The main goal of the study was to investigate high school advanced placement calculus teachers' subject matter and pedagogical perceptions by examining the following questions: What are the teachers' perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus? Additionally, the sampling technique used shed some light on the question: Are these teachers' perceptions associated with their participation in a calculus reform project focused on staff development? A multi-case study approach involving detailed examination of six teachers (three had participated in a calculus reform project and three had not participated in any calculus reform project) was used. The data collected and analyzed included questionnaires, interviews, observational fieldnotes, videotapes of classroom instruction, journals, and written instructional documents. Upon completion of the data collection and analysis, detailed teacher profiles were created with respect to the questions above. The results of this study were then generated by searching for similarities and differences across the entire sample as well as comparing and contrasting the group of project teachers and the independent teachers.

The teachers in this study perceived calculus as a linearly ordered set of topics in which the concept of limit formed the backbone for appreciating and understanding all other calculus topics. The teachers felt the intuitive understanding of limits was essential to further understanding of calculus. Nevertheless, little classtime was devoted to
developing an intuitive understanding. Furthermore, little emphasis was given to drawing connections between limits and subsequent calculus topics. The independent teachers devoted considerable time to discussing formal epsilon-delta definition and arguments. The complex relationship between teachers’ perceptions and classroom practice appeared to be affected by the significant influence of the teachers’ goals of preparing students for the advanced placement exam and college calculus and the authority given to the calculus textbook. Differences between the group of independent teachers and the group of project teachers were found related to the following factors: (a) commitment to the textbook, (b) planning, (c) use of multiple representations, (d) attitude toward graphing technology, (e) classroom atmosphere, (f) examinations, (g) appropriate level of mathematical rigor needed for teaching calculus, and (h) the stability of perceptions. These factors, however, were not fully attributed to participation in the calculus reform project.
Teachers' Perceptions of the Concept of Limit, the Role of Limits, and the Teaching of Limits in Advanced Placement Calculus

by

Linda M. Simonsen

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APPROVED:

Redacted for Privacy

Major Professor, representing Science and Mathematics Education

Redacted for Privacy

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Dean of Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

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Linda M. Simonsen, Author
First, I would like to thank the advanced placement calculus teachers who gave of their time and energy to participate in this study. I recognize that it is not easy to let a stranger into one’s mathematics classroom. I appreciate their trust in me and their support of educational research.

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# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>I THE PROBLEM</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>Statement of the Problem</td>
<td>4</td>
</tr>
<tr>
<td>Significance of the Study</td>
<td>7</td>
</tr>
<tr>
<td>II REVIEW OF THE LITERATURE</td>
<td>9</td>
</tr>
<tr>
<td>Introduction</td>
<td>9</td>
</tr>
<tr>
<td>Mathematics Teachers' Subject Matter and Pedagogical Perceptions</td>
<td>10</td>
</tr>
<tr>
<td>Development of Mathematics Teachers' Perceptions</td>
<td>11</td>
</tr>
<tr>
<td>Discussion of Development of Mathematics Teachers' Perceptions</td>
<td>19</td>
</tr>
<tr>
<td>Change in Mathematics Teachers' Perceptions</td>
<td>20</td>
</tr>
<tr>
<td>Change Due to Teacher Education Programs</td>
<td>22</td>
</tr>
<tr>
<td>Change Due to Experience</td>
<td>38</td>
</tr>
<tr>
<td>Change Due to Inservice, Workshops, and Special Programs</td>
<td>47</td>
</tr>
<tr>
<td>Discussion of Change in Mathematics Teachers' Perceptions</td>
<td>58</td>
</tr>
<tr>
<td>Students' Perceptions of Limits</td>
<td>59</td>
</tr>
<tr>
<td>Conclusion</td>
<td>63</td>
</tr>
<tr>
<td>III DESIGN AND METHOD</td>
<td>67</td>
</tr>
<tr>
<td>Subjects</td>
<td>67</td>
</tr>
<tr>
<td>Methods</td>
<td>69</td>
</tr>
<tr>
<td>Data Collection</td>
<td>70</td>
</tr>
<tr>
<td>Questionnaire</td>
<td>73</td>
</tr>
<tr>
<td>Interviews</td>
<td>74</td>
</tr>
<tr>
<td>Classroom Observations</td>
<td>76</td>
</tr>
<tr>
<td>Videotaped Classroom Instruction</td>
<td>77</td>
</tr>
<tr>
<td>Journals</td>
<td>77</td>
</tr>
<tr>
<td>Written Instructional Documents</td>
<td>78</td>
</tr>
<tr>
<td>The Researcher</td>
<td>78</td>
</tr>
<tr>
<td>Chapter</td>
<td>Page</td>
</tr>
<tr>
<td>---------</td>
<td>------</td>
</tr>
<tr>
<td>Data Analysis</td>
<td>79</td>
</tr>
<tr>
<td>IV RESULTS</td>
<td>82</td>
</tr>
<tr>
<td>Introduction</td>
<td>82</td>
</tr>
<tr>
<td>Individual Subject Profiles</td>
<td>87</td>
</tr>
<tr>
<td>Independent Teacher Profile, Ryan</td>
<td>87</td>
</tr>
<tr>
<td>Scholastic and Professional History</td>
<td>87</td>
</tr>
<tr>
<td>Portrait of the Calculus Classroom</td>
<td>88</td>
</tr>
<tr>
<td>Perceptions of Calculus and the Teaching of Calculus</td>
<td>91</td>
</tr>
<tr>
<td>Perceptions of the Concept of Limit</td>
<td>102</td>
</tr>
<tr>
<td>Perceptions of the Role of Limits in the Teaching of Calculus</td>
<td>103</td>
</tr>
<tr>
<td>The Teaching of Limits</td>
<td>105</td>
</tr>
<tr>
<td>Summary of Ryan</td>
<td>115</td>
</tr>
<tr>
<td>Independent Teacher Profile, Russell</td>
<td>118</td>
</tr>
<tr>
<td>Scholastic and Professional History</td>
<td>118</td>
</tr>
<tr>
<td>Portrait of the Calculus Classroom</td>
<td>120</td>
</tr>
<tr>
<td>Perceptions of Calculus and the Teaching of Calculus</td>
<td>124</td>
</tr>
<tr>
<td>Perceptions of the Concept of Limit</td>
<td>130</td>
</tr>
<tr>
<td>Perceptions of the Role of Limits in the Teaching of Calculus</td>
<td>131</td>
</tr>
<tr>
<td>The Teaching of Limits</td>
<td>136</td>
</tr>
<tr>
<td>Summary of Russell</td>
<td>148</td>
</tr>
<tr>
<td>Independent Teacher Profile, Richard</td>
<td>151</td>
</tr>
<tr>
<td>Scholastic and Professional History</td>
<td>151</td>
</tr>
<tr>
<td>Portrait of the Calculus Classroom</td>
<td>153</td>
</tr>
<tr>
<td>Perceptions of Calculus and the Teaching of Calculus</td>
<td>156</td>
</tr>
<tr>
<td>Perceptions of the Concept of Limit</td>
<td>162</td>
</tr>
<tr>
<td>Perceptions of the Role of Limits in the Teaching of Calculus</td>
<td>163</td>
</tr>
<tr>
<td>The Teaching of Limits</td>
<td>166</td>
</tr>
<tr>
<td>Summary of Richard</td>
<td>177</td>
</tr>
</tbody>
</table>
TABLE OF CONTENTS (Continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Summary of Calculus Teachers’ Perceptions of Limits and the</td>
<td></td>
</tr>
<tr>
<td>Correlation to Classroom Practice</td>
<td>271</td>
</tr>
<tr>
<td>Association Between Calculus Teachers’ Perceptions of Limits and</td>
<td></td>
</tr>
<tr>
<td>Participation in a Calculus Reform Project</td>
<td>272</td>
</tr>
<tr>
<td>Stability of Calculus Teachers’ Perceptions of Limits</td>
<td>278</td>
</tr>
<tr>
<td>V DISCUSSION AND CONCLUSIONS</td>
<td></td>
</tr>
<tr>
<td>Introduction</td>
<td>281</td>
</tr>
<tr>
<td>Summary and Discussion of Main Findings</td>
<td>282</td>
</tr>
<tr>
<td>Goals for the Teaching of Calculus</td>
<td>282</td>
</tr>
<tr>
<td>Influence of the Calculus Textbook</td>
<td>284</td>
</tr>
<tr>
<td>Calculus as a Linearly Ordered Subject</td>
<td>285</td>
</tr>
<tr>
<td>The Role of Limits in the Teaching of Calculus</td>
<td>286</td>
</tr>
<tr>
<td>The Concept of Limit in the Teaching of Calculus</td>
<td>288</td>
</tr>
<tr>
<td>Association Between Calculus Teachers’ Perceptions of Limits and</td>
<td></td>
</tr>
<tr>
<td>Participation in a Calculus Reform Project</td>
<td>292</td>
</tr>
<tr>
<td>Summary of Calculus Teachers’ Subject Matter and Pedagogical</td>
<td></td>
</tr>
<tr>
<td>Perceptions of Limits and the Correlation to Classroom Practice</td>
<td>299</td>
</tr>
<tr>
<td>Limitations of the Study</td>
<td>300</td>
</tr>
<tr>
<td>Implications and Recommendations for Mathematics Education</td>
<td>302</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>311</td>
</tr>
<tr>
<td>APPENDICES</td>
<td>318</td>
</tr>
<tr>
<td>Appendix A Discussion of Advanced Placement Calculus Examination</td>
<td>319</td>
</tr>
<tr>
<td>Appendix B Description of the Dick and Patton (1990) Textbook</td>
<td>327</td>
</tr>
<tr>
<td>Appendix C Advanced Placement Calculus Questionnaire: Part A</td>
<td>331</td>
</tr>
<tr>
<td>Appendix D Advanced Placement Calculus Questionnaire: Part B</td>
<td>332</td>
</tr>
<tr>
<td>Appendix E Description of the Leithold (1986) Textbook</td>
<td>334</td>
</tr>
<tr>
<td>Appendix F Description of the Larson and Hostetler (1986) Textbook</td>
<td>339</td>
</tr>
<tr>
<td>Appendix G Description of the Larson, Hostetler, and Edwards (1990)</td>
<td>344</td>
</tr>
<tr>
<td>Textbook</td>
<td></td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Ryan’s Recommendations for Success in Calculus</td>
<td>94</td>
</tr>
<tr>
<td>2.</td>
<td>Ryan’s Diagram of Calculus, Prior to Beginning of School</td>
<td>95</td>
</tr>
<tr>
<td>3.</td>
<td>Ryan’s Diagram of Calculus, Planning to Teach Limits Unit</td>
<td>97</td>
</tr>
<tr>
<td>4.</td>
<td>Ryan’s Diagram of Calculus, After Teaching Limits Unit</td>
<td>98</td>
</tr>
<tr>
<td>5.</td>
<td>Ryan’s Diagram of Calculus, End of Data Collection Period</td>
<td>99</td>
</tr>
<tr>
<td>6.</td>
<td>Ryan’s Graph Used for the Introduction to Limits</td>
<td>107</td>
</tr>
<tr>
<td>7.</td>
<td>Ryan’s Graph for $f(x) = (1/100)x + 4.96$ Before Discussion of Definition of Limit</td>
<td>111</td>
</tr>
<tr>
<td>8.</td>
<td>Ryan’s Graph for $f(x) = (1/100)x + 4.96$ After Discussion of Definition of Limit</td>
<td>113</td>
</tr>
<tr>
<td>9.</td>
<td>Russell’s Diagram of Calculus, Prior to Beginning of School</td>
<td>125</td>
</tr>
<tr>
<td>10.</td>
<td>Russell’s Diagram of Calculus, Planning to Teach Limits Unit</td>
<td>126</td>
</tr>
<tr>
<td>11.</td>
<td>Russell’s Diagram of Calculus, After Teaching Limits Unit</td>
<td>127</td>
</tr>
<tr>
<td>12.</td>
<td>Russell’s Diagram of Calculus, End of Data Collection Period</td>
<td>128</td>
</tr>
<tr>
<td>13.</td>
<td>Russell’s Drawing of Limit Definition of Derivative</td>
<td>132</td>
</tr>
<tr>
<td>15.</td>
<td>Russell’s Drawing of $f(x) = (x^2 - 4)/(x - 2)$</td>
<td>139</td>
</tr>
<tr>
<td>16.</td>
<td>Richard’s Diagram of Calculus, Prior to Beginning of School</td>
<td>158</td>
</tr>
<tr>
<td>17.</td>
<td>Richard’s Diagram of Calculus, End of Data Collection Period</td>
<td>159</td>
</tr>
<tr>
<td>18.</td>
<td>Richard’s Graph of the Derivative Definition</td>
<td>164</td>
</tr>
<tr>
<td>19.</td>
<td>Richard’s Graph of the Limit Definition</td>
<td>169</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>------------------------------------------------------------------------------</td>
<td>--------</td>
</tr>
<tr>
<td>20.</td>
<td>Trey’s Diagram of Calculus, Prior to Beginning of School</td>
<td>187</td>
</tr>
<tr>
<td>21.</td>
<td>Trey’s Diagram of Calculus, Planning to Teach Limits Unit</td>
<td>188</td>
</tr>
<tr>
<td>22.</td>
<td>Trey’s Diagram of Calculus, After Teaching Limits Unit</td>
<td>189</td>
</tr>
<tr>
<td>23.</td>
<td>Trey’s Diagram of Calculus, End of Data Collection Period</td>
<td>190</td>
</tr>
<tr>
<td>24.</td>
<td>Trey’s Graph Used for Introduction to Limits</td>
<td>198</td>
</tr>
<tr>
<td>25.</td>
<td>Trey’s Graph of the Formal Definition of Limit</td>
<td>202</td>
</tr>
<tr>
<td>26.</td>
<td>Trey’s Graph Used for the Introduction to Continuity</td>
<td>203</td>
</tr>
<tr>
<td>27.</td>
<td>Terry’s Diagram of Calculus, Prior to Beginning of School</td>
<td>217</td>
</tr>
<tr>
<td>28.</td>
<td>Terry’s Diagram of Calculus, Planning to Teach Limits Unit</td>
<td>218</td>
</tr>
<tr>
<td>29.</td>
<td>Terry’s Diagram of Calculus, After Teaching Limits Unit</td>
<td>219</td>
</tr>
<tr>
<td>30.</td>
<td>Terry’s Diagram of Calculus, End of Data Collection Period</td>
<td>220</td>
</tr>
<tr>
<td>31.</td>
<td>Terry’s Graph Used for Introduction to Limits</td>
<td>228</td>
</tr>
<tr>
<td>32.</td>
<td>Terry’s Graph of the Formal Definition of Limit</td>
<td>231</td>
</tr>
<tr>
<td>33.</td>
<td>Terry’s Graph Used for the Introduction to Continuity</td>
<td>234</td>
</tr>
<tr>
<td>34.</td>
<td>Tom’s Diagram of Calculus, Prior to Beginning of School</td>
<td>244</td>
</tr>
<tr>
<td>35.</td>
<td>Tom’s Diagram of Calculus, Planning to Teach Limits Unit</td>
<td>246</td>
</tr>
<tr>
<td>36.</td>
<td>Tom’s Diagram of Calculus, After Teaching Limits Unit</td>
<td>246</td>
</tr>
<tr>
<td>37.</td>
<td>Tom’s Diagram of Calculus, End of Data Collection Period</td>
<td>247</td>
</tr>
<tr>
<td>38.</td>
<td>Tom’s Graph Used for Introduction to Limits</td>
<td>253</td>
</tr>
</tbody>
</table>
CHAPTER I

THE PROBLEM

Introduction

Mathematics teachers commonly view their subject matter as a static, linearly-ordered, fixed body of knowledge (National Council of Teachers of Mathematics [NCTM], 1989; National Research Council [NRC], 1989). The traditional philosophy is that mathematics is best learned by memorizing facts, rules, and procedures and then applying these ideas to a set of exercises. The result of this instructional method is that the students of these teachers acquire an inadequate understanding of the meaning of mathematical processes and come to view mathematics as an accumulation of facts and pointless rules to be filed away or just used to pass a future test (Thompson, 1984). Additionally, the traditional mathematics educator views the role of the teacher as one who implements goals which are heavily influenced by a given section in the textbook (Leinhardt & Smith, 1985; Sullivan & Leder, 1992). Learning is then assessed through paper-and-pencil tests where ability to select or compute the correct answer is considered knowledge (Schram, Wilcox, Lanier, & Lappan, 1988).

This traditional view of teaching and learning mathematics has been challenged by current reflection and research in mathematics and mathematics education (Mathematical Sciences Education Board [MSEB], 1990; McKnight et al., 1987; NCTM, 1989; NRC, 1989). As a result, a significant change has been emphasized in the attitude of mathematics teachers toward mathematics subject matter and a new emphasis in teaching
methods is recommended. In particular, mathematics should be perceived by educators and their students as a dynamic, everyday activity of analyzing and describing the numerical and spatial aspects of our world. Viewing mathematics as a dynamic human activity means that one values *doing* mathematics over accumulating facts about mathematics (Schram et al., 1988). Traditional teacher-centered mathematics instruction is not conducive to the development of this dynamic type of mathematical knowledge. In the reform view, educators need to encourage students to *do* mathematics, to determine ways of making sense of mathematics, to develop procedures to solve new problems, and to build models to understand mathematical situations.

Leaders in mathematics education, including the NCTM (1989, 1991) and the NRC (1989), have called for a reorganization of the mathematics curriculum to focus on concept development and problem-solving. In particular, the recommendations to teachers include: (a) reducing the amount of time devoted to drill-and-practice of computational skills, (b) engaging students in challenging problem-solving situations, (c) creating a classroom atmosphere where questioning, exploration, reasoning, and justification are encouraged and expected, and (d) using the power of computing technology to free students from tedious computations, allowing them to concentrate on problem-solving processes (NCTM, 1989).

Implementation of a conceptually-based, problem-solving approach to mathematics instruction requires teachers to have a conceptual understanding of mathematics, to value conceptual understanding, and to know how to help students gain that understanding (Clark & Peterson, 1986; Romberg & Carpenter, 1986). Teachers need to be able to comprehend how various mathematical concepts interrelate with the larger field of mathematics and its applications. A change, therefore, is required in teachers’ perceptions and beliefs about what it means to know and do mathematics as well as in the way mathematics is organized and taught. For many teachers, the limitations of their knowledge about mathematics and teaching mathematics constrains their ability to teach
conceptually (Ball, 1990a; 1990b; Tirosh & Graeber, 1989). Thus, teacher-educators and researchers need to examine perceptions and beliefs that prospective or experienced teachers bring to teacher education programs, workshops, or other special programs focused on perceptions and beliefs consistent with the reform effort (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; McDiarmid, 1990; Schram, 1991; Wilcox, Schram, Lappan, & Lanier, 1991; Wood, Cobb, & Yackel, 1991).

Part of the call for change in mathematics education involves the reformation of calculus curricula and teaching. Calculus is possibly the most dominant mathematics class taught with respect to higher level educational and social importance. Hughes-Hallett and Gleason (1992) point out that calculus is one of the greatest achievements of the human intellect. For the past three centuries, the power of calculus has been demonstrated by illuminating questions in mathematics, engineering, the physical sciences, and the social and biological sciences. More specifically, one practical value of calculus is its ability to reduce complicated problems to simple rules and procedures. This characteristic has created a traditional view of teaching calculus as nothing but simple rules and procedures without an emphasis on conceptual understanding. Approximately one million students study calculus each year in the United States, and the general feeling of calculus reformers is that "No one is well served by the present (traditional) course" (Steen, 1987, p. xi). Calculus curricula and teaching, however, are deeply rooted in tradition and as a result are extraordinarily resistant to change.

Douglas (1986), NCTM (1989), Steen (1987), and Tucker (1990) have focused on applying to calculus the NCTM (1989, 1991) and NRC (1989) general recommendations for changes in mathematics. These specific calculus recommendations are intended to explore opportunities for revitalizing calculus instruction and to develop a new calculus curriculum that restores the practical value of calculus by focusing on (a) conceptual understanding rather than procedural knowledge, (b) practical applications, (c) a classroom atmosphere of interactive exploration of the fundamental ideas, and (d) use of
technology to free students from algebraic manipulations and allow students to concentrate on and investigate the underlying concepts using multiple representations. The recommendations from the NCTM and National Council of Supervisors of Mathematics [NCSM] (1986) suggest that training sessions, implementation of appropriate materials, and ongoing support are key components of an effective calculus reform effort. The NCTM and NCSM, however, caution that any effective reform effort is dependent on consistency of teachers’ perceptions with reform recommendations.

Statement of the Problem

The existing research is not definitive, but it does lead to the speculation that the most important factor leading to the development of mathematics teachers’ subject matter and pedagogical perceptions and beliefs is past experiences in mathematics classes (Ball, 1990a; 1990b; Marks, 1990; Thompson, 1984; Tirosh & Graeber, 1989). Participation in workshops or special programs may be another factor leading to the development of subject matter and pedagogical perceptions of mathematics teachers (Carpenter et al., 1989; Putnam, 1987; Wood et al., 1991); only limited research exists, however, on the effects of such programs in mathematics education. Thus, for our understanding of what mathematics teachers’ subject matter and pedagogical perceptions are, and the identification of the factors that lead to new developments and/or changes within these areas, the results of these studies must be viewed as formative.

Past research on mathematics teachers’ subject matter and pedagogical perceptions has focused primarily on elementary mathematics teachers and a narrow range of mathematics topics. For example, much of the research has dealt with teachers’ understanding of remedial or lower level mathematics topics such as whole numbers, fractions, multiplication, and division. The most advanced topic addressed in the research

Describing the behavior of functions is central to the study of calculus. In particular, this description involves how the outputs of a function act or change relative to given inputs. The concept and language of limits is fundamental to communicating the behavior of a given function. Although the concept of limit permeates the calculus curriculum, it has proven to be extremely difficult for calculus students to understand (Davis & Vinner, 1986; Orton, 1983a, 1983b; Tall & Vinner, 1981). Student difficulties with understanding the concept of limit poses a significant problem as this concept is fundamental to understanding other calculus topics (e.g., differentiation and integration).

In the past 15 years, significant progress has been made in understanding students’ cognition about limits (Davis & Vinner, 1986; Orton, 1983a; 1983b; Tall & Schwartzzenberger, 1978; Tall & Vinner, 1981; Williams, 1991). For example, Tall and Schwartzzenberger found that student difficulties regarding limits may arise from the casual meaning of the terms used. For example, the word “close” means near, but not coincident. Research by Graham and Ferrini-Mundy (1989) demonstrated that although students were capable of evaluating limits of continuous functions, they showed little intuitive or geometric understanding of limits. In brief, the above research has demonstrated that students’ understandings of limits as well as other major calculus concepts are exceptionally unrefined. Research has not examined teachers’ understandings of limits specifically; thus, it is only possible to speculate how teachers actually understand or conceptualize limits, the role of limits, and the teaching of limits in calculus.

The present study presents the results of an investigation of calculus teachers’ subject matter and pedagogical perceptions of limits by examining the following: What are teacher’s perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus? Additionally, the sampling technique used sheds some light on the
question: Are these teachers' perceptions associated with their participation in a calculus reform project focused on staff development? Each question was significant and sufficiently broad to serve as a base for a whole study. The decision to study these questions collectively was made so that a more comprehensive representation of the teachers' subject matter and pedagogical perceptions of limits could be portrayed. The interrelations among the questions would have been lost if each question was studied in isolation.

The objective of this study was to examine teachers’ subject matter and pedagogical perceptions of calculus in light of current reform. Thus, it is important to define the specific dimensions of these domains in a way that is appropriate for focus upon the teaching of mathematics. The term *perceptions* refers to the sum of a person’s ideas and beliefs concerning the given topic. Examining *subject matter* perceptions refers to explorations of teachers’ understanding of mathematics, including their understanding of mathematics principles, concepts, procedures, assumptions, and proofs as well as the explicitness and the connectedness of their understanding. *Pedagogical* perceptions are defined as teachers’ conventional understanding of instruction and learning theory specific to the teaching of mathematics. Thus, examination of pedagogical understanding refers to analyzing teachers’ ways of working with pupils in mathematics classes, their repertoire of strategies for helping students learn mathematics, and teachers’ decisions on particular plans or courses of action to teach mathematical concepts, procedures, and their connections both within the subject and to applications. Examination of pedagogical understanding also includes the specific mathematics examples being used as well as the explanations given. Additionally, pedagogical understanding includes teachers’ ideas of what it means to learn mathematics, as well as their own roles as educators of mathematics.
Significance of the Study

A significant portion of the extant research indicates that a relationship exists between teachers’ perceptions and instructional practices. Therefore, knowledge of teachers’ existing perceptions is crucial for effective mathematics reform. Past research gives us a formative understanding of mathematics teachers’ subject matter and pedagogical perceptions, but has not addressed the perceptions of calculus teachers. This study is the first to provide an initial description of the subject matter and pedagogical perceptions of calculus teachers.

Calculus reform is an important component of the overall mathematics reform endeavor. No research exists, however, on teachers’ subject matter and pedagogical perceptions of calculus in general or of specific calculus topics. Examining teachers’ perceptions of calculus in its entirety would be an enormous undertaking. Because the language of limits is fundamental to the conceptual understanding of calculus, teachers’ subject matter and pedagogical perceptions of limits serve as a basis for describing the subject matter and pedagogical perceptions of calculus. Thus, this study is an important starting point in portraying teachers’ subject matter and pedagogical perceptions of calculus. The results of this study serve as a basis for future research on understanding teachers’ subject matter and pedagogical perceptions of calculus in general, or of other specific calculus topics.

The ultimate goal of calculus reform is the development of appropriate students’ perceptions. The results of this study provide information on whether calculus teachers’ subject matter and pedagogical perceptions of limits are demonstrated in their teaching practices. Knowledge of the relationship between teachers’ perceptions and instructional practices allows for more effective attempts to alter students’ perceptions. Clearly, teachers’ classroom practices are the most direct influence on students’ calculus perceptions. If classroom practice is related to calculus teachers’ subject matter and
pedagogical perceptions, then efforts to influence calculus teachers’ subject matter and pedagogical perceptions would be the most logical focus of those pursuing the revitalization of calculus.

Furthermore, half of the sample considered for this investigation participated in a calculus reform project focused on staff development. Consequently, it was possible to reach some preliminary conclusions concerning the feasibility of calculus reform projects of a particular type in achieving the goals of the calculus reform movement. The findings of this research provide information on whether the perceptions of teachers prior to any involvement in a calculus reform project were consistent or inconsistent with the goals of the reform effort. Based on the necessity for teachers’ subject matter and pedagogical perception consistency with the goals of the reform effort, information regarding both initial perceptions and perceptions following a reform effort are essential for designing effective reform projects.
CHAPTER II
REVIEW OF THE LITERATURE

Introduction

Leaders in mathematics education, including the NCTM (1989, 1991) and the NRC (1989), have called for a reorganization of the mathematics curriculum around concept development and problem-solving. The most prominent examples of these recommendations to teachers include: (a) reducing the amount of time devoted to drill-and-practice of computational skills, (b) engaging students in challenging problem situations, (c) creating a classroom atmosphere where questioning, exploration, reasoning, and justification are encouraged and expected, and (d) using the power of computing technology to free students from tedious computations, allowing them to concentrate on problem-solving processes. Specifically, for calculus, the NCTM (1989) reiterates these suggestions in the following statement:

Instruction should be highly exploratory and based on numerical and geometric experiences that capitalize on both calculator and computer technology. Instructional activities should be aimed at providing students with firm conceptual underpinnings of calculus rather than at developing manipulative techniques. (p. 180)

The NCTM recommendations are in direct correspondence with the agendas for calculus reform of the Mathematical Association of America (MAA) (Douglas, 1986; Steen, 1987; Tucker, 1990). These agendas recommend that calculus instruction should be focused on (a) conceptual understanding rather than procedural knowledge, (b) practical applications, (c) a classroom atmosphere of interactive exploration of the fundamental ideas, and (d) use of technology to free students from algebraic manipulations and allow them to concentrate on and investigate the underlying concepts using multiple representations.
Implementation of these conceptually-based recommendations in the calculus classroom requires teachers to have a conceptual understanding of calculus, to know why understanding concepts is important, and to know how to help students gain that understanding. Making these types of changes in the classroom often requires an alteration in the teacher's existing subject matter and pedagogical perceptions. Thus, part of the job of teacher educators and researchers is to examine what perceptions and beliefs learners bring to teacher education programs, workshops, or other special programs focused on perceptions and beliefs consistent with the reform effort.

Mathematical reform as well as a growing awareness of the role that mathematics teachers' perceptions play in teaching has lead researchers to address a number of related questions: What factors lead to the development of mathematics teachers' subject matter and pedagogical perceptions? How do mathematics teachers' perceptions evolve? How can mathematics teachers' perceptions be affected? The first section of this review examines those studies which specifically address these questions.

This study will involve the investigation of teachers' subject matter and pedagogical perceptions of limits; thus, it is important to also include the literature on perceptions of limits in this review. No research exists, however, regarding teachers' subject matter and pedagogical perceptions of limits. Thus, studies of students' perceptions of limits will be reviewed to help develop formative understanding of how teachers actually conceptualize limits, the role of limits, and the teaching of limits in calculus.

Mathematics Teachers' Subject Matter and Pedagogical Perceptions

The theoretical framework of teachers' understanding discussed by Shulman (1986) provides a guide for review of teachers' subject matter and pedagogical perceptions. This framework consists of three domains of knowledge: (a) subject matter
knowledge, (b) curricular knowledge, and (c) pedagogical content knowledge. Curricular knowledge refers to understanding the curricular alternatives available for instruction. None of the studies in this review discuss curricular knowledge as a separate identity, but only as a part of the teachers' pedagogical perceptions. However, because all of these types of knowledge appear to be interconnected, it is difficult to differentiate where one ends and another begins. Thus, the studies reviewed involve the separate examination of mathematics teachers' subject matter and pedagogical perceptions, as well as the intersection of the two.

**Development of Mathematics Teachers' Perceptions**

To understand how mathematics teachers' subject matter and pedagogical perceptions develop and change, it is important to examine general learning theory on how one develops and changes such perceptions. According to Rumelhart and Norman (1978), there are three basic schema-learning processes which lead to the development and change of knowledge, perceptions, and beliefs: accretion, tuning, and structuring (restructuring). *Accretion* refers to the “daily accumulation of information,” *tuning* refers to “actual changes in the very categories we use for interpreting new information,” and *restructuring* occurs when “new structures are devised for interpreting new information and imposing a new organization on that already stored” (p.39). Based upon Piaget, Farnham-Diggory (1992) states that accretion might be described as *assimilation*, or the incorporation of new knowledge and perceptions into existing knowledge, conceptual, and belief structures. In addition, tuning and restructuring might be described as *accommodation*, or the transformation of existing knowledge, perceptions, and beliefs to better fit new information. These theories represent the way mathematics teachers learn and develop their subject matter and pedagogical knowledge (Thompson, 1984).
This section includes the review of five studies which have examined the factors that lead to the development of mathematics teachers' subject matter and pedagogical perceptions and beliefs. The first three deal solely with analysis of the development of preservice teachers' subject matter knowledge. Tirosh and Graeber (1989) explore the misconceptions regarding multiplication and division of whole numbers and decimals that preservice teachers bring with them to elementary teacher education programs, suggesting that teacher-educators instruct in such a way that their students reach cognitive conflicts with their previously held perceptions and beliefs. In two articles, Ball (1990a, 1990b) extends this research to the topics of division of fractions, zero, and algebraic equations. In a fourth article, Marks (1990) remains focused on the teaching of fractions; however, pedagogical content knowledge is explored, rather than examining subject matter knowledge. Finally, Thompson (1984) examined the factors that lead to the development of both subject matter knowledge and pedagogical content knowledge among junior high school mathematics teachers.

The overall goal of Tirosh and Graeber (1989) was to provide insight into the sources of prospective teachers' beliefs about multiplication and division, and the relationship among beliefs, conceptual skills, and performance on given word problems. It was hypothesized that the primitive model associated with multiplication is repeated addition, and that the exclusive acceptance of this model may lead to the belief that “Multiplication always makes bigger.” The literature described two primitive division models, a partitive model and a measurement model. Using the primitive measurement model, one seeks to determine how many times a set quantity is contained in a larger quantity. Using the partitive model, an object is divided into equal parts, thus exclusive acceptance of this model may lead to the belief that “Division always makes smaller.”

The study sample included one male and 135 female preservice teachers enrolled in one of six mathematics content or methods classes for elementary education majors in their third year of university study in either the winter or spring of 1986 (Tirosh &
The preservice teachers were asked to justify their responses to six explicit true/false questions regarding the misbeliefs, "Multiplication always makes bigger" and "Division always makes smaller." One-half of the questions was computational in nature, and the other half was strictly conceptual. Data were also collected on the preservice teachers' computational and word problem-solving skills.

Two statements were related to the misbelief "Multiplication always makes bigger." Approximately 87% of the prospective teachers responded correctly to both, whereas, only 3% responded incorrectly to both statements. In general, about 10% of the preservice teachers were considered to hold the misbelief explicitly (Tirosh & Graeber, 1989). However, the data on the prospective teachers' performances in writing expressions for word problems suggested that "many" preservice teachers were implicitly influenced by this misbelief. In fact, when the operator in the word problem was a decimal less than one, approximately 50% of the prospective teachers responded incorrectly with a division expression. On the other hand, if the operator was a whole number, approximately 90–95% of the preservice teachers responded correctly with a multiplication expression. These results were reinforced by similar findings in the interview data.

The majority of the preservice teachers, however, responded incorrectly to the following statement: "In a division problem, the quotient must be less than the dividend." The data on prospective teachers' performances in writing expressions for word problems confirmed the pervasive nature of this misbelief. Approximately 45% of the prospective teachers wrote incorrect multiplication expressions for the division problems with decimal divisors less than one. These results were supported by the data collected from the interviews, and the computation exercises (Tirosh & Graeber, 1989).

The results thus suggested that preservice teachers successfully used procedural knowledge when justifying specific examples, but used the primitive models of the operations when they were responding to general statements or making generalizations
about procedures. From these results, it was generalized that "The percent of the American preservice teachers whose performance in solving word problems appears to be influenced by the misbeliefs is larger than the percent who explicitly hold these misbeliefs" (Tirosh & Graeber, 1989, p. 91). This may be explained by the justification that conceptual knowledge, although faulty, was developed from procedural knowledge. From a constructivist point of view, a learner who has already constructed his/her own conceptual knowledge may find further meaning unnecessary.

The analysis reported in Ball (1990a) examined prospective elementary and secondary mathematics teachers’ knowledge and reasoning in mathematical pedagogy at the point of entry into a formal teacher-education program. Specific focus was directed at the teacher candidates’ knowledge of division. In addition, an examination on what they believed makes a statement true or reasonable in mathematics was also conducted. The sample consisted of 10 elementary and nine secondary education students who had enrolled in their first education course. The secondary education students were mathematics majors or minors, and the elementary education students were majoring in either elementary education or child development.

All of the subjects were interviewed on three different mathematical contexts: division of fractions, division by zero, and division with algebraic equations. The prospective teachers were asked to explain or generate representations of problems in each context. Probes were developed to examine the prospective teachers’ ideas about what counts as mathematical justification for a solution to a problem. Additional probes were developed as needed to obtain clarifications of the subjects’ thoughts. All interviews were audio-taped and transcribed, the initial analysis of which led to a set of response categories for each context. The questions were also cross-analyzed for several dimensions. For the purposes of this study, the responses to the division questions were coded in two dimensions for: (a) correctness and (b) the nature of the justification provided by the teacher candidate (Ball, 1990a).
The questions for division of fractions included: “How would you solve $\frac{13}{4}$ divided by $\frac{1}{2}$? Can you come up with the real-world situation to represent this problem?” Only five prospective teachers (all mathematics majors) were able to generate an appropriate representation; however, 17 of the 19 were able to correctly calculate the problem. Although the correct representations did not come easily, they all used the measurement model of division. The most frequent incorrect representation involved the error of division by two instead of one-half (Ball, 1990a).

The questions for division by zero included: “Suppose a student asked you what seven divided by zero is. How would you respond?” “Suppose the student asks you why that is? What do you mean by undefined?” Of the 19 teacher candidates, 12 responded by stating rules (five were incorrect), five explained the actual meaning behind division by zero, and two did not know (Ball, 1990a).

The questions for division in algebraic equations included: “Suppose a student asks for your help with the following, if $x/0.2 = 5$, then $x = \_\_\_\_$. How would you respond? Why?” Only one elementary teacher candidate tried to discuss the meaning of the equation. Fourteen of the prospective teachers, including all nine of the secondary teacher candidates, focused on the mechanics of manipulating algebraic equations, and most of them “explained” the problem by restating the procedural steps to solve such equations. Four elementary teacher candidates could not solve the problem themselves (Ball, 1990a).

These results indicated that the difficulties experienced by the prospective teachers represented a very narrow understanding of division. As established from previous studies, the teacher candidates were viewing division in only partitive terms. The teacher candidates’ knowledge was generally fragmented, since the responses to each question involved specific bits of mathematical knowledge. Thus, the prospective teachers’ knowledge of division seemed focused less on conceptual understanding and more upon memorization and rules. Past studies suggested that once rules are forgotten, they are not
easily retrieved without the concepts to support them. This study demonstrated that what prospective teachers learned in their previous mathematics classes was most likely inadequate subject matter preparation for teaching mathematics for understanding (Ball, 1990a).

In a second study, Ball (1990b) focused on what subject matter knowledge prospective elementary and secondary teacher education students possessed on the topic of division of fractions. The sample included 217 elementary education majors and 35 mathematics majors who were prospective high school teachers. Each of the subjects was at the initial stages of a formal teacher education program at one of five sites: Dartmouth College, University of Florida, Illinois State University, Michigan State University, or Norfolk State University.

The design of the study was longitudinal. A questionnaire was administered at repeated intervals to all 252 prospective teachers. A smaller “intensive” sample of the prospective teachers was closely observed and interviewed throughout their preservice program and during their first year of teaching. The questions posed in both the questionnaires and interviews were “grounded in scenarios of classroom teaching and woven with particular subject matter topics” (Ball, 1990b, p. 451). On the questionnaire, the prospective teachers were asked to select all story problems that applied to a given fraction-division problem. The options included “None of these” and “I’m not sure,” as well as four different story problems in which only one response was correct. The interview task involved asking prospective teachers how they were taught to divide fractions, and if they could come up with a model to help students understand $1 \frac{3}{4}$ divided by $\frac{1}{2}$.

The results indicated that prospective teachers’ substantive understanding of mathematics was rule bound and “compartmentalized.” Although “very few” secondary teacher candidates and no elementary teacher candidates were able to generate appropriate representations of a division problem, most of them were also unable to select the correct
representation when the answer was presented to them in a multiple-choice format. However, virtually all of the candidates were able to successfully calculate $1 \frac{3}{4}$ divided by $\frac{1}{2}$ using invert and multiply. None of the preservice teachers derived correct representations with ease. The most frequent incorrect response to this problem involved a model that represented division by two instead of $\frac{1}{2}$. Most of the prospective teachers used round food models like pizza or a pie in their proposed models. This result was not surprising since most of the pictorial representations of fractions in school textbooks involved round models. Ball (1990b) suggested that teacher errors resulted from the difficulties of everyday language confounding mathematical language. Awareness of this confusion is important for perspective teachers. In conclusion, it was asserted that prospective teachers considered division only in partitive terms, not in terms of the measurement model. This finding was similar to those of Tirosh and Graeber (1989, 1990).

Marks (1990) wanted to (a) present a description of fifth grade teachers' pedagogical content knowledge of the equivalence of fractions, then based on these descriptions, (b) give suggestions how to modify these general perceptions, as well as (c) suggest revisions for teacher education to improve teachers' understanding of their own pedagogical content knowledge. The sample consisted of six experienced teachers and two novice teachers. The criteria used to segregate the levels of experience were not specified. Two experienced teachers scored low on a composite scale of mathematics, whereas the others scored high. The instrument used was developed for the Teacher Assessment Project at Stanford University in 1986 and consisted of eight task-based interviews. Investigation of math pedagogical knowledge was the principal focus of the interviews. In particular, the eight tasks included planning a lesson, critiquing a classroom videotape, and diagnosing and remediating student errors. The results demonstrated that teachers' pedagogical content knowledge was composed of four principal components: knowledge of subject matter for instructional purposes, student understanding of the
subject matter, media for subject matter instruction, and subject matter instructional processes.

The investigation by Thompson (1984) sought to identify whether teachers' professed beliefs, views, and preferences about mathematics and mathematics teaching were positively correlated with their instructional practices. A case study method was deemed most appropriate for this inquiry. Three junior high school mathematics teachers were selected from a group of 13 teachers who had participated in a pilot study previously conducted by the researcher.

Each of the teachers in the study was observed daily for a period of four weeks. Additionally, in the last two weeks, each teacher was interviewed for approximately 45 minutes following the observed lesson. To become more familiar with the social context of the mathematics classroom before starting the more directed interviews, the first two weeks were limited to observations. This decision also gave the researcher time to generate conjectures about the teachers' subject matter perceptions, and allowed for a better sense of direction for the interviews. It was intended that this procedure would avoid any potential influence upon the beliefs and views of the teachers. The inferences made by the researcher were then tested for accuracy by the interviews. To allow the researcher to concentrate on one case study at a time, none of the teachers were observed simultaneously. Each lesson was audiotaped, providing a resource which could then be used during teacher interviews to assist in recall of the events of the lesson (Thompson, 1984).

At different times throughout the case study, each teacher was also asked to respond in writing to six tasks. The first five tasks involved: (a) teachers' views of the relative importance of goals in mathematics instruction, (b) the relative emphasis to be attributed to several instructional objectives, (c) the relative importance of several pedagogical practices, (d) common concerns for unsatisfactory student progress in mathematics, and (e) the more valuable types of information used in judging teaching
effectiveness. Confrey (1978) developed the sixth task, which consisted of six bipolar dimensions that may be used for assessing perceptions of mathematics in relationship to the general characteristic qualities of mathematics. The data were reviewed on a daily basis, thus new data were always examined in light of the tentative hypotheses and previously developed inferences, which provided new foci for subsequent observations and interviews. The teachers felt that the major influence upon their views of mathematics and mathematics teaching was their previous experience in school mathematics (Thompson, 1984).

In general, the discussion of the results involved examination of aspects of the three teachers’ perceptions and behaviors on which they most differed. These results included the differences in the teachers’ conceptualizations of mathematics and mathematics teaching, the “integratedness” of the teachers’ beliefs, and the teachers’ reflections on their instructional actions, the subject matter, and their beliefs. It was found that the relationship between teachers’ perceptions of mathematics and their instructional decisions was very complex. Many factors appeared to interact with the teachers’ perceptions of mathematics and teaching practices, including pedagogical beliefs that were not specific to mathematics and perceptions of the social and emotional make-up of their students. In some cases, the views and beliefs specific to the teaching of mathematics were overshadowed by these other views (Thompson, 1984).

Discussion of Development of Mathematics Teachers’ Perceptions

There is reason to be skeptical of the results from research on the factors which lead to the development of mathematics teachers’ subject matter and pedagogical perceptions. No one can be certain exactly when, where, or how these perceptions were first developed; thus, the results of these studies must be taken as only speculative in nature. Most of the investigators were interested in what subject matter or pedagogical
perceptions teachers possessed in mathematics, in the hope that this knowledge would provide insights into the sources of their perceptions.

The results of the studies of the development of subject matter and pedagogical perceptions seemed to reflect that the most important factor leading to conceptual development was past experiences in mathematics classes (Ball, 1990a, 1990b; Marks, 1990; Thompson, 1984; Tirosh & Graeber, 1989). Most of the studies suggested that teachers experienced difficulties in making mathematical connections because the focus of their preparatory mathematics classes was directed at procedural rather than conceptual knowledge (Ball, 1990a, 1990b; Tirosh & Graeber, 1989). However, this conclusion appeared to be more of an assumption than it was an actual result stemming from the data. From the data, it was evident that many of the subjects demonstrated a weakness in conceptual knowledge, but the cause of this weakness was not demonstrated.

These studies of development suggested that research is needed to clarify what factors change mathematics teachers’ subject matter perceptions. Specifically, do these perceptions change when they confront a conflict with their original perceptions (Tirosh & Graeber, 1989); are they dependent upon differences in course work requirements (Ball, 1990b); are they different when a more integrated approach in teacher preparation is used (Marks, 1990); or do they change due to the grade level being taught, the academic aptitude of students, and the difficulty of mathematical content (Thompson, 1984)?

Change in Mathematics Teachers’ Perceptions

In the previous section on conceptual development, it was noted that Thompson (1984) suggested that a need existed for examination of the stability of teachers’ pedagogical content and subject matter knowledge. Perhaps of even greater importance, it was suggested that there was a need to explore those factors which generated change in teachers’ pedagogical content and subject matter knowledge.
Change in teachers’ perceptions and instructional methodologies may be hampered by the fact that future mathematics teachers have had over 2,000 hours of traditional teacher observation training before they take their first mathematics methods course (Lortie, 1975). This factor alone could make change difficult since prospective teachers have seemingly already shaped their perceptions about what it means to teach and learn mathematics prior to their own formal instructional coursework. Thus, part of the job of teacher educators and researchers is to examine what perceptions and beliefs learners bring to teacher education programs, workshops, or other special programs focused on perceptions and beliefs consistent with the reform effort.

To assist in making the changes recommended by educational leaders it is important to examine the general literature of educational change. Vossniadou and Brewer (1987) state that little has been said about the mechanisms through which observed changes in knowledge occur. Posner, Strike, Hewson, and Gertzog (1982) asserted that in order for accommodation (i.e., reorganization of perceptions) to occur, the following conditions must be met: (a) some dissatisfaction with the existing conceptual framework must exist, (b) alternate perceptions must be both intelligible and initially plausible, and (c) alternate perceptions must be seen as useful or valuable. Nussbaum and Novick (1982) suggested a similar three-part instructional sequence designed to encourage learners to make desired conceptual changes. The sequence begins with an exposing event which encourages learners to explore their own perceptions. This event is followed by a discrepant event which helps produce a cognitive conflict, hopefully leading to dissatisfaction with current perceptions. The sequence finishes with a resolution in which the alternative perceptions are made plausible and intelligible, thus encouraging the desired conceptual shift.

In this review, several studies are considered which employed these methods in their search for the factors which lead to change in teachers’ subject matter and pedagogical perceptions (Civil, 1992; Schram, 1991; Tirosh & Graeber, 1990; Wilcox et
al., 1991). In this section, research articles are considered in three sections representing varying factors of change: (a) change due to teacher education programs, (b) change due to experience, and (c) change due to inservice programs, workshops, and special programs.

**Change Due to Teacher Education Programs**

Studies considered in this section were organized based on the topics covered and the methodology used in a described teacher education course. The first two, from Wilcox et al. (1991) and Schram (1991), examined change in both preservice teachers’ pedagogical content and/or subject matter knowledge as a result of participation in a number theory course conducted under the “community of learners” methodology. In turn, Borko et al. (1992) examined change in preservice teachers’ pedagogical content and subject matter knowledge as a result of taking part in a “Concepts in Mathematics” course taught with a constructivist format. Then, both Civil (1992) and Tirosh and Graeber (1990) examined the effect of cognitive conflict teaching on preservice teachers’ pedagogical content and/or subject matter knowledge. The sixth study reviewed, McDiarmid (1990), involved an investigation of change in preservice teachers’ pedagogical content and subject matter knowledge as a result of participation in a course that required observation of experienced teachers. The final paper, Cooney, Wilson, and Shealy (1992), focused on the change in secondary preservice teachers’ pedagogical content and subject matter knowledge as a result of taking an intense course on functions.

The introduction in Wilcox et al. (1991) discussed the development of a “community of learners” with shared responsibility for learning mathematics. The purpose was to determine whether establishment of a classroom, where conjectures, questions, and collaborations exist, may engender a different view of mathematics among students. It was felt that these ideas were equally valid for teacher education programs. Thus, this
study involved an examination of an “intervention study” in an elementary teacher education program. The main feature of the intervention was the establishment of a community of learners focused upon the goal of creating a more conceptual level of knowledge about mathematics, teaching, and learning mathematics among preservice teachers. The subjects were 23 elementary teacher candidates enrolled in a sequence of three “nontraditional” mathematics courses on number theory, geometry, and probability and statistics, and also in a methods course, a curriculum seminar, and field experience.

Classroom observations, student questionnaires, and samples of prospective teachers’ work were the primary sources of data. In an “intensive” sample of four teacher candidates, data included tape-recorded interviews, observations of student teaching, interviews with mentors and instructors, and periodic observation and interviews during the candidates’ first year of teaching. The data were analyzed in two ways, including: (a) the ways in which the community was constructed and (b) the ways in which the community contributed to changing the preservice teachers’ beliefs about the meaning of mathematics, how it is learned, and the role of the teacher in the classroom (Wilcox et al., 1991).

Results indicated that students who entered the preservice education program were unprepared for the collaboration and shared-responsibility concept of the community of learners in mathematics. The types of problems posed did not themselves lead to singular algorithmic solutions, thus students relied on each others’ insights. Over the three-course sequence, the observational data indicated an increasing reliance on the collaboration of members in small groups. The nature of the mathematical tasks as well as the students’ desires seemed to be the main factors that dictated the organization of a small group at any given time. The teachers’ role was that of a guide. When the instructor was confident that the investigations in each group produced understanding; then the class was brought together to discuss the findings of each group. This shift away from the instructor as the sole authority was possibly the most significant development. The intervention increased
confidence in the ability of prospective teachers to apply their knowledge in both the second and third courses in the mathematics sequence (Wilcox et al., 1991).

The results demonstrated considerable evidence that the beliefs and behavior of the prospective teachers had changed; however, some uncertainty was evidenced on whether the intervention was sufficiently powerful to alter their deeply held beliefs about the learning of mathematics. In the first questionnaire, the prospective teachers indicated that group work would probably aid "slow" learners. However, in the final questionnaire every prospective teacher valued group work for one of the following reasons: (a) communicating about mathematical ideas, (b) talking with others to clarify one's own understanding, (c) increased willingness to take mathematical risks within small groups, (d) observing the multiple ways in which diverse learners approached a problem situation, (e) learning how to work collaboratively, and (f) developing independence as learners. In conclusion, subject matter knowledge was linked with the development of pedagogical content knowledge because the learning of mathematics was embedded in the context of learning to teach and the prospective teachers were given the chance to reflect on their learning process (Wilcox et al., 1991).

Schram (1991) was part of the same longitudinal intervention study in an elementary teacher education program discussed in Wilcox et al. (1991). The purpose established by Schram was to examine what the prospective teachers actually learned about mathematical content, given the opportunity to experience mathematics in a community-of-learners-environment which encouraged risk-taking. The specific focus of this paper was to examine the prospective teachers' views and ideas about number theory concepts and how they changed.

The subjects for this study were two of the 23 teacher candidates participating in the sequence. Kim and Andrea were chosen because they represented different perspectives of what the prospective teachers brought to these mathematics classes. Kim entered the class expecting to find another traditional math class and felt completely
comfortable with her view of mathematics. Andrea came to class with the desire to think about mathematics differently, take the “mystery” out of it, make connections, and increase her understanding. The primary data source included three interviews which took place before, during, and at the end of the first course. The interviews were designed to examine what the prospective teachers learned about number theory content and relationships, and ways of mathematical thinking.

Schram (1991) concluded that participation in one carefully designed mathematics course was not sufficient to change most prospective teachers’ beliefs that mathematics is an “abstract, mechanical, and meaningless series of symbols and rules” (p. 32). Andrea entered the course with a good mathematical background, in addition to which she had a desire to learn mathematics using a new approach. She finished the course with a “new mathematical world” and appeared to be eager to learn more. Kim entered the course with a weak mathematical background and a desire to learn mathematics in the traditional way. She struggled with the course, but by the end she was thinking about mathematics differently. However, she was not confident and her knowledge was very fragile.

The results of this study were not surprising, however, the findings may have been important. Although an increase in knowledge could be anticipated after a concept has been taught, the stability of this knowledge was found to be delicate. Changing mathematical ways of thinking is difficult and requires time. Careful attention needs to be given to the development of courses that take into account what knowledge prospective teachers bring to teacher education courses. Intense desire, incredible patience, and an abundance of support are required for change to occur (Schram, 1991).

Borko et al. (1992) examined emergent knowledge, beliefs, thinking, and actions related to understanding mathematics and the teaching of mathematics among novice teachers, as well as investigating the impact of teacher education experiences on the process of learning to teach. This study focused on the thorough analysis of a classroom lesson in which a student teacher provided an unsuccessful conceptual justification of the
standard division-of-fractions algorithm, directing specific attention to: (a) the student teacher’s beliefs about good mathematics teaching, knowledge as related to the division of fractions, and beliefs about learning to teach; and (b) the treatment of division of fractions in a mathematics methods course.

The student teacher who was analyzed, Ms. Daniels, had an extensive mathematics background since she had completed the first three years at the university as a mathematics major. Ms. Daniels maintained a “C” average, at least until her junior year, throughout most of her mathematics coursework, and was not admitted into the secondary teacher education program because of her grades in mathematics. She then decided to major in elementary education with an emphasis in mathematics. Consequently, Ms. Daniels tested out of the first two quarters of the course sequence “Concepts in Mathematics,” which examined elementary number concepts and operations. She received a grade of “B” in the third course of the sequence that dealt with elementary geometric topics. Ms. Daniels’ teaching placements included a self-contained sixth-grade classroom, a second-grade classroom, a mathematics classroom in a junior high, and finally, another sixth-grade classroom. The teaching episode which is discussed by Borko et al. (1992) occurred at her final teaching placement.

The principal data sources for the participant’s knowledge and beliefs were interviews given at the beginning, middle, and end of each school year for a two-year period. The format included open-ended questions which covered vignettes, comparisons of presentation of topics in different textbooks, descriptions of how students would teach a particular subject, statements on how students would evaluate student learning, or reactions to student homework assignments. The division of fractions received special attention in many of the interview questions. The interview data were supplemented by responses to a questionnaire developed by the National Center for Research in Teacher Education (NCRTE), as modified for the study (Borko et al., 1992). One particular item on the questionnaire was to illustrate what $1\frac{1}{4}$ divided by $\frac{1}{2}$ meant. The questionnaire
was administered several days prior to each interview. All written work for the mathematics methods course was also collected as data. Knowledge about the novice teacher’s thinking and actions in the classroom was obtained by week-long visits to the classes of the participant toward the end of her first, third, and fourth teaching placements.

Overall, Ms. Daniels’ perception of good mathematics teaching included many elements that are compatible with current views of effective mathematics teaching. To develop her personal beliefs about the characteristics of good mathematics teaching, as do most novice teachers, Ms. Daniels drew on three experiences: (a) as a student in mathematics courses, (b) her methods course, and (c) student teaching. Initially, her beliefs seemed to originate in personal experiences of school mathematics, but as the year progressed she seemed to rely more heavily on her experiences from the mathematics methods course (Borko et al., 1992).

Ms. Daniels believed that good mathematics teaching included making mathematics relevant and meaningful to students by the use of applications. Her experiences in the methods course seemed to reinforce this belief, as well as her belief that making mathematics fun is a component of good teaching. Over time, Ms. Daniels realized that having fun was not sufficient. In addition, she was not capable of providing the applications or illustrations of the explanations that she wanted. Specifically, in the interviews on the subject of division of fractions, Ms. Daniels was unable to explain or represent the process meaningfully. In fact, there was little evidence of conceptual understanding at all. All of her responses to students in the classroom seemed to turn into suggestions for remembering the algorithm rather than a conceptual explanation of the algorithm. This lead to what was labeled the “teaching episode” by Borko et al. (1992). Ms. Daniels was interviewed on the topic of how to explain division of fractions three times during the year and each time her response was vague, reflecting a global description, one that used pizzas as a concrete example. The authors concluded that Ms.
Daniels' beliefs about good mathematics teaching were not implemented in this case because she lacked conceptual understanding of division of fractions.

In conclusion, when discussing explanations of why Ms. Daniels was unsuccessful during the "teaching episode," it must be recognized that the topic of division of fractions is difficult to learn and to teach. As suggested by Ball (1990a, 1990b), the majority of students entering elementary and secondary preservice teacher education programs are not able to generate appropriate representations for division of fractions. Thus, university mathematics courses must downplay rote learning of computational techniques and stress meaningful learning of mathematics. From Borko et al. (1992), it may be seen that Ms. Daniels may have had less opportunity to explore elementary number concepts and operations than other novice teachers because she earned the credits for the "Concepts of Mathematics" course by successfully passing an examination. By allowing Ms. Daniels to pass out of this course, the university teacher education program was unable to foster her acquisition of conceptual understanding of subject matter and it served to reinforce her belief that her knowledge of mathematics was sufficient for teaching mathematics at the elementary level.

Civil (1992) discussed the impact of a course that emphasized small group discussion, based upon a focus intended to give prospective teachers a chance to express their fears and concerns about teaching particular mathematics topics in their teaching careers. The course, "Mathematics for the Elementary Teacher," was devised with the intent to create cognitive conflict, enhance reflection, and make prospective teachers responsible for their own learning. Civil organized and taught this course during an eight-week period in the summer. The goal was to examine the initial understandings and beliefs about mathematics held by eight preservice elementary teachers, as well as any changes in these understandings and beliefs.

A total of five seniors, one junior, and two graduate students were enrolled in the class and constituted the all-female sample for this study. It was stated that students
enrolled in summer school offerings tended to be weaker academically, which was felt to be the case among the subjects, though evidence of this claim was not presented. The data sources included observations, informal conversations, written homework, essays and diaries, audiotaped interviews with one or two students at a time, and audiotaped small-group discussions. Three types of tasks were used to address the research goal: (a) problem-solving type tasks, (b) tasks based on “things they have always known,” and (c) tasks aimed at creating cognitive conflict (Civil, 1992).

All of the prospective teachers shared the view that the role of the teacher was to tell the students what to do. In fact, they had very specific ideas on how and what to teach, and these concepts were discussed by Civil (1992). The first idea was to “Teach linearly, one thing at a time, and practice” (p. 7). Some of the prospective teachers were unaware of the importance of making connections between topics in mathematics. The second idea, “Showing, telling the student how to do something” (p. 7) was apparent in many of the prospective teachers’ comments on the work of children or peers. The third idea, “Teaching skills is a priority” (p. 8), played a major role in their agenda. Finally, “Efficiency” (p. 10) seemed to be the thrust in some of the preservice teachers’ thinking about teaching mathematics. In traditional school mathematics, a certain amount of content has to be covered, and whether or not every child understands this content is not considered to be the job of the teacher.

