation. (RS05, pre-instruction interview)

> Mathematics always has a fixed outcome in spite of evolving over complicated and lengthy time...in my view, the major difference between math and science is that science may not always attain identical outcome over a lengthy process (S17, pre-instruction questionnaire)

The two statements seemingly hint mathematics goes after preset routes in which surveying, testing, and guessing play no role in the making of mathematics.

Progress and validity of mathematical knowledge is closely related to ways of mathematical thinking. Participants were asked to address how mathematical knowledge developed and whether there was a rule for the development of mathematics. As seen in Table 6, 13 of the 44 respondents (30%) considered that growth of mathematics is subject to human demand, as evidenced in the following quotation:

> Mathematics was developed when people pursued a more convenient life. The more human demanded, the more mathematics progressed. (S03, pre-instruction questionnaire)

> "Demand is the mother of invention." (quotation original) People need math to make life more convenient, therefore math was invented. (S15, pre-instruction questionnaire)

Table 6

Major Pre-Instruction Responses Regarding Mathematical Development

How Mathematical Knowledge Develops	Number of responses	Percentage
Related to human demand	13	30%
Developed in proper order	7	16%
Developed by following certain rules	13	30%
No rule	8	18%
No answer	3	7%

RS01 also viewed the origins of mathematics to be caused by resolving daily problems. During an interview, he interpreted it in the instance of nails and boats.

> Taking nails as an example, in the beginning nails were made of bamboo, weren't they? Then people had to seek a better substitute. Boats were made of wood at the outset. How can we make it more durable? A lot of problems were coming out. Then boats were made of iron, iron is heavy. How can we make it float on the sea? People began to think of these issues. (RS01, pre-instruction interview)

These statements fully reflected the utility of mathematics that had occupied much weight in their understanding of the development of mathematical knowledge. The chief vision held by them was that mathematics mainly flourished along with human progress. Given that abstract mathematics plays an increasingly significant role in the field of modern mathematics, it is important to know how participants look at the rise of abstract mathematics. When exploring the issue, several interviewees were asked to think about whether mathematics exists parallel to, or has nothing to do with, human demand. Respondents in most cases showed a lack of understanding of this concern.

> I seemingly have heard about that, but was not quite clear. It could be. (RS01, pre-instruction interview)

> You have to ask them [mathematicians] about why they created mathematics little to do with practical demand. It could be that each mathematician has his or her own process. Newton could be thought of in this way because of being hit by an apple, triggering fresh ideas. It could be starting something, then coming up with an idea. (RS06, pre-instruction interview)

> Certain developments of mathematics may not fit human demand. This is probably because some mathematicians are more curious or bored. (RS08, pre-instruction interview)

The responses reveal that these respondents lacked an appreciation of pure mathematical thinking. In their minds, mathematical knowledge must serve practical purposes only. Seemingly, abstract mathematics is a product of boredom or unexpected events.

When asked if there is any rule for the development of mathematical knowledge, eight (18%) responded that there should be no rule for it. Their reasons were varied.

> No rule. The more humans demanded, the more mathematics progressed. (S03, pre-instruction questionnaire)

> There should be no rule because it was thought out by the people. (S24, pre-instruction questionnaire)

> Mathematical knowledge may be developed, extended or innovated, along with changing times and diverse human thought. There is no rule for its development. It could be extending to a more general scope, or tracing back to the most primitive origins, such as trigonometric function. (S11, pre-instruction questionnaire)

S11 demonstrated a relatively thoughtful comprehension about this concern, in which mathematics seeks fundamental conceptions as well as generalization. Nevertheless, he was not one of the selected interviewees, hence there was no way to make further investigation.

On the other hand, there were thirteen participants (30%) claiming that the development of mathematical knowledge should follow certain rules. Yet, many students seemingly misunderstood the question. Some considered the rules for development as addition, subtraction, multiplication, division, formulas, and theorems; they tended to interpret mathematical knowledge as operations. Most did not give precise explanations to defend their positions; nonetheless, two respondents associated the development of mathematical knowledge with nature.

> I guess it should follow the universal rule. Think about the application of mathematics; it is around us and applied to the universe. Why do mathematicians have opportunity for developing? It is because nature is creating their curiosity. (S09, pre-instruction questionnaire)

> Development of mathematical knowledge is mainly coming from the survey of nature. A system is developed when concepts of number are built up. During the process of development, I feel it always follows the rules of nature. (S25, pre-instruction questionnaire)

The two statements bridging a link between mathematics and nature convey a widespread view that mathematics is the key to revealing secrets to the revolution of the universe. The nine randomly sampled students were therefore asked to address this issue: Is the revolution of the universe always following mathematics, or is mathematics merely a tool invented for describing the universe? All respondents supported the latter; whereas only some optimistically held that mathematical models could be applied to all natural phenomena.

Another chief vision expressed by respondents (7 or 16%) was that mathematical knowledge always progressed in order, where new concepts were built on old ones.

> Development of mathematics should follow a fixed procedure. New conceptions must be based upon old ideas. (RS02, pre-instruction interview)

> Math is a defined item. You must define basic numbers first then operate on them by four fundamental operations of arithmetic. It slowly evolved to geometry in this way, deriving by definitions. I guess it proceeds gradually because you must have a foundation first, then slowly develop to higher level. (RS05, pre-instruction interview)

It seems that in their minds, mathematics proceeds according to logical steps and mathematical knowledge is firmly erected on an established foundation. Nevertheless, a seemingly contradicted finding was that five of the nine randomly selected interviewees professed that mathematical knowledge could be fallible:

> [New mathematical knowledge could overthrow the old one.] It could be that previous researchers did not think of that much, or did not absorb that much. It could be that new methods were thought out through incorporating scientific things. (RS06, pre-instruction interview)

Even still, it should be noted that three of the five respondents were unable to support their position. Even one respondent, when questioned to elaborate further, soon changed his mind by saying:

> Overthrow is less likely possible. Old theorems must have their own sense, their own examples. You cannot overthrow it unless the original theorem is incorrect. (RS01, pre-instruction interview)

The fact suggested that the participants in the present study normally held that mathematical knowledge is reliable; it is created by following a logically proper sequence and hence built upon a stable base. Though some of them claimed mathematical knowledge is unsound, as a rule they were unable to confidently defend themselves.

Main Features of Pre-Instruction Views

The purpose of the pre-instruction questionnaire was to create a profile of the participants' pre-instruction views about mathematical thinking, in particular, mathematics in general. Students' written responses were further validated and elaborated upon with follow-up individual interviews, pursuing interviewees' lines of thought and interpreting their thinking appropriately. Contrary to an analysis with respect to each questionnaire item, the aim of this section is to provide an overall comparison of all participants' responses to reveal the interrelationship among all concerns and sketch their major characteristics of pre-instruction views.

By summarizing their accounts, the participants were more likely to:

1. Have a narrow understanding about mathematics and thinking.
2. Lack recognition of the importance of individual persistence in doing mathematics.
3. Associate creativity with multiple approaches.
4. View mathematics as an abstract subject.
5. Believe mathematical knowledge is logically developed.

Though the participants had learned and thought about mathematics over years, the findings indicated that they demonstrated a superficial knowledge about the essence of mathematical thinking in particular, mathematics in general. In response to the issue of mathematical thinking, most failed to achieve a comprehensive insight. They tended to perceive thinking on mathematics merely as a process for reaching answers. Despite their claim that mathematical thinking is the process of logical thinking or reasoning, on the basis of the interview transcripts, they tended to see logic as a fixed route that mathematical operations should follow and some even confessed they never experienced any logical thinking during the process of learning mathematics. The participants also showed incomplete comprehension about how mathematicians think. A majority of students considered that mathematicians' flexibility is due to solid knowledge background and more problem-solving experiences. Only two exhibited insight into the mathematicians' work in which the situations such as getting stuck, guessing, testing, and discussing are typical. Note that eight respondents made no comment on the essence of mathematics, seemingly the most difficult item of all. For those responding to this concern, they were more likely to see mathematics as a study of numbers by employing mathematical operations. Moreover, participants counted mathematical outcomes as absolute and unchangeable, which could be shaped by conventional school mathematics training. Plus, some participants held a pragmatist view toward the discipline, emphasizing the utility of mathematics in daily life. With few exceptions, participants on the whole failed to illustrate mathematics appropriately.

Nearly one-third (14) of participants confessed they would directly seek outer assistance or skip them when facing challenging mathematics problems. Such a high ratio suggests participants lacked individual persistence while doing mathematics. The phenomenon was also evident in the participants' views that mathematicians are always able to solve problems in the easiest and quickest manners and their successful is due to solid knowledge base or more practice.

Mathematicians' endurance was rarely addressed. Only four claimed that mathematicians are more likely to think hard on tasks. Participants' shortage of recognizing the significance of persistence may be attributed to their past learning. They mostly were trained to solve routine problems, which normally can be worked out in few steps. In this manner, while facing predicaments, they tended to recall formulas, similar problems and what teachers said rather than believe in their own capability.

An overwhelming majority of the participants viewed doing mathematics as an activity involving creativity. This claim though in a way can be seen as a proper understanding, a slightly different vision emerged while investigating their responses in more detail. Most respondents interpreted creativity in doing mathematics as multiple ways for deriving answers. This view was also in line with participants' responses that mathematicians are flexible problem solvers, capable of demonstrating diverse approaches while attacking a problem. Interpretation in this way, however, only reflected a one-dimensional view on this concern. It should be noted that, in addition to generating multiple solutions, mathematical creativity also applies to looking for valuable open problems, excavating hidden patterns, making and testing plausible conjectures, specializing, generalizing and so on. Thus, unlike the participants' static and single dimension vision, creativity in mathematics is dynamic and multi-dimensional (Tall, 1991).

Participants did not address the abstraction of mathematics until they were asked to compare it with other disciplines, such as art and science. In their minds, abstraction is one special characteristic of mathematics different from other subjects; science and art possess observable objects upon which one can operate but mathematics does not. The operational objects of mathematics are symbols, yet seemingly not concrete enough to them. Moreover, one participant claimed that the abstraction of art is visible and can be appreciated through eyes, not the case for mathematics. On the other hand, however, it appears that the participants lacked an understanding of the significance of abstract thinking in mathematics. They neither voiced the importance of abstract thinking ability nor demonstrated appropriate recognition of the significance of abstract mathematics. The latter could be influenced by their pragmatist view, stressing the utility of mathematics in daily life. As a whole, the participants showed an incomplete comprehension of the abstract components of the discipline.

A large portion of participants held that mathematical thinking is a way of logical reasoning, based upon definition, proceeding deductively, toward a reliable product. Thus, nearly one-third of the participants believed that the development of mathematical knowledge should follow certain rules. Among them, many indicated mathematical knowledge grows logically and progresses in proper sequence; new concepts are built upon old ones. A connection was made between thinking and development of mathematics in their minds. Consequently, some regarded mathematical knowledge as flawless, absolute truth, and unlike scientific investigation. This perspective was also evident in their thoughts on mathematicians' mathematical intuition. Many respondents claimed mathematicians tend to think about problems rigorously and methodically.

Connection Between Earlier Learning and Current Views

In the present study, participants were asked to complete their mathematics biographies, describing their learning experiences in the past. Though it is generally held that an individual's views of doing mathematics is greatly shaped by their experiences in earlier periods, empirical evidence concerning the case of Taiwanese college students is sparse. This section therefore proposes to search for potential links between the two respects by contrasting their professed views on mathematics as expressed in their biographies and questionnaire. Since only 34 students returned their mathematics biographies, the following analysis is merely based upon available data sources.

An investigation was made to check the degree of consistency between the two data sources. For avoiding possible biases, each data source was independently read and analyzed to highlight main traits. If professed responses on both data sources mostly exhibited parallel views regarding issues of interest, it was marked as moderate consistency (e.g., the respondent reported initiative learning behavior in the mathematics biography and meanwhile demonstrated thoughtful understanding of mathematical thinking on the questionnaire). Otherwise, it was marked inconsistent. It was found that 17 of the 34 (50%) participants demonstrated a moderate consistency between their claims to the two data sources and seven (21%) showed inconsistency. Nevertheless, 10 (29%) were hard to categorize and were unidentified as a result.

Of the 17 respondents considered as consistent, seven were more likely to hold an active perspective of learning mathematics as well as express a vivid view about mathematical thinking. For instance, S09's response on the mathematics biography showed a thoughtful knowledge of the role of mathematics:

> While attending to elementary school, my thought changed. Having a good math may not rely upon strong ability of mental calculation, but on thinking of the concept. Math not only is closely related to our daily life, but occupies much weight in our professional learning. Math is not for taking an exam in order to obtain high score, but for application. If it is only for obtaining scores, it is worthless to learn math. (S09, math biography)

Conversely, he expressed an active view on questionnaire about the definition of mathematical thinking and his strategies reacting to challenging tasks:

> Mathematical thinking can be applied to many things, such as; (1) counting money, (2) architecture particularly requires it, (3) writing computer programs is also mathematical thinking. Thinking of logic, thinking of the application of calculus. (S09, pre-instruction questionnaire)

> My instant reactions are: (1) thinking on the content first, (2) pondering why I feel it difficult, (3) searching for key concepts, (4) discussing with classmates, (5) inquiring to teachers. (S09, pre-instruction questionnaire)

Another participant S25, confessing in the math biography that mathematics attracts him by its processes of discovering and solving problems, displayed a comprehensive understanding about mathematicians' work.

> Mathematicians usually carry out a survey with respect to problems, boldly make a conjecture, test it repeatedly by using known and unknown, then constantly verify result when all done. (S25, pre-instruction questionnaire)

S25's illustration in some way featured the empirical aspects of doing mathematics. On the other hand, 10 respondents exhibited a relatively conventional vision either on the mathematics biography or pre-instruction questionnaire. For instance, RS05, confessing a passive habit of learning mathematics in the mathematics biography, claimed on the questionnaire:

> Mathematics is a result-oriented study. It is correct as long as answers can be acquired. (RS05, pre-instruction questionnaire)

> Mathematics is a way of leading to answers...as long as the direction [of doing mathematics] is correct, the path leading to answer would be rather short. (RS05, pre-instruction questionnaire)

RS05's main focus seemingly was always in the pursuit of answers.

In contrast, seven participants' responses to the mathematics biography were found to be inconsistent with written responses on the questionnaire. S35 confessed that his mathematical capability is just average and complained his interest was depressed by a semi-forced means of school mathematics. He seemingly experienced a discontented period of learning mathematics. Whereas, when asked whether solving problems is an activity involving creativity or a predetermined procedure, he was in favor of the former saying:

> Learning mathematics should be subject to means. The best way is thinking and applying it with flexibility and ingenuity. I personally hold that following predetermined procedure is a conservative behavior. (S35, pre-instruction questionnaire)

Moreover, while responding to what mathematics is, he claimed that all natural phenomena are implicated in mathematics, a vibrant perspective toward the discipline. Conversely, RS02 expressed a thoughtful vision about mathematics in the mathematics biography but was found to be limited in the questionnaire. For instance, he professed in the mathematics biography that,

> I agree what people say, "mathematics is for fostering one's concept of logical thinking." (quotation original) This implies learning mathematics is really to make one think on problems, rather than memorizing alone. I read this sentence in a book, "performing out known stuff is knowledge; applying it further to unknown stuff is wisdom." (quotation original) I guess this is what mathematics is about. (RS02, math biography)

RS02 appeared having an insight into the essence of mathematics. However, he likely associated mathematics with a study of numbers and mathematical thinking with dealing in relationships among numbers, a relatively conservative conception. Furthermore, he acknowledged in the interview that he never experienced the merit of improving logical thinking during the period of learning mathematics. Consequently, RS02's views of mathematics and thinking appeared to be incongruent and disconnected.

As a result of short and brief statements or contradicted descriptions, 10 participants' responses to the mathematics biography and the questionnaire were unidentifiable as to congruence. With one exception, none of them were in the random sample, therefore not interviewed. Thus, further investigation was not possible to validate their written responses. Still many respondents were unable to demonstrate their intrinsic thought clearly and concurrently through writing. As in one case, RS03 expressed inconsistent written responses on two data sources (lively point of view in the mathematics biography, but lacked detail on the questionnaire), yet displayed a highly dynamic and active vision of mathematical thinking during the interview. To avoid a careless misinterpretation, such vague responses were left unanalyzed.

Post-Instruction Views

This section reports findings germane to the participants' post-instruction views on mathematical thinking, after experiencing the 18-week problem-based course. First of all, the classroom setting is depicted to sketch a holistic picture of the experiences in the calculus class, setting the stage for a better understanding of the post-instructional results. The second section focuses on post-instruction views of concerns. Analysis is on an aspect-by-aspect basis, wherein the respondents' account of each post-questionnaire item will be parsed independently, followed by a contrast of their pre- and post-instruction statements on the same subject. Lastly, an overall investigation summarizes any noteworthy change in the participants' views following the course.

The Classroom Context

The chief goal of the problem-based learning in this class was to problematize the topics in two different ways. First, to arouse curiosity and elicit interest, students were frequently questioned to consider the importance of a topic and key concepts. Several handouts regarding the historical background and origins of topics served as auxiliary material for increasing students' understanding. Further, they were urged to propose credible approaches to solve the historical problems, followed by public investigation of these ideas. All students were encouraged to challenge and revise ideas proposed by someone else. During the initial lessons, perhaps because of the unfamiliarity and traditional conservative atmosphere in Taiwanese classrooms, students were less likely to take the initiative in responding to questions. The researcher/instructor therefore called on specific students or in random to increase their involvement. The situation gradually improved in the following weeks until the end of semester. When called on to answer questions, students were more at ease; some even tended to actively respond to the researcher/instructor's questions. Meanwhile, to avoid the case of active students occupying too much class time, the researcher/instructor also created opportunities for the silent students to participate in public discussion via calling on them. Regardless of correctness, all students were asked to defend their answers. When respondents were not able to give any answer, they would be further encouraged to make plausible guessing in order to stimulate critical thinking.

