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Title: A Study of Time Series Interpolation and Decimation With Signal Processing Applications

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John L. Saugen

Decimation and interpolation is shown to be an effective technique for reducing the storage requirements for bandlimited time series. Interpolation and decimation is theoretically developed using z-transform theory and graphically interperted. A sequence of data acquired by sampling a sonar signal is used to demonstrate the effects of decimation. Using a finite impulse response linear phase filter-decimator system it is shown that storage requirements for the sonar signal can be reduced by a factor of at least eight.

A Study of Time Series Interpolation and Decimation With Signal Processing Applications

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Thomas Fredrick Kraemer

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Redacted for Privacy

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A STUDY OF TIME SERIES INTERPOLATION AND DECIMATION WITH SIGNAL PROCESSING APPLICATIONS:

I. Introduction

In the acquistion of data a device called a transducer converts a physical phenomenon into a continuous analog signal -- a function of time. Frequently this analog signal occurs as a time varying voltage. An analog to digital convertor system can be used to sample this signal at equal intervals in time and convert these samples into a form suitable for manipulation by a digital computer. If the interval between samples is short enough, it is theoretically possible to deduce the properties of the original signal given only the samples. The reciprocal of the time between samples is known as the sampling frequency. Occasionally it is necessary to sample a signal at one frequency and process it digitally as if it were sampled at another frequency. For example, it may be desired to compute the value of the original signal at points between the samples or it may be desired to eliminate samples to reduce space required for storage.

The process of estimating intermediate values is called interpolation and the process of eliminating sample points is called decimation. The two processes may be combined to form an interpolation decimation system.

After a preliminary discussion concerning sampling and time series, interpolation and decimation systems are defined and analyzed. A sonar signal is used for an example application.

Problem Definition:

First, relations are derived which relate the z-transform of a given sequence to the z-transform resulting after the sequence is operated on by a discrete system. Of primary concern are the relationships applicable to the processes of interpolation and decimation. The discrete Fourier transform (DFT) operation is also studied.

Second, decimation is applied to a given class of sampled analog signals as a means of reducing storage requirements while still being able to recover a specified portion of the signal.

Review of Pertinent Sampling Theory:

Consider a continuous time signal $\hat{x}(t)$ with the Fourier transform

$$\hat{X}(w) = \int_{-\infty}^{+\infty} \hat{x}(t) \exp(-jwt)\, dt \quad .$$

The signal is sampled to produce the sequence

$$x(n) = \hat{x}(nT) \, , \quad -\infty < n < \infty$$

where T is the sampling period. The z-transform of the sequence x(n) can be defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \quad .$$

The z-transform evaluated with $z = \exp(jwT)$, i.e. $X(\exp(jwT))$, is called the Fourier transform of the sequence x(n).

The Fourier transform of the sequence x(n) is related to the Fourier transform of $\hat{x}(t)$ by [19]

$$X(\exp(jwT)) = \frac{1}{T} \sum_{k=-\infty}^{\infty} \hat{X}(w + k\frac{2\pi}{T}) .$$

If the signal $\hat{x}(t)$ is bandlimited, that is $\hat{X}(w) = 0$ for $|w| \geq \Omega$, and if $T \leq \pi/\Omega$, then

$$X(\exp(jwT)) = \frac{1}{T}\hat{X}(w) , \quad -\frac{\pi}{T} \leq w \leq \frac{\pi}{T} .$$

Assuming that $\hat{x}(t)$ is bandlimited as defined above, then the original continuous time signal can be recreated uniquely from the samples x(n) through the interpolation formula [19]

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k) \left\{ \frac{\sin\left(\frac{\pi}{T}(t - kT)\right)}{\left(\frac{\pi}{T}(t - kT)\right)} \right\} .$$

In many digital signal processing problems the sequence x(n) is given, corresponding to a sampled analog signal with sampling period T. A new sequence y(n) can be derived such that $y(n) = \hat{x}(n\tilde{T})$, where $\tilde{T}$ is a different sampling period, by defining $t = n\tilde{T}$ in the above interpolation formula. Thus y(n) may be computed from x(n) without resampling the analog signal. This equation is difficult to utilize because of the infinite summation.