To gain insight into how prospective teachers might react to student work, three tasks were given in which they were asked to comment on student work. Five general themes were manifest from these comments. The first theme, as noted by Civil (1992), was one of “surprise and disbelief” (p. 11) that students could actually come up with their own ways of solving mathematical problems without being taught the rules first. It was evident that the thought of being creative in mathematics was foreign to them. A second comment involved the idea that if the prospective teacher thought something was difficult, then the students would naturally find it confusing. The third comment involved the
importance of giving praise in an appropriate manner. The prospective teachers’ concern for giving praise to prevent student frustration is valid, however, poor subject matter knowledge by the preservice teachers may lead to inappropriate praise. Another topic derived from the comments entailed the prospective teachers’ lack of appreciation for multiple approaches to the solution of mathematical problems. Some of the prospective teachers had the idea that it was more important to just “get it done” (p. 15) as opposed to being creative. The final theme discussed the prospective teachers “conforming to the school way” (p. 15). The prospective teachers were familiar enough with the school culture to know what may be expected from them as a teacher, therefore, many of their concerns directly reflected this concern.

Three major factors played an important role in challenging the prospective teachers beliefs about mathematics: (a) the characteristics of the traditional school mathematics class, (b) their personal experience with this approach, and (c) the course in which they were presently enrolled. Civil (1992) was not surprised that a couple of her successful prospective teachers thought about teaching mathematics the way they had been taught. She acknowledged, however, that it was difficult to understand why most of the prospective teachers insisted on teaching in the very style that had failed with them. Desire to conform was an expected result, since these prospective teachers were taught the same way over the last 12 or more years. Their prospective teaching careers already presented too many unknowns for them to feel comfortable putting aside the traditional teaching style. Moreover, their views were even less likely to change insofar as traditional methods were used in their other college courses.

A definite hope exists that courses, such as the one described by Civil (1992) as well as others, could lead to the development of a more reflective and critical approach in prospective teachers’ thinking about teaching and learning mathematics. The overall comments about what it meant to “do” mathematics indicated growth as reflective learners. Evidence showed preservice teachers’ desires to move toward a model of
teaching that reflected their learning process in this course. However, Civil was quite skeptical, observing that not only would preservice teachers have to change their deeply rooted perceptions, but they would have to do so in the presence of many personal conflicting messages derived along the way as a result of: (a) their K-12 experience, (b) their field experience, (c) their education courses, (d) their cooperating teachers, and (e) community experience.

Tirosh and Graeber (1990) investigated the instructional strategy of conflict teaching as a means of probing the preservice elementary teachers' misconception that in division problems, the quotient must be less than the dividend. Research defined conflict teaching as an approach to help students discuss and reflect on their errors and misconceptions, eventually bringing students to the realization that their perceptions were inadequate and in need of change. Once preservice teachers were faced with the inconsistencies between their calculations and their misconceptions, it was hoped that this "cognitive conflict" might help preservice teachers resolve the inconsistencies, and thus form a more accurate perception of division. It was also hypothesized that changes in the perceptions could be expected to bring about changes in performance on word problems, as discussed in Tirosh and Graeber (1989).

In spring 1986, three pretest instruments were administered to one male and 57 female preservice elementary teachers enrolled in either a mathematics content or methods course at a large southeastern university. The 21 female preservice teachers who responded incorrectly to the statement, "The quotient must be less than the dividend," and completed correctly 3.75 divided by .57, were the subjects selected for this study. Three paper-and-pencil instruments were administered in pre- and post-test form during regularly scheduled class sessions. The first instrument, computation using decimal numbers, required the preservice teachers to calculate four multiplication and division problems. The second instrument, statements about division, presented preservice teachers with seven true/false statements about multiplication and division. The responses to these
statements had to include a justification of their answers. The final instrument, writing expressions for word problems, involved giving preservice teachers 21 word problems, to which they were asked to write an expression that would lead to the solution of each problem. The post-test for each instrument included similar formats with different numbers and four similar word problems (Tirosh & Graeber, 1990).

In addition to the three paper-and-pencil instruments, the subjects were interviewed individually for 35 to 50 minutes and asked to elaborate upon their perceptions in the following ways: (a) verbalize their perceptions of division, (b) explicate their perceptions about the relative size of the dividend and the quotient, (c) recognize the inconsistency between their expressed conviction and the results of a computation with a decimal divisor less than one, and (d) reflect on the sources of their original misconception. All of the interviews were audiotaped and transcribed, conducted over a three-week period at least three weeks prior to administration of the post-test instrument. From examination of the preservice teachers’ perceptions of division, 12 of the 21 gave only a partitive interpretation of division. Three subjects gave a measurement interpretation as well as a partitive interpretation. Three subjects were unable to describe division in any way, except possibly through written algorithms (Tirosh & Graeber, 1990).

When the preservice teachers were confronted with their ideas about the dividend and the quotient, three of them explained that their incorrect responses on the written statement were due to the fact that they were thinking about whole numbers. Three subjects agreed with their initial beliefs, however, they did express some uncertainty. The interviewer then attempted to draw attention to the misconceptions of the remaining 15 subjects by asking them to calculate four divided by 0.5. Three prospective teachers used the standard algorithm, and thus immediately recognized their inconsistency. Four others realized an inconsistency after a second prompting question. It was not until after asking the remaining eight prospective teachers to calculate four divided by ½, using the measurement model, that seven of the subjects realized the contradiction. One subject
never did reach a conflict. Her comments reflected the conviction that “Mathematics is a series of meaningless rules that have nothing to do with reality” (Tirosh & Graeber, 1990, p. 103).

In conclusion, it was apparent that many of the subjects focused extensively on the domain of whole numbers, which perhaps caused them to overlook the equivalence-preserving nature of the “move the decimal points” part of the standard division algorithm. The fact that the teachers felt that they had mastered the algorithmic procedures may have made their misconceptions more resilient. In agreement with the theoretical framework, the interviews confirmed that the primitive partitive model of division, the extensive time spent on only using whole numbers in school, and the standard long division algorithm were the bases of the misconception “Division always makes smaller” (Tirosh & Graeber, 1990).

McDiarmid (1990) designed an early field experience that was intended to challenge prospective elementary teachers’ beliefs about teaching and learning mathematics. Specifically, teacher education students were brought face-to-face with their initial assumptions about addition and subtraction of integers. It was intended that this focused field experience would force prospective teachers to rethink, and possibly change, their understanding and beliefs on their subject matter and the way it was taught.

The sample consisted of prospective elementary teachers in an “Exploring Teaching” course offered by McDiarmid (1990) at Michigan State University. As a group, the subjects observed Deborah Ball, an experienced teacher who “Teaches in ways that were likely to challenge their assumptions and beliefs” (p. 14). Ball conducted her third-grade classroom in such a way that the students felt comfortable expressing their thoughts and understandings about mathematics without being ridiculed by their classmates. It was inferred that the observations were conducted to minimize disturbance of the classroom, though the methods of observation were not described. Prior to each observation session, the prospective teachers were asked to write and discuss the topic to
be studied by the third graders: addition and subtraction of integers. Prior to and following each of the classroom observations, the prospective teachers interviewed Ball on her plans and goals for the lesson, as well as her reactions to and rationale for specific events that occurred during the lesson. Additionally, Ball, McDiarmid, and the prospective teachers as a group developed a clinical interview that the prospective teachers then used to explore student understanding of the addition and subtraction of integers. Finally, the prospective teachers, each attempted to teach the same topic to “someone they knew” (p. 14) and wrote a case study on the experience. The study involved a total of four hours of discussion with Ball and 10 hours of classroom observation. The entire process took place over a period of four weeks.

Though individual prospective teachers’ responses were discussed, McDiarmid (1990) seems to have generalized unique anecdotal events from the group as a whole, appearing to select the most memorable events that occurred throughout the four weeks. One such experience that McDiarmid found unnerving was that most of the prospective teachers were unaccustomed to providing explanations of their answers in mathematics. When one prospective teacher was asked to explain his answer, he just repeated his first explanation more slowly. It was suggested that this prospective teacher must have learned this technique from observing some of his own teachers. Thus, McDiarmid observed that the results of this study did not imply that the beliefs of the prospective teachers had changed. It was concluded that the only reliable test of change in beliefs would be an examination of what the prospective teachers actually did in their classrooms. Evidence indicated that many prospective teachers reconsidered their initial beliefs; however, skepticism about the long-term effects of the course was acknowledged.

The purpose of the study described by Cooney et al. (1992) was to determine the relevance of research on the beliefs of mathematics teachers. Since preservice teachers’ knowledge and beliefs about mathematics and pedagogy are often too naive to support the current reform movement in mathematics education, materials “rooted in the notion of
constructivism" (p. 2) represent an attempt to nurture and provide the foundation for the further education of prospective teachers. A brief description of the elements of research on teachers' beliefs that were helpful in the design of the teacher education materials, referred to as the “Function Unit,” was provided. The goal of the study was to observe the impact of these materials upon secondary education preservice teachers who studied the unit.

The “epistemological foundation” for these materials involved the assumption that “A person learns through modeling their environment and consciously and subconsciously adjusts this model as they encounter resistance through further experience, particularly in the context of social interaction” (Cooney et al., 1992, p. 5). Therefore, the unit encouraged prospective teachers to be active and autonomous learners. The Functions Unit had three mathematical content goals for the prospective teachers: (a) to develop an understanding of function from an informal, intuitive viewpoint; (b) to connect the concept of function to real-world and applied situations; and (c) to develop a deeper understanding of functions beyond the secondary school mathematical content areas.

The design of the activities for this unit was based on six principles, outlined below. The prospective teachers should have the opportunity to (a) reflect on their mathematical learning experiences and understanding, (b) construct and organize mathematical ideas, (c) generate and evaluate mathematical interpretations, (d) assess students' mathematical understandings, (e) appreciate the historical development of mathematics and the role of mathematics in society, and (f) experience and practice innovative teaching methods, particularly those involving the use of technology. Various activities were used to generate and evaluate mathematical interpretations, including vignettes, card-sort activities, data-collection and modeling activities, and interview tasks. A graphing device of some sort was required for many of the activities (Cooney et al., 1992).
The study sample was made composed of 20 preservice secondary mathematics teachers enrolled in an undergraduate mathematics education class that used the Function Unit materials. Two of the researchers, Wilson and Shealy, observed all of the class sessions, with Wilson concentrating upon four specific teachers. All of the written assignments, tests, and field notes from the observations provided the bulk of the data for this study. Each of the four specified teachers participated in seven one-hour interviews throughout the course, each with a different format and a distinct objective. The first three interviews gathered information on the ways the prospective teachers conceptualized functions, mathematics, and mathematics teaching prior to participation in Function Unit activities. The tasks engaged preservice teachers in mathematical problem-solving (including the use of graphing calculators). The last four interviews provided data on how preservice teachers changed throughout the course, with the fourth and fifth interviews directed specifically at reactions to particular teaching episodes. The sixth interview asked the preservice teachers to cluster segments from their previous interviews, and the final interview was designed to probe further into the themes that emerged from the previous interviews, directed at gathering perceptions of the course and bringing closure to the process by asking the subjects for any final comments (Cooney et al., 1992).

At the beginning of the course, two of the preservice teachers demonstrated a narrow view of mathematics and two of them communicated a more open and flexible view. All four exhibited a dualistic approach to the teaching styles they believed they would use. For example, one prospective teacher described mathematics nontraditionally as a "way of thinking," and another prospective teacher continuously referred to the usefulness of mathematics, however, both of the prospective teachers stated that they thought their teaching style would be traditional and didactic (Cooney et al., 1992).

Throughout the course, it was manifest that there was substantial growth in understanding of functions occurred among the preservice teachers. Not only did the activities facilitate deeper understanding, they helped the prospective teachers reflect on
how their perceptions were connected to teaching issues. One teacher stated that the course activities helped her “become aware of and reflect upon important teaching issues, such as how students understand specific mathematical concepts differently than teachers, and how this might effect teaching” (Cooney et al., 1992, p. 13).

Most of the preservice teachers could not see the usefulness of mathematics at the beginning of the course. However, by the end they demonstrated an increased cognizance of its usefulness. Nonetheless, one prospective teacher communicated relativistic perceptions of mathematics while maintaining a dualistic notion about mathematics in general as demonstrated by her excitement about how the activities in the Function Unit helped her understand mathematics, but she explained that she would not generally expect her students to understand why all of the procedures worked. This perception led to a contradiction between what was an effective strategy for the prospective teacher and what she saw as an effective strategy for her students (Cooney et al., 1992).

In conclusion, the strategies used in the Function Unit were effective aids to help preservice teachers develop an understanding and increased knowledge of key mathematical connections involving functions. The strategies also helped the prospective teachers develop an appreciation of the power and usefulness of functions. These aspects coincided with what was considered essential to the reform movement in mathematics education. The preservice teachers were enthusiastic about using activities from the unit for their own enhancement of mathematical understanding. In contrast, their conflict with use of such a unit and beliefs about the nature of teaching of mathematics may prevent them from significantly changing their teaching approach (Cooney et al., 1992).

Results obtained from studies on change due to teacher education programs provided evidence of many positive changes in teachers’ perceptions about mathematics. First, the teachers communicated an increased desire to change (Borko et al., 1992; Civil, 1992; Cooney et al., 1992; Tirosh & Graeber, 1990; Wilcox et al., 1991). In addition, they demonstrated expanded awareness of the useful of mathematics (Cooney et al., 1992;
Schram, 1991) and increased confidence as mathematical problem-solvers (Wilcox et al., 1991), as well as a greater ability to articulate their understanding of mathematics (Schram, 1991). Despite these positive findings, several of the studies reported that, though the teachers seemed to have changed as a result of their particular teacher education course, the prior beliefs and perceptions of these teachers appeared to be deeply rooted (Civil, 1992; Cooney et al., 1992; McDiarmid, 1990; Schram, 1991; Wilcox et al., 1991). In fact, the results obtained by Civil (1992) revealed that most of the prospective teachers insisted on teaching in the very style that had failed them in the first place. The findings from Cooney et al. (1992) echoed this observation insofar as the preservice teachers manifested a certain contradiction in what methods worked for them in the “Function Unit” and what they believed would work for students in their classrooms.

The research discussed in this section highlights a number of implications for future research. First, further research is needed to examine what prospective teachers actually bring to teacher education courses (McDiarmid, 1990; Schram, 1991; Wilcox et al., 1991). This information would be helpful for the design of favorable curricular changes to help prospective teachers change undesirable perceptions positively. Another suggestion for future research involves the examination of the support structures needed by prospective teachers during their first years of teaching. Civil (1992) contended that without follow-up support, the seeds planted in teacher education courses may not germinate.

Change Due to Experience

The three studies in this section were organized based on the methodologies used. The first two studies, Leinhardt and Smith (1985) and Borko and Livingston (1989), examined the differences between a set of expert teachers and a set of novice teachers. Leinhardt and Smith focused on change in elementary teachers’ subject matter knowledge
as a result of experience. Borko and Livingston extended this type of research to include secondary teachers and the exploration of change in teachers' pedagogical content knowledge. The final study in this section, Sullivan and Leder (1992), examined changes in elementary teachers' pedagogical content and subject matter knowledge by means of a longitudinal study, following preservice teachers through their first year of teaching.

The relationship between expert and novice fourth-grade teachers' subject matter knowledge and/or classroom behavior was explored by Leinhardt and Smith (1985). In particular, fraction knowledge, was explored in depth for natural teaching settings. The goal of this study was to examine the nature, level, and use of this specific subject matter knowledge among a set of teachers. The sample consisted of eight fourth-grade mathematics teachers, four experts and four novices, chosen from a sample of 12 expert and four novice teachers who participated in a three-year study of teaching expertise.

Two forms of data were analyzed. The first instrument was a card sort, consisting of 40 math problems randomly selected from a fourth-grade text. The teachers were also required to provide a rationale for their sorts. Second, videotapes of three expert teachers giving lessons on reducing fractions were examined in detail. These three teachers were deliberately selected by the researchers because they used the same teaching style and text, taught the same topic, and used similar examples while differing in their organization of subject matter knowledge (Leinhardt & Smith, 1985).

Despite the high levels of student success (as determined by student test scores) for all the teachers, the outcome of the card sorts and fraction knowledge interviews confirmed the original categorization of knowledge levels as stated in Leinhardt and Smith (1985). From the math topic card sort it was found that the novices made categories for every one or two problems and noted little differentiation in problem difficulty. In contrast, the more experienced and competent teachers sorted the cards into approximately 10 categories and ordered the topics by difficulty through to decimals, thus exhibiting a more refined hierarchical knowledge structure.
One of the most striking results of the interviews was a difference in the ability of expert teachers to express an algorithm. This difference was primarily attributed to the "low knowledge," or teachers’ lack of understanding of underlying mathematical concepts and relationships. To further explore this disparity, the three videotapes of expert teachers on reducing fractions were viewed to make comparisons of knowledge used during actual classroom teaching. To make comparisons, brief summaries of the lesson flow and semantic nets for the subject matter presentation of each teacher were prepared (Leinhardt & Smith, 1985). Semantic nets were also created for the textbook organization of concepts on equivalent fractions. Though the textbooks seemed to lack critical information on how the concepts and relationships discussed should be applied in problem-solving situations, it was felt the publisher’s decision was intentional. The text provided the general framework of the concept, that could then be further elaborated by individual teachers.

Substantial differences in the teachers’ content emphasis and elaboration were found. Specifically, substantial differences were found in the levels of conceptual information presented, the degree to which procedural algorithmic information was presented, the way each of the teachers entered the topics, and the use of representation systems (i.e., number line, regional, and numerical). It was concluded that the use of semantic nets permitted a systematic means to display the teachers’ system of knowledge, which consisted of both their conceptual understanding and the way they transmitted it to their students. Deficiencies in conceptual understanding could be determined by the semantic nets and then used to focus upon teacher remediation and change in subject matter and pedagogical knowledge (Leinhardt & Smith, 1985).

Borko and Livingston (1989) examined the nature of pedagogical expertise by comparing the mathematical thinking and actions of expert and novice teachers. The goal of this study was to examine the relationship between teachers’ knowledge structures and the improvisational characteristics of interactive teaching practices. Additionally, this
study explored the patterns of difference between novice and expert teachers in terms of specific changes in their knowledge structures. The sample included four student teachers (novices) and each of their cooperating teachers (experts). All of the student teachers were enrolled in a Masters’ Certification Program at a large eastern university. Two of the student teachers were prospective secondary mathematics teachers and two were elementary teachers. The cooperating teachers were chosen based on the recommendation of building administrators and county teacher center coordinators, and their students were selected based on the strength of their mathematical background and performance in mathematics methods courses. Due to the illness of one elementary school cooperating teacher, the findings reported in this study were from only three pairs of novice and expert teachers.

For one full week of instruction, each participant was observed teaching mathematics. Each student and cooperating teacher were interviewed about their instructional plans prior to teaching, and on their reflections about the lesson following teaching. This type of interview process was used on consecutive lessons to trace the evolution of sequences of instruction. Specifically, the interview examined what influence the classroom events and the participant reflections had on succeeding plans and teaching. The elementary participants were observed only during mathematics periods, whereas secondary participants were observed for two sections of the same course on each day. Preobservation interviews were semistructured and focused on asking participants: (a) to discuss the nature of the lesson, (b) how they planned for the lesson, (c) what they thought about when they planned the lesson, and (d) what factors influenced their plans. Postobservation interviews focused on asking the participants to reflect on: (a) prominent features of the lesson, (b) unexpected occurrences, (c) changes from plans, and (d) reasons for any changes. All the interviews were taped and transcribed. Photocopies of the written plans, content notes, and texts were gathered to supplement the data (Borko & Livingston, 1989).
When examining patterns across expert teachers' planning, it was found that all had engaged in several levels of planning. The goals of planning involved establishing a general content, developing a curriculum sequence for the course, and constructing a timeline for content coverage. Shortly before the actual teaching events, all three teachers made decisions about the details of instruction. However, final decisions about the specific examples to be used were determined during actual teaching rather than before teaching began. None of these expert teachers had written lesson plans; they merely described mental plans for their individual lessons (Borko & Livingston, 1989).

Unlike the similarities in expert teachers' planning styles, differences were noted in their individual teaching styles. Ellen described her course (analytic geometry) as "strictly them (the students) taking notes and answering questions . . . , lecture-type situations and demonstration" (Borko & Livingston, 1989, p. 481). In contrast, both Scott (an elementary teacher) and Randy (a calculus teacher) characterized their lessons as inductive. Despite these differences, however, the actual lessons shared many characteristics indicative of their expertise, such as skill in keeping the lesson on track and accomplishing their objectives, allowing student questions to lead discussions, simultaneous generation of illustrative examples to reinforce concepts, and achieving an appropriate balance between content-centered and student-centered instruction. The expert teachers' post-lesson reflections were focused and concise. Focus was largely directed at students' understanding of the material and rarely on classroom management.

Examination of the patterns across novice teachers' planning found that the prospective teachers had mental plans and agendas for their lessons. In fact, their agendas were flexible with respect to many of the same instructional elements used by the expert teachers, including timing, pace, instructional examples, and problems for the students. This finding seemed to contradict previous results which had found novice teachers not as "planful" as experts (Borko & Livingston, 1989). Perhaps this was due to the fact that the novice teachers selected had strong mathematical content knowledge and pedagogical
preparation. Despite similarities in the planning agendas of novice and expert teachers, it was discovered that the processes by which they were created were quite different. Essentially all of the planning completed by novice teachers was short-term in nature. Furthermore, to plan content presentations, the novices primarily considered the textbook, the teacher's manual, and their own experiences in learning the materials.

It was evident that the novice teachers were not as effective as the experts at translating their plans into actions. All three of the novices had difficulty when students' questions or comments led them to attempt explanations for which they were not prepared. Moreover, their explanations lacked connectedness across the curriculum. Time also seemed to be a factor. The novice teachers' post-lesson reflections primarily addressed concerns related to their own effectiveness as teachers, as well as student involvement and classroom management. Their reflections appeared to be more dependent on the events of that day, as opposed to the more global concerns of the expert teachers. The novices were equally as focused on their reflections as the experts (Borko & Livingston, 1989).

From these results it was conjectured that teachers' pedagogical thoughts and actions change with experience. It was noted that several of the difficulties encountered by the novice teachers were due to the fact that it was the first time they taught this particular body of knowledge. Moreover, it was found that the expert teachers' propositional structures for pedagogical content knowledge included a reservoir of powerful explanations, demonstrations, and examples for representing subject matter to their students. Thus, the analysis of teaching as a complex cognitive skill justified the several differences in the planning, teaching, and post-lesson reflections of expert and novice mathematics teachers that were considered. It was conjectured that novice teachers' cognitive schemata were less elaborate, interconnected, and accessible than those of experts. Additionally, the pedagogical reasoning skills of the novice teachers was quite immature (Borko & Livingston, 1989).
Sullivan and Leder (1992) were concerned that the classroom practices of beginning teachers differed from those recommended by researchers, teacher educators, and curriculum developers. Specifically, it was believed that novice teachers tended to be conservative, instrumental, authoritarian, and make predominant use of drill and practice type activities. Sullivan and Leder sought to identify those factors that influenced the mathematical thinking and teaching of mathematics of beginning elementary teachers.

The subjects for this study consisted initially of all 10 preservice teachers enrolled in a teacher education program, and each of whom participated in a class in which the researchers were lecturers. Seven of the 10 who secured a teaching job following graduation became the subjects of this investigation. All subjects completed a three-year teacher education program, and practice taught for eight weeks during each year of the program. For each program year, about one-third of the courses were tertiary academic subjects and the rest addressed the preparation of teachers. Mathematics education courses made up about 8% of the requirements for the program. One of the seven subjects was a mathematics major (Sullivan & Leder, 1992).

A case study approach was deemed appropriate due to the fact that it was impossible to isolate the variables and study them quantitatively. The framework for data collection, as adapted from Clark and Peterson (1986), consisted of the following components: (a) teachers’ thoughts (in particular, the effects of their backgrounds and attitudes) and their beliefs and understandings about what mathematics teaching should be; (b) factors that restricted beginning teachers from implementing their desired teaching styles and the salient factors that influenced the way they taught; and (c) teachers’ actions, as determined from both the classroom observation and self-reported records of classroom practices. Prior to the major data collection period, the subjects were asked to complete a survey regarding their educational background, beliefs about teaching, and their goals and aspirations. The three major components of the data collection included interviews, classroom observations, and surveys (Sullivan & Leder, 1992).
Individual interviews occurred once during their final year of the teacher education program, three times during the first year of teaching, and twice during the second year of teaching. The interviews were structured about the subjects' attitudes, their beliefs about mathematics and mathematics teaching, their concerns about themselves and teaching, self-reported teaching practices, the most salient influences on their teaching, the types of planning in which they engaged, their goals and aspirations, and an evaluation of the teacher education program. Also, once each year, a card sort accompanied the interviews. Sets of influences were written individually onto cards and the stack of cards was given to each subject. Each subject removed the cards that contained a factor that had no influence, added any factors that were not listed, and then arranged the cards in order from the most influential to the least. Although the card sort may have biased the prospective teachers, it was used only to supplement other instruments used (Sullivan & Leder, 1992).

Classroom observations occurred three to four times during the final year of the teacher education program and twice during the first year of teaching. The particular elements that were examined carefully included the quality of explanations provided by teachers, the level of questions the teachers directed at students, the extent to which manipulatives were used, and the extent to which the lessons were child-centered (Sullivan & Leder, 1992).

To compare the consistency of the interview responses from the seven subjects, open-ended surveys were completed by 120 student teachers during the teacher education program, and then again by 55 of the 85 students who had secured teaching jobs following graduation. These surveys examined respondent beliefs about mathematics teaching, asked for an evaluation of the teacher education program, and requested subjects to summarize their concerns, goals, aspirations, and teaching practices (Sullivan & Leder, 1992).

The general results from the 120 student teacher surveys and the 55 first-year teacher surveys demonstrated that the teachers were more confident about teaching
mathematics after one year of experience. The card sorts also provided an indication of the way the subjects' perceptions changed over the first year. During training, most of the subjects considered practice teaching as well as the mathematics curriculum units to be important. By the end of the first year of teaching, all of the teachers selected the materials and texts as influences as well as their own ideas and personalities. This confirmed the concept that the materials available influenced both the content taught and the methods used. The remainder of the results involved the discussions of two subject case studies, chosen because the two selected represented a range of teaching styles and because these two mentioned the reactions of students more directly in their responses to the interview questions than had the others (Sullivan & Leder, 1992).

In conclusion, the important factors reported from the card sorts and surveys did not emerge as the most important factors found in the data collected from within the components of the case study framework described earlier. The response of the students to instruction was the most influential factor cited by the seven case-study subjects during planning, teaching, and review. This response was evidenced by the observation that all classroom events were planned and interpreted in terms of the response of the pupils. The teachers appeared to be more teacher-directed, gave more explicit directions, emphasized completion of work over comprehension, and avoided student-centered activities or problem-solving tasks.

In summary, the major changes that were found based on experience included an increase in teachers' conceptual knowledge (Leinhardt & Smith, 1985) and the connections of this knowledge to lesson preparation (Borko & Livingston, 1989; Leinhardt & Smith, 1985). Sullivan and Leder (1992) extended the connections of this knowledge to dealing with students' responses. In general, based on inconsistent definitions of expert status, it is difficult to draw comparisons across these studies. The description of an expert teacher given by Leinhardt and Smith (1985) was based on consistent growth scores among their students. The criteria provided by Borko and
Livingston (1989) included being a cooperative teacher, recommended by either the building administrator or the county teacher-coordinator. In many cases, just the number of years taught was the only means for determining level of expertise.

The implications of the investigations on change due to experience involve recommendations for future research, as well as suggestions for teacher education programs. Leinhardt and Smith (1985) suggested that semantics nets should be used to chart the conceptual understanding of teachers and prospective teachers. These nets could then be used as the focus of remediation of, and eventually change in, subject matter perceptions. Based on the knowledge that since teachers are heavily influenced by materials and textbooks (Leinhardt & Smith, 1985; Sullivan & Leder, 1992), then the marginal notes in teacher manuals should be expanded. Borko and Livingston (1989) recommended limiting the number of different courses that prospective teachers have to prepare; rather, it was proposed that the prospective teachers offer the same course several times. This proposal was based on the increase of successful planing implementations each time a course was taught.

Change Due to Inservice, Workshops, and Special Programs

This section reports three studies that focused on change in experienced elementary teachers' pedagogical content knowledge as a result of some type of intervention. Carpenter et al. (1989) examined this change as a result of teachers participating in a cognitively-guided instruction which provided teachers with knowledge derived from research on children's' thinking. Wood et al. (1991) examined this change in one subject who was involved in a second-grade teaching project that emphasized problem-centered instruction, pair collaboration, and whole class discussion. Finally, Putnam (1987) examined this change as a result of participation in several live and simulated tutoring sessions.
The purpose of the study by Carpenter et al. (1989) was to investigate whether provision of teacher access to explicit knowledge derived from research on children’s thinking in mathematics would influence teacher instruction and student achievements. This study drew on remarkably consistent findings from research conducted on the development of addition and subtraction concepts and skills among young children. The goal of the study was to provide teachers with detailed knowledge on children’s thinking, addressing the following questions: Do these teachers employ different instructional practices in their classroom than teachers who were not provided with this information? Do they have different beliefs about teaching mathematics, about how students learn, and about the role of the teacher in facilitating that learning? Do the students of these teachers have higher levels of achievement or confidence in their ability, or have different beliefs about themselves and mathematics than the students of teachers who were not provided with this information?

The study sample consisted of 40 first grade teachers (39 women and one man) and their students from 24 different schools in and adjacent to Madison, Wisconsin. The subjects volunteered to participate in a four-week summer inservice program, to be observed during periods of mathematics instruction the following year, and to complete necessary questionnaires and interviews. Twelve students (six girls and six boys) were randomly selected from each case to serve as target students for observations and interviews (Carpenter et al., 1989).

Half (n=20) of these teachers were randomly assigned, by schools, to the Cognitively Guided Instruction (CGI) treatment group. These teachers participated in an 80-hour workshop taught by Carpenter and Fennema during the first four weeks of summer. The goal of the first half of the workshop was to help teachers understand how children develop addition and subtraction concepts and to provide them with access to knowledge in terms of how to classify word problems, identify students strategies in solving these problems, and how to best apply these strategies. The last half of the
workshop focused on teacher-designed instruction, based on the following principles derived from research: (a) instruction should develop understanding by stressing relationships between skills and problem-solving; (b) instruction should be organized to facilitate active construction of their own knowledge among students; (c) each student should be able to relate problems, concepts, or skills learned to those already possessed; and (d) instruction should be based on what each child knows. The teachers were not given a specific written assignment, but were asked to plan a unit to teach during the following year, as well as a year-long plan for instruction based on the principles of CGI (Carpenter et al., 1989).

The other half of the teachers (n=20) constituted a control group, consisting of participating in two two-hour workshops held in September and February during the instructional year. These workshops focused on non-routine problem-solving and required no year-long planning activity. The goal of these workshops was to give the control group teachers a sense of participation in the project, not to provide a contrasting treatment. These workshops were taught by a graduate student member of the CGI staff (Carpenter et al., 1989).

Two observation systems were constructed for a minimum of 16 days of classroom observations during the following school year: one focused on teachers and the second focused on students. For both systems, the observers used a 60-second time-sampling procedure involving observations for 30 seconds and the coding of behaviors and activities for the following 30 seconds. The observation categories, as selected from the literature, included: setting (whole-class, medium group, small group, teacher/student alone), teacher behavior (focused on process rather than answer), expected strategy (direct modeling, advanced counting, derived facts, recall, multiple strategies), and content. The teacher wore a wireless microphone and the observer listened through earphones to help understand the teachers’ private interactions with students (Carpenter et al., 1989).
To assess the teachers' knowledge of their own students, three measures were constructed. For each measure, teachers were asked to predict their target student solution strategies. The scores were based on the match between their predictions and the actual student response. The three measures included: (a) knowledge of number fact strategies (five items), (b) knowledge of problem-solving strategies (six problems), and (c) knowledge of problem-solving abilities (eight problems). Teachers' beliefs about learning and teaching addition and subtraction were measured using four 12-item scales: (a) the role of the learner; (b) relationship between skills, understanding, and problem-solving; (c) sequencing of mathematics; and (d) the role of the teacher. For each item the teachers responded using a five-point Likert-type scale (Carpenter et al., 1989).

At the beginning of the year, student achievements were measured using the Mathematics subtest of the Iowa Test of Basic Skills (ITBS), Level 6, as a pretest. During the spring, the Computation subtest of the ITBS, Level 7, as well as a mixture of standardized problem-solving skills and experimenter-developed problem-solving tests were used for the post-test. Additionally, interviews were conducted to determine the strategies the students used to solve certain problems. Student confidence was measured using a 12-item instrument that involved asking the students to verbally answer whether or not they could solve each problem. Students were interviewed on their beliefs using the same four-belief constructs as used for the teachers. A 16-item interview was used with a three-point Likert-type scale (yes, no, or maybe). Student reports of attention and understanding were assessed through interviews using questions that were developed from research. The student interviews were conducted in the spring by trained interviewers. Within two days of these interviews, trained graduate assistants conducted the teacher interviews (Carpenter et al., 1989).

The results of this study were organized into four sections representing the four major analyses conducted: (a) how observations of CGI and control teachers' classrooms differed, (b) differences between CGI and control teachers' knowledge and beliefs,
(c) effect on student achievement, and (d) effect on students' beliefs and confidence (Carpenter et al., 1989).

Classroom observational analysis involved computing the means, standard deviations, and t-tests between groups for 27 categories and both the teacher and student observation systems. The categories were grouped into four sections: setting, content and lesson phase, behavior, and strategy. No significant differences were found between the CGI and control teachers' classroom settings for either of the two systems. Although the CGI and control students spent the same amount of time engaged in addition and subtraction, a significant difference was found in content emphasis between the CGI and control teachers for these topics, \( p < .01 \) (Carpenter et al., 1989). The CGI teachers spent significantly more time on word problems, whereas the control teachers spent significantly more time on number-fact problems. The control teachers and students spent significantly more time on review; however, no difference was found in the time spent on development, controlled practice, or seat work. In the teacher behavior section, it was found that CGI teachers posed problems to students more frequently and also listened more often to the process used by students to solve problems, \( p < .01 \). The control teachers focused more frequently on the answer to the problem, \( p < .05 \). Though there were no overall differences apparent between the CGI and control classes for the strategies students actually used during class, the CGI teachers allowed students to use a variety of different strategies more frequently than the control teachers, \( p < .05 \).

Means, standard deviations, and three t-tests (for the three measures previously discussed) between groups for the test scores on teachers' knowledge were presented tabularly. As expected, the CGI and control teachers differed significantly in their knowledge of student strategies for both number facts and problem-solving, \( p < .01 \). However, no significant difference was found in their knowledge of student problem-solving abilities. Pre- and post-test means and standard deviations for the CGI and control teachers' beliefs were also presented tabularly. In addition, Group X time analyses of
variance (ANOVA) were calculated to examine the treatment effects for each of the four teacher belief scales from pre- to post-test. Results indicated that the CGI teachers were significantly more cognitively guided (i.e., able to identify student strategies and know best how to apply them) than were control teachers. This result was demonstrated by a significant time-by-treatment interaction after treatment for the second and fourth scales and total of all four scales (respectively, \( p < .01; p < .05; p < .01 \), as previously described; Carpenter et al., 1989).

Controlling for prior mathematics achievement as measured by the pretest, six analyses of covariance (ANCOVA) between groups were computed for students. The CGI students demonstrated a higher level of recall of number facts, scoring higher on the Complex Addition and Subtraction Word Problem test, and using correct strategies significantly more often than did control class students, \( p < .05 \). Means, standard deviations, and four t-tests between groups were presented for students’ confidence and beliefs. CGI students were more confident of their ability to solve mathematics problems than were control students, \( p < .05 \). From results identical to those for teachers, the CGI students were significantly more cognitively guided in their beliefs than control students, \( p < .01 \), and reported a greater understanding of mathematics than control students, \( p < .05 \). No difference was found between CGI and control students in the extent to which they were attentive during mathematics classes (Carpenter et al., 1989).

In general, the results of this study provided a representation of teachers’ knowledge and beliefs, and classroom instruction practices, as well as students’ achievements and beliefs that were consistent with the assumptions and principles of CGI. Two major CGI themes were validated by these results: (a) instruction should develop understanding by stressing the relationship between skills and problem-solving, with problem-solving as the focus of instruction; and (b) instruction should build upon students’ existing knowledge. In contrast to control teachers, the CGI teachers communicated beliefs that were more consistent with the principle that problem-solving
should be the main focus of instruction. Additionally, based upon standardized tests of computational skills, the study documented that a focus upon problem-solving does not necessarily result in a decline in performance in computational skills (Carpenter et al., 1989).

The article by Wood et al. (1991) discussed the importance of understanding the processes involved as teachers make changes in their teaching of mathematics to accommodate the transformations suggested by research. The purpose of the study was to examine the process by which a second-grade teacher made changes in her beliefs about teaching mathematics while participating in a specific teaching project, determining whether the changes, as measured, were due to reflection, and/or resolution of conflict between traditional teaching methods and the goal of the construct of mathematical meanings by children, the principal emphasis of the project. This project was conducted during one full school year, providing learning opportunities which were significantly different from those in a traditional mathematics classroom. Three interrelated aspects were presented as equally important: (a) problem-centered instructional activities, (b) pair collaboration, and (c) whole-class discussions. All instructional activities were designed to be open-ended, challenging the students to create their own knowledge and to devise their own solutions.

This case study examined one female second-grade teacher from a rural elementary school in the midwest with 15 years of elementary teaching experience. Based upon ITBS test results, the 20 students in the selected class were not significantly different from students in any of the other five second-grade classes. Every mathematics lesson of the project classroom was videotaped for the entire year. During pair collaboration, two cameras were focused on four pairs of students, and these eight students were followed the entire year. The data source for the evolving nature of subject mathematical understanding included the video recordings (and resultant transcriptions), ethnographic field notes, and copies of all the work performed by subjects (Wood et al., 1991).
Open-ended interviews were conducted with the project teacher during weekly meetings. Data from the project meetings also included discussions of the teacher's concerns and questions about the project. Detailed procedures for implementing the curriculum were not provided for the teacher, thus she had to make immediate classroom decisions based on her knowledge. Over the course of the year-long project it was evident from the data that the project teacher gradually reorganized her beliefs for three major factors: (a) the teacher's role in mathematics instruction, (b) teaching mathematics as a process of negotiation rather than imposition, and (c) the overall teaching and learning of mathematics (Wood et al., 1991).

Not only were the philosophy and methods imposed by this project quite different for the project teacher, they were equally unconventional for the children. The initial discussions between the teacher and students about the new teaching and learning style were crucial to the success of the project. As the teacher became familiar with listening to and respecting the explanations given by the second-graders, she discovered that their thinking was much more sophisticated than she had previously assumed. Recognizing that her role was no longer to be the sole source of knowledge, she became more of a facilitator in the development of students' mathematical understanding. Eventually, the teacher realized that it was important not to impose her methods upon the students, but rather create conditions in which conflict resolution, negotiation of meaning, and considering the perspective of others occurred. This change did not come easily. Even with the support of the project staff, temporary acceptance of incorrect answers from students created a significant contradiction for the teacher with respect to her beliefs about the nature of mathematics. In her initial attempt to resolve this conflict she tried "funneling" student discussions until correct answers were reached. Reflecting on this procedure, the teacher realized that she was still "telling" rather than teaching. A final resolution was reached when she began to teach by negotiation rather than by imposition, conveying to the students that they expected them to resolve the conflicts that arose from
differences over answers. The children were then able to make sense of the situation in terms of their existing knowledge (Wood et al., 1991).

In conclusion, a definite essential relationship between pedagogical practice and teachers' learning was demonstrated by this classroom teaching experiment. What the project teacher learned contrasted sharply with her previous traditional practices. The project teacher's "change" was best expressed by her seemingly paradoxical comments, "I am not teaching (mathematics) anymore," stated at the beginning of the year, and "I am teaching (mathematics) more this year than ever before," stated toward the end of the year (Wood et al., 1991, p. 609).

The stated purpose of the Putnam (1987) study was to observe experienced teachers in tutorial interactions, determining what diagnostic strategies they used and how they changed their instruction based on these diagnoses. The results led to a major shift in the focus of the study. It was found that diagnosis was not a significant part of teacher interactions, rather the teacher's knowledge of the content and the ways it was structured and sequenced for students seemed to be most critical. The sample consisted of four second-grade and two first-grade teachers from public schools in the San Francisco Bay area, each with at least 10 years of experience teaching elementary school. Each teacher tutored one second-grade student, selected based upon performance on a 12-question addition test. Students who made systematic computational errors were identified as possible candidates for the tutoring. Arrangements were then made to tutor one of the students for two 20-minute sessions in the classroom either before or after school, each of which session was strictly timed as well as videotaped. The goal as stated in the written instructions was "To teach the student to add numbers, each having up to three digits, with renaming (carrying)" (p. 18). The teachers were able to use any procedures, techniques, or manipulatives they believed to be appropriate.

Directly following each tutoring session, the researcher conducted a stimulated recall session for each teacher. The videotape was reviewed and was stopped at certain
points to ask questions and discuss the teacher’s choice of methods. The purpose was to gain insight into teaching strategies and goals by allowing the teacher to clarify actions. These loosely structured review sessions were audiotaped and used to augment the data from the tutoring sessions (Putnam, 1987).

Upon completion of the live tutoring sessions, the teachers tutored six computer-simulated students. Each teacher received a set of written instructions describing the task and how various moves worked. The simulated student was designed to make systematic errors working addition problems posed by the teacher. Descriptions of the specific procedures and resulting errors were discussed. The teacher continued to work with the student until both felt that the skill of addition was mastered. To assess the accuracy of the teachers’ diagnoses, a retroactive “prediction task” was presented to each teacher, who was then asked to work four problems as the simulated student would have worked them prior to tutoring. After each simulated student was tutored, except for the first “practice student,” the researcher asked the following questions: (a) What was the student’s problem; (b) when you decided what the problem was, how did you try to teach her/him; and (c) if you had been working face to face with the student, how would your approach have been different? Directly following the simulated tutorials, the researcher conducted a simulated recall session similar to that used after the live tutor sessions. The computer-recalled characteristics were used in the same manner as the videotapes from the live tutoring sessions (Putnam, 1987).

Results indicated that when students made errors, the teacher’s primary subgoal was to obtain the correct answer to the problem. In fact, getting the correct answer was the only subgoal of a teacher following 58% of the errors. In most cases no attempt was made to teach general addition facts, procedures, or concepts. During the live tutoring, it was evident that teachers did not respond to student errors by searching for the exact nature or extent of the students’ difficulties, an approach which would have been suggested by the diagnostic/remedial approach. This concern for moving students toward
the correct answer without diagnosis of the source of the error was also evident in the simulated tutoring sessions (Putnam, 1987).

Although the teachers differed in how systematic they were in varying problem features, they all demonstrated a consistent sequence of problem types, thus presenting the notion of "curriculum scripts," described as "a loosely ordered but well defined set of skills and concepts students are expected to learn, along with the activities and strategies for teaching this material" (Putnam, 1987, p. 17). Differences among teachers' questioning styles reflected the overall goals for teaching addition. The goal of having students carry out the steps in the standard written algorithm for addition was common for all the teachers, however, they differed considerably in what aspects of the addition algorithm were emphasized. Some teachers had algorithmic goals, while others emphasized conceptual understanding. The simulated tutoring reflected the overall goals inferred from each of the live tutoring sessions.

The teachers' primary goals were not to diagnose the nature of student difficulties accurately, as evidenced by the teachers' performance for the retroactive prediction task. In fact, the computer calculated that teachers provided accurate descriptions of students difficulties in 40% of the cases, partially accurate descriptions in 23% of the cases, and incorrect or no descriptions in 37% of the cases (Putnam, 1987). In addition, it was stated that the teachers often could not recall the student difficulties. The evidence obtained through performance of the retroactive prediction task reinforced the fact that the teachers relied more heavily on what goals the students could achieve, rather than upon arriving at accurate and complete diagnoses of the source of student difficulties.

Putnam (1987) concluded with a lengthy discussion on how teachers moved through a "curriculum script" when teaching a particular topic. The exact nature of this script varied for different subject-matter areas and for different topics. Thus, it was defined to fall in the category of pedagogical content knowledge. The curriculum scripts used by these teachers during tutoring were extensions of those used in the classroom.
In summary, both of the studies by Carpenter et al. (1989) and Wood et al. (1991) demonstrated that teachers gradually changed (i.e., reorganized) their perceptions about mathematics and teaching mathematics due to staff development interventions. This result included the role of becoming more of a facilitator, recognizing the process of negotiation rather than imposition (Wood et al., 1991), and the acceptance of a larger variety of strategies for problem-solving (Carpenter et al., 1989). The findings on change due to inservice, workshops, and special programs provides a reiteration of the fact that change did not come easily. Although the subjects consistently communicated beliefs that problem-solving should be a principal focus, they recognized the difficulties of immediate implementation based on the sharp contrast of this approach to traditional practices (Carpenter et al., 1989; Wood et al., 1991).

The major implication shared by all of the studies on change due to inservice, workshops, and special programs was the need to emphasize building classrooms in which students take an active learning role. The development of workshops and special programs designed to make teachers aware of the importance of a problem-solving emphasis is vitally important (Carpenter et al., 1989). In addition, Putnam (1987) suggested that teachers be made aware of several problem-solving strategies, as well the many misconceptions the students may have regarding different strategies and how they could be diagnosed. Thus, investigations considered in this section provide reiteration of the need to building upon existing knowledge and the need for research on finding ways to best guide and support teachers as they encounter change.

Discussion of Change in Mathematics Teachers’ Perceptions

The samples considered in this section on conceptual changes in mathematics teaching were predominantly female, and the majority of the studies reviewed involved elementary teachers. In the section on change due to inservice, workshops, and special
programs, the teachers for each of the samples taught only first or second grades, thus
findings from these samples cannot be generalized across mathematics teachers as a whole.
Nonetheless, since the studies considered are the only ones of this type that have been
conducted, their results and implications are important for a formative knowledge of the
impact inservice, workshops, and special programs have upon mathematics teachers' perceptions.

Analogous to the samples predominantly confined to elementary teaching, the
mathematical topics that were examined were remedial. Most of the research reviewed
was concerned with teachers' understanding of whole numbers and fractions,
multiplication and division, and general middle school mathematics. The highest level
mathematical topic examined was functions. Thus, to obtain a more integral view of
mathematics teachers' subject matter and pedagogical perceptions, research involving
higher level mathematics concepts must be conducted.

Students' Perceptions of Limits

The limit concept plays a fundamental role in the development of calculus. Not
only is the language of limits essential to communicating the behavior of functions, but the
concept of limit serves as a prerequisite for such other calculus topics as the definitions of
differentiation and integration. Although the concept of limit permeates the calculus
curriculum, it has proven to be extremely difficult for calculus students to understand.
Significant progress has been made in understanding students' perceptions over the past
15 years. While the present study is concerned with teachers and teaching, no research
exists that could be used to guide understanding of teaching subject matter and
pedagogical perceptions of limits. Therefore, studies of students' perceptions of limits
have been included to assist in understanding perceptions of limits in general.
Arguably the most in-depth study on students' understanding of the concept of limit was completed by Williams (1990, 1991). A questionnaire was administered to 341 second-semester college calculus students, from which 50 students volunteered to participate in further study. Based on responses to the questionnaire, 10 students who were judged to clearly and unambiguously fit into a particular category (viewpoint of limits) were chosen to participate. From review, the four viewpoints included the limit as motion, reachable, bound, and approximation.

The participating students were interviewed on five occasions throughout a three-part instructional sequence. The first phase of the sequence included an interview that was used to explore the components of a student model of limits. During the second phase, students were given two opposing descriptions of limits, and were interviewed upon three occasions. The students were asked to explain each viewpoint and choose which one was best suited to their view of limits. For example, in one of these interviews the students were presented with the following two statements reflecting a difference whether or not a limit was a boundary beyond which a function could not go (Williams, 1990):

Statement 1: A limit is like a point or line you can't cross over. So mathematically, a limit is a point or number that a function cannot cross over, or get bigger than. It's the largest or smallest the function can get as x approaches t.

Statement 2: The term limit has different meaning when you use it mathematically than when you use it in ordinal speech, like the term "speed limit." It isn't really a boundary or something you can't cross over. The function can assume values much larger or smaller than its limit point, and even when you get close to limit, it can assume values both larger and smaller than the limit. What matters is that for all x close enough to t, the values of f(x) are arbitrarily close to the limit. (p. 295)

Students were then given a set of problems and asked to explain each from the alternative viewpoints they had just read. During the last phase of the sequence and the final
interview, students were asked to explain why their viewpoint of limits had or had not been changed during the course of the study.

The results with respect to the first three viewpoints (the limit as motion, unreachable, or as a bound) are applicable to the objectives of the present study. Prior to participation in the instructional sequence, nine of the 10 students thought of the limit as unreachable. By the final sequence session, seven of those nine had changed their viewpoints. Initially, four of the 10 students thought of the limit as a boundary, but by the end of the sequence only one retained this belief. No change occurred in any of the students' dynamic view of the limit (i.e., the limit as motion). In many cases it was determined that the students felt that all of the viewpoints were acceptable, depending upon the nature of the specific problem. Certain counter-examples to this viewpoint were considered to be insignificant exceptions. Seven of the students placed great faith in using graphs as a means to understand the concept of limit. Finally, Williams (1990) found that students chose the model of limits which was the easiest for them to understand and work with, whether or not it happened to be correct.

Another well-documented analysis of students' perceptions of limits was that provided by Davis and Vinner (1986) for a study involving 15 second-year calculus students. Though this study investigated students' perceptions of limits of sequences rather than limits of functions, the results pertaining to the intuitive vocabulary which the students used to describe the limit of a sequence may also be applied to the limit of functions. The research assumption was that students held non-mathematical images of such terms as limit. For example, students frequently confused the word “limit” with bound, as for a speed limit that could not be reached. This casual meaning of the term limit may interfere with the understanding of the concept in a mathematical context.

Tall and Vinner (1981) were also concerned with the casual (i.e., non-mathematical) meaning of the term “limit.” A questionnaire was given to 70 first-year college students with either A or B grades in A-level mathematics. The students were
presented a formal limit notation and then asked to define limit. Additionally, the students were given a formal limit problem and solution and were then asked to discuss the meaning of such a problem and answer. Fifty-four of the students used the dynamic or motion approach when exploring the definition question using language including “gets close to,” “approaches,” and “tends to.” Tall and Vinner believed that this approach might lead students to assume that at a given point the function can never equal the limit. This fact could create a potential cognitive conflict for the students.

Several studies have investigated students’ perceptions of other calculus topics, as a result deriving some conclusions regarding the concept of limit. For example, in the context of studying students’ understanding of integration, Orton (1983a) found that students had poor conceptions of the power of limiting processes in mathematics. In fact, few students realized that the limit of a sequence of successive approximation to the area under a curve would actually give the correct area.

In the exploration of students’ understanding of the central concepts of calculus, Graham and Ferrini-Mundy (1989) demonstrated that although students were capable of evaluating the limits of continuous functions, they showed little intuitive or geometric understanding of limits. Additionally, some of the students sampled did not understand formal limit notation, but were nonetheless able to correctly evaluate limits when this notation was used.

Hart (1992) investigated the influence of supercalculator implementations used in conjunction with an experimental curriculum emphasizing multiple representations of students’ perceptions of differential and integral calculus. Sixty-four interviews (33 experimental and 31 traditional) were conducted, 24 of which were analyzed. Paper and pencil tasks dealing with various calculus topics were also administered. Three tasks dealt specifically with the concept of limit. The reported results were not topic specific; however, important information regarding students’ perceptions are appropriate to the present review. For example, an experimental student group showed greater facility with
graphical and numerical representations and exhibited better ties among the representations than the traditional group. In addition, the factors that influenced each student's choice of representations appeared to be the teachers' preferences for certain representations and which representation was most often used during instruction.

In summary, review of students' perceptions of limit suggests that students appeared to view a limit as how the function moves when \( x \) is moved toward a target value. This dynamic view represents the limit as a process rather than as a number or as the result of a process. Students also demonstrated confusion concerning whether limiting values could be reached. As with functions, familiarity may play a role in students' understanding of the limit concept. Non-mathematical ideas such as "speed limit" may influence how students interpret such terms as limit when used mathematically.

A significant proportion of completed research indicates that a relationship exists between teachers' perceptions and instructional practices. Furthermore, teaching instructional practices have been shown to have the most direct influence on students' perceptions. For example, specific to the study of calculus, this influence is demonstrated in the conclusion reached by Hart (1992) that a student's choice of representations of calculus topics appeared to be the same as the representations most commonly used by the teacher. This link between students' and teachers' conceptualization indicates that the results from research on students' perceptions of limit will be valuable to an understanding of teachers' perceptions of limit.

Conclusion

The research conducted on mathematics teachers' subject matter and pedagogical perceptions is limited in extent. However, what research has been completed is crucial to a formative understanding of what mathematics teachers' perceptions actually are and what factors may affect these perceptions. The studies included in the present review
shared the common objective that it was important to investigate teachers' perceptions in order to build upon these existing perceptions or change if they were poorly formed or found to be misconceptions. In addition, if future research efforts are to investigate effectively mathematics teachers' subject matter and pedagogical perceptions, the factors which have been found to affect existing perceptions must be recognized in the design of the study.

This review revealed that the most important factors leading to the development and change of mathematics teachers' subject matter and pedagogical perceptions were past experiences in learning mathematics. For several of the studies considered, this "past experience" is confined to those high school and college mathematics courses for preservice teachers entering a teacher education program. But experienced mathematics teachers also appeared to be affected by previous mathematics courses. However, these teachers' past experiences also included classroom materials (e.g., textbooks), planning for mathematics lessons, teaching mathematics lessons, and participating in workshops, inservices, or special programs. All of these factors have been shown to positively influence teachers' understanding of mathematics, awareness of mathematics' usefulness, and/or the ability to teach mathematics. It is noted that several of the studies indicated that though the teachers' perceptions were affected positively, their prior perceptions appeared to be deeply rooted. Thus, future research must not only examine what perceptions the teachers express, but also the stability of these perceptions.

In consideration of the stated findings deduced from the research reviewed, several problems intrinsic to the research methodologies are causes of concern for the results and implications of these studies. To construct the most accurate representation of mathematics teachers' subject matter and pedagogical perceptions, these concerns must be recognized in the design of future research. The primary concerns include: (a) choice of some of the instruments that were used, (b) lack of generalizability of the sample, and (c) narrowness of the mathematical topics being covered.
One source of concern regarding the instruments used for these studies may be the ubiquitous presence of the interview technique as the primary means of gathering data in the absence of observations to determine if what the teachers said was actually demonstrated in the classroom. This type of data can only be as accurate as the ability of the subject to articulate what it is he or she knows. Additionally, at times it was difficult to determine the level of bias introduced by the researcher. In particular, the critical lack of detail regarding interviews protocols and methods in certain studies is a cause for concern. It is fortunate that some of the studies incorporated classroom observations in addition to teacher interviews. The use of multiple data sources allowed for “triangulation” of the findings; thus adding validity to the conclusions.

Card-sort tasks were the instrument of choice for assessing subject cognitive structures in several of the studies. This method of inquiry presents a problem since it provides a number of pitfalls that were not addressed by the researchers. The subjects lose considerable flexibility when a preassigned list of terms must be used to describe the structure of their subject matter. Additionally, card sorts could easily bias the subjects or serve as a treatment. Thus, one must question the validity of the use of this method. Note that it was reassuring to observe that Sullivan and Leder (1992) claimed to have used card sorts only as a means to supplement other instruments in use. To eliminate potential researcher bias, the topics which are included in an instrument to obtain information on teachers’ subject matter and pedagogical perceptions should be left open to the teacher.

The abundance of samples of convenience, the high number of female subjects, and the fact that the overwhelming majority of teachers surveyed were elementary mathematics teachers are cause to question the generalizability of the findings of studies which reflect these characteristics to a broader population. In fact, in the section on change due to inservice, workshops, and special programs, the teachers for the samples were confined to only first or second grades. It is obvious that these samples were not generalizable across mathematics teachers as a whole. Nonetheless, these studies are the only ones of
this type, so the results and implications remain important for formative knowledge of the
impact inservice, workshops, and special programs have upon teachers’ perceptions.

Analogous to the samples confined to elementary teachers, the mathematical topics
that were examined were often remedial in nature. Most of the research reviewed dealt
with teachers’ understanding of whole numbers and fractions, multiplication and division,
and general middle school mathematics. The function was the highest level of
mathematical topic subject to investigation. Thus, to obtain a more complete picture of
mathematics teachers’ subject matter and pedagogical perceptions, research must be
conducted involving higher level mathematical concepts.

The present study is concerned with the examination of high school calculus
teachers’ subject matter and pedagogical perceptions of limits. Though the concept of
limit plays a fundamental role in the development of calculus, the review of the literature
on students’ perceptions of limits has demonstrated that the concept is extremely difficult
for students to understand. Research has not examined teachers’ perceptions of limits
specifically; thus, based on the research on students’ perceptions of limits, how teachers
actually understand or conceptualize limits, the role of limits, and the teaching of limits in
calculus remain areas confined to speculation.
CHAPTER III

DESIGN AND METHOD

This study presents an investigation of mathematics teachers’ subject matter and pedagogical perceptions of limits through an examination of the following: What are high school calculus teachers’ perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus? In addition, a sampling technique has been used to provide answers to the question: Are these teachers’ perceptions associated with their participation in a calculus reform project focused on staff development?

Given the broad focus and the exploratory nature of these questions, a multi-case study approach involving detailed examination of a small group of high school calculus teachers was considered to be the most appropriate method. Detailed descriptions of the sample, research methods, and proposed data collection and analysis procedures are presented in the following sections.

Subjects

Six high school advanced placement (AP) calculus teachers served as the sample for this study. The decision to select AP calculus teachers was made to provide consistency in the breadth and depth of the calculus material taught in each of the classrooms surveyed (see Appendix A for a discussion of AB and BC advanced placement calculus). Many calculus teachers are currently involved in calculus reform and more teachers will probably become involved in the near future. The sample for this study was chosen to reflect this difference in involvement in the calculus reform movement.

Three of the teachers, hereafter referred to as “project teachers,” were chosen based upon participation in a calculus reform project focused on staff development. The remaining three teachers, hereafter referred to as “independent teachers,” were chosen
from a list of AP calculus teachers not participating in a calculus reform project. Priority was given to the geographic location of the teacher in relationship to the researcher. Both the project and independent teachers were chosen because they taught in the state in which the research was conducted and were willing to participate in the investigation. This sample of convenience was chosen by the researcher based on the need for accessibility to AP calculus classes in the extensive data collection process. No reason existed for believing that the teachers in the sample differed significantly with respect to teaching concepts or practices from other AP calculus teachers.