While teaching, the instructor/researcher acted more like a leader than a director at this point, establishing a problematic setting to promote students' curiosity and involvement. Moderate wait time (at least 10 seconds) was used to slow down the teaching pace in an effort to allow students more thinking time. Rather than presenting material in a clear and organized fashion, the course was also arranged to demonstrate the give-and-take struggle of problem solving in which the researcher/instructor behaved like a novice working problems from scratch, suggesting various plausible approaches, and asking students to evaluate the possibility and validity of each approach. For instance, contrary to direct instruction of the concept of a tangent line, students were first asked to propose their definition of the tangent line to a curve. They usually considered a tangent line to a curve as a straight line "touching" the curve at a specific point. According to their descriptions, the researcher/instructor drew a figure and then asked the students to decide whether the figure on board fit their criteria. Thereafter, the researcher/instructor drew another curve with a sharp point on board and students were asked to decide the tangent line at that sharp point. At that moment, they would have difficulty deciding which straight line was desired because, as such, there are an infinite number of straight lines satisfying their definition. They then realized that a more complete definition was required and defining the tangent line as the limit of secant lines is necessary and reasonable. All in all, as opposed to serving as a content transferor, the instructor/researcher intended to function as a multiple-role lecturer leading students to explore, investigate, and even interpret the essence of mathematical thinking on their own.

Additionally, 12 historical problems gleaned from a variety of sources, differing in routine text exercises in nature, were assigned to students as weekly homework. All problems were related to and preceded the particular topic with the intent of eliciting the students' thinking and laying the groundwork for the coming topics. Students were asked to solve the problems or seek relevant information on their own. Discussion with classmates was allowed, but copying of others' work without effort was strictly prohibited. Moreover, they were also asked to submit all used approaches, regardless of appropriateness. Grading was based on their elaboration as well as the correctness of answers. After collection of their answers, several students were invited to present their approaches and the class was requested to evaluate the plausibility and validity of the approaches. For instance, for the problem of deriving the area of a circle, students proposed various methods such as cutting a circle into triangles, rectangles, and even more complex geometrical shapes. The class then assessed the adequacy of each method and decided which approaches might fully achieve the goal. In this manner, the class became a small mathematical community in which all members communicated ideas with each other and attained a commonly accepted criterion as a result.

Because of their difficulty, two problems, Napier's logarithm and Perrault's tractrix problem, were worked both individually and in groups. Prior to entering group discussion, students were asked to hand in individual solutions to the two assignments, either complete or incomplete. They then participated in group discussion to share ideas with group members and have their thinking scrutinized in public. The grouping was random and there were five or six persons in each group. Following group work activities, all students handed in a final version of the solution representing their overall effort. Aside from working together outside class, participants were given 40 minutes in class to communicate with one another and to better prepare their final work. Students were expected to work together with group members outside class, but it appeared most were not doing so. As such, the two in-class discussions were, in most cases, the only chance for interaction among members. During in-class group discussion, students were requested to report their current progress toward the problem to members and work together to select one method which was most likely to resolve the problem. The researcher/instructor was then walking around the class to monitor the advancement of each group and assure the group discussion was moving forward properly. Students got no hints from the researcher/instructor. Rather, they had to explain the strategies that they adopted and why they decided to do so. The researcher/instructor did not explicitly pinpoint students' mistakes unless they were not on the right track at all. After group discussion, students were asked to keep working on the problem and submit their final work at the next class meeting on the basis of previous personal thinking and the conclusion of group discussion. Similarly, when answers were collected, students exhibiting elaborative thinking (whether correct or not) were invited to share their approaches at the board with classmates and further questioned by others or the researcher/instructor. According to students' instant responses in class, assignments were so tough that many were beyond their capacity; they also confessed that inviting students' insight provided alternative windows for thinking.

Though the entire course in most cases progressed on the basis of original design, several unforeseen occasions and events affected execution of the designed curriculum. During the initial class meetings, students' reactions to the teaching style, materials, and assigned problems were not as expected. They seemingly struggled to capture the main points of emphasis and were reluctant to respond to the instructor/researcher's questioning. To avoid rushing and save time for students to acclimate, the curriculum was presented more slowly. Plus, due to the influence of two typhoons, four class meetings were cancelled. As a result, the course schedule fell about a week behind the original plan. By taking the continuity and completeness of curriculum into account, some topics (introduction to differential equation, Mean Value Theorem for integrals, and the trapezoidal and Simpson's rule) were removed.

On the other hand, students were supposed to hand in homework solutions weekly, except for two problems worked through cooperative learning. Nevertheless, because most students could not finish some assigned problems on time, due dates were extended at times and, consequently, some planned tasks (Leibniz's problem of refraction, L'Hopital's pulley problem, and the shroud of Turin) had to be deleted. In addition, given students' curiosity about and interest in historical misconceptions regarding infinite series, the instructor/researcher accordingly decided to replace one problem (Newton's chain rule) by Euler's mistakes on infinite series to show evidence of potential fallacy in mathematical thinking.

Post-Instruction Views on Mathematical Thinking

As reported, at the outset of the course, the participants were likely to (a) have a cursory understanding about mathematics and thinking, (b) lack recognition of the importance of individual persistence in doing mathematics, (c) associate creativity with multiple approaches, (d) view mathematics as an abstract subject, and (e) believe mathematical knowledge as logically developed. This section analyzes their post-instruction views about these issues.

Written responses on the post-instruction questionnaires were initially used to categorize participants' views on each item. Special emphasis was placed on overall differences as well as individual distinctions during the course. Both respondents' written and oral statements served as evidentiary data for further interpretation and demonstrating changes in their views of mathematical thinking and mathematics.

Participants' post-instruction responses on mathematical thinking demonstrated wide variety. Several key distinctions between participants' pre- and post-instruction views of the definition of mathematical thinking are shown in Table 7. Compared to previous responses, fewer subjects referred to mathematical thinking as merely a process of solving problems or recalling formulas; none saw mathematical thinking as involving calculations and/or operations alone; all were able to propose definitions of mathematical thinking, whereas four had been silent on this concern at the start of the course. On the contrary, some shifted to emphasize the significance of understanding problems and rationale of theorems; some instead perceived approaching problems in one's own ways or from diverse facets as critical.

Table 7

Comparison of Major Responses Regarding Mathematical Thinking

Mathematical Thinking	Pre-number of responses	Post-number of responses
A process of solving problems or deriving answers	9	3
Logical thinking and reasoning	12	6
Calculation and operation	4	0
Recalling formulas	4	1
No answer	4	0
Understanding the rationale	0	4
Thinking from multiple facets or angle, integrating diverse thinking	0	4
Using one's own way	0	4
Understanding the problem	1	4

As noted, fewer students considered mathematical thinking as a route leading to the solution of problems. For instance, participant S15 reworded:

> I forget what I wrote last time. Seemingly no answer?! [but] I got some ideas right now. I think mathematical thinking could mean that attaining reasonable answers through logic of making sense and reasonable generalization. In sum, it is a process of solving problems by means of reasonable ideas and procedure. (S15, post-instruction questionnaire)

S15 repeatedly stressed the role of reasonableness. By reasonableness, on the basis of his footnote, he meant *evidential and meaningful facts*; he was seemingly inclined to value the importance of logical sense in doing mathematics. Still others expressed this view. S05 previously defined mathematical thinking as multiple solutions, yet afterward reworded it as:

> Mathematical thinking is a mode of thinking possessing reasonableness, persuasion, and logic. When solving problems, either processes or answers must have three features such as *reasonable, persuasive and logical* [italics added]. (S5, post-instruction questionnaire)

Participant S22, who earlier claimed mathematical thinking was a process of constantly pursuing the only true answer, changed his focus by saying:

> I don't think there is a fixed pattern for mathematical thinking. Sometimes [we] must jump out from subjective recognition and *measure it from objective angles* [italics added] for attaining correct results. (S22, post-instruction questionnaire)

S22's concern appeared to shift from mathematical product to process; S26 also endorsed this view by declaring that human intuition is unreliable because relationships among numbers may not fit one's logical thinking. The aforementioned findings suggest, in addition to validity of answers, reliability of procedures looms larger in these participants' minds.

Another main difference between pre- and post-instruction views was a tendency to stress flexibility in doing mathematics. Four considered mathematical thinking as attacking problems from manifold facets or integrating diverse thinking, while four others emphasized individual creativity. S06 professed during the pre-instruction stage that mathematical thinking is nothing but recalling formulas; he later reworded it as understanding problems and looking for alternative ways to derive answers. RS03, initially interpreting mathematical thinking simply as thinking of solving problems, gave a more detailed description of the construct:

> Just like problems in assignments, many were never seen or heard before. We were required to prove or guess why this is so. In the process of answering these questions, [I] would employ a variety of ideas to write or guess. These ideas are mathematical thinking. (RS03, post-instruction questionnaire)

When asked to explain further, RS03 claimed:

> I spent much time thinking on those problems, thinking a lot. Various methods are used to solve them...roughly like two persons knowing how to do may use different approaches though their starting point is the same. So a problem can be worked out in multiple ways. (RS03, post-instruction interview)

Respondent S32 was silent on this item at the outset, but highlighted two features of mathematical thinking this time:

> Thinking for solving problems. [Mathematical thinking is] *thinking in one's own way and approaching from multiple facets* [italics added], identifying appropriate routes for achieving correct answers. (S32, post-instruction questionnaire)

This perspective was also endorsed by others, such as S10's claim:

> [Mathematical thinking] is how you apply your own methods to derive answers or prove and locate relevant formulas. For instance, handouts tell us a lot of stories about mathematical discovery. How did they [mathematicians] think of it, attain inspiration, and even prove it? The process involved in their thinking should be mathematical thinking. (S10, post-instruction questionnaire)

Some class handouts, describing mathematicians' ideas of attacking problems, seemed to make S32 re-conceptualize the essence of the construct. S17 had formerly proposed mathematical thinking as solving a problem in the easiest way; he later held that mathematical thinking is solving problems by using *unusual* or *weird* ideas. Note that in class several peculiar approaches employed by Archimedes for deriving area and volume were introduced, and students were seemingly impressed, as will be discussed later.

Despite the noteworthy distinctions cited above, still others demonstrated little or no difference in defining mathematical thinking. RS02 insisted that mathematical thinking is pondering the relationship among numbers; RS05 and S25 firmly held that mathematical thinking is a mode of logical reasoning for achieving answers; on pre- and post-instruction questionnaires, S26 concurrently addressed three traits of mathematical thinking: changing, unchanging, and simplifying.

Following the analysis of the participants' definitions of mathematical thinking, it is important to ascertain how they dealt with difficulties with the problem. Table 8 compares pre- and post-instruction responses regarding instant strategies to make progress toward a solution.

Table 8

Comparison of Major Responses Regarding Strategies To Unfamiliar Situations

Instant Strategies to Unfamiliar Situations	Pre-number of responses	Post-number of responses
Keeping thinking on it for a while	8	11
Looking for relevant materials	9	6
Asking for help	3	1
Understanding the problems	3	6
Going back to basic concepts	5	1
Recalling similar problems or formulas	4	5
Testing alternative approaches to work out	4	4
Skipping it	2	2

A major distinction was that slightly more respondents expressed a willingness to try other strategies (e.g., thinking on their own, understanding the problem) prior to seeking outer assistance (looking for relevant materials or asking for help). For instance, when stymied, participant S01's previous first reaction was to check out relevant materials or ask for help; nonetheless, he instead declared:

> *Keep thinking on it for one or two days* [italics added]. If you cannot figure it out, then go check out relevant material. If unavailable, then go ask someone else. If nobody could do it, leave it for a while and then pick up and think again. (S01, post-instruction questionnaire)

Compared to his pre-response, S01 exhibited more individual persistence in struggling with demanding tasks. S26, who claimed on the pre-questionnaire that he would set up a fixed time limit for solving a problem, also showed more elaborative thinking on this concern:

> First of all, [I would] make a judgment to categorize the problem and then review all relevant basic definitions; or discussing with people adept in this respect, proposing my opinion about this problem and bottleneck that I encounter, and then absorbing others' point of views. (S26, post-instruction questionnaire)

S26's situation was also seen in others' responses. With some exceptions, students in this class were generally allowed to share ideas with classmates or group partners and convey any information they had. Participants, as such, were able to witness how others worked on or approached problems. In this manner, individuals may adjust their strategies or perspectives of doing mathematics in certain ways. Interviewee RS07 used to skip difficult problems outright or simply do some random thinking, yet affirmed on the post-questionnaire that she would no longer skip any problem without effort because, as she further explained during the interview:

> I can see that those guys have a lot of pleasure in doing mathematics. Even though doing it wrong, they still quite enjoyed it. I have a high regard for them…I feel they paid much attention. It could be that they are so interested in it. They are willing to spend time on it. (RS07, post-instruction interview)

When interviewed, RS07 admitted her conception of doing mathematics was affected by some other participants' patience and professed that, while responding to the post- questionnaire, she really wanted to exhibit her progress in this respect.

Students devised multifarious strategies to cope with predicaments, yet neither written nor oral responses evinced any significant upgrading of individual persistence in doing mathematics. Likewise, six of the nine chosen interviewees displayed little change on this item; a potential hidden cause for this shortage merits further investigation.

During instruction, the participants learned several ancient mathematicians' approaches to specific problems; thus it was important to probe again their thoughts on how mathematicians think about problems. A contrast of their responses to this concern yielded one unchanging response: the mathematician is good at attacking a problem from multiple facets and diverse angles, as shown in Table 9.

One typical viewpoint was interviewee RS09's interpretation. She professed on the post-questionnaire that mathematicians would not be simply satisfied by achieving the answer. Upon further inquiry, she explained:

> When mathematicians think, their approaches would not be rather quick and may not be correct. Until encountering a mistake, they began to know what could or could not be used. Laypersons would stop at finding out the answer. But for mathematicians, even if they obtain the answer, they still wonder: What other approaches can achieve this answer? (RS09, post-instruction interview)

Table 9

Comparison of Major Responses Regarding How Mathematicians Think

How Mathematicians Think	Pre-number of responses	Post-number of responses
Thinking from diverse angles/alternative approaches	8	7
Being able to find a quickest way	8	1
Possessing solid knowledge	3	1
Starting from basic concepts	5	1
No answer	6	2
Verifying their ideas	0	4
Doing thought experiments	2	5
Being creative and associated	4	10
Being imaginative	1	5

On the contrary, several distinctions surfaced. As reported earlier, at the outset, several participants tended to think the mathematician's brain was filled with a variety of ready solutions for use, yet they paid little attention to mathematicians' effort. On the basis of responses on post-instruction questionnaires, however, some of them shifted their focus. For instance, two participants, considering mathematicians might have some kind of intuition or presentiment regarding problem solution, professed instead:

> Mathematicians won't be stuck on a fixed method. [They] may think from multiple facets, even try approaches that they have never known. (S13, post- instruction questionnaire)

> Mathematicians' ways of thinking do not follow a preset rule. They may employ their imagination, attack problems from multifarious facets...they are able to approach from alternative angles. (S31, post-instruction questionnaire)

The participants were more likely to stress mathematicians' struggle for identifying the solution. The assertion was also evident in the fact that, with two exceptions (S4 and S16), far fewer participants viewed mathematicians as efficient problem solvers always being able to find the quickest solution or possessing solid mathematical background. For instance, S21 previously held that mathematicians could solve a problem in a rather quick and neat way, whereas he reworded:

> I feel mathematicians' ways of thinking should be like when seeing a mathematical equation, [he] would analyze its association with other formulas and decide whether to do calculation directly or reorganize it first. (S21, post- instruction questionnaire)

Though S21's current statement centered on formulas and calculation, compared to his former point of view, he exhibited a more sophisticated thought in which mathematicians' struggles were taken into account.

On the other hand, none of the participants recognized the role of justification in mathematicians' work during pre-instruction; four identified the concern after instruction. S17, who once saw mathematicians as efficient solvers, shifted his focus:

> Mathematicians may use some proofs to verify the answer and make it be accepted by public. Laypersons, nevertheless, merely employ all known methods to attain the answer and feel it should be right but without further justification. (S17, post-instruction questionnaire)

Participant S15, aside from endorsing a similar view, added that mathematicians tend to be more curious about problems. He recognized that not all problems can be solved, but mathematicians' desire to try and see what they really can do is strong (a perspective congruent with RS09's statement).