Sampling at twice the highest frequency contained in an analog signal, $\hat{x}(t)$, will theoretically guarantee that the signal can be reconstructed from

(a)

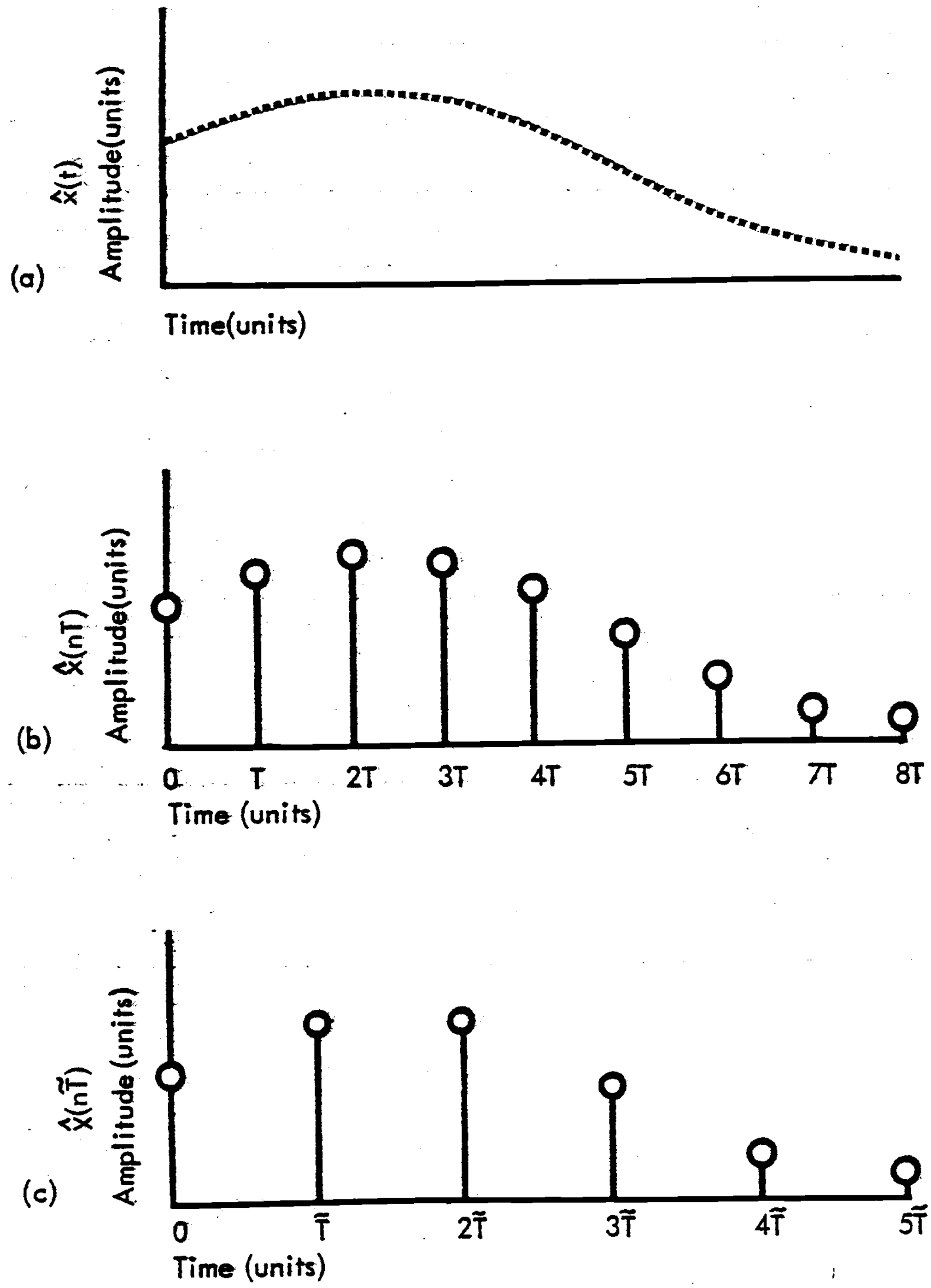

Fig. 1 (a) Graph of an analog signal. (b) Signal sampled every T units of time. (c) Signal sampled every $\tilde{T}$ units of time. At the instants in time indicated by circles the signal's amplitude is converted in to a form suitable for computer processing.

the set of sample values. However, if a band limited signal is sampled at twice the bandwidth of the signal, then it can be shown that the original signal $\hat{x}(t)$ can be reconstructed from the reduced set of sample values provided the frequency band is known.