The background of the calculus reform project qualification for "project teacher" status is as follows. In the fall of 1992, AP Calculus teachers across the United States were invited to apply to participate in a calculus reform project focused upon teacher development. Upon acceptance, Hewlett-Packard (HP) gave a set of HP-48GX graphing calculators with peripheral equipment to support instruction in at least one AP Calculus class at each school. Consistent with NCTM and NCSM (1986) recommendations, this calculus reform project included three components: (a) teacher training sessions, (b) a textbook reflecting the goals of calculus reform, that is, *Calculus, Volume I* (Dick & Patton, 1992; see Appendix B for an outline of this text), and (c) continuing teacher support.

The project teacher training session involved participation in a two-week regional inservice during August, 1993. During the inservice, teachers were introduced to the contextual problems of the calculus reform curriculum and given formal instruction on use of the HP-48GX. The calculus reform textbook emphasized practical applications and the use of technology to present multiple representations for conceptual understanding of calculus concepts. Throughout the inservice, teachers were also given the opportunity to explore, reflect upon, and discuss multiple techniques for solving the textbook problems and any problems they generated in their inservice work. In addition, participants had a chance to reflect upon their first quarter implementation of the calculus reform curriculum.
in the classroom, and to share experiences with their peers in a follow-up meeting held in December, 1993.

Methods

A multi-case study methodology was developed for this study. As discussed in Merriam (1988), the initial phase of this type of methodology involved the treatment of each of the six individual cases as a comprehensive case study. Comparisons across the individual cases were then drawn to determine any differences and similarities that existed. Miles and Huberman (1984) stated that “By comparing sites or cases, one can establish the range of generality of a finding or explanation, and at the same time, pin down the conditions under which that finding will occur” (p. 151).

Qualitative research methods, which provided for multiple means of observing the same event, were employed. This approach allowed the researcher to obtain a detailed representation of calculus teachers’ subject matter and pedagogical perceptions of limits through regular interaction with each subject. Questionnaires, interviews, observational field notes, videotapes of classroom instruction, journals, and written instructional documents for each of the six teachers composed the data collected for the investigation. Data were then analyzed using a constant comparative method, as described by Bogden and Biklen (1992). This method involved identification of the primary categories or themes emerging from the data, using this information to guide further data collection and analysis. The use of several methods of data collection allowed for the triangulation of the findings, thus providing greater confidence in the results as a whole.
Data Collection

Review of the literature suggested that experience, planning, and teaching affect mathematics teachers' subject matter and pedagogical perceptions. Moreover, findings from the literature suggested that any change in perceptions may not be long term in nature. In consideration of the importance of these issues, the data collection and ongoing analysis which took place from August to December, 1993, was conducted in four phases. The first phase was used to establish a baseline of teachers' perceptions prior to the teaching experience by collecting data immediately before the beginning of the school year. Because teacher planning may affect perceptions, the second phase was designed to capture teachers' subject matter and pedagogical perceptions of limits as they were involved in planning for teaching the unit on limits. The focus of the third phase was directed at both the effect of the teaching experience upon perceptions, as well as the effect of perceptions on teaching practices. This phase documented teachers' subject matter and pedagogical perceptions of limits as the unit on limits was actually taught. The final phase of data collection and analysis had two goals. The primary goal was to obtain information on teachers' views of the role of limits in the teaching of calculus. The secondary goal was to check the persistence of teachers' subject matter and pedagogical perceptions of limits as subsequent topics related to the concept of limits were taught.

In the first phase, prior to the beginning of the school year in August, a written questionnaire was given to all six teachers. Responses to the questionnaire were analyzed immediately following the administration of the instrument. This analysis served to guide the initial interview with each of the six teachers, which was conducted within 24 hours of completion of the instrument. The purpose of the first questionnaire and interview was to provide information to help establish a baseline concerning the teachers' initial subject matter and pedagogical perceptions of limits prior to the beginning of the school year.
Following establishment of the baseline, the teachers moved into the planning phase for teaching the unit on limits. During this phase, the researcher observed each of the six classrooms approximately once each week from the beginning of the first semester until the introduction of limits. Dependent upon the particular teacher, this phase lasted from one to four weeks. Each of the observations were analyzed prior to subsequent data collection so that the emerging categories or themes could be used to guide the next observation.

All observations (in this phase as in all phases) were followed by informal interviews conducted at the convenience of the instructor. These interviews were used to obtain the teacher's perspective regarding the specific class observed, to the end of developing a more complete picture of classroom activities and goals. Thus, the interviews provided additional data concerning teachers' subject matter and pedagogical perceptions of limits.

The final data collected and analyzed in the planning phase were derived from the second administration of the questionnaire to all six teachers, approximately one week prior to beginning the unit on limits. The analysis of the questionnaire, as well as the analysis of the previous questionnaire, observations, journal entries, and written instructional documents, served as guides for the second interview, conducted within the same week. The purpose of the second questionnaire and interview was to obtain evidence regarding the teachers' subject matter and pedagogical perceptions of limits during preparation for teaching the unit on limits.

When the teachers entered the next phase and began teaching the unit on limits, observations followed by informal interviews occurred approximately twice weekly. Since there was an overlap for the two weeks the teachers were teaching the unit on limits, direct observations of all classes in this unit were not possible. Thus, all classroom lessons on limits were videotaped. As before, each observation (videotaped and live) was analyzed as quickly as possible.
Data collection and analysis for the teaching phase ended with the third administration of the questionnaire to all six teachers in the sample upon completion of teaching the unit on limits. The analysis of the questionnaire, as well as the analysis of the previous questionnaires, observations, videotapes, journal entries, and written instructional documents, were used to direct the third interview. This interview was conducted approximately one week after completion of the unit on limits. The purpose of the third questionnaire and interview was to gain information on the teachers' subject matter and pedagogical perceptions of limits directly after teaching the limits unit.

In the final phase, subsequent to the teaching of limits, at least two observations of lessons which incorporated the concept of limit (e.g., the introductions to derivatives and integration) occurred. These observations were analyzed similarly to those described for the earlier phases and were used to verify, modify, or negate emerging themes or categories. This final phase concluded with the completion of the questionnaire by all six teachers during the first week of March, 1994. Responses to the questionnaire were analyzed immediately. The final questionnaire was used to obtain information regarding the teachers' subject matter and pedagogical perceptions of limits after teaching subsequent topics which incorporated the concept of limit. Informal telephone interviews with each teacher occurred at the end of data collection. These interviews were used primarily to obtain details regarding the scholastic and professional history of the teachers. This information was useful for the development of individual teacher profiles.

Throughout data collection and analysis, as described above, the data were subject to ongoing examination by constantly identifying recurring themes in teachers' subject matter and pedagogical perceptions of limits that emerged from the data. Future data collection sessions were then designed in light of what was previously found. The final data analysis involved an "analytic refinement" of the existing themes or categories by a thorough reexamination of the data in search of instances that did not fit existing themes or categories. When such instances were encountered, the researcher "refined" the theme
or category, a process which was continued until all themes or categories were deemed acceptable.

**Questionnaire**

The questionnaire was administered to the three project teachers on the first day of the two-week inservice, and to the three independent teachers at some point in August, 1993. The questionnaire was given to all six teachers in the sample a second time one day prior to beginning the unit on limits, a third time upon completion of the unit on limits, and a fourth time during the first week of March, 1994.

The questionnaire consisted of two components, the first of which involved having the teachers diagram their overview of the important concepts and relationships in calculus (see Appendix C). This component was used to obtain information regarding the teachers' beliefs about the role of limits in the teaching of calculus. To remove any possible source of bias imposed by the researcher, this methodology was intentionally open-ended. For example, a list of topics given by the researcher for the teacher to sort or diagram may have been quite different from the topics the teacher would have selected independently. Moreover, if a list of calculus topics was provided for the teacher, the concept of limit would have been on the list, possibly biasing the teacher's beliefs about the role of limits in teaching calculus. During the earliest administration of the questionnaire, this diagramming component was administered first since the second component of the questionnaire focused on the concept of limit and its prior placement could have potentially biased the teacher's responses to the first part of the questionnaire.

Prior to any data collection, it was recognized that a testing effect could occur based upon the nature and frequent use of this questionnaire. This fact was considered when the questionnaire data were collected and analyzed. For example, the researcher asked the following type of questions in the interviews following the analysis of the
questionnaires: "Did the act of previously diagramming your overview of calculus have any influence on what you have diagrammed now? On your teaching?"

The second component of the questionnaire was developed by the researcher and refined following extensive discussions with a group of experienced AP calculus instructors (see Appendix D). To establish content validity, each question was listed along with the description of what the question was designed to measure. The questions and descriptions were then given to four mathematics educators who reached 100% agreement on the connections between each question and the given description. However, based on the concern that the formal limit notation used in question two may lead to a formal limit response, the following question was added to the beginning of the second component: "Describe in your own words what a limit is." In the conduct of this study, questions one through four were used to assess teachers’ understanding of limits, questions three, four, and five were used to address teachers’ views about the teaching of limits, and questions four and five were used to gather data regarding teachers’ beliefs about the role of limits in teaching calculus.

Interviews

Three formal interviews were conducted with each of the six teachers. These interviews were used to validate the researcher’s interpretation of the responses to the questionnaires, as well as to substantiate analysis of the existing data. The formats for the three 30-minute (approximate) audio taped interviews were similar in nature. The questions were open-ended and guided by analysis of the completed questionnaire and, when possible, analysis of previous observations, journal entries, and written instructional documents.

The first interview was conducted the day following completion of the first questionnaire in August, 1993. The protocol for this interview was based on analysis of
the completed questionnaire. The diagram and responses to the questions served as visual stimuli for the discussion. Some examples of the questions asked include:

- Explain what you have diagrammed on this paper.
- Do you feel you have listed the important concepts and relationships in calculus?
- How did you choose the given concepts and relationships?
- Have you ever thought about calculus this way before?
- What level of rigor do you use in teaching the concept of limit?

The second interview occurred the first day each teacher began to teach the unit on limits, and the third interview occurred within one week following completion of teaching the unit. The protocols for the second and third interview were guided by analysis of the completed questionnaires as well as by analysis of previous observations, videotapes (applicable to third interview only), journal entries, and written instructional documents.

Several informal interviews with each of the teachers were held throughout the data collection period. Though the initial proposal for this study stated that informal interviews would be conducted directly following each of the classroom observations, it was not possible since all of the teachers had to teach another course directly following the observed class. Thus, the majority of the informal interviews took place via telephone conversations. These interviews occurred after each observation as well as during scheduling calls. All comments made by the teachers were transcribed and used as data. These interviews focused on the teacher’s perspective of classroom activities and goals. The data gathered from these interviews included items such as the teacher’s impressions about what was happening in the classroom, reactions to the flow of particular lessons, and discussions of student responses to the materials or activities. Some examples of the questions asked include:

- How do you feel about today’s lesson?
- What was your basis for selecting the materials used for today’s class?
What can you tell me about yesterday’s lesson?
What can you tell me about tomorrow’s lesson?
How do you feel the class is going this quarter?
How would you characterize this class in terms of other AP calculus classes you have taught? (e.g., best, favorite)
Are your goals for this class the same or different for other classes you have taught or are teaching?

**Classroom Observations**

The researcher observed each of the six classrooms at least once a week from the beginning of first semester until the introduction of limits, twice each week during the approximate two-week unit on limits, and at least twice following instruction of the limits unit. Due to an overlap in the lessons on limits for all six teachers, direct observations of all of these classes was not possible. Thus, whenever possible all classroom lessons on limits were videotaped, including the lessons being observed. Following the unit on limits, at least two observations of lessons which incorporated the concept of limit (e.g., the introduction to derivatives and integration) occurred. Additionally, each teacher was observed teaching in two noncalculus classes at least twice throughout the semester to allow the researcher to discriminate between general teaching routines and those which were related specifically to the concept of limit, the teaching of calculus, or the effects of the calculus reform project.

Extensive field notes on the teachers’ general attitudes, styles, actions, interactions, and verbal communications were recorded by the researcher. In addition, all student responses as well as the overall dynamics of the classroom were documented. Data analysis of the transcribed field notes and informal interview sessions occurred upon completion of each observation period. In brief, this analysis involved generating recurring themes and was used to create a context from which to view the next classroom observation. Thus, each successive observation was planned in light of the previous observation.
Videotaped Classroom Instruction

Since it was not feasible for the researcher to observe every lesson given on limits by each teacher, whenever possible all lessons on limits were videotaped. The sections on limits requiring videotaping lasted approximately two weeks. To minimize disturbance, the video camera was set up as a permanent fixture in the classroom. The videotapes were transcribed as soon as possible and analyzed in conjunction with and in similar fashion to those for classroom observations.

Journals

Each teacher was asked to keep a weekly journal on the events of the calculus class throughout the data collection period. During the two-week period while teaching the concept of limit and on the days of the introductory lessons for derivatives and integration, the teachers were asked to keep a daily journal. The teachers were told that the journal entries needed to include personal and student reactions to the material discussed, comments on the choice of activities used, and their perceptions of student learning. The researcher collected available journal entries on a weekly basis. The journal entries provided information regarding the classes that were not actually observed or videotaped, as well as those that were. The information obtained from the journals was used to supplement or refine the emerging categories or themes derived from the analysis of the observations.

The researcher also kept a journal for each teacher, identifying information regarding the developing categories or themes used to guide further data collection and analysis. This journal also included personal insights, feelings, or concerns with regard to each particular case. This documentation of the research process was useful in the development of each individual case.
Written Instructional Documents

All written material used in the classroom throughout the course of this study was also collected as data. In particular, these materials included a weekly collection of all lesson plans, handouts, overhead transparencies, and exams used throughout the data collection period. These documents helped in the examination of overall classroom activities, providing a general description of the individual teachers. Additionally, the information obtained from these documents was used to support or negate the categories or themes developed from the analysis of the observations.

Research indicated that mathematics teachers are heavily influenced by given sections in the textbook; thus, the textbooks for each of the six teachers were also collected. A description of the goals of each textbook and an outline of the topics covered can be found in Appendices B, E, F, and G. To test the strength of textbook influence, the sections covered in the textbook and actual teaching practices were compared.

The Researcher

It must be acknowledged that although this study promised to use a variety of data collection techniques, the primary instrument for data collection and analysis was the researcher. Thus, it is important to note that the researcher has been involved in teaching college calculus for the past six years, and has worked with calculus reform projects focused on teacher development for the past two years. Furthermore, the researcher was involved in an earlier two-week calculus reform project regional inservice, but was not involved in the two-week regional inservice used for the present study. It is therefore acknowledged that although these past experiences were helpful with respect to recognition of the teachers' subject matter and pedagogical perceptions of limits, they may also cause some concern with respect to the existence of potential personal biases. Hutchinson (1988) suggested that the researcher must not only be aware of personal
preconceptions, but must also attempt to transcend them in the effort to see the situation from a new perspective. Thus, the researcher maintained a detailed log of personal reflections on the ongoing data collection and analysis process. This log included all thoughts, feelings, insights, and decisions regarding each teacher included in the sample. The journal entries were used to help the researcher recognize and transcend potential personal biases when analyzing the data.

Data Analysis

The data obtained for each case was examined in an ongoing inductive and qualitative manner by the continual identification of recurring themes from the teachers’ subject matter and pedagogical perceptions of limits as reflected by the data. The researcher adopted the general strategies of the Bogden and Biklen (1992) analysis-in-the-field mode. The strategies of this comparative method of constant analysis included identifying the primary categories and themes for each type of data, thoroughly reexamining the data as a whole to confirm or refine the preliminary conclusions, and then designing future data collection sessions in view of what was found in previous data collection sessions. Hutchinson (1988) proposed that the researcher seek patterns by comparing parallels among incidents, between incidents and categories, and among categories. For example, episodes that occurred during observations were compared and used to create categories for further investigation. Existing categories were then compared with the next episode as well as with each other. This type of analysis forced the researcher to expand on the emerging categories or themes continuously by searching for structures, stability, causes, and relationships to other findings.

It is important to indicate what mode of data collection received the most emphasis. It was the intent of this study to give first consideration to what the teachers “actually did” in the classroom; thus, the greatest emphasis was given to data collected
during classroom observations and the videotaped classroom instruction. The additional modes of data collection included what teachers "said they did" and was used to provide support for what actually happened in the classroom.

Initially, each teacher was treated as an individual case and the final data analysis for each case involved an "analytic refinement" of the existing categories or hypotheses. This process consisted of searching for instances in the data that did not fit these categories or hypotheses. When such instances were encountered, the researcher "refined" the categories or hypotheses. This process continued until all categories or hypotheses were considered acceptable.

A detailed profile of each teacher's subject matter and pedagogical perceptions of limits was developed upon completion of the data collection and analysis. Tentative teacher profiles were created throughout the data collection period following each phase of the following time sequence: (a) prior to the beginning of the school year, (b) in preparation and planning for teaching the unit on limits, (c) during teaching of the unit on limits, and (d) during teaching of subsequent topics which incorporated the concept of limit. These profiles detailed each teachers' perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus.

The first phase, prior to the beginning of the school year, produced a baseline for teachers' subject matter and pedagogical perceptions of limits. This conceptual baseline was set aside during data collection and analysis for the remaining three phases. Upon completion of the detailed teacher profiles, the baseline was compared to the conceptualization from each of the other phases. This comparison enhanced the researcher's point of view regarding the effect of experience, planning, and teaching upon teachers' subject matter and pedagogical perceptions of limits as well as the stability of these perceptions.

Following development of the individual teacher profiles, these profiles were examined collectively in search of patterns of similarities and differences across the entire
sample. The resulting comprehensive profile provided an initial description of the calculus teachers' subject matter and pedagogical perceptions of limits. To address whether the teachers' perceptions of limits were demonstrated in their teaching practice, comparisons were drawn between teachers' professed perceptions and those derived from analysis of classroom data. The level of consistency between each teacher's professed perceptions and teaching practices was used to determine the impact of perceptions on teaching practices. To develop hypotheses regarding the influence of calculus teachers' subject matter and pedagogical perceptions of limits on teaching practices, individual levels of consistency were then compared across the sample.

The question of whether teachers' subject matter and pedagogical perceptions of limits were related to participation in a calculus reform project focused on staff development was addressed by the development of general profiles for each of the two groups, project and independent teachers. A general profile for each of the two groups was drawn from analysis of the similarities and differences within the three individual teacher profiles in each group. These general profiles were then compared and contrasted to assess the impact of teacher involvement in a calculus reform project.
CHAPTER IV

RESULTS

Introduction

As discussed in Chapter III, six AP calculus teachers, including three project and three independent teachers, volunteered to participate in the present study. Project teachers were selected based upon participation in a calculus reform project focused on staff development, whereas independent teachers were selected based upon proximity to the research location, a large university in the Northwest U.S.

All participants in the calculus reform project were AP calculus teachers. The first component of this project involved participation in a two-week inservice in August, 1993. Initial contact with the subjects occurred on the first day of this two-week inservice. Four teachers from the state in which the research was conducted participated in the two-week inservice, completing the first questionnaire on the first day of the two-week inservice, followed by personal interviews with the researcher. One teacher was dropped from the study since she had never taught calculus. The experience of the project teachers ranged from 10 to 21 years teaching high school mathematics and from 4 to 12 years teaching AP calculus.

The three independent teachers were selected from a list of experienced AP calculus teachers in the Northwest U.S., each within a 100 mile radius of the research institution. The first three teachers who were contacted expressed interest in participation and were thus selected as the sample of independent teachers for the study. The first teacher was called because he taught in the same city in which the researcher lived, whereas there was no specific order of calling determined for the next two teachers. During initial telephone contacts, a date prior to the beginning of school was arranged for the completion of the first questionnaire and interview. The experience of the selected
independent teachers ranged from 6 to 26 years teaching high school mathematics and from 3 to 21 years teaching AP calculus.

The high schools of all of the participating teachers were located within 100 miles of the research institution. Additional demographic information for each of the subjects is presented in individual teacher profiles included in this chapter. To protect the anonymity of the subjects, pseudonyms are used. All male names were chosen to protect the sole female participant in this study. The pseudonyms chosen for the project teachers were Terry, Trey, and Tom, and the pseudonyms for the independent teachers were Russell, Richard, and Ryan.

All subjects participated in every aspect of first-phase data collection prior to the beginning of the school year. Completion of the questionnaire took from 20 to 30 minutes and follow-up interviews lasted from 30 to 45 minutes. All teachers, except Richard, completed each of the three subsequent questionnaires. Richard completed only the first and final questionnaire because he felt the other two questionnaires were redundant. Details regarding the significance of the uncompleted questionnaires can be found in the profile for Richard. Each of the subsequent formal interviews lasted approximately 30 minutes.

Each teacher was observed on approximately eight occasions during the data collection period. Dependent upon individual situations, observations were conducted either once or twice for each of the subjects prior to teaching the unit on limits. The unit on limits lasted from 8 to 17 days, which coincided with the limit chapter in given textbooks (see Appendices B, E–G for outlines of the chapter on limits for textbooks used). Though each teacher was observed at least twice each week, as originally proposed, such scheduling conflicts as school assemblies, testing, illness, and inservice days precluded the possibility of additional observations. The number of observations during this phase ranged from three to eight days. In most cases, since each of the lessons
on the concept of limit was also videotaped, the low number of observations did not pose a problem.

At the beginning of the study, all of the subjects agreed to be videotaped when videotapes of each lesson throughout the unit on the concept of limit were requested. However, just prior to teaching this unit, Richard and Tom declined to videotape any of their classes. In the case of Richard, the researcher was able to observe each of his lessons on the concept of limit. Due to scheduling conflicts, or to last-minute cancellations of scheduled observations, the researcher was only able to observe Tom three times during the unit. To compensate for the lack of videotaped lessons, numerous informal interviews occurred via telephone. The intent of this study was to give more weight to what the teachers actually did in the classroom. In the case of Tom, however, the findings may be weak insofar as the analysis was derived from what he said he did in the classroom rather than observations of what he did. In addition, each teacher was observed twice on subsequent lessons related to the concept of limit. Specific details from the classroom observations and videotaped lessons is provided below in the individual teacher profiles.

Data collected and analyzed using the previously mentioned constant comparative method were used to develop the individual teacher profiles, each of which is presented in the following seven sections: (a) scholastic and professional history, (b) portrait of the calculus classroom, (c) perceptions of calculus and the teaching of calculus, (d) perceptions of the concept of limit, (e) perceptions of the role of limits in the teaching of calculus, (f) the teaching of limits, and (g) summary. A description of the content of each section and the primary data sources analyzed are provided in the succeeding paragraphs.

Each teacher profile begins with a scholastic and professional history, providing details regarding academic preparation, community, and school situations. The majority of this information was obtained through informal interviews conducted near the end of the data collection period. In addition, this section contains details regarding teacher participation in each facet of the data collection process.
The second section, portrait of the calculus classroom, was developed primarily from classroom observations and videotapes, and is provided to present a clear image of the classroom atmosphere of each teacher, as interpreted by the researcher.

The third section, perceptions of calculus and the teaching of calculus, provides a description of the teacher's perceptions of calculus content and pedagogy. The primary data sources used include all questionnaires and follow-up interviews, as well as instructional materials and plans. Though the diagrams of each teacher's overview of the important concepts and relationships in calculus were initially designed to provide information regarding the role of limits in calculus, they were also used to provide information regarding their general perceptions of calculus. Classroom observations were used to provide additional information regarding each teacher's perceptions of the teaching of calculus. This section also includes a discussion of the text and supplemental materials used in the calculus class. Additional information about the textbooks used by each teacher can be found in Appendices B, E–G.

Section four describes each teacher's perceptions of the concept of limit. The primary data sources used to develop this section include the questionnaire responses to items one and two (see Appendix D) and the follow-up interview responses corresponding to these questions. Additional information regarding the teacher's perceptions of the concept of limit, as demonstrated by the actual teaching of limits, is provided in section six.

Section five describes each teacher's perceptions of the role of limits in the teaching of calculus. This section was developed primarily from responses to questionnaire item five (see Appendix D), follow-up interviews, and classroom observations of subsequent topics that involved the concept of limit. Additional information was also provided from classroom observations, videotapes, informal interviews, journals, and written instructional documents. Each teacher's beliefs about the appropriate level of rigor with respect to limits in high school AP calculus classes is also discussed in this section.
Section six describes the teaching of limits for each of the subjects. This section was developed primarily from classroom observations and videotapes of teaching the concept of limit. The questionnaire responses to items three and four (see Appendix D), follow-up and informal interviews, journals, and written instructional documents were also used to develop this section. Once again, it should be noted that, with the exception of the profile for Tom, the greatest emphasis was given to what the teachers actually did in the classroom as opposed to what was written or verbally communicated.

Each teacher profile concludes with a summary, highlighting the principle findings from each of the previous profile sections. In addition, the summary contains a discussion of whether the stated perceptions of each teacher matched those demonstrated in classroom teaching practices. Finally, this section addresses the level of persistence of each teacher’s perceptions throughout the data collection period.

Following the development of the individual teacher profiles, the profiles were examined collectively in search of similarities and differences across the sample. These results, which correspond to the first research question proposed for this study, are presented in four sections: (a) overview of calculus teachers’ subject matter and pedagogical perceptions of calculus, (b) calculus teachers’ perceptions of the concept of limit and the correlation to classroom practice, (c) calculus teachers’ perceptions of the role of limits and the correlation to classroom practice, and (d) summary of calculus teachers’ perceptions of limits and the correlation to classroom practice. Group profiles were then developed for the project and independent teachers. These profiles were compared and contrasted. The results, which correspond to the second research question proposed for the study, are presented in two sections: (a) association between calculus teachers’ perceptions of limits and participation in a calculus reform project and (b) stability of calculus teachers’ perceptions of limits.
Individual Subject Profiles

Profiles for the independent teachers are given first, followed by those for the three project teachers.

Independent Teacher Profile, Ryan

Scholastic and Professional History

Ryan has been teaching high school mathematics since graduation six years ago from a large Northwest U.S. university with a triple major in mathematics, education, and computer science. Ryan described the reason he wanted to become a mathematics teacher:

I have been a math teacher for five and one-half years. Honestly, the first motivation for getting into teaching was coaching. However, as my experience expanded, I became increasingly enlightened regarding the logical, deductive thought processes I had learned and taken for granted. I wanted to inspire this nature of mathematical thought.

Ryan described his overall academic preparation as an “eight and one-half on a scale of one to 10.” He felt that the “career [teaching] component” was lacking in his preparation but the “logical and theoretical basis was superb.” Ryan stated that since graduation, he had taken “a few graduate-level courses” from a small local university and had also participated in AP calculus workshops, calculator institutes, and several departmental retreats.

Ryan was currently teaching at a high school located in a wealthy, predominantly white suburb of a large metropolis. The student population exceeded 1,300 in the top three grades. In addition to teaching AP calculus, Ryan taught AP computer science throughout the year. He stated that since student teaching he had taught mostly AP
calculus and computer science. Ryan was also the varsity girls’ basketball coach. Ryan’s office was connected to the high school computer lab. This connection corresponded with his position as the “technology coordinator” at the school. On several occasions during the informal interviews, Ryan was observed helping both students and faculty with their computer problems. His pleasant and knowledgeable demeanor made him very approachable.

Ryan was selected for participation in this study from a list of AP calculus teachers in the state. Ryan was contacted by telephone the first week in August and he agreed, without hesitation, to participate in the research project. During this conversation a time was arranged for the completion of the first questionnaire and an interview prior to the beginning of the academic year. The interview took place on September 4, three days prior to the beginning of school, conducted in Ryan’s office. Ryan was focused on the task and tended to provide precise responses. The fact that the interviews were being audio taped did not appear to affect Ryan. In fact, he appeared confident in his answers and pleased that someone cared enough to listen to his comments.

Ryan’s organized nature made him a cooperative study participant. Each lesson on limits was videotaped, questionnaires were completed on time, journal entries were well thought out, and scheduling was convenient and free of conflicts. Ryan appeared to respect research in mathematics education as well as he had a healthy attitude about mathematics teaching. He stated that his basic philosophy of education was to “provide opportunities for students to improve their world through self-motivated and/or self-initiated learning.”

Portrait of the Calculus Classroom

Ryan had been involved with teaching calculus for the prior three years. His enthusiasm for the course was demonstrated by his actions continuously throughout the
data collection period. Approximately 38 students, including eight females and 30 males, were initially enrolled in Ryan's calculus class. These students were described by Ryan as “highly motivated, college-bound, with a GPA [grade point average] of 3.6 and up.” Ryan suggested that approximately 32 students, as projected from participation during the previous year, would be in the class at the end of the year. Of those 32 students, roughly 15 students would take the BC exam and eight would take the AB exam (see Appendix A for a discussion of AB and BC advanced placement calculus). At the end of the data collection period, 25 students remained in Ryan's calculus class and Ryan was uncertain at that point which level AP exam each student would be taking.

Ryan's calculus classroom was very crowded. Every desk in the room was occupied and there was little room even for walking through the aisles. The very organized, quiet, and professional atmosphere of his class, however, provided a sense that everyone had ample space. Since the chalkboard at the front of the class was too small and the side board was inaccessible, Ryan used the overhead projector as his sole instructional tool. The general atmosphere of the classroom was friendly and orderly. The instruction was predominantly lecture-based with occasional group activity. Ryan described his basic course organization as “lecture 60%, evaluation (tests and quizzes) 20%, group work 10%, and independent work 10%.” Ryan created a challenging environment for the students and the students reacted to this environment with enthusiasm and intensity.

Each calculus lesson began similarly. The students visited freely until Ryan put a “warm-up” transparency on the overhead and began to take roll. Nearly all of the students immediately began to work independently on the 5 to 10 problems listed on the transparency. The only discussions for the next 10 to 15 minutes involved questions regarding the visibility of the problems and directions from Ryan suggesting that the problems as well as the answers be written down to facilitate later review. The students did exactly as they were told and assiduously copied down every word from the overhead.
During this "warm-up" period the only movements observed occurred when the students got up to sharpen pencils. The class remained relatively quiet. While some students shared problem-solving methods, students primarily worked as individuals.

The "warm-up" activities appeared to be well conceived. Though the use of these activities was routine, the purpose of each warm-up was quite different. Some of the activities were used as review, some as the introduction to new material, and some were patterned upon AP exam questions. Even the type of questions asked varied, from true-false to problem-solving formats. In all cases, the students appeared motivated and worked diligently to find solutions. In fact, this activity seemed more like a competition between students to find the solutions first, and then to provide the best justification of how solutions were obtained.

Depending on the purpose of the warm-up activity, Ryan would next spend 20 to 30 minutes covering students' solutions to the problems as well as introducing new material for the day, typically from the next section in the textbook. Based on Ryan's desire that students be able to work the more difficult text problems, he would spend the next five minutes verbally completing the easy exercises in the text. For example, one day he stated, "Real quickly now we are going to do about three-fourths of your assignment for tonight. Turn to page 101. We need to find all the points where the following equations are not differentiable." The class, in a choral response, then recited all the solutions out loud at Ryan's prompting. The students seemed quite interested and were able to arrive easily at correct solutions. Ryan would then choose one of the more difficult problems in the text exercises and work it for the class. On one occasion, after completing the exercise, he announced to the class, "Oh, this was the hardest homework problem. I'm sorry I did it."

The final 10 minutes of class was dedicated to homework problems from the previous day or any other questions that the students posed. Typically, Ryan first verbally responded to the questions, then he would go into greater written detail, dependent upon
the levels of comprehension among the students. Ryan gave the impression that questions on homework problems should be addressed prior to class and not eliminate time from the new lesson. However, Ryan did create an atmosphere in which students appeared to feel comfortable asking questions as well as challenging his responses. Frequently, Ryan answered their questions with the wrong answer so that the students had to catch his mistakes. In several instances, if Ryan purposely made an inane mistake, the students would respond “No!” in unison, or they would laugh.

The class period rarely ended with students working on their assignment. In fact, one day Ryan handed out the new assignment and said, “Here is your assignment. For the first and probably the last time this year, you have some class time to work on it. Yes, it looks like you have four minutes.” The class period concluded with watching “The Roar,” a student broadcast of daily and upcoming school events. Though the class occasionally missed this presentation because Ryan was discussing the calculus topic of the day, none of the students broke the discussion to inform him of the time.

The overall structure of Ryan’s classroom was regimented. The students were aware of what was expected of them and they routinely complied. Ryan required that each student possess a spiral notebook, which would contain the “warm-up” problems and all homework. Each week, Ryan scanned the notebooks and he and each student reached mutual agreement upon a score. In addition, a quiz was given every Thursday on all material covered to that point. Though Ryan consistently gave the impression that calculus was difficult, he challenged the students to succeed in the course.

Perceptions of Calculus and the Teaching of Calculus

Ryan stated that his past teaching experiences helped him form the following three primary goals for teaching calculus: (a) prepare my students for the AP exam, (b) prepare my students for college, and (c) inspire an appreciation for the deductive nature of
mathematics. Each of the three goals was demonstrated in Ryan’s teaching. The extent to
which each goal was covered is discussed below.

Ryan believed that most students could learn calculus as long as the mathematical
symbolism did not get in their way. In fact, he frequently discussed this, as “Ryan’s
Theorem,” with his class:

Just as with what I refer to as Ryan’s Theorem, I believe that most kids are
fully capable of understanding most of the concepts of calculus. Too often,
the symbolism and use of shorthand in writing is actually what gets in the
way.

To prevent students from being distracted by the symbolism, Ryan valued a discovery-type
approach to learning calculus. Ryan described his role in helping the students with this
intuitive approach to learning calculus:

I try to help students discover calculus as we are learning it. I believe that
they take on more ownership of it in this way. As we discover things they
take on more and more pride in their understanding. As the different areas
of math are used to build the new knowledge, they begin to understand and
appreciate the connections in math.

The desire to motivate students to learn calculus was evident in Ryan’s teaching
style throughout the entire data collection period. His motivational technique used during
the first week of school was the most notable. On the first day of class, Ryan read letters
written by calculus students from previous years during their last week of school. These
letters were specifically written to the new group of calculus students, containing such
suggestions as the need to do the homework, helpful hints for the AP exam, connections
between particular calculus concepts, and information about Ryan. Ryan felt that these
letters provided an excellent incentive for students to succeed in his calculus course.

The first day of Ryan’s class continued as he extended a welcome to the students
and announced, “In this course, I will attempt to inspire you to a sense of appreciation of
Welcome to probably the most difficult and demanding course of your high school career. Calculus is unlike any class you have taken before. The beauty of calculus does not lie in computation and manipulation skills, although you will be drilled extensively on these. The beauty of calculus does not lie in the memorization or fact retrieval, although you’ll have to do plenty of this as well. The beauty of calculus, more than even any other math class, lies in communication and reasoning, thinking and problem-solving. In the realization that calculus is based in theory and that, because it is theoretical, we are not constrained by the real world. Nearly everything we do this year will touch on the concept of infinity. Your mind will be pushed to think in a whole new manner.

Ryan provided closure to this motivational lesson by distributing a handout on his recommendations for success in his class (Figure 1). The handout reflected Ryan’s organized and punctual nature, and also demonstrated Ryan’s demand for student commitment to the class.

Though Ryan appeared to feel comfortable in his need and desire to challenge and motivate his calculus students, he also demonstrated a concern that the students fear him. During an informal interview the second week of class, Ryan vocalized this concern: “The students seem really intent on paying attention and giving a good effort. Though I think this is mainly due to their fear of me—and then maybe the material.” Subsequently, during data collection, he brought up this concern again, in one circumstance, demonstrating it in a reverse manner. Instead of this student fear being motivational, he commented “Maybe I scare kids into not being successful, I don’t know?”

Ryan stated that the calculus course this year was “intended to cover the AP calculus (AB) syllabus (see Appendix A for outline of the AP (AB) calculus topics), starting with limits, working through derivatives, and ending with integral applications.” These intentions, as well as his decision to closely follow the order of the textbook, were also demonstrated in his diagram (Figure 2) of the important concepts and relationships in
Calculus

My recommendations for success in this class:

1) Be here everyday and on time!

2) Take good notes. (Rule of thumb: If I put it on the board, you put it on your paper.)

3) After class, review your notes. Read them over 2 or 3 times and then try to rewrite them from memory.

4) Pay attention to the details. In life, if's and but's are candies and nuts, but in math if's and but's are important.

5) Challenge yourself to do the hardest problems on the page. Do more than I assign.

6) Commit at least an hour of each day to this class. Use the hour to review, to do the homework, or to look ahead.

7) Let me know when you will be missing class before you miss it. Often, I will be able to give you the days notes and assignment.

8) Focus on understanding the deduction of the concept first. You'll find that the problems are generally easier this way.

9) Don't "cop out" on your commitment to this class. By sticking with this class you're telling me and yourself that you want to learn Calculus—when things are tough, don't forget this!

10) Act like a responsible adult taking a college class (after all that's what you're doing).

Figure 1. Ryan’s Recommendations for Success in Calculus.

calculus. With regard to the topics Ryan chose to diagram, in the first interview he observed, “I’ve always looked at calculus as a three-tier thing: (a) pre-calc stuff which includes limits, (b) differentiation, and (c) integration. I make a distinct transition in class. Each section is a different world.” Ryan’s choice of words, “Each section is a different world,” did not mean that each of the sections was disjointed from the others. Ryan
INTRODUCTORY CALCULUS
- FUNCTIONS
- GRAPHICAL ANALYSIS
- DISCRETE CONCEPT OF INFINITY
- CONTINUOUS CONCEPT OF INFINITY
- LIMITS
- ALGEBRAIC MANIPULATION
- SEQUENCES/SERIES

DIFFERENTIAL CALCULUS
- THE DERIVATIVE AS A LIMIT
- THE DERIVATIVE AS A SLOPE
- THE DERIVATIVE AS A RATE OF CHANGE
- RELATED RATES

INTEGRAL CALCULUS
- INTEGRAL AS REVERSE DERIVATIVE
- SUMMATION
- NUMERICAL INTEGRATION
- DEFINITE INTEGRAL
  - FUNDAMENTAL THEOREM OF CALCULUS
- TRICKS TO INTEGRATION

VECTOR CALCULUS
- ALGEBRA OF VECTORS
- GEOMETRY OF VECTORS
- CALCULUS OF VECTORS

Figure 2. Ryan’s Diagram of Calculus, Prior to Beginning of School.
provided a more precise description of his views of these three tiers with regard to the
organization of his teaching as well as student understanding:

There’s a lot of what I call “fundies” [fundamentals] to calculus, not things [that] aren’t taught in calculus, but [which] are crucial to getting through calculus. For example, the concept of infinity in a discrete sense. I tell the students to pick a number as close to one as they can and I can find one closer. When we start the section on differential calculus, I always start with the derivative as a slope. This is a good way to understand definition of derivative, then I look at the definition as limit.

After differential calculus, most kids are fine—there is not a lot of art in differential calculus. Given a problem, there is a recipe to complete it. Integral calculus is completely different, it requires creativity, insight, and reflection. It’s a different world! The Fundamental Theorem is key to this transition, however, most kids don’t understand it. They can use it, but it is a very abstract theorem. It is a little above their math ability from a thinking standpoint.

It is interesting to note that his comment with respect to differentiation, “Given a problem, there is a recipe to complete it,” does not appear to coincide with his desire for students to “discover” calculus. This comment, however, was not discussed in greater detail.

Although Ryan viewed calculus as a three-tiered course, he strongly asserted that “Calculus builds on itself more than any other class.” Throughout the entire data collection, Ryan continued to diagram the important concepts and relationships in calculus in a similar “tiered” fashion, with a few additional sections (Figures 3–5). The format of each diagram merely involved a list of discrete topics arranged in three to five tiers. The organization of the tiers was linear. Connections between the topics were not illustrated.

Though the overall structure of the course appeared to be organized relative to the text, *Calculus: Alternate Third Edition* (Larson & Hostetler, 1986), preparation for the AP exam seemed to partially guide the amount and type of emphasis given to particular topics. Ryan communicated a desire for students to obtain a conceptual understanding of
PRE-CALC FUNDAMENTALS
- SEQUENCES/SERIES (DISCRETE)
- REAL NUMBER CONCEPT OF INFINITY
- LIMITS
  - DEFINED
  - UNDEFINED
- COORDINATE GEOMETRY
- NOTATION & GREEK ALPHABET
- THE ABILITY TO INCORPORATE "FUZZY LOGIC" AND INTUITION INTO A STRICT MATHEMATICAL STRUCTURE
- FUNCTIONS

DIFFERENTIAL CALCULUS
- THE DERIVATIVE AS A SLOPE
- THE DERIVATIVE AS A LIMIT
- THE DERIVATIVE AS A GENERIC RATE-OF-CHANGE
- DIFFERENTIABILITY & CONTINUITY
  - CONNECTED VIA LIMITS

INTEGRAL CALCULUS
- THE INTEGRAL AS REVERSE DIFFERENTIATION
- NUMERICAL APPROXIMATIONS FOR AREA
- THE DEFINITE INTEGRAL
  - THE FUNDAMENTAL THEOREM
- CREATIVE SUBSTITUTION TECHNIQUES

Figure 3. Ryan's Diagram of Calculus, Planning to Teach Limits Unit.
PRE-CALCULUS
- FUNCTIONS
- LIMITS
- CONCEPT OF INFINITY
- A HIGH LEVEL OF RIGOR
- DISCRETE SEQUENCES & SERIES
- POLAR & PARAMETRIC EQUATIONS

DIFFERENTIAL CALCULUS
- DERIVATIVE AS A LIMIT
- DERIVATIVE AS A SLOPE
- DERIVATIVE AS A GENERIC RATE OF CHANGE
- MEAN VALUE THEOREM
- THE DERIVATIVE TESTS
- CURVE-SKETCHING
- APPLICATIONS

INTEGRAL CALCULUS
- THE INTEGRAL AS ANTI-DERIVATION (INDEFINITE)
- THE INTEGRAL AS A SUM (DEFINITE)
- DEFINITE INTEGRATION IS SAME AS INDEFINITE INTEGRATION
- TRICKS TO INTEGRATION
- SEQUENCES AND SERIES/CONVERGENCE AND DIVERGENCE

Figure 4. Ryan's Diagram of Calculus, After Teaching Limits Unit.
FUNDAMENTALS
- ALGEBRA SKILLS
- CONCEPT OF INFINITY (DISCRETE)
- CONCEPT OF INFINITY (CONTINUOUS)
- TRIGONOMETRIC RELATIONSHIPS

LIMITS
- CONCEPT OF
- DEFINITION (RIGOROUS)
- MANIPULATION
- PROPERTIES

DIFFERENTIAL CALCULUS
- CONCEPT AS RATE OF CHANGE
- DEFINITION & ALTERNATE DEFINITION OF DERIVATIVE
- PROPERTIES & SHORTCUTS
- APPLICATIONS

INTEGRAL CALCULUS
- FUNDAMENTAL THEOREM OF CALCULUS
- DEFINITION OF INTEGRAL AS INFINITE SUM
- APPLICATIONS

FUN STUFF
- POLARS, PARAMETRICS, VECTORS, MULTIPLE-VARIABLES

Figure 5. Ryan’s Diagram of Calculus, End of Data Collection Period.
calculus. However, his desire was intermixed with the necessity to successfully prepare his students to take the AP exam, which additionally requires a procedural understanding of calculus. His desire to teach both conceptually and procedurally did not appear to pose a problem for Ryan. Though Ryan was certain his students would be adequately prepared for the AP exam, he asserted that “If you were to ask my kids they would tell you that they understand calculus a lot better than they can do calculus.”

Throughout the data collection period, the significance of the AP exam was demonstrated in a variety of ways. In some cases, the AP exam was used for motivational purposes. For example, Ryan described creative substitution techniques as “one of the worst parts of calculus because there is not a lot of conceptual learning involved.” Thus, Ryan gave motivation for memorizing trigonometric identities by saying, “There are certain things in calculus that can be made easier and when you take the AP exam you will want to know and use these shortcuts.” In some cases, to keep the terminology consistent with the AP exam, Ryan would deviate from the book:

Some of you are making a good reading of the book, but there are some differences in what they classify and what I classify. For example, I call the discontinuities as removable, infinite, and jump. The AP exam will ask you to classify this way so I will stick with these three.

Though the students would not be taking the AP exam for more than seven months, Ryan continually coached them for this exam. He would ask them questions as if they were taking the exam and ask or tell them how to answer it. “Don’t let the semantics of the (AP) exam bother you. If they ask you to list three reasons why a function may not be differentiable, you should know how to respond.” Ryan also told the students that he expected them to know how to solve the most difficult problems in the text because they most reflected the problems given on the AP exam. As an example, one day Ryan placed the following difficult homework problem on the overhead, asking the students to find the solution using their homework and notes:
\[
\lim_{x\to3} \frac{\sqrt{x + 1} + 1}{x - 3}
\]

As the students worked on the problem, he told them:

My assumption is that all of you can work this problem because no one came to me. You better understand that I am not going to give you the easy problems on the quizzes and then have you screw up on the AP exam because they have all the hard problems on there.

Ryan organized his course by creating an outline of the material the students should know as prerequisite knowledge to calculus. Ryan commented that “To reinforce this [prerequisite knowledge], I do a lot of review intertwined into the normal curriculum.” Part of this organization involved Ryan addressing speculated misconceptions before they potentially became problems. In the first interview, Ryan had stated: “Especially early in the year, I try to clear up any misconceptions regarding simple algebraic properties. I also try to do the same thing with regard to proofs.” For example, one class period began with the placement of a transparency containing a proof which arrived at an obviously incorrect conclusion on the overhead. The steps of the proof, however, appeared to make sense at the intuitive level. After the students were given some time to figure out the reasoning for the proof, Ryan addressed their possible misconceptions and briefly brought up the deductive nature of mathematics:

Notice that this is a nice proof, but the conclusion is obviously wrong. I'm showing you this because in this class there will be many proofs that may appear to be over your head, so you'll let it go and then just accept the conclusion. It is important that you know how to follow the steps and not readily accept a given proof.

In conjunction with Ryan’s desire for students to gain an intuitive sense of calculus concepts, he felt that the students best acquired this type of knowledge by approaching particular concepts visually.
I believe that students learn exceedingly well through the use of pictures. This is why I started limits from a graphical standpoint. I try to pick a variety of problems that touch on misconceptions of many students before they actually become misconceptions.

Although we do not use trig (algebraically based) until after derivatives and integrals are covered, I use the graphs of trig functions when building conceptual and visual understanding.

Though Ryan did not use graphing technology for instructional purposes, he did use prepared graphic transparencies. In addition, he encouraged the students to use their graphing calculators to help them understand their homework problems.

Perceptions of the Concept of Limit

Ryan's perception of the concept of limit as a way to describe the behavior of a function remained stable throughout the entire data collection period. For example, he consistently responded to the questionnaire item, "Describe in your own words what a limit is," with "A limit is a description of the behavior of a function around a point or an end." Analogously, in response to each of the four questionnaire completions, Ryan described the meaning the formal limit notation

\[ \lim_{x \to a} f(x) = L \]

as "The behavior of \( f(x) \) around the point where \( x = a \) is described by \( L \)."

To the questionnaire item asking when he felt that he really understood the meaning of the formal limit notation, as given above, Ryan stated that he understood limits intuitively quite a bit earlier than when they had been formally introduced. In fact, he claimed that his intuitive understanding developed from his junior high years, adding, however, that he did not understand limits formally until after he took calculus and was enrolled in advanced analysis classes. He consistently stated, "Fundamentally and
conceptually, I understood limits in junior high. I was always fascinated by the concept of infinity. In a more formal sense, I didn’t understand limits until after calculus.”

**Perceptions of the Role of Limits in the Teaching of Calculus**

In response to the questionnaire item, Ryan repeatedly considered the limit to be “the most important concept to understanding calculus,” adding that “though it is crucial to understanding calculus, it is not fundamental to the doing of calculus.” During the interview, this perception was discussed in greater detail in reference to his understanding of a limit when he was in calculus.

Ryan: Understanding a limit in the mathematical sense [epsilon-delta definition of limit] was not really important to me in order to do calculus problems. I could use the power, product, and quotient rule and all that stuff fine, but I never really realized that embedded in all that was the limit when I was in school.

Researcher: How important is the limit in your teaching of calculus?

Ryan: I think if you were to ask my kids they would tell you that they understand calculus a lot better than they can do calculus. So, I think its absolutely crucial that they understand [the concept of limit].

From Ryan’s diagrams of the most important concepts and relationships in calculus, as given in Figures 2–5, the concept of limit was revealed to be linked with other calculus topics, but it was difficult to interpret from the diagrams whether connections between limits and other calculus topics were actually perceived. Ryan, however, demonstrated this connectedness in the interview:

We never finish a concept, we’re continuing to expound on it throughout the year. Limits, we’ll finish the chapter on limits in September, but we
won't finish limits ever throughout the year, like L'Hopital's rule, derivative, integral. That's one of the things I like about calculus, you're never done with it, there's a structure to it, you're building on it continuously. Later in the year the students can look back and say "Goll, I sure am glad I understood limits. They didn't seem important then, but now they are!" That's a beautiful thing about calculus.

Ryan believed that an intuitive understanding of limits was fundamental to understanding the derivative as well as other calculus concepts. The importance of the limit was demonstrated by how he taught the introduction to derivatives, as follows:

OK, how about the definition of derivative. Now, I don't think that it is vitally important to have everything memorized. What is the derivative? It is the slope. Just by that fact you should easily remember the definition of the derivative.

[They discuss the formal definition of derivative,]

\[
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Why does the derivative incorporate a limit? Because I want that distance to be able to get smaller. So, the distance between the points approaches zero.

Ryan then continued with the lesson without further discussion of the connection between the concept of limit and the definition of derivative more thoroughly.

Ryan's response to the questionnaire item which asked for the appropriate level of rigor with respect to the concept of limit, stressed that to answer this item, the focus of the given calculus class would have to be examined. For the first two completions of the questionnaire, Ryan responded similarly:

I feel a high level of rigor helps students understand strict structure and the significance of the concept. For an appreciation of mathematics, true understanding and high rigor are crucial. The focus of the class determines the rigor necessary.
After teaching the concept of limit, Ryan also responded in a similar manner, "Again, the goal must be kept in mind. If the goal is mathematical understanding, high rigor is fundamental. However, if the goal is application oriented (with a business flavor in particular), rigor is not important." The following statement from an interview following the first questionnaire completion provides evidence of the focus of Ryan’s calculus class:

Students find little use for and reason to learn the epsilon-delta definition of limits. I try to emphasize the importance of thinking and reasoning skills. I want students to realize the cause and effect nature of functions, like that a change in x causes a measurable change in y. Only in mathematics is this cause-effect relationship so concrete. In many ways, this makes it, cause-effect not epsilon-delta, easier to grasp and understand.

Given the previously stated goals Ryan had for his calculus course, he believed that both intuitive and rigorous mathematical definitions needed to be taught and understood in his AP calculus course.

The Teaching of Limits

Ryan began covering the eight-day unit on limits the second day of the school year. Since it was an excellent place to combine the learning of new material with the review of pre-calculus material, Ryan felt that it was best to start the calculus course by teaching the concept of limit. The sequence of the topics covered coincided with the given section in the limits chapter of the textbook (see Appendix F, outline of Chapter Two). Two quizzes, each of which were of half a class period in duration, were given during this eight-day time period. No unit exam was given on limits. The first examination occurred after the unit on differentiation, approximately four weeks into the term.

Ryan stated that he relied on students’ intuitive notion of the idea of limit to guide them to the solution during the initial investigation of finding limits for various graphs of functions. In a journal entry, Ryan discussed his introduction to limits:
I intentionally avoid defining a limit early in the exploration of it. This, I believe, helps students gain a complete conceptual understanding. The manipulative skills that we use in evaluating limits are one of the best algebra reviews I can imagine.

Specifically, prior to the beginning of the school year, Ryan had described his favorite way to explain what a limit was to a beginning calculus class as, “Ignore the specific and focus on the abstract. Thus, I tell them don’t worry about what happens at the point, look at what happens around the point. The limit at a point is a description of the function.” Throughout the entire data collection period Ryan consistently stated that his favorite way to explain limits was by clearly stating to the class that “A limit is a behavior of a function (or graph of a function) in the immediate vicinity of a specific point, not at the point.” During a follow-up interview, Ryan was somewhat more specific about his concern for possible misconceptions students may develop for the “limit as a behavior”:

Often times when a kid is asked to look at a limit at $x = 3$, they look just at three, they look at the specific point. They have a hard time understanding that the limit is that point that’s open. I tell students to think about what’s happening “around” that point, not at the point. I tell them to focus on the abstract. I guess “focus on the abstract” is almost an oxymoron.

On the first day of teaching the unit on limits, Ryan’s lesson reflected exactly what he had discussed in response to questionnaire items, journal entries, and the interviews. He began the class with an overhead transparency of a graph containing a variety of types of discontinuities (Figure 6). He then had the students anticipate the limit at various designated $x$-values. For example, he began the class by asking the students: “What is the limit at $x = -4$? Remember, we are not necessarily concerned with the value of the graph at that point, but what is the limit in the vicinity of that point?” This question led to a discussion on why the limit at $x = -4$ was two and not one, and the students were continually reminded to focus on what was happening around the point $x = -4$ and not at
just the point. Ryan continued to look at the limit of the graph at \( x \) equals 5, 1, and 2 with a similar discussion of looking at the vicinity of the point and not the at the point itself.

The entire first day on the unit of limits was focused on the graphical interpretation of a limit. The examples used primarily involved the interpretation of hand-drawn graphs. The students were asked collectively to give the limit of the graphs at various points.

Ryan began the second day on limits by writing the following problem on the overhead,

\[
\lim_{x \to 3} 2x = ?
\]

Ryan asked, “What would you do to find this answer?” Several of the students said, “Graph it.” Ryan relied, “Why? Because that is what we did yesterday?” Many of the students responded with six as the answer to the limit. Ryan suggested that the response
could be graphed, but in this case an easier method would be to use "direct substitution." The majority of the exercises for this textbook section involved direct substitution. Based on the simplicity of these types of problems, Ryan spent the remainder of the day having students evaluate limits, verbally responding in class to all of the exercises in this section of the textbook. The order of the methods used to solve these exercises were direct substitution, factor and use direct substitution, and graphing. This order was also indicated by the order for the type of problems listed in the textbook. One student commented that "There are too many rules."

Though Ryan spent little time in class finding limits using a table of values, in response to questionnaire item four (see Appendix D), he consistently stated that a table of values would be most convincing to his students in support of the statement

\[
\lim_{x \to 0} \frac{\sin(x)}{x} = 1.
\]

Throughout the data collection period, Ryan offered five arguments that supported this statement, including: (a) L'Hopital's Rule, (b) analysis of a table of values, (c) exploration of the relationship between \( x \) and \( \sin(x) \), (d) the Squeeze Theorem, and (e) sector area. The two arguments listed consistently were L'Hopital's Rule and the table of values. Though Ryan felt that the table of values would be more convincing to his students, Ryan listed L'Hopital's Rule as the argument which was the most convincing argument to him each time the questionnaire was completed. He explained that "Although [L'Hopital's Rule] is very abstract, I built great confidence in it during my undergrad days."

L'Hopital's Rule requires understanding of differentiation and it is not covered in the textbook until after differentiation (see Appendix F, Section 7.7). Since L'Hopital's Rule is a required topic for the AP (BC) exam, it was assumed that Ryan covered the topic in class. However, the lesson on L'Hopital's Rule was not directly observed since it was covered after conclusion of the data collection period.
Though Ryan wanted his students to understand limits intuitively, either graphically or using a table of values, later in the unit he appeared to want students to be able to first find the limits algebraically. The following dialog demonstrates how, toward the end of the unit on limits, Ryan wanted his students to work limit problems:

Ryan: The only problem we have with this limit

\[
\lim_{{x \to 2}} \frac{x - 3}{{x - 2}}
\]

is when the denominator is equal to zero.

Terry: Isn't it best to just graph it?

Ryan: Yes and no. What if you are walking down the street and you don't have the calculator and someone asks what the limit is. That happens to me all the time. It is better to be able to factor it, and find the limit using algebra.

Chris: Why can't we just graph it?

Ryan: There is nothing wrong with graphing, but graphing is a crutch. Try to use the crutch at times when it is prudent to use [it]. If you can't figure out the problem, maybe graphing will give you insight about the problem. That kind of analysis is what I'm trying to get you to do.

Ryan: Now you can tell what direction you are going. That is doing it with the table which is fine, and graphing is fine. But can we figure out how to do this with algebra? [For this problem they are trying x-values 2.01, then 2.001.]

Well, as \( x \) goes to two the limit of \( x - 3 \) goes to negative one, and as \( x \) goes to two the limit of \( x - 2 \) goes to zero.

So, intuitively you should be able to see that from the positive side of two, as the bottom goes to zero and the top goes to negative one, you can tell its
Ryan then informed the students that graphing calculators would not be allowed on the AP exam, however, this was not brought up in response to the example given above. (See Appendix C for information regarding the future use of graphing calculators on the AP exam.) Throughout the data collection period, Ryan was never observed discussing the future use of graphing calculators on the AP exam with his students, nor was this topic discussed with the researcher.

For the following couple of classes, Ryan continued to discuss the algebraic evaluation of finite as well as infinite limits. Throughout this time period, he also incorporated the section of the text on continuity. As Ryan discussed what it meant to be continuous throughout the entire unit on limits, much of the material found in the continuity section appeared to be review. For example, at the same time and in the same manner he was providing the intuitive introduction to limits, he intuitively discussed continuity. Connections between limits and continuity were discussed daily.

The next section of the textbook, as well as the sixth day of the unit on limits, involved the epsilon-delta definition of a limit. In the initial interview, Ryan had discussed the difficulty in providing motivation for this topic:

As I begin my “discussion” of epsilon-delta, I get on my high horse and try to motivate the conceptual understanding of the definition. I believe that many students are not confused by the definition’s concept. I believe that they truly understand it. I believe that the bigger stumbling block is the language and symbolism used in the definition.

Ryan attempted to motivate the students for the epsilon-delta definition in the following way:

We are going to talk about something now that will have very little use in your calculus career, but to give you motivation it makes you analyze things analytically. It also will be on the AP exam. It gives you skills. The
reality is that you won't use these specific problems again, but you will use the analytical skill again.

Ryan's actual lesson on the definition of limit began with the an extensive algebraic and graphical discussion of a specific linear example. Even after giving the example, Ryan continued to try to provide motivation for working these types of problems:

![Graph of f(x) = (1/100)x + 4.96](image)

Figure 7. Ryan's Graph for \( f(x) = (1/100)x + 4.96 \) Before Discussion of Definition of Limit.

Here is a graph (Figure 7) of

\[
f(x) = \frac{1}{100}x + 4.96.
\]

How close do I have to be of four one way on the function (on the x-axis) to be 0.1 from five in the other direction (y-axis). The x-variable changes the y. How much do I have to change x to affect a 0.1 change in y?