A contrast of their visions also revealed slightly more of the participants were likely to realize the mathematician's thinking process. They tended to recognize the importance of doing thought experiments when struggling for answers. For example, S18, referring to mathematicians' capability of doing more problems, turned to hold that a mathematician would break a problem into parts and constantly dwell on it, in which guessing is required. Participant S11, previously referring to mathematicians as more associated and creative, delineated a more comprehensive account on the post-instruction questionnaire:

> When solving, the mathematician definitely identifies clues from problems first, followed by focusing on plausible approaches, and then finally achieves the answer. The mathematician's ways of thought are more associated and careful than laypersons'—being able to locate clues by making oblique references—to avoid mistakes; [namely] *making bold conjectures followed by cautious verification* [italics added]. (S11, post-instruction questionnaire)

The conception of mathematicians as creative problem solvers also drew numerous participants' attention. As shown in the Table 9, 10 participants (versus four at the beginning) considered mathematicians as creative, able to develop individual methods, and able to generate unusual ideas. It appears mathematicians are not only regarded as flexible problem solvers, approaching problems from multiple angles, but also as sophisticated thinkers, creating idiosyncratic thoughts for reaching answers. Furthermore, interviewee RS07 previously considered mathematicians as people more able to think from the basic, then paid attention to the mathematicians' creative thinking and associative ability:

> Mathematicians' enthusiasm is like a mania. A series of common numbers in laypersons' eyes could become numerous kinds of varieties and logical combinations. It's incredible. (RS07, post-instruction questionnaire)

Interviewed further, she explained:

> I forget who [he or she is]. I was told a story about a mathematician near death who could create a bunch of combinations when he saw numbers on license plate. He was quite sick then and on the way to hospital, but saw a four-digit number on a license plate. He could organize it through calculation, permutation, and combination...I feel they are so cool, super cool. You also mentioned that they would do some things when seeing these stuffs and do some other things when seeing those stuffs. (RS07, post-instruction interview)

RS07 seemed impressed by the mathematicians' persistence. Plus, RS03 considered the ability to make connection among mathematical ideas as culture issue:

> Experience should be related to cultural background. His or her job does matter. For example, if he or she is a physicist, he or she would be more likely to employ physical concepts. As for our Chinese mathematicians, they rarely approached [mathematical problems] by using physical concepts. I guess it is related to cultural background and the engaging job. (RS03, post-instruction interview)

RS03's argument could be related to class events in which several Chinese and Greek mathematicians' mathematical thoughts were introduced and compared. Archimedes' peculiar approaches in deriving area and volume drew students' attention and interest. As such, more participants were inclined to see mathematicians as imaginative. For instance, RS02, an interviewee initially proposing that mathematicians tend to derive relationships of numbers through logical reasoning, highlighted their capability of association and imagination instead. In the interview, he cited Newton and Archimedes:

> Just like capability of association, many figures had discovered calculus but not specifically until Newton. I consider imagination more important because of Archimedes. I feel he is so strange. He derived the volume of a sphere by means of lever...How did he think of it? Plus, he transferred a circle into a triangle. I feel his imagination is quite strange. (RS02, post-instruction interview)

RS02 was additionally asked whether Archimedes' idea was accessible through deductive reasoning; he labeled this sort of thinking as imaginative rather than logical and confessed this (Archimedes' mathematical approaches) is the cause for changing his mind.

These above findings suggest that, near the end of the course, students perceived mathematicians as (a) ingenious thinkers, rather than skillful problem solvers and (b) resourceful innovators, as opposed to erudite content experts. It is therefore noteworthy to investigate if this view shift exerted any potential effect on their conceptions of the processes of doing mathematics.

Having witnessed historical eastern and western mathematicians' approaches, the participants tended to see mathematicians as imaginative and creative figures. It is reasonable that, as such, the participants would be more inclined to view doing mathematics as activities linked with individual originality in contrast to pursuing fixed steps.

Table 10

Comparison of Major Responses Regarding Creativity in Mathematics

Involving Creativity/Following Preset Procedures	Pre-number of responses	Post-number of responses
Involving creativity	26	15
Following preset procedures	3	1
Both	5	18
Depends (on individuals or occasions)	8	9
No answer	2	0

Still, as highlighted by Table 10, the results differ. Though most participants no longer considered doing mathematics as merely rigid, fewer believed creativity was the most essential element. Quite a portion of participants shifted to a neutral position (requiring both creativity and following preset procedure).

Two respondents formerly viewing mathematics as fixed activities changed their beliefs. For instance, interviewee RS01 previously believed doing mathematics as following predetermined steps by saying:

> Why does 1 + 1 = 2? Can't it be 3? This is defined by ancestors...Our current thought could see it as 0 if 1 + 1 was previously defined as 0. *I think [doing mathematics] is following known procedure* [italics added]. (RS01, pre-instruction interview)

RS01 apparently paid attention to arithmetic components of mathematics in the beginning, but shifted to focus on another respect, on the post-instruction questionnaire:

> My point of view is both [creativity and following preset procedures] are required. When seeing an unfamiliar problem, I may not necessarily use formulas in the text. [I] could put my own creativity into the thinking process to increase power of persuasion. (RS01, post-instruction questionnaire)

Asked the reason for the change, he explained:

> The problems you gave us! [I] would make a guess to reach the answer if I cannot figure it out. Then start to think again...creativity is involved in the process of guessing. (RS01, post-instruction interview)

It appears RS01 found that the strategies like following preset procedures or fitting formulas did not work when solving assigned tasks and began to value the role of creativity.

As noted, many participants took a neutral stance on the issue of creativity, an important phenomenon calling for further investigation. S19, who initially considered the issue as dependent on individuals without explanation, gave a thorough description on the post-instruction questionnaire:

> Both have advantages and disadvantages. Thinking would become more delicate if on the basis of personal creative thinking. The disadvantage of doing so [solving problems by using creativity] is several obstacles would occur during solving processes; yet creativity is stronger. [On the other hand] Thinking would be weaker if solving the problem by going after predetermined procedures. The advantage of doing so [following predetermined procedures] is the extant problem can be solved neatly. (S19, post-instruction questionnaire)

S19 seemingly indicated that both approaches may lead to the answer; nevertheless, while creative thinking is original yet challenging, following routine steps is rigid but efficient. Another participant S12 also viewed the issue as individual-dependent at the outset, but he subsequently interpreted creativity in another light:

> Both have their own merit. Correct answers may be more safely achieved if a routine approach is adopted. However, the result would not be attained if stuck somewhere in the midst of solving process...the result probably can be obtained if [we] work from backward or approach it in unusual ways. (S12, post-instruction questionnaire)

S12 tended to see creativity as the key for escaping a predicament. Interviewee RS05 previously regarded doing mathematics mostly as activities for pursuing answers. On the post-instruction questionnaire, however, he professed:

> I think both are required. Because every mathematics problem can derive its answer in different angles and methods. But each angle has a set of procedures that ought to be followed. (RS05, post-instruction interview)

RS05 additionally claimed that his point of view was somewhat influenced by other classmates' primitive ideas proposed in class; as such, he became more aware of the role of creativity. Yet, in the interview, he still insisted on the necessity of following preset steps. To him, creativity functions more as an entry for approaching the task; predetermined rules follow. Moreover, some participants associated creativity with the development of mathematical knowledge. For instance, S17 and RS08, both initially holding the view that doing mathematics mostly involves creativity, slightly shifted their position to neutral and made a connection between creativity and knowledge in the making, as evident from S17's claim on the post-instruction questionnaire:

> I think both arguments are reasonable, but personal creativity occupies more weight. Where does mathematics begin if without creativity? Where are current diverse [mathematics] ideas from? Following predetermined procedures is according to the predecessor's methods, a more efficient way. (S17, post- instruction questionnaire)

RS08 also endorsed this view by saying:

> Mathematics would not make any progress without creativity. New stuff will be coming out when [you] think of something that others have never thought. Mathematics is the stuff about thinking. It would not make any progress if merely by fitting fixed formulas in. (RS08, post-instruction questionnaire)

On the post-instruction questionnaire, RS08 saw solving problems by following preset steps as only for daily application. Aforementioned findings indicate that though many participants shifted to holding that both creativity and routine procedures are needed, the ways they interpreted creativity varied.

Furthermore, the number of participants seeing this concern as dependent slightly increased. As seen earlier, some participants in this category shifted to hold a neutral view; others remained unchanged. S34 and RS03, either before or after the instruction, firmly believed fixed steps were required for beginners, who may thereafter develop individual strategies on the basis of experiences and ingenuity. They appeared to consider the two counterparts as essential components of a learning cycle. By contrast, four viewed the issue as doer-dependent. S10 previously professed that doing mathematics should involve creativity, what with the multiple approaches for solving a problem, yet shifted his focus to the solver:

> I think it depends on the individual. Some people are born to be good at logical thinking. His [or her] process of thinking and solving could differ from others'. In this manner, it might be that certain special ideas emerge and subsequently a simpler and easier understanding method is created to solve mathematical problems. Everybody's thinking processes would be not much different if all followed known procedures. (S10, post-instruction questionnaire)

Two participants (S7 and S8) split problems into two counterparts: those requiring originality and those that should follow extant rules:

> For certain problems, a little bit of personal creativity could be required (for example, some problems about deriving area), but for some other problems, thinking according to mathematical definitions and rules is demanding. (S08, post-instruction questionnaire)

As compared to the neutral argument mentioned earlier, participants viewing the issue as doer- or occasion-dependent, as a whole, demonstrated a dichotomous perspective in which mathematics learning processes, doers, or problems all can be classified into distinct levels.

Analysis of these participating college students' pre-instruction account of the essence of mathematics indicated that, as reported earlier, they heretofore paid scant attention to this aspect, despite years of doing mathematics. This finding also suggests the participants' vision regarding phases of mathematical thinking had somehow shifted. Probing this theme again was reasonable to look for any potential link between their images of mathematical thinking and mathematics as a whole. Table 11 shows all respondents who addressed this concern. Note the number of participants (nearly one-third) who still considered mathematics as a subject dealing with relationships among numbers.

Table 11

Comparison of Major Responses Regarding The Essence of Mathematics

What Mathematics Is	Pre-number of responses	Post-number of responses
A subject of studying numbers	9	12
A tool of daily life	7	1
A subject possessing infallible knowledge	5	0
No answer	8	0
A subject for studying science and nature	5	10
A discipline involving logic, thinking, and reasoning	1	8
A subject involving operation (of equations)	1	5

For instance, RS01, originally claiming on the pre-instruction questionnaire that mathematics is a thinking process of fitting in appropriate formulas, professed that "mathematics is for resolving human's recognition of numbers via appointing the length and area of objects." Upon further inquiry, he explained:

> People are always interested in unknown stuff. When having an object, how do you represent it in number? What is its quantity? People will make a rule to work it out. (RS01, post-instruction interview)

RS01 appeared to interpret the chief aim of mathematics as quantifying observable objects. Likewise, three participants in this category also associated mathematics with symbols. S21 exhibited a concurrent position, firmly believing mathematics is a subject involving operations of numbers and symbols; S33 also expressed a similar account on the post-instruction questionnaire. S29 tended to see mathematics as a sort of logic combining words, numbers, and symbols. It seems that numbers and symbols (especially the latter) loomed large in some participants' minds in which mathematical activity is mostly executing operations involving these two elements. Compared to numbers, the symbol is a more abstract and puzzling element that confuses mathematics learners, as evident in another participant's claim:

> Mathematics is *an abstruse and changeable game of symbols, but without any rules* [italics added]. It's lovable but also hateful! (S30, post-instruction questionnaire)

Nonetheless, it should be noted that though a large portion of participants associated mathematics with numbers, their description could be a mere rhetorical response rather than well thought out. Several interviewees expressed this point of view on the post-instruction questionnaire yet interpreted the issue from other angles. The rhetorical meaning of mathematics in Chinese (as "the scholarship of numbers") might have directed their responses.

Moreover, several chief differences were found in the participants' statements. Firstly, fewer participants claimed mathematics as a tool in daily life. In the beginning, seven respondents interpreted mathematics from a pragmatic standpoint in which the discipline functions as a practical tool for improving human life; afterwards they changed their focus to various aspects. For instance, one of the seven participants professed on the post-instruction questionnaire:

> Mathematics is a kind of science as well. A variety of scientific theories are derived from mathematics, and much stuff in daily life is related to mathematics, such as the golden ratio for appreciating the sense of beauty. (S03, post-instruction questionnaire)

Instead of mentioning practical utility of mathematics, S3 cited the example of the golden ratio, one of topics in this course, to support his argument. While stressing an intimate relationship between mathematics and ordinary life, he still seemingly tended to view mathematics from alternative facets. Participant S33, originally making a connection between mathematics and shopping, claimed mathematics is the mother of science, involving numbers, symbols, and operations. S13 previously described mathematics as a virtual subject hard to use in daily life; nevertheless, he shifted his concern in the end when asked to define mathematics:

> Mathematics cannot be entirely on the basis of imagination; certain techniques and rules for problem-solving should be memorized. Not everyone may have a sense regarding mathematics...In addition to memorizing, realizing rationale is necessary for generalizing known knowledge. (S13, post-instruction questionnaire)

S13 was more likely to construe the discipline in terms of doing and thinking aspects of mathematics. Secondly, similar to S13's shift, more participants (eight as compared to one at the outset) stressed logic of thinking and reasoning in mathematics when describing its essence. S25, formerly viewing mathematics from a metaphysical aspect, regarded the discipline as a training of rationality and logic, whose theorems are outcomes of reasoning. S15 responded vaguely on the pre-instruction questionnaire (mathematics is a concrete subject), yet showed a more wide-ranging understanding:

> Mathematics, I currently feel, consists of several parts: imagination and creation of art, combining theory and practice of science, built on ancestors' discovery and creation in history, further integrating logical processes. The end product is mathematics. (S15, post-instruction questionnaire)

S15's extensive post-instruction vision is noteworthy; however, he was not one of the selected interviewees, precluding further investigation. Of nine interviewees, RS09 was the only one who consistently viewed mathematics as a discipline of thinking. She earlier claimed that mathematics is like a racing game in the brain and professed on the post-instruction questionnaire:

> Mathematics is an auxiliary tool helping me think. It is a tool that I mostly rely upon in life because, by learning mathematics, I tend to use "mathematical fashion" [quotation original] to think when facing problems… mathematics is a tool for resolving thought problems. (RS09, post-instruction questionnaire)

In the interview, RS09 reconfirmed her written response and devalued the practical utility of mathematics. As compared to other interviewees, she exhibited a more thoughtful perspective regarding the issue. Asked about commonality and distinction between mathematics and art, she described certain mathematical approaches as an art of philosophical principle and took Archimedes as a supportive example:

> Because I feel Archimedes' ideas are filled with philosophical principles, like deriving volume by means of leverage principle. I feel there is a philosophical principle behind it. Just like two different figures yet with identical volume. I feel it makes me feel that they are presented in different forms but possess the same nature. So you may not look at the surface or a single facet; you can only know holistic stuff after parsing and integrating it. So Archimedes' stuff gives me a feeling of philosophical principle. (RS09, post-instruction interview)

RS09 also stressed that when she worked on assigned problems, her thinking was guided by the aforementioned philosophical feeling.

Furthermore, results indicated more participants tended to make a connection between mathematics and other disciplines; it is the key to studying nature and other subjects. For instance, RS03, initially viewing mathematics as a scholarship exploring principles, expressed more delicate thinking on the post-instruction questionnaire:

> Mathematics is the root of everything...mathematics can be used to explain certain physical phenomena and justify these phenomena. Human civilization would therefore make more progress through applying the data of these phenomena to technique or life. (RS03, post-instruction questionnaire)

In interview, RS03 resolutely held that mathematics is the base of all things and the value of mathematics is subject to its power and utility. Most participants in this category espoused similar beliefs, whereas S09 interpreted the concern in terms of learning respects rather than application:

> Mathematics can improve mental power and even resolve a lot of things. Mathematics is the root. Other subjects cannot be learned very well without a good learning of mathematics. (S09, post-instruction questionnaire)

As on his pre-instruction view, S09 drew a picture locating mathematics in the center connected with other disciplines (e.g., science and art) to demonstrate its vital role.

Another notable finding was that no participant claimed mathematical knowledge as infallible. Five professed initially that such knowledge is time-independent truth due to its rigorous nature, yet all reinterpreted the issue from other aspects. For instance, S17 formerly opined: "mathematics always has fixed outcome in spite of evolving over complicated and lengthy time," then changed his mind by saying:

> Mathematics is solving problems by means of operations or peculiar methods...*a lot of mathematical methods and ideas are not yet justified* [italics added] and there is no way to know [it is] right or wrong or "how" [quotation original] to solve problems. (S17, post-instruction questionnaire)

S17's change appeared quite drastic in that he later held a doubtful attitude toward mathematical methods and ideas, with no supportive example on the questionnaire, leaving the reason for the alteration unknown. Nevertheless, by referring to others' account in this respect, a plausible guess can be made. Interviewee RS05 initially saw mathematical outcomes as fixed and the process as a predetermined route leading to known results, yet on the post-instruction questionnaire he no longer endorsed this perspective and emphasized the importance of objectivity instead. Asked to clarify the distinction, he explained:

> I think the major influence should be Euler's idiot approach to infinite series. It is quite subjective. From our point of view, it is so funny. Your subjectivity would ruin you if too subjective or without similar experience...you ought to be objective and objectivity is subject to certain amount of experiences. Euler would not make that ignorant mistake if he had more experiences in doing problems of infinite series. (RS05, post-instruction interview)

RS05 became more likely to value the role of thinking processes and confessed his previous view was shaped by past school exams on which students always get partial credit for valid answers. Note that several instances regarding past mathematicians' fallible thinking and struggle to develop new ideas were introduced in class, serving as supplementary material for drawing students' attention to rigor in mathematical thinking. In certain ways, some participants' focus on mathematics had seemingly transformed from ultimate product to intermediate process.

These findings suggest participants were likely to adjust their visions of mathematics and its thinking in diverse ways. Examples they cited were mostly from classroom episodes or auxiliary materials. As shown earlier, several reevaluated the notion of mathematical knowledge as absolute truth and became aware of potential fallibility of mathematical thinking; both could exert a certain degree of influence on their conceptions of the development of mathematics. This section thus investigates how participants interpreted the issue 18 weeks later.