An Important Question:

The option exists to use either digital or equivalent analog methods to perform interpolation and decimation. Analog methods are inherently real time. That is if a signal is input to an analog system, then the output signal will be in the same time reference as the input. When using digital techniques computation speed becomes important. It is assumed that there is a certain amount of time between each sample in a sequence. Thus any digital computation performed on the sequence must result in an output value for every input value to be a real time digital process. For this reason any digital process for interpolation and decimation must be computationally efficient. For example, an analog signal can be sampled at twice its highest frequency, digitally filtered, and then decimated. Alternately, the signal could be analog bandpass filtered, and then sampled at a lower rate. Most analog filters have highly non-linear phase characteristics whereas a digital filter can be designed with linear phase. If the filtering requires tremendous amounts of computation, then analog methods could be required.

Principle Results:

Closed-form expressions are derived in Chapter 2 which relate an interpolated and decimated sequence to the original sequence. These expressions provide a convenient picture of many aspects of signal processing. For example, it is shown that the discrete Fourier transform is essentially a filter and the hopping discrete Fourier transform is a decimation of this filter output.

The techniques are applied to the processing of sonar signals. A typical sonar signal was sampled at more than twice the highest frequency contained in the signal. Digital methods of decimation were applied to the sonar sequence. A decimator and filter are designed which retained every eighth sample. Therefore an average length sequence of 8192 points could be stored in 1024 locations. A tremendous savings in terms of computation results for any further signal processing. For example, a fast Fourier transform of the original sequence requires about 100,000 arithmetic operations whereas the fast Fourier transform of the decimated sequence requires about 10,000 operations. Results of this study are shown in Chapter 3.

Results of Literature Survey:

The need for efficient translation between various sampling rates, that is interpolation and decimation, has been established by Rabiner and Crochiere [1]. One application for interpolation and decimation is digital narrow-band filtering where it is more efficient computationally to decimate a sequence, filter it and then interpolate it than to directly filter the original sequence [2 - 4].

The processes of sampling rate reduction, often called decimation, and sampling rate increase or interpolation has been examined from several points of view. Bellanger et al. found that efficient implementations of low pass finite impulse response filters could be obtained by a process of first reducing the sampling rate of an arbitrary signal, filtering, and then increasing the sampling rate back to the original frequency [3]. Crochiere and Rabiner developed a general theory of sampling rate reduction and increase which they refer to as decimation and interpolation [1]. All methods presented in the literature attack the problem of implementing a low-pass filter.

A general interpolation decimation system consists of an interpolator, followed by a filter and a decimator. A problem is to design the system to give some desired response for a given input. It is possible to only approximate many desired responses. Most of the previous research concerning interpolation and decimation has been devoted to approximating these responses with finite impulse response systems. Much has been published concerning the use of finite impulse response systems [1,4,9,11,13] . A computer program has been published for designing optimum linear-phase finite impulse response filters [9]. This program is capable of designing a wide variety of low pass, high pass, band pass and band stop filters as well as differentiators and Hilbert transformers. Many examples of its use are shown in books by Peled and Liu [10] and Rabiner and Gold [11] . The theory behind the program which uses the Remez exchange algorithm is discussed by Rabiner and Gold.

Several specialized techniques have been published for performing interpolation and decimation. Bellanger et al. has successfully used a technique called half-band filtering [3,16] . A method which is similar to half-band filtering, called octave band filtering, is discussed by Nelson, Pfeifer and Wood [12] . Hilbert filtering for decimation is discussed by Rabiner and Gold. Sabri and Steenaart recently developed a generalized matrix representation of the Hilbert filtering process [21] . The method appears to be effective when only finite numerical accuracy is available for computation. A summary of these methods is formulated in Appendix B.

II. Theory of Interpolation Decimation Systems

Definition of a system:

A system is defined here as a mapping of some complex valued input sequence f(n), where the index variable n is a positive integer, into an unique complex valued output sequence g(n). This mapping is denoted by

$$f(n) \longrightarrow g(n), \quad n = 0, 1, 2, \ldots . \tag{1}$$

Several comments should be made about this definition. Initial conditions in the system are not allowed. The word unique is required in the above definition due to the characteristics of complex numbers. A complex number may be represented as a sum of a real part and an imaginary part. For example, the complex number z may equal x + j y where x and y are real and j is the square root of -1. An alternative representation of z is in polar form. That is z = r exp(jw), where r is the magnitude of z and w is the angle of z. The identities which relate r, w, x and y are

$$r = \left| \sqrt{x^2 + y^2} \right|$$

$$w = \tan^{-1}\left(\frac{y}{x}\right)$$

A problem exists since the $\tan^{-1}$ function is multivalued. It is impossible to distinguish between angles which differ by an integer multiple of 360 degrees. Furthermore, if only the tangent of w is known then y/x can take on two values.