When I took an analysis class from Professor Maanen, we had plenty of these questions. He would say, "I don't know who would come up with this question, it must be the devil...", then he would tirade around and really get into it by saying "It must have come from hell." You can tell how much motivation is needed to do this.
Ryan continued the example by demonstrating algebraically that if he wanted $|f(x) - 5|$ to be less than 0.1, then by direct substitution $|x - 4|$ had to be less than 10. He then discussed how this example was related to the definition of limit:

What do you suppose all of this has to do with? This is what it means to be a limit. I have never said exactly what a limit is. I am telling you mathematically the definition of a limit looks very confusing, extremely confusing. You'll be thinking, “Oh my gosh, I don’t know what epsilon is and I can’t even draw a delta.” Don’t get stuck in the symbols. We just did the definition of a limit and you understood it. So let’s put symbols in for what we did. [They discuss that $a = 4$, $L = 5$, $\varepsilon = 0.1$, and $\delta = 10$.]

Here’s what it means to be a limit. In order for $L$, in this case five, to be the limit as $x$ goes to $a$ of $f(x)$, you must be able to find a delta, in this case that was 10, for any epsilon, which in this case was 0.1. In layman terms it says that regardless of if someone challenged me to get so close, I can get that close. I could do it. I could find delta if you gave me any epsilon. Remember the first day I told you that $0.9$ (repeating) = 1. It's the same idea here, because I can get as close to the limit as I want and no matter what you say I can get closer. So, this is kind of the idea of the definition of a limit. Here is the definition of a finite limit.

The limit of $f(x)$ is equal to $L$ as $x$ approaches $a$ means that for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever the $|x - a| < \delta$.

Finally, Ryan discussed the definition algebraically in terms of the previous example again. In an attempt to provide an intuitive graphical representation of the definition of limit, Ryan then applied the symbolism of the definition of a limit to the graph of the previous example (Figure 8).

These attempts to inspire an appreciation for the deductive nature of mathematics in his students did not appear to motivate the students. The following example demonstrates that Ryan finished the lesson by attempting to provide greater motivation for learning the definition of a limit.
Figure 8. Ryan’s Graph for \( f(x) = \frac{1}{100}x + 4.96 \) After Discussion of Definition of Limit.

Are you OK on this stuff? It may not seem important, but there is lots of importance to this in calculus. This kind of analysis we are going to do all through the year. Now even if you are an engineer or a mathematician, you probably won’t be asked to state the definition of a limit. But, you may be asked, “What is a derivative?” You will be processing a limit when you do a derivative because limits are the foundation of derivatives. Certain careers will ask you to do a derivative. So, can you get through a calculus class without knowing this definition? Yes. Can you get through this one? No. I will test you on this. Many times I will test with a fill-in-the-blank kind of thing.

The next segment of the lesson for that class period involved a discussion of how to prove the existence of a limit. Ryan explained that to prove that a limit exists, one had to show that the definition of a limit fit the particular situation. More specifically, he stated, “To prove something has a limit, I have to show that it works for every epsilon. There are infinite epsilons so you can’t try all the choices. So, you have to treat it with a
generic epsilon.” Ryan worked through the proof for a linear function similar to the example given in the text, and then discussed the type of proofs they would have to do for the homework. The following day, Ryan answered some of the students’ questions regarding the epsilon-delta proofs, however, little time was devoted to them. Ryan reminded the students to see him before school if they needed additional help.

The formal investigation of limits ended with a review lesson on limits and continuity. In an informal interview, Ryan stated that he intentionally did not present the definition of limits at infinity directly following the lesson on the definition of finite limits. Rather, the investigation of limits at infinity would take place a few weeks later, “After a sufficient gestation period for finite, epsilon-delta definition of limits.” Ryan suggested that “The presentation of infinite limits [limits at infinity] provides a nice extension and review.” Though Ryan indicated that the motivation for the delay was his decision; it was noted that the textbook covered limits at infinity only after differentiation. Ryan appeared to follow the organization of the textbook, but provided justification for doing so. Ryan’s teaching of limits at infinity duplicated the format used for finite limits.

Throughout teaching the unit on limits, Ryan discussed mathematical symbolism in great detail with his students. He was concerned that many students do not succeed in understanding the concept of limit because they get “too caught up in the symbolism.” In his journal, Ryan commented:

> Many argue that true beauty of mathematics lies in this symbolism. Indeed, as the first day of school I mentioned this. Frankly, I think that symbolism itself is dramatically under-taught. Where is \( |x - c| < \delta \)? Answer: Within delta of \( c \). Most kids have a lot of trouble making this and other connections, between the symbols and the meaning.

Ryan investigated his students’ understanding of limits by giving two quizzes during the unit on limits, in addition to which one-third of the midterm examination involved the concept of limit. Half of the first quiz consisted of problems in which the
students needed to find the limit of a given graph at certain points, whereas the second half contained a few pre-calculus review problems and six problems for which students had to directly evaluate the limit of a given equation by direct substitution. The second quiz included the definition-of-a-limit, fill-in-the-blank problem, as Ryan indicated earlier in the course. In addition, half of the quiz involved the evaluation of limits, with the remainder involving identification where given functions were discontinuous. All of the limit problems on the midterm exam involved an algebraic evaluation of the limit. The use of graphing calculators was not permitted on the quizzes and exam. However, graphing calculators were allowed for the completion of homework.

Summary of Ryan

Ryan viewed calculus in three tiers: limits, differentiation, and integration. Throughout the entire data collection period his diagrams of the most important concepts and relationships in calculus consisted of a linear list of discrete topics. The list of topics closely resembled the outline of his textbook, as well as the outline of the topics required for the AP exam. The diagrams did not illustrate connections between the given topics. However, Ryan strongly asserted that all of the topics were interrelated as “Calculus builds on itself more than any other class.”

Ryan discussed three primary goals he had for teaching calculus: (a) prepare for the AP exam, (b) prepare for college, and (c) inspire appreciation for the deductive nature of mathematics. Each of the three goals was demonstrated in his lecture-style of teaching, however, the extent to which they were covered varied. Preparation for the AP exam appeared to be the primary goal revealed in Ryan’s teaching practices. Based upon the purpose of the AP exam, the goal of preparation for college coincided directly with preparation for the AP exam goal. Throughout the entire data collection period, Ryan demonstrated enthusiasm and appreciation for the calculus content. Discussions involving
appreciation for the deductive nature of mathematics were largely teacher-directed, however, it was apparent from his teaching practices that he wished to instill the same appreciation for calculus in his students.

Ryan stated that his calculus course was specifically designed to cover the AP (AB) syllabus. Though not directly related to the AP (AB) syllabus or the outline of the textbook, Ryan stated that he valued discovery learning using an intuitive visual approach. Ryan's view of teaching calculus can be summarized as the necessity for computation, manipulation, and memorization, but the true beauty of calculus lies in "communication, reasoning, thinking, and problem-solving."

Though the structure of Ryan's teaching closely followed the order of textbook sections, emphasis given to particular topics was guided by the AP exam. In fact, Ryan would deviate from the textbook when necessary to keep the terminology and methods consistent with the AP exam. Ryan's role appeared to be as AP exam "coach," and he frequently used the AP exam as a motivational tool. For example, he had students work the most difficult problems in the textbook because these problems most closely resembled problems on the AP exam. Though Ryan's teaching did reflect the goals he professed for the class, preparation for the AP exam appeared to be the most fundamental goal.

Ryan considered the concept of limit to be the "most important concept to understanding calculus." He perceived limits as a way to describe the behavior of functions, and felt that limits were connected to all other calculus topics. Thus, Ryan continued to expound on limits throughout the whole course. Nonetheless, he did observe that the concept of limit was "not really important to me in order to do calculus problems." Ryan felt an intuitive notion of limits would be helpful throughout the course, however, the appropriate level of mathematical rigor was dependent on the focus of the class. One of the goals of Ryan's class did contain mathematical rigor with respect to the epsilon-delta definition of limit as it involved preparation for the AP (BC) exam. Though it was not necessary that all of his students obtained this level of understanding, Ryan felt that to
acquire true understanding of the concept of limit, a high level of mathematical rigor was critical. Moreover, the desire for a high level of mathematical rigor was tied to Ryan’s third goal: to inspire an appreciation for the deductive nature of mathematics.

Based on the course set-up, Ryan envisioned that students may not appreciate the concept of limit as it was covered at the beginning of the school year, but that later in the course students would look back and understand the importance of limits. Thus, Ryan stated that his primary objective was for students to obtain an intuitive understanding of limits as it was fundamental to understanding other calculus topics. Though the classroom observations revealed that Ryan initially taught new topics from an intuitive perspective, this objective was somewhat in conflict with Ryan’s desire for students to evaluate limits algebraically. For example, though Ryan spent considerable time on the intuitive notion of limit, from both graphical and numerical representations, later in the unit he stated to the students that when evaluating limits, it was better to do it algebraically since graphing was a “crutch.” This statement may have been derived from the fact that graphing calculators could not be used on the AP exam. Similarly, the use of graphing calculators was not allowed on Ryan’s exams. As further reflections of AP exam needs, Ryan’s exams involved primarily the evaluation of limits and identification of points of discontinuity.

Ryan’s classroom practices also reflected his desire that students obtain an intuitive understanding of the concept of limit. In fact, the connections between the intuitive notion of limit in relation to continuity was examined daily. Ryan also used the teaching of limits as a place to incorporate algebraic and trigonometric review.

Ryan used the fact that the concept of limit was connected to other calculus topics as a motivational tool for the students. However, problems were apparent from the lack of motivation among students when Ryan covered the epsilon-delta definition of limit. His attempts to inspire an appreciation for the deductive nature of mathematics among his students appeared to be without serious result. In fact, Ryan relied upon the AP exam as his motivational standby. Furthermore, the intuitive notion of limits, not the formal
epsilon-delta definition of limit, was used in Ryan’s classroom lessons for other calculus topics. Throughout the entire data collection period, to make connections between calculus problems as well as topics, Ryan both planned ahead and looked behind.

Throughout the entire data collection period, Ryan’s perceptions of limits, the role of limits in calculus, and the teaching of limits remained stable. Ryan noted that he had understood limits intuitively since junior high. However, he felt that he really understood formal limit notation only from college analysis. Ryan was the only teacher in the study who did not mention that the act of teaching limits for the first time enhanced his understanding of the concept.

Though the translation of Ryan’s perceptions of limits and the role of limits in calculus into classroom practice was relatively strong, preparation for the AP exam may have posed a threat to complete translation. Ryan valued the connections between limits and other calculus topics as a motivational tool. Topics that were included as motivation solely because of the AP exam appeared to provide some degree of frustration to Ryan. He frequently stated that the level of mathematical rigor necessary for a calculus course was dependent upon the focus of the course. Though Ryan desired to inspire an appreciation for the deductive nature of mathematics among his students, and he appeared to thoroughly enjoy his role as “coach” for the AP exam, his perceptions of limits and the role of limits in calculus may have been portrayed differently in the classroom in the absence of preparation for the AP exam as a major focus.

Independent Teacher Profile, Russell

Scholastic and Professional History

Russell has taught high school mathematics since graduation from a small college in the Northwest U.S. in 1968. He was a mathematics major with an emphasis in
economic theory. He wanted to become a mathematics teacher because he “wanted to coach high school athletics” and he had “older relatives that were secondary teachers (my wife was an elementary teacher).” Russell asserted that his overall academic preparation was “very solid in mathematics,” however, in the same sentence he expressed the concern that “Most teachers my age have not had the support to stay current with technology and its use in the classroom.” Russell has also taken a few summer collegiate courses on the teaching of mathematics throughout the past 26 years, but has not pursued a graduate degree of any type.

Centrally located in a large metropolis, Russell’s high school had a student population of over 1,100 in grades 10–12. The student body was approximately 85% white with a 15% mixture of Asian and African-Americans. One wing of the school was devoted to the mathematics department.

Russell’s teaching assignment for the year in which this study took place included one section of pre-algebra and one section of AP calculus. Throughout the 26-year period that Russell had been teaching at this school, he taught all of the math courses offered at the school, ranging from general math to AP calculus. Though his duties were limited due to educational budget cuts throughout the State, Russell was also the current department chair. Since becoming a mathematics teacher, Russell had been a member of NCTM, took part in a discrete mathematics conference, participated on several textbook adoption committees, and periodically worked with curriculum development in mathematics. Russell’s job responsibilities also included assignment as head football coach.

Russell was selected for participation in the study from a list of AP calculus teachers in the state. Russell was contacted by telephone during the first week in August, 1993 and he enthusiastically agreed to participate in the study. During this conversation, a time was arranged prior to the beginning of school for the first questionnaire completion and interview. The completion of the questionnaire and interview took place on September 1, one week prior to the beginning of school.
The first questionnaire completion process and interview were conducted in Russell’s classroom. Russell appeared to be focused upon completion of the questionnaire, but the interview was interrupted on several occasions. Visitors kept coming into the room to engage in friendly conversation. Russell is a congenial man, so some of these conversations continued for extended periods of time. In addition, Russell was in a hurry during the interview because he needed to get to football practice. Russell appeared anxious to begin the football season and he talked about it frequently with enthusiasm and devotion throughout the fall term.

Throughout the entire data collection period Russell was very cooperative. Items were given to the researcher on time, phone calls were immediately returned, and Russell was always willing to discuss the events of the day. Russell conveyed his willingness to participate in any study that would better mathematics education. He commented on his basic philosophy of education:

I believe in public education as the responsibility of the entire community. Opportunities must be provided for every student equally. I believe that almost every student can learn the mathematics necessary for survival in our society above the service-level type jobs, provided they are properly motivated and taught. I believe that education is the key that will unlock the opportunities that are beyond the reach of a large segment of our population.

Portrait of the Calculus Classroom

Russell had taught calculus for the past 21 years. The mathematics department supported the need for this course and anticipated offering the course in the foreseeable future. In the year of the present study, 28 students, characterized by Russell as follows, participated in the calculus class:

Approximately 50% of the students are taking calculus because they think that it is the “thing to do” to get where they want to go in college. The
rest of the students have a genuine love for the material beyond what it can get for them in the future.

Each of the large classrooms in the mathematics wing at Russell's high school were set up identically. Desks were placed in six rows with six or seven chairs in each row. Russell's desk was centered in the front of the room, with four file cabinets off to one side. The only usable chalkboard extended over the entire front wall, while the other walls were composed of either windows or bulletin boards.

Of the 28 students in Russell's AP calculus class, 24 were studying for the AB exam and four for the BC exam. Basically, the BC exam extends the AB exam to contain series-and-sequence (see Appendix C for a complete description). The class was set up in such a way that Russell taught the AB students and the BC students worked mostly on their own. On a typical day, Russell began class by taking roll, and then asked the BC students to come to the front of the room for a question-and-answer session usually lasting between 10 to 15 minutes. During this time, the AB students worked quietly on assignments from the previous day or other personal concerns. A few of the AB students would try to follow what Russell was discussing with the BC students. Russell occasionally would ask the AB students to follow along with the discussion, but for the most part the AB and BC students were taught separately. Upon completion of the question-and-answer session, the BC students either went to a study hall or to the back of the room to worked on their assignment. Given the circumstances, this system seemed to work quite well for Russell and his students.

After the BC student question-and-answer sessions, Russell spent the rest of the period lecturing to the AB students. The greater part of each lecture involved the introduction of new material, or new examples to demonstrate a lecture previously given. However, some of this time was devoted routinely to review of homework problems. Since Russell consistently lectured until the end of the class period, the students were not given any class time to work on new assignments.
Russell taught with confidence and was very knowledgeable about calculus, though his teaching style was not student interactive. Even his attempts at humor rarely received a response. The following is an example of Russell's humor: "Graphing seems to be the easiest way and I tend to like doing the easiest thing. I don't know, maybe it's my age?" Frequently, the students seemed to be lost in their own thoughts. For example, during a particular teaching episode four students were observed with their head down, 10 students were working on something at their seats, while the remaining seven were attentive and passively participating.

The students in Russell's class did not ask many questions. In fact, they did not respond to the teacher during the majority of the time when they were observed. Prior to the beginning of class one day, Russell communicated that he was well aware of his students' unresponsiveness.

Russell: I'm not going to videotape today. I'm in my coaching clothes and I don't really want to be on tape.

Researcher: It doesn't matter what you are wearing. I just want to be able to hear what you say and what the students are saying.

Russell: It's just review today—besides they [students] don't respond much and they really don't ask any questions. It's too late now anyway, I don't want to disrupt anymore of the class time.

Part of the students' unresponsiveness to Russell's teaching was credited to the fact that Russell did not give them much time to answer his questions. Consider the following example, taken from observation of a review lesson:

Russell: Let's look at the assignment. We can do some of these in our heads. [They discuss problem number 25.] What will delta equal?
Students: (No response.)

Russell: What delta will work?

Students: (No response.)

Russell: Epsilon over?

Students: (No response.)

Russell: One.

This lack of classroom interaction was consistent since Russell posed mostly rhetorical questions, then proceeded to give the answers to all of the problems for the given assignment.

The serious nature of Russell’s class may be attributed to the nature of the AP calculus students. These students were well-disciplined, though strict ground-rules for classroom deportment were never observed or discussed. The only statement regarding discipline that was observed was Russell “warning” them of the importance of doing their homework:

But, unless you are some kind of a math god or goddess of some type, if you don’t do the homework in here from now on I can guarantee that you will get absolutely crashed. You won’t have a chance. You can almost get by without much until now, but I can almost guarantee disaster in the future if you don’t keep up. Even if you think you are sick, come to school anyhow. You can not miss class and you have to do the homework.

Along with this serious tone, Russell also had a reassuring nature about him. At first he challenged the students and warned them that calculus was going to be tough. Upon the completion of an exam, however, he reassured them that they were fine even though they were in the “fog.” One example was Russell’s comment to the class prior to a quiz: “I can promise you, that no matter how you do on this exam, I can promise you that you will all have smiles on your face when you are through.” After the quiz, Russell
commented, "If you don't have this done then you're probably a little woozy on this stuff, or in a fog. That's OK."

Perceptions of Calculus and the Teaching of Calculus

Throughout the data collection period Russell continuously viewed calculus as a linearly-ordered set of topics. Figure 9 illustrates Russell's first diagram of his overview of the important concepts and relationships in calculus. In the interview following analysis of the first questionnaire Russell declared that the diagram included just what was "on the top of his head." However, the topics listed were arranged in sections that coincided with given chapters in the calculus textbook (see Appendix G for textbook outline). Furthermore, when Russell was asked, "What is the basic organization of your calculus course?" he responded with, "It is textbook-driven for the most part." Connections between the concept of limit and other calculus topics were illustrated as limits repeatedly appeared in other sections of the diagram.

Though to more informal degrees, Russell continued to demonstrate his linearly-ordered view of calculus in the other three diagrams of his overview of important concepts and relationships in calculus (Figures 10–12). Moreover, the influence of the textbook was reiterated in several of the informal interviews and journal entries. The following example shows that the influence of the text was also revealed from classroom observations: "What is the title of Chapter Three? Holy cow, differentiation. We're going to follow the text as close as possible. They start off by talking about tangent lines."

Each of Russell's diagrams also represented an emphasis upon introductory topics intuitively. After presentation of an intuitive notion of the topic, Russell typically listed a formal notion of the concept, ending each section with applications.
I. REVIEW
   A. Functions (Algebra of)
   B. Absolute Value and inequalities
   C. Graphing relations (transformations)
   D. Trig

II. LIMITS / CONTINUITY
   A. Intuitive approach
   B. Formal definition for \( \lim_{x \to a} f(x) = L \) (Epsilon-Delta)
   C. Graphing Calculator – TI-81 and its use for finding limits.

III. DERIVATIVE
   A. Intuitive approach wrt to slope of tangent and rate of change
   B. Formal limit definitions
   C. Chain Rule and other formulas (sum, product, etc)
   D. Applications
      1. Max/Min.
      2. Related Rates
      3. Diff. Approximations
      4. A tool for graphing
      5. Mean Value / Rolle’s Thm
      6. Intermediate Value Thm
      7. Newton’s Method (program on TI-81)
   E. Use of the TI-81 as an alternative to the derivative.

IV. INTEGRATION
   A. Intuitive approach
   B. Algebraic methods with \( \int_{a}^{b} f(x) \, dx = L \)
   C. Fundamental Thm (1st & 2nd)
   D. Methods of Integration
      1. Formulas
      2. Substitution
      3. Parts
      4. Trig subst.
      5. Partial Fractions
   E. Applications
      1. Area
      2. Volumes (shells, discs, washers, “slabs”)
      3. Work
      4. Fluid Force
      5. Centroids (Thm of Pappus)

V. OTHER APPLICATIONS USING PARAMETRICS AND POLAR
   A. Length of curve
   B. Areas
   C. Volumes

VI. APPLICATIONS IN ECON.
   A. Marginal Cost / Revenue
   B. Elasticity
   C. Generating cost and revenue curves from data.

Figure 9. Russell’s Diagram of Calculus, Prior to Beginning of School.
An intuitive understanding of the following limits:

\[ \lim_{{x \to a^-}} f(x) = L \quad \text{with} \quad \lim_{{x \to a^+}} f(x) = L \quad \text{and} \quad \lim_{{x \to a^+}} f(x) = L \]

\[ \lim_{{x \to a}} f(x) = \pm \infty \quad \text{(along with one-sided limits)} \]

\[ \lim_{{x \to \pm \infty}} f(x) = L \]

\[ \lim_{{x \to \pm \infty}} f(x) = g(x) \quad \text{(linear and non-linear asymptote)} \]

Differential calculus with applications

- Related rates
- Motion problems
- Application in economics
- Exponential growth (decay)
- Differential approximations

Integral calculus with applications

(Riemann Sums)
(Methods of Integration)

Areas
- Acceleration \( \Rightarrow \) velocity \( \Rightarrow \) distance
- Volumes (disc, shells, etc.)
- Lengths of curves
- Work
- Fluid Force
- Centroids

Integration of technology

(TI-81, 82, 85 calculators, computer - Math Exploration toolkit)

as aids to problem solving.

Figure 10. Russell’s Diagram of Calculus, Planning to Teach Limits Unit.
Advanced Placement (AB Level)

Limits

\[
\lim_{x \to c} f(x) = L \quad \lim_{x \to \pm \infty} f(x) = \pm \infty
\]

Also - one sided limits and continuity

Derivatives and their applications - Max/Min Related rates - graphs
- Economic applications as well as in science
- Exponential growth and decay.
- Motion problems

Integral Calculus and applications
- Areas / Volumes / Length of curves in Cartesian, Polar and parametric.
- Acceleration, velocity, distance
- Moments
- Force
- Hooke's Law (Springs)
- Work Calculations
- Simple differential equations

Use of Technology (TI-81, 82, 85 as a problem-solving tool).

Figure 11. Russell's Diagram of Calculus, After Teaching Limits Unit.
AP (AB)

Notation and becoming comfortable with symbolism.
Riemann Sums and their limits as a definition for the definite integral.
1st & 2nd Fundamental Thm for integral calculus
Mean Value Thm (Ave. Value)
Approximating methods (Trap. Rule, Simpson's Rule)
Using TI-81, 82, 85 with programs
Improper Integrals
Applications for integration - Areas, Volumes,
Surface areas, lengths of curves. All in Cartesian, parametric & polar form.

Figure 12. Russell's Diagram of Calculus, End of Data Collection Period.

The principal text used by Russell's class was Calculus with Analytic Geometry: Fourth Edition (Larson, Hostetler, & Edward, 1990). This particular text devotes an entire chapter (Chapter Two) to the study of limits. In both the first and final interview, Russell reported that he may supplement this text with exercises and examples found in various other available calculus textbooks. He also discussed using the TI-81 calculator
and Math Toolkit software as supplemental tools for the class. Russell stated that these tools may help "to look at alternative methods of solution as opposed to the traditional textbook approach."

At the end of each major textbook section, the students were expected to solve problems from a Calculus Mastery Test, which were older AP readiness exams that Russell had on file from the early 1980s. The tests were given to help the students prepare for the AP exam. Russell commented to the class as they had just finished the unit on limits, "The reason we do these things is that these questions are very much like the AP exam questions. You may not be able to do all of these yet."

Russell stated that his three primary goals for teaching the class were: (a) prepare for the AP test, (b) encourage students to discover and appreciate the beauty of calculus as the entrance into higher mathematics, and (c) incorporate technology into the problem-solving process. Though each of these three goals was demonstrated in Russell's classroom practice, the degree of focus varied considerably.

Russell affirmed that he wished students to gain an intuitive understanding of calculus concepts first. In the first interview he stated:

I will spend a lot of time on the intuitive approach to topics in calculus. I try not to even say the word limit during the introduction to this topic. I will use easy functions, for example, \((1 + 1/x)^\infty\). I will make tables with the TI-82 or a computer Math Toolkit. They will recognize this goes to \(e\). So, intuitively they expect \((1 + 0)^\infty = 1\).

This specific example, using a graphical tool for instructional purposes, did not occur in any of the classroom observations or videotapes gathered during this investigation. Observations did indicate his desire to approach limits in an intuitive manner by using hand-drawn graphs of discontinuous functions. He also discussed a desire to use an intuitive approach to introducing derivatives. This desire was suggested by the following claim: "I try to use an intuitive approach to derivatives, slopes of lines. I talk about a
speedometer and average velocity as opposed to instantaneous rate of change; not only
slopes of tangent lines, but also velocity and acceleration.”

Throughout the data collection process, Russell required students to memorize
basic calculus facts so they would be able to call on these facts quickly at appropriate
times. For example, in many of his classroom discussions he conveyed to the students that
it was important for them to memorize a large collection of graphs. Thus, they would
have the opportunity to access the graphs from memory at any given time.

We want to have an enormous catalog of graphs in our head, thus we can
quickly do translations like $y = \sqrt{x}$, $\sqrt{(x - 3)}$, $\sqrt{x}$, ... Understanding
translations and transformations can help us build on our knowledge at any
time.

His objective that students memorize certain calculus topics may be a result of his
previously stated course goal to prepare students for the AP exam.

**Perceptions of the Concept of Limit**

Russell generally discussed the concept of limit in a dynamic way, using motion
toward a limit. For example, each time he responded to the questionnaire item asking him
to describe what a limit is in his own words, he stated that “It describes the behavior of a
function close to a finite value or at values that are large without bound” and “The limit,
say $L$, is a number the function approaches in value when the function is evaluated for
appropriate numbers close to $a$.”

Russell was also consistent throughout the data collection period in responses to
the questionnaire item asking when he first felt that he really understood the meaning of
the formal limit notation,

$$\lim_{{x \to a}} f(x) = L.$$
Each response indicated that Russell felt he first understood limits “as an undergraduate in elementary calculus.” However, he observed that he did not really understand limits formally until his graduate level classes in analysis. He stated, “I thought I had a real solid understanding of the limit concept [prior to graduate level analysis], although I was too weak in my math skills to actually prove very many limits using the definition.” It was after Russell had finished teaching the unit on limits that he added to his questionnaire response, “Teaching calculus also helped me understand limits.” Russell elaborated on this response in the interview that followed: “When I teach limits I have to put myself in my students’ shoes. I understood [the concept of] limit better when I had to teach limits with the proofs.”

Russell viewed the importance of the formal limit notation, given above, as “A means of formalizing an abstraction,” also describing the notation as “A clever and inspired means of formalizing a rather simple intuitive concept.” He followed up on this response in an interview by stating the limit was “A necessary concept to establish the validity of calculus.” Russell discussed his perception of the formal limit notation by stating, “I have a good, friendly feeling for limits.”

Perceptions of the Role of Limits in the Teaching of Calculus

Russell viewed the concept of limit as “extremely important” in teaching calculus. Though every diagram Russell contributed throughout the data collection period indicated that he viewed the concept of limit as one of three major sections of calculus (Figures 9–12), classroom observations of subsequent topics involving the concept of limit revealed that Russell also recognized the connections between the concept of limit and other calculus topics. In fact, each connection Russell made between the concept of limit and other calculus concepts solely involved an intuitive understanding of the concept of limit.
The following example illustrates the intuitive notion of the connection of the limit to the derivative. Russell drew a graph for \( f(x) \) on the chalkboard and added the secant and tangent lines at suggested times in the example (Figure 13). The following discussion between Russell and approximately one-quarter of the class occurred directly after a student, Ted, had correctly drawn a tangent line to a curve Russell had drawn on the chalkboard.

Russell: OK, here is a curve. Ted would have probably drawn a tangent like this. [Russell draws the secant line.] Does that look like Ted’s tangent line?

Students: No.

Russell: [Russell then draws the tangent.] What about that? Is that closer to what Ted would have drawn?

Students: Yes.

Russell: We’re going to write an equation of this line. What do we need to draw a line?

Figure 13. Russell’s Drawing of Limit Definition of Derivative.
Students: Two points.
Russell: A point and ?
Students: A slope.
Russell: [Writes equation of line on the chalkboard.]
\[ y - f(x) = ???(x - c) \]

OK, now the other line is called the secant line. We can find the slope. Right? The slope is very easy to get [writing the following:]
\[ \text{slope} = \frac{f(c + \Delta x) - f(c)}{\Delta x} \]

What if I make the \( \Delta x \) shorter?

Students: (No response.)
Russell: Wouldn't I get a new secant line?
Students: (No response.)
Russell: Isn't the slope of this secant closer to the slope of the tangent line?
Students: (No response.)
Russell: What would it take for the secant line to become the tangent?
Ted: Have the \( \Delta x \) be zero.
Russell: OK, we'll call the slope of this tangent line \( m \). Thus, the limit of the slope of the secant line [writing]
\[ m = \lim_{x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} \]

Does this make pretty good sense to you?
Students: (No response.)
Russell: Is this a natural type thing?
Students: (No response.)
Russell: Nothing could be as clear in my mind, I don’t think, to see how if we cause this \( \Delta x \) to go to zero, that the secant line, the slope becomes what we’ve been looking for. So, we’ll call it \( m \). That’s it — let’s do a problem, a problem that you would have no chance of doing if you didn’t have a little bit of intuitive idea of limits in the back of your mind. Keep in mind that the problem we are going to do here, you will be able to do in about 10 seconds in a couple of days.

Though Russell wanted his students to possess an intuitive notion of limits to make connections to other calculus topics, he also wanted to hold the students accountable for using the formal epsilon-delta definition of limit to prove the limit of linear functions. For example, in response to the questionnaire item asking what level of rigor he felt was appropriate, Russell responded consistently with:

Students should be able to write an epsilon-delta argument for linear functions:

\[
\lim_{x \to 4} (2x + 3) = 11.
\]

1) For all \( \varepsilon > 0 \) there exists \( \delta(\varepsilon) > 0 \) such that
\[ 0 < |x - 4| < \delta(\varepsilon). \]

2) Choose \( \delta, \, \delta(\varepsilon) \leq \varepsilon/2. \)

3) Show that it works:
\[
\begin{align*}
|x-4| &< \varepsilon/2, \\
2|x-4| &< \varepsilon, \\
12|x-4| &< \varepsilon, \\
12x + 3 - 111 &< \varepsilon, \\
12x - 81 &< \varepsilon, \\
12(x - 4) &< \varepsilon.
\end{align*}
\]

In addition, in each of the follow-up interviews, Russell confirmed that students should be able to write a rigorous epsilon-delta proof for linear equations.
I would expect the students to be able to write an epsilon-delta proof for a linear limit. I test them on that—just the linear. I tell them I don’t care how they get delta, but must show that it works. I show proofs for polynomials with inequalities and things like that, but they don’t have to be able to write them. But I expect them to be able to write this linear proof and a proof for finite limit at \( \infty \) later or when we get to integral calculus.

Russell never discussed the importance of these “epsilon-delta proofs” when calculus was taught. In fact, the following example taken directly from Russell’s lecture demonstrates that Russell felt that this theoretical approach to limits did not play an important role in understanding other calculus topics.

But always keep in the back of your mind that you will do fine in integral and differential calculus if you can not do an epsilon-delta proof. You will still get correct answers, and the magic of what calculus apparently does will still be at your command.

Furthermore, Russell never mentioned that this formal understanding of limits was important for the AP exam, and rather did just the opposite. He discussed with the class the insignificance of this type of knowledge:

Would it help you feel better if I told you that if that stuff [formal definition of limit] has you mystified, all is not lost in calculus. As a matter of fact, you can be an epsilon-delta wimp and you can get everything right from now on. As a matter of fact, you can actually do fairly well in calculus and not even be able to evaluate limits very well.

At the end of the data collection period, Russell asked the researcher, “What are the colleges deciding to do with limits? Do they think they will continue to be important to teach?” These questions prompted a short discussion of the various calculus reform projects that were being used at a variety of institutions and the fact that little research had been done on the effects of such implementations. Russell responded, “If research came out saying it’s not important to teach, I could be persuaded that the students only need an intuitive knowledge of limits. Thus, I wouldn’t teach as much rigor.”
The Teaching of Limits

After four weeks of review of pre-calculus material, Russell spent a total of 11 class periods covering the unit on limits. Four of the days were devoted entirely to student reviews or taking quizzes or unit exams. The remaining seven days involved lessons corresponding to sections in Chapter Two of the textbook (see Appendix G for outline of Chapter Two). However, Russell did not follow the specific order of the sections. In addition, though each section in Chapter Two was covered, the majority of the lesson time for the remaining seven days was given to the discussion of the formal definition of limit. Little class time was devoted to techniques of evaluating limits, however, several homework problems on this topic were given. Furthermore, only minimal amounts of class time were devoted to the concept of continuity.

In both the questionnaires and interviews, Russell indicated his desire to teach limits from an intuitive perspective. This approach was demonstrated in Russell’s teaching during the first day of the limits unit. The unit on limits was introduced as follows:

We’re going to talk about limits today. This is really the first discussion of anything that is really quite different from what you have done in the past in mathematics. What I am going to do is talk about an intuitive feeling for limits at first and you are all going to be able to answer all of the questions I ask regarding limits very easily.

Russell then briefly discussed that the need for formalizing the concept, stating that

It [the concept of limit] is all going to be perfectly obvious and then what’s going to happen is that the formal definition is going to cloud the issue for a bit. But, there is a need for some formality in talking about a relatively simple mathematical concept.

However, Russell did not discuss why this need for formality existed. Russell’s desire for an intuitive feeling for limits, coupled with his need to cover limits in a more theoretical
sense, was a theme that ran through the instruction of the entire unit on the concept of limit.

Specifically, when Russell was asked about his favorite way to explain a limit to a beginning calculus class, he brought up the intuitive notion of the “evil twins” (Figure 14). “I use a rather intuitive approach. Two “evil” twins approaching a value (a) on the x-axis from each side and agreeing to what they anticipate the value of the function will be when they actually arrive at (a), although they never will.” This particular intuitive “evil twins” approach was demonstrated in classroom observations directly following Russell’s introduction. Throughout the entire unit on limits, Russell brought up this intuitive “evil twins” image repeatedly when students experienced trouble understanding limits.

The entire first day of the unit on limits was taught in an intuitive manner. In addition to the “evil twins” approach, Russell had the students examine the limit of various functions graphically. Though Russell did not teach using graphing technology, he did attempt to verbally help some of the students use their own graphing calculators to validate what he was discussing using the chalkboard. The following segment of Russell’s lesson illustrates the intuitive graphical approach to finding limits.

Russell: Let’s look at

\[ \lim_{x \to 2} \left( \frac{x^2 - 4}{x - 2} \right) \]

(Figure 15 illustrates the graph Russell drew on the chalkboard).

This happens to be the graph of that function. If you don’t believe it, take out your machine, your TI-81 or whatever, and put it in and graph the thing. It should look like this line. But, you may not see this hole. Can you get the thing to show up on your screen?

Ted: Yep.
Tonya: Yeah.
Russell: Can you see the hole? What happens at two?
Ted: There is a hole.
Russell: Are you sure?
Tonya: Yeah.
Ted: The trace says so.
Russell: Oh, you traced it? What does it say at two?
Ted: Y is nothing.
Russell: So, there is a "nothing" there, OK.
Russell continued with this problem, having some of the students spend a considerable amount of time using their calculators to plug in numbers approaching the limit as he created a table on the chalkboard. Over half of the students in the class did not have calculators and appeared disinterested in the students who were actually working the problem on their calculators. In fact, these students were willing to accept Russell's drawing as well as the other students answers as "truth" and waited for Russell to get on with the discussion of the problem. Russell finished the problem by "cleverly factoring the polynomial and plugging in two."

Russell continued the lesson with another example involving an intuitive approach to finding the limit using both graphical and numerical representations. A few of the students who had graphing calculators often initiated the use of graphical representations,
even if Russell planned on finding the limit using another method. This type of student initiation is demonstrated in the following segment of Russell’s lesson:

Russell: How about the

$$\lim_{x \to 0} \frac{\sin(x)}{x}$$

First of all, would you ever try to plug a zero into this mess?

Ted: No.

Russell: We can’t factor this like the last one. Does anyone have a guess at this limit?

Ted: One.

Russell: How did you get this?

Ted: I graphed it on my calculator and at zero it looks like it goes through one.

Russell: OK, everyone who has a calculator plug this in and make sure you are in radian mode. If you have a TI-85, take out your manual and find out how to make a table of values. But, anyhow, we could just replace $x$ with any number close to zero. Let’s start with positive side, like $x = 1$.

(He then had the students plug in values for $x$ getting closer and closer to zero, and filled out a chart on the chalkboard.)

This problem opened up a rich discussion and interaction between Russell and some of the students, but an exchange which posed a dilemma insofar as more than one-half of the students did not have calculators and were left out of the discussion, which was focused upon the image and values the students were obtaining using their graphing calculators. One student communicated a concern because he did not have a graphing calculator:

Lumpy: I don’t have a graphing calculator?
Russell: That is OK. You can use a TI, but it doesn't prove anything. For those of you who do not have a calculator, it's OK. Tomorrow I will prove this limit using something called the Squeeze Theorem. So, we have explored this limit graphically and with a table and the answer seems to be one. We haven't proved that, but we believe it to be one. By looking at the graph we can predict the limit to be one. Tomorrow we will prove it.

In the questionnaire, Russell was asked to list the different arguments he had seen in support of the statement:

$$\lim_{{x \to 0}} \frac{\sin x}{x} = 1.$$  

Each time the questionnaire was administered Russell consistently responded to this question with two methods: (a) graphing $\sin(x)/x$ close to zero, and (b) the “Squeezing” Theorem (for definition of the Squeeze Theorem see Appendix G.) Russell felt that the “graphical approach with the TI-81” was most convincing to his students, however, he felt that the “Squeezing” Theorem was most convincing to him. The second day of the unit on limits was not observed, and the videotape machine was not working properly and the lesson was thus not videotaped. However, Russell’s journal entry for the following day and a subsequent informal interview indicated that this limit was covered again using the Squeeze Theorem.

Russell concluded this segment of the unit on limits by stating:

We've looked at all kinds of types of limits now. First we had the easiest kind. The kind you just plug it in. Then we had some we had to factor first and then what? Plug it in? Then we find one we can't factor and cancel so we plug in values close to zero on both sides to find the limit and then we have some that have no limits.

In the final minutes of the class period, as students were putting their books away, Russell stated that now that they had an intuitive understanding of limits, they would need to
become familiar with the formal definition of limit. It was unclear how Russell knew that the students actually had an intuitive understanding. He then very quickly wrote the definition on the board (for the formal definition of limit see Appendix G). From this day on, the principal focus of the lessons on limits shifted away from the intuitive notion of limits toward the formal definition of limit.

The remainder of the lessons on limits were taught more formally. Russell’s teaching style became more lecture-oriented without the student interaction witnessed during the first lessons. The principal motif running through Russell’s next three lessons on limits was using the epsilon-delta definition of limit to prove the limit of linear and quadratic functions. These “epsilon-delta proof” lessons included the following elements: (a) Russell frequently informed the students that they would be held accountable for reproducing the definition of limit proof, given a linear function; (b) Russell stated that the students were not expected to understand the proof, just reproduce it; (c) Russell was not concerned with how the students obtained the delta, just how to “plug and chug” to show that it worked; and (d) Russell reassured the students that the ability to use the definition of limits to prove the limit of a linear or quadratic function may not be important. The following section includes examples of each of these elements by examining the typical “epsilon-delta proof” lesson.

Russell began the following segment of the lesson by referring the students to the sketchy epsilon-delta proof left on the board from the lecture of the previous day:

OK, you are going to have to write a proof like this. My goal is that after you write it you will actually understand what you wrote. We’ll do one together. In general, now we have this function:

$$\lim_{x \to 3} 2x - 1 = 5.$$  

If we are asked to verify this using the definition, prove that this limit is truly five by definition, what did we say yesterday that we are going to
have to do? (No student response was given during a short waiting period.)

At least for the time being it is going to have to be a lock-step proof. I am going to ask you to write down three things specific and you will be asked to do a problem just like this on a test. My goal is that it will actually make some sense, and we will do enough with limits in here that I think, for many of you, it will probably start making some sense. But always keep in the back of your mind that you will do fine in integral and differential calculus if you cannot do an epsilon-delta proof. You will still get correct answers, and the magic of what calculus apparently does, will still be at your command.

He continued by covering the first of the three tasks, writing down the limit definition for the stated problem:

So, anyhow try to recall what we actually have to write down. You are going to have to write down the limit definition in terms of a specific problem. We will get used to these new symbols (he begins writing the definition in terms of this problem)

\[
\forall \varepsilon > 0 \exists \delta(\varepsilon) > 0 \text{ s.t. } 0 < |x - 3| < \delta(\varepsilon) \rightarrow |2x - 1 - 5| < \varepsilon
\]

Now that looks like a lot of symbolic mumble-jumble. If this ever starts making any sense to you, you can look at that and just read it. (Russell reads it word for word.)

For arbitrarily small epsilon we can find a delta that is a function of epsilon that determines a neighborhood of three that is sufficiently small that any x in that neighborhood, excluding three itself, has an f value that is within epsilon units of five. That’s what it says.

Although the students appeared to be very lost, with their eyes glazed over, he continued with the second and third tasks, finding the delta and showing that it works.

Now the second thing we have to do is that we actually have to find the delta that’s a function of epsilon. We don’t care how you get it. Whether you look at your neighbor’s paper, come up and give me money, how much is it worth?
(A student responds with $4.)

OK, come up and give me $4 and I'll tell you. Correct answers are $4.
So, $\delta(\epsilon)$ is less than or equal to what? Ted, what is it?

Ted responds, "Um, it would be $\epsilon/2."

$\epsilon/2$. OK. Ted seemed to know the answers so I asked him. We don’t care
where you get it from. Well, I actually do care, but what is important is to
show that your delta works ... Plug them in and see if they work. That’s
all we are doing here. We are going to take this delta and plug it in and see
if it works.

Russell did not care how the students found the delta. On the following day Russell was
covering another one of these epsilon-delta proofs and a student asked him how he got the
delta. Russell responded, "We found this number by algebra weight lifting. More
importantly, now we have to show that this delta works."

Throughout each of the epsilon-delta proof discussions, Russell was aware of the
fact that the students were not totally convinced in their ability to reproduce what was
being shown to them. Russell would then present his reassuring manner and try to make
them feel more at ease by stating, "Now, I don’t know if this makes any sense to you, it
didn’t make any sense to me when I took it. I knew I would survive though because I was
only there to play football."

In an informal interview the evening after this example lesson took place, Russell
discussed his thoughts regarding the teaching of limits to the point reached by the class:

It’s probably going OK, they understand the intuitive concept [of limit] but
I’m not sure of much more. I gave them the standard disclaimer [about
limits]: You’ll be able to calculate it, and understand it intuitively, but you
may not understand proofs though.

In the following evening informal telephone interview, Russell discussed his means of
motivating students to learn how to do an epsilon-delta proof:
Like I’ve said to my students before, in my years of teaching this stuff I have never ever had a student who could not reproduce this on the limit quiz with 100% accuracy. I tell them, “I don’t think you want to be the first who can’t do it.” Although I’m not very confident about how many people knew the heck what was going on when they did it. So, we’ll be back to this. I’ll have to cover this again.

After the third full day of lessons involving the formal epsilon-delta definition of limit, Russell suggested to the researcher that it was time to “downplay” the definition. The following example was taken from his journal entry for the day:

Tomorrow we will cover some of the theorems on limits, like the limit of \( f(x) + g(x) \) is equal to the limit of \( f(x) \) plus the limit of \( g(x) \). We won’t do any of the proofs. I think I’m going to downplay the definition for now. Maybe it’s not important—may not need to be giving it full attention.

The following day Russell did cover the theorems he mentioned, however, the discussion was brief. He mentioned to the class that there were some theorems to help with understanding limits and then he worked a short example, following which he added, “Go back to page 79 and . . . never mind, I’ll read the theorems to you.”

For the introduction to continuity and one-sided limits, Russell returned to the “evil twins” intuitive approach, however, this approach was soon followed with rigorous definitions. The lesson also involved proving one right-hand limit using the definition. Finally, the intuitive approach to infinite limits was almost nonexistent. Russell proved one infinite limit, however, he informed the students that they “would not be held responsible for this proof, only [for] finding infinite limits at finite values.”

Russell investigated his students’ understanding of limits twice throughout the unit by means of a quiz and a chapter exam. The quiz problems were split evenly into three parts: (a) finding the limit at various points on a given graph, (b) evaluating limits for a set of equations, and (c) proving the limit of a linear equation using the definition of limit. The chapter exam contained 14 limit problems in which the students had to: “Answer the following with a number, \(-\infty, +\infty\), or does not exist.” Twelve questions involved finding
the limit or determining continuity at given points on the graph. A few problems asked the students to give pieces of the formal definitions of continuity and one-sided limits. The use of graphing calculators was not permitted on either the quiz or chapter exam.

In both the first and final interviews, Russell mentioned the importance of the use of graphing calculators in his calculus course. Contrary to his comments, Russell did not use graphing technology as an instructional tool in the calculus class throughout the data collection period. The mention of graphing calculators appeared to be Russell's way of acknowledging the importance of this tool, however, he did have difficulty justifying the use or non-use of the tool. The following example demonstrates a discussion he had with students directly following the quiz given in the class. Russell was trying to justify his decision to ban graphing calculators from the quiz by giving an example of a problem where the calculator helped to provide an immediate solution. Apparently realizing that the students did not understand his concern, he launched into a discussion verifying the solution algebraically.

Heather: Why couldn't we use our calculators on the quiz?
Steve: It would have been easier with the TI.
(A few more students join in.)
Russell: Now, the reason we can't use calculators on these things is—well, there's nothing wrong with using a calculator—ah, but, on this one,

$$\lim_{{x \to 0}} \frac{\tan(x)}{x},$$

just try this and see what happens. We just need to look at what type of function this is—is this odd or even?

Steve: Even
Russell: Yes, this function is even because \( f(-x) = f(x) \). So, with a calculator you could create a table and find
Russell: This problem is easy if you split it up. Write it like this (He writes the following on the chalkboard.):

\[ \lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \frac{\sin(x)}{x} \cdot \lim_{x \to 0} \frac{1}{\cos(x)} \]

(He continues on with the problem.)

The students either appeared to accept this explanation or, since they made no further comments on this matter, got lost in the discussion. Russell brought up his view of the appropriate use of graphing calculators in an informal interview shortly after this incident.

Russell: I would absolutely allow students to solve this

\[ \lim_{x \to 0} \frac{\tan(x)}{x} \]

problem or a problem like

\[ \lim_{x \to 0} \frac{\sin(3x)}{x} \]

with their TI-81 if that was the concept I was testing. I wouldn't put either of these problems on the test if they hadn't seen it \( \lim_{x \to 0} \frac{\sin(x)}{x} \) before.

To test [whether they understood] this concept, I would not allow them to use their calculators.

Researcher: So, what is your policy regarding the use of calculators on exams?

Russell: My test are two parts, one without calculators and one with calculators, so if this problem

\[ \lim_{x \to 0} \frac{\sin(3x)}{x} \]

was on the calculator section that solution would be fine. I encourage them to use the calculators in any way they please to get the answers. All tests have both calculator and non-calculator sections.
answers. All tests have both calculator and non-calculator sections.

Throughout the entire data collection period, the students were not allowed to use their calculators on the exams. None of the exams collected by the researcher were comprised of two parts, as Russell had suggested in the interview.

Summary of Russell

Throughout the data collection period, Russell diagrammed the most important concepts and relationships in calculus as linearly ordered sets of discrete topics. Russell acknowledged that his organization of the content was “textbook driven for the most part.” However, Russell deviated from the given order of the topics in his textbook, in addition to which the emphasis given to each topic varied considerably and did not directly correspond with the typical teach-one-section-a-day rule. The only time Russell completely deviated from the textbook was when he gave practice worksheets at the end of each unit to help the students prepare for the AP exam.

Preparing for the AP exam was one of Russell’s three primary goals for teaching his calculus class. The second goal was to encourage students to discover and appreciate the beauty of calculus as an entrance to higher mathematics. His third goal was to incorporate technology into the problem-solving process. Though each of these goals was demonstrated by classroom practices, they were not demonstrated consistently. The primary focus of Russell’s classroom practices was to directly cover the material required for an AP (AB) calculus course (see Appendix A for outline of the AP (AB) calculus course.)

Throughout the data collection period, Russell’s perceptions of limits remained stable, and the concept of limit was consistently perceived dynamically. For example, he discussed the limit as “A number the function approaches in value when the function is
evaluated for appropriate numbers close to \( a \).” He described the formal limit notation as “A clever and inspired means of formalizing a rather simple intuitive concept.” Russell felt that he first understood the concept of limit in college calculus, however, he observed that he did not really understand limits formally until his graduate-level courses. He felt that his math skills were too weak to prove limits using the formal epsilon-delta definition prior to his graduate enrollment. Russell also felt that teaching the concept of limit had helped him to understand limits.

Russell’s perceptions of the role of limits in the teaching of calculus also remained stable throughout the data collection period, and that the concept of limit was “extremely important” to teaching calculus. Though his diagrams illustrated the concept of limit as one of the three major sections of calculus, Russell recognized the connections between the concept of limit and other calculus topics. Each connection Russell made between the concept of limit and other calculus concepts involved an intuitive understanding of the concept of limit. However, Russell also wanted to hold his students accountable for using the formal epsilon-delta definition of limit to prove the limit of linear functions.

Inconsistent with his belief that the concept of limit was important to teaching calculus, on several occasions Russell discussed with both the students and the researcher that the formal epsilon-delta definition of limit was not important to an understanding of other calculus topics. In fact, Russell was unsure as to why it was important to even teach more than just an intuitive understanding of the concept of limit. These types of comments led the researcher to question Russell’s ability to achieve his second goal, that of encouraging students to appreciate the beauty of calculus.

Though Russell’s perceptions of the teaching of limits remained constant, the translation of his perceptions of and the role of limits in calculus into classroom practices were inconsistent. In many instances, Russell appeared to experience a conflict in what he thought was the best approach and what he actually did in the classroom. For example, though in both the questionnaires and interviews Russell had reiterated his strong desire to
teach limits from an intuitive perspective, only the first day of the lesson on limits focused on examining limits from an intuitive perspective, using the intuitive “evil twins” approach and such multiple representations as graphing and the creation of tables of values. Russell introduced the intuitive “evil twins” approach whenever students had problems understanding limits, but these discussions were always brief. In addition, after the lessons of the first day, Russell rarely used multiple representations, and the majority of the lessons provided involved the formal epsilon-delta definition of limit and the use of this definition to prove the limits of linear and quadratic functions. These epsilon-delta proof lessons included four elements: (a) Russell frequently informed the students that they would be held accountable for reproducing the definition of limit proof given a linear function; (b) Russell stated that the students were not expected to understand the proof, beyond the ability to reproduce it; (c) Russell was not concerned with how the students obtained the delta, just how to “plug and chug” to show that it worked; and (d) Russell reassured the students that the ability to use the definition of limits to prove the limit of a linear or quadratic function may not be important.

Russell consistently stated the importance of the use of graphing calculators in his calculus course and encouraged students to use their own graphing calculators during the lessons. However, in contrast to these comments, Russell never used graphing technology as an instructional tool in the classroom. In addition, students were not allowed to use graphing calculators on exams, though Russell had stated to the researcher that his exams would include both calculator and non-calculator sections. Furthermore, Russell felt that it was more important for students to memorize a collection of graphs for immediate recall. Thus, Russell’s observed classroom practices were inconsistent with his third goal of incorporating technology into the problem-solving process.
Independent Teacher Profile, Richard

Scholastic and Professional History

For the past 16 years Richard has taught every course in the high school mathematics curriculum, from general mathematics through calculus. His interest in mathematics dated from his college experience at a large university in the Southwest U.S. Richard stated that upon college graduation in 1975 he decided to become a mathematics teacher because he “enjoyed mathematics and people,” adding that though his undergraduate major was mathematics, he had “no real specialization.” In 1981, Richard completed a Master of Arts in mathematics education from the same university, and described his overall academic background briefly as “good.” Richard also mentioned that he worked on a doctorate (Ph.D.) at a large university in the Northwest U.S. for one year in the mid-1980s.

The high school where Richard taught during the period of this study was situated in a college town with a population of approximately 45,000. Over 1,000 students were enrolled in the top three grades at this predominantly white high school. At the time of this study, Richard was teaching first year algebra, geometry, and AP calculus, and the school had decided for the first time to run a block schedule during the year in question. Each of the classes met each day for 90 minutes. Furthermore, the courses were now only one semester long. Richard’s other duties at the school include track coach, committee work, and coordinator of the math club. Richard introduced the name of the math club to his calculus class as SPAM, “Students for Peace through Awareness of Mathematics.”

Richard was selected for participation in this study from a statewide list of AP calculus teachers. The first week in August, Richard was contacted by telephone and asked to participate. At first, he was somewhat hesitant about participating. His concern seemed to stem from his uncertainty regarding the effect of the implementation of block
scheduling at his school. Finally, he stated that if the block schedule did not bother the researcher, then it probably would not bother him. During the initial telephone conversation, a time was arranged prior to school for the completion of the first questionnaire and interview. This completion of the questionnaire and interview took place on the second of September, one week prior to the beginning of school.

The first questionnaire completion and interview took place in Richard's classroom, when he was busy organizing for the beginning of school. He was focused on the task-at-hand, but a few interruptions from other teachers entering his classroom interrupted his train of thought. Richard's responses to the questionnaire were quite brief, and it appeared that Richard was in a hurry to continue with his organizational work for the school year.

Throughout the data collection period, Richard was willing to have his lessons observed and scheduling observations with Richard was easy. However, he did not cooperate in providing all requested types of data. For instance, none of the classroom lessons were videotaped. The researcher compensated by observing every lesson Richard delivered on the concept of limit. In addition, Richard did not make entries in a journal, but was willing to discuss his thoughts during class each day. While the students were taking a break during class, Richard would visit with the researcher for 5 to 10 minutes. Moreover, Richard decided not to complete the second and third questionnaires. When the researcher went to his classroom to pick up the second questionnaire, Richard stated, "Hey, I noticed that this form was the same as the one I filled out before. Maybe we talked about this before sometime, but I really don't see any sense in filling it out again. So, just say I'd put the same thing." The necessity for completing all four questionnaires was explained to him. Richard then replied, "Oh, OK. I'll fill it out this weekend and mail it to you." The following week the researcher went to Richard's classroom to pick up the questionnaire and also give Richard the third questionnaire, since he had completed teaching the unit on limits. The second questionnaire was still not completed. Richard
stated that he did not feel the need to fill out all these questionnaires, observing that he would complete the final questionnaire at the end of data collection. The fourth questionnaire was completed and returned promptly.

Portrait of the Calculus Classroom

Richard had taught AP calculus for the past 12 years, and a high level of school support for AP classes motivated him to plan to teach AP calculus for several years in the foreseeable future. At the time of the study, due to the high number of students taking the course, Richard was teaching two sections of AP calculus. The section that met in the morning contained 27 students, but the researcher observed the afternoon section with 15 students for reason of scheduling demands; four of the six teachers observed for this study taught AP calculus in the morning. Richard commented that he taught both classes in exactly the same way and that in his opinion there were no differences between the students in one class and the students in the other. Class selection for the students was merely a scheduling choice.

Richard's classroom was large, with approximately 35 student desks. The seven female and eight male students in the afternoon calculus class tended to sit bunched up in three or four groups near the front of the classroom. Richard described his AP calculus students as "Excellent! They are well prepared for the most part. They are enthusiastic, they like math and have done well to be in calculus in high school." Richard's desk was centered in the front of the room. In the back of the room was a recliner, a stereo, and bookshelves. The only usable chalkboard in the room was at the front and Richard used it exclusively for instructional purposes as well as announcements.

The atmosphere of Richard's class was relaxed and easy-going, and Richard and the students appeared to feel comfortable with this type of structure. Richard once commented to the researcher that "My classes are not that structured so you can come at
any time. Don’t worry about being late if you are coming from another observation. I don’t think anything could really disturb us.” Often throughout the class period, other students would visit the classroom. For example, one day a non-calculus student came in, walked to the back of the room, sat in the recliner and read a book. Thirty minutes later he was snoring and the class just laughed but paid little attention to him. In addition, Richard had a couple of students as graders. These students were constantly in his calculus class grading and frequently asking Richard questions on how to grade particular questions.

As previously mentioned, Richard’s high school was shifting to block schedules, for the first time, the year this study was conducted, and Richard’s calculus class met from 1:20 p.m. until 2:50 p.m. Richard did not seem to be affected by the change in scheduling early in the data collection period. At the end of the data collection period, however, Richard was unhappy about not having calculus during the second semester. The AP exam was in May and Richard felt detached from the students; thus, he worried about the amount of preparation they would get. He commented on the block schedule,

I don’t like it. It’s too much too fast. I felt I couldn’t cover as much. They may be OK for the AP exam. We may spend more time on the review this year. We’ll get together on weekends and in the evening. I just feel detached from the kids right now because I don’t have them in class—I don’t like it.

Each day, Richard’s calculus students patiently lined up outside the locked classroom door waiting for him to let them into the classroom. Class started at 1:20 p.m. and Richard usually unlocked the door at 1:22 p.m. For the first 10 to 30 minutes of the period, students organized their assignments while Richard answered individual questions and prepared for the lesson. Following the initial organization and question-and-answer period, Richard covered the new lesson. This segment of the class period lasted from 10 to 60 minutes, most typically lasting for 30 minutes. However, this period did not consist
wholly of lectures. The students appeared to be interested in the new material Richard was delivering, but they were equally interested in getting Richard off-track and on to unrelated discussions. For example, they would have non-mathematical discussions on such subjects as biking or why the water in the drinking fountain was yellow. They would also have mathematical discussions which were only partially related to the lesson at hand. For example, they spent time discussing googleplex and infinitesimals. Frequently, discussions of mathematicians were brought up in class. For example, after discussing how calculus classes were conducted in college the following discussion took place:

Richard: This [proofs using the epsilon-delta definition of limit] is really intimidating but that’s how mathematicians are.

Student: It seems that mathematicians don’t want anyone to know what is going on in mathematics.

Discussions about the make-up of mathematics also came up in discussion, as demonstrated in the following snapshot of the classroom discussion of infinity.

Richard: Now, what about infinity times zero?