Table 12

Comparison of Major Responses Regarding Mathematical Development

How Mathematical Knowledge Develops	Pre-number of responses	Post-number of responses
Related to human demand	13	13
Developed in proper order	7	6
Developed by following certain rules	13	19
No rule	8	9
For solving problems	1	4

The first two statistics shown in Table12 display little distinction between participants' pre- and post-visions of how mathematics evolved in history. In both cases, there were 13 respondents considering development of mathematical knowledge as subject to human demands; almost as many held that mathematical knowledge progressed in proper order. With few exceptions, those expressing this sort of view maintained a similar stance throughout the course. Interviewees also exhibited a belief that mathematical knowledge was inherited from ancestors' creation and accumulated gradually.

Besides, the number of respondents believing there are rules for the development of mathematical knowledge had considerably increased. Interviewee RS05 did not explicitly respond to the issue to start, yet on post-instruction questionnaire claimed:

> Development of mathematics began from definitions. First of all, numbers, units, and operational symbols were defined; then these tools were used to explain problems that occur in daily life. *Fundamental definitions are rules to be followed* [italic added]. (RS05, post-instruction questionnaire)

In the interview he further explained:

> [For example,] you want to find an area, you probably need to define the magnitude that you want, then you need numbers to do operations, then the formula of area is derived by us. You use a fundamental definition to derive, to extend. Following the definition, we need to search out the method and then solve our problems by means of the method. We have to go along with fundamental definition because it would go extremely wrong if the definition is incorrect. (RS05, post-instruction interview)

For RS05, definitions are the base of mathematical knowledge as well as the grounds for formulas; the scope of mathematical fields is identified as soon as chief concepts are defined. Contrary to RS05's interpretation of mathematics, S25 considered mathematics more pragmatically:

> Mathematical knowledge is a system developed from people's desire to solve certain problems; meanwhile, [it is] a descriptive language of a lot of natural phenomena. Its developing processes follow some natural rules, proven axioms, and formulas. (S25, post-instruction questionnaire)

Either on pre- or post-instruction questionnaire, S25 concurrently manifested identical views on this issue, suggesting his firm belief of mathematics as a discipline for study of nature. Others responded to this item in terms of thinking aspects. S09 claimed on the post-instruction questionnaire development of mathematical knowledge should follow "thinking rules" but without additional explanation. His "thinking rules" could be manifested by S05's description:

> Discovering→Deriving→Proving. A lot of mathematical knowledge comes from discovering certain particular characteristics followed by deriving and proving. (S05, post-instruction questionnaire)

Notice how S05 and S09 previously gave vague accounts of mathematical knowledge following "universal" or "world" rules; both came to associate this construct with elements of mathematical thinking. Plus, among participants believing there is a rule for developing mathematical knowledge, S04's response was particularly notable. He formerly held that growth of mathematical knowledge is subject to rules of formulas, yet reworded: "the rule is constantly overthrowing preceding incorrect theorems and creating fresh ideas." His vision regarding the concern seemed to shift from a rigid status to a flexible one.

As shown above, nine participants professed that no rule exists for the evolution of mathematics; nevertheless, most did not give an explicit description.

Of the nine, two attributed this "no-rule" situation to mathematicians. As S10 claimed:

> There should be no rule! A lot of mathematics is dependent on mathematicians' demands and thought out by sudden inspiration. Just like many mathematical formulas and rationale, a lot of inspirations are originated by problems in daily life. (S10, post-instruction questionnaire)

Another participant S32 also held a similar perspective:

> Mathematicians notice a certain phenomenon or think of numerical values of certain stuff, (area of a circle, volume of a sphere, parabola and the like), then try to turn this stuff into understandable symbols. Developing mathematical knowledge, I think, follows no rule; it is subject to mathematicians' creativity and imagination. (S32, post-instruction questionnaire)

Compared to their counterparts (those believing there is a rule) viewing mathematical knowledge as context-imbedded intellectual heritage across generations, S10's and S32's accounts seemingly conveyed a belief in which mathematical thoughts are fortuitous ideas created by mathematicians' ingenuity. In this problem-based course, the mathematician's role was often addressed. It appears this class feature was recognized by some participants' minds.

Summary of Participants' Post-instruction Views

After exposure to an 18-week problem-based calculus course, participating students were better able to describe their perceptions of mathematical thinking in particular, mathematics in general. Still, their responses were diverse. Major findings about any change between these Taiwanese college students' pre- and post-instruction conceptions can be summarized as the participants: (a) were less likely to perceive mathematical thinking merely as a process of solving problems, (b) displayed little difference in individual persistence when encountering unfamiliar situations (c) as a rule, saw mathematicians as creative figures, as opposed to efficient problem solvers, (d) tended to consider doing mathematics as a mixed activity involving creativity and preset steps (e) were less likely to consider mathematics as a daily tool or a rigid subject (f) believed the development of mathematical knowledge is mainly subject to human demand and following certain rules.

The first concern of this study was how Taiwanese college students interpreted the essence of mathematical thinking after an 18-week problem-based course. Contrast of their responses yielded an image of participants shifting their focus of the process of mathematical thinking. In the beginning, they were inclined to consider mathematical thinking as the route of identifying solutions and deriving answers; calculation, recalling, and fitting formulas are the main modes. However, far fewer held this sort of view in the end. Participants normally shifted their concern to other aspects, such as understanding a problem and the rationale it is based on, thinking from diverse angles, and approaching the problem in one's own way. They seemed to pay more attention to the nature of mathematical thinking rather than emphasizing its superficial function of solving problems.

A somewhat unexpected result is that fewer respondents mentioned logical thinking and reasoning on post-instruction questionnaires. As noted earlier, a large portion of students tied mathematical thinking to logic at the outset; upon further inquiry, none could defend their positions and meanwhile confessed they had never experienced the merit. Namely, their pre-instruction responses probably were more like an intuitive, rhetorical conception; they reevaluated their recognition regarding the construct when answering post-instruction questionnaires and thus addressed it in other ways.

As Schoenfeld (1985) indicated, when stuck, an important feature of good problem solvers is the ability to think straight, avoid "wild goose chases" and figure out a way to cope with a quandary. Thus successful problem solving in some sense is subject to the degree of individual perseverance. Pre-instruction data showed that when seeing an untried or demanding task, participants mainly tended to look for external assistance (referring to relevant materials or asking someone else) immediately or adopted conservative fashions (e.g., recalling similar problems or formulas). Post-instruction responses demonstrated how slightly more of the participants were willing to try on their own, while somewhat fewer would seek external help directly. Nonetheless, neither written nor oral responses manifested significant upgrading of individual persistence in doing mathematics, though several interviewees confessed they would be more likely to discuss with others instead of giving up quickly.

Two major images of the mathematician held by the participants in the beginning were that mathematicians were: (a) good at attacking problems from diverse angles and (b) usually able to find the quickest, easiest way. The former was retained till the end of semester; the latter almost vanished (only one respondent endorsed this view). It appears the participants still saw mathematicians as flexible mathematics thinkers but abandoned the conception that mathematicians are efficient problem solvers. Several respondents previously held that mathematicians should have ready tools in their brains at their disposal; these responses showed little awareness of the mathematicians' struggle and effort. Whereas the participants later began to notice thought experiments in mathematicians' minds.

The most prominent trait of the participants' post-instruction responses regarding mathematicians was that mathematicians are creative, associative, and imaginative, able to concoct individual approaches and generate unusual ideas. Combining previous findings, mathematicians apparently were not only regarded by participants as flexible problem solvers, solving problems from multifarious angles, but also as sophisticated thinkers, creating idiosyncratic strategies for deriving answers. Moreover, the role of justification in mathematics seemingly loomed large in some participants' minds as well. Four students (versus none at the outset) expressed that, in addition to generating ideas, mathematicians would make an effort to verify their ideas.

Nearly 60% of the participants initially professed that problem-solving in mathematics mostly involves creativity. As noted, the students were impressed by mathematicians' creative imagination; therefore it was expected that more participants would be inclined to appreciate the role of creativity in mathematics. Post-instruction results, however, indicated otherwise. Far fewer firmly saw creativity as the most vital component in doing mathematics: on the contrary, far more tended to believe that doing mathematics is a pursuit requiring fixed procedures as well as originality. Likewise, a certain portion (20%) continued to consider this concern as individual- or occasion-dependent, a complicated issue. It appears the participants' reactions on this aspect merit further exploration, which will be discussed later.

Similar to their pre-instruction views, participants' chief image regarding mathematics was as a subject of studying numbers. Plus, mathematical symbols seemingly occupied more weight in some respondents' minds; hence, more participants viewed mathematics as a subject involving operations. However, on the post-instruction questionnaire and interviews, they hardly stressed mathematics' practical function in day-to-day life. Instead, more addressed it as a discipline exerting far-reaching influence on the study of science and nature. Participants tended to view the utility of mathematics in a broader window.

They no longer professed mathematical knowledge as absolute truth. According to interviewees' claims, mathematical knowledge could be fallible due to immature development of the topic at the time or mathematicians' personal subjective prejudice. Mistakes caused by mathematicians' subjective prejudice appeared to impress many participants. In the interview, examples about mathematicians' mistakes were cited as supportive evidence. Few participants linked mathematics to logical thinking at the pre-instructional stage (though many associated mathematical thinking with logic at the time); on the post- instruction questionnaire, many more viewed mathematics as a discipline pertinent to logic, thinking, and reasoning.

While acknowledging how mathematical knowledge could be unsound, most participants believed the development of mathematics is intimately related to human demand and follows certain rules. This sort of view was coherent with the aforementioned conception of mathematics as a discipline studying science and nature. In their minds, mathematical knowledge is a heritage constantly accumulated over the course of time and therefore must follow rules. Yet they seemingly interpreted rules in different ways. Some saw mathematical operations as the rules; others thought mathematics may not deviate from mathematical definition. Still others deemed the development of mathematics is subject to rules of thinking. One participant even claimed that the law of mathematical development is continually overthrowing preceding incorrect theorems and creating fresh ideas.

Among those arguing that no rules govern the evolution of mathematics, few gave specific examples. On the basis of limited evidence, these usually attributed progress of the discipline to mathematicians' ingenuity and inspiration, which is unpredictable.

The Course Features And Participants' Post-Instruction Views

The purpose of the present study was to explore the interrelationships between a problem-based calculus course and Taiwanese college students' perspectives of mathematical thinking. The two previous sections investigated the participants' pre- and post- instruction views respectively, on the basis of their written data sources (mathematics biographies, open-ended pre- and post-instruction questionnaires and oral responses of nine randomly selected interviewees). Some findings hint that the participants' account of mathematical thinking in particular (and mathematics in general) had changed. Such alteration in their conceptual framework of mathematical thinking may or may not be related to designed course features. This section therefore delves into the potential link between curriculum and participants' post-instruction views.

Post-instruction data sources indicated participants (a) were less likely to perceive mathematical thinking merely as a process of solving problems, (b) displayed little difference in individual persistence when encountering difficulties with the problem, (c) as a rule, saw mathematicians as creative figures, as opposed to efficient problems solvers, (d) tended to consider doing mathematics an activity involving creativity and preset steps, (e) were less likely to consider mathematics merely as a daily tool or a rigid subject, and (f) believed the development of mathematical knowledge is mainly subject to human demand and follows rules. Besides questionnaire responses and interview transcripts, discussion in this section is additionally based on students' in-class reflection papers describing their intuitive feeling and thoughts respect to certain topics taught at the time. Furthermore, for demonstrating a holistic picture of the interviewees' notion, interview transcripts are presented in the form of dialogue when necessary.

Weekly Problems

The main doctrine of this problem-based course was to problematize course topics in an effort to elicit learners' intrinsic desire for knowing and thinking mathematics. To reach that aim, doing and discussing assigned twelve weekly problems during the instructional stage constituted a major classroom activity throughout the course. Contrary to normal textbook exercises, the twelve tasks were demanding, differing from routine problems in nature, designed to situate students in predicaments and let them experience blocking progress toward solutions. Furthermore, unlike traditional homework with a purpose of helping the students review and practice knowledge and skills, several problems were done in reverse order; students received homework prior to instruction with planned topics. The purpose of this method was to assist learners in building a basic conceptual framework prior to entering the formal curricula. Depending on its difficulty, each problem was allocated a moderate time period, from one to three weeks, for participants to finish. Participants were allowed to communicate and discuss relevant ideas out of class, yet copying others' work without effort was prohibited.

Post-responses indicated the challenging feature of these tasks seemingly impressed participants in diverse ways. As noted, many participants initially tended to conceptualize mathematical thinking as merely an action of solving problems; they interpreted it otherwise afterward. RS03's post-instruction account indicated:

> Just like problems in assignments, many were never seen or heard before. We were required to prove or guess why this is so. In the process of answering these questions, [I] would employ a variety of ideas to write or guess. These ideas are mathematical thinking. (RS03, post-instruction interview)

He further added:

> I spent much time thinking on those problems. Think a lot. Various methods are used to solve them...kind of like two persons knowing how to solve [the problems] may use different approaches though their starting point is the same. So a problem can be worked out in multiple ways. (RS03, post-instruction interview)

RS03 usually demonstrated tremendous eagerness in attacking assigned problems. In the pre-instruction interview, he admitted having never thought about the essence of mathematical thinking. Nevertheless, as compared to his pre-instruction statements, he showed more thoughtful thinking on this construct.

In addition, the physical feature of some assigned problems in a way influenced RS03's use of approaches for resolving a dilemma. Asked about strategies for escaping from predicaments, he proposed physical conceptions and geometrical methods as the keys. He also confessed his thinking was affected by homework problems, some involving physics. Note that, of the 12 tasks, three (Napier's logarithm, the tractrix problem, and volume of a sphere) were to some extent relevant to physical ideas, leading students to build a link between mathematics and other disciplines. The following dialogue with RS02 can also manifest the inference:

> Researcher: You were more likely to see mathematics as a part of culture?
> RS02: Mathematical development would influence science and technology. A lot of tests need to be measured by mathematics. Some jobs cannot be done without certain equations, like deriving the slope of a tangent line; a bunch of physical phenomena could never be explained without it.
> Researcher: Why do you have this sort of thinking for now?
> RS02: A lot of homework…the tractrix problem just like physics.
> Researcher: You learned the closer relationship between mathematics and science from homework?
> RS02: Yes! (RS02, post-instruction interview)

Interviewee RS07 manifested similar thinking:

> Researcher: In your opinion, how does a mathematician create a mathematical theorem?
> RS07: Just keep calculating!
> Researcher: Could calculation alone derive the theorem? How do they come up with an idea at the beginning?
> RS07: I don't really know about this. It could be from physics. I do feel physics involves lots of stuff. I had never thought in this way, yet I do think so afterward. (RS07, post-instruction interview)

These descriptions provide a probable answer of why many participants adjusted the image of mathematics as a daily tool and took on the notion that it is the root of other scientific disciplines. In the pre-instruction stage, as reported, a large portion of participants believed doing mathematics mostly followed fixed procedures, whereas their concept shifted to a neutral stance considering mathematical problem solving as an activity implicating individual creativity and pursuing known steps. By referring to interview transcripts, a potential connection could be identified between students' view-shifting and homework problems. RS07 originally professed that, for attaining high scores, following routine processes is required and students are always taught to do so. Nevertheless, he reworded creativity as more critical in mathematical problem solving though going along with predetermined procedures is also essential in certain spots, as evident in the dialogue below:

> Researcher: Why do you like to think creativity is more important now?
>
> RS07: How to say?...This (creativity is more important) should be because of those problems you gave. Because those problems have been done by original authors. But the teacher [the instructor/researcher] gave them to us and did not worry we might copy the solution down...[I] feel that you gave us problems and asked us to think, to study, and to keep searching for better methods than those of former people. (RS07, post-instruction interview)

In RS07's mind, a need to complete appointed tasks compelled her to devise individual strategies. In many cases a quick solution was inaccessible, forcing her to think carefully about a problem. In a way, she re-evaluated her original thinking that following routine procedures is the main key for achieving answers.

Another interviewee RS01, who earlier held a rigid notion about mathematical thinking, also showed a notable difference. In the pre-instruction questionnaire, he professed:

> The correct answer is prescribed by people. Methods consented by public are also the norm for proving facts. (RS01, pre-instruction questionnaire)

When asked to elaborate further, RS01 confirmed this view by saying:

> Why does 1 + 1 = 2? Can't it be 3? This is defined by ancestors…Our current thought could see it as 0 if 1 + 1 was previously defined as 0. I think [doing mathematics] is following known procedure. (RS01, pre-instruction interview)

However, he adjusted his idea indicating that both (creativity and preset steps) are necessary:

> Researcher: Your response is more neutral this time. Is creativity quite important? Why?
> RS01: The problems you gave us! [I] would make a guess to reach the answer if I cannot figure it out. Then start to think again.
> Researcher: You would make a guess to reach the answer? Is creativity involved in the process of guessing?
> RS01: Sure! (RS01, post-instruction interview)

In the interview, RS01 also endorsed the view that guessing is a rather essential entry when mathematicians establish some mathematical fact; this is where creativity comes into play. He seemingly reflected his experience of solving assigned problems on mathematicians' work and further took this experience as a basis for interpreting the role of creativity in problem solving. A similar perspective can also be seen in RS08's description. Like others, RS08 once viewed doing mathematics chiefly as following predetermined steps, yet afterward he saw originality as critical:

Researcher: You emphasized the former [following predetermined procedures] last time but stressed both this time?
RS08: Mathematics cannot be improved without creativity. New stuff will be forthcoming if [we are] able to think of something never thought by others. Mathematics is all about thinking stuff. It would not make any progress if all were based upon planned steps, such as identifying formulas and fitting them in.
Researcher: Did it take creativity to solve our problems?
RS08: Sure! Our assignments were never seen before. They were so difficult, which should involve creativity. Working them out was somewhat like earning extra bonus.
Researcher: Which problems do you think employed your creativity? Which one most impressed you?
RS08: Probably that rabbit problem [Fibonacci sequence]. That was derived by myself and came up with some numbers to reveal the relationship (RS08, post-instruction interview).