Definition of a linear system:

Given any two input sequences to a system, $f_1(n)$, $f_2(n)$ and their corresponding outputs $g_1(n)$, $g_2(n)$ then a system is linear if and only if

$$a\, f_1(n) + b\, f_2(n) \longrightarrow a\, g_1(n) + b\, g_2(n) \qquad (2)$$

where a and b are complex constants independent of the index variable $n = 0, 1, 2, \ldots$.

Definition of a shift-variant system:

Given a system with the mapping of eq. (1) then the system is shift-invariant if and only if

$$f(n-m) \longrightarrow g(n-m), \quad n = 0, 1, 2, \ldots \qquad (3)$$

for all positive integers m where $f(n-m) = 0$ for $n-m < 0$.

Definition of an Eigenfunction:

An Eigenfunction, $e(n)$, $n = 0,1,2,\ldots$, of a system is a function such that

$$e(n) \longrightarrow H(z)\, e(n), \quad n = 0, 1, 2, \ldots \qquad (4)$$

where $H(z)$ is a complex valued function of the complex constant z.

Eigenfuction for a linear-shift invariant system:

Theorem 1:

For a linear-shift invariant system, an eigenfunction is

$$e(n) = z^n \quad (5)$$

Proof:

Let

$$z^n \longrightarrow g(n) .$$

From the linearity assumption we may multiply the input and output by the constant z^{-m},

$$z^{-m}z^n \longrightarrow z^{-m}g(n)$$

or
$$z^{(n-m)} \longrightarrow z^{-m}g(n) . \quad (6)$$

From the shift-invariance assumption we must have

$$e(n-m) \longrightarrow g(n-m) .$$

Thus
$$z^{(n-m)} \longrightarrow g(n-m). \quad (7)$$

But from equation (6) and (7) it is clear that

$$z^{-m}g(n) = g(n-m) . \quad (8)$$

From the definition of a system, the output function is unique for any given input function. An unique solution to eq. (8) is given by

$$g(n) = H(z)\, z^n \quad (9)$$

since substitution of eq. (9) into eq. (8) gives

$$z^{-m}H(z)z^n = H(z)z^{(n-m)} .$$

Therefore an eigenfunction is

$$e(n) = z^n \ .$$ Q.E.D.

Definition of an impulse response and transfer function:

The impulse sequence is defined as

$$d(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases} \tag{10}$$

If d(n) is applied as the input sequence to a linear, shift invariant system then the output is defined as the impulse response h(n).

$$d(n) \longrightarrow h(n), \quad n = 0, 1, 2, \ldots \tag{11}$$

Theorem 2:

Given a linear shift invariant system with the eigenfunction $e(n) = z^n$, then H(z) as defined in eq.(4) may be called the system's transfer function and is given by the expression

$$H(z) = \sum_{k=0}^{\infty} h(k)\, z^{-k} \tag{12}$$

Proof:

From the definition of an impulse response,

$$d(n) \longrightarrow h(n) \ ,$$

and from the shift invariance and linearity assumptions,

$$d(n-m) \rightarrow h(n-m),$$

$$z^m d(n-m) \rightarrow z^m h(n-m),$$

where z is a complex constant and m is an integer, $0 \leqslant m \leqslant n$.

Using the linearity assumption again and multiplying the input and output by $z^n z^{-n}$ (= 1).

$$z^n z^{-n} z^m d(n-m) \rightarrow z^n z^{-n} z^m h(n-m)$$

$$z^n d(n-m) z^{-(n-m)} \rightarrow z^n h(n-m) z^{-(n-m)}$$

This mapping must be true for all values of $m < n$.

From the definition of linearity the following must be true.

$$z^n \sum_{m=-\infty}^{n} d(n-m)\, z^{-(n-m)} \rightarrow z^n \sum_{m=-\infty}^{n} h(n-m)\, z^{-(n-m)}$$

But from the definition of d(n-m)

$$\sum_{m=-\infty}^{n} d(n-m) z^{-(n-m)} = 1 \quad .$$

Thus

$$z^n \rightarrow z^n \sum_{m=-\infty}^{n} h(n-m)\, z^{-(n-m)} \quad .$$

Letting $k = n - m$, then

$$z^n \rightarrow z^n \sum_{k=0}^{\infty} h(k)\, z^{-k} \quad .$$

From eq. (4) and the definition of a system

$$H(z) = \sum_{k=0}^{\infty} h(k)\, z^{-k} \qquad \text{Q.E.D.}$$

A Linear Shift Invariant System:

Using the above results, a linear shift invariant system is representable in terms of its impulse response h(n) as

$$y(n) = \sum_{i=0}^{n} h(i)\, u(n-i)\ , \quad n = 0,1,2,\ldots \tag{13}$$

equivalently

$$y(n) = \sum_{i=0}^{n} u(i)\, h(n-i)\ , \quad n = 0,1,2,\ldots$$

where u(k) is the input sequence, $k = 0,1,2,\ldots$ and y(n) is the output sequence.