Student: It must be zero, because anything times zero is zero.

Richard: But, infinity is not a number, so maybe this rule doesn’t apply.

Student: Math is evil because they teach us something and then the next year they say, “Ha, ha, that really wasn’t true. Now it’s this way.”

At the completion of teaching each lesson segment, Richard gave the assignment and explained to the class that the students should “Work on some of these problems and see how they feel to you and make sure you have an understanding of what is going on.”
Dependent upon the time of the day, the students would either take a break or move directly into working on their new assignment.

A 5 to 15 minute break was an every day occurrence in Richard’s class period, during which time the atmosphere of the classroom shifted to individual and social events. The typical setting began with one student turning on the stereo, another student reading a book, and a couple of students asking Richard questions about the lecture. The rest of the students went into the hall. A few minutes later a couple of students returned to the room and began copying what had been written by Richard on the chalkboard. Though the atmosphere of the class was loose and easy-going, students still appeared to be motivated and many of them looked like they were working on their new assignments. Richard either helped students with questions, conversed with his graders, or just visited with the students throughout the break. When Richard ended the break, the students all returned to their seats and began working on their new assignment.

The allotted time set aside for students to work on their homework in class ranged from 30 to 60 minutes each day. As the students began to work on their assignments, Richard would discuss the amount of detail he expected from them on each problem. The atmosphere of the classroom continued to be relaxed as Richard walked around the room helping individual students. One student continued to play with the stereo, but he was also working on his assignment. In fact, when Richard was out of the room the other students directed questions to the student playing with the stereo. He would help them and then head to the back of the room again. The students continued to work on their assignments until the end of the class period.

**Perceptions of Calculus and the Teaching of Calculus**

Richard discussed the four major goals he had for teaching AP calculus this year with the researcher, as follows: (a) to teach as much content successfully as possible,
(b) to help students enjoy the experience, (c) to prepare students to apply what they had learned, and (d) to prepare students for the AP exam. Observed classroom lessons provided evidence for these goals, each of which is discussed in greater detail throughout the remainder of this profile.

As previously mentioned, Richard completed only two diagrams of the important concepts and relationships in calculus (Figures 16–17). Although one diagram was completed prior to the beginning of school and one was completed after Richard was finished teaching the entire calculus course, the general format of the diagrams was almost identical. Both diagrams illustrated calculus in four major sections, including algebra skills, limits, differentiation, and integration. According to the diagrams, it was important to have a solid foundation in algebra, which Richard described as “the alphabet of calculus.” From this foundation, the understanding of limits, the backbone for all other calculus topics, was to be derived. Both diagrams demonstrated that the connections between limits and other calculus topics was important, appearing as the foundation of calculus connections. The only distinct difference between the diagrams involved the illustration of the relationship between the derivative and the integral. The arrow in Figure 17, at the end of data collection, illustrates a one-way relationship between the derivative and the integral, whereas Figure 16, at the start of the school year, shows that the relationship between differentiation and integration involves limits and works in both directions. Richard described his interpretation of the diagrams in a similar fashion. For example, Richard described Figure 17 as follows:

Foundation in algebra is very important. The algebra seems to be more of a problem than anything else. Then my feeling is that the whole basis of calculus is the concept of limit. To help them understand it better, we’ll do some epsilon-delta proofs. Coming out of limits are differentiation and integration. Students must understand the connection of limit with these topics—limits for rate of change and summation. My feeling is you need to give kids a good foundation.
Figure 16. Richard’s Diagram of Calculus, Prior to Beginning of School.
Figure 17. Richard’s Diagram of Calculus, End of Data Collection Period.
As a complement to the previously stated goals for teaching calculus, Richard felt that it was important for his students to think about and write about what they considered to be the beauty of calculus. When Richard distributed the first writing assignment, he explained that “Writing is really a part of everything, so there will be an essay on the take-home exams. I won’t grade your grammar because I am incapable and I don’t care how long it is, a couple pages or paragraphs.” After reading the student essays on the beauty of calculus, Richard discussed his impressions:

The idea of beautiful mathematics is really a neat essay. I have read a couple of your essays and they are really good. A lot of people took beauty in the way mathematics looks in the way the formulas and theorems look on paper. That is true. I was thinking more of the beauty of the form it takes, like patterns.

Richard also observed that he felt that calculus had more meaning in his life because of meditation. He frequently discussed his personal attachment for this practice and suggested that students may experience the same thing. For example, after discussing the limit definition of the derivative, Richard ended the class by saying, “What you are going to do is find the power of the derivative going through your body.” This discussion may have been Richard’s attempt at achieving his goal of helping students enjoy the mathematical experience.

Coinciding with his goal to prepare students to apply what they had learned in class, Richard frequently stated that he wanted his students to become problem-solvers. His idea of problems, however, was mostly in reference to solving mathematical equations as compared to contextual applications. The following example, which occurred early in the course during review, demonstrates Richard’s perception of problem-solving.

Richard: I would also like to talk about the application problem [of the Intermediate Value Theorem] Amanda brought up. The purpose of those types of
problems is to get you to think, and to be problem-solvers.

Amanda: I just felt like I didn’t understand the whole Intermediate Value Theorem enough to apply it. Maybe it was just my fault.

Richard: We don’t always have all the answers in life, so we have to work with what we know. For example on this problem, \( f(x) = x^2 + x + 5 \), show the roots exist. What I would like you to get at in this class is if you don’t know exactly what to do with a problem, you take what you know and apply it.

The textbook, *The Calculus with Analytic Geometry* by Liethold (1986), was a main component of the way Richard organized his AP calculus course. His lessons followed the text section-by-section, and he frequently used the teaching examples given in the book. His students, however, encouraged him to also use other examples not in the text. In fact, one day Richard was asking questions on an example in the book and a student responded with the correct answer and then stated, “It’s in the book, you are just teaching what’s in the book lately.” In most cases, Richard conceded to their requests. Richard did not have any written lesson plans, thus he supplemented with his own examples. Occasionally, Richard was asked by the researcher to discuss other materials he may have used to supplement the text, but none were ever acknowledged.

Richard saw no advantage to supplementing his class with the use of graphing calculators. In fact, the following comment by Richard suggested that in some cases using technology was a disadvantage: “Some of the technology really bothers me because many teachers feel that you just need to plug numbers into those things and the students will understand what a limit is. This is not going to help kids apply it.” In addition, Richard occasionally brought up the insignificant role of graphing technology in the classroom. For example, one day the students were discussing using a graphing calculator and Richard stated, “I’m going to show you how to graph faster than the calculator. You can
beat everybody.” This comment may have been based on Richard’s goal of preparing students for the AP exam, however, no further discussion ensued regarding this issue.

**Perceptions of the Concept of Limit**

Prior to the beginning of school, Richard’s perception of limits appeared tentative, however, it is possible that the questions being asked were so open-ended that Richard was encouraged to think about limits more globally. For example, when Richard was asked to describe in his own words what a limit was, he responded with “instantaneous rate of change.” Richard may have been thinking about limits as they were connected to derivatives. This connection may be seen in the diagram given in Figure 16, which also indicated that Richard not only thought of the limit as instantaneous rate of change, but also as connected to continuity and antidifferentiation. Richard continued to be abstract to some degree in his description of what he thought about when he saw the limit notation,

\[
\lim_{x \to a} f(x) = L
\]

He vaguely responded, “It had a profound impact. Great’s expanding mankind’s mathematical possibilities. There are many important applications.”

At the end of the data collection period, Richard’s diagram (Figure 17) continued to demonstrate his understanding of limits as it was connected to other concepts in calculus. The responses to the questionnaire items, however, focused more closely upon the limit. Richard described the concept of limit in a dynamic manner stating that “A limit is what an expression approaches as a given variable gets close to a given number or value.” He went on to discuss the limit notation,

\[
\lim_{x \to a} f(x) = L
\]

as a “simple concept—tough definition.”
Richard felt that he did not understand the limit notation and definition until he took calculus in college. Further mathematical training also contributed to his understanding of limits. At the end of the data collection period, Richard also added that he did not really understand the formal limit notation, definition, and proof until he had to teach it.

Perceptions of the Role of Limits in the Teaching of Calculus

Throughout the data collection period Richard continually remarked that the concept of limit was "extremely" important in teaching calculus, and his diagrams of the important concepts and relationships in calculus (Figures 16–17) demonstrated his belief in the importance of limits. Both diagrams suggest that it is important first to have a firm foundation in algebra that allows one to understand limits. The diagram then demonstrates that all other calculus topics stem from the concept of limit.

The important role of limits is also demonstrated in Richard's lessons for subsequent topics involving the concept of limit. In each of them he consistently alluded to the connection between limits and the new concept. For example, the following snapshot of a classroom lecture demonstrates the way Richard connected the concept of limit to the introduction of derivatives:

Richard: OK, derivatives. Do you guys know of a guy named Liebnitz? He was more into using the idea of limit in calculus. Now, one of the things Newton studied was laws of motion. He was trying to do some things with equations and he needed to find a way to explain and find rates of change. We are going to start with looking at the derivative in a graphical sense, and then we will break away from it. We don't study things that don't change at all, so remember that. [Richard writes the following on the chalkboard:}
Derivative (instantaneous rate of change)

Where have you dealt with rate of change before?

Students: Slope.

Richard: We only have one point. This is interesting because we only have one point and we are talking about change. How many points does it take to find a slope?

Students: Two.

Richard: [Richard draws a graph finding the tangent line to a curve (Figure 18) and lists the equation for finding the slope of the tangent line.]

\[ m = \frac{\delta y}{\delta x} = \frac{f(x_0 + \delta x) - f(x_0)}{\delta x} \]

How can we make this slope be more exact?

Figure 18. Richard’s Graph of the Derivative Definition.
Pat: Take the limit as that section $[\delta x]$ goes to zero.

Richard: Yes. This is now one of the reasons the limit is such an important concept. It helped in the derivative and it will come up again.

Does anyone have a favorite quadratic equation?

Ian: X squared.

Richard: Ian, you’re a simple man with simple ideas, I like that. Let’s use the limit definition to find the derivative for this one.

Richard worked a few more problems with the students using the limit definition of derivative and then gave an assignment covering the same material. At the end of the hour he told the students that they could soon forget all about this limit definition because he would teach them a “short-cut” to finding a derivative. The students then wondered why they needed to continue with the assignment. Richard replied,

Because in this section you will have a lot of problems to work that help you work on your algebra skills, but more importantly it should etch in your brain that this is a rate of change. This should help you use this knowledge at another time.

Though observations of Richard’s subsequent lessons involving the concept of limit demonstrated the importance of connecting the intuitive notion of limit to other calculus topics, he also demonstrated that a high level of rigor in teaching limits was important in teaching calculus. Richard observed: “I do epsilon-delta proofs rather informally because of the level of students I get in my classes.” In the follow-up interview, Richard discussed why he felt it was important to cover epsilon-delta proofs in his class:

For people to get a better feel for limits—the epsilon-delta proofs seem to give them either a better feeling instantaneously or later on in the next
course or other courses, such as advanced calculus in college. I feel that epsilon-delta proofs make the definition sink in a little bit better.

However, Richard did not comment on the beauty of mathematics at this point. At the end of the data collection period, Richard responded similarly regarding his desired level of rigor by stating that it was important for students to “know the epsilon-delta definition and have the ability to go through a proof, [but] not very rigorously. I have great students though. In college I would not teach the proof.” Richard did not comment on the AP exam at this point, and rather stated that he had more confidence in his students as opposed to college students because they were accelerated math students and not typical of the math students found in college calculus. However, Richard added that his students may not understand the proof.

They may understand what a limit is, but I don’t know if they understand how the definition really works. I’d like for them to understand that things need to be well-defined, so epsilon-delta proofs will help. However, a lot of kids have difficulty understanding then—so we try to do it informally. My students are on an advanced track so they can do it. If I was teaching a college course with students from a variety of backgrounds, I may need to do epsilon-delta proofs as an extra only.

The Teaching of Limits

Following two complete weeks of pre-calculus review lessons, Richard spent eight days covering the unit on limits. The last two days of this unit were devoted to review and the unit exam. The organization of the first six days of the unit corresponded directly with given sections in Chapter One of the textbook (see Appendix E for an outline of Chapter One). Richard’s only deviation from the one-section-per-day format was covering all of the continuity sections in one day toward the end of the unit. All of the material given in Chapter One of the textbook was covered during this unit. There was no evidence of incorporating supplementary activities into the lessons was observed.
Prior to the beginning of school, Richard responded on the written questionnaire that his favorite way to explain limits to beginning calculus students was to give them a simple example, such as

$$\lim_{x \to 5} 2x = 10,$$

as $x$ gets close to five, $2x$ gets close to 10. Richard responded that he would then show the students a graph. Seven months later, Richard responded identically to this questionnaire item, listing the same example. In both cases Richard commented, “I just try to make it real simple—what a limit is. Once they understand this I give them the epsilon-delta definition.” Richard never discussed that it was important for his students to obtain an intuitive notion of limits first. He just commented that it was important to keep evaluations of the limit “simple” with reference to types of functions at the beginning.

On the first day he presented the concept of limit, Richard captured all the students attention with the statement, “Today we start calculus!” One student joked with Richard by saying, “I’ve dreamt about this for a long time.” The students in the class laughed, but at the same time they did appear to be excited and attentive as Richard’s excitement about calculus was clearly indicated to them. Richard continued with his description of the lesson for the day, “Today we will talk about limits. Now, I’m not going to give you an assignment on limits today, I’m just going to give you the definition of limit. Then we’ll just let it swell in your brain.” The classroom atmosphere appeared to be different from the atmosphere during other observations. The students were more attentive, on-task, and focused on what Richard was telling them. Richard then stated, “We’re going to do something that they hardly do in the college anymore.” A student asks, “Why do we do it then?” Richard responded, “Because you’re special. You’ll also need to know it for the AP exam.” In an informal interview after class, Richard was asked what he was referring to when he said he was going to do something they hardly do in college anymore. He stated that “I don’t mean limits, but we are going to prove some of this stuff. I don’t think
they do these epsilon-delta proofs in college anymore.” In the same interview, Richard later said that “The proofs were important for the students to know for the AP exam and maybe other higher math courses.”

Richard’s actual introduction to the concept of limit was similar to the examples he gave in the written questionnaires. Richard did not, however, present the examples from a graphical perspective, as was suggested by his responses to the questionnaire items. The following snapshot was taken directly from Richard’s lesson introducing the concept of limit:

Richard: OK, it’s about time to start limits. You can just sit back and soak it up, because remember I told you that you won’t have an assignment on this today. We’re going to do this again tomorrow, but you better take notes today. To start off, just to give you guys a feel for what the limit is. If I just asked you what is the limit of $2x + 1$ as $x$ goes to three?

Chris: Seven.

Lisa: It has to be less than seven.

Richard: OK, how about this one,

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Cory: Four.

Richard: How did you get four?

Cory: I just factored it and then put the number in.

Richard: So, what is a limit?

Chris: Something you can get closer and closer to, but you can never reach it.

The last comment led Richard directly into an apparently confusing discussion on how infinitesimals made calculus more rigorous. The inferred confusion came from such
Richard ended this discussion by stating, "You guys are getting off track, just think of getting closer and closer to something."

Apparently Richard felt that the students were comfortable now with the idea of limit because directly following the above example, he began his discussion of the formal definition of limit. This discussion included a graphical interpretation, but was not tied to the previous examples in any way. As Richard drew the graph (Figure 19) on the chalkboard, he first explained without using epsilon and delta, "As $x$ approaches $a$, the value of the function approaches $L$." He then continued with a written formal definition of limit using textbook notation (see Appendix E for the textbook formal definition of limit) and stating, "I'm going to put this [definition of limit] up with the notation and then I will explain the notation." The students were moaning and very vocal. Their comments included, "I don't understand anything you are writing," and "What is this delta and epsilon anyway?" Richard just continued writing and consoled them by saying, "Now if you understand this right now you scare me. We are going to talk about it more in detail."

Then Richard explained his diagram:

Figure 19. Richard's Graph of the Limit Definition.
Just think of delta and epsilon as being really, really small numbers. Think of this \([a \pm \delta]\) as a distance. What this means is that you can’t be at that point in the neighborhood. So putting that zero \([\delta > 0]\) in there means that you can never actually get to the limit because it \([\delta]\) never is zero. So by the definition you can get close but not there. Now back to the definition, so no matter how small the epsilon is, you have to find a delta that works. If it happens for every single epsilon, then we have a limit.

Following discussion of his diagram, Richard stated, “To see how this definition really works, it may be easier to see when it doesn’t work.” Richard drew a graph of a function that had a jump discontinuity at \(x = a\) and asked the class, “Does everybody accept the fact that this doesn’t have a limit at \(x = a\)?” Most of the students appeared to agree with Richard, however, one student asked, “Couldn’t it have two limits?” Not addressing the misconception, Richard continued to explain how the definition of limit failed for this example. After this example, Richard decided he had covered enough for the day: “We will stop now to let this soak in.”

The formal definition of limit was still on the chalkboard the following day. Richard began the lesson by having a student read the definition aloud, following which another student volunteered to reproduce the picture associated with the definition of limit on the chalkboard. Richard then told the class that they would be held accountable for reproducing the formal definition of limit symbolically and graphically.

Now on your first test there will be a question that asks you to define limit and draw a picture to represent it. So, you must make sure this soaks into your brain. This is a really abstract concept and even after we prove some stuff today you may not really understand it. My feeling on this is that we will discuss this enough so you have some feeling and basis on what a limit is because it is what all of calculus is based on.

This discussion appeared to be Richard’s motivation for doing proofs using the definition of limit. The remainder of the day, how to accurately write proofs using the definition of limit for both linear and quadratic functions was discussed. No examples involving the intuitive exploration of limits, either graphically or numerically, were given. The following
Richard: These proofs are like proofs you dealt with before. Remember how you thought it was easier to work backwards because it was easier. You will start with $|f(x) - L| < \epsilon$ and work your way back up to

$$\lim_{{x \to 5}} 2x - 3 = 7.$$ 

So, it goes like this:

$$|2x - 3 - 7| < \epsilon$$

$$-\epsilon < 2x - 10 < \epsilon$$

$$-\epsilon/2 < x - 5 < \epsilon/2$$

So, let $\delta = \epsilon/2$.

So, for all $\epsilon > 0$,

$$|x - 5| < \epsilon/2,$$

$$|2x - 10| < \epsilon$$

$$|2x - 3 - 7| < \epsilon$$

Thus, we have $|f(x) - L| < \epsilon$.

So, you just need to find the delta, not actually do the proof.

Adam: Then there doesn’t seem to be any point in doing the proof.

David: Do we have to go through the proof every time?

Richard: No, you are doing five problems and only one will involve doing the proof. I suggest that you work with someone on this, and do number 17 before you do number 15 because it’s a little weird to prove it. Work on these types of linear problems first and
then we’ll get back together to go over how to do the quadratic ones.

Now does anyone have an idea what the relationship is between the epsilon and delta?

Jerry: Slope.

Richard: Good. You are just looking at the relationship between the distance epsilon and the distance delta. That’s just the slope.

From this point, Richard allowed the students to work on the assignment for the remaining 40 minutes of class. Though Richard did not address Adam’s comment regarding the point of these proofs, Adam repeated his question 30 minutes later, as Richard was walking around the room helping individual students with their assignment.

Adam: If you were given this proof to do, why would you do it if you already did it backwards? There doesn’t seem to be any point in proving it forwards if you already did it backwards. What’s the purpose?

Richard: Because it forces you to work with some stuff like inequalities. This is just one of those things with absolute values.

Chris: This is easy, all you have to do then is write it backwards.

Adam: I get it, I just don’t see the point.

Richard: It’s not as easy as it seems because you have to find the delta, and you have to understand it.

Adam did not appear to be satisfied with Richard’s response; nonetheless, the purpose of these proofs was not explained in any greater detail throughout the remainder of the lessons on limits.

The majority of the third day was spent on the assignment proving the limit of a linear or quadratic function using the limit definition. The students put their solutions on
the chalkboard and they went over them with the class. The only new material covered was a brief discussion of the limit theorems found in the text, for example:

\[
\lim_{x \to a} c = c, \quad \lim_{x \to a} x = a, \quad \text{and} \\
\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)
\]

Richard proved a few of the limit theorems in class, but he told the students that he did not expect them to reproduce them. The assignment given involved the evaluation of both one- and two-sided limits of a variety of equations. Richard offered extra credit points to the students if they proved any of the theorems listed in the textbook.

Richard finally discussed solving limits using numerical representations during the fourth lesson in the unit. For example, when the students were having difficulty with one of the homework problems, finding

\[
\lim_{x \to 0} \frac{|x|}{x},
\]

Richard helped them by stating, “Sometimes it’s just crunching the numbers from both the left and the right. Don’t think that they are so difficult. Try some numbers. If you just try some numbers you can see where it is going.” One of Richard’s responses to the questionnaire item asking for the different arguments he has seen to support finding

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1
\]

included crunching numbers and creating a table of values. Richard also responded consistently with the Squeeze Theorem. Richard felt the table of values was most convincing to him, but the proof resulted in increased confidence. At the end of the data collection period he wrote, “What was most convincing perhaps is just plugging numbers into a calculator—except it doesn’t feel like proof!” Similarly, on both questionnaires, Richard acknowledged that the students would probably be more convinced by just punching numbers into the calculator and creating a table of values. During the interview
following completion of the first questionnaire, Richard discussed what he did in the classroom,

I just do both [the table of values and Squeeze Theorem] and then they end up fairly convinced. It’s hard to say which one they believe more. I’m the one discriminating the information. We look at a table and it shows [the limit] is going to one. We will sometimes even look at the graph. But the proof gives everybody the feeling that, “OK, now it must be true.” At that point in calculus they’re ready for proofs. Maybe they weren’t so ready for them in geometry?

Richard did discuss the Squeeze Theorem later in the unit on limits, however, he rarely used graphical representations to examine particular limits. In fact, he gave the impression to the students that it was better to find the limit algebraically than graphically. Though the particular exercise discussed in this case is unimportant, the following example demonstrates Richard’s response when a student, Miles, explained to the class how he solved a particular exercise algebraically.

Richard: Good job. I like the way Miles did it because he did it algebraically. (Richard then demonstrated the exercise graphically.)

As x approaches two from the left that means I’m getting closer and closer to three. Notice you don’t even need a graph, did you use a graph at all Miles?

Miles: Just to check my answer.

Richard: Good. It’s better to be able to find the limits algebraically.

Richard’s fifth lesson included a brief discussion of infinite limits. Richard stated that he wanted the students to “Check if the limit is going to positive or negative infinity and then state does-not-exist along with whether it is positive, negative or both.” Though this lesson lasted less than 30 minutes, Richard was more willing to examine infinite limits
graphically than he was for finite limits. The particular graphical interpretation he did in class happened to be a quick sketch of a graph reaching toward infinity along the $x = 0$ asymptote. Richard stated that the class only needed to respond with positive or negative infinity on their homework. He did not care how they obtained the limits, except for one problem: "When you get to number 31 you’ll need to show it algebraically.”

Richard concluded the unit on limits following the text with a lesson on continuity. Richard first allowed the students to state what they felt it meant to be continuous. The students’ ideas of continuity included items such as, “no breaks in the graph” and “you don’t have to lift your pencil.” Richard guided the students to make connections between the limit and continuity. For example, after stating that in order to be continuous at a point, first that point had to exist, the rest of the lesson proceeded as follows:

Richard: Can anyone draw a function that exists at $a$, but is not a continuous function?

(A student draws a graph of a jump discontinuity on the chalkboard.)

How could we demonstrate this in the definition? What else isn’t true on this function here?

David: Could you say something about limits?

Richard: Yes, what?

David: Something like the limit from the left has to equal the limit from the right.

Richard: Yes, good job. So the next part says $\lim_{x \to a} f(x)$ exists.

Does anyone know a function such that there is a limit, $f(a)$ exists, and yet still isn’t continuous?

(The class comes up with one with a little prompting.)
How can we make it continuous?

Chris: Pick up the point and move it over.

Richard: Yes, this is what is called a removable discontinuity. If you can do this to a finite number of points. So, we just need to write \( \lim_{x \to a} f(x) = f(a) \)

Richard then briefly reviewed some related theorems in the text and a few example problems, one of which was from the textbook. The students were then given 50 minutes to work on the assignment.

Throughout the unit, Richard occasionally brought up the topic of the AP exam, although usually not for motivational purposes. The majority of the time when the AP exam was discussed in class, it involved some discrepancy between textbook notation and the notation used on the AP exam. The following quote is an example of the type of discussion involving the AP exam:

Here is something that our book does different than the advanced placement exam. The advanced placement exam will say that any limit that goes to \( \infty \) is considered DNE [Does Not Exist]. This book treats \( \infty \) and \(-\infty\) in the same manner as real numbers.

Richard’s exam for the unit on limits was a take-home test consisting of 23 individual items. The students were given class time to work on this exam and Richard provided some assistance. Just as Richard had discussed in class, the exam contained a question asking the students to “Write the epsilon-delta definition of a limit and make a sketch showing \( f(x), L, a, \varepsilon, \) and \( \delta \).” Richard also had the students define continuity and apply the Intermediate Value Theorem. The students were allowed to use their text, so the purpose of these questions was unclear. The greater part of the exam included six items which involved finding values where the listed function was continuous or discontinuous, and 13 items involving direct evaluation of the one- or two-sided limit of a
given function. Richard finished grading the exams one week after they were turned in and commented that he was pleased with the students' understanding of limits.

**Summary of Richard**

Richard's overview of important concepts and relationships in calculus was presented in four major sections, including algebra skills, limits, differentiation, and integration. It was of the utmost importance to him that his students possess a solid foundation in algebra. With this foundation established, Richard perceived the concept of limit as forming the "backbone" for all other calculus topics. In addition, the relationship between the major sections of differentiation and integration involved the concept of limit. Throughout the entire data collection period, Richard discussed the importance of the concept of limit in teaching calculus, an importance further confirmed by Richard's classroom teaching practices. In each of his lectures dealing with subsequent topics involving the concept of limit, he alluded to the connection between limit and the new topic.

Richard felt that it was important for his students to first obtain an intuitive understanding of the concept of limit. To assist his students, he commented that it was important to keep the evaluation of limit problems "simple" at the beginning of the unit. Richard began the unit on limits with a few simple problems involving the evaluation of limits. Obtaining an intuitive notion of limits, however, appeared to be defined by the student capability to evaluate simple limits for continuous functions. In fact, the brief introduction to simple limit problems did not involve any graphical or numerical interpretations. After spending time on these few examples, Richard began to discuss the rigorous epsilon-delta definition of limit. Though this discussion involved a graphical interpretation of the definition of limit, the discussion was not tied to the previous simple examples in any way.
Though only an intuitive notion of limits was important for the connection of the concept of limit to other calculus topics, Richard believed that a high level of rigor in dealing with limits was important for teaching calculus. Understanding the epsilon-delta proofs was to help the definition of limit “sink in a little better.” Although Richard believed the rigorous treatment of limits was no longer necessary in college calculus courses, he felt that his advanced placement calculus students were sufficiently mathematically mature to grasp the meaning of the epsilon-delta definition of limit. Richard held his students accountable for reproducing the formal definition of limit symbolically and graphically. In addition, the students needed to be able to reproduce an epsilon-delta proof of the limit of a linear function at a particular value. Though questioned several times by students, Richard never really explained the purpose of these proofs. In discussion with the researcher, Richard did state that the proofs were important for the students to know for the AP exam and for other higher level mathematics courses.

Richard had four major goals for teaching his calculus course: (a) to teach as much content successfully as possible, (b) to help students enjoy the experience, (c) to prepare students to apply what they have learned, and (d) to prepare students for the AP exam. Observed classroom lessons provided evidence for the pursuit of each of these goals. For example, coinciding with his desire to teach as much content as successfully as possible, Richard’s lessons on the concept of limit corresponded directly to given sections in the limit chapter of the textbook, and Richard frequently used the precise examples given in the textbook. When the students requested additional examples, however, Richard supplemented with his own problems. In partial correspondence to his goal of helping students enjoy the mathematical experience, Richard demonstrated that it was important for his students to think and write about the beauty of calculus.

Parallel to his goal to prepare students to apply what they have learned, Richard stated that he wanted his students to become problem-solvers. However, problem-solving to Richard did not mean solving application problems, but solving unfamiliar mathematical
equations. Furthermore, Richard's unit exam was for the greater part involved with the evaluation of limit problems and not the application of limit problems. After grading the exam, Richard commented that he was pleased with his students' understanding of limits.

The goal of preparing students for the AP exam was demonstrated in classroom practices, though it seemed secondary to the goal of teaching as much content as successfully as possible. The topic of the AP exam was not brought up for motivational purposes. The majority of the time the subject of the AP exam arose in class, it involved some disparity between notation used on the exam and that used in the textbook. Though Richard did not appear to be "coaching" his students for the exam, he did mention that review sessions for the exam would occur outside of class during the month before the exam.

Though he stated that both he and his students would be most convinced about the existence of a limit using a table of values, Richard rarely used multiple representations throughout the unit. Except for the examination of infinite limits, Richard seldom used graphical representations to examine particular limits and usually gave the impression that it was better to find the limit algebraically than graphically. Moreover, Richard saw no advantage to supplementing his class with the use of graphing calculators, observing to his class that he could show them how to graph faster than the calculator.

Though Richard's perception of limits appeared to be tentative and global at the beginning of data collection, he consistently described the concept of limit in a dynamic manner. Richard felt that he first understood the formal limit notation and definition in college calculus, observing that teaching formal limit notation, definition, and proof had helped him to thoroughly understand the concept of limit.

The translation of Richard's perceptions of limits and the role of limits in calculus into classroom practices can be termed only a qualified success. Richard's major goal was to cover as much calculus content as successfully as possible. To accomplish this goal, he covered each section in the order it was given in the textbook. Though Richard valued the
connection between the concept of limit and other calculus topics methodically, following
the order of the sections in the textbook may have posed a threat to a complete translation
of this goal.

Project Teacher Profile, Trey

Scholastic and Professional History

Following graduation from a large university in the Northwest U.S., Trey taught
mathematics for the past 10 years, including courses in pre-algebra, algebra, geometry,
calculus, introduction to computers, basic programming, and Pascal programming. Trey
started college as a chemical engineering major and shifted to education during his senior
year. Due to his high school chemistry experience and his academic background in
chemical engineering, he initially decided to teach chemistry. During student teaching,
however, he made the decision to change to mathematics. Trey felt confident about his
preparation in mathematics. One of Trey’s siblings taught high school mathematics, which
may also have influenced his switch from chemistry to mathematics teaching. Trey
described his overall academic preparation as “Great in content, but lacking in curriculum
instruction—like how to teach the content to average and below-average students.” Trey
has developed the following basic view of education throughout his teaching: “Given the
time and opportunity, all students can learn—some simply have to work harder than
others. I also believe that understanding and being comfortable with mathematics’
concepts is vital in today’s society.”

Approximately 720 students, predominantly white, attended the top three grades at
the high school where Trey taught, located on the edge of a city with a population of
approximately 100,000. At the time of the investigation Trey was teaching half-time.
Other than participating in usual staff duties, Trey’s job responsibilities for the year
included teaching first year algebra, calculus, and Pascal programming. Trey has remained active in the mathematics education community by attending several state and national NCTM conferences, as well as participating in a Woodrow Wilson Institute.

Trey was selected for the study sample based on his participation in the calculus reform project previously described. Trey was invited to take part in the present investigation on the first day of the two-week inservice accompanying this project. His initial reaction to the invitation was tentative, which may be explained by the extensive data collection techniques used for this study. However, after a brief period of hesitation, he willingly agreed to participate and, in a humorous aside, proposed: “You can observe my class if you give guest lectures every other day.” This comment seemed to lighten the mood and relieved Trey from his apparent nervousness. The “guest lectures” request never came up again throughout the data collection period.

The completion of the first questionnaire and interview took place on the first and second day, respectively, of the two-week inservice portion of the calculus reform project. The inservice was held approximately three weeks prior to the beginning of the school year. During the interview, Trey commented that he may have been somewhat careless on his responses to the questionnaire items since it was given toward the end of the day and he was tired. He felt relieved, however, that he would be able to clarify his responses to the items during the interview.

Trey was a sincere person with a refreshingly humorous personality. These traits proved to be an asset both in the classroom and as a participant in this study. He made the researcher’s job quite pleasant as he always made himself and his calculus class accessible. All classes were videotaped, journal entries were made, and phone calls were promptly returned.
Portrait of the Calculus Classroom

At the time of the investigation, Trey was in his fifth year of teaching calculus. Though he intended to teach it again the following year, he described the school’s level of support for calculus instruction as “Slim to none—in fact, we will be lucky to offer it next year.” Seven female and nine male students were in Trey’s calculus class during this investigation. According to Trey, compared to previous groups, these students were exceptional. Frequently he described them as “Terrific!” and “The best group of students I’ve ever had.” He also commented that this class was his favorite of the day. Trey felt confident that his students would be prepared for the AP exam; however, he did not necessarily encourage them to take the exam. Of the 15 students, Trey estimated that seven would take the AP exam. Trey mentioned that if fewer students took the exam he would not be surprised. He commented, “I often tell my students not to take it. I feel that it is good for them to take calculus again in college.” He did not discuss with the researcher why taking calculus again in college would benefit his students, other than providing a good review.

Trey’s calculus classroom was arranged in a familiar fashion. Trey’s desk was typically centered in the front of the room and the students were in rows of desks facing his. Differing from typical classrooms, however, Trey had two overhead projectors set up in front of the class. One was set up specifically for the HP-48GX calculator-projection device and one for Trey to write on or to place pre-made transparencies. This set-up was consistent with the way the room had been arranged for the two-week calculus inservice taught in the summer. This classroom set-up persisted throughout the entire data collection.

Trey’s calculus class met for 46 minutes, on three days of the week, and for 86 minutes on another. Most of the observations for this study took place during the extended 86-minute class period. Trey described his organization of the calculus class
period as follows: “We usually spend 10 to 15 minutes covering the previous day’s homework, sometimes longer, and then we go through the new material and several examples.” This basic organization was confirmed from classroom observations. On a typical day, the students would enter the classroom and begin visiting with each other or visiting with Trey until the class started. Trey then spent the first 20 minutes of class covering student questions about the assigned homework. Dependent on the length of the class period, the next section of time, between 20–45 minutes, was devoted to introducing new material and working several related example problems. The class period ended with a discussion of the new assignment. Occasionally, the students were given a short period of time at the end of class to begin working on the new assignment.

Trey’s classroom teaching style was reliant upon frequent class discussions. The discussions appeared to be encouraged by the use of calculators insofar as the students continuously shared with one another the graphs that they had produced on their calculators. In addition, they frequently asked questions of each other and of Trey throughout the entire class period. At all times, Trey was receptive and willing to answer student questions. His teaching and responses to student questions were clearly and correctly stated. Trey told the students if he was unsure of the answer to a question, and that he and the students would have to check on it. Trey’s lectures on new material were openly structured. Students frequently and actively participated by producing graphical images on their own calculators and asking questions. In addition, Trey frequently had students demonstrate their work to the class and produce justification for their particular solutions.

Trey covered both the review and new material very quickly. This rapid pace was in part due to his teaching style and partially to Trey’s concern for the amount of material to be covered in preparation for the AP exam. As indicated by the following journal entry, he did recognize the fact that he needed to slow down. “I postponed the test to Friday.
We need a day to tie up loose ends. The calculator is frustrating for some students. I need to slow down and not try to explain while students are working on calculators.”

Trey was very responsive to his students’ needs. In a journal entry Trey wrote: “Several students stayed after class to get a review on limits. We need to spend some time tying ends together.” Though he was disappointed with how the students performed on their quizzes and tests, the following journal entries illustrate that he accepted the blame for not going over certain material in sufficient detail.

I just graded a set of quizzes and was really disappointed in

\[ \lim_{x \to a} \quad \text{when} \quad a \neq 0. \]

We should have done more examples when \( a \neq 0 \).

Also, asymptotes,

\[ \lim_{x \to 3} f(x) = \infty. \]

The tables on continuity were good and the questions on continuous [functions]—no mistakes.

Overall, I’m disappointed in the grades. I must pay more attention to homework and unspoken questions. I would have guessed that the quizzes would have been a lot better.

Trey stated in class that he planned to cover the items they had difficulty with one more time. He also placed some of the responsibility with the students by discussing in class the importance of doing the homework. The following segment was taken from the classroom observation as he was handing back the quiz:

One of the things I am going to do is have homework be turned in on time. The other thing, I read your comments on the back of this page about what you liked or disliked about the class. The comment that was most frequent was to have more examples about what is being assigned. Make sure that I give you enough examples, I will try to remember.
Overall, Trey had good rapport with the students. Though the students were often noisy, they appeared interested and on-task. The topic of college came up quite frequently during the class. Even at the beginning of the year, the students were focused on figuring out where to go to school and what it would be like. The following example refers to studying calculus in college.

Trey: OK, we’re going to start Chapter Three. We’re not going to use the calculator today, so take out a piece of paper for notes. This is the differential calculus you’ll see in college.

Student: Is college calculus harder than all other math we’ve taken?

Trey: Some people think it is easier, some harder.

Perceptions of Calculus and the Teaching of Calculus

Trey perceived calculus in a linear ordering of related concepts divided into three major sections: (a) limits, (b) differentiation, and (c) integration. This linear ordering of concepts was probably best demonstrated by his initial diagram of the overview of the important concepts and relationships in calculus (Figure 20). When he was asked to explain this diagram he commented that it was guided by the order of topics in the text:

Trey: We start out with a review, but I think that was calculus . . . so I started out with limits, from here you go to series and sequences—but then limits goes to derivatives as for defining what a derivative is. From derivative we go off to applications of derivative, then to integrals, application of integrals. Then you could go on to differential equations or series and sequences.

Researcher: How did you chose these topics?
Trey was referring to the text, *Calculus* by Larson and Hostetler (1990), that he used prior to participation in the calculus reform project. As a required component of the calculus reform project, Trey used the Dick and Patton (1992) calculus text as his principal text throughout the data collection period. However, he did continue to supplement his teaching with additional examples from the Larson and Hostetler textbook.

Subsequent subject matter diagrams (Figures 21–23) demonstrate that after using the new textbook for planning to teach limits, teaching limits, and teaching subsequent topics involving the concept of limits, Trey continued to view calculus as a linearly ordered set of topics. In addition, these diagrams continued to illustrate Trey’s perceptions of calculus as a three-tiered combination of limits, differentiation, and integration. Offshoots from the differentiation and integration tiers always included applications. However, the diagrams differed in the number of concepts related to each tier. For example, in the diagram drawn during the planning phase (Figure 21), Trey illustrated more specific offshoots from the “limits box,” but the offshoots for differentiation and integration remained approximately the same. Similarly, in the diagram sketched after teaching the limits unit (Figure 22), additional specific offshoots from the “derivatives box” were evident, and the offshoots from the “limits box” seemed to become more general again. In both cases the “integration box” remained basically the same, subject to mention of problem-solving. The final diagram (Figure 23) appeared to demonstrate calculus as an integrated concept. For example, the arrows extending from the “understanding of limits box” illustrated connections between limits, the derivative and the integral, a possible demonstration of increased emphasis of the concept of limits in the teaching of calculus. Moreover, the final diagram appeared to be more formal than the prior diagrams.
Figure 20. Trey’s Diagram of Calculus, Prior to Beginning of School.
Figure 21. Trey's Diagram of Calculus, Planning to Teach Limits Unit.
Figure 22. Trey's Diagram of Calculus, After Teaching Limits Unit.
It was interesting to note that throughout the entire period, Trey illustrated an one-way connection between the derivative and the integral, which may be directly related to the order of concepts in the calculus textbook. Trey noted that he felt quite comfortable with his ability to teach the calculus course, though he felt that the first time he had taught calculus it was "very difficult." Trey stated his goals for teaching calculus this year: "I
would like my students to enjoy calculus enough to continue to study math in college. I would also like my students to see an appreciation for the mathematics they have learned and labored over."

Trey appeared dedicated to following the goals and ideas established in the new text, and followed the book very closely. Though he sometimes appeared to doubt the authors' reasoning regarding notation, the examples used, and the exercises given, he gave the authors the benefit of the doubt. He appeared to trust that if the authors felt it was important, then it was important. For example, Trey decided to have the students write polynomial expressions in Taylor's form because that was the form the book used.

Trey: OK, here is the general form of an equation. First we want to (a) write it in the slope-intercept form:

\[ y = \frac{2}{3}x + 2 \]

(They discuss slope.) Then (b) put into point-slope form at (3,4):

\[ y - 4 = \frac{2}{3}(x - 3) \]

and finally (c) write it in the Taylor form at \( x_0 = -4 \):

\[ f(x) = \frac{2}{3}(x - 4) + f(-4) \]
\[ f(x) = \frac{2}{3}(x + 4) - \left(\frac{2}{3}\right) \]

That's your Taylor form.

Student: Why do we use this [Taylor's form]?

Trey: Because it is used in the book and I want you to be familiar with it.

Though Trey did not appear to look ahead in the text to understand the authors' motivation behind using the Taylor's Form, he appeared to trust that the authors included the topic for a reason.
Trey frequently shared his concerns with the researcher regarding participation in the calculus reform project. Trey felt that his students would gain a better conceptual understanding of calculus concepts as reflected in the goals of the calculus reform project. Yet, he worried that conceptual knowledge in the absence of considerable procedural knowledge may not be sufficient for success in college calculus. His concern appeared to be based on the fact that some of his students would be taking the AP exam and many of them would take calculus in college. In addition, Trey felt very much a part of the project and frequently discussed ways to improve the calculus reform curriculum. The following examples illustrate some of the suggestions Trey shared with the researcher:

The graphs in Section 3.2 are too small. This forces the students to write in their book. Also, it would be nice to have a xerox of the water thing [an example of rate of change given in the textbook] too. The numbers aren’t close enough.

Tell Tom [Dick] that he needs to make a bank of overheads to go with his book so we don’t have to blow up the pages every time. The other thing I noticed was that on page 89 they should have the graphs further apart.

Trey shared these suggestions with other teachers in the project at the December follow-up support meeting. At this meeting it was apparent that Trey was happy to be a part of this investigation. He was often overheard at the meeting discussing his videotaped classes and other items pertaining to the data collection with other participants in the calculus reform project who were not involved in this study.

**Perceptions of the Concept of Limit**

Judging from responses to questionnaire items as well as demonstrations from classroom observation and interviews, Trey’s perception of the concept of limit was very stable. Throughout data collection, Trey perceived the concept of limit in a dynamic way.
Trey first described the concept of limit as “a value a graph approaches, but never gets there.” In response to the succeeding questionnaires, he described a limit to be “a value (defined or undefined) that a function approaches as you get closer and closer to it.”

Trey claimed he really understood the concept of limit when he taught it. For example, the questionnaire asked: “Looking back on your own mathematics education, when did you feel that you really understood the meaning of the statement

\[
\lim_{x \to a} f(x) = L.
\]

Trey consistently responded with “The second or third year I taught it,” and reiterated this conviction in an interview:

During my first year [of teaching calculus] I understood it [formal limit notation], but not really, really until the second year. This will be my fifth year and now I feel very comfortable with the [limit] terminology. I had to teach it once to really understand it.

After teaching the limits unit using the new textbook he added to his response, “When I taught the definition of limit by looking at walking along the graph blindfolded.” Trey stated that he had never taught limits using this approach before participating in the calculus reform project. In the follow-up interview, he commented that he felt his understanding of limits continues to evolve each year he teaches. In fact, he added the following statement to his response to this question in the final questionnaire completion: “Actually my understanding of the notation continues to evolve as I review and teach it year to year.”

Perceptions of the Role of Limits in the Teaching of Calculus

As previously observed, the diagrams of Trey’s overview of the important concepts and relationships in calculus (Figures 20–23) demonstrated that he perceived the role of limit to be very important in the teaching of calculus. In fact, the diagrams
appeared to demonstrate increased emphasis upon the importance of the concept of limit in the teaching of calculus. Other modes of data collection also indicated an increase in the importance of limits. For example, prior to participation in the calculus reform project, Trey defined the importance of the concept of limit in teaching calculus as “It is important for the AP exam and to discuss the definition of derivative. But, most of it is quickly forgotten.” In the interview following the completion of the first questionnaire he stated:

We do ease them into it. There isn’t a lot of materials from [the] limit chapter that they need to carry on to the other chapters in order to do derivatives, but in order to talk about the derivative we need to have a little bit of it, an idea of continuity and limits.

After teaching the limits unit, however, he responded: “It is more important to have a conceptual understanding of limits and continuity than to be able to grind out limits algebraically; important in discussing derivatives and rate of change.” In fact, throughout the remainder of data collection, Trey continued to discuss with the researcher the importance of a good conceptual understanding of limits for understanding other calculus topics, in addition to which he appeared to minimize the significance of simply evaluating limits. At the end of the data collection period he stated, “The concept [of limit] is very important, especially the visual idea of dividing a length into a lot of delta-x’s, but doing a lot of numerical calculations is only so-so.” Furthermore, in a telephone conversation at the end of data collection, Trey discussed his excitement regarding the role of limits in his calculus class this year. Without researcher prompting, he offered the following comment,

I think limits is much more a part of calculus now than it ever has been for me. I think graphing calculators helps make limits much easier to understand. You know, visually and graphically, [students] get a better intuitive feel for it. Thus, connections can be more easily made and it nicely ties all of calculus together.
Trey felt that an intuitive notion of limits was needed for an understanding of other calculus concepts. However, as demonstrated from classroom observations of the teaching of subsequent topics involving the concept of limit, these connections appeared to be less significant. For example, in the “introduction to derivative” lesson, Trey understood the connection or need for limits and discussed looking at the limit of the secant line becoming the tangent line. This “lecture,” however, was only eight minutes long. Though this connection may have been revisited in more detail at another time, it is doubtful that the students were able to catch the limit connection from just the following short discussion Trey provided on that day.

Trey: Now, how do we find the slope of the tangent line?

(Not expecting a students response he discussed the formula for calculating slope from algebra.)

But if we are just at one point, how do you find what the slope is? What we will do is look at the limit. If you find two points and look at them getting closer and closer.

Maggie: Yeah, but that wouldn’t be a tangent then.

Trey: Yes, but we will be looking at the limit.

(He doesn’t appear to be satisfied with this, but he goes on. He starts with the linear form, then shows the general form, point-slope form, and slope-intercept form of the equation.)

This particular lesson did not involve graphical interpretation of the definition of the derivative. Without prompting in a telephone interview, Trey stated that he did look at the graphical interpretation of the definition of derivative using a few examples the following day. This lesson, however, was neither observed nor videotaped since scheduling difficulties prevented direct observation. Trey did not videotape as he felt this lesson was no longer a part of the introduction to derivatives.
Though Trey felt the intuitive notion of limit was important in the teaching of calculus, throughout the data collection period Trey felt that the role of rigorous epsilon-delta proofs was insignificant. Trey also felt that the formal limit definition deserved little attention. Prior to participation in the calculus reform project, Trey expressed a desire for his students to be more rigorous with the definition of limit; however, he felt that covering a lot of rigor with respect to limits was senseless. The following interview response indicates why he felt this way.

I would like my students to be more rigorous with it, but we don’t do it because it’s right at the beginning of the year and they get blown away if it is very rigorous. So we don’t do the formal definition and there isn’t a lot of time to do it, but we kind of hit the important concepts.

After teaching the unit on limits, Trey observed that “I think the concept of continuity and limit is important to understand. Evaluating limits traditionally is less important and deserves little rigor.” At the end of data collection, he basically reiterated this view stating that the appropriate level of rigor was “a deep intuitive understanding, using a graphing calculator to do the numerical calculations.”

The Teaching of Limits

Trey spent the first two to three weeks of the school year covering the pre-calculus review material found in Chapter One of the textbook (see Appendix B for an outline of Chapter One.) The next 10 class periods were devoted completely to the unit on limits. The lessons given on eight of the class days were sequenced identically to the order of the topics in the limits chapter of the textbook (see Appendix B for an outline of Chapter Two.) Briefly, the order of the major topics was as follows: examining one-sided limits, estimating limits numerically and graphically, formal definition of limit, graphical interpretation of the formal limit definition, limit proofs, continuity, analyzing
discontinuities and asymptotic behavior, consequences of continuity, and horizontal asymptotes and end behavior. Two of the 10 days were devoted to a quiz and unit exam.

Trey frequently expressed his excitement about teaching calculus concepts in a new way. In a telephone conversation with the researcher the evening before Trey was to begin the unit on limits, he expressed this excitement.

Trey: I am nervous.
Researcher: Are you nervous because I am coming to observe the class?
Trey: No, it [the new calculus text discussion of limits] is just so different from how I have been teaching [limits].
Researcher: Is that good?
Trey: Yes, the students should be able to understand what is going on better. I’m nervous, but yet I am excited to teach this way.
Researcher: What way is that?
Trey: You’ll see tomorrow I guess.
Researcher: I guess you are right.

The next day Trey appeared enthusiastic in his instruction on the introduction to limits. Trey’s introduction to limits followed the text closely. Thus, he started with a visual picture of the limit process, using a transparency made from a copy of the graph found on the first page of the limits chapter in the text (Figure 24). The following segment includes an extract from Trey’s discussion of the diagram:

From the book here are two people walking along this curve here and they are blindfolded and they can’t see where they are going. Now if you talk about this left-hand person he is trying to predict what the value of \( f(a) \) is. So he walks toward along this curve until he falls in this hole. As he walks along the curve he tries to predict what \( f(a) \) is. That’s called the left-hand
limit. (He then wrote the left-hand limit notation on the overhead.) Now the other guy walks along from the right-hand side, and he is walking down this direction . . . .

Figure 24. Trey's Graph Used for Introduction to Limits.

Trey followed this discussion by reviewing the first example in the textbook, which involved visually determining the limits of $f$ as $x$ equals $b$, $c$, and $d$.

When Trey was asked about his favorite way to explain a limit to a beginning calculus class, he responded differently during the various stages of data collection. Prior to participation in the calculus reform project he indicated that his favorite way to explain limits was using "The story of the chemical engineer and chemist, and getting close enough for practical purposes." He discussed his response in more detail in the follow-up interview:

I had this story in engineering school. The chemical engineers go half the distance, and then half . . . , but the chemists just stand there because they know they can never get there. The chemical engineers say, "Well, we can get close enough for practical purposes." Thus, approaching the limit.
During the planning stage prior to teaching limits he discussed his favorite way to explain limits to students as "approaching the speed of light or the greatest speed your car can go on an open freeway." He did use this explanation briefly at the beginning of the lesson on the introduction to limits.

What we are going to do today is talk about limits. You are somewhat familiar with the concept of limit, like in science. (He discussed driving a car in Montana as fast as you can.) There still is some limit. If you put your foot down as far as it would go, how fast could you go? What is the limit? There has to be some kind of limit.

After teaching the unit on limits, as well as during the final completion of the questionnaire, Trey responded that his favorite way to explain limits to a beginning calculus class was exactly the technique he used in the classroom, which was the method used in the textbook. "I enjoyed using the graph on page 63 (Figure 24) to introduce limits, . . .if two people walk blindfolded along the function f(x), the limit of f(x) as x approaches a is the last measurable value the people reach before they fall into the hole."

Following the order of the textbook, Trey's next lesson involved the next two textbook examples. The first of these examples involved estimating the

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \]

numerically, and the second example involved estimating the same limit graphically. The following is an example of the numerical approach as seen in the classroom observation:

Trey: Get your calculators out. If we are just given a function

\[ f(x) = \frac{\sin x}{x} \]

and say we want the limit of f(x) as x approaches zero. Well, you can't really just determine this
[limit] because you get 0/0 and you can’t evaluate that and you get what is called indeterminant form—and indeterminant form means that we can not determine what it is from that evidence. So, we need to find some other ways to look at [the limit]. There are two ways we are going to look at it today, numerically and graphically. So, we want to use our calculator and try some numbers that are really close to zero, from both the positive and negative sides. We are going to evaluate this numerically at each point and that is why it is nice to have these calculators.

(They put the equation into their calculators.)

What we want to do is go into SOLVE [a command on the HP-48G]. I want to start with the value x=.1, highlight y and then solve. Do you get .9993?

Students: No.

Trey: Are you in radians? We should all be in radians. We could just make an initial guess that this limit is one, but let’s put something closer in to see if it does something tricky as we get closer?

(They do and they see that it is definitely getting closer to one.)

So, putting in .001 is close enough for me to believe that the limit is actually one.

(Some of the students claim to be putting in .00000001 and getting something like .99999999999 so they definitely believe one to be the limit.)

Let’s try the negative values -.01, -.001, . . . and you get .993, .99934, .9999 . . .

David: I put in six zero’s before the one and I got one.

Students: Yeah, so did I.
Trey very briefly and vaguely explained the accuracy of the calculator. The students accepted his explanation. Following this discussion the students took a short break. After the break, Trey went through the graphical approach of this example.

It was obvious from Trey's response to the initial questionnaire that he had thought about this example before participation in the calculus reform project. He explained four different arguments he had seen to support this example: (a) the Squeeze Theorem, (b) L'Hopital's rule, (c) a numerical approach using a calculator, and (d) a graphical approach using a calculator. Thus, even before participation in the calculus reform project, Trey discussed the use of multiple representations. In addition, throughout the data collection period, he felt that the argument that was most convincing to him, the graphical approach using the calculator, was also most convincing to his students. Thus, though Trey followed the order of the textbook, it cannot be assumed that his perceptions regarding the teaching of limits had necessarily changed.

The next major section of the textbook, as well as Trey's next three classroom lessons, involved primarily the intuitive notion of the formal definition of limit. Trey began the lesson by writing the definition of limit using the textbook notation (see Appendix B for the textbook definition of limit) on the overhead and verbally discussing what the notation meant. From this point Trey provided a graphical interpretation of the formal limit definition using the graph provided in the textbook (Figure 25).

Examination of field notes indicates a direct correspondence between the classroom lessons and the textbook throughout this entire section. The only other pattern that developed was that though at times Trey found the textbook interpretation of certain concepts ambiguous, he taught them using the proposed approach. For example, a portion of the exercises in this section asked students the following: "Given a limit of the form

\[ \lim_{{x \to a}} f(x) = L, \]
find a suitable positive value delta that would guarantee \(|f(x) - L| < .001\) whenever \(0 < |x - a| < \delta\).” Trey felt that finding delta using the graphing calculator was ambiguous, nevertheless he taught it anyway. In his journal, he indicated that he was not quite sure of the purpose or meaning of this approach.

The idea of setting vertical range \(L + \varepsilon, L - \varepsilon\) and finding delta seems ambiguous. Should the line come in at the corners of the calculator screen or straight across, or doesn’t it matter?

This type of problem was given as part of the homework assignment, however, Trey did not put this type of problem on either the quiz or exam.
Throughout data collection, Trey felt that to find limits the students were most convinced by graphical and numerical arguments. Though he examined topics using other representations, he routinely used these two types of arguments. Typically, when introducing new topics, Trey worked through a graphical or numerical example first and then gave a formal definition of the concept. For example, for his introduction to the final section in the textbook, the definition of continuous, as suggested in the text, he began with the graphical approach, enlarging the exact figure used in the text (Figure 26).

Figure 26. Trey’s Graph Used for the Introduction to Continuity.

The following segment is a snapshot of his introduction to continuity:

Trey: If you have a limit at $a_2$ and $f(a_2)$ exists, therefore the function is continuous at the point $a_2$. Let’s look at $a_3$. Well, there is a limit at $a_3$ but $f(a_3)$ is undefined. So it is not continuous.

Student: What if I was going to do it on my calculator?
Trey: It would just say undefined. (He did not demonstrate this on the calculator.)

So, let’s look at what we need for the formal definition. We need to check three things—does the limit exist, is the function defined, and does the limit equal the value of the function at that point?

Trey then wrote the formal definition of continuous on the overhead, and to check students’ understanding of the concept of continuity, directly followed with the next example given in the text.

As previously noted, Trey covered the topics in the limits chapter rapidly. For the most part the students appeared comfortable with this pace. However, Trey occasionally overlooked students’ comments and continued with a discussion without direct response to questions. For example, during one lesson a student had raised an important insight regarding continuity prior to discussion. Though Trey heard the student’s comment, he did not acknowledge it during the class period. This example occurred when they were working an exercise in the text that required examining a graph, finding the limit, and deciding whether or not the function was continuous at that point.

Trey: What happens if you put in negative two?

Students: Nothing.

Trey: OK, then we say that it is undefined. (There was a long pause before the next question.) OK, what is the limit from the left?

Students: Zero.

Trey: From the right?

Students: Zero.

Trey: So, is there a limit?

Students: Yes.
Trey: It is?

Students: Zero.

Trey: Is it continuous?

Students: No.

Then, as Trey continued on to another value in the table, a student vocalized an important insight regarding continuity.

Student: So, if there isn’t a limit, then it looks like it won’t be continuous.

Trey: Ah, yeah . . . . What happens at negative three?

Student: If there isn’t a limit, then there is no reason to even think about if it is continuous.

Trey continued to have the students help him fill in the table and the student’s insight appeared to go unnoticed. However, without prompting, Trey did mention it in both the informal interview after the observation as well as in the following journal entry: “When we talked about limits from the left, right, and continuous functions, the use of the tables sparked some interesting questions regarding the relationship between limits and continuity, and the existence of limits.”

Throughout the unit on limits, Trey was otherwise generally responsive to the needs of his students. For example, since several students stayed after school to work on limits, he felt that they may need to spend more time on review topics and looking at connections. In addition, at the end of the unit on limits Trey realized that it was important to examine the limitations of the calculator. The following example reflects this recognition, as he was discouraged with the way his students performed on given quiz problems since the majority of them did not realize the limitations of the calculator’s graphing screen.
Trey: I have asked Jeff to come up to the board and write the solution to problem two on the quiz. The difficulty with this problem is that many of you were unable to get the whole graph. (Trey puts the function into his calculator for Jeff, some of the students do it with him.)

Jeff: For the zeros I factored out $x^2$, for the discontinuities I just used my calculator, I knew they had to be asymptotes because nothing canceled so there weren't any holes. I just used the calculator to graph it and I zoomed out.

Trey: Show that please, because that is where people had problems. (Jeff works on zooming out on the overhead calculator.)

Jeff: The bottom part isn't showing so I have to zoom-out again until the bottom part shows up. I had to zoom out three times to get that bottom part.

Trey: Most people didn't zoom out far enough to get that part. Jeff, how did you know to keep zooming out when most of the people didn't?

Jeff: Because when I looked at the equation I knew that there had to be a large zero, so I kept zooming until I found it.

Trey: Good, that is good number sense. You have to remember to use the information that you have found to draw your graph. One of the things I realized was that we didn't cover very many incomplete graphs on your calculator. We need to look more at the limitations of the calculator.