With their challenging nature, assigned problems appeared to compel participants to discard conventional notions of mathematics as mainly recalling/installing formulas. Meanwhile, participants established a concept that individual originality was indispensable in problem-solving. Note that besides RS08, who cited the Fibonacci sequence as the most impressive problem, another five interviewees endorsed this option as well. This problem was presented in a four-stage fashion in which the first two are easy to access, the last two more demanding. Compared to other harder tasks, Fibonacci sequence problem appeared to successfully draw upon participants' eagerness.

The Curriculum

Aside from doing and discussing weekly problems in and out of the classroom, the curriculum sequence played a significant role in this problem-based course. As shown in Appendix B and C, the arrangement of curriculum differed from the conventional sort. A typical calculus curriculum is usually executed in such an order that limits of functions appear first. laying the groundwork for the following topic, concept of derivatives. Soon after work with differentiation, the indefinite integral is introduced as the inverse operation of the derivative. Lastly, definite integrals are introduced and solved by techniques of indefinite integrals on the basis of the fundamental calculus theorem. From a deductive standpoint, this sequence is smooth and most convenient because all topics are connected and carried out in a logical order, in which each theme functions as a stepping stone to the next. All the same, such an arrangement may be inappropriate for a problem-based curriculum. The aim of this problem-based course was to problematize relevant concepts, eliciting learners' curiosity and desire to know and solve problems. This problematization would play an insignificant role if the curriculum progressed in the above-mentioned fashion. In a deductive mode, students get an understanding of every topic as preliminary knowledge and accept it without doubt. They have no need to question the whole curriculum structure because learning preceding topics is the preparation for later ones, but not for resolving real problems. Problematization is diminished in such a deductive way of instruction.

Curriculum design in this problem-based course was intended to help students grasp the real need of studying the topic taught—for mathematical knowledge itself. not just for subsequent units. For instance, instead of introducing the concept of limit in the early stage, derivation of the circular area was the first task for students to explore. They were not allowed to employ the circular area formula. Yet, they were asked to solve the problem by basic geometry, in order to experience the historical struggle regarding the problem. The handout describing how ancient mathematicians attacked the problem followed, and students were asked to compare their approaches with past figures. It was assumed that, in this manner, not only is the necessity of concept of limit naturally planted in students' minds. but also they have the opportunity for contrasting diverse mathematical approaches and thinking genres.

Post-instruction responses indicated that their images of mathematicians' thinking had undergone a significant shift. A large portion of students considered mathematicians as efficient problem solvers possessing solid background knowledge and a number of ready solutions kept in mind at their disposal. This impression, however, in a way was altered near the end of semester. More participants were likely to see mathematicians as creative, associative, and imaginative thinkers; post-instruction notions could be related to handouts showing great eastern and western mathematicians' approaches in history—e.g., Liu Hui, Zu Chongzhi, Archimedes, Fermat. The following dialogue with RS05 may manifest the inference:

> Researcher: Among the handouts, which one impressed you most?
> RS05: Liu Hui's derivation of the area of a circle and Archimedes' pursuit of the circular area. I feel it [Archimedes' approach] was too incredible! Turning a circle into a triangle and verifying by using *double reductio ad absurdum*. I am gonna give you my head if I am able to do so. (RS05, post-instruction interview)

RS05's exaggerated wording reflected a widespread point of view held by participants. Following the first assignment (deriving the area of the circle without using any circular area formula), Chinese mathematician Liu Hui's approach (partitioning a circle into triangles and rearranging them to form a parallelogram) and Archimedes' proposition in the book, *Measurement of a Circle*, (area of a circle equals a right-angle triangle with one of the sides about the right angle equal to the radius and the other to the circumference) were presented in class. Liu Hui's account is mostly intuitive, easily understood and accepted; Archimedes' tack is purely deductive and harder to understand. The thought of favoring Liu Hui's idea was evident in RS03's description on his in-class reflection report:

> Comparing Archimedes and Liu Hui, I like Liu's approach better because it is expressed in an understandable way, yet Liu's method relies on intuition very much. A persuasive fashion should make readers feel it as a matter of course. (RS03, 1st in-class reflection paper)

The most unimaginable part of Archimedes' proposition is his transformation of a circle into a right-angle triangle. Note that a participating student happened to propose a plausible explanation of Archimedes' idea. He regarded a circle as a combination of infinite numbers of concentric circles with varying radii. He then straightened the circumferences of all concentric circles and piled them up (from the longest to the shortest) to form a right triangle. Archimedes' approach thus became attainable; this account was demonstrated on the board and appeared to attract the audience's impression and interest. On another handout, Archimedes employs the leverage principle to derive the volume of a sphere, which is a peculiar approach drawing students' attention also. Exposure to these diverse ways of thinking seemingly led some participants to reevaluate their original views of mathematicians. Asked how mathematicians think, RS02 expressed an alternate perspective in the post-instruction interview:

> Researcher: How do mathematicians think? You mentioned logical reasoning last time, but this time you refer to imagination and association.
>
> RS02: Like association...several people discovered calculus, but it did not become concrete until Newton. The reason that I think imagination is more important is because of Archimedes. I feel he is so strange. Just like the former problem requiring [us] to find the volume of a sphere, he uses leverage principle to find it. I still don't quite understand that problem. Then I feel he is so strange. How could he think of that? Turn a circle to a triangle?! I feel his imagination is so strange.
>
> Researcher: Do you think these approaches are attainable by logical reasoning?

RS02: Laypersons probably don't know how to attain [the solutions] by logical reasoning. Even when you know the answer already, you don't know how to associate with it. Using leverage principles to derive volume is so odd to me. Besides, how did he establish the whole idea, even he got this idea at the outset? So I feel he needed imagination.

In addition to Archimedes' ideas, Zu Chongzhi's method of three-dimensional partition for finding the volume of a sphere impressed others like RS07. In general, on the basis of instances cited, some course handouts, to some extent, urged students to re-evaluate their notions of the mathematician's mode of thinking.

A significant finding in the post-instruction responses was that the participants no longer claimed mathematical knowledge is infallible. Interviewees were further questioned about the fallibility of mathematical knowledge, and all gave a definite answer: new found facts can supersede old knowledge, except fundamentals. Meanwhile, as shown in Table 12, a considerable portion of participants considered that development of mathematical knowledge is in proper order and follows certain rules; their seemingly inconsistent responses therefore warranted further investigation. For increasing students' awareness of potential fallibility of intuitive thinking, several instances were presented in class, such as "comparing the magnitude of 1 and 0.999...", Euler's mistake in computing infinite series "1 + 2 + 4 + 8 + 16 +...", and multiple historical approaches to summing the alternative series "1 – 1 + 1 – 1 + 1 – 1 + ..." It appeared these examples had come into play in students' conceptions. RS05 formerly held that in mathematics the result is the most critical and the rest is mere detail, yet he turned to stress the role of objectivity on his post-instruction questionnaire:

> Researcher: Is there any difference in your description (about what mathematics is)?
>
> RS05: I think the major influence should be Euler's idiotic approach to infinite series. It is quite subjective. From our point of view, it is so funny. Your subjectivity will ruin you if too subjective or without similar experience. You ought to be objective, and objectivity is bound to a certain amount of experiences. Euler would not make that ignorant mistake if he had more experience in doing problems of infinite series.
>
> Researcher: So from that example, you feel objectivity is very important?
>
> RS05: You must have objectivity and a certain amount of experience to attain objectivity. Euler would not make that error if he had a lot of experience in doing problems of infinite series. (RS05, post-instruction interview)

In another dialogue, RS05 talked about the evolution of mathematical knowledge:

> Researcher: Do you think new mathematical facts could supersede old ones?
>
> RS05: I feel it is possible, but current [mathematical] definitions are getting more rigorous. It seems not to happen anymore.
>
> Researcher: Like Euler's stuff, we currently consider it wrong. In your opinion, is this because of the criteria in a different era?
>
> RS05: Criteria would be different over the course of time. Current criteria are subject to the amount of knowledge. (RS05, post-instruction interview)

RS05 attributed fallibility of mathematical knowledge and thinking to the change of criteria in different time periods; RS09 also took this stance:

> Researcher: Can new mathematical knowledge displace old, if it develops in proper order?
>
> RS09: It could! *Because mathematics is invented* [italics added], the followers would create better tools. Thus, it may supplant [old knowledge].
>
> Researcher: For example?
>
> RS09:……(pondering)

> Researcher: Like Euler's mistake in infinite series, does it fit your account?
> RS09: Roughly! And like the example: $1 - 1 + 1 - 1 + 1 - 1$ …(RS09, post-instruction interview)

When interviewed, RS09 repeatedly emphasized that mathematics *per se* is a subject of thinking; thus the update of knowledge is normal. Non-interviewees' concepts of this issue may be manifested by the in-class reflection papers describing their feeling regarding three invalid historical approaches to summing the series "$1 - 1 + 1 - 1 + 1 - 1$ … ." A typical response was that intuition is unreliable, as seen in S19's claim:

> When knowing the answer, [I] found that the series really can not be calculated by number rules…Working problems can only continue by following a set of rules; Otherwise, it is dangerous to think of doing via human intuition. (S19, 2nd in-class paper)

Still others harbored doubts about mathematicians' work:

> [I] had never expected well-known mathematicians could make mistakes too! But the concepts of infinite series do easily confuse people. I realize that the infinite series cannot be solved by using [the concept of] finite terms. (RS04, 2nd in-class reflection paper)

Participant S34 expressed on the 2nd in-class reflection paper that mathematicians must search for correct answers by means of many invalid assumptions. This perspective may explain why in the post-instruction stage participants tended to abandon the image that mathematicians are efficient problem solvers.

One particular trait for this problem-based course, noted earlier, was that the course sequence was generally arranged in chronologically historic order. Students first learned concepts of definite integration and then the idea of derivative, followed by knowledge of limit. The fundamental theorem of calculus appeared later to integrate two major branches: integration and differentiation. A handout pertinent to developing the fundamental theorem was also presented in class to reveal how this theorem emerged over the course of time. The purpose of this design was to help students experience and be aware of the processes of formation of this human scientific discipline and remove the impression of mathematical propositions as ready tools handed down from above. Despite nearly half of the participants' claim that the development of mathematics should follow certain rules, two interviewees expressed contrary views and defended their positions by citing the handout of the fundamental theorem of calculus:

> Researcher: Why does [the development of mathematics] not follow any rule?
> RS07: Because they tried to find the area of a land. Because nobody thought of it, and the problem was handed to them [Newton and Leibniz]. So it should be no rule.
> Researcher: [repeating RS07's statement on the post-instruction questionnaire] Successors would keep studying on the basis of the teacher's theory. If they did not make it, someone else would study it again to see whether it could be invalidated. What do you mean?
> RS07: This could be related to previous answers. [I] feel you gave us problems, asked us to think and then study them. Keep searching for better methods than those of former people. If we do not make it, someone else will pick and do it. Just like the fundamental theorem of calculus. At the outset, some people only finished it partway, though they had nearly reached the answer, but they didn't know [the truth] until a certain guy [Newton]. (RS07, post-instruction interview)

RS07 seemingly indicated the evolution of mathematical ideas was subject to personal judgment and effort, lacking universal laws and further cited the handout concerning the fundamental theorem of calculus as most impressive, whereas RS02 interpreted the issue from another standpoint:

> Researcher: Does the development of mathematics have fixed process? You said, "Yes!" last time, but "No!" this time.
> RS02: I am not sure for now. Like differentiation and integration, limit is usually taught first, isn't it? But differentiation and integration were invented before it [limit]. However, current teaching is in reverse order. In terms of its theory, limit ought to appear first, with differentiation and integration following. Yet its development started from integration, so I feel it did not follow a fixed process. (RS02, post-instruction interview)

It appears the inconsistency between the historical development of calculus and logical sequence of its theoretical rationale confused RS02, compelling him to reconsider the relevant epistemological issue.

Interaction In Class

Other than the aforementioned designed homework problems and curriculum, for achieving this goal of problematizing, a dynamic two-way classroom interaction was essential. By frequent questioning, the instructor/researcher was more a leader than a director guiding students to think about problems and evaluate approaches to solving them. Students were encouraged to raise and/or answer questions, even challenge others' answers. Given the conventional classroom culture in Taiwan, students normally act as passive content receivers. The task was therefore not easy to accomplish; not until two or three weeks later did students become more comfortable with this method of instruction.

For avoiding potential bias, participants were not directly asked to voice their feeling about the instructor/researcher's teaching. Hence relevant information could only be attained from interviewees' responses to other questions. RS07 previously had an aversion to study mathematics and expressed superficial thoughts on related issues in the pre-instruction interview. Asked in post-instruction interview to propose a good way to learn mathematics, she claimed:

> For now, I am willing to discuss and study with others because [I] was very lazy before. During high school, you never studied mathematics, because you were just given a problem and asked to do it. If you could not work it out, teachers would explain, and you just memorized it. That's all processes. [This method] has advantages and disadvantages. Sometimes it was good if you understood, yet you were dead if you could not figure it out. *But in current mathematics, you get a chance to think how inflection point can be derived. You don't have to memorize it...so realizing it becomes the most important* [italics added] (RS07, post-instruction interview)

A similar reaction was seen in RS05's response:

> Researcher: For now, what do you think is the best way to learn mathematics?
> RS05: Establish concepts and promote experiences.
> Researcher: How to establish concepts and promote experiences?
> RS05: Pay attention to lecture! Lots of stuff cannot be accepted without doubt, but I must think from multiple angles. More verification or refutation... In the past, I always believed and memorized what teachers said, later applying it to many problems...[I] used to do exercises passively. *Yet for now, [I] am more likely to stress thinking. Sometimes you pose some questions to us, and I often listen up and ponder where the problem is.* (RS05, post-instruction interview)

RS07 and RS05 both seemed to begin comparing two contrary fashions of learning mathematics and realizing that making sense of key concepts is critical. Besides professing his feeling about the teaching approach in this course, in the post-instruction interview, RS05 further emphasized the influence of colleagues' presentation:

> If answers cannot be reached, [I'll] listen to classmates' demonstration. [I] feel their thinking is so flexible...I am not good at thinking mathematically, unable to quickly identify conflicts or do inverse thinking. (RS05, post-instruction interview)

> Many classmates were able to create their own methods. So I am more likely to comprehend the flexibility of mathematics...I am so curious how they attain this sort of method. How could this make it? But [I] never tried. After experiencing careful verification, it seems the degree of flexibility [in mathematics] gets higher. Self-creation is possible. That sort of creativity is much more than I had ever known. (RS05, post-instruction interview)

RS05 repeatedly acknowledged his own shortcomings in mathematics and exhibited an appreciation for colleagues' creativity. He also confessed classmates' presentation caused him to consider both creativity and fixed procedures as essential.

Collaborative Learning

Of the twelve assigned problems, two (Napier's logarithm and tractrix problem) were carried out by means of cooperative learning. Students were randomly divided into four- or five-member groups. For the two problems, each student had to hand in individual work prior to entering group discussion. Following the group activity, each student then submitted a solution again representing a personal final record. Besides two 50-minute open discussions in class, students were encouraged to share ideas out of class. It was hoped this manner would foster their experiences of verbalizing and communicating mathematical ideas (typically lacking in Taiwanese classrooms).

On the basis of most participants' responses, collaborative learning provided them an opportunity to hear and consider colleagues' thinking and approaches to tasks. As such, they were more likely to see flexibility and pleasure in doing mathematics. As RS07 claimed:

> Researcher: [Repeat questionnaire responses] You were told that mathematical thinking is interesting. What does that mean?
> RS07: Yes! I could feel those guys having much fun in thinking mathematics. Even though they did it wrong, they still enjoyed it. I admire them… they were willing to spend time on it [doing mathematics]. I am not that kind. (RS07, post-instruction interview)

With one exception (RS09), interviewees expressed their liking for this activity. Still, by referring to their solutions, the effect of cooperative learning was seemingly not as significant as it appeared. Students were requested to submit final manuscripts by integrating their own approaches as well as group members' ideas. Yet a considerable number of students showed little meaningful work in the first manuscript, while the second revealed a high degree of similarity. It appeared they tended to rely on certain members' answers rather than thinking on their own prior to group discussion. This phenomenon not only was manifested in the instructor/researcher's in-class observation but also in RS09's statement:

> Researcher: What do you think about group discussion?
> RS09: It was not much fun for my team, because they did not make any effort at all. I don't like that kind of feeling.
> Researcher: People say you may get some feedback when explaining to others.
> RS09: But it was not fun at all when everybody just listened to you without any reaction. (RS09, post-instruction interview)

RS09's feeling to a great extent was in line with the instructor/researcher's survey, in which several passive students were not aggressively engaged in the activity. As a whole, the design of cooperative learning failed to accomplish its expected goal. The reason could not be classified, as will be discussed in the next chapter.