Definition of Infinite and Finite Impulse Response Systems:

A system with the impulse response h(n) is called a finite impulse response system (FIR system) if $h(i) = 0$ for $i < 0$ and $i \geq N$ where N is a finite integer. An infinite impulse response system (IIR system) has an impulse response such that $|h(i)| > 0$ for some $i > Q$ where Q is any positive integer.

For a finite impulse response system the summation in eq. (13) will never have more than N terms since $h(i) = 0$ for $i > N$. Thus a finite impulse response system can be realized by the direct evaluation of eq. (13).

Definition of the z-transform:

Given some function u(n), n = 0, 1, 2, ..., where u(n) = 0 for n less than 0, then the z-transform of u(n) is defined as

$$U(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n} \tag{14}$$

Notice that the transfer function of a system is the z-transform of the impulse response. (eq. (12))

Right Shifting Property of the z-transform:

Theorem 3:

If a function f(n), n= 0, 1, 2, ..., has the z-transform F(z) and a new function is defined as y(n) = f(n-m) where m is a positive integer and f(n-m) = 0 for $n-m < 0$ then

$$Y(z) = z^{-m}F(z)$$

Proof:

Substituting y(n) into eq. (14)

$$Y(z) = \sum_{k=0}^{\infty} f(n-m)\, z^{-k}$$

$$= f(-m) + f(1-m)z^{-1} + \ldots + f(-1)\, z^{-m+1}$$

$$+ f(0)z^{-m} + f(1)z^{-m-1} + \ldots$$

But since $f(-m) + f(1-m)z^{-1} + \ldots + f(-1)\, z^{-m+1} = 0$

$$Y(z) = z^{-m}(f(0) + f(1)\, z^{-1} + \ldots\,)$$

$$= z^{-m}F(z)$$

Q.E.D.

Convolution Property of z-transforms:

Theorem 4:

Given the system represented by

$$y(n) = \sum_{i=0}^{\infty} h(i)\, u(n-i)\ , \quad \text{where } u(k) = 0 \text{ and } h(k) = 0 \text{ for } k < 0,$$

then $$Y(z) = H(z)\,U(z) \tag{15}$$

Proof:

$$y(n) = h(0)\,u(n) + h(1)\,u(n-1) + \ldots$$

Thus $$Y(z) = \sum_{n=0}^{\infty} \{h(0)\,u(n) + h(1)\,u(n-1) + \ldots\}\ z^{-n}$$

$$= h(0) \sum_{n=0}^{\infty} u(n)\,z^{-k} + h(1) \sum_{n=0}^{\infty} u(n-1)\,z^{-k} + \ldots$$

Using the right shifting theorem,

$$Y(z) = h(0)\,U(z) + h(1)\,z^{-1}U(z) + \ldots$$

$$= (h(0) + h(1)\,z^{-1} + \ldots\,)\,U(z)$$

$$= H(z)\,U(z)$$

Q.E.D.

Z-plane Plotting Conventions:

A three dimensional plot may be constructed to represent a z-transform function H(z) as shown in Fig. 2. Two axes represent the real and imaginary part of the complex variable z and the third axis represents the magnitude of H(z) where the magnitude of H(z) is defined as

$$|\,H(z)\,| = \left|\sqrt{\text{real}(H(z))^2 + \text{imag}(H(z))^2}\right| \tag{16}$$