In his journal entry for the day, Trey also acknowledged the need to look at the limitations of the calculator: “They did fine when the graph was given, domain and range, but they didn’t think to find it. Too dependent on the calculator but they seem to have the concepts. We need to look more at the limitations of the calculator!” The limitations of the calculator were not discussed in detail during the remainder of the data collection period.
Trey’s overall view of the new project approach to limits was dualistic in nature. Though he seemed quite positive toward the new approach and he felt the students had a more visual understanding of limits, he worried that they may not have had the ability to find them algebraically. “We have been spending a lot of time in calculator activities, and very little on the algebraic evaluation of limits. Will we be sorry later? Or will it be covered during l’Hopital’s rule?” This concern was also demonstrated in his final journal entry, in which he worried that they may regret it that they did not spending much time on the algebraic evaluation of limits and had spent more time on calculator activities.

I feel in general that students have a better visual understanding of limits and less understanding of the process of finding limits algebraically. I will emphasize asymptotes—vertical and horizontal—when we talk about graphing and first and second derivative tests with derivatives. We also touch on limits again with l’Hopital’s rule, which really makes the “old” way of evaluating limits somewhat obsolete.

Trey held his students accountable for the material on limits by giving one short quiz toward the beginning of the unit and an exam at the end of the unit. The questions Trey asked on the exam reflected the goals of the calculus reform project. In fact, all of the questions on the exam were cut-and-pasted directly from the examples or exercises in the new text.

Summary of Trey

Throughout data collection, Trey viewed calculus as a linearly ordered set of topics divided into three tiers: (a) limits, (b) differentiation, and (c) integration. Trey commented that he thought of calculus as it was portrayed in the textbook. As the study progressed, Trey’s diagrams of the important concepts and relationships in calculus increasingly demonstrated the connections between various calculus topics. In particular,
though the order of the topics remained linear, increased emphasis appeared to be given to the concept of limit connections.

Trey described his goals for teaching calculus as, "I would like my students to enjoy calculus enough to continue to study math in college. I would also like my students to see an appreciation for the mathematics they have learned and labored over." Although it was impossible to determine whether Trey's students would continue to study mathematics in college, it was possible to observe that the students in Trey's class were active participants in the experience. Trey's class was characterized by frequent and open discussion and the students appeared to be quite interested. In part, this classroom atmosphere may be attributed to the use of graphing calculators and a great part of the discussions pertained to their use. Trey demonstrated enthusiasm for teaching calculus and expressed to his students that he wished they could develop the same appreciation for mathematics.

Trey was dedicated to following the goals and ideas established in the new textbook, and followed the book example-by-example throughout the unit on limits. Though he occasionally doubted the reasoning expressed in the text, he appeared to trust that the authors included given topics for sound reasons. Typically, he did not even look ahead in an attempt to understand the motivation for the presentation of certain topics, and though Trey taught these topics, he frequently expressed his uneasiness about certain presentations and did not hold his students accountable for them on the unit exam. Trey's teaching style was also quickly paced, but since he was responsive to student need, the students appeared to feel comfortable with this pace. When students asked for additional examples or review time, Trey always obliged. Though Trey's lectures for subsequent topics involving the concept of limit were brief, in discussions with the researcher, he consistently observed that the role of limits were extremely important to the teaching of calculus. In fact, Trey appeared to demonstrate increased emphasis for the importance of the concept of limit in the teaching of calculus as the study progressed. This increased
emphasis is best demonstrated by comparing his statement at the beginning of the study, "There isn't a lot of material from the limit chapter that [the students] need to carry on to the other chapters," to his statement at the end of the study:

I think limits is much more a part of calculus now than it ever has been for me. I think graphing calculators helps make limits much easier to understand. You know, visually and graphically, they get a better intuitive feel for it. Thus, connections can be more easily made and it nicely ties all of calculus together.

Trey felt that a good conceptual understanding of limits was important for understanding other calculus topics. He felt that the formal epsilon-delta definition of limit deserved little attention. In fact, although he expressed a desire for his students to be more rigorous with limits, he felt that the role of rigorous epsilon-delta proofs was insignificant and that there was little sense in giving it much class time.

Throughout the unit on limits, Trey's classroom practices consistently matched his professed perceptions. He introduced the limit concept from an intuitive graphical perspective, and the graph he used for the introduction to limits was taken directly from the text. Trey used the textbook diagrams frequently, and routinely used graphical and numerical arguments to find limits in that he felt that the students were most convinced by the graphical argument. Trey discussed the use of multiple representations even before participating in the calculus reform project. It was interesting to note that though Trey consistently used graphical representation, his discussions regarding the limitations of the calculator were quite brief.

Trey perceived the concept of limit dynamically and interpreted the concept consistently. He stated that he had to teach the concept of limit before he really understood the concept. After teaching the unit on limits using the new textbook, Trey commented that his understanding of limits continued to evolve every year that he taught.
Trey typically spoke in positive terms about the calculus reform project. He frequently expressed excitement about teaching the various calculus topics in a “new” way. He also felt a part of the project as he shared helpful suggestions and some concerns about the project with the researcher. By the end of the data collection period, many of these concerns diminished as the reasoning in the text became apparent to him.

Project Teacher Profile, Terry

Scholastic and Professional History

Terry graduated from a small liberal arts college in the Northwest U.S. in 1970 with a major in mathematics and a minor in economics. The year after his college graduation, Terry realized that he wanted to teach. “In 1971, after my BA, I joined the Peace Corps. My assignment was to teach and develop mathematics curriculum. This was my first exposure to teaching. I really enjoyed it and I decided to become a teacher.” Though Terry never went through a teacher education program, he has been teaching high school mathematics as well as computer applications and programming ever since.

Terry continued his educational development by obtaining a Master of Arts in Mathematics in 1981 from another small private college in the Northwest U.S. Terry commented that since he obtained his degree in 1981, he has “taken over 60 hours of postgraduate courses at various institutions.” With respect to his recent professional activities related to mathematics, Terry is an active member of the state council of teachers of mathematics and currently serves as an area representative. In addition, he has been active in the state math leaders conference for the past eight years, as well as serving as exhibit chairman for a Northwest Mathematics Conference. Terry considered his overall academic preparation as “Very good! A mixture of liberal arts and mathematics.”
At the time the present study was conducted, Terry was teaching at a high school situated in a small, predominantly white rural community with a population of approximately 500, located 30 miles from a large metropolis. The population was approximately 250 students in the top three grades. Terry referred to the students in his school as “motivated and not lacking much financially.”

In addition to teaching AP calculus, Terry taught computer applications, geometry, and advanced algebra throughout the course of the year, as well as serving as mathematics department chair. Terry enjoyed teaching and stated that his main responsibility as a teacher was “to guide the students to be self-learners,” an approach equally demonstrated by his basic philosophy of education:

I have always seen my role as a teacher to be that of a guide for my students, to challenge them, to involve them, and to excite them about learning mathematics. I strive to make my classes non-threatening and free of conflict. Education is free for the taking. One should learn from all his experiences. Education is on-going, never ending.

Terry was selected for the study sample based upon his participation in the previously discussed calculus reform project. On the first day of the two-week project inservice, Terry was invited to take part in the current investigation. With no apparent hesitation, Terry accepted this invitation. However, after discussing the extensive data collection techniques involved in the study, Terry was concerned that he would not be able to participate since he would have a student teacher during the second term. Since the majority of the data collection took place during the first term, use of a student teacher for the second term did not pose a research problem. As the year developed, the student teacher decided not to teach any portion of the calculus course because he felt uncomfortable with the material.

The first questionnaire completion and interview took place on the first and second days of the two-week inservice, respectively. The approximate timing of this inservice
was three weeks prior to the beginning of school. During this phase and throughout the entire data collection period, Terry was a helpful and conscientious participant. Scheduling was uncomplicated and all data collection items were completed on time.

Portrait of the Calculus Classroom

Terry had been involved with teaching calculus for the past 11 years. In the year the study took place, 18 students including five females and 13 males, were enrolled in Terry's calculus class. These students were described by Terry as "well rounded honors students." Terry also stated that the students in his calculus classes have always been "Good students, though rarely exceptional. In the past six years I have had one national merit scholar." Terry commented that most of his calculus students would take calculus again at the college level, and he stated that "I encourage them to do so." The school supported the need for a calculus class, and Terry observed that "I'm encouraged to teach the class no matter how small the enrollment."

Terry's desk and bookshelves were located in the back corner of his classroom. Roughly 25 desks appeared to be haphazardly scattered about the room, however, throughout the course of the class period the students would rearrange the desks to form groups of three or four. This type of organization was also observed in Terry's non-calculus classes. At the front of the room Terry had placed two overheads. One overhead was used for writing and transparencies, and the other overhead was solely used for a HP-48G projection device. This overhead projector set-up was identical to the model used for the two-week inservice.

Twice each week Terry's calculus class met for approximately 48 minutes and twice a week they met for extended class periods of 96 minutes. All observations took place during the extended class periods, thus the subsequent listed times were based on a
213

96-minute period. Terry described the basic organization of his calculus classroom in the following way:

I usually have a class starter (maybe a quiz). I don’t like to spend much time on assignment questions. I encourage students to ask on their own. We do some group work. We do some discussion or discovery. We have daily assignments.

Though Terry valued this type of classroom organization, it was not an accurate portrayal of what was actually observed during the study. Terry spent considerable time on individual student’s questions. Terry commented that the students in his calculus class were “varied in ability,” consequently he felt he had to spend a considerable amount of time making sure that everyone understood each of the presented concepts.

On a typical day the students entered the room soon after the five-minute warning bell and most of them went directly to their seats and took out their book and HP-48G calculator. Some of the students began to play games on their calculator, some of them began to work on their assignment, and some of them discussed calculator functions with one another. A probable explanation for this behavior was that most of the observations took place at the beginning of the school year when the calculators were new to the students.

The class period began officially with the Channel One Whittle News television program lasting from 10 to 15 minutes. During this time, several of the students asked Terry questions regarding homework and the mechanics of the calculator. In fact, this type of individual question-and-answer period usually extended into the first 20 to 30 minutes of the class period.

After the news and individual question-and-answer period, Terry typically discussed the assignment from the previous day with the entire group for from 10 to 45 minutes. The length of time was based on the number of questions the students asked as well as the amount of material Terry had to cover for the day. Capturing all of the
students attention to initiate this segment of the class period was often somewhat difficult.

The following snapshot is an example of Terry’s attempt to get the group’s attention:

Terry: OK guys, let’s talk about the assignment a little bit. Should we collect it? (Some of the students moan and groan.)

Jeremy says we should. (The students make lots of noise.)

Shhhhh, it’s time for calculus now. Hey, hey, hey ...

Student: Let’s do number seven.

Terry: Ok. Hey you guys, shhhhh. Hey, let’s look at number seven. (Some of the students are now trying to pay attention and are even giving dirty looks to the students who are continuing to make noise.)

First of all, you all should have your book open. Has anyone done this? Someone should come up to the board and do it. (Finally most of the students are quiet, though some are still not paying attention.)

The next period of time was devoted to covering new material. Terry’s shortest lesson, the introduction to limits, lasted for 13 minutes. Terry’s usual lecture for new material lasted for durations from 15 to 60 minutes. The longer lessons were combined with an individual question-and-answer period. For example, Terry would explain how to use the graphing capabilities of the calculator and then he would walk around the room fielding individual questions. Then he would discuss a new topic and follow it with another individual question-and-answer period. During the final 10 to 30 minutes of the period, Terry presented new assignments and allowed the students to work on them until the period ended. Terry continually walked around the room fielding individual and group questions until the bell rang and the students left the room.
In general, Terry permitted a relaxed atmosphere in the class, yet it was not out of control. At all times, several student discussions were taking place. The students frequently asked questions of Terry as well as other students. In several instances, Terry asked individual students to help other students who were having a problem with a given concept. For example, one student, Marcy, asked, “How do you use the SOLVER on this calculator?” As Terry was helping another student, he said “Chris could you please explain to Marcy how to use the SOLVER?” Chris explained it verbally and then Terry reiterated what Chris said. The following illustration also demonstrates students addressing questions from one another:

Julie: How do you find the “alligators” [< and >] on the calculators?

Dave: Just hit PROGRAM key, and then hit TEST.

Julie: Oh, that is sweet. Way to go Dave!

Furthermore, some of the students consistently addressed Terry by his first name. For example, Terry mistyped an equation into the calculator and a student responded, “You’re wrong Terry. You forgot the absolute values.” Regardless of the absence of standard formalities, the discussions were generally productive and on-task.

Occasionally, the discussion was not constructive. Terry frequently had to remind students not to visit and that they were in the class to do calculus. For example, some comments Terry made in class include, “Melissa, this not the time for you to be writing in your yearbook,” and “Hey, do you guys who are visiting in the back know how to do this problem? I suggest you listen then.” In addition, the students in Terry’s class frequently attempted conversations with the researcher. For instance, one day a group of four girls sitting near the researcher asked, “You probably know, can you quit a sorority at any time?” There appeared to be constant chatter in the classroom. However, everyone, including Terry, appeared comfortable with this type of easy-going atmosphere.
Terry’s calculus class also experienced a number of interruptions, most of which originated from outside the classroom. These distractions included other teachers walking to the front of the room to talk to Terry, students coming to talk to individual students in the class, as well as occasional visits from the high school counselor and librarian. It was interesting that these type of interruptions appeared to be quite common and accepted at this particular high school. On several occasions, however, the interruptions came from the calculus students themselves. For example, distractions included students returning from college recruitment meetings to individual students asking for the “potty pass.” One day a student spilled a can of pop in class, creating an unnecessary delay in Terry’s lesson. Observations of non-calculus classes verified that these type of interruptions were not unique to Terry’s calculus class. Perhaps, it was that this type of atmosphere was a school climate problem.

Perceptions of Calculus and the Teaching of Calculus

Prior to participation in the calculus reform project, Terry’s overview of the important concepts and relationships in calculus is demonstrated in Figures 27–30. Though the appearance of the diagrams changed throughout the data collection period, Terry consistently perceived calculus as a linearly ordered problem-solving tool. From Figure 27, Terry regarded calculus as taking an application, feeding it through a “function machine,” applying the techniques of calculus, and arriving at a solution which possibly led to further questions, permitting the cycle to be restarted. In the first interview, Terry described his thoughts regarding this diagram:

I was thinking of a calculus class. We have certain problems which involve calculus and can be put in function format, then we operate on them using differentiation or integration tools, and then get a solution to it. Sometimes the solutions lead to other questions. This is a continuous loop. This is a good overall view. I usually think of calculus this way.
Figure 27. Terry's Diagram of Calculus, Prior to Beginning of School.
Figure 28. Terry’s Diagram of Calculus, Planning to Teach Limits Unit.
THE INCREDIBLE PROBLEM SOLVING MACHINE.

Figure 29. Terry's Diagram of Calculus, After Teaching Limits Unit.
Figure 30. Terry's Diagram of Calculus, End of Data Collection Period.
Similarly, Figures 28–29 also demonstrated Terry’s view of calculus as feeding “problems beyond the scope of pre-calculus mathematics” through a “calculus machine.” However, the machine in Figure 28 presents the concept of limit as the primary tool. Terry explained that he was preparing for the unit on limits and that limits were “the most important part of all of calculus.” Figure 29 appears to give the concept of limit equal weight to pre-calculus mathematics and calculus topics, the differential and integral. It is interesting to note that the diagram illustrates that limits are not a part of calculus. However, Terry explained that limits are important to “appreciate other calculus topics such as the derivative.” Figure 30 appears to illustrate the connections and relationships which exist within the “calculus machine,” representing the limit as connected to derivatives, integrals, and infinite sequences and series. In addition, the connection between the derivative and integral was demonstrated. Terry commented that he “always viewed calculus this way,” but that he was unable to diagram it exactly as he would view it.

Terry discussed four major goals for teaching calculus: (a) to prepare students for upper level college mathematics, (b) to prepare students for math-related careers, (c) to prepare students to be problem-solvers, and (d) to develop thinkers. Terry did not specifically mention the goal of preparing students for the AP exam, however, this goal may be incorporated into the goal of preparing students for upper level college mathematics. Terry did estimate that approximately half of his students would take the AP exam, although he does not force this issue and rather minimizes emphasis upon preparing for the exam. The amount of emphasis given to the AP exam as well as any of the other previously stated goals is discussed in the following sections.

Besides the four previously stated goals, Terry rarely communicated the daily expectations he had for his students. In most of the informal interviews, Terry primarily chose to discuss his teaching style and the use of the graphing calculator. Occasionally, he would mention particular opinions he had with respect to the calculus reform project. For
example, after covering the chapter on linear functions, Terry’s journal entry was as follows: “The students did well with this section of the book. It was a good review. They didn’t see any motivation behind the Taylor form, and these problems seemed to present the greatest problem.” Generally, Terry’s comments regarding the calculus reform project were positive. For example, during a phone conversation after the observation of the introduction to derivatives, Terry stated that “The students have a better understanding of derivatives. Last year they wouldn’t have been able to understand it as well.” When asked what he meant, Terry responded: “The graphing calculators just help them to see it better.” Although this appeared to be a positive statement regarding the use of the calculators, Terry did not actually describe what he expected his students to know with respect to derivatives.

As a required component of the calculus reform project, Terry used the Dick and Patton (1992) calculus text as his main text. Although discussed in greater detail in the following sections, it should be observed that Terry closely followed the new textbook. However, he did continue to supplement his teaching with the book he used for calculus the previous year, using it primarily for example problems. Terry also discussed supplementing the text with other texts focused on conceptual learning.

Perceptions of the Concept of Limit

Terry’s perception of the concept of limit appeared to change in two ways throughout the data collection period. First, prior to participation in the calculus reform project, Terry described the concept of limit dynamically by describing a limit as “the value a function approaches when it approaches a value of x.” After teaching the unit on limits, Terry responded using a quantified form with “The limit L of a function at a point a does not depend on the function value at a, but exists if we can get values of the function as close to L as we wish by restricting x to a sufficiently small interval about a, but excluding
Coinciding with this change from a dynamic to a quantified conceptual approach, he also changed the language in which he discussed the concept of limit. In fact, by the end of the data collection period, he appeared to use language that reflected the flavor of the text: “If we restrict the input of a function to a sufficiently small interval about a specific value we get an output close to the limit of the function.” In addition, after starting to teach limits he appeared to be stronger in his convictions and more confident in his understanding.

Terry commented that he really did not understand the meaning of the statement:

$$\lim_{x \to a} f(x) = L,$$

until he started to teach it. When Terry responded to what he thought about when he saw this notation, prior to participation in the calculus reform project, he had again responded dynamically with, “When the value of $x$ gets close to the constant value ‘$a$’ the value of $f(x)$ approaches the value of ‘$L$’.” After teaching the unit on limits, however, Terry changed the language style he used to describe the notation. The new style reflected the flavor of the new textbook, specifically the use of graphical representation. Nevertheless, he continued to use a dynamic form when he responded with “I think about moving closer and closer (narrowing in) on the point where $x = a$, zooming in on a point on a graph and looking at the behavior around that point.” Similarly, in his final questionnaire completion, he stated that “I see a picture of a magnified region of a line representing some function. As we zoom in on a point we increase the magnification until we are sufficiently close to the point.” Thus, Terry perceived the concept of limit in both a dynamic and quantified manner. The change in language style used to describe the concept of limit was consistent throughout all modes of data collection.
Perceptions of the Role of Limits in the Teaching of Calculus

Terry felt that the concept of limit was extremely important to teaching calculus. Even prior to participation in the calculus reform project, he had mentioned the need to intuitively understand limits in order to “appreciate the derivative and other calculus concepts.” Questionnaire responses invariably indicated the importance of the concept of limit:

The concept of limit is essential [to teaching calculus]. The “limit process” transforms elementary mathematics into calculus. Using the “function machine” as a model, the limit machine generates new calculus formulas from old pre-calculus formulas (i.e., slope of a line approaches slope of a curve).

In an informal interview, Terry reiterated what he had asserted in his questionnaire responses, stating that “The limit process reformulates elementary mathematics into calculus. Without the limit process, calculus just becomes a new collection of formulas. Both the derivative and integral are based on it.”

As previously discussed, the significant role of limits was also conveyed in Terry’s diagrams of the important concepts and relationships in calculus. Though Terry’s first diagram (Figure 27) did not specifically list the concept of limit, in Terry’s discussion of the diagram the concept of limit was mentioned as “important to understanding calculus.” Each of the subsequent diagrams (Figures 28–30) listed the concept of limit specifically as part of a “calculus machine,” used to solve “problems beyond the scope of pre-calculus mathematics.” Terry stated that Figure 30 probably best demonstrated his view of the relationship of the concept of limit to other calculus topics. Though it appeared as though the “calculus machine” analogy was no longer used, Terry continued to describe calculus as a machine.

Terry’s view of the connection between the concept of limit and other calculus topics was also demonstrated in his instructional practices, as illustrated by his lesson on
the definition of the derivative. Terry composed overhead transparencies demonstrating the limiting process of finding the tangent line to a given curve. In a telephone interview directly following this lesson, Terry observed that "I just kept laying one [transparency] on top of each other until one of my students responded with, 'Hey, that's a limit'." He was quite enthusiastic in his discussion of this lesson. He thought "The students really understood the limit definition of the derivative" with this type of lesson. Though that particular lesson was not observed or videotaped, the lesson for the following day was observed, involving a review of the definition of derivative lesson as well as extending it to discussion of the derivative as a rate of change. During this observation, students asked questions regarding the limit definition of derivative.

Throughout data collection, Terry felt that the most appropriate level of rigor with regard to the concept of limit was "an intuitive approach." Terry discussed this approach by calling for "An intuitive understanding. I want my students to have a picture of what a limit is. This may not be the traditional epsilon-delta picture, but it must be a picture they can relate to—a picture of the limiting process." Terry further discussed that he believed "A high level of rigor is important using a very visual-graphical approach," suggesting that this approach meant "using a graphing calculator to look at the epsilon-delta definition [of limit]." Moreover, he suggested that this approach may be the best way for students to understand limits. Finally, Terry stated that what he expected from his students was the ability "to understand limits at an intuitive level. Sometimes we go through the formal epsilon-delta arguments, but most kids don't get a good grasp of that, so I feel happy when they just understand the graphical approach." Terry did not expect his students to work through formal epsilon-delta proofs using the definition of limit, however, he felt that they visually and intuitively understood the definition of limit.
The Teaching of Limits

Terry spent just over three weeks on Chapter One pre-calculus topics (see Appendix B for an outline of Chapter One) in preparation for the unit on limits. Terry intended to start the unit on limits a week earlier, however, the students' pre-calculus quiz results were bad and he thus needed to review for an additional week. Considering the 96 minute class periods as one and one-half days, Terry spent a total of 10 days on the limits unit. One day was devoted to the limit unit examination and one day was given to review. Again, the sequence of the major topics discussed on the other eight days coincided with the major sections of Chapter Two in the textbook: (a) what are limits, (b) definition of limit, (c) continuity, and (d) analyzing discontinuities and asymptotic behavior. Terry also appeared to cover each of the subsections (see Appendix B for an outline of Chapter Two). Though in most cases Terry covered the subsections in textbook order, he occasionally skipped topics to cover them at the end of the unit. Terry did not provide any explanation to the students or the researcher concerning the order of the various topics. For example, Terry stated in class one day, “On page 76 there are some properties on limits.” A student asked, “Why are we going back?” Terry responded, “Well, we didn’t touch on this and I thought we better touch on this.” Directly following the discussion of the properties, Terry told the class, “Another thing we didn’t cover was the ‘Squeezing’ Theorem on page 77. Everybody read over that and see if you understand what they are saying.” The students were given time to read it and then Terry spent time showing how the theorem worked using an example.

Starting with the completion of the first questionnaire and interview, Terry consistently maintained that his favorite way to explain the concept of limit to beginning calculus students involved an intuitive graphical approach. For example, he provided the following response on the second (planning to teach) questionnaire:
I like to have students graph $y = x^2$ on their calculators and zoom in on the graph in the $x = 1$ region. They see that the graph goes through $y = 1$ because $y = 1^2 = 1$. "No biggie" they say. Then I have them graph $y = (x^2 - 1)/(x - 1)$ and zoom in around the region $x = 1$. Here we discuss how the function is undefined at $x = 1$, but the graph looks like $y = 2$ when $x = 1$. Which brings us to the limit of $(x^2 - 1)/(x - 1) = 2$.

After teaching the unit on limits, as well as at the end of the data collection, Terry indicated his favorite way to explain the concept of limit to beginning calculus students similarly:

Graphically. The most useful example (user-friendly) to use when first introducing limits is the limit of a value of $x$ where the function has a removable discontinuity. At that point I like to zoom-in (using a graphing calculator) and demonstrate that as the $x$-values approach the undefined value the $y$-values constrict around a value.

Coinciding with his questionnaire responses, Terry's actual lesson on the introduction of limits involved an intuitive graphical examination of the limit around a variety of discontinuities. It was interesting that the visual picture Terry used happened to be a redrawn image of the graph found at the beginning of the limit chapter in the text (Figure 31). Terry then had the class examine the values of the limit of the function when $x$ equals $a$, $b$, $c$, and $d$. Following the order of the book, Terry then examined one-sided limits with respect to the graph.

Adhering to the order of the topics in the text, Terry's next two lessons involved the presentation of the textbook example,

$$\lim_{x \to 0} \frac{\sin x}{x} = 1,$$

first using a numerical argument, then creating a table using the SOLVER on the calculator. This numerical approach was followed the next day with the graphical argument, using the graphing calculator. After each of these arguments, Terry commented in his journal, respectively, that "The students seemed to follow the material easily" and
"The students seem to be getting a sound intuitive understanding of limits." He did not discuss how he became aware of such student understanding. It is possible that he based this assumption on the students providing correct answers to his questions throughout the lesson.

Again, starting from the first questionnaire completion, Terry responded consistently to the questionnaire item involving what different arguments he had seen to support the following statement:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$ 

Terry invariably answered with two methods: the Squeeze Theorem and graphically. In most cases he included the numerical method of investigation with the graphical. He wrote, "Before graphing utilities were available, these graphs were generated numerically—so I guess numerically also." In all circumstances, Terry felt that a "graphical approach using a graphing calculator" argument was most convincing to both
him and his students. He discussed the graphical approach in greater detail during each of the interviews. For example, in the first interview he maintained,

We do graphing by using tables and substituting values in for $x$. Graphing is most convincing to me and my students. For some reason some students aren’t convinced that the graph gives all the data, so then if they also see it in a table they are more convinced. Some kids are just number crunchers. So, I use both [the Squeeze Theorem and graphing], but graphing is my favorite.

Terry also declared that the graphical approach was more convincing because it “doesn’t involve all of that mystical math, they can see the results.” Thus, Terry covered the example in the manner he found most convincing for both him and his students.

The next lesson followed the definition of limit section in the textbook quite closely. Terry provided motivation for the definition of limit by reviewing a limit example from the textbook exercises,

$$\lim_{x \to 0} \frac{|x^2|}{\sqrt{(x + x^2 + 4) - 2 - .25x}},$$

both numerically and graphically. This particular example had a right-hand limit which is defined, but an undefined left-hand limit. The students had a difficult time trying to find the left-hand limit as the function oscillated towards infinity near the limiting point. The class discussion of this graph led to the definition of the limit in the following way:

Terry:  So, on this problem, [the limit] appears to what? Does it exist or not?

Students: Not exist.

Terry:  For awhile [the limit] was approaching something. Do you think something funny is going on with the calculator? Can you depend on it?

Students: Yes.
Student: Yes. What else can we depend on?

Terry: Remember the accuracy or inaccuracy of the calculator. So up to a certain error range we thought we had a limit. So, within a certain accuracy we had a limit. What we really need to do here is look at the formal definition of a limit. Here's what a limit is according to the gospel of your book. On page 71 you see something like this:

\[
\lim_{x \to a} f(x) = L
\]

if and only if for all \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that

\[|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.\]

(Terry reads it aloud.) The function \( f \) has a limit \( L \) at \( x = a \) if and only if for all epsilon greater than zero there exists a delta greater than zero such that the distance between \( f(x) \) and \( L \) is less than epsilon whenever the distance between \( x \) and \( a \) is less than delta and greater than zero.

Student: Overload, overload!

Terry: That's what I thought when I first saw this.

From this point in the lesson, Terry immediately proceeded to discussing the graphical interpretation of this definition using the exact diagram given in the book (Figure 32).

Terry: What the graph is trying to say is you have a function and the limit is going to be within these epsilon units, and if it is within this, then there has to be this delta window for \( a \). (Pause.)

Given a vertical view, if there is a limit, then there should be a horizontal view where that \( x \) is going to be. In other ways you should be able to box in this graph.

Student: I don't follow you.

Terry: Let's look at an example. Page 72, example four.
Figure 32. Terry’s Graph of the Formal Definition of Limit.

Terry closely guided the students through the example. The objective, according to the text, is that given an epsilon one can set the vertical range, using graphing technology, from $L - \varepsilon$ to $L + \varepsilon$. Once the vertical range is set, the challenge is to rescale the horizontal range such that the graph of $f(x)$ enters from the left and leaves only from the right (with possible exceptions of a hole or jump). Thus, the well-behaved graph “fits” the given epsilon. By looking at the corresponding values of the horizontal range, the student has found the delta that works for the given epsilon.

After completion of the first example, Terry did another example from the textbook. Making connections to previous representations, this example was once again finding the limit of $\sin(x)/x$. The following segment is from the discussion during presentation of this example:

Terry:

Look at example five. They [the textbook authors] go back to that function we started off with a couple days ago, $\sin(x)/x$. Do you remember what this limit came out to be?
Students: One.

Terry: Suppose we wanted to find a limit within a certain amount of accuracy, say \( \varepsilon < 0.005 \). We can do this graphically by doing what we just did. Let's use a horizontal zoom.

Student: Wait, now in order for this to work does it have to go through the sides [of the calculator screen]?

Terry: Yes.

Student: So, you want the delta to be as big as it can and still fit?

Terry: Yes.

Student: How did you get .005, did you just chose any number?

Terry: Yes, we [he and the book] just chose .005.

(They enter the equation into the calculator, and they look at the standard graph because they can't remember what it looks like.)

We want to set the window to what?

Student: .995 to 1.005.

Terry: (They graph it individually and Terry graphs it on the overhead projection device.) Oh my gosh, look what happened. It's all scrunched up. Go to your zoom screen and notice the HZIN [horizontal zoom-in]. Push it.

(Everyone does this.)


Terry: So, we want to keep doing this until the line goes through the sides.

Student: It has to go through the sides?

Terry: Yes. Now it worked so we need to find out for what delta it worked. You could use trace or we
could get out of here and just look at the domain and range values.

(The students find it.)

Students: Cool!

Given the previous example, the researcher’s impression at this point was that the students were really more interested in finding the correct technique for solving this type of problem than in actually understanding what was happening. Their questions always seemed to be procedurally-oriented instead of conceptually-based. Terry spent a considerable amount of time on problems that involved this type of graphical interpretation of delta. As indicated by the students’ questions, however, Terry was having difficulty getting this point across to his students. In fact, the next day the majority of the time was spent on going over the assignment given on these types of problems. Terry also observed that the students had difficulty with this type of problem on the exam.

In a journal entry written directly after the graphical interpretation of the definition lesson, Terry wrote that he felt the “Students seem to have a clearer idea of what the epsilon-delta arguments are all about using the graphical zoom-in approach.” The following day, Terry covered the related homework problems in great detail. In addition, he deviated from the text to some degree to briefly demonstrate a traditional epsilon-delta proof for a linear function. Following the daily lesson, he commented that the “Students aren’t real clear on proofs but feel comfortable with epsilon-delta on graphing calculators.” Terry felt he spent more time this year than he ever has in the past on limits. He commented on why the increased time was spent by stating that “In previous years we discussed the graphical meaning of epsilon-delta, but didn’t have the tool (calculator) to actually narrow in on the delta.”

Terry continued to follow the organization of the text throughout the remainder of the limits chapter. As previously discussed, in several instances he briefly covered the material and then had the students read the text to obtain further understanding. For
example, though Terry consistently brought up continuity throughout the unit on limits, for the lesson on continuity Terry simply had the students open their textbooks to page 82, read the definition, and then worked the given example from the text. The following segment illustrates the dialogue that took place during the lesson on continuity, for which Terry again used the diagram given in the text. In fact, this time he did not make a transparency, but had the students look at the diagram in their textbook (Figure 33).

![Terry's Graph Used for the Introduction to Continuity.](image)

Figure 33. Terry's Graph Used for the Introduction to Continuity.

**Terry:** If you turn to page 82, they talk about continuity and stuff on this page. Notice what they write down. Notice that in order for continuity to exist they have three little laws; the value exists, the limit exists, and [the value and the limit] are the same. Then you have continuity at a point. To understand this, look at example 11 on page 84 (Figure 33). Where is it continuous?

**Students:** A-two and a-six

**Terry:** OK. It is obvious at these points because?
The students answered correctly by commenting that all of the “three little laws” were satisfied and they continued discussing this example. The lesson ended with Terry assigning a group of problems for the students to work during the remainder of the class period.

Analogous to his desire for students to obtain a graphical interpretation of limits, Terry’s exam was composed predominantly of graphical interpretations of given limits. In fact, more than 80% of the exam involved interpreting a graph to obtain the limit. Though on most problems students could choose any method for finding the limit, only one of the problems had to be solved numerically since Terry had asked for a table of values. In addition, these values could have been obtained from a graph if the student chose to do so. The exam appeared to be well thought out and corresponded to what Terry indicated he expected his students to know. Furthermore, the questions mimicked the exercises in the text. Terry listed his test scores as nine A’s, seven B’s, and one C. Overall, Terry indicated he was happy with the results. He commented in his journal:

Students were able to interpret limits from graphs, but had more difficulty making graphs from given limits of an interval. They also had a little difficulty estimating a delta from a given epsilon graphically and a bit of difficulty with infinite limits.

Summary of Terry

Terry consistently viewed calculus as a linearly ordered problem-solving tool. Given a complicated problem, one would feed it through the “calculus machine” to find a solution. The obtained solution possibly led to additional problems, thus starting the whole cycle over again. Though the concept of limit was recognized as part of this machine, it was not illustrated as a part of calculus. Terry stated that the concept of limit was important to an appreciation and understanding of other calculus topics.
In the year of the study, Terry had four major goals for teaching calculus: (a) to prepare students for upper level college mathematics, (b) to prepare students for math-related careers, (c) to prepare students to be problem-solvers, and (d) to develop thinkers. Though Terry did not specifically mention the goal of preparing students for the AP exam, it was incorporated into the goal of preparing students for college mathematics. Terry's classroom was characterized by constant student discussion. Observation of these discussions revealed that each of the goals was to some extent addressed in the classroom. Though much of the discussion appeared to be initiated because of the graphing calculator, discussion of college mathematics and math-related fields of study were frequent topics discussed. The problem-solving goal was met as Terry followed the textbook example-by-example and increased problem-solving was a stated emphasis of the calculus reform project. It was difficult to determine to what extent Terry was devoted to his goal to develop thinkers. However, Terry did use an inductive approach, having students examine graphs on their own calculator and then arrive at generalizations.

During the course of the study the manner in which Terry discussed his perception of limits, as well as the language he chose to discuss the concept of limit, changed. Prior to participation in the calculus reform project, Terry described the concept of limit in a tentatively dynamic manner. After teaching the unit on limits, Terry began to use a quantified approach to describe the concept of limit. At the end of the data collection period, Terry used a combination that was both dynamic and quantified. As time progressed, the language he used to describe the concept of limit changed to reflect the flavor of the text and Terry gradually appeared more confident in his descriptions. Finally, from the beginning of the study, Terry commented that it took teaching the concept of limit to really understand it.

Terry felt that the intuitive notion of limits was extremely important in the teaching of calculus. At the beginning of the study, Terry suggested that the concept of limit was needed to appreciate other calculus topics. During the study, Terry would add that the
concept of limit was important to an understanding of other calculus topics. The connections to other topics and importance of the intuitive concept of limit was also reflected in his classroom practices. Terry felt that it was important that his students visually and intuitively understand the definition of limits, however, the formal epsilon-delta arguments were insignificant. He felt that most students could not handle the rigorous epsilon-delta arguments, thus he would be pleased if they were able to understand the graphical approach.

Terry's classroom lessons were consistent with his professed teaching strategy of introducing calculus topics using an intuitive graphical approach. Terry felt that this type of approach to the concept of limit would be most convincing to his students. Terry used a variety of representations throughout the unit on limits, but consistently appeared to give the most emphasis to the graphical approach. In fact, even his unit exam involved questions that primarily could be solved graphically. These problems could also be solved using other representations, such as algebraic manipulations or numerically.

Terry followed the text quite closely and often taught directly from the text. He had students read a portion of the text during the lesson and then they would discuss it. In addition, he would closely guide students through every step of some of the textbook examples and exercises. Furthermore, he would occasionally have the class examine the graphs in their text. Students were observed writing directly on the graphs in the textbook.

Terry felt that he spent more time on limits in the year of the study than he had in the past. He attributed the increased time to use of the graphing calculator, since it provided a way to easily examine the graphical meaning of the epsilon-delta definition of limits. Overall, Terry's comments regarding the calculus reform project were positive.
Scholastic and Professional History

In 1967, Tom graduated with a liberal arts degree from a small college in the Northwest U.S., majoring in mathematics with a minor in architecture. From 1970 to 1973, Tom pursued a degree in education from a larger Northwest U.S. college. Tom stated that it was after this period of professional development that he wanted to teach mathematics, “I wanted to work with students—and [mathematics] was the area [in which] I had the most knowledge and understanding.” Though this comment appeared to make the importance of subject matter, mathematics, seem secondary to his desire to work with students, Tom felt that he had a “good background to teach mathematics,” but he briefly described his overall academic preparation as “diverse, fits and starts.” For example, he discussed his only real memory of past calculus classes as “Some were good, some bad.” Moreover, he felt that his background was appropriate to be able to “talk to students about constant career and job changes and the broad background that is needed.”

Tom was in his twenty first year of teaching mathematics. The mathematics courses he had taught ranged from remedial seventh grade math through AP calculus. Tom was currently teaching pre-algebra, first year algebra and AP calculus in a high school located in a rural logging community of approximately 10,000 residents. At this high school there were 850 students, predominantly white, in grades 10–12. The only other job responsibility Tom discussed was supervision of a study-hall.

Since Tom became a mathematics teacher he has participated in several related professional activities. He acknowledged that he was a successful grant writer, based upon his selection to participate in the previously mentioned calculus reform project. He has also participated on the District Curriculum Committee, and had taken part in the Mathematics and Science Teacher Improvement Project and two Woodrow Wilson
Summer Workshops. The details of these workshops were not discussed. Furthermore, Tom briefly discussed being an active member in a few other State Department of Education groups throughout his teaching career.

Tom was selected for the sample of this study based on his participation in a particular calculus reform project focused on teacher development. On the first day of the two-week inservice accompanying this project, Tom was invited to take part in this investigation. He accepted the invitation without reservation. Since classroom observations were a necessary part of the data collection, Tom first had to check with his principal.

The first questionnaire completion and interview took place on the first and second days of the two-week inservice, respectively. The timing of this inservice was approximately three weeks prior to the beginning of school. During the interview, Tom commented that it was summer and his mind was not directly focused on calculus yet. He cautioned that his responses may be quite different once he was in the flow of the school year. In the first interview he stated:

I look at the calculus class and I forget half of it, summer and all—and then I get back to it and think, oh yeah, that’s important and this is important. But the questionnaire answers and diagram was just what was at the top of my head. I’ll probably add more theoretical and textbook stuff when my head is in it during teaching—it’s so easy to get my mind off on other things today.

Though Tom was pleasant and outwardly cooperative, his casual nature made it difficult to schedule classroom observations. The best way to schedule an observation time with Tom was to call the night before the desired time because scheduled observations were frequently canceled due to pep assemblies, staff meetings, or quizzes. This scheduling problem made it difficult to arrange observations with the other teachers participating in the study. Equally difficult was his preference not to have any of the
classes videotaped. Though this decision was possibly due to an equipment availability problem, it was also apparent that Tom did not choose to have his lessons videotaped.

Overall, the difficulties regarding data collection for Tom did not appear to be avoidance tactics, but more due to his preparation style, based upon planning only one day in advance. Tom frequently stated that he was behind schedule and often commented on his last-minute planning. The following is an example of a phone conversation that occurred the morning of a requested observation time:

Researcher: Will this morning work out for me to come to your class, or will tomorrow be better?

Tom: Well, I may finish up Section 2.4 today on limits, but we need to finish up talking about grades—review asymptotes. I’m not really sure what the plan is for the rest of the week. I’m not even sure what the plan is for tomorrow.

Researcher: I’ll come today then since I have another observation I can do tomorrow.

Tom: OK. I know my eyes are open, but I don’t think I am awake. See you later.

Tom spent half of the class period finishing up Section 2.4 and understanding asymptotes, and the remaining 25 minutes were spent covering any questions the students had on old homework problems.

Tom’s basic philosophy of education seemed to reflect this nature as he wished to help students prepare for a good life. He stated that his role as a teacher was “to teach students so they understand the material—the level should be high enough to challenge them, but attainable. To get them ready for life. To try to teach them to overcome failure. Work at socialization.”
Portrait of the Calculus Classroom

Teaching AP calculus had been a part of Tom's life for the past 10 years, and he anticipated that it would remain part of his routine for years to come. Seventeen students signed up for AP calculus the year this study took place, however, Tom felt three of the students should not have been there. In fact, toward the beginning of the data collection period, Tom expressed doubts about the abilities of the entire class. For example, after the first exam, Tom commented that "The test was not good. I am down on their attitude. I don't think they are bright. This is the poorest senior class in awhile. We need to have a big talk on what it means to be an A or B student." At the end of the data collection period, there were 15 students in the class and Tom described his remaining calculus students as "bright and somewhat hard working, however, many of them want to get through the class, get the grade, and are not interested in learning the material." He also discussed the level of support for the calculus class as "Fair—people know it is tough and that it is the most advanced math we have—but they have no concept of what it is."

Tom's desk was placed off to the side in the front of the classroom. Chalkboards extended across the entire front and back of the classroom, though Tom only used the front chalkboard. One side of the room contained a large bulletin board, on which Tom kept a giant calendar with all assignments and exams for each of his classes. The other side wall was filled completely with windows. The room contained approximately 40 desks arranged in six rows.

At the beginning of the school year, Tom's class was set up with an overhead projector in the front of the room for the graphing calculator projection device. This set-up was similar to the set-up of the two-week inservice. As time passed, however, the overhead projector disappeared from the room. One day Tom wanted to demonstrate a graphical representation of a limit and he spent five minutes trying to find an overhead projector. Tom commented on his lack of calculator use in an informal interview. "I
don’t really need my overhead calculator anymore. The students work on their
calculators. Some of them are better than me at it. They don’t need much help from me,
they help each other.”

Tom’s calculus class met every morning for 46 minutes. Generally, Tom was at
his desk going over the material to be covered in class when students entered the calculus
class and took their assigned seats. The students would visit quietly and take out their
homework to be handed in for the day. As soon as the bell rang Tom would stand up and
ask if there were any questions on the homework. Some days Tom felt the need to give
the students a “pep-talk” on the importance of completing their homework. For example,
one day he stated to the students, “Most of you get the surface of these mathematical
concepts, but you really just need to dig a little deeper.” Each day approximately 20
minutes was devoted to going over the assignment from the previous day as well as any
other questions the students might have. Generally, the class was quite attentive and
asked questions regarding how to obtain solutions to the homework problems. Tom often
appeared to be working the problems for the first time, since he would occasionally take
large amounts of time to figure out how to work the problem. Sometimes the students
were asked to put their answers on the board. At the end of this time period, the students
would hand in their homework.

After going over students’ questions on the homework at the beginning of the
class period, Tom spent from 5 to 25 minutes lecturing on new material. Considering all
the classroom observations by the researcher, Tom’s average lecture time was 12.2
minutes. This length of lecture was typical throughout the observation period. Based on
the comments and actions of the students, it was assumed that Tom seldom lectured and
possibly conducted the class more like an independent study. The reader is reminded that
only seven observations took place and that none of the lessons were videotaped.

Towards the end of the data collection period, Tom commented that his basic
organization of the calculus course involved mostly lecture and some small group work.
He remarked that he planned to move toward some individual projects, “but this is a slow moving future direction.” Other than working in small groups to do homework, the researcher never observed small group work or individual projects.

Each of the observed days also included a period of time at the end of the class, when the students were allowed to work on the new assignment. Throughout this period, the atmosphere of the class was quite relaxed and the students were on-task for the most part. The students would frequently ask Tom questions. If more than a few students wanted help with a particular problem, then Tom would work the problem on the chalkboard. Tom would often remain at his desk throughout this segment of the class period and answer questions from there. The students also interacted well with each other. They were constantly asking each other questions and comparing graphs they obtained with their calculators. As the end of the class period drew near, the students became more interested in talking about things other than calculus. For example, one day the discussions also involved the calculator, chemistry, college, Spring break, the football game, and study hall.

Perceptions of Calculus and the Teaching of Calculus

Tom had a general view of calculus as demonstrated in his first diagram of its important concepts and relationships (Figure 34). Tom interpreted his diagram by explaining that calculus was “Relating the infinitely infinitesimal, that’s what calculus is!” He continued this explanation by discussing a science fiction story he was reading that was to some effect related to the idea of the infinitely infinitesimal. He then went on with his discussion of the diagram by stating that parts of the diagram were listed based on his past calculus courses:
Figure 34. Tom’s Diagram of Calculus, Prior to Beginning of School.

I put down acceleration, speed, and velocity because it was a really neat concept when I was taking calculus 20-some years ago. There are lots of neat relationships [in calculus], and anybody can relate to it. This leads to real world problems such as auto industry (expansion) and heat flow.

After this discussion, Tom responded that he felt he did not necessarily list all the important concepts in calculus. This diagram represented his instantaneous overview of
calculus while he was in what he described as his "summer mode." He suggested that this
diagram would be modified when he was actually teaching. "I partially forget all the
concepts in calculus when I am not teaching it. If you give this to me during the school
year it may be different—it won't change, things will just be added to it while I'm teaching
it." These missing elements included topics such as limits, differentiation, and integration.

The remaining three diagrams (Figures 35–37) completed by Tom throughout data
collection, while actually teaching calculus, were quite different from the first. Each of
these three diagrams characterized the important concepts in calculus as limits,
differentiation, integration, the idea of rate-of-change, and the applications coincident to
each of these concepts. In fact, each time Tom completed a new diagram, it appeared to
increasingly demonstrate the connections between the concepts. For example, in the
diagram completed for the planning to teach limits phase (Figure 35), Tom merely listed
linearly the important concepts in calculus. After teaching the unit on limits, Tom
illustrated that a relationship existed between differentiation and integration (Figure 36).
For this diagram, Tom moved away from a linear illustration, but for the final diagram
(Figure 37) Tom moved back to a linear illustration of calculus. In Figure 37, Tom
demonstrated the Fundamental Theorem of Calculus in which it was the connector
between differentiation and integration. Moreover, in this final diagram the arrows were
conscientiously placed to indicate the connections between topics. In a brief informal
telephone interview following the completion of the final diagram, Tom commented, "I
like the picture. Well, I had more time to think about this diagram because we were on
break."

The major goal Tom had for teaching calculus was to cover an appropriate amount
of material so that the students would be successful in future calculus endeavors. He
stated this goal in the following way:
Limits

The idea of instantaneous change
differential calculus

Fundamental Theorem of Calculus

Finding areas and volumes of odd shapes
integral calculus

Convergence and divergence sequences

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Figure 35. Tom's Diagram of Calculus, Planning to Teach Limits Unit.

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Figure 36. Tom's Diagram of Calculus, After Teaching Limits Unit.
I want to cover approximately two terms of college calculus, teach basic understanding of content and ideas—have my students step into the second or third term of college calculus with no trouble or loss of continuity—score three or better on the AP exam.
Specifically, Tom stated that he not only wanted the students to learn the mathematics, he wanted them to be able to communicate it. In an informal interview following a classroom observation, Tom stated:

I think the students need to be able to not only do the math, but to communicate it in writing and orally. That's why I had them get into groups yesterday to explain their answers in writing. I used the analogy of the difference between a good and a great football team is their ability to communicate.

This analogy was of interest to the class as there were four football players in the class and they had a big game that night. Tom discussed this concern for increased student communication throughout much of the data collection period. After grading an exam in which the students were allowed to use calculators, he mentioned the following in an informal telephone interview:

There were lots of complaints on the exam. The kids are hung up on that the HP [graphing calculator] will do it all. So, then the test took a long time. I'm trying to get them away from thinking they have to use the calculator for everything. I'm trying to wean them away from just one number or one word answers. By end of the year they'll understand what I want, I hope.

The extent to which each of his goals for teaching calculus was covered in Tom's teaching is discussed throughout the following sections.

Participation in the calculus reform project required Tom to use the Dick and Patton (1990) textbook, *Calculus* (see Appendix B for a description and outline of the textbook contents). Tom said that he supplemented this textbook with that from the previous years, *Calculus: Alternate 3rd Edition* (Larson & Hostetler, 1986). The type of supplement was not discussed in detail; however, Tom did comment that he used other calculus texts for extra problems and examples.
Perceptions of the Concept of Limit

Throughout the entire data collection period, when Tom was asked to describe the concept of limit in his own words, he responded with a dynamic description of the concept. For example, at the end of the data collection period, he stated, "As x gets closer to a specific value or infinity or negative infinity, f(x) gets closer to a value L called the limit."

When Tom was asked to describe what he thought about when he saw the formal limit notation, \[ \lim_{x \to a} f(x) = L, \]
he responded analogously with a dynamic interpretation. In fact, during the first questionnaire completion, Tom merely responded to this question with "See the answer to [question] above." In response to this question in the other three questionnaire completions, Tom responded similarly with, "I visualize in my mind a line. As I get closer to a specific value of x, the line gets closer to some value f(x), which I associate with a boundary line L, the limit of the function at the point a."

Tom's responses to when he really felt he understood the meaning of the notation seemed to refer to a formal quantified understanding of the epsilon-delta definition of limit, not an intuitive dynamic interpretation. Throughout the data collection period, Tom stated that teaching the concept of limit was required to really understand it. For example, in the first questionnaire completion, Tom responded "I don't remember my understanding of the statement before I started teaching calculus." In the interview following the completion of this questionnaire, Tom reiterated this comment, stating that "Some of my past calculus classes were good, some bad, but teaching it brought it all together." After teaching the unit on limits, Tom still commented that it required teaching to really understand it: "The first couple of years I taught calculus and I had to explain to students who were having trouble understanding the concept. When I saw some of the different
view point students had and I needed to explain the concept coming from their point of view.” Similarly, at the end of the data collection period, Tom communicated that he felt he really understood the meaning of the formal limit notation “during my first term of teaching calculus. We spent a third to half of the term on the idea of limits, using supplementary books on limits.”

**Perceptions of the Role of Limits in the Teaching of Calculus**

In response to both questionnaires completed prior to teaching the concept of limit, Tom indicated that he felt the intuitive limit was “fundamental” to teaching calculus. He stated that “I feel it is one of the basic intuitive ideas needed in the formation of the foundation of calculus.” After teaching the concept of limit, Tom added that though one could probably do calculus without understanding the concept of limit, one could not “understand calculus without understanding limits.” Tom extended his answer in the interview by stating,

I don’t think you can get a fundamental understanding of the basic concept of where calculus comes from without it. You can teach calculus without it, but you can’t teach the understanding of calculus without it. If you don’t have limits, then you don’t have calculus.

When Tom was asked what he meant by “understanding limits,” he responded “understanding limits intuitively in an informal sense.”

Tom’s desire that his students intuitively understand limits directly correlated with his belief that the appropriate level of rigor with respect to limits was “short of epsilon-delta proof.” Each completion of the questionnaire reproduced the same response. After teaching the unit on limits Tom discussed this response in greater detail in an informal interview.
Tom: I know the stress on [formal] limits is less today than it was 20-30 years ago. Advanced calculus may need to spend more time on [formal] limits. They used to spend more time on epsilon-delta proofs. They thought it was important to prove everything about limits. I don’t think in a basic course in calculus that it is so important. If you are a college math major it may be more important, but not in a high school course.

Researcher: What exactly do you feel is the appropriate level of rigor for teaching your calculus class?

Tom: They need to know that there are proofs for [limits and limit theorems], and maybe even see one, but not have to reproduce on a regular basis and not even see them on a regular basis. They need to know what a limit is. They need to know all the different forms a limit can take . . . to infinity, to zero, symbolic manipulation, graphical representation, real world examples—just short of epsilon-delta. Either skip induction proofs altogether—or do just the simple ones.

As previously noted, Tom’s diagrams of the important concepts and relationships in calculus (Figures 34–37) became progressively more organized with respect to demonstrating the connections between listed calculus concepts. Even in Tom’s last two diagrams (Figures 36–37), however, the concept of limit appeared to be connected only to differentiation and not necessarily to integration or any other topic. Though the researcher scheduled a classroom observation on the day Tom said he would cover the definition of derivative, the actual 12- minute lecture that took place involved a presentation of the derivative as a rate of change. This lecture corresponded with the order of the textbook (see Appendix B for detailed outline). The concept of limit was not mentioned throughout the lecture, though an intuitive understanding of limits would have been helpful for understanding instantaneous rates of change. In an informal interview, Tom stated that he did cover the definition of the derivative the following day. He felt that the students “understood the idea of the limit of slopes of the secant lines. They
could see how limits relate to the derivative.” The reader is reminded that coverage of this relationship in class was not directly observed.

Furthermore, though the lesson on the formal definition of the definite integral was not observed by the researcher, Tom stated that he discussed the formal definition in class. The textbook contained a section which involved the formal discussion of the definite integral as the limit of the Riemann sum (see Appendix B for information on Section 6.2). Based upon Tom’s comment and his pattern of closely following the textbook, it was presumed that the concept of limit was indeed presented at this juncture. The detail in which it was covered and the type of connections made, however, were not observed.

The Teaching of Limits

Tom spent approximately three weeks prior to teaching the unit on limits reviewing pre-calculus material. Seventeen school days passed from the time Tom started the unit on limits until the unit exam on limits was given. Due to illness, Tom had a non-mathematics substitute teacher in his class for five of these days. Three days were devoted to either review or working on worksheets. Two days involved the administration of a quiz and an exam. The remaining seven days contained lessons on the given sections in the textbook (see Appendix B for outline of Chapter Two). Though a description of the format of these class periods was given in the previous section, the reader is reminded that most of each class period was spent with Tom answering questions about either old or new homework assignments.

Prior to teaching the unit on limits, Tom said that his favorite way to explain a limit to a beginning calculus class was numerically, “Taking the function and plugging in numbers and forming a chart and from that chart making a graph.” This response replicated Tom’s desire to first intuitively explore limits in a dynamic manner. This desire was demonstrated throughout the data collection period. The actual intuitive method used
in the classroom when presenting the concept of limit was first a graphical approach, followed by a numerical approach. In the final two questionnaires, Tom responded with “I like to start with a drawing so the students will have an intuitive feel for what a limit is.” He discussed that he liked the drawing from the textbook (Figure 38).

![Graph of limit](image)

Figure 38. Tom’s Graph Used for Introduction to Limits.

Tom did introduce the concept of limit to his students intuitively by looking at a graph, using the graph given in the textbook (Figure 38). In an informal interview, Tom commented that he just had the students look at the graph in the textbook and they discussed the limits at the different points. Though the first observation of Tom’s lesson on the concept of limit did not occur until the second day of the unit, the examination of the textbook diagram was repeated as the students were still uncertain about the meaning of a limit. The lesson began with the following classroom discourse:
Tom: Today, I want to talk some more about limits. Yesterday I gave you just three problems to do. We sort of defined a right-hand and left-hand limit, and we discussed the limit at each point in the diagram in the book. I think we even finished the day by saying that if the left-hand [limit] equals the right-hand [limit] then we have a limit. Does everyone understand?

Ryan: What page are we on?

Shannon: What is a limit again?

Bill: All you want us to do is graph these things right?

Tom: Hang on to your homework for a little bit. Let’s look at a graph again. We need to look at what it means to have a limit.

Tom drew individual points of discontinuity from the diagram in Figure 38 on the chalkboard. The review of these limits was brief, but from their correct responses the students appeared to be understanding. For example, each time Tom asked what the limit was at a particular point, several students provided the correct answer and briefly justified why their answer was correct.

Tom quickly jumped to a discussion on finding limits numerically using the calculator. His lesson corresponded directly with the next example, the numerical estimation of

$$\lim_{x \to 0} \frac{\sin x}{x},$$
given in the textbook. This numerical discussion involved examining the limit from both the positive and negative side by creating a table of values. The students were motivated and seemed to enjoy helping Tom fill in the table he had drawn on the chalkboard. An interesting discussion occurred at the end of this lesson when a student asked, “Why can’t we just graph it?” Tom responded, “What I want you to do for the homework is to make tables. I don’t really like what the author did on this part, but I didn’t have time to make out a worksheet. But I don’t like the questions the author asked.” Tom avoided the
question about graphing and just put up eight problems on the board and stated, “Here, just do these. You must create a table like we did on the board.” One student responded, “This just looks like a tedious but easy bunch of calculator problems.” This comment was not acknowledged and the students went on working on the homework.

Prior to teaching the unit on limits, Tom responded with the Squeeze Theorem and a table of values as two arguments to the questionnaire item: “What are the different arguments you have seen in support of the statement

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 ?$$

He suggested that he would be most convinced by the Squeeze Theorem and that the students would be more convinced by the table of values. It was not until after teaching the unit on limits that Tom added a graphical approach “using a calculator to make a graph and zoom in and show as x approaches zero, (sin x)/x gets close to one.”

Throughout the data collection period, Tom continued to feel that the Squeeze Theorem was most convincing to him, however, he also discussed both “the graph and table method” as most convincing to his students. In his discussion of the graphical method, however, Tom consistently stated his concern that some students would “fall into the trap of, if the calculator says it, then it must be true.” This concern was demonstrated in the lesson involving the graphical interpretation of the same example given above. Though the students were able to graphically find the limit of the example to be one, their comments to each other indicated they had a misconception as to why the limit was one. Tom was focused on a couple of students and appeared to miss some of the following comments heard by the researcher.

**Tom:** How do we figure out what the limit is from this graph? Does anyone remember what we said the limit was from yesterday?

**Bob:** One.
Dustin: What does it mean that the limit is one?

(As Tom continues with plotting the function a couple of other students in the class respond incorrectly to Dustin.)

Alan: It means that the y-value never gets higher than one.

Joe: You see, one is the highest point on the graph so it is the limit at zero.

Though Tom incorrectly felt that the students successfully discovered the limit on their own, at this point in the lesson Tom appeared to be more concerned with explaining the limitations of the calculator.

Tom: The other thing that happens is that everything is going fine, but you can’t tell that there is a hole at $x$ equals zero. Because of the constraints of the mechanical drawing you may not always get the correct limit. You could fall into some bad choices if you don’t really understand what the limitations of the machine are.