Summary

Post-instruction data indicated that, while responding to what mathematical thinking is, participants still claimed that mathematical thinking means figuring out a way to reach answers, with their wording differing in some way. They tended to conceptualize mathematical thinking as solving problems in one's own way, multiple approaches, or peculiar ideas. In addition, participants were more likely to value logical sense in doing mathematics at this time. Several respondents also cited mathematical thinking as a way of exploring rationale of formulas and intuition alone as unreliable, suggesting justification began to loom larger in their minds.

Participants' strategies reacting to difficult problems generally showed wide diversity. In addition to looking for relevant material and asking for assistance, more students emphasized they would try to understand a problem, identify all knowns and unknowns, then make a plan. Moreover, several participants exhibited more willingness to discuss with others, yet neither written nor oral responses showed any significant improvement of individual persistence while doing mathematics.

Contrasting the answers of how the mathematicians think yielded an unchanged point of view: mathematicians are good at attacking a problem from multiple facets and diverse angles. Nonetheless, at this point they emphasized the mathematician's imagination and creativity, and were less conscious of the mathematician's approach as most convenient and quickest. Note that, following a recognition of mathematicians' imagination, though the majority of participants held that doing mathematics involved individual creativity, most tended to take a neutral stance—both creativity and preset procedures are essential.

It was also found that the participants' epistemological belief regarding mathematics had been affected in some way. By contrasting responses, several distinctions emerged. While a majority of the students continued to viewed mathematics as a fundamental subject (involving numbers, operations and logic) for exploring other disciplines, no participant claimed mathematical knowledge as absolute truth. Asked about the possibility of new mathematical facts superseding old ones, no interviewees were in doubt; all defended their answers by citing examples given in class. According to them, mathematical criteria evolve over the course of time, and validity of mathematical knowledge must be constantly examined. The findings seemingly suggested that, after an 18-week problem-based calculus course, some students' views of mathematical thinking in particular, mathematics in general, had shifted in certain aspects.

CHAPTER V
DISCUSSION AND IMPLICATION

The purpose of the present study was to explore relationships between a problem-based calculus course, using historical problems, and Taiwanese engineering majors' views of mathematical thinking. Particular foci include: (a) how college freshmen interpreted mathematical thinking and to what extent their notions in this respect related to previous learning experiences; (b) in what aspects, if any, participants' conceptual framework regarding mathematical thinking changed, and (c) the interrelationships among course features and participants' shift in their concepts of mathematical thinking.

This chapter consists of five sections aiming to interpret the findings to answer the questions of interest. The first section analyzes students' pre-instruction views of mathematical thinking, with additional focus on the influence of past learning. The second investigates post-instruction views—i.e. in what aspects and to what extent, if any, those changes in views of mathematical thinking related to pre-instruction views or any traits of this problem-based course. Implications and limitations of the present study are explicated in the third and fourth sections, respectively; the last section proposes recommendations for prospective relevant research.

Pre-instruction Views of Mathematical Thinking And Past Learning Experience

Students' pre-instruction views of mathematics in this study were in line with Schoenfeld's report (1983a, 1983b, 1988, 1989). By comparing two protocol analyses in a mathematical problem-solving class, Schoenfeld (1983a) explored how college students' beliefs influenced their problem-solving behavior and further indicated their empirical beliefs about mathematical problems as playing a more significant role in learning mathematics than knowledge. To understand the origins of students' empirical beliefs, Schoenfeld conducted subsequent studies (1988, 1989) probing high school students' beliefs about mathematics, their learning of mathematics, and how classroom teaching influenced their concepts in this respect. It was found that the context of "practice makes perfect" not only augmented students' empiricism but also shaped inaccurate views of what mathematics should be done because, according to Schoenfelds' observation, students were normally asked to do routine exercises that can be completed in few steps and were expected to reproduce a pre-existing solution. As such, they tended to assume mathematical results as intact truths passed on "from above." Students then typically viewed mathematics as a set of rules and procedures in which each problem was supposed to be resolved by following specific algorithms.

Despite cultural and educational differences between the two sites, on the basis of mathematics biographies and pre-instruction interview transcripts, Taiwanese college freshmen in the present study manifested a striking resemblance to their American counterparts of years ago. They professed in the mathematics biography that more practice is the most efficient way to learn, and they were more likely to stress the significance of basic definitions and principles, yet they were less prone to a holistic conceptual understanding. In spite of several participants exhibiting an active disposition and enthusiasm toward the subject, a fitting appreciation of the value of mathematics was lacking in their responses. As a rule, they demonstrated a pragmatist point of view, associating the importance of mathematics with professional training or with utility in daily life. The merit of mathematics as an intellectual discipline was barely addressed in their mathematics biographies.

The influence of past learning experiences was also evident in students' accounts of mathematics and mathematical thinking. With an overwhelmingly mechanical training, participants had hardly paid any attention to thinking about the essence of mathematical thinking and mathematics, they typically associated such thinking with the process of deriving answers and mathematics with a subject of dealing numbers by quantifying daily objects. Problem solving in mathematics, as they saw it, was mostly a rigid activity that follows preset procedures; creativity thus plays a relatively minor role. While some respondents addressed mathematical thinking as involving logical reasoning, they also confessed in the interview that they scarcely experienced this advantage. To them, the conventional belief that mathematics may promote capability in logical reasoning was seemingly a rhetorical and superficial understanding. In the beginning, over half of the participants (26, 59%) claimed that doing mathematics involved creativity. Further referring to written and verbal responses, however, yielded meager supportive evidence. They tended to interpret creativity as "multiple-solution" but failed to demonstrate a more profound appreciation or give precise instances to defend their positions. It is likely that they accepted this widespread public image of mathematics without a second thought. As Schoenfeld (1989) indicated, students' perception regarding mathematics may refer to that which takes place in classrooms as well as to the mathematics that takes place outside them. Namely, they might hold diverse beliefs about mathematics in school, in everyday life, and in the abstract, thus contributing to varying views about mathematics in general, mathematical thinking in particular, and even on learning of mathematics, as Schoenfeld stressed.

Though initially several participants exhibited enthusiasm toward the subject, it was found that appreciation of the value of mathematics was missing from their responses in the mathematics biography; they normally referred to the significance of mathematics for professional training or utility in daily life, a pragmatist viewpoint. On their pre- instruction questionnaire, either in the responses explicating mathematical thinking or mathematics, recognition of mathematics as an intellectual discipline was absent and, on the basis of the mathematics biographies, it was apparent that their teachers played a critical role in their learning. Several respondents cited a mathematics teacher as the most influential figure during their past learning careers and confessed their performance and disposition in mathematics to a great extent were subject to teachers' instructional approaches. This finding was congruent with Ford's study (1994) that aimed to identify to what extent the belief of teachers was reflected in the beliefs of their students, reporting a high resemblance between the two. Teachers in Ford's study believed that problem solving is primarily the application of computational skills in everyday life, and their students' beliefs about the nature of problem solving were for the most part consistent with the views held by their teachers. Despite students' outstanding performance in the international test (e.g., IEA, 2000), mathematics education in Taiwan has long been criticized for its cramming method and exam driven instruction. Participants' rigid notions were probably by-products of this long-term affection; while the present study paid little attention to this issue, further investigation is merited.

Changes In Views of Mathematical Thinking

Post-instruction data sources hint, in certain aspects, the participants' interpretation of mathematical thinking and mathematics differed from their previous ones. The changes in their accounts may be affected by several conditions, such as previously preexisting conceptual scheme, personal learning habits or experiences, unexpected out-of-class events, and course features in the present study. Some factors were beyond the investigation of the researcher, and since this study is exploratory rather than causal, this section merely examines to what extent and in what ways, if any, their changes in views were related to (a) pre-instruction views and (b) the problem-based course. The following discussion is mainly based on the nine selected interviewees' relevant data sources to avoid misinterpretation.

Interrelationship Between Pre- and Post-Instruction Views

It is assumed that, in such a problem-based course, initial notions would be related to the participants' changes in their views. For instance, students holding a vivid perception of concerned topics may be more apt to improve their existing thinking to accommodate other changes; on the other hand, those who exhibited rigid thought are comparatively reluctant to make any alteration. Still, the data suggest this appealing assumption should not be accepted without question. When entering the course, the overwhelming majority of interviewees espoused cursory or superficial beliefs about several issues on mathematical thinking and mathematics, whereas it was found that in the post-instruction their views differed in some respects demonstrating a variety of trends in their changes.

Among the nine interviewees, RS09 alone expressed a positive disposition toward mathematics as well as a thoughtful account about mathematical thinking at the beginning. During the pre-instruction, she steadfastly believed that mathematics is an intellectual subject challenging one's way of thinking and conceptual understanding is the key element in mathematical thinking; as such, mathematics can be flexibly expressed in diverse forms. Near the end of the semester, she became better able to convey her thinking by drawing a picture and flowchart on the questionnaire to explain the essence of mathematical thinking and describe how she copes with a predicament. In the post-instruction interview, RS09 repeatedly stressed flexibility, even associated mathematics and philosophy, claiming that in some cases, mathematicians' approaches involved certain philosophical principles. This sensation sometimes guided her ways of mathematical thinking. Afterward, RS09 professed a dynamic notion that mathematical knowledge is mainly a human invention and thus unstable. As such, it could be superseded and updating knowledge is a normal activity in mathematics. It appeared both RS09's active pre-instruction views and open mind contributed to her enriched conception.

Three participants (RS02, RS05, and RS07), who professed rigid, cursory views of mathematical thinking and mathematics, changed to express more thoughtful accounts on several issues. Despite seemingly adequate recognition of mathematical creativity, RS02 originally espoused superficial and inconsistent beliefs about logic and thinking. He afterward displayed an appreciation of mathematical thinking in the post-instruction interview: "A beautiful mathematical equation is a kind of art too." He meanwhile proposed an interesting idea that a powerful formula may not necessarily be a good tool because it will hamper further progress. In his mind, an exceptional mathematical proposition ought to "sting but not stun" mathematical development. Moreover, asked how mathematicians think, RS02 replaced the original response of mathematicians' logical reasoning by stressing their imagination. Interviewee RS05 at the outset resolutely held a cursory view of mathematics as a result-oriented subject in which outcomes are essential and processes detail. He subsequently changed his mind and valued the significance of objectivity in mathematical procedure. Similarly, RS05 considered that, to mathematicians, mathematics is an art. As for RS07, she formerly confessed her lack of patience in doing mathematics and demonstrated superficial conceptions in both questionnaire and interview; yet she recognized the delight of mathematical problem solving and appreciated its association with other disciplines.

In spite of holding inflexible views similar to those of above-cited participants, four interviewees (RS01, RS04, RS06, and RS08) exhibited relatively minor changes in their post-instruction responses. One shared notion was that they began to perceive creativity as fundamental in mathematical thinking. Besides this notion, their conceptual understanding of the research issues was as a rule congruent with their former views; few noteworthy insights could be seen in their accounts.

Of the nine interviewees, RS03's case was peculiar because, similar to RS09, he demonstrated an extraordinary enthusiasm toward doing mathematics at the start yet showed a lack of understanding on several items. He confessed (in pre-instruction interview) he had paid attention to these issues. In the post-instruction questionnaire and interview, he was better able to articulate his beliefs. Nonetheless, unlike RS09 professing considerate depiction of mathematical knowledge and thinking, he still failed to plot a holistic picture of mathematics in general, mathematical thinking in particular. For instance, with his outstanding mathematical performance in solving assigned tasks, RS03 was asked to imagine how a mathematician develops theorems. He was unable to verbalize this issue clearly, merely stressing constant doing and calculating. He further firmly espoused a pragmatist belief that mathematics' value is totally subject to its utility and mathematical knowledge without application doomed to be a mistake. Despite their commonality of an active perspective regarding doing mathematics, RS03 and RS09 expressed diametrically opposite views in this aspect.

The interrelationships between pre- and post-instruction views were by no means straightforward. Those professing similar pre-instruction views on mathematical thinking developed varied notions over time, some manifesting noteworthy differences in perspective, others not. Individuals showing keen disposition toward doing mathematics could hold contrary epistemological beliefs regarding mathematics in general, and mathematical thinking in particular (e.g., RS03 and RS09). On the basis of interview transcripts, it was found that the interviewees, originally exhibiting cursory views that ended with little change, were those less likely to verbalize their thoughts clearly and completely. On the other hand, most eloquent interviewees tended to gain productive insight into the essence of mathematical knowledge and thinking. Thompson (1984) proposed the conception of "reflectiveness" and "integratedness" as potential causes of the inconsistency between teachers' mathematical belief and instructional behavior. This idea may also be applied to probing students' view-shift.

Interrelationship Between Course Features and Post-Instruction Views

Participants were apt to grow individual views on mathematical thinking in assorted ways, but several major trends emerged in post-instruction responses. They generally tuned up the conventional image that mathematical thinking simply is a computational process for deriving answers, yet tended to value the role of creativity. As such, they seemingly had a better understanding of the mathematicians' vocation. Participants also showed an appreciation for the flexibility in doing mathematics, as opposed to following predetermined steps. Moreover, they were convinced that mathematical knowledge progressed in proper order, mainly subject to human demands, meanwhile claiming fallibility. Post-instruction data imply these notions were to some extent related to two noteworthy aspects of the designed curriculum: weekly problems and historical approach.

Instead of direct presentation from the outset, students in this study were situated in an ill-structured environment where target problems were assigned early for exploration without hints. Required tools and concepts were no longer at their ready disposal; they were forced to develop their own strategies, seek relevant material, or discuss with peers. In this way, problem solving involved not only fitting formulas and theorems but also generating individual tactics for getting out of predicaments; mathematics became a minds-on as well as hands-on activity. According to interviewees' oral reports, this fashion of instruction, to a great extent, differed from past learning in mathematics and likely urged them to reflect on the meaning of doing mathematics. RS06, initially considering herself proficient at mathematics, complained homework problems were so strange that she barely had a clue of beginning them and even began to doubt her own mathematical ability. When seeing other colleagues' peculiar methods on the board, RS06 tended to better understand the subtleness of mathematical thinking.

With the challenging nature, homework problems appeared to stimulate particpants to create and test their naïve strategies. It was often seen that the participants invited to present on the board demonstrated varied approaches (though some were complicated or problematic), drawing the audiences' attention and admiration. With miscellaneous modes of thinking, participants were inclined to foster an appreciation of flexible thinking, as RS05 claimed in the post-instruction interview. Nevertheless, the difficulty of the tasks may also have created a by-product. As reported, the effect of collaborative learning in the present study was not as noteworthy as expected, and participants' recognition of the significance of individual perseverance in mathematical thinking hardly improved. These undesired outcomes could be attributed to a requirement for seeking entries or achieving solutions of problems because less-confident participants might quickly ask for assistance (e.g., discussing with others or looking up relevant material). The inference was evident in the six of nine interviewees' claim: of the twelve tasks, the Fibonacci sequence was most impressive due to its ease of access. The issue of problem posing deserves further investigation along this line.

Findings also suggested participants' post-instruction notions were related to a vital trait of this problem-based calculus course: *history*. For problematizing mathematical topics, students were first urged to weigh the merits of studying a topic, followed by a historical background introduction. All 12 weekly tasks were relevant to origins of topics at the time. Ancient mathematicians' key ideas and varied approaches were presented to reveal historical obstacles and struggles. In order to foster and improve participants' appraisal ability, students were requested to compare and contrast various mathematicians' modes of thinking as opposed to merely learning them individually (e.g., Liu Hui's, Archimedes', and Seki Kowa's derivation of the area of a circle; Fermat's, Descartes', and Barrow's finding tangents; Zu Chongzhi's and Archimedes' ways of computing the volume of a sphere). Of the cited mathematicians, on the basis of their responses, participants were impressed by Archimedes' work. His peculiar ideas of deriving circular area (turning a circle to a right triangle) and computing the volume of a sphere by using leverage principles amazed these students. All interviewees mentioned how Archimedes' unusual approaches attracted their attention and confessed they were more likely to comprehend the role that originality plays in mathematical thinking. RS02 even replaced his previous idea that mathematicians reason logically, contending they likely employ imagination because those weird ideas were seemingly inaccessible merely by logic. Nevertheless, an unforeseen side-effect emerged upon further examination. To probe more deeply interviewees' intrinsic perception of mathematical ways of thinking, interviewees were asked to propose conditions for good mathematical methods; all but one cited *simple*, *precise*, and *easily understood* as essential components. In terms of the criteria, they claimed Archimedes' idiosyncratic ideas, though impressive, may not have been good, because of their instinctive inconceivability. On the contrary, they preferred the Chinese mathematicians' concrete approaches that were more accessible. It was assumed, by exposure to these contrary approaches, participants would recognize the necessity of abstraction as well as the obviousness of concrete operations. This outcome revealed one restriction of the present study with an attempt to promote the participants' disposition of appreciating the esthetics of mathematical thinking. This restriction could be due to cognitive obstacles or cultural barriers. Follow-up investigation or cross-cultural study may help to resolve the question.

Post-instruction responses indicated that the majority of the participants considered mathematical knowledge as increasing gradually and tied to human demand. This notion may have referred to the handouts depicting the needs and origins of calculus concepts in history, such as the construction of tangent lines, logarithmic functions. Yet subjects also proposed that mathematical knowledge is potentially unsound, a seemingly contradictory perspective. Upon additional inquiry, all interviewees cited a historical impediment of understanding the concept of infinite series as supportive. As reported, the participants were to a great extent astonished by Euler and other mathematicians' mistakes on summing up infinite series by unconsciously using algebraic rules. Recognition of the latent fallibility of mathematical thinking might have urged the participants to reevaluate their widespread and rooted image that mathematics is the most infallible scientific discipline. Interviewees instead contended the validity of mathematical knowledge was time-dependent, subject to criteria at the time. It appears advanced realization regarding the processes of mathematical thinking also increased individuals' insight into epistemological facets of mathematics.