The last couple of pages in the text reaffirm what you guys have just discovered. It’s always better when you discover it on your own.

Though Tom had stated that using a graph was important for students to obtain an intuitive understanding of limits, throughout the unit on limits Tom appeared to be somewhat skeptical of finding the limits using a graph. This skepticism, however, was not just because of his concern regarding the limitations of the calculator. For example, when Tom gave the students homework problems from the textbook involving graphical interpretation of the limit he said to the class, “It was pointed out to me that on the AP exam you are going to have to interpret graphs even though you have this nice calculator.”

In an informal interview prior to this example, Tom asked the researcher what types of questions would be on the exam if students could use the graphing calculator. Thus, it is
probable that Tom assigned the graphical interpretation problems based on this discussion, though he was skeptical of finding limits graphically. In fact, Tom appeared to give the students the impression that more faith could be placed in numerical as well as algebraic representations.

The issue regarding the limitations of the calculator came up quite frequently in Tom's classroom. But though it was frequently discussed, it never appeared to be resolved as the discussions continued throughout the entire data collection period. The following is an example of such a discussion:

Tom: Well, my graph doesn’t appear to be showing the spike like the book. It just may be the way the calculator is programmed.

Dustin: Well, how are we to know? I mean on a test, what if this happens and we don’t know?

Tom: That is why it is important to know the limitation of the calculator. We discussed that this would be a problem right from the beginning.

At this point Tom does not mention the textbook emphasis of multiple representations for the interpretation of limits. Possibly because of this lack of explanation, the students never appeared to be satisfied with Tom’s response. They wanted to be able to trust and believe in the graph the calculator produced. One student, Shannon, frequently became frustrated because she could not be sure of her answer as interpreted from her calculator. The following is an example of one of Shannon's frequent comments regarding this issue, “That’s just fine. I just don’t understand why it won’t tell us the truth, whether it is undefined or not. They should make a calculator that always tells the truth.” Tom responded to Shannon by saying, “We’ve talked about this before. You must understand that this machine has limitations.”
As previously mentioned, Tom made the decision not to videotape his lessons on limits. In addition, scheduling conflicts had prevented the researcher from conducting daily classroom observations. Thus, the lessons on the formal definition of limit and continuity were not directly observed. Informal telephone interviews occurred at least twice weekly, though minimal information was obtained. In general, Tom stated that he was following the textbook at an “easy-pace and having to do lots of review for this group of students.” The majority of the information provided in the informal telephone interviews was how Tom would improve on the textbook and his interpretation of his students’ understanding. For example, he stated,

We just reviewed all day today. We’ll have to spend more time on given an epsilon to find the delta. I don’t like the way it is done in the book. Next time I’ll do it backward and find the delta first and then teach them the proofs. Basically, limits are not going well for the kids. Especially the epsilon-delta proofs. I hate to even mention the word proof or they’ll say, “Where will we use it?” Two students did come up with reasons for using them, but I can’t remember them right now.

The intent of this section was to analyze what Tom actually did in the classroom and not what he said he did. Thus, this section is perforce incomplete.

It was interesting to note that although Tom appeared to follow the textbook example-by-example, he consistently expressed doubts about the authors’ reasoning and reservations regarding the textual presentation of the material. Though based upon the fact that Tom appeared to prepare for his lessons less than one day in advance, he usually did not supplement the lesson in any way. In addition, as previously noted, he frequently commented on a lack of need for certain exercises and the need for supplementary problems. It was interesting to note that the same types of problems he had been reluctant to assign for homework appeared on the end-of-the-unit examination on limits.

Tom administered one quiz halfway through the unit on limits and one exam at the end of the unit on limits. The quiz contained six problems in which the students were
asked to “Use a table of numeric values to find if the indicated limits do or do not exist. If
the limits exists state the limit, if not why not.” Although the quiz involved only one
method of evaluating limits, the exam contained a balance of several representations for
demonstrating understanding of the concept of limit. For example, beside having to create
a table to find the limits, the students were asked to sketch graphs satisfying a given set of
limit notations. Moreover, the students were also asked to determine the limits of a given
graph at specific points, discuss the continuity of given equations, and evaluate limits of
difficult expressions that would require a calculator. None of the problems involved just
plugging in the value to find the limit or using algebraic manipulations to simplify the
expression so one could then just use the value to find the limit. The problems on Tom’s
exam did reflect the goals of the textbook, and the questions on the exam were merely
cut-and-pasted directly from given problems in the textbook. It was difficult to determine
if these questions actually reflected Tom’s desired learning goals for his calculus students.
The questions did match the content of the lessons and the homework problems appeared
to be guided by given textbook sections. The questions did not, however, appear to
match Tom’s expressed concern regarding textbook organization and exercises.

Summary of Tom

Prior to participation in the two-week summer inservice portion of the calculus
reform project, Tom claimed that he was in “summer mode” and that his perception of
calculus and the concept of limit would be modified when he was actually teaching again.
Each time Tom completed the questionnaire it appeared to be increasingly more well
thought out and demonstrated connections between topics. Basically, Tom perceived
calculus in a linear manner divided into three sections: (a) limits, (b) differentiation, and
(c) integration.
Tom’s overall goal for teaching calculus was to cover an appropriate amount of material so the students would be successful in future calculus endeavors. The endeavors included having his students step into the second or third term of college calculus and scoring a three or better on the AP exam. In addition, Tom wanted his students to be able to communicate mathematically. It was difficult to determine the extent to which these goals were covered in Tom’s classroom since it was not possible to speculate on the success of obtaining the first goal from the evidence obtained. Based on the observations, the goal of writing and communicating mathematically was addressed by Tom only minimally. Tom’s class, however, was characterized by constant communication between students and between students and teacher. The students were constantly asking Tom questions and comparing their calculator graphs with those obtained by others.

Throughout data collection, Tom discussed the concept of limit dynamically. A dynamic response was also given when he described the formal limit notation. He appeared to be referring to the quantified notion of the definition of limit when he claimed that he did not remember understanding the formal limit notation before he started teaching calculus. Helping students understand the concept of limit in the first couple of years Tom taught calculus, helped his understanding greatly.

Tom felt the intuitive limit was “fundamental” to teaching calculus. In fact, he viewed limits as “one of the basic intuitive ideas in the formation of the foundation of calculus.” He stated that one could probably do calculus without understanding limits, however, one could not “understand calculus without understanding limits.” In direct correlation with Tom’s desire for students to understand limit intuitively was his belief that the appropriate level of rigor was “short of epsilon-delta proof.” Though the connections between the concept of limit and other calculus topics could not be directly observed, Tom discussed the connections he had made in his lessons. Since his nature was to follow the text, it was assumed that the connections were actually brought up in the classroom lessons.
As previously noted, Tom followed the text quite closely. He frequently claimed that he intended to cover each topic in the text at an "easy pace and having to do lots of review for this group of students." This routine of following the text example-by-example throughout the unit on limits was somewhat contradictory to the fact that Tom frequently shared his doubts regarding the authors' reasoning behind certain textual examples and exercises. In addition, he also had reservations regarding the authors' presentation of material. Tom appeared to prepare for each lesson shortly before teaching it; thus, it was possible that he did not look far enough ahead in the text to acquire an appreciation for the authors' reasoning. Furthermore, Tom's unit exam on limits contained some of the precise exercises he had previously denounced. The exam questions were merely cut-and-pasted directly from the text.

Tom felt that it was important to first teach calculus topics intuitively. In fact, he said that his favorite way to introduce limits was first numerically, followed by graphical representation. In Tom's actual teaching, he first introduced the concept of limit graphically and then demonstrated the numerical approach. This order of representation was identical to the order of the text. In fact, Tom used the graph found on the first page of the limits chapter to start his lesson.

Though Tom felt that a more rigorous argument was needed to convince him of a limit, he felt that students were more convinced by a numerical or graphical representations. Thus, he typically used these types of representations. However, Tom, was concerned about relying on graphs to obtain the limit. He stated that students may "fall into the trap of, if the calculator says it, then it must be true." It was possible that this skepticism of graphical representations was a result of the limitations of the calculator. Nevertheless, Tom appeared to give the students the impression that more faith could be placed in numerical as well as algebraic representations.
All of the teachers in this study perceived calculus as a linearly ordered set of topics. In the context of this study, perceiving calculus as linearly ordered means that the teachers’ primary focus was on the sequence of topics to be taught in order to support the understanding of topics which followed. For example, teachers believed that the concept of limit must be taught prior to the concept of differentiation because understanding limits was a prerequisite to understanding differentiation. In this study the teachers’ perceived set of calculus topics was divided into three main tiers: limits, differentiation, and integration. In some cases, a review section on pre-calculus topics was incorporated as a beginning tier. Subsections containing related applications of differentiation and integration were also frequently mentioned. Terry deviated slightly from the rest of the group of calculus teachers by viewing calculus as a linearly ordered problem-solving tool. His view also divided calculus into the three tiers, however, he additionally discussed feeding complicated problems through the “calculus problem-solving machine” in order to obtain solutions or possibly lead to more problems. Coinciding with the perception of calculus as a linearly ordered set of topics, each teacher’s perception of calculus mirrored the section order in the textbook.

All of the teachers recognized limits as a principal component of calculus. For example, Richard felt that a solid foundation in algebra was of utmost importance to doing calculus, but the concept of limit formed the backbone for understanding all other calculus topics. Ryan best summed up the teachers’ view of connections found in the linearly ordered calculus by simply stating that “calculus builds on itself more than any other class.”

The calculus teachers in this study shared three primary goals for teaching calculus: (a) to prepare students for the AP exam, (b) to prepare students for college calculus, and (c) to inspire students to appreciate mathematics. Preparing students for the
AP exam was considered essential for each of the teachers. The reader is reminded that the sample was chosen because the teachers were teaching an AP calculus course. The amount of emphasis given to this goal, however, varied considerably. For example, Ryan viewed his role as that of an AP exam coach and he frequently used the AP exam as a motivational tool for his students. Terry, on the other end of the spectrum, minimized the emphasis of the AP exam and suggested that only a few of his students would even take the exam. The amount of emphasis the rest of the teachers gave to this goal was somewhere between the emphasis given by Ryan and Terry. None of the teachers required their students to take the AP exam. However, they all felt it was important to cover the material necessary for successful completion of the exam for those few students who would take the exam. For example, Trey discussed that he did not necessarily encourage his students to take the AP exam, but he wanted to leave the option open for his students. In fact, he recommended that his students take calculus again in college regardless of their AP exam score. Russell, Terry, and Trey all felt that taking calculus over in college would help to reinforce the students' conceptual understanding of the calculus material and give them a stronger background for a variety of possible academic majors. Ryan and Tom, however, were more focused on having their students obtain advanced placement in college calculus than on having their students retake calculus in college.

Preparing students for college calculus was the main goal demonstrated in these teachers' classroom teaching. This goal appeared to coincide with the goal of preparing students for the AP exam as the motivation behind taking the AP exam was to give students the opportunity for advanced placement in college calculus. The majority of the students in each of the teacher's classes would more than likely be required to obtain college calculus credit. In all cases, preparing students to either obtain college calculus credits or be successful in college calculus possibly validated what the teachers actually did in the classroom.
The third major goal, inspiring students to appreciate mathematics, was primarily revealed in the classroom by the teachers’ enthusiasm for the subject matter. Discussions involving the beauty of the deductive nature of mathematics, however, were teacher-centered. Thus, it was difficult to determine whether students were also inspired by this beauty. Terry, Ryan, Trey, and Richard each suggested that using a discovery learning approach helped students become problem-solvers and fostered an appreciation for mathematics. Problem-solving to these teachers, however, meant using guided discovery to arrive at solutions to unfamiliar problems. As one example, Richard’s idea of problem-solving was merely being able to algebraically solve unfamiliar mathematical equations. None of the teachers dealt extensively with applications. Regardless, it was difficult to determine what type of appreciation for mathematics the teachers desired for their students. Equally difficult to determine was what each teachers’ idea of the beauty of calculus actually meant for their students. The following quote by Ryan describes what the majority of the teachers in this study perceived the beauty of calculus to be: “[There is a need for] computation, manipulation, and memorization, but the beauty [of calculus] lies in the communication, reasoning, thinking, and problem-solving.” The organization of the sections in the textbook appeared to have an impact on the amount of time devoted to either computation, manipulation, memorization, or to communication, reasoning, thinking, and problem-solving. The emphasis in Ryan’s textbook, as well as the textbooks of the other independent teachers, was computation and manipulation.

Each of the established major goals this group of teachers had for teaching calculus were demonstrated in their classroom teaching. In some cases, however, the individual teachers appeared to have difficulty completely meeting their goals due to existing conflict between the goals. For example, the strength of the goal regarding preparation for the AP exam may have minimized the desire to incorporate technology in the calculus course. Graphing calculators were not allowed on the AP exam. This conflict may be resolved when graphing calculators are allowed on the AP exam in 1995. Additional goals were
mentioned by some of the teachers. Terry, Trey, Tom and Russell commented on the goal of incorporating technology into the calculus classroom. The goal of incorporating technology coincided with the emphasis of technology in the calculus reform curricula. Thus, the project teachers, Terry, Trey, and Tom all incorporated technology into the classroom. An inconsistency existed, however, between Russell’s desire to incorporate technology and what actually occurred in his classroom. He allowed the students to use technology on their assignments. He did not, however, teach using technology nor allow the use of technology on exams.

Two of the teachers, Richard and Tom, mentioned the importance of having their students write and communicate mathematically. Observations of classroom teaching, however, revealed that the goals of writing and communication were addressed only minimally. For example, Richard merely had his students write a paragraph on “What is calculus?” on the limit examination.

Calculus Teachers’ Perceptions of the Concept of Limit and the Correlation to Classroom Practice

The significant influence of the textbook was noticeable throughout the two-week unit on limits. For each of the teachers, the individual lessons corresponded directly with the given section in the text. In most cases the same examples given in the textbook were used for instruction. In addition, the majority of the assignments were given directly from that particular section’s exercises. Moreover, the amount of emphasis given to a particular topic within the limits chapter of the textbook appeared to be guided by the AP exam as well as the textbook.

The teachers viewed the concept of limit as a way to describe the behavior of functions. Two methods are typically used by teachers to describe this behavior: dynamic and quantified. Briefly, the dynamic method emphasizes motion and the quantified
method emphasizes measure. When asked to discuss the concept of limit, the teachers consistently responded in a dynamic fashion emphasizing the motion of "moving closer to" or "approaching." For example, Russell described a limit as "A number the function approaches in value when the function is evaluated for appropriate numbers close to a." Terry was the only teacher who described the concept of limit in both a dynamic and quantified way. For example, at the end of the data collection period Terry described a limit in a quantified manner, "The limit L of a function at a point a does not depend on the function value at a, but exists if we can get values of the function as close to L as we wish by restricting x to a sufficiently small interval about a but excluding a." At the beginning of the study he only discussed the concept of limit in a dynamic, rather than quantified, way. The calculus reform curricula frequently discussed the concept of limit in both a dynamic and quantified manner. The textbook's discussion is a possible explanation for why Terry discussed the concept of limit using both notions interchangeably at the end of the study. This conclusion may also be supported by the fact that the language Terry used to talk about the concept of limit also changed to reflect the language of the calculus reform textbook.

Prior to teaching the unit on limits, each of the teachers discussed the importance of teaching limits from an intuitive perspective first. Consistent with their professed perceptions, all of the teachers began the unit on limits by examining the concept of limit from an intuitive perspective. The amount of time each teacher devoted to this primarily teacher-centered intuitive approach to limits, however, varied considerably. Some teachers spent 10 minutes on an intuitive discussion pertaining to the concept of limit while other teachers spent three days. The three project teachers spent the most time intuitively examining limits replicating the sections in the textbook. Each of the project teachers introduced the concept of limit using the graph provided on the first page of the chapter on limits in their textbook. Following the order of the textbook, the project teachers spent two to three days covering textbook examples which involved examining
limits intuitively using both graphical and numerical representations. Two independent teachers, Ryan and Russell, also began the unit on limits using intuitive graphical representations. Following a brief 10 minute intuitive introduction to limits, Ryan spent considerable time on the algebraic evaluation of limits of given functions. He also devoted a modest amount of time (less than 10 minutes) to relating the intuitive idea of limits to the algebraic evaluation of limits. Russell moved directly to discussing limits more formally after his short 10 minute intuitive introduction to limits. Richard was the only teacher who did not use a graphical approach to introduce the concept of limit. His idea of teaching limits intuitively meant keeping the problems “simple” on the first day. Obtaining an intuitive notion for the concept of limit appeared to mean to Richard that students would be able to easily evaluate the limits of continuous functions. After a brief introduction to the concept of limit Richard moved directly to discussing the formal epsilon-delta definition of limit. Though he did not use a graphical approach to introduce the concept of limits, Richard provided an intuitive graphical interpretation during his discussion of the formal definition of limit. This lesson, however, was not directly tied to his lesson on the simple evaluation of continuous limits.

All textbooks used by the teachers in this study contained a section on the formal epsilon-delta definition of limit. Accordingly, all teachers in this study devoted class time to the discussion of the formal definition of limit. In all cases, this discussion involved a graphical intuitive interpretation of the epsilon-delta definition of limit. The individual teachers differed, however, in the amount of time spent using the formal epsilon-delta definition for proving the limits of various functions. It was maintained by most of these teachers that the appropriate level of rigor needed in calculus was dependent on the focus of the class. Though all of these teachers appeared to have the same focus and goals for their calculus class, the perceptions with respect to the level of mathematical rigor split the group of teachers into two sets, the independent teachers and the project teachers.
The independent teachers felt that a high level of mathematical rigor was critical to understanding limits. More specifically, Ryan and Richard suggested that understanding the epsilon-delta proofs would help the definition of limit “sink in a little better.” Richard continued by stating that while high mathematical rigor may not be important in college calculus, the high school AP calculus students were mathematically mature enough to handle the rigor. Russell, on the other hand, felt it was important to hold the students accountable for the epsilon-delta proofs. However, he also discussed that understanding the formal epsilon-delta definition and proofs was not important for the understanding of other calculus topics. At this juncture Russell deviated from the given order of topics in the textbook. He spent over three days immersed in lessons involving epsilon-delta definition of limit and proofs. Throughout these three days he continually emphasized three items: (a) that students would be held accountable for reproducing an epsilon-delta proof for a linear limit, (b) that they were not expected to understand the proof but be able to reproduce it, and (c) that they were not to be concerned where the delta came from in the proofs but they needed to show that it worked. Russell also reassured the students by stating that being able to do the proofs was not that important for understanding the rest of calculus. In fact, Russell was unsure of his reason for actually teaching epsilon-delta proofs. Though Russell never actually mentioned it, it is possible that his goal of inspiring students to appreciate the beauty and structure of mathematics was his motivation. More probable explanations for this view were the goal of preparing students for college calculus or his own history of learning the concept of limit. Russell discussed possibly changing what he taught if colleges decided that epsilon-delta arguments were no longer important in the teaching of calculus. Ryan and Richard used the goal of preparing students for the AP exam as a means of providing motivation for covering epsilon-delta proofs. It is interesting to note that only a small portion of the AP (BC) calculus topics (see Appendix A) include the topic of rigorous epsilon-delta arguments. Regardless, all of the independent teachers held their students accountable for reproducing the formal
epsilon-delta definition of limit both symbolically and graphically. Additionally, all independent teachers held their students accountable for reproducing an epsilon-delta proof for a linear limit.

The project teachers also held their students accountable for the symbolic and graphical representation of the formal definition of limit. Epsilon-delta proofs, however, received little attention in class. Epsilon-delta arguments were also given minimal attention in the calculus reform textbook. Paralleling the contents of the textbook, the project teachers focused on the intuitive graphical representations of the formal epsilon-delta definition of limit and proofs. It is interesting to note, however, that prior to participation in the calculus reform project each of the project teachers already felt the epsilon-delta arguments were insignificant for the beginning study of calculus. Terry felt it was important for students to visually and intuitively understand the formal definition of limit, but he thought formal epsilon-delta arguments were insignificant for the study of calculus. He felt that most students could not understand the rigorous epsilon-delta arguments and he would be happy if they could just understand the graphical interpretation. Thus, his goal was for students to understand the underlying concept rather than developing mathematical rigor. Though Trey wished his students could be more mathematically rigorous, he felt the epsilon-delta arguments were insignificant for his calculus class and that giving it much class time was senseless. At the end of the study, however, Trey no longer discussed his desire for his students to be more mathematically rigorous and appeared to feel comfortable with his decision to minimize the coverage of the epsilon-delta arguments. Throughout the entire study, Tom also suggested that the appropriate level of mathematical rigor with respect to limits was "just short of epsilon-delta proof." Based on the design of this study, it was difficult to determine why the project teachers felt the significance of the epsilon-delta arguments should be minimized prior to participation in the calculus reform project. It is possible that because they held this view they were motivated to join the calculus reform movement. Another explanation
may be that they felt that mathematical rigor could be postponed for those students participating in further mathematical study.

Calculus Teachers’ Perceptions of the Role of Limits and the Correlation to Classroom Practice

All of the teachers in this study consistently felt that the concept of limit was extremely important in the teaching of calculus. Additionally, all of the teachers agreed that the concept of limit was critical to the appreciation and understanding of other calculus topics. Both Ryan and Tom commented that the concept of limit was not necessarily critical to developing algorithmic skill in solving certain types of calculus problems, but one could not “understand calculus without understanding limits.” More specifically, all of the teachers asserted that the intuitive notion of limit was fundamental for student understanding of calculus. Tom’s quote regarding the concept of limit as “one of the basic intuitive ideas in the formation of the foundation of calculus” appeared to reflect the perceptions of all the teachers in the study. Though collectively the teachers spent a great deal of time on covering the formal epsilon-delta definition and proof of limits, none of the teachers mentioned the role of formal epsilon-delta arguments in other calculus topics. In fact, all six teachers maintained that only an intuitive notion of limits, rather than the formal definition, was fundamental to understanding other calculus topics.

Little emphasis was given to drawing connections between the concept of limit and other subsequent calculus topics. Classroom observations of the introduction to differentiation and integration revealed that attempts at making connections between limits and other calculus topics involved an intuitive, but not formal, understanding of limits. Observations also revealed that though the relationship between limits and other topics were discussed in the classroom, the time devoted to teaching this relationship was quite brief. Only one observation occurred, however, during each teachers’ introduction to
differentiation and the introduction to integration. Thus, it was possible that further connections were made in subsequent unobserved classes. The data from this study suggested that though the teachers were aware of the relationship between limits and other calculus topics, the discussion of these relationships in the classroom were brief and incomplete.

Summary of Calculus Teachers’ Perceptions of Limits and the Correlation to Classroom Practice

The extent of relationship between these teachers’ perceptions of limits and actual classroom practice varied. Though none of the teachers demonstrated a complete match between their perceptions of limits and classroom practice, three of the teachers, Ryan, Trey, and Terry, demonstrated relatively strong relationships between the two. In Ryan’s case, preparation for the AP exam may have posed the biggest road block to complete translation of his goals. For example, although he felt the intuitive notion of limit was most critical for understanding other calculus topics, he also felt that finding limits using a graph was a “crutch.” Graphing calculators were not be allowed on the AP exam in the year this study took place and thus, according to Ryan it was better for students to find the limits algebraically. Though Trey and Terry at times doubted the reasoning behind information provided in the new textbook, they faithfully followed the text. It was difficult to determine if they felt that following the textbook was part of their commitment to the project or not. Regardless, using a new textbook for the first time, may have made it difficult to completely translate their perceptions of limits to classroom practice.

The other three teachers, Russell, Richard, and Tom, demonstrated some inconsistencies between their perceptions and classroom practice. Russell appeared to have an inner conflict between what he thought was best for his students and what he actually did in class. For example, in an interview Russell discussed focusing on the
intuitive notion of limit in great detail. However, he only spent a small portion of one
lesson covering limits intuitively. Additionally, he consistently talked about the
importance of incorporating technology into the classroom with the researcher, but he
never used technology in class and rarely discussed its use with his students. Richard
valued the relationship between limits and other calculus topics. However, he
methodically followed his textbook. The influence of the text may have posed a threat to
translation in the classroom since the text appeared to minimize the relationship between
limits and other calculus topics. Finally, Tom appeared to struggle with the goals of the
new textbook. Though he covered the sections in the text he frequently shared his doubts
with his students concerning the authors’ goals. For example, when giving a homework
assignment he wrote a list of problems on the board and stated, “I’m not sure what the
author was getting at with these problems [in the textbook]. Work the following instead.”
Additionally, though Tom felt the students were more convinced of the existence of a limit
using graphical and numerical representations, he was concerned about the students
relying too much on graphs to obtain the limits. He frequently told students to place more
faith in numerical and algebraic representations. It was difficult to determine why this
disparity between Tom’s professed goals for the class and classroom practice existed.
Once again, however, it may stem back to the fact that the focus of his own preparatory
mathematics courses was possibly on procedural knowledge rather than conceptual
understanding.

Association Between Calculus Teachers’ Perceptions of
Limits and Participation in a Calculus Reform Project

The sampling technique used in this study generated information regarding the
association between teachers’ perceptions of limits and participation in a calculus reform
project. By comparing and contrasting the general profiles for the group of independent
teachers and the group of project teachers, differences in the following eight factors emerged from the data: (a) devotion to the curricula, (b) planning, (c) use of multiple representations, (d) attitude towards graphing technology, (e) classroom atmosphere, (f) examinations, (g) appropriate level of mathematical rigor needed for teaching calculus, and (h) the stability of perceptions. These factors, however, may not be fully attributed to participation in the given calculus reform project. It is possible that some of these factors could be attributed to the project teachers' prior beliefs and practices. In fact, some of these factors possibly contributed to project teachers' decisions to participate in the project in the first place. Given the design of this study, however, prior teacher beliefs were difficult to determine. The following paragraphs contain discussions of each of the factors listed above, as well as indicate the extent of plausible correlation between the teachers' perceptions and participation in the calculus reform project.

All of the teachers in the study taught lessons in the order of the given sections in the textbook. The project teachers, however, demonstrated a stronger, possibly blind, commitment to the textbook's contents. For example, throughout the unit on limits the project teachers followed the textbook example-by-example. Each of the teachers often taught directly out of the textbook by having all of the students refer to a given page or graph. Terry occasionally had the students read portions of the textbook during class. Class discussions about the material directly followed the class reading. This type of devotion to the given section in the textbook may be attributed to using the textbook for the first time. All the independent teachers had used their textbook for two years or more. Additionally, the teachers may have felt that it was their job as a participant in the calculus reform project to implement the goals as represented in the calculus reform textbook. Thus, they may have felt an obligation to follow the text as part of their commitment to the project.

Possibly correlating with the devotion to the curricula and project factor, the project teachers' planning tended to be short term. In many cases, the individual teachers
could not explain, and sometimes doubted, the authors' reasoning behind particular presentation or sequencing of material. These teachers appeared to trust the authors' reasoning and taught the material exactly the way it was done in the text. The teachers rarely looked ahead in the textbook to try to ascertain the authors' motivation for the presentation or to attempt understanding the reform curricula. Again, this result may be explained by the teachers' assumed legitimacy and authority of the required textbook or commitment to the project. Additionally, the majority of the lessons on limits appeared to be prepared shortly before the class period in which it was taught, particularly in the case of Tom. When attempting to schedule classroom observations Tom frequently did not know what he was planning to teach for the upcoming week. On the other hand, the independent teachers appeared to plan ahead and knew exactly what they would or would not be covering.

As discussed previously all of the teachers in this study felt that it was important to first teach the concept of limit from an intuitive perspective. Additionally, all of the teachers mentioned the importance of the intuitive notion of limit in the teaching of calculus. However, none of the teachers mentioned the importance of a formal epsilon-delta definition of limits in connection with other calculus topics. Furthermore, all of the teachers felt that students would be more convinced of the existence of a limit using either algebraic manipulation, graph or a table of values rather than the formal epsilon-delta limit definition. Classroom observations revealed that the project teachers spent considerable more time than the independent teachers examining limits intuitively. The project teachers spent up to three days whereas the independent teachers spent between 10 minutes to a full class period. Again, the amount of time the project teachers spent on examining limits intuitively was a direct reflection of the number of sections in the textbook. Though two of the independent teachers, Russell and Ryan, examined limits using graphical and numerical representations, the time spent on these representations was less than one full class lesson. Additionally, as the unit progressed Ryan appeared to experience conflicts in
his desire for examining limits intuitively. He suggested that using graphical representations to find limits was a "crutch," and discussed with his students that the best way to find limits was algebraically. Russell told the students that it was important for them to have immediate recall of several graphs in their head, and consequently further minimized the use of graphical representations. The most probable reason for Ryan's focus on algebraic procedures and Russell's desire for memorization of graphs is the goal of preparation for the AP exam since graphing calculators could not be used on the exam. The project teachers, on the other hand, consistently used a variety of representations, graphical, numerical, and symbolic. Again, the use of multiple representations reflected the goals of the calculus reform project and the given sections of the textbook. Each new topic was introduced using these multiple representations. The goals of the calculus reform project were developed to match the NCTM (1989, 1991) and NRC (1991) general recommendations for change in mathematics which included the use of multiple representations to develop an underlying understanding of mathematical concepts.

As in the use of multiple representations, the project teachers' and the independent teachers' focus differed regarding the incorporation of graphing technology. All three of the project teachers and one independent teacher, Russell, discussed the importance of incorporating graphing calculators into the classroom. As a requirement of the calculus reform project all students in the project teachers' classes had graphing calculators. These calculators were being used every day by all of the students in the project teachers' classroom and some of the students in Russell's classroom. The amount of calculator use for teaching in the classroom varied for each of the individual project teachers. Russell never used the calculator as a teaching tool. All observations, however, revealed continued teacher support for its use by students.

One difficulty that became evident during classroom observations of the project teachers was the lack of teacher explanation to their students regarding the limitations of the calculator. Each observation revealed that the project teachers covered this topic
occasionally in class, and they commented that this issue was discussed in detail later in the year. As the year progressed, the teachers discovered the need to teach students how to strategically use the calculator as another problem-solving tool.

Several students in the independent teachers' classrooms also brought graphing calculators to class. The nature of calculator use, however, was quite different in the independent teachers' classrooms as compared with the project teachers. Though Russell acknowledged the importance of using technology in the classroom, he rarely used them in class. Lack of motivation and time appeared to be the primary reason for his lack of use. A few of Russell's students, however, continued to use their personal calculators throughout the data collection period. As previously mentioned, Ryan felt viewing limits graphically was important, but he encouraged students not to rely on their graphing calculators. Ryan did not use graphing technology while teaching. One of Ryan's major focuses for the course was preparation for the AP exam, so a logical explanation for not using graphing technology was the influence of the AP exam. Richard saw no advantage to graphing calculators at all. He felt the students would become too dependent on the calculators for simple manipulation. Moreover, he felt he could teach them to graph faster than the machine anyway. Richard possessed a lack of motivation, time, and desire to even consider the idea of incorporating technology into his lessons.

The general atmosphere in the classrooms of the project teachers was quite different from that of the independent teachers. Each of the independent teachers' lessons were primarily teacher-centered and the students responded directly to the teacher. The project teachers' lessons also tended to be predominantly teacher-centered. However, they frequently used class discussions, driven by teacher questioning, as well as lectures to introduce lessons. In fact, throughout the class period students were constantly asking questions of other students as well as the teacher. As evidenced by the interviews, the project teachers attributed this type of atmosphere to the continual use of graphing calculators by students in the classroom. Additionally, the focus on multiple
representations of mathematical concepts naturally led to guided discovery lessons which may have also attributed to the differences in classroom atmosphere. The classroom atmosphere mimicked that found in the summer two-week inservice portion of the calculus reform project because the participants also frequently asked questions of both the instructor and other participants.

A difference between the group of project teachers and the group of independent teachers also surfaced in the types of questions asked on the limit unit examination. The examinations given by the independent teachers contained primarily questions involving the algebraic evaluation of limits. On either a quiz or the unit exam the students were also asked to prove that a limit existed for a linear function using the formal epsilon-delta definition of limit. Additionally, the students were also asked to reproduce the epsilon-delta definition of limit both symbolically and graphically. Furthermore, graphing calculators could not be used on any of the independent teachers' examinations. Again, this type of assessment may also be used because it mimicked the teachers' perceived goals of AP calculus and format of the AP exam.

In contrast, the project teachers' examinations included questions that allowed for the use of a variety of representations to find limits. Additionally, graphing calculators could be used on the project teachers' exams. Problems that involved merely algebraic evaluation of limits were not found on these exams. Each of the project teachers' exam questions, however, were taken directly from the textbook. In many cases, the problems were copied, cut, and pasted directly from the text. In fact, Terry and Tom put the exact problems on the exam even though they had doubted the authors' reasoning just days earlier. Finally, investigation of Terry's unit exam revealed that although a variety of representations could be used to solve the problems, all but one question could be solved using a graphical representation.

Perceptions regarding the appropriate level of mathematical rigor with respect to the formal definition of limit further separated the group of independent teachers from the
group of project teachers. As previously discussed, the independent teachers felt that a high level of mathematical rigor was critical to understanding limits. More specifically, they felt that understanding the epsilon-delta proofs would help the definition of limit “sink in a little better.” Though the independent teachers discussed that understanding the formal epsilon-delta definition and proofs was not important for the understanding of other calculus topics, they focused on this topic for a significant amount of class time. Additionally, they felt it was important to hold their students accountable for reproducing epsilon-delta arguments for linear functions. The project teachers, however, felt the epsilon-delta arguments should receive little attention. They felt that it was important for students to visually and intuitively understand the formal definition of limit but formal epsilon-delta arguments were insignificant to the understanding of the concept of limit.

As discussed previously, the independent teachers’ perception of the importance of epsilon-delta arguments was attributed to the following factors: preparation for the AP exam, belief that the structure and beauty of mathematics should be addressed, preparation for college calculus, and possibly previous personal experiences in calculus classes. Again, it was difficult to determine why the project teachers all felt the epsilon-delta arguments were insignificant as they appeared to hold the perception prior to participation in the calculus reform project. It is possible that the beliefs held prior to participation in the project may have motivated the teachers to participate. The project teachers did, however, appear to feel more comfortable with and confident in this perception at the end of the study. This increased comfort may have been attributed to the fact that the epsilon-delta arguments did not receive much emphasis in the calculus reform curricula.

Stability of Calculus Teachers’ Perceptions of Limits

Except for Ryan, all of the teachers in the study commented that teaching the concept of limit enhanced their understanding of it. In fact, in most cases the teachers
could not remember understanding limits formally before they started teaching calculus. The teachers felt that they obtained an intuitive notion of the concept of limit in college calculus. Ryan claimed he understood limits intuitively since junior high school, but did not understand limits formally until college analysis courses. Russell commented that he first understood limits formally in graduate level courses. He felt his mathematics skills were too weak to prove limits using epsilon-delta proofs before graduate school. This result may be an explanation for the conflict between the desire to teach rigorous epsilon-delta arguments in high school because it is traditional, but not expecting the students to fully understand because the ability to understand will come much later in their mathematical studies.

The independent teachers’ perceptions of limits remained stable throughout the entire study. Some aspects of the project teachers’ perceptions of limits, however, appeared to change as the study progressed. For example, all of the diagrams of the important concepts and relationships in calculus produced by the project teachers changed to reveal more relationships between limits and other calculus topics. Additionally, as time progressed the project teachers also discussed the increased importance of the intuitive notion of the concept of limit as the topic that links all of calculus together. Though this particular calculus reform curriculum emphasized the relationship between limits and other calculus topics and the teachers discussed the importance of limits, the project teachers’ classroom practice appeared to minimize these relationships and continued to represent calculus in a more linear fashion.

One project teacher, Terry, changed from describing the concept of limit in a strictly dynamic manner to using both a dynamic and quantified manner. Additionally, the language Terry used changed to reflect the flavor of graphical representations, examining calculus using a visual approach. Terry also commented that he felt he spent more time on limits in the year of the study than he had any other year he taught calculus. He attributed
the increased time spent on limits to the graphing calculators as the calculator provided an easy way to examine the graphical meaning of calculus topics.

Possibly the best example of the change in the project teachers' perceptions of limits can be seen in two quotes taken from Trey. At the beginning of the study Trey commented to the researcher, "There isn't a lot of material from the limit chapter that they [the students] need to carry on to the other chapters." At the end of the study Trey stated to the researcher, "I think limits is much more a part of calculus now than it ever has been." Furthermore, Trey commented that after teaching from the new textbook he felt his understanding of limits continued to evolve every year he teaches the topic.
CHAPTER V

DISCUSSION AND CONCLUSIONS

Introduction

The objective of this study was to investigate high school AP calculus teachers’ subject matter and pedagogical perceptions of limits. To do so, the study examined the following question: What are teachers’ perceptions of the concept of limit, the role of limits, and the teaching of limits in calculus? Moreover, the sampling technique used in this study sheds some light on the question: Are these teachers’ perceptions associated with their participation in a calculus reform project focused on staff development?

The results to the first question were generated by examining six teacher profiles and searching for similarities and differences across the sample. It is recognized that conclusions obtained from single teacher profiles are not generalizable to the population of all AP calculus teachers. The conclusions found in this study, however, were generalizations formed across six teacher profiles and thus provide conjectures that may be examined further with a larger sample of calculus teachers. The results to the second question were formed by developing profiles for two groups: (a) project teachers, and (b) independent teachers. These profiles were then compared and contrasted in order to develop an understanding of the association between the teachers’ subject matter and pedagogical perceptions of limits and participation in a calculus reform project.

The following sections contain a summary and discussion of the main findings. Finally, the chapter concludes with comments regarding the limitations of the study, directions for future research, and implications of the study for mathematics education.
Summary and Discussion of Main Findings

Goals for the Teaching of Calculus

To understand AP calculus teachers' subject matter and pedagogical perceptions of limits, it is essential to know the primary goals the teachers in this study held for teaching calculus: (a) to prepare students for the AP exam, (b) to prepare students for college calculus, and (c) to inspire students to appreciate mathematics. The goal of preparing students for the AP exam comes as no surprise as this goal is the purpose for offering an AP course (see Appendix A for a discussion of AP calculus). The level of support given to this goal varied among the teachers in this study and in some cases was quite extensive. Though none of the teachers required their students to take the AP exam, they all felt it was important to cover the content needed to successfully complete the exam for those few students who would take the exam. This result echoed the findings of Webb (1992), which stated that large scale assessment has a major influence on what and how teachers teach mathematics. Webb discussed a national survey of schools (Romberg, Zarinnia, & Williams, 1989) that reported how eighth-grade mathematics teachers were influenced by mandated testing. The results indicated that eighth-grade mathematics teachers had concerns similar to those of the calculus teachers in this study. Specifically, the teachers spent considerable time preparing students for the tests. They felt compelled to adhere to the content required for the test and thus modified their teaching to reflect what was to be tested. The predominant importance of preparing for the AP exam will be reiterated throughout the remainder of this chapter.

Though all of the teachers felt that part of their job was to prepare students for college calculus, some of the teachers focused on college preparation rather than preparation for the AP exam. These teachers felt that re-taking calculus in college would reinforce the students' conceptual understanding of the material and strengthen their
backgrounds for a variety of possible academic majors regardless of their AP exam scores. According to the literature, this reasoning seems to be partially justified. Research has shown that students who have had a full year of secondary school calculus are more successful in a first semester college calculus course than those students who have no exposure to calculus (Ferrini-Mundy & Gaudard, 1992). The same study cautions, however, that these students may have a “false confidence” (p. 68). The study suggests that procedural, technique-oriented high school calculus may predispose students to focus on the procedural aspects of calculus that they have learned to value. Thus, they may be less open to and appreciative of the more theoretical developments of topics such as the definition of derivative and integral. The results of this study provided support for this concern.

Helping students to receive credit for college calculus or to be successful in this course may have validated what the teachers actually did in the classroom, both on a daily basis and for the entire course. On a daily basis, the teachers used the AP exam or college calculus as a way to motivate students to learn certain topics. Validation for the entire course may have been provided by the students’ AP exam scores or college calculus grades. As evidenced by the teachers in this study, advanced placement calculus teachers may be working in isolation at their high schools. Thus, this external accountability may be one of the few measures the teacher has regarding success as a calculus teacher.

The third major goal stated by all of the teachers in this study was inspiring students to appreciate mathematics. However, it was difficult to determine exactly what type of appreciation for calculus the teachers desired for their students. Correspondingly, it was difficult to determine when, during classroom lessons, the teachers were consciously trying to inspire this appreciation in their students. As with the goals of preparing students for the AP exam and college calculus, the teachers may have used this goal as a motivational tool for covering certain calculus topics, such as the epsilon-delta arguments, that they found difficult to otherwise justify. It seemed that some of the
teachers were attempting to maintain an oral tradition to "pass on the beauty of calculus" to their students. That is, the teacher's own calculus instructor may have discussed calculus as "beautiful," so they are now passing on this notion of calculus as a beautiful subject to their students. Additionally, it is likely that the mathematics training of these teachers included the appreciation of the formal structure of mathematics, including definitions, theorems, logic, and mathematical rigor (NRC, 1989). Thus, classroom discussions involving the formal structure of calculus may have been perceived as part of the teachers' job of teaching calculus.

**Influence of the Calculus Textbook**

Existing research asserts that elementary and lower-level secondary mathematics teachers are greatly influenced by the course textbook (Leinhardt & Smith, 1985; Sullivan & Leder, 1992). However, literature on the teaching of calculus suggests that the emphasis of the textbook is minimized, particularly at the college level (Ferrini-Mundy & Graham, 1991). Although a de-emphasis of the textbook may hold true for college calculus instructors, the results of this study suggest that the textbook plays a major role in the teaching of AP calculus in high school. For each of the teachers in this study, individual lessons corresponded with given sections in the textbook. In most cases, the same examples and exercises presented in the text were also used for instruction and homework. Moreover, all of the teachers were aware of, and appeared comfortable with, their reliance on the textbook. Further discussion regarding the extent of teacher commitment to the textbook are addressed in subsequent sections.
Calculus as a Linearly Ordered Subject

All of the teachers in this study perceived calculus as a linearly ordered set of topics. For each of the teachers in this study, calculus was divided into three main topic areas: limits, differentiation, and integration. These topic areas were perceived as linearly ordered in the sense that understanding limits was a prerequisite to understanding differentiation, which in turn was a prerequisite for understanding integration. This finding supports the observations of the NCTM (1989) and NRC (1989) which stated that mathematics teachers commonly viewed their subject matter as a linearly ordered, fixed body of knowledge. In contrast, though not observed in this study, a teacher holding an integrated view of calculus may view the three topics in a spiraling way. For example, a teacher holding this view may believe that in order to understand differentiation one must understand the relationship between integration and differentiation. Thus, the topics may be taught informally during the initial presentation, with increasing formality on subsequent presentations. This spiraling approach to calculus is demonstrated in the calculus reform curriculum of Hughes-Hallett and Gleason (1992).

The origin of teachers’ perceptions of calculus as a linearly ordered subject is difficult to determine. Existing research suggests that past experiences in mathematics classes may have led to the development of the teachers’ perception (Ball, 1990a; 1990b; Marks, 1990; Thompson, 1984; Tirosh & Graeber, 1989). Each of the teachers in this study had only one experience with an introductory calculus course. In all cases, this experience was a traditional college calculus course which had been approached in a linearly-ordered fashion. Based on the results of this study, however, it is also plausible that the teachers’ perception of calculus as linearly ordered were influenced by the order of topics in the calculus textbook and by the order of topics in the AP syllabus. Due to the influence of the AP exam on each of the teachers in this study, it is also possible that the
topics suggested for students' understanding (see Appendix A) may have impacted the teachers' perceptions of calculus.

The results of this study not only suggest that calculus itself was perceived as linearly ordered, but also support the literature stating that calculus is frequently considered to be the culmination of many years of linearly ordered, mathematical study (Steen, 1987). The claim was that calculus not only builds sequentially on itself, but also builds on all previous courses of mathematical study. The teachers in this study felt that a solid foundation in algebra was of utmost importance to doing calculus. To these teachers, doing calculus meant being able to successfully solve procedural calculus problems. Given the focus on preparation for the AP exam, much attention was paid to helping students obtain a procedural understanding of calculus. The emphasis on procedural understanding is discussed in subsequent sections.

The Role of Limits in the Teaching of Calculus

Although the teachers in this study focused on procedural aspects of calculus, they also expressed a desire for students to obtain a conceptual understanding of the subject. In fact, the teachers felt that understanding the concept of limit facilitated an understanding of all other calculus topics. In particular, all of the teachers asserted that it is the intuitive, not the formal, notion of limit which is fundamental for student understanding of calculus. This perception is consistent with Orton's (1983a) suggestions that an initial approach to calculus concepts should be informal, focusing on intuitive, numerical and graphical representations. Additionally, this perception corresponds with the constructivist views that mathematical learning must include conceptual activities in which students integrate new knowledge into their existing knowledge base. With respect to the role of formal epsilon-delta arguments in the teaching of calculus, the constructivist perspective is discussed in more detail in subsequent sections.
Calculus reform curricula have been developed recently that coincide with the teachers' assertion that it is the intuitive, not the formal, notion of limit that facilitates an understanding of other calculus concepts (Hughes-Hallett & Gleason, 1992). One widely-used calculus reform curriculum, developed by the Harvard Project, uses a spiral approach to present calculus material. The initial presentation of topics focuses on developing an intuitive notion of the calculus concepts, whereas subsequent presentations seek to formalize students' understanding. The formal epsilon-delta definition of limits plays no role in the development of other calculus concepts in this text. In fact, unlike traditional calculus texts, epsilon-delta arguments are found nowhere in this text.

Despite the fact that teachers felt an intuitive understanding of limits was essential to further understanding of calculus, little effort was made to draw connections between the concept of limit and other calculus topics. Classroom observations of teachers' introduction to differentiation and integration revealed that brief attempts to connect limits and other calculus topics involved an intuitive, and not formal, understanding of limits. Only one observation occurred, however, during teachers' introductions to differentiation and integration. Thus, it is possible that further connections were made in subsequent unobserved classes. However, the possibility that further connections were made is unlikely given that the introduction to these topics seems the most opportune time for addressing these connections. The teachers in this study progressed quickly to finding derivatives of functions procedurally, and devoted little time to developing conceptual understandings of the derivative. The goal of preparing for the AP exam may be one explanation for this lack of emphasis on conceptual understanding. Tall and Vinner (1981) proposed that the amount of time calculus teachers typically spend on teaching the connections between limits and differentiation would be brief. They state, "Very soon the standard formulae for derivatives are derived and the general notation of limit recedes into the background" (p. 155). Thus, since it was found that non-AP calculus teachers also devote little attention to the connections between limits and subsequent calculus topics, it
is possible that the goal of preparing students for the AP exam may not be the only cause of the brevity of these teachers' brief attempts to make connections between limits and other calculus topics.

Ball (1990a; 1990b) and Tirosh and Graeber (1989) suggest that teachers have some difficulty making clear mathematical connections due to the focus of their preparatory mathematics courses on procedural rather than conceptual knowledge. This result might explain why these calculus teachers failed to devote the time needed to make connections between limits and other calculus topics. However, the data from this study does not directly imply this explanation, since the exact nature of the teachers' preparatory mathematics courses is unknown. Regardless, the brief amount of time spent on developing the relationship between limits and the derivative may provide insight into the finding that students have difficulty interpreting how secant lines "sliding" along a curve lead to the tangent line interpretation of limits (Orton, 1984). Additionally, research suggests that without an intuitive understanding of the connection between limit and derivatives, students tend to view the derivative of a function as the *equation* of the tangent line and not the slope of the tangent line at a given point on the curve (Amit & Vinner, 1990).

**The Concept of Limit in the Teaching of Calculus**

The teachers in this study perceived the concept of limit as a way to describe the behavior of functions. Though each of the teachers was able to describe limits in both dynamic (emphasizing motion) and quantified (emphasizing measure) ways, in the classroom and during interviews, the teachers in this study primarily discussed the concept of limit in a dynamic way. The quantified description of the concept of limit was only used by teachers during classroom lessons on the formal definition of limit. This finding supports Tall and Vinner (1981) and Williams (1990) findings that students usually discuss
and think of the concept of limit in a dynamic manner. The finding is not surprising to the researcher, as personal anecdotal evidence suggests that when mathematicians discuss the concept of limit they also do so in an intuitive, dynamic manner.

All of the teachers in the study discussed the importance of introducing the concept of limit from an intuitive perspective. Their classroom practice reflected this goal, as the teachers began the unit on limits by examining intuitive, graphical representations of the concept of limit. The amount of time devoted to intuitive interpretations of limits, however, varied considerably. The project teachers spent two to three days discussing intuitive concepts of limit, whereas each of the independent teachers devoted only between 10 to 20 minutes on the first day of the unit on limits to intuitive approaches.

The results of this study suggest that the greatest influence on teachers’ behavior in the classroom was the textbook. Each of the project teachers introduced the concept of limit in a similar manner, by using the graph provided on the first page of the chapter on limits in their textbook. The textbook then provided several examples involving the examination of intuitive graphical and numerical representations of limits. Correspondingly, the teachers covered each example in class. This reliance on the textbook, as well as the use of multiple representations, is discussed in the sections that follow.

Following the brief intuitive introduction to the concept of limit, the independent teachers spent considerable time on the algebraic evaluation of limits of given functions. A modest amount of time (less than 10 minutes, if any) was spent relating the intuitive idea of limits to the algebraic evaluation of limits. Again, the influence of the textbook may have played a minor role in these teachers’ decisions to spend considerable time on the evaluation of limits. Perhaps another plausible explanation may be the teachers’ goal of preparing students for the AP exam. The teachers’ focus on procedural, rather than intuitive understandings of limits may provide some insight into the literature on students’ perceptions of limits. This literature suggests that although students are capable of
producing correct answers to limit problems, there appears to be a mismatch between their intuitive notion of a limit and the answers they produced (Fischbein, Tirosh, & Melamed, 1981). Further analysis of the independent teachers’ focus on procedural understanding is contained in subsequent sections.

All the calculus textbooks used by the teachers in this study contained a section on the formal, epsilon-delta definition of limit. Accordingly, all the teachers in this study devoted class time to the formal definition of limit. In all cases, the introduction to the epsilon-delta definition of limit involved an intuitive, graphical interpretation. Once again, the independent teachers and the project teachers were separated with respect to their perceptions of the level of mathematical rigor needed by their students, as well as the amount of class time spent using the formal epsilon-delta definition to calculate the limits of various functions. The independent teachers felt that a high level of mathematical rigor was critical to understanding limits. In fact, they spent up to three days immersed in lessons involving the epsilon-delta definition of limit and related proofs, and they all asked their students to reproduce an epsilon-delta proof for a linear limit. Contrary to the considerable time spent on epsilon-delta arguments, the teachers indicated both to the researcher and their students that understanding the formal epsilon-delta definition of limits was not needed to understand other calculus topics.

Additionally, two of the independent teachers recognized that the epsilon-delta arguments were not being taught for understanding. Tall (1992) warned that presenting formal definitions and proof by means of logical development only teaches students advanced mathematical thought but not the process of advanced mathematical thinking. Additionally, in their exploration of students’ understanding of calculus concepts, Graham and Ferrini-Mundy (1989) found that second-semester calculus students did not understand formal limit notation. Gass (1992) further warned that if the epsilon-delta definition is attempted early in the course, “one runs the risk of confusing, discouraging, and alienating a sizable portion of the class” (p. 9). It is possible that the three goals the
teachers held for teaching calculus provided motivation for teaching, and holding their students accountable for, epsilon-delta arguments. It is interesting to note, however, that only a small portion of the AP (BC) calculus topics (see Appendix A) address the topic of rigorous epsilon-delta arguments. Thus, the goal of preparing for the AP exam may not be justified. It is possible that the teachers wished to inspire in their students an appreciation for the beauty of calculus. In this case, the beauty of calculus may have been perceived as part of the triumph of mathematics to formalize, using deltas and epsilons, the intuitive ideas of limits. Forcing students to duplicate epsilon-delta proofs of limits of linear functions may fall well short of imparting to students the magnitude of this triumph. Nevertheless, it may represent teachers’ tribute to the history and beauty of the logical structure of calculus.

A more probable explanation for the focus on mathematical rigor was the goal of preparing students for college calculus. As evidence of this explanation, one of the independent teachers claimed that if colleges decided that epsilon-delta arguments were no longer important in the teaching of calculus, then he would possibly not focus on the epsilon-delta arguments in his classroom. Thus, it is plausible that because these teachers were taught the epsilon-delta arguments in their own calculus course they felt it was important to teach this topic to their calculus students. This finding is consistent with research indicating that the most important factors leading to the development of elementary and lower level secondary mathematics teachers’ subject matter and pedagogical perceptions are their past experiences in mathematics classes (Ball, 1990a; 1990b; Marks, 1990; Thompson, 1984; Tirosh & Graeber, 1989).

The project teachers, on the other hand, only held their students accountable for the intuitive, graphical representation of the formal definition of limit. The epsilon-delta arguments received little or no attention in classroom lessons. It is interesting to note that the project teachers also viewed epsilon-delta arguments as insignificant with respect to introductory calculus prior to participation in the calculus reform project. It is possible
that this shared view was one of the factors that initially motivated them to join the calculus reform project. Researchers have yet to identify the factors that influence calculus teachers to participate in calculus reform projects. The project teachers’ reasons for de-emphasizing the epsilon-delta arguments were that their students would not need this understanding of limits in order to understand subsequent calculus topics. Additionally, the results indicated that the teachers may have felt that mathematical rigor could be postponed until further subsequent mathematics classes.

**Association Between Calculus Teachers’ Perceptions of Limits and Participation in a Calculus Reform Project**

The results of this study revealed differences between the independent teachers and project teachers in the following areas: (a) commitment to the textbook, (b) planning, (c) use of multiple representations, (d) attitude toward graphing technology, (e) classroom atmosphere, (f) examinations, (g) appropriate level of mathematical rigor needed for teaching calculus, and (h) the stability of perceptions. The reader is reminded that these factors may not be fully attributed to participation in the calculus reform project. It is possible that some of these factors could be attributed to the project teachers’ prior beliefs and practices. In fact, some of these factors, such as attitude toward graphing technology and appropriate level of mathematical rigor needed for teaching calculus, may have contributed to the project teachers’ initial decision to participate in the project. Given the design of this study, however, prior teacher beliefs were difficult to determine. The difference between project and independent teachers’ belief in the level of mathematical rigor needed for the teaching of mathematics was discussed in the previous section. The following paragraphs contain discussions of the other seven factors, as well an indication of the plausibility of a correlation between the teachers’ perceptions and participation in the calculus reform project.
As discussed previously, all of the teachers in the study taught lessons in the order suggested by the textbook. The project teachers, however, demonstrated a stronger, possibly blind, commitment to the textbook’s contents and order. This reliance on the approach and examples of the textbook may, in part, be attributed to the fact that project teachers were using the textbook for the first time, whereas the independent teachers had all used their textbooks for two years or more. Brown (1974) found a similar strong reliance on geometry and second year algebra texts. The teachers and students in the Brown study were heavily dependent on the content and order of topics in the textbook. In fact, similar to the findings of this study, the teachers rarely, if at all, presented topics or examples not contained in the textbook. Another plausible explanation for the commitment to the textbook demonstrated in this study is that the teachers may have felt that it was their job (or obligation) as a participant in the calculus reform project to incorporate the lessons and examples presented in the calculus reform textbook.

Associated with the commitment to the textbook and project factor, the project teachers’ planning tended to be short term. Though the individual teachers could not explain, and sometimes doubted, the authors’ reasoning behind a particular presentation or sequencing of material, they appeared to trust the authors’ reasoning and presented lessons as they were presented in the text. This result further supports the idea that these teachers felt committed (or obligated) to utilize the material in the calculus reform textbook. However, the teachers rarely looked ahead in the textbook to ascertain the authors’ motivation for a presentation or to understand the overall goals of the reform curricula. Part of the difficulty may have stemmed from the fact that there was not enough time to cover all of the sections in the textbook during the summer inservice. Additionally, the lack of time available for planning classroom lessons may have compelled the teachers to trust the authors’ reasoning. In Brown and Baird’s (1993) discussion of the influence of teachers’ beliefs and attitudes on classroom instruction, it was stated that in order for teachers to choose to teach according to the visions of mathematics reform,
they must believe that the mathematics and teaching described in the materials are indeed valuable. The project teachers appeared to give the authority and legitimacy of what and how calculus was to be taught to the authors of the calculus reform textbook. Thus, the findings of this study may cause one to doubt Brown and Baird's assumption. It is possible that teachers lease authority and legitimacy to the authors only the first time through the text and might be challenged in future versions of the course.

The independent teachers, on the other hand, gave the appearance of planning ahead, knowing exactly what they would or would not be covering. For these teachers, however, it was at least the third time through the text. It is probable that the structure and sequence of the lesson plans were easily remembered or recorded in previous years.

All of the teachers in this study discussed the importance of the intuitive notion of limit in the teaching of calculus. Furthermore, all of the teachers felt that students would be more convinced of the existence of a limit using a graph or a table of values. The research of Williams (1991) supported this claim, stating that "Students as a whole showed great faith in graphing as a means of understanding the limit concepts, doing limit problems, and justifying statements about limits" (p. 233). Though the independent teachers examined limits using graphical and numerical representations, the time spent on these representations was less than one full class period. Some of the reasons given by independent teachers for minimizing the use of graphical representations include: (a) using graphical representations to find limits was a "crutch," (b) the best way to find limits was algebraically, and (c) it is important for students to have immediate recall of several graphs. The most probable reason for focusing on algebraic procedures and memorization of graphs was the goal of preparing for the AP exam. Graphing calculators were not allowed during the exam, and thus, these skills were needed. Thompson (1984) warned that if teachers focused on the memorization of facts and algebraic procedures, students may obtain an inadequate understanding of the meaning of mathematical processes. Thompson also stated that the students of these teachers may eventually view
mathematics as nothing more than an accumulation of facts and rules to be filed away or used only for passing future tests. The fact that independent teachers focused on algebraic procedures and minimized their use of graphical representations may also explain why college calculus students are able to evaluate limits of continuous functions, but show little intuitive or geometric understanding of their results (Graham & Ferrini-Mundy, 1989).

The project teachers, on the other hand, consistently used a variety of representations: graphical, numerical, and symbolic. Again, the use of multiple representations reflected the textbooks presentation of each topic. The textbook incorporated the NCTM’s (1989, 1991) and NRC’s (1991) general recommendations for change in mathematics, which included the use of multiple representations to develop an understanding of mathematical concepts. Preparation for the AP exam and college calculus appears to provide the motivation for covering symbolic representations. Though reliance on the textbook was considered the primary reason for these teachers’ emphasis on multiple representations, access to and the emphasis on technology may have contributed to the consistent use of graphical and numerical representations. It is also possible that the use of multiple representations to enhance students’ understanding of mathematical ideas was part of these teachers’ teaching philosophy prior to participation in the project.