Limitations of the Study

As an exploratory investigation, the present study was limited in several respects. First of all, extrapolation of the findings in this study to subjects beyond its participants is speculative. The students were Taiwanese freshman engineering majors in a public technological college, not representative of any larger population of undergraduates. Extending suggestions made in the present study, even to a limited population of Taiwanese college freshmen, could be problematic because, in Taiwan, technological college learning experiences, especially in mathematics, are quite dissimilar from those enrolled in academic colleges. Thus, the two counterparts may hold dissimilar visions of concerned issues. As a study examining individual views on mathematical thinking, the effect of such a distinction should be treated cautiously.

Second, the analysis was initially based on all students' written data (mathematics biographies, questionnaires, in-class reflection papers) to profile and generate patterns of their views. To validate written responses and investigate further, a second round of analysis was made by referring to oral data obtained from nine randomly selected interviewees. To avoid misinterpretation caused by limited data source, the investigations therefore heavily relied on interview transcripts. Though there was no evidence to suggest the 20% random sample was different from all other participants in any particular respect, it is still improper to claim the interviewees' accounts can be unquestionably applied to non-interviewees. While larger involvement may shed more light on participants' conceptions, by taking practical possibility into account, the 20% sample should be a moderate sample approximately representing the whole.

Third, the present study primarily probed the individual's views of mathematical thinking. Given the intimate interrelationship between mathematical thinking and knowledge, participants' notions regarding holistic mathematics were examined as well to sketch a comprehensive picture of respondents' internal thought. Note that factors contributing to shape one's conceptions by no means were clear-cut; visions about other relevant subjects affected the formation of their views of mathematical thought. Students' perspectives on other scientific disciplines were not formally tabulated, but their images of science and art were treated as supplementary evidence. Scope and strength of interpreting students' accounts were thus restricted.

Fourth, it should be reminded that an alteration of an individual's views or beliefs in such a limited period of time is unrealistic. Prior to this study, participating students had been experienced at least 12 years of exam-driven instruction and certain beliefs of doing mathematics had formed. The effort made in this study at most challenged their existing thinking and motivated them to accommodate alternative thoughts in learning mathematics. A permanent or a radical change in students' views on mathematical thinking is unattainable without subsequent examination.

Last and probably most critical, given the dual role the researcher/instructor played in the present study, unforeseen biases were inevitable. As an instructor, on the basis of personal belief and interest, he designed and implemented the problem-based curriculum throughout. Meanwhile, as a researcher, he assessed the outcome of the designed curriculum, based on identical belief and interest. The seemingly conflicting positions might have hindered the researcher/instructor's insight into the key issue and even lead him to a state of illusion, making a wholly impartial report in the present study nearly unattainable. Nonetheless, this drawback was to a certain extent assuaged by avoiding excessive value judgment and reporting findings as neutral as possible. Moreover, the researcher/instructor's researcher's diary and tape-reviewers' checklist about practical teaching indicated from time to time the designed curriculum was not fully implemented, especially at the start. The reported outcome as such might not totally reflect the expected consequences of designed curriculum. Strengths and weaknesses addressed here thus cannot directly be inferred to relevant investigation.

Implications of the Study

As an exploratory investigation, the present problem-based study may hold implications for developing college students' views on mathematical thinking. The following discussions mainly center on several issues, including past learning experience and pre-instruction views, the role problem-based courses play in improving students' notions about mathematics in general and mathematical thinking in particular, and the historical approach utilized in this problem-based course.

Pre-instruction data concurrently indicated participants' conceptions regarding mathematical knowledge and thinking mostly reflected earlier learning. Despite some expression of a certain degree of dissatisfaction about prior mathematical practice, most participants were less apt to resist extant inadequate perspectives shaped by prior experience. The existing visions may have imposed an effect on a subsequent mathematical career in general and this problem-based course in particular.

"Experience modifies human beliefs… To make the best possible use of experience is one of the great human tasks" (Polya, 1954, p. 3). This noteworthy interconnection between preceding incidents in mathematics and pre-instruction views on mathematical ideas justifies using a problem-based course in building on individual concepts of mathematical thinking. Problem-based learning confronts students with problematic, ill-structured situations and urges them to identify required information for achieving a solution. Such a course may enable learners to question extant beliefs and build their own mental network of experiences via personal association or interpretation. Findings in the present study suggested participants tended to derive their own conclusions regarding concerned issues. Factors contributing to post-instruction accounts were neither straightforward nor linear. As indicated, pre-instruction views functioned subtly in developing post-instruction visions; students holding analogous notions may end with unparallel perspectives. Individual cognitive schemes seemingly come into play in the midst of forming new conceptual framework. It was found that the participants who had formerly depicted considerations plainly (regardless of adequacy) would likely demonstrate more thoughtful accounts afterward. On the other hand, those initially giving short, vague responses as a rule exhibited minor change in their perceptions subsequently. It is argued here that the degree of *integratedness* and *reflectiveness* of an individual's cognitive scheme may exert a significant influence on accommodating and assimilating outward messages.

While challenging in nature, weekly problems used in the present study not only positioned students in a problematic situation but also provided opportunities to take varied perspectives. Findings indicated, via solving problems, students acquired thoughtful insight into the intrinsic essence of mathematical thinking, yet an appreciation of personal perseverance in doing mathematics was not evident in their responses. This undesired outcome might relate to two aforementioned critical components: *problem design* and *problem implementation*, interrelated processes in a problem-based course that balanced needs of students and curriculum (Torp & Sage, 1998).

In terms of problem design, though several weekly problems had been used in a pilot study and found not totally inaccessible to college freshmen, participants likely considered non-routine problems as demanding or ill-defined. As such, individuals tended to identify with different formulations depending on personal interpretation, interest, and involvement. Namely, on the basis of past experience in solving routine problems, some participants saw resolution of problems as practically unfeasible and became reluctant to engage in time-consuming problem-solving activities. Another critical issue regarding the problem design was use of historical problems, which will be discussed later.

As for the implementation of the problems, one barrier was that the participants were used to viewing the purpose of solving a problem merely as seeking answers. With the final answer as the ultimate concern, any expectation of recognizing the significance of persistence was impracticable. For productively promoting perseverance in doing mathematics and eliciting thought, developing the students' sense of "the solution is more than the answer, just as the problem is more than the question" (Lampert, 1990, p. 40) is required. Torp and Sage (1998) advocated problem-based learning and listed five benefits (increases motivation, makes learning relevant to real world, promotes higher-order thinking, encourages learning how to learn, and requires authenticity). However, these benefits may not have emerged without paying attention to the context of implementation. In a problem-based course aiming to improve individuals' conceptions, the context of problem implementation is far more fundamental. For instance, criteria of acceptability of solution must be clearly understood by students; assessment should reflect requirement. It is appropriate to say that the extent to which a problem-based course succeeds is subject to the degree to which its context is established. The idea of using historical problems in mathematics teaching has received increasing attention among relevant scholars (Avital, 1995; Barbin, 1996; Ernest, 1998; Furinghetti, 1997; Horng, 2000; Katz, 1997; Rickey, 1995; Siu, 1995a, 1995b; Swetz, 1995a, 1995b). In contrast to telling mathematical stories to draw interest and improve attitudes (both are merely related to the affective domain), using historical problems in the classroom is an advance in the ability to benefit not only students' affective domain but also their cognitive one. Mathematical concepts have continually evolved and been revised. Wisdom behind great endeavors may provide insight into mathematical thought. As Ernest (1998) indicated, "Mathematicians in history struggled to create mathematical processes and strategies which are still valuable in learning and doing mathematics" (p. 25). The outlook theoretically justifies the appealing idea of incorporating history into mathematics teaching; findings of this study suggest negative as well as positive effects of using history in mathematics instruction. Being aware of ancient great figures' mathematical approaches, participants fostered an admiring disposition toward mathematicians' originality and broadened their vision of changeability and abundance in mathematical thinking. In the meantime, they were also apprehensive about the potential uncertainty of mathematical thinking from their recognition of well-known mathematicians' *obvious mistakes* (from today's viewpoint). In this manner, posing historical problems to students not only may *problematize* but also *humanize* mathematical concepts. Philippou and Christou (1998), exploring the extent to which mathematical history alters prospective teachers' attitudes or views of mathematics, reported a pre-service teacher's claim similar to the findings in this study:

> History of mathematics provided me with a variety of interesting, new, experience… The course showed me that *mathematics is, at least sometimes, a human activity. I felt more confident when I realized that even great mathematicians did mistakes as I frequently do* [italics added] (p. 202).

Despite afore-cited inspiring outcomes, it should be noted that participants' responses also uncovered another perspective. On the basis of interviewees' claim, it appeared mathematicians' ingenious thinking was barely reachable in these college freshmen's eyes; the ideas are inspiring but practically inapplicable. To them, the ease of use and intuitive understanding are essential conditions for good mathematical methods. They tended to distance themselves from those mathematics masters, whose subtle thought is distant and unreachable. Attempts to empower students to recognize intellectual delicacy of mathematical thinking seemingly did not fulfill its aim. This defect may be an inevitable restriction of historical material, learner-dependent issue, or even instructional challenge. Additional investigation may shed light on this doubt.

Recommendations for Future Research

The importance of students' views of mathematics and learning mathematics has been documented, yet a study of the interrelationships between a problem-based course and subjects' views on mathematical thinking are rare. The present study was intended to function as an initiator bringing forth relevant issues and drawing researchers' attention to this respect. The recommendations are based on critical concerns and findings.

For achieving a more precise knowledge about the influence of a problem-based course on learners' views of mathematical thinking, an experimental design involving control groups is suggested. Taking a purposeful sample (e.g., students exhibiting significant or insignificant differences in their views) into account may be shed more light on the issue of interest. Despite several optimistic outcomes reported, findings in the present exploratory study cannot be interpreted from a causal-effect angle. Since a control group was lacking and the dual character of the researcher/instructor, potential biases were inescapable. In addition to using a control group, it is also advised that, if possible, the researcher/instructor' dual role should be avoided—i.e., the researcher is better as an objective observer monitoring progress rather than an actual executor—but this expectation may not be reached without a well-designed curriculum including appropriate guidance and prescription of instruction.

This concern elicits a second important direction, the curriculum. Within the framework of problem-based learning, as reported, the curriculum in the present study mostly followed a historical approach. In spite of the use of history in teaching mathematics as long being promulgated, experimental and systematical examination of its effect is sparse. The present investigation may also be a touchstone of this appeal. As discussed earlier, integrating historical material into the course produced inspiring outcomes, along with undesirable side-effects. Researcher must clarify whether this by-product is some inevitable restriction or an improvable phenomenon. Other than establishing the curriculum on a historical ground, a problem-based course might entail a plethora of approaches and generate diverse consequences. In what ways and to what extent assorted curricula may exert influence on learners' conceptions merits further investigation.

The third research issue is the employment of a questionnaire. Numerous instruments have been developed and conducted to survey students' beliefs about mathematics but, to the best of the researcher's knowledge, a single questionnaire exclusively aimed to probing college students' views of mathematical thinking is nil. The open-ended type used in the present study was developed in reference to relevant instruments and was used in a pilot study. However, as in several cases, the instrument was still incapable of successfully and satisfactorily eliciting respondents' internal thought regarding concerned issues. For better sketching of respondents' conceptual framework, a revision of the current instrument is mandatory.

Fourth, the present study found not only that some participants' notions of mathematical thinking was enriched but also that their views of mathematical knowledge became more thoughtful. This result seemingly suggests that individual epistemological views regarding mathematics as a whole can be somewhat improved by a course of this kind. Given those inextricable relationships between mathematical thinking and knowledge, it is thus prudent to increase the scope of investigation to the interrelationship between a problem-based, historical approach course and students' epistemological views of mathematics. The extension may provide alternative angles that delineate students' ideas and thus shed more light on their thoughts in a more holistic point of view.

Lastly, as indicated in the first chapter, the merit for studying and improving students' views of mathematical thinking is built upon a potential influence of students' inner conceptions on their learning behavior in mathematics. Though the present study did not aim to investigate the effect of this problem-based course on students' thinking skills, several instances suggested that students were able to develop their own strategies to solve nonroutine problems. For instance, when asked to derive the area of a circle without using formulas, one student proposed an idea which may explain Archimedes' approach (equating the area of a circle to a right-angle triangle). He regarded the area of a circle as a combination of infinitely many concentric circles and then straightened the circumferences of all concentric circles to form a right-angle triangle by piling them up, as shown in Figure 1.

Figure 1
One student's idea of converting the area of a circle to the area of a right triangle

Consequently, one critical issue for future research is to consider in what ways and to what extent students, whose views on mathematical thinking are enriched, alter their learning modes in subsequent mathematical courses. If they do adjust strategies to accommodate their shift in view point, the outcome justifies the purpose of this line of study and succeeding effort is required. Contrarily, if expected effects do not appear, follow-up investigation should identify hidden causes and in turn produce feedback to the relevant research. Though some researchers see beliefs as major determinants and guides of one's behavior (e.g., Brown & Cooney, 1982; Harvey, 1986), not all researchers endorse the view that beliefs offer greater insight into human behavior than other construct, such as knowledge, as Pajares (1992) indicated. The aforementioned cyclic manner expectantly may enrich the knowledge of the perplexing impact of individuals' internal conceptions on their external performance in doing and thinking mathematics.

Developing students' mathematical thinking ability is one of the major goals of mathematics education (NCTM, 1991). However, with the potential effect of an individual's own beliefs, this goal cannot be reached without a moderate understanding of the nature of mathematical thinking. Any theory of the psychology of mathematical thinking must be studied in the wider context of human mental and cultural activity (Tall, 1991). Thus, as no one absolute way of thinking about mathematics exists, there are alternative ways for investigating and improving one's views on mathematical thinking. The present study hopefully can bring forth valuable issues for public discussion to gain extensive attention to the study of this construct.

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Appendix A
Course Goals

I. Purpose: The purpose of this course is, through a problem-based fashion, to provide each student with opportunities to learn first semester of calculus including the concepts of definite integral, limits of function, differential and their applications. In addition to strengthening students' ability to manipulate mathematical concepts and formulas to solve relevant problems, a special emphasis is placed upon mathematical thinking through problem solving.

II. Goals: On completing the course, it is expected that each student will be able to:

1. develop a holistic understanding of calculus including the origins and growth of concepts of integral calculus, differential calculus, and limit.

2. apply learned knowledge to more open-ended problem-solving activities, such as finding the area of a circle without the formula as well as solve routine problems, such as finding the area of a bounded area by definite integral.

3. use high order thinking in doing mathematics through solving diverse levels of problems and comparing the alternative approaches to resolving a problem.

4. foster an appreciation of mathematical knowledge is a crystallization of human thoughts and its quasi-empirical, social construction aspects of thinking.

APPENDICES

Appendix B
Course Schedule

(This schedule presents the topics as they were taught.)

Week	Topics	Weekly Problems
1. Mon. Thu.	What is calculus? Alternative historical approaches to finding the area of a circle.	1. Finding the area of a circle
2. Mon. Thu.	The bounded area under a curve Riemann sum and definite integral.	2. Fibonacci sequence 3. Computing the sum of 1−1+1−1+1−1+…
3. Mon. Thu.	Sequences and their limits, Infinite series Infinite series (cont)	4. Euler's mistake on infinite series
4. Mon. Thu.	Tangents to a curve The limit of a function	5. Descartes and Fermat's approaches to finding the tangent line to a curve
5. Mon. Thu.	The limit of a function (cont.) Properties of limits Continuity	6. Compare the magnitude between 1 and 0.999…
6. Mon. Thu.	Techniques of differentiation Derivatives of trigonometric, exponential, and logarithmic functions	7. Napier's logarithm*
7. Mon. Thu.	Rates of change The chain rule	7. Napier's logarithm* (cont.)
8. Mon. Thu.	Implicit differentiation Related change and applications	8. Fermat's approach to find extreme values

9.	Midterm exam	
10. Mon. Thu.	Extreme values of a continuous function The Mean Value Theorem (MVT)	
11. Mon. Thu.	First-derivative test Concavity and the second-derivative test	9. The curve of "witch of Agnesi"
12. Mon. Thu.	Curve sketching Optimization in the physical sciences and engineering	
13. Mon. Thu.	L'Hopital rule Antidifferentiation, the Fundamental Theorem of Calculus (FTC)	10. The Tractrix problem*
14. Mon. Thu.	FTC (cont.)	10. The Tractrix problem (cont)*
15. Mon. Thu.	Integration by substitution	
16. Mon. Thu.	Area between two curves Finding the volume of a solid	11. Finding the volume of a sphere inscribed in a cylinder
17. Mon. Thu.	Volume by disks and washers Volume by shells	11. Finding the volume of a sphere inscribed in a cylinder
18.	Final exam	12. Finding the volume of a sphere

* The problem was solved in cooperative group activity.

Appendix C

Scope and Sequence

(The scope and sequence present the content as they were taught.)