One of the recommendations for revitalizing the teaching of calculus focuses on the use of technology. The use of technology frees students from algebraic manipulation and allows them to investigate concepts using multiple representations (Douglas, 1986; NCTM, 1989; Steen, 1987; Tucker, 1990). Project and independent teachers differed with regard to the use of graphing technology. As a requirement of the calculus reform project, all students in the project teachers’ classes had graphing calculators. These calculators were an integral part of the project teachers’ classrooms and they were used frequently by the students in these classes. Commitment to the calculus reform project may partially explain the project teachers’ focus on the technology. It is more likely that
these teachers were interested in using graphing technology prior to participating in the calculus reform project. The calculus reform textbook, however, appeared to play a critical role in providing direction for the use of graphing calculators in these teachers’ classrooms.

A few students in the independent teachers’ classrooms used graphing calculators. The calculators were never used by the teachers in any of the classes observed or taped in this study. Again, preparation for the AP exam may have been the most logical explanation for not using graphing technology. As previously mentioned, the independent teachers focused on developing students’ algebraic skills. Research supports efforts to implement technology-intensive calculus courses, as they have been shown to have positive effects on student understanding without hindering their acquisition of algebraic skills (Beckman, 1988; Dick, 1990; Heid, 1988). Yet, it is doubtful that these teachers were familiar with these research results. Equally doubtful is the idea that knowing the results of these studies would affect their stance regarding the use of graphing technology in their classroom lessons. In the case of one independent teacher (Richard), a lack of motivation, time, and desire to consider the idea of incorporating technology into classroom lessons are more probable explanations for the lack of technology use.

The general atmosphere of the project teachers’ classrooms was quite different from that of the independent teachers. Each of the independent teachers’ lessons were generally teacher-centered and the students responded directly to the teacher. This traditional, teacher-centered method of instruction continues to dominate most mathematics classrooms (Brooks & Suydam, 1993). The teachers’ own prior experiences in traditional, teacher-centered environments may explain why these teachers taught in this manner. The project teachers, on the other hand, frequently used class discussions, driven by teacher questioning, to introduce lessons. Throughout the class period, students were asking questions of other students and of the teacher. As illustrated by the interviews, the project teachers attributed this type of atmosphere to the continual use of graphing
calculators by students in the classroom. This result supports Dick’s (1992) findings that the calculator brings about an increase in discovery activity on the part of students. Additionally, the classroom discourse found in project teachers’ classrooms resembled that reported by Heid and Baylor (1993). These researchers described situations in which students challenged one another to prove results using technology by generating appropriate examples and non-examples. A classroom atmosphere that encourages exploration and discussion corresponds to the recommendations for revitalizing calculus instruction. It is possible, however, that since the teachers were novice “reform” teachers and they lacked experience with the curriculum, more responsibility for learning was given to the students.

Differences between project and independent teachers also appeared in the types of questions asked on the limit unit examination. The examinations given by the independent teachers contained primarily questions involving the algebraic evaluation of limits. Furthermore, graphing calculators could not be used on any of the independent teachers’ examinations. These conclusions support and extend to higher level mathematics teaching the results reported by Schram, Wilcox, Lanier, and Lappen (1988) that traditional mathematics educators typically assess learning through paper-and-pencil tests where the ability to compute the correct answer is considered knowledge. This form of assessment is consistent with the teachers’ goal of preparing students to take the AP exam since graphing calculators were not allowed on the AP exam and the questions involving the concept of limit were mostly procedural in nature. Again, Thompson (1984) cautions that the students of these teachers may come to view mathematics as an accumulation of facts and rules to be used solely for the passing of future tests. This concern is relevant as the students may have viewed the goal of the class strictly as preparation for the AP exam. Research has shown that classroom testing sends a strong message to students regarding both the content and the kinds of thinking (in this case procedural or conceptual) that are valued (Kirkland, 1971; Wilson, 1994).
In contrast, the project teachers’ examinations included questions that encouraged the use of a variety of representations to find limits. Additionally, students could use graphing calculators on the project teachers’ exams. Problems requiring merely the algebraic evaluation of limits were not found on these exams. Each of the project teachers’ exam questions, however, were taken directly from the textbook. In some cases teachers doubted the authors’ reasoning behind problems while teaching them, yet only days later used the same problems on their exams. It is possible that the project teachers’ commitment to the goals of the calculus reform project prompted them to use the textbook questions on the exam. More probable, however, is the fact that technology-active questions are difficult to write and that teachers may not have had enough confidence, experience, or time to write new exam questions.

All but one teacher in the study commented that teaching the concept of limit enhanced their own understanding of the concept. Though the teachers felt that they developed an intuitive understanding of the concept of limit in college calculus, they did not understand limits formally until college analysis or graduate level courses. The independent teachers’ perceptions of limits appeared to remain stable throughout the entire study. The most probable reason for this stability is that the teachers were comfortable teaching calculus and it is quite likely that they taught it in the same way during the study year as they in previous years.

As time progressed, however, the project teachers, equally experienced at teaching calculus, expressed a change in their perceptions of limits and attributed the increased importance of the intuitive notion of the concept of limit as the topic that ties all of calculus together. This change was most probably brought about by the significant influence of the calculus reform text. The textbook emphasized the importance of limits in the teaching of calculus and thus the teachers may have gradually developed an increased awareness of the limit’s importance. Gradual change in teachers’ subject matter and
pedagogical perceptions of mathematics due to participation in special reform programs is well-documented (Carpenter et al., 1989; Putnam, 1987; Wood et al., 1991).

Summary of Calculus Teachers’ Subject Matter and Pedagogical Perceptions of Limits and the Correlation to Classroom Practice

The results of this study demonstrate that the relationship between calculus teachers’ subject matter and pedagogical perceptions of limits and their instructional practice is very complex. In this regard, the findings support Thompson’s (1984) research. The main factors affecting instructional practice in this study were the teachers’ goals of preparing students for the AP exam and college calculus, and the significant influence of the textbook. Thompson (1992) suggests that teachers’ goals for teaching AP calculus should be considered part of the teacher’s perception of calculus teaching. Analysis of the data in light of Thompson’s assertion indicate a greater degree of consistency between the teachers’ perceptions of calculus teaching and their classroom practice because the teachers’ goals in part define the teachers’ perceptions. Conflicts existed, however, between some of the goals teachers held for teaching limits and their perceptions of teaching limits. For example, the goal of preparing for the AP exam conflicted with the expressed desire to explore limits using graphing technology because graphing calculators were not allowed on the AP exam. Therefore, defining the goal of preparing students for the AP exam as part of the teachers’ perception of calculus teaching, as proposed by Thompson (1992), would lend consistency to the relationship between the teachers’ perceptions and classroom practice. Nevertheless, this defining terminology does not alleviate potential conflicts between teachers’ goals for teaching AP calculus and their perceptions of calculus teaching and classroom practice.

The significant influence of the textbook appeared to cause some inconsistencies between the project teachers’ subject matter and pedagogical perceptions of limits and
their classroom practice. As previously discussed, although these teachers periodically doubted the goals and reasoning behind the information provided in the new textbook, they faithfully followed the text. This apparent commitment to the calculus reform project, at least during their initial experience with the text, may have hindered the translation between teachers' perceptions of limits and classroom practice.

Limitations of the Study

The findings of this study are limited in generalizability to this population of calculus teachers for a variety of reasons: the representativeness of the sample, the limited material available for establishing a baseline for the group of project teachers, the limited number of classroom observations of subsequent topics involving the concept of limit, various modes of data collection which may have biased the teachers in the sample, and potential researcher bias as data collector.

It is obvious that one should be cautious in generalizing conclusions obtained from a sample of six teachers to a general population. In fact, the reader is reminded that for logistical reasons, the six teachers selected for participation in this study represented a small geographical area. Few reasons exist, however, to suggest that the characteristics of these teachers are remarkably different from calculus teachers elsewhere. A study involving a larger sample of teachers who were representative of the calculus teaching force will certainly strengthen the generalizability of the results. This type of study, however, would have been impractical based on logistics and the lack of related literature needed to guide such a study. Perhaps a large representative sample may now be used for exploration of the various hypotheses generated by this study.

The sample of the study contained three teachers who were participating in a calculus reform project and three teachers who were not. The only data collected prior to participation in any part of the calculus reform project were obtained through a
questionnaire and interview. It must be recognized that information gathered from teachers' reminiscence during interviews may be unreliable. Without direct observation of classroom practice, conclusions drawn from interview data on prior events may misrepresent actual events. The information obtained for the baseline, however, was primarily used to examine the stability of teachers' perceptions and thus did not substantially affect the major conclusions of the study. The methodology of future research investigating change in teachers' subject matter and pedagogical perceptions as a result of participation in a reform project should involve classroom observations prior to participation as well as during and after participation in the project.

Each of the teachers in the sample was observed twice during subsequent topics involving the concept of limit. Thus, the conclusions regarding the correlation between the teachers' perceptions of the role of limits in the teaching of calculus were generated in part from direct observation of two days' lessons: introductions to differentiation and integration. Lesson plans and classroom materials were collected and analyzed in order to compensate for the limited number of observations. While this information added to the data concerning what the teachers said they did in the classroom, it did not add to information about what they actually did. In order to effectively assess the role of limits in the teaching of calculus, observations need to be made in many class periods throughout the calculus course. Daily observations of six teachers for an entire year by one researcher is not feasible. A more suitable alternative would be to videotape daily classroom instruction and observe weekly.

Some of the modes of data collection may have biased the teachers in the sample. For example, prior to observations of subsequent topics involving the concept of limit the teachers were aware of the researcher's interest in the role of limits in the teaching of calculus. The reader is reminded that connections between limits and subsequent calculus topics were evident in each of the two observations on the introduction to differentiation and integration. As discussed previously, the extent to which the connections were made
was minimal. Additionally, except for the first completion of the questionnaire, the teachers were aware that the focus of this study was on the concept of limit when they completed their diagrams of the important concepts and relationships in calculus. All of the teachers, however, emphasized the importance of the concept of limit on the first completion of the questionnaire. This limitation could be eliminated in future research by utilizing a topic-independent questionnaire and conducting extended classroom observations.

Finally, a variety of data collection techniques were used in this study. However, the primary instrument for data collection and analysis was the researcher. It is acknowledged that although the researcher's past experiences with calculus reform were helpful with respect to the analysis of teachers' perceptions of limits, they may cause some concern with respect to the existence of potential biases. The reader is reminded that the researcher took steps to protect against researcher bias by maintaining a detailed journal of personal thoughts, insights, and decisions throughout the data collection and analysis. The journal entries were useful in helping the researcher recognize and transcend potential biases.

Implications and Recommendations for Mathematics Education

The preceding discussion of the study's limitations provided a number of recommendations for future research in mathematics education regarding appropriate methodology for assessing calculus teachers' subject matter and pedagogical perceptions. The overall findings of this study suggest additional directions for future research, as well as implications for teacher education, curriculum development, classroom practice, and the current calculus reform movement.

Mathematics educators participating in the current calculus reform movement are concerned with the content to be taught, as well as the way it should be taught. They are
calling for an increasing focus on conceptual understanding and meaningful learning. The subject matter and pedagogical perceptions that the AP calculus teachers in this study possessed with respect to limits causes one to question how well they could effectively teach calculus for conceptual understanding and meaningful learning. The findings of this study reveal that the significant influence of the AP exam, the influence of college calculus, the influence of the calculus textbook, the perception of calculus as a linearly-ordered collection of topics, the emphasis on procedural understanding, the contradictions between what was perceived as important and the amount of class time devoted to it, and the focus on formal epsilon-delta arguments are some of the factors which may impede AP calculus teachers' capacity to teach in this manner.

The significant influence of the AP calculus exam on teachers' behavior has important implications for classroom practice and calculus reform, as well as for the AP exam itself. The implications, however, appear to be double-edged. On one hand, the AP exam has been vilified by some in the calculus reform movement. To these individuals, the AP exam is considered an obstacle to true reform, preventing teachers from trying new things in the classroom. Though the AP calculus course description guide states that the primary concern is the development of an intuitive understanding of the concepts of calculus, the exam itself appears to reflect the importance of procedural understanding. The actual questions on the AP exam appear to send a powerful message to the teachers in this study since a primary focus of their teaching of calculus was preparing their students to be successful on the AP exam. Given the influence of this exam, the designers of the AP exam must develop questions that reflect the goals of the calculus reform movement if true reform is to happen.

On the other hand, if the influence of the AP exam is as significant as purported, then the exam itself could be used as a catalyst for change. Evidence of whether change in the AP exam affects change in the classroom may be gleaned as a result of the 1995 AP exam. Though the AP calculus course syllabus will remain the same, the 1995 AP exam
will change to incorporate graphing calculators. Further investigation is needed on what changes the addition of graphing calculators will have on what teachers think and do in the classroom, particularly those teachers who have not previously incorporated technology due to their goal of preparing for the AP exam. Furthermore, one wonders what changes would occur in the teachers’ subject matter and pedagogical perceptions of calculus if the AP calculus course syllabus were changed to reflect the goals of the calculus reform movement. Given the significance of the AP exam for AP calculus teachers, there is a need for continuous examination of the match between the questions on the exam and the goals of the current calculus reform movement.

Concerning the teachers’ goal of preparing students for college calculus, there are important implications for teacher education. Given this significance, communication and articulation are needed between institutions of higher education and secondary schools regarding the teaching and learning of calculus. There is much to be gained from substantive communication regarding curriculum development and instructional strategies for the teachers of both institutions. For example, if a particular college is adopting one of the current calculus reform texts, it may be advantageous for the collegiate educators to disseminate information regarding this adoption to the local high schools. Perhaps cooperative partnerships between college and high school instructors would be useful in articulating the goals of further mathematical study. These partnerships may also help the AP calculus instructors feel less isolated and provide validation for what they do in the classroom.

The data reported in this study suggest that the teachers’ past experience in college calculus may have impacted what they thought and did in the classroom. This finding is consistent with existing research (Ball, 1990a; 1990b; Marks, 1990; Thompson, 1984; Tirosh & Graeber, 1989). Therefore, for the preparation of future high school calculus teachers, it is critical that the collegiate calculus course be taught in the manner in which these pre-service teachers should teach their high school calculus courses. Ideally,
although perhaps less feasible, pre-service mathematics education programs could include a course on the teaching of calculus. During this course the pre-service teachers might be given the opportunity to reflect on what topics are important for the teaching of calculus as well as how the topics are interconnected. The results of this study also imply that calculus teachers should be given greater opportunities to reflect on the beauty of calculus and what it means to inspire students to appreciate this beauty. Reflecting on the structure and beauty of calculus may inspire teachers to recognize how mathematical thinking and knowledge grow and are accepted by the mathematical community.

The significant influence of the order of the sections in the textbook has important implications for classroom practice and the development and implementation of calculus reform curricula. First, if teachers are going to rely primarily on the text for their instructional approach, then the text can be a powerful vehicle for reform. This statement does not mean that it is possible or desirable to write teacher-proof curricula. However, it implies that if the desire is for calculus to be perceived as a connected and integrated subject with an emphasis on conceptual learning and applications, then the textbook should be written with this goal in mind.

The commitment to the calculus reform textbook demonstrated by the project teachers in this study implies that the overall goals, or the “big picture,” as well as suggestions for best implementing these goals in the classroom, should be addressed in each section of the text. This change to the textbook is especially important considering the project teachers’ limited opportunities to plan for instruction. Thus, they may have never developed the “big picture” until the end of the course, if at all. Perhaps a longer or additional preparation period may be useful for teachers trying reform calculus curricula. This reliance on the textbook also has ramifications for teacher inservice. If curriculum developers want their materials to be implemented with respect to the spirit in which they were developed, then more time should be spent during teacher inservice examining the “big picture,” including the goals and motivation behind various concepts, methods, and
connections between the topics and textbook sections. Just as discussions should also address appropriate methods for implementing these goals, research should examine the type of support teachers need in this time of reform. Additionally, longitudinal studies that follow project teachers through their second experience with the curricula would be useful. Specifically, this type of study may indicate whether or not the teacher was able to develop the “big picture.” Additionally, this type of study may provide information regarding the stability of teachers’ perceptions. It would also be interesting to see if the teachers continued to be committed to the textbook. Furthermore, one could investigate whether teachers’ questions regarding the authors’ reasoning persist or disappear. Examining how teachers resolve or choose to handle this conflict could provide better support for implementing reform curricula.

Developers of calculus reform curricula should not underestimate the gravity of teachers’ perceptions of calculus as linearly ordered. Therefore, this result should be given serious consideration in the development of materials. For example, if the desire is to implement a spiraling curriculum in which calculus topics are introduced informally the first time through with increased formalization during subsequent presentations, then the designers of the curriculum need to be aware that the teachers’ perceptions of calculus as linearly ordered may hinder this implementation. The teacher may have difficulty deciding what needs to be said during the initial presentation of a concept and what needs to saved for later. Again, implementation of this type of curriculum may benefit from inservice training that addresses this concern.

The teachers in this study professed that the intuitive understanding of the concept of limit was critical to understanding other calculus concepts. The time the independent teachers spent helping students develop this type of understanding of limits, however, was brief. These teachers quickly moved to procedural and formal discussions of limits. In the project teachers’ case, they spent several days helping students develop an intuitive understanding of limits. However, they were closely following the textbook.
Additionally, in all cases, the time spent on relating this intuitive notion of limit to subsequent calculus topics was minimal. In interpreting these results, it is quite plausible that the teachers really perceived that an intuitive understanding of limits is critical to understanding other calculus topics. It is possible that they were unaware of the discrepancy between this perceived importance and the actual amount of class time devoted to this topic. Additionally, one could not conclude that this conflict was absent from the project teachers since they were following the textbook regardless of their perceptions. Almost certainly, the teachers felt they were effectively prompting an intuitive understanding of limits, although the brevity of class time devoted to it by the independent teachers was inconsistent with their stated importance of the goal. If this is the case, how does one go about exposing this inconsistency to teachers? Thompson (1984) suggests that the extent to which experienced teachers’ perceptions are revealed in their classroom practice depends greatly on the teachers’ tendency to reflect on their perceptions, subject matter, instruction, and students. Participating in an inservice course that encourages reflection may benefit these experienced calculus teachers. It is possible that the process of reflection will expose them to the conflict between the stated emphasis of a particular topic or goal and the amount of class time devoted to it. Further research is needed to identify similar discrepancies between teachers’ perceived importance of an instructional goal and amount of class time spent in pursuit of that goal. Research exploring the factors that influence this discrepancy would also be beneficial.

The results of this study have important implications regarding the examination of the role of the formal definition of limit and epsilon-delta arguments in the teaching of calculus. Much of the discussion regarding the development of calculus reform curricula has centered on streamlining the traditional syllabus to create a “lean and lively” course (Douglas, 1986). Coinciding with this objective is the school of thought that holds formal definitions of limits and epsilon-delta arguments to be inappropriate for beginning calculus students. In fact, Halmos (1990) concludes that “Very few people advocate epsilons and
When considering the teachers’ subject matter and pedagogical perceptions of limits found in this study, as well as the findings of research on students’ misunderstandings of limits (Davis & Vinner, 1986; Graham & Ferrini-Mundy, 1989; Hart, 1991/1992; Orton, 1983a; Tall & Vinner, 1981; Williams, 1989/1990, 1991), one must question whether teaching the formal epsilon-delta definition of limit or rigorous epsilon-delta arguments are appropriate for AP calculus or first semester college calculus. Recommendations that formal epsilon-delta definitions and arguments be delayed until further mathematical study are related to constructivist notions of the proper goals of education. For instance, von Glaserfeld (1987) states that traditional teachers dispense the truth, whereas constructivist teachers guide students’ conceptual organization of mathematical experiences. When teaching epsilon-delta definitions and arguments, the teachers in this study appeared to be dispensing truth rather than guiding students’ conceptual organization of the topic. Confrey (1990) states that when one applies constructivism to the teaching of mathematics, one must reject the assumption that information can be simply passed on to a set of learners and expect that it results in understanding. The teachers in this study, however, did not necessarily expect students to understand epsilon-delta definition and arguments. Additionally, understanding of this topic was not considered necessary for understanding subsequent calculus topics. Thus, the questions remain. When should formal epsilon-delta definition and arguments be taught? Is it possible to teach this topic for understanding of most students at this level? The answers to these questions are certainly beyond the scope of this study. Further investigations of this issue would be beneficial for mathematics educators striving to determine appropriate content for a “lean and lively” elementary calculus. Analogously, further research is needed to examine whether intuitive notions of limits and other calculus concepts actually lead to conceptual understanding of calculus at the introductory level.

An article written jointly by the NCTM and NCSM (1986) states that any effective reform effort is dependent on consistency of teachers’ perceptions with reform
recommendations. In this study the project teachers’ perceptions of limits and the teaching of limits included the following characteristics consistent with the recommendations of the calculus reform movement: the use of multiple representations when examining calculus topics, implementation of graphing calculators into lessons, and classroom atmospheres that encourage classroom discussion. Though these characteristics parallel the goals of the calculus reform project, the methodology used in this study made it difficult to determine whether credit for these observations could actually be given to their participation in the calculus reform project. It is possible that decisions to participate in a reform project may have been based on some of these characteristics. Regardless, the results from the teachers in this study advocate a calculus reform project of this type in achieving the goals of calculus reform. Longitudinal research methods that account for examination of a teacher’s classroom practice prior to, as well as during, participation in a calculus reform project would provide evidence on the possibility of teacher change. Additionally, if the project teachers and independent teachers were indeed separated from the beginning based on interest in updating classroom practice or incorporating graphing technology, then it may be important to focus on requiring teachers to participate in lifelong learning such as staff development projects. Research would be necessary to determine if such a requirement would be beneficial to either teacher or student understanding of high school calculus.

In addition to the previously mentioned implications, the results of this study also provide information that may further the understanding of the relationship between teachers’ perceptions and classroom practice. First, the findings of this investigation supplement the body of literature indicating that teachers’ subject matter and pedagogical perceptions are not related in a simple way to their classroom practice. In Thompson’s discussion (1992) of the literature, many sources of influence appear to suggest the existence of a complex relationship. This study provides evidence that one may add preparation for the AP exam, preparation for college calculus, and commitment to a new
calculus reform textbook to this list of sources influencing classroom practice. Secondly, findings in this study suggest that the task of deciding whether teachers' perceptions of teaching are congruent with their classroom practice itself is not simple. When developing an understanding of teachers' perceptions of teaching mathematics, the literature states that the goals the teachers have for their mathematics program are to be considered part of their perception of teaching mathematics (Thompson, 1992). The results of this study demonstrated that conflicts existed between the teachers' goal of preparing for the AP exam and their perceptions of teaching limits. If the goal of preparing for the AP exam is a conflicting part of the teachers' perception of teaching limits but is a driving goal behind classroom practice, then are the teachers' perceptions considered congruent to classroom practice? Given the influence this goal appeared to have on the AP calculus teachers in this study, it may be beneficial to conduct a similar study with calculus teachers for which this goal is not applicable.

Finally, though the results of this study appear to provide insight into the findings on students' misunderstandings, little is known about the extent to which teachers communicate their mathematical perceptions to their students. Several questions regarding the possible influence of teachers' perceptions of mathematics on students' perceptions of the subject arise from the findings of this study. For example, what influence does the amount of class time spent on particular topics have on students' perceptions? What impact do the sources that influence teachers' classroom practice have on the students' perceptions? Plausible as it may seem that teachers' perceptions and classroom practice influence students' perceptions, the relationship between teachers' and students' perceptions of calculus is largely unknown and a viable avenue of future research.
REFERENCES


Dick, T. P. (1990). Super calculators as calculus laboratory instruments. Unpublished manuscript, Mathematics Department, Oregon State University, Corvallis, OR.


APPENDIX A
DISCUSSION OF ADVANCED PLACEMENT CALCULUS EXAMINATION

The following discussion was taken directly from the booklet Advanced Placement
Course Description: Mathematics - Calculus AB, Calculus BC (College Entrance
Examination Board, 1992). This discussion includes a description of the Calculus AB and
Calculus BC examinations, the philosophy of The College Board regarding the
examinations, the topics that should be covered in a calculus course designed to prepare
students to take the examination, the format of the examination, and the philosophy of The
College Board regarding the use of calculators on the examination.

An Advanced Placement course in mathematics consists of a full
academic year of work in calculus and related topics comparable to
courses in colleges and university. It is expected that students who take
an AP course in calculus will seek college credit or placement, or both,
from institutions of higher learning. The AP Program offers description
of two calculus courses and examination for each course. The two
course descriptions and the two corresponding examinations are
Calculus AB and Calculus BC.

Both courses described here represent college-level mathematics for
which most colleges grant advanced placement and credit. Most
colleges and universities offer a sequence of several semester courses in
calculus, and entering students are placed within this sequence
according to the extent of their preparation as measured by the results
of an AP Examination or other criteria. Appropriate credit and
placement are granted by each institution in accordance with local
policies. At many institutions Calculus AB is given a full year’s credit.
The content of Calculus BC is designed to qualify the student and credit
one semester beyond that granted for Calculus AB. In designing which
AP Examination a student should take, the student and teacher should
weigh carefully the student’s preparation in calculus in relation to the
AP course descriptions and the mathematics curriculum of the student’s
prospective college. Many colleges provide statements regarding their
AP policies in their catalogs.

In establishing a school curriculum, teachers should carefully consider
not only content but also the level of sophistication at which new
concepts are introduced. Although intuition is extremely important in
mathematics, understanding of some mathematical concepts is best
acquired by formal treatment.
Both Calculus AB and Calculus BC are primarily concerned with an intuitive understanding of the concepts of calculus and experiences with its methods and applications. The expanded content of Calculus BC requires some additional knowledge of the rudiments of the theoretical tools of calculus. Use of the word "intuitive" is not meant to suggest a reduction of either clarity of concept or precision of expression. Rather, it attempts to distinguish between a calculus course that emphasizes precise proofs of all theorems - rigor in the formal sense - and a calculus course that states definitions and theorems correctly but that frankly defers some proofs until later. In deciding which course to offer, many teachers will find it helpful to seek advice from departments of mathematics in secondary schools that have had experience with the program and from colleges in their vicinity or to which their students may be applying.

Comparison of Topics in Calculus AB and Calculus BC

If students are to be adequately prepared for the Calculus AB Examination, most of the year's course must be devoted to the topics of differential and integral calculus. In addition to the topics covered in Calculus AB, the Calculus BC course included other topics such as infinite series. Column I below lists topics that should be covered in the Calculus AB course or in preparation for it. It is assumed that each topic in Column I will have been studied as a prerequisite for or as a part of the Calculus BC course. The topics in Column II are additional topics that are to be covered in Calculus BC but are not included in Calculus AB.
A. Functions and Graphs
   1. Properties of functions
      a. Domain and range
      b. Sum, product, quotient, and composition
      c. Inverse functions
      d. Odd and even functions
      e. Periodic functions
      f. Zeros of a function
   2. Properties of graphs
      a. Intercepts
      b. Symmetry
      c. Asymptotes
      d. Relationships between the graph of \( y = f(x) \) and the graphs of 
         \( y = kf(x) \), \( y = f(kx) \), \( y = |f(x)| \), 
         \( y-k = f(x-h) \), and \( y = f(|x|) \).

B. Limits and Continuity
   1. Finite limits
      a. Limit of a constant, sum, product, or quotient
      b. One-sided limits
      c. Limits at infinity
   2. Nonexistent limits
      a. Types of nonexistence
      b. Infinite limits
   3. Continuity
      a. Definition
      b. Graphical interpretation of continuity and discontinuity
      c. Existence of absolute extrema of a continuous function on a closed interval \([a,b]\)
      d. Application of the Intermediate Value Theorem

C. Differential Calculus
   1. The derivative
      a. Definition
      b. Derivative formulas

Column II
(Calculus BC)

g. Vector functions
h. Parametric equations
i. Conversion between polar and rectangular coordinates

e. Parametrically defined curves
f. Graphs in polar coordinates

d. Rigorous \( \varepsilon-\delta \) definitions

c. Rigorous definitions
(1) Derivations of
d/dx (x^n) = nx^{n-1}
for a positive integer n and
d/dx (sin x) = cos x
(2) Derivatives of elementary
functions
(3) Derivatives of sums, products,
and quotients
(4) Derivatives of composite
functions (chain rule)
(5) Derivatives of implicitly
defined functions
(6) Derivatives of higher order
(7) Derivatives of inverse functions
(including inverse trigonometric
functions)
(8) Logarithmic differentiation
(9) Derivatives of
vector functions
and parametrically
defined functions

2. Statements and applications of theorems about derivatives
a. Relationship between differentiability
   and continuity
b. The Mean Value Theorem
c. L’Hopital’s Rule for the indeterminate
   forms 0/0, \(-\infty/\infty\), and \(0^\infty\)
d. L’Hopital’s Rule for
   the indeterminate
   forms 0^0, 1^\infty, \infty^0,
   and \(-\infty - \infty\)

3. Applications of the derivative
a. Geometric applications
   (1) Slope of a curve; tangent and
       normal lines
   (2) Increasing and decreasing
       functions
   (3) Critical points
   (4) Concavity
   (5) Points of inflection
   (6) Curve sketching
   (7) Differentials and linear
       approximations
   (8) Newton’s method for
       approximating zeros of functions
b. Optimization problems
   (1) Relative and absolute maximum
       and minimum values
   (2) Extreme value problems
c. Rate of change problems
(9) Tangent lines to
    parametrically
    defined curves
323

(1) Average and instantaneous rates of change
(2) Velocity and acceleration in linear motion
(3) Related rates of change

(4) Velocity and acceleration vectors for motion on a plane curve

D. Integral Calculus
1. Antiderivatives (indefinite integrals)
   a. Techniques of integration
      (1) Basic integration formulas
      (2) Integration by substitution
         (change of variables)
      (3) Simple integration by parts
   b. Applications of antiderivatives
      (1) Distance and velocity from acceleration with initial conditions
      (2) Solutions of \( f(x)dx = g(y)dy \)
         (separable differential equations)
      (3) Solutions of \( y' = ky \) and applications to growth and decay

2. The definite integral
   a. Definition of the definite integral as the limit of sums
   b. Properties of integration
   c. Approximations to the definite integral
      (1) Rectangles (Riemann sums)
      (2) Trapezoids (Trapezoidal Rule)
   d. Fundamental theorems
   e. Applications of the definite integral
      (1) Area under a curve; area between curves
      (2) Average value of a function on an interval

   (3) Parabolas (Simpson’s Rule)
   (4) Composite functions defined by integrals
   (5) Area bounded by polar curves
(3) Volumes of solids with known cross sections, including solids of revolution (disc and washer methods) about the x-axis, the y-axis or a line parallel to either axis
(4) Volumes of solids of revolution (shell method) about the x-axis, the y-axis or a line parallel to either axis

(6) Length of a path, including parametric curves
(7) Work as an integral, with either force or displacement as a variable, e.g., Hooke’s Law (conversion of units not required)

f. Improper integrals (as limits of definite integrals)

E. Sequences and Series
1. Sequences of real numbers; convergence
2. Series of real numbers
   a. Convergence
      (1) Tests for convergence: comparison (including limit comparison), ratio, root, and integral tests
      (2) Absolute and conditional convergence
   b. Special series
      (1) Geometric series
      (2) Alternating series and error approximation
      (3) p-series
3. Power series
   a. Manipulation of series, e.g., addition of series, substitution, term-by-term differentiation and integration
   b. Convergence
The Examinations

The AP Calculus Examinations are three hours long and seek to determine how well a student has mastered the concepts and techniques of the subject matter of the corresponding course. Each examination consists of (1) a multiple-choice section testing proficiency in a wide variety of topics, and (2) a problem section requiring the student to demonstrate the ability to carry out proofs and solve problems involving a more extended chain of reasoning. In the determination of the grade for each examination, the two sections are given equal weight. As the examinations are designed for full coverage of the subject matter, it is not expected that all students will be able to answer all the questions.

Use of Calculators

Beginning in May of 1993, both the Calculus AB and Calculus BC Examinations contain questions for which a scientific calculator (nongraphing, nonprogrammable) is necessary or advantageous. Students will be expected to bring to the examinations a scientific calculator from an approved list.

While scientific calculators are certainly useful, it is anticipated that graphing calculators will soon become the norm in calculus classrooms. AP Calculus teachers are encouraged to make immediate use of this technology to improve their students’ understanding of calculus and to educate them in the creative use of this powerful tool for problem-solving. Requiring or even allowing graphing calculators on the AP Examinations at this time, however, would create equity problems that might outweigh the benefits of encouraging their use. As access and
familiarity increase, the equity problems will disappear, making the introduction of graphing calculators on future AP Examinations both possible and desirable. The AP Calculus Development Committee has set May, 1995 as a target date for introducing problems on the Calculus AB and Calculus BC Examinations for which a graphing calculator (without symbolic algebra capabilities) would be necessary or advantageous. It is hoped that AP Calculus teachers will explore and discuss the implications of this technology in the meantime so that the examinations can continue to reflect the curriculum rather than direct it.
APPENDIX B
DESCRIPTION OF THE DICK AND PATTON (1990) TEXTBOOK

Preface Synopsis

The entire introductory course in calculus is being reexamined under the closest scrutiny that it has received in several years. Through a special funding initiative, the National Science Foundation (NSF) has made resources available for a variety of calculus curriculum revision efforts to be tried and implemented. The Oregon State University Calculus Curriculum Project is one of these NSF-funded efforts, and this book is one of the major results of this project.

A brief glance at the Table of Contents might suggest that the text does not differ radically from the traditional calculus text in terms of major topics. This is as it should be - calculus reform will not change the importance and vitality of the major ideas of calculus, and any wholesale departure from those ideas should be viewed with great skepticism. What is possible is a fresh approach to these important ideas in light of the availability of modern technology. In particular, the technology can invite us to change or adopt new emphases in instruction. The four major themes of the Oregon State Project are (a) making intelligent use of technology, (b) using a multiple representations approach to functions, (enhancing visualization and approximation, and (d) emphasizing problem-solving and mathematical modeling.

Chapter Two discusses limits and continuity. Numerical and graphical approaches receive equal, if not more emphasis than symbolic techniques. The rigorous epsilon-delta definitions are included, but their meanings are explained with reference to their numerical and graphical consequences, rather than heavy emphasis on proofs. For example, the definition of continuity of a function has a dynamic interpretation in terms of the scaling of a graphing window. The "δ-hunt for a particular ε becomes a search for a certain horizontal scaling, given a vertical scaling. Numerically, ε’s and δ’s can be interpreted as output and input tolerances.
Table of Contents

0. Introduction: What is Calculus?

1. Real Numbers and Functions - The Language of Calculus
   1.1 Absolute value, set and interval notation
   1.2 Functions - Notation, terminology, and representations
   1.3 A dictionary of functions
   1.4 Making new functions from old

2. Limits and Continuity

Much of calculus concerns describing the behavior of functions, meaning how the outputs act or change relative to the inputs. Limits provide a very effective terminology for describing the behavior of a function, and in this chapter, we introduce the language of limits both visually and through the idea of error tolerances. We’ll show how limits can be estimated both numerically and graphically. Then we will use the language of limits for discussing the notion of continuity and its consequences, and for describing the asymptotic behavior of a function. The idea of limit also underlies the concept of derivative, as we’ll see in the next chapter.

2.1 What are limits?
   Examination of left- and right-hand limits
   Estimating limits numerically and graphically

2.2 Definition of limit
   Formal definition of limit
   The function f has a limit L at x=a, written
   \[
   \lim_{x \to a} f(x) = L
   \]
   or \( f(x) \to L \) as \( x \to a \)
   if and only if the following condition holds:
   Given any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \)
   whenever \( 0 < |x - a| < \delta \).

   Graphical interpretation of the formal limit definition
   Limit proofs - the epsilon-delta machine
   New limits from old
   The Squeezing Principle
   An application of the Squeezing Principle

2.3 Continuity
   Definitions of continuity
Combining continuous functions
Examples of continuous functions

2.4 Analyzing discontinuities and asymptotic behavior
Skips, holes, jumps, and poles
Vertical asymptotes
Consequences of continuity
    Intermediate Value Theorem
    The Extreme Value Theorem
    Intermediate Zero Theorem
Horizontal asymptotes and end behavior

3. The Derivative
3.1 Linear functions
3.2 What is the derivative?
3.3 Computing and estimating derivatives
3.4 Derivative formulas, notations, and properties
3.5 New derivatives from old

4. Derivative as Measurement Tool
4.1 Physical interpretation of derivative - rate of change
4.2 Best linear approximations
4.3 Using the derivative to analyze function behavior
4.4 Higher order derivatives

5. Applications of the Derivative
5.1 Problem-solving and mathematical modeling
5.2 Solving optimization problems - finding the extrema
5.3 Implicit differentiation and related rates
5.4 Parametric and polar equations

6. The Integral
6.1 Definite integrals - terminology and examples
6.2 Computing and estimating definite integrals
6.3 Indefinite integrals - antiderivatives
6.4 The fundamental theorem of calculus
6.5 Numerical integration techniques

7. Differential Equations
7.1 What is a differential equation?
7.2 Exponential and logarithmic functions
7.3 Applications of the exponential model
7.4 Methods of integration

8. Integral as Measurement Tool
8.1 Using definite integrals to measure area
8.2 Using definite integrals to measure volume
8.3 Using definite integrals to measure arc length
8.4 Integration using polar coordinates

9. Applications of the Integral
9.1 Using definite integrals to measure averages
9.2 Using definite integrals to make physical measurements
9.3 Improper integrals
9.4 Using definite integrals to measure probability

10. Sequences and Approximations
10.1 Examples of sequences
10.2 Convergence and divergence of sequences
10.3 Finding limits - indeterminate forms
10.4 Root-finding approximation methods

11. Series and Function Approximation
11.1 Series
11.2 Convergence tests for series
11.3 Ratio and root tests - power series
11.4 Taylor polynomials and series
Diagram your overview of the important concepts and relationships in calculus.
APPENDIX D
ADVANCED PLACEMENT CALCULUS QUESTIONNAIRE: PART B

1. Describe in your own words what a limit is.

2. Looking back on your own mathematics education, when did you feel that you really understood the meaning of the statement:

\[ \lim_{{x \to a}} f(x) = L \]

Describe in your own words what you think about when you see this notation.

3. What is your favorite way to explain what a limit is to a beginning calculus class?
4. What are the different arguments you have seen to support the following statement:

\[ \lim_{{x \to 0}} \frac{{\sin x}}{{x}} = 1 \]

Which of these arguments was most convincing to you?

Which of these arguments is most convincing to your students?

4. How important is the concept of limit in teaching calculus?

What level of rigor do you feel is appropriate?
APPENDIX E
DESCRIPTION OF THE LEITHOLD (1986) TEXTBOOK

Preface Synopsis

I have endeavored to achieve a healthy balance between the presentation of elementary calculus from a rigorous approach and that from the intuitive and computational point of view. Bearing in mind that a textbook should be written for the student, I have attempted to keep the presentation geared to a beginner's experience and maturity. I desire that the reader be aware that proofs of theorems are necessary and that these proofs be well motivated and carefully explained so that they are understandable to the student who has achieved an average mastery of the proceeding sections in the book. If a theorem is stated without proof, I have generally augmented the discussion by both figures and examples, and in such cases I have always stressed that what is presented is an illustration of the content of the theorem and is not the proof.

Chapter One contains a thorough discussion of limits and continuity. The definition of limit is stated in the form "if a then b" rather than its logical equivalent "b whenever a," and this form of the definition is used throughout the text. Proofs of some theorems on limits appear in Supplementary Sections 1.9 and 1.10. The treatment of continuity has been modified, and there is an added discussion of continuity of the trigonometric functions. Some additional geometrical interpretations are included.

Table of Contents

Prelude

Chapter Zero Topics in Precalculus
0.1 Real numbers and inequalities
0.2 The number plane and graphs of equations
0.3 The distance formula, the circle, and the midpoint formulas
0.4 Equations of a line
0.5 Functions
0.6 The trigonometric functions
0.7 An application of the tangent function to the slope of the line
The two fundamental mathematical operations in calculus are differentiation and integration. These operations involve the computation of the derivative and the definite integral, each of which is based on the notion of limit. The treatment of limits, along with that of continuity, appears in Chapter One. The idea of a limit of a function is first given step-by-step motivation, which brings the discussion from computing the value of the function near a number, through an intuitive treatment of the limiting process, up to a rigorous epsilon-delta definition.

1.1 Graphs of functions
1.2 The limit of a function
The formal definition of limit
Let \( f \) be a function that is defined at every number in some open interval containing \( a \), except possibly at the number \( a \) itself. The \textit{limit of} \( f(x) \) \textit{as} \( x \) \textit{approaches} \( a \) \textit{is} \( L \), written as
\[
\lim_{{x \to a}} f(x) = L
\]
if the following statement is true:
Given any \( \varepsilon > 0 \), however small, there exists a \( \delta > 0 \) such that
if \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \)

1.3 Theorems on limits of functions
1.4 One-sided limits
Formal definition of one-sided limits
1.5 Infinite limits
Formal definition of infinite limits
Definition of vertical asymptotes
1.6 Continuity of a function at a number
Definitions of continuity
1.7 Continuity of a composite function and continuity on an interval
More definitions of continuity
1.8 Continuity of trigonometric functions and the Squeeze Theorem
1.9 Proofs of some theorems on limits of functions (Supplementary)
1.10 Additional theorems on limits of functions (Supplementary)

Chapter Two The Derivative and Differentiation
2.1 The tangent line
2.2 The derivative
2.3 Differentiability and continuity
2.4 Theorems on differentiation of algebraic functions
2.5 Rectilinear motion and the derivative as a rate of change
2.6 Derivatives of the trigonometric functions
2.7 The derivative of a composite function
2.8 The derivative of the power function for rational exponents
2.9 Implicit differentiation
2.10 Related rates
2.11 Derivatives of higher order
2.12 The differential

Chapter Three Extreme Function Values and Techniques of Graphing
3.1 Maximum and minimum function values
3.2 Applications involving an absolute extremum on a closed interval
3.3 Rolle’s theorem and the mean value theorem
3.4 Increasing and decreasing functions and the first-derivative test
3.5 Concavity and points of inflection
3.6 The second derivative test for relative extrema
3.7 Limits at infinity
   Formal definitions of limits at infinity
3.8 Asymptotes of a graph
   Definition of horizontal asymptotes
3.9 Applications to drawing a sketch of the graph of a function
3.10 Further treatment of absolute extrema and applications
3.11 Applications of differentiation in economics and business (Supplementary)
3.12 Numerical solutions of equations by Newton’s method (Supplementary)

Chapter Four The Definite Integral and Integration
4.1 Antidifferentiation
4.2 Some techniques of antidifferentiation
4.3 Differential equations and rectilinear motion
4.4 Area
4.5 The definite integral
4.6 Properties of the definite integral
4.7 The mean-value theorem for integrals
4.8 The fundamental theorem of calculus
4.9 Area of a region in the plane
4.10 Applications of integration in economics and business (Supplementary)

Chapter Five Applications of the Definite Integral
5.1 Volume of a solid of revolution: Circular-disc and circular-ring methods
5.2 Volume of a solid of revolution: Cylindrical-shell method
5.3 Volume of a solid having known parallel plane sections
5.4 Work
5.5 Length of arc of the graph of a function
5.6 Center of mass of a rod
5.7 Centroid of a plane region
5.8 Centroid of a solid of revolution (Supplementary)
5.9 Liquid pressure (Supplementary)

Chapter Six Inverse Functions, Logarithmic Functions, and Exponential Functions
6.1 Inverse functions
6.2 Inverse function theorem and the derivative of the inverse of a function
6.3 The natural logarithmic function
6.4 Logarithmic differentiation and integrals yielding the natural logarithmic function
6.5 The natural exponential function
6.6 Applications of the natural exponential function
6.7 Other exponential and logarithmic functions

Chapter Seven Inverse Trigonometric Functions and Hyperbolic Functions
7.1 The inverse trigonometric functions
7.2 Derivatives of the inverse trigonometric functions
7.3 Integrals yielding inverse trigonometric functions
7.4 The hyperbolic functions
7.5 The inverse hyperbolic functions (Supplementary)

Chapter Eight Techniques of Integration
8.1 A synopsis of integration formulas
8.2 Integration by parts
8.3 Integration of powers of sine and cosine
8.4 Integration of powers of tangent, cotangent, secant, and cosecant
8.5 Integration by trigonometric substitution
8.6 Integration of rational function by partial fractions when the denominator has only linear factors
8.7 Integration of rational functions by partial fractions when the denominator contains quadratic factors
8.8 Integration of rational functions of sine and cosine
8.9 Miscellaneous substitution
8.10 Numerical integration
8.11 Integrals yielding inverse hyperbolic functions (Supplementary)
8.12 Use of a table of integrals (Supplementary)

Chapter Nine Indeterminate Forms, Improper Integrals, and Taylor’s Formula
9.1 The indeterminate form 0/0
9.2 Other indeterminate forms
9.3 Improper integrals with infinite limits of integration
9.4 Other improper integrals
9.5 Taylor’s formula
Chapter Ten  Polar Coordinates and the Conic Sections
10.1  The polar coordinate system
10.2  Graphs of equations in polar coordinates
10.3  Area of a region in polar coordinates
10.4  The parabola and translation of axes
10.5  The ellipse
10.6  The hyperbola
10.7  Rotation of axes
10.8  A unified treatment of conic sections and polar equations of conics
10.9  Tangent lines of polar curves (Supplementary)

Part 2  Infinite Series
Chapter Eleven  Sequences and Infinite Series of Constant Terms
11.1  Sequences
11.2  Monotonic and bounded sequences
11.3  Infinite series of constant terms
11.4  Four theorems about infinite series
11.5  Infinite series of positive terms
11.6  The integral test
11.7  Alternating series
11.8  Absolute and conditional convergence, the ratio test, and the root test
11.9  A summary of tests for convergence or divergence of an infinite series

Chapter Twelve  Power Series
12.1  Introduction to power series
12.2  Differentiation of power series
12.3  Integration of power series
12.4  Taylor series
12.5  The binomial series
APPENDIX F
DESCRIPTION OF THE LARSON AND HOSTETLER (1986) TEXTBOOK

Preface Synopsis

What is calculus? We begin to answer this question by saying that calculus is the reformulation of elementary mathematics through the use of a limit process. If limit processes are unfamiliar to you this answer is, at least for now, somewhat less than illuminating. From an elementary point of view, we may think of calculus as a "limit machine" that generates new formulas from old. Actually, the study of calculus involves three distinct stages of mathematics: precalculus mathematics (the length of a line segment, the area of a rectangle and so forth), the limit process, and new calculus formulations (derivatives, integrals, and so forth).

On the following two pages we have listed some familiar precalculus concepts coupled with their more powerful calculus versions. As you proceed through this text, we suggest that you come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as follows: precalculus (Chapter One), the limit process (Chapter Two), and new calculus formulas (Chapter Three). This cycle is repeated many times on a smaller scale throughout the text. We wish you well in your venture into calculus.

Chapter Two contains the presentation of limits. Limits are defined intuitively in Section 2.1, and the section now contains a strategy for finding limits. Section 2.2 discusses techniques for evaluating limits. Continuity is introduced in Section 2.3 and infinite limits in Section 2.4. Finally, the chapter closes with an $\epsilon-\delta$ approach to limits. Some of the more technical proofs of theorems in Chapter Two are given in the appendix.

Table of Contents

1. The Cartesian Plane and Functions
   1.1 The real line
   1.2 The cartesian plane, the distance formula, and circles
   1.3 Graphs of equations
2. Limits and Their Properties

The notion of a limit is fundamental to the study of calculus. For this reason it is important for you to acquire a good working knowledge of limits before moving on to other calculus topics. In this chapter, we will discuss limits in two stages. In the first four sections, we build an intuitive understanding of limits by examining various kinds of limits, ways to evaluate limits, and the role of limits in defining continuity. This background is followed, in Section 2.5, by a more rigorous development of the limit concept. This two-stage approach follows closely to the historical evolution of limits. In fact, the formal definition of limits in section 2.5 was developed after much of the early work in calculus was already complete.

2.1 Introduction to limits

Informal definition of a limit

If $f(x)$ becomes arbitrarily close to a unique number $L$ as $x$ approaches $a$ from either side, then we say that the limit of $f(x)$, as $x$ approaches $a$, is $L$, and we write

$$\lim_{x \to a} f(x) = L$$

A strategy for finding limits

Properties of limits

2.2 Techniques for evaluating limits

One-sided limits

The existence of a limit

2.3 Continuity

Definition of continuity

Properties of continuous functions

Intermediate Value Theorem

2.4 Infinite limits

Definition of infinite limits

Definition of vertical asymptotes

Properties of infinite limits

2.5 $\epsilon$-$\delta$ definition of limits

The statement

$$\lim_{x \to a} f(x) = L$$

means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$. 


Formal definition of one-sided limits
Formal definition of infinite limits
The Squeeze Theorem

3. Differentiation
3.1 The derivative and the tangent line problem
3.2 Velocity, acceleration, and other rates of change
3.3 Differentiation rules for constant multiples, sums, and powers
3.4 Differentiation rules for products and quotients
3.5 The chain rule
3.6 Implicit differentiation
3.7 Related rates

4. Applications of Differentiation
4.1 Extrema on an interval
4.2 The mean value theorem
4.3 Increasing and decreasing functions and the first derivative test
4.4 Concavity and the second derivative test
4.5 Limits at infinity
   Definition of limits at infinity
   Definition of horizontal asymptote
   Properties of limits at infinity
4.6 A summary of curve sketching
4.7 Optimization problems
4.8 Business and economics applications
4.9 Newton's method
4.10 Differentials

5. Integration
5.1 Antiderivatives and indefinite integration
5.2 Integration by substitution
5.3 Sigma notation and the limit of a sequence
5.4 Area
5.5 Riemann sums and the definite integral
5.6 The fundamental theorem of calculus
5.7 Numerical integration

6. Applications of Integration
6.1 Area of region between two curves
6.2 Volume: The disc method
6.3 Volume: The shell method
6.4 Work
6.5 Fluid pressure and fluid force
6.6 Moments, centers of mass, and centroids
6.7 Arc length and surfaces of revolution

7. Logarithmic and Exponential Functions
   7.1 The natural logarithmic function
   7.2 The natural logarithmic function and differentiation
   7.3 The natural logarithmic function and integration
   7.4 Inverse functions
   7.5 Exponential functions and differentiation
   7.6 Integration of exponential functions: Growth and decay
   7.7 Indeterminate forms and L'Hopital's rule

8. Trigonometric Functions and Inverse Trigonometric Functions
   8.1 Review of trigonometric functions
   8.2 Graphs and limits of trigonometric functions
   8.3 Derivatives of trigonometric functions
   8.4 Integrals of trigonometric functions
   8.5 Inverse trigonometric functions and differentiation
   8.6 Inverse trigonometric functions: Integration and completing the square
   8.7 Hyperbolic functions

9. Techniques of Integration
   9.1 Review of basic integration formulas
   9.2 Integration by parts
   9.3 Trigonometric integrals
   9.4 Trigonometric substitution
   9.5 Partial fractions
   9.6 Summary and integration tables
   9.7 Improper integrals

10. Infinite Series
    10.1 Introduction: Taylor polynomials and approximations
    10.2 Sequences
    10.3 Series and convergence
    10.4 The integral test and p-series
    10.5 Comparisons of series
    10.6 Alternating series
    10.7 The ratio and root tests
    10.8 Power series
    10.9 Representations of functions by power series
    10.10 Taylor and Maclaurin series

11. Conics
    11.1 Parabolas
    11.2 Ellipses
    11.3 Hyperbolas
11.4 Rotation and the general second-degree equation

12. Plane Curves, Parametric Equations, and Polar Coordinates
12.1 Plane curves and parametric equations
12.2 Parametric equations and calculus
12.3 Polar coordinates and polar graphs
12.4 Tangent lines and curve sketching in polar coordinates
12.5 Polar equations for conics
12.6 Area and arc length in polar coordinates
APPENDIX G
DESCRIPTION OF THE LARSON, HOSTETLER, AND EDWARDS (1990) TEXTBOOK

Preface Synopsis

What is calculus? We begin to answer this question by saying that calculus is the reformulation of elementary mathematics through the use of a limit process. If limit processes are unfamiliar to you this answer is, at least for now, somewhat less than illuminating. From an elementary point of view, we may think of calculus as a "limit machine" that generates new formulas from old. Actually, the study of calculus involves three distinct stages of mathematics: precalculus mathematics (the length of a line segment, the area of a rectangle and so forth), the limit process, and new calculus formulations (derivatives, integrals, and so forth).

On the following two pages we have listed some familiar precalculus concepts coupled with their more powerful calculus versions. As you proceed through this text, we suggest that you come back to this discussion repeatedly. Try to keep track of where you are relative to the three stages involved in the study of calculus. For example, the first three chapters break down as follows: precalculus (Chapter One), the limit process (Chapter Two), and new calculus formulas (Chapter Three). This cycle is repeated many times on a smaller scale throughout the text. We wish you well in your venture into calculus.

In the Fourth Edition, all examples, theorems, definitions, and prose have been revised - or at least considered for revision. Chapter Two still contains the presentation of limits as described in the Third Edition. Chapter Two begins with a new introduction to the tangent line problem. Section 2.1 has a new example dealing with the e-δ definition applied to a quadratic function. Section 2.3 begins with a summary of a strategy for finding limits.

Table of Contents

1. The Cartesian Plane and Functions
1.1 Real numbers and the real line
1.2 The cartesian plane
2. Limits and Their Properties

The concept of the limit of a function is the primary idea that distinguishes calculus from algebra and analytic geometry. Section 2.1 begins with a brief discussion of the way a limit will be used later (in Chapter Three) to solve the tangent line problem. The section then gives an informal description of the idea of limit and is followed by a theoretical definition— the so-called "ε-δ definition."

Section 2.2 discusses properties of limits. In this section it is important that you become familiar with several types of functions whose limits are easily found. For instance, the limit of \( f(x) = x^2 \) as \( x \) approaches two is simply \( f(2) = 4 \). Then, in Section 2.3, we use the properties discussed in Section 2.2 to find the limits that are not so straightforward.

Section 2.4 introduces the notion of continuity. Informally, when we say that a function \( f \) is continuous on an interval \((a, b)\), we mean that the graph of \( f \) has no holes, gaps, or jumps on the interval.

The last section in the chapter discusses infinite limits and vertical asymptotes. (Limits at infinity and horizontal asymptotes are discussed later in the text, in Section 4.5.)

2.1 Introduction to limits

The notion of a limit is fundamental to the study of calculus. For this reason it is important for you to acquire a good working knowledge of limits before moving on to other topics in calculus.

The tangent line problem

Informal definition of a limit

If \( f(x) \) becomes arbitrarily close to a single number \( L \) as \( x \) approaches \( a \) from either side, then we say that the limit of \( f(x) \), as \( x \) approaches \( a \), is \( L \), and we write

\[
\lim_{{x \to a}} f(x) = L
\]

Limits that fail to exist
A formal definition of limit

Let $f$ be a function defined on an open interval containing $a$ (except possibly at $a$) and let $L$ be a real number.

The statement

$$\lim_{x \to a} f(x) = L$$

means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Using the $\varepsilon-\delta$ definition of limit

2.2 Properties of limits
A strategy for finding limits
Limits of algebraic functions
Limits of trigonometric functions

2.3 Techniques for evaluating limits
Cancellation technique
Indeterminate forms
Rationalization technique
The Squeeze Theorem

If $h(x) \leq f(x) \leq g(x)$ for all $x$ in an open interval containing $c$, except possibly at $c$ itself, and if as $x \to c$ the limit of $h(x) = \lim_{x \to c} g(x) = L$, then the limit of $f(x) = L$.

Limits of trigonometric functions

2.4 Continuity and one-sided limits
Definitions of continuity
One-sided limits
Properties of continuous functions
Intermediate Value Theorem

2.5 Infinite limits
Definition of infinite limits
Definition of vertical asymptotes
Properties of infinite limits

3. Differentiation
3.1 The derivative and the tangent line problem
3.2 Velocity, acceleration, and other rates of change
3.3 Differentiation rules for constant multiples, sums, powers, sines, and cosines
3.4 Differentiation rules for products, quotients, secants, and tangents
3.5 The chain rule
3.6 Implicit differentiation
3.7 Related rates

4. Applications of Differentiation
4.1 Extrema on an interval
4.2 Rolle’s Theorem and the Mean Value Theorem
4.3 Increasing and decreasing functions and the first derivative test
4.4 Concavity and the second derivative test
4.5 Limits at infinity
   Definition of limits at infinity
   Definition of horizontal asymptote
   Properties of limits at infinity
4.6 A summary of curve sketching
4.7 Optimization problems
4.8 Newton’s method
4.9 Differentials
4.10 Business and economics applications

5. Integration
5.1 Antiderivatives and indefinite integration
5.2 Area
5.3 Riemann sums and the definite integral
5.4 The fundamental theorem of calculus
5.5 Integration by substitution
5.6 Numerical integration

6. Logarithmic, Exponential, and Other Transcendental Functions
6.1 The natural logarithmic function and differentiation
6.2 The natural logarithmic function and integration
6.3 Inverse functions
6.4 Exponential functions: Differentiation and integration
6.5 Bases other than e and applications
6.6 Growth and decay
6.7 Inverse trigonometric functions and differentiation
6.8 Inverse trigonometric functions: Integration and completing the square
6.9 Hyperbolic functions

7. Applications of Integration
7.1 Area of region between two curves
7.2 Volume: The disc method
7.3 Volume: The shell method
7.4 Arc length and surfaces of revolution
7.5 Work
7.6 Fluid pressure and fluid force
7.7 Moments, centers of mass, and centroids
8. Integration Techniques, L’Hopital’s Rule, and Improper Integrals
   8.1 Basic integration formulas
   8.2 Integration by parts
   8.3 Trigonometric integrals
   8.4 Trigonometric substitution
   8.5 Partial fractions
   8.6 Integration by tables and other integration techniques
   8.7 Indeterminate forms and L’Hopital’s Rule
   8.7 Improper integrals

9. Infinite Series
   9.1 Sequences
   9.2 Series and convergence
   9.3 The integral test and p-series
   9.4 Comparisons of series
   9.5 Alternating series
   9.6 The ratio and root tests
   9.7 Taylor polynomials and approximations
   9.8 Power series
   9.9 Representations of functions by power series
   9.10 Taylor and Maclaurin series

11. Conics
    11.1 Parabolas
    11.2 Ellipses
    11.3 Hyperbolas
    11.4 Rotation and the general second-degree equation

12. Plane Curves, Parametric Equations, and Polar Coordinates
    12.1 Plane curves and parametric equations
    12.2 Parametric equations and calculus
    12.3 Polar coordinates and polar graphs
    12.4 Tangent lines and curve sketching in polar coordinates
    12.5 Area and arc length in polar coordinates
    12.6 Polar equations for conics and Kepler’s Laws