Scopes Titles of topics	Sequences List of subtopics
What is calculus?	The origins of calculus; the area of a bounded region; the instant velocity; the tangent to a curve; Zeno's paradox; intuitive concept of limit (handout: the story of calculus)
Alternative approaches to finding the area of a circle	Archimedes, Liu Hui, and Seki's methods (handout: the story of pi)
The bounded area under a curve	Calvalieri's (Zu's) indivisible principle; infinitesimal; area as a limit of sum
Riemann sum and definite integral	Riemann sum; definite integral; area as an integral; properties of the definite integral; distance as an integral
Sequences and their limits	Sequences; Fibonacci sequences; the limit of a sequences; bounded, monotonic sequences (handout: Fibonacci number and golden ratio)
Infinite series	Definition of infinite series; historical obstacles of infinite series; general properties of infinite series (handout: historical obstacles of infinite series)
Tangents to a curve	The rise of the concept of a tangent line; Descartes and Fermat's approaches to finding the tangent line to a curve (handout: Barrow, Descartes, and Fermat's methods of finding the tangent)
The limit of a function	The intuitive notion of limit; one-sided limit; formal definition of limit (handout: the idea of infinite small)
Properties of limits	Computations with limits; using algebra to find limits; special limits involving sine and cosine

Continuity	The intuitive notion of continuity; definition and properties of continuity; the relationship between derivative and continuity; intermediate value theorem
Techniques of differentiation	Derivative of a constant and power function; basic rules for finding a derivative; higher derivative
Derivatives of trigonometric, exponential, and logarithmic functions	Derivative of trigonometric function; definition and derivative of exponential; logarithmic functions; Napier's logarithm; Euler's derivation of *e* (handout: Napier's logarithm; story of *e*)
Rates of change	Rate of change; average and instant rate of change; relative rates of change
The chain rule	Introduction to the chain rule; extended derivative formulas; justification of chain rule
Implicit differentiation	General procedure for implicit differentiation; derivative formula for the inverse trigonometric function; logarithmic differentiation
Related rate and applications	Applications of related rate
Extreme values of a continuous function	Extreme value theorem; relative and absolute extrema; optimization; max-min methods by Fermat and Leibniz (handout: max-min methods by Fermat and Leibniz)
Mean value theorem	Mean Value Theorem (Cauchy), Rolle's Theorem
First-derivative test	Increasing and decreasing functions; the first-derivative test
Concavity and the second-derivative test	Concavity; inflection points; second-derivative test
Curve sketching	Curve sketching by the first-derivative test and second-derivative test

Optimization in the physical sciences and engineering	Optimization procedure; Fermat's principle of optics, Snell's law (handout: Fermat's principle of optics and Snell's law)
L'Hôpital rule	Indeterminate form; L'Hôpital rule; Bernoulli families
Antidifferentiation, The Fundamental Theorem of Calculus (FTC)	Barrow, Newton, and Leibniz's approaches to FTC; antiderivative of a function; indefinite integral; the first and second FTC (handout: Barrow, Newton, and Leibniz's approaches to FTC)
Integration by substitution	Newton's use of integration by substitution; substitution with indefinite and definite integral
Area between two curves	Calvalieri's (Zu's) indivisible principle; area by vertical and horizontal strips
Finding the volume of a solid	Method of cross section; volume of double vault (handout: Archimedes and Liu Hui's approaches to finding the volume of a solid)
Volume by disks and washers	Archimedes' method for finding the volume of paraboloid, volume of revolution; disk method, washer method
Volume by shells	Method of cylinder shells

Appendix D
Weekly Problems

1. (Week 1) Design a method showing that the area of a circle with radius r is πr^2.

2. (Week 2) Leonardo de Pisa, also was known as Fibonacci, was one of the best mathematicians of the Middle Ages. His book, *Liber Abaci*, published in 1202, introduced the famous rabbit problem:

> The first month after birth, the rabbits are adolescents and produce no offspring. However, beginning with the second month, the rabbits are adults, and each pair produces a pair of offspring every month. The sequence of number describing the number of rabbits is called the *Fibonacci sequence*.

(a) Please fill in the table below the number of pairs of rabbits in the 5th, 6th, and 7th month.

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th	11th	12th
1	1	2	3								

(b) Can you derive the number of pairs of rabbits in the 12th month?

(c) Let a_n denote the number of pairs of rabbits at the end of nth months, a_n=?

(d) Calculate the value of a_{n+1}/a_n. Do you discover any pattern?

3. (Week 2) Answer the following two questions.

(a) Calculate the value of the infinite series 1 – 1 + 1 –1 + 1 –1 + 1…

(b) What follows are three different approaches to deriving the sum of "1– 1 + 1 – 1 + 1 –1 + 1…." However, they reach different answers. Which one is the correct one? Why? Defend your answer.

4. (Week 3) Leonard Euler was the greatest mathematician of the 18th century. He derived several sums of infinite series by his excellent algebraic computation ability. Nonetheless, he once derived an odd result as shown below:

Since $x+x^2+x^3+\ldots+x^n+\ldots=\frac{x}{1-x}$ and $1+\frac{1}{x}+\frac{1}{x^2}+\ldots+\frac{1}{x^n}+\ldots=\frac{x}{x-1}$

Summing up the two series, we then have

$$\ldots+\frac{1}{x^2}+\frac{1}{x}+1+x+x^2+\ldots=\frac{x}{1-x}+\frac{x}{x-1}=\frac{x}{1-x}-\frac{x}{1-x}=0\ ,$$

If x is positive, the above result shows an infinite sum of positive numbers equal to 0. How could this happen? Can you identify the mistake that Euler made?

5. (Week 4) Please find the slope of the tangent of the curve $f(x) = x^2$ at the point (1,1) by using Fermat, Descartes, and Barrow's approaches, respectively.

6. (Week 5) Compare the magnitude between 1 and .9999.... Which one is larger? Why?

7. (Week 6, 7) John Napier (1550-1617), a Scotsman, is credited with being the inventor of logarithm. His idea can be illustrated as follows: Let a line segment AB be of fixed length and the line DE be of infinite extent. Consider a point C to move on AB with a velocity numerically equal to the distance CB and a point F to move on DE with a constant velocity equal to the initial velocity of C, as in Figure 1. Napier defined the logarithm of CB to be the length of DF. Namely, let DF = y *and* CB = x , then

$$y = \mathrm{n}og\, x \ (\mathrm{n}og\, x = \mathrm{Nap}\ log\, x)$$

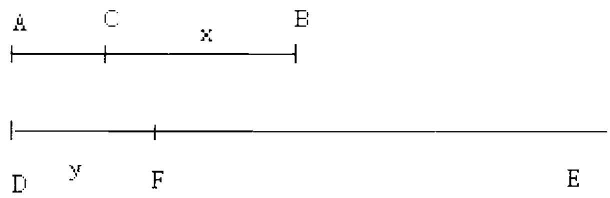

Figure 2 Napier's logarithm

According to the above definition, it can be seen that the value of y increases as x decreases. Please show that (a) the value of x decreases in a geometric progression while the value of y increases in an arithmetic progression; (b) $y = nog\, x^{a} = a\, nog$

x。

8. (Week 8) Fermat's method of finding the extreme value of functions was most similar to the modern approach. Please use his idea to find the absolute value of the function $f(x) = 3x^2 - x^2$ on [0.3].

9. (Week 11) Let $y = \dfrac{a^3}{x^2 + a^2}$

(a) Derive critical points, extreme values, inflection points, and asymptotes of y.

(b) Sketching the graph of y on the basis of results in (a).

(c) There is a "dynamic" method for sketching the graph of y. Can you derive it?

10. (Week 13, 14) When Leibniz was inventing the calculus in Paris in 1676, Claude Perrault placed his watch in the middle of a table and pulled the end of its watchchain along with the edge of the table. Perrault asked: What is the shape of the curve traced by the watch?

(a) Please sketch this curve.

(b) How do you derive the equation of this curve? Show your work in detail.

11. (Week 16, 17) Among his mathematical discoveries (or inventions), Archimedes was proudest of the finding that the ratio between a circular cylinder's volume and its inscribed sphere and the ratio between the surface area of a circular

cylinder and its inscribed sphere are both 3:2. This beautiful relationship was carved on his tombstone. To comprehend his enjoyment further, please show that the ratio between the volume of a circular cylinder and its inscribed sphere is 3:2 without applying formulas.

12. (Week 18) Derive the volume of a sphere without using Zu Chongzhi and Archimedes' methods.

Appendix E
Audiotape Reviewer's Checklist

Directions

I. Purpose of the checklist

The checklist asks you to describe various aspects of my teaching recorded in the selected audiotape. Your answers will enable me to keep the instruction and study on the right track. There are no right or wrong answers. Please feel free to let me know your opinion.

II. How to answer each item

There will be 10 questions below. For each question, circle only one number corresponding to your answer. For example

I asked students to think the importance of mathematical concepts.	Always	Often	Sometimes	Seldom	Never
	5	4	3	2	1

If you think that I **always** asked the students to think the importance of mathematical concepts, please circle '5.' Contrarily, if you think that I **never** asked the students to think the importance of mathematical concepts, please circle '1.' You may also choose the number 2, 3, 4 if one of these seems like a more appropriate answer.

Body of the checklist

Basic information of the tape

Date:_______________ Topic:____

Problematizing mathematical concepts	Always	Often	Sometimes	Seldom	Never
1. I asked the students to think why the topics are important.	5	4	3	2	1
2. I asked the students to think the key concepts of the problem.	5	4	3	2	1
3. I encouraged the students to propose the plausible approaches for solving the problem.	5	4	3	2	1

Metacognitive teaching	Always	Often	Sometimes	Seldom	Never
4. I acted as a novice working problems from scratch.	5	4	3	2	1
5. I asked the students to evaluate the possibility and difficulty of approaches.	5	4	3	2	1
6. The students witnessed and experienced the cost of an incorrect approach.	5	4	3	2	1

Interaction in the classroom	Always	Often	Sometimes	Seldom	Never
7. I encouraged the students to make plausible guessing.	5	4	3	2	1
8. The students were invited to share their ideas with class on board.	5	4	3	2	1
9. I encouraged the students to question and challenge other students' presentations.	5	4	3	2	1
10. I established a student to student interaction environment.	5	4	3	2	1

Appendix F
Mathematics Biography

Please describe your past experiences in learning mathematics in 300 words, including any significant events or people influencing your dispositions or attitudes toward learning mathematics, how important mathematics is to you, and how you evaluate your own capability and performance in doing mathematics.

Appendix G
The Pilot Study Questionnaire

(The following questions present as they were used in the pilot study.)

1. What is your instant strategy when facing unfamiliar mathematics problems?

2. How do you proceed when stuck on a mathematical problem?

3. Some mathematics education researchers argue that students can be guided to discover mathematical properties on their own with little instruction. Nevertheless, still others consider this approach suitable only for students good at mathematics. What is your viewpoint about this?

4. In your opinion, what is the best way for learning mathematics?

5. How do you correctly judge your approach when doing mathematics?

6. In your understanding, what is mathematical thinking? Please explain your answer with examples.

7. Some hold that solving mathematical problems is a thinking activity involving personal creativity; others argue that getting correct answer requires following predetermined, known procedures. What is your opinion about this? Why? Please defend your answer with examples.

8. It is widely believed that one can correctly answer mathematical problems only when appropriate understanding is reached. Is it possible for you to do a problem right, yet without adequate understanding? Why?

9. In your understanding and imagination, is there any difference between a mathematician' way of thinking and a layperson's?

10. In your understanding and imagination, how do mathematicians conduct their research? Do they solve problems alone or discuss with others? Which way do you think is better? Why?

11. How much time do you spend obtaining the correct answer if you understand a mathematics problem rightly and are confident of solving it? How soon would you give up when you think yourself incapable?

12. What are necessary conditions for a person to be mathematically capable?

13. Some consider the purpose of mathematical knowledge as pursuing absolute truth. Nevertheless, others argue that mathematical knowledge is a symbolic operation system following particular rules. Still others see mathematical knowledge as a product constructed by mathematicians. What is your viewpoint about this?

14. People usually distance themselves from mathematics because following fixed rules is required for producing mathematical knowledge. Nonetheless, some hold that the establishment of mathematical knowledge is a highly creative activity, the most fantastic part. What is your viewpoint about this?

15. Mathematics is usually seen as the mother (or servant) of science because it can afford the nutrition for development of science. Some even argue that mathematics is a perfect science. Its theories, unlike other natural science, are not superseded by new discovery—i.e., mathematical truths are infallible once its theories are established. What is your viewpoint about this? In your opinion, is there any difference between the nature of mathematics and that of science?

16. Some consider mathematical knowledge as "discovered" (existing in Nature). Nevertheless, still others hold that mathematical knowledge is "invented" (a tool created by humans to describe Nature). What is your opinion about this?

Appendix H
The Study Questionnaire

(The following questions present as they were used in the present study.)

1. In your understanding, what is mathematical thinking? Please explain your answer with examples.

2. When you are stuck on an unfamiliar mathematics problem, what is your instant reaction to and strategy for this?

3. In your understanding and imagination, how do mathematicians think while solving a problem? Is there any difference between a mathematician's way of thinking and a layperson's?

4. Some hold that solving mathematical problems is a thinking activity involving personal creativity; others argue that getting correct answers requires following predetermined, known procedures. What is your opinion about this? Why? Please defend your answer with examples.

5. In your opinion, what is mathematics? What makes mathematics differ from other disciplines (e.g., science, art)?

6. In your opinion, how does mathematical knowledge develop? Does the development of mathematical knowledge follow any rule? Please defend your answer with examples.

Appendix I
In-Class Reflection Reports

1. (Week 2) Of the alternative approaches proposed by Liu Hui, Seki Kowa, and Archimedes to deriving the area of a circle, in your point of view, which is the best? Do you think Liu Hui's idea is a rigorous proof?

2. (Week 5) What is your thinking about the mistakes made by past mathematicians' approaches to deriving the sum of infinite series (e.g., 1-1+1-1+1-1+1... and 1 + 2 + 4 + 8 + 16 + 32 +...)?

3. (Week 10) What is your thinking about the deficiency in Fermat's method for finding the tangent line to a curve?

4. (Week 16) Please compare and contrast ancient Chinese (Liu Hui and Zu Chongzhi) and Archimedes' approaches to deriving the area of a circle and the volume of a sphere to sketch the distinct features of mathematical thoughts between the both sites.

Appendix J
Sample Interview Protocols

1. What do you mean by ......?
2. Could you be more specific?
3. In your opinion, is mathematical thinking equivalent to thinking on solving mathematics problems?
4. Do you think mathematicians are smarter than laypersons?
5. Are mathematicians born to be good at mathematics?
6. What makes a good problem solver?
7. Are you a good problem solver? Why?
8. What are important factors in mathematical thinking?
9. Is the development of mathematical knowledge logical? Why? Could you give me an example?
10. In your imagination, how mathematicians generate ideas?
11. In your opinion, is mathematical knowledge infallible? Why?
12. Have your views of mathematical thinking changed during the course? In what aspect?
13. Why did you change your responses?

Appendix K
Time Schedule for Data Collection

Month	Week	Data of Collection
2001, September	1	Pre-instruction questionnaire Mathematics biography
	2	Pre-instruction interview transcripts (20% random sample) Weekly problem 1; In-class reflection report I
	3	Pre-instruction interview transcripts (20% random sample) Weekly problem 2 and 3
	4	Weekly problem 4
October	5	Weekly problem 5; In-class reflection report II
	6	Weekly problem 6
	7	
	8	Weekly problem 7
November	9	(Mid term)
	10	Weekly problem 8; In-class reflection report III
	11	
	12	
	13	Weekly problem 9
December	14	
	15	Weekly problem 10
	16	Post-instruction questionnaire; In-class reflection report IV
	17	Post-instruction interview transcripts (20% random sample) Weekly problem 11
January	18	Post-instruction interview transcripts (20% random sample); Weekly problem 12; Final

Appendix L
Participants' Data Source Information

Code	Sex	Mathematics Biography	Pre-Questionnaire	Post-Questionnaire
S01	M	✔	✔	✔
S02	M	✔	✔	✖
S03	M	✖	✔	✔
S04	M	✔	✔	✔
S05	M	✔	✔	✔
S06	M	✔	✔	✔
S07	M	✔	✔	✔
S08	F	✔	✔	✔
S09	M	✔	✔	✔
S10	M	✔	✔	✔
S11	M	✔	✔	✔
S12	M	✔	✔	✔
S13	M	✔	✔	✔
S14	F	✔	✔	✔
S15	M	✖	✔	✔
S16	M	✔	✔	✔
S17	M	✖	✔	✔
S18	M	✔	✔	✔
S19	M	✖	✔	✔
S20	M	✔	✔	✔
S21	M	✔	✔	✔

S22	M	✗	✓	✓
S23	M	✓	✓	✓
S24	M	✗	✓	✓
S25	M	✓	✓	✓
S26	M	✓	✓	✓
S27	M	✓	✓	✓
S28	F	✓	✓	✓
S29	M	✗	✓	✓
S30	M	✓	✓	✓
S31	M	✗	✓	✓
S32	M	✗	✓	✓
S33	M	✗	✓	✓
S34	M	✓	✓	✓
S35	M	✓	✓	✓
RS01	M	✓	✓	✓
RS02	M	✓	✓	✓
RS03	M	✓	✓	✓
RS04	M	✓	✓	✓
RS05	M	✓	✓	✓
RS06	F	✓	✓	✓
RS07	F	✓	✓	✓
RS08	M	✓	✓	✓
RS09	F	✓	✓	✓

*The mark " ✓ "means the data source was collected and "✗" represents the data source was not submitted.