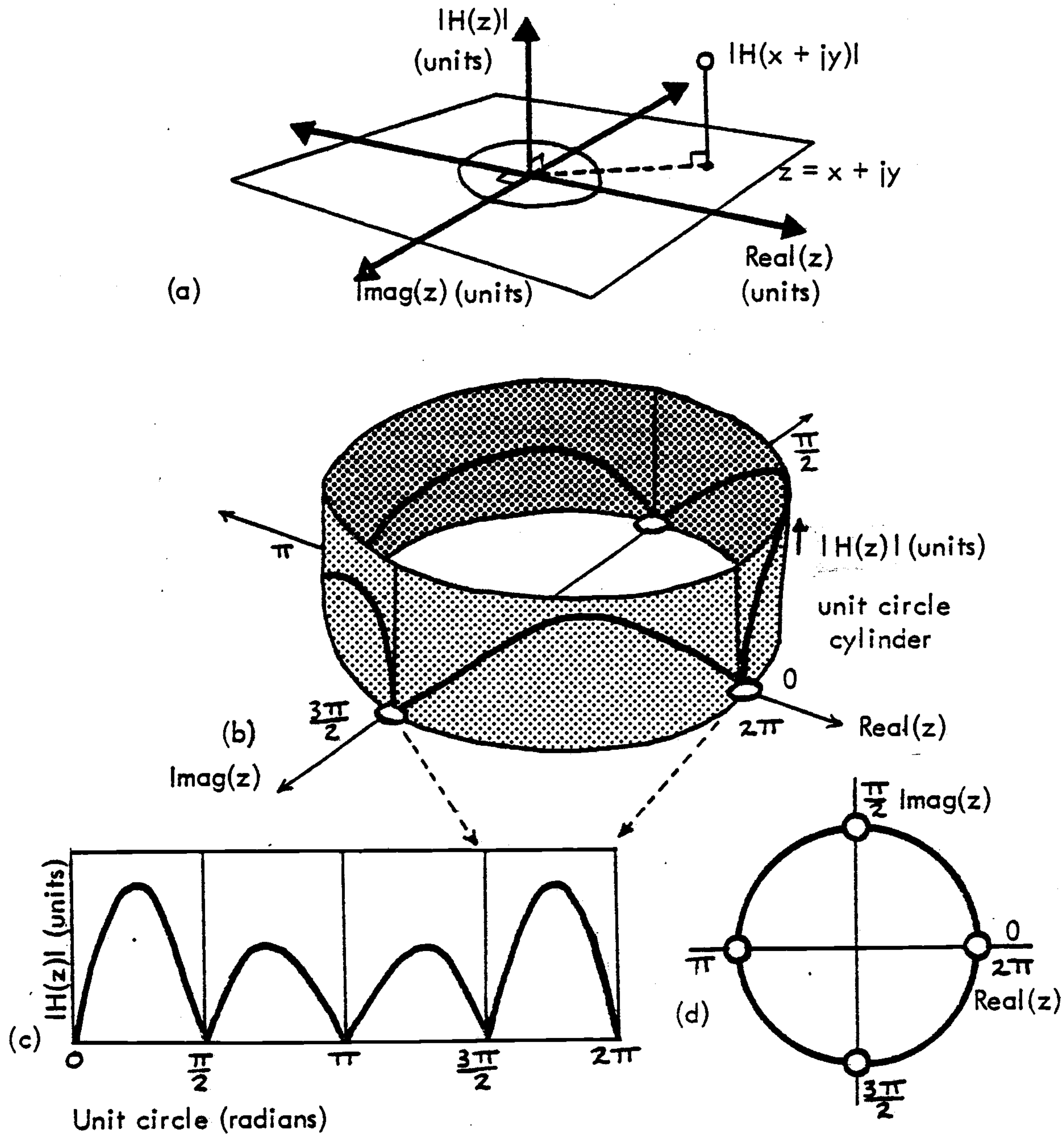

Fig. 2 Z-plane plotting conventions. (a) Three dimension plot of |H(z)| and the real and imaginary parts of z. (b) |H(z)| plotted on a cylindrical section of radius one. (c) Cylindrical plot projected on a flat surface. (d) Unit circle plot of values of z where |H(z)| = 0 .

Similarily a phase plot may be defined as

$$\arg(H(z)) = \tan^{-1}(\mathrm{imag}(H(z))/\mathrm{real}(H(z))) \quad (17)$$

As stated previously caution must be exercised with eq. (17) since the $\tan^{-1}$ function is multiple valued. If imag(H(z)) and real(H(z)) are known then arg(H(z)) can be one of many angles each differing by a multiple of 360°.

For convenience, the magnitude of H(z) evaluated for z = exp(jw) is plotted as shown in Fig. (2c). The z-plane is plotted as shown in Fig. (2d). The location of the zero values of |H(z)| are called "zeroes" and are indicated by the symbol "O".

Complex Conjugate Property of the z-plane:

For a real sequence, its z-transform can be expressed as a polynomial in z. The theory of polynomi als requires $H(z) = H^*(z^*)$ where the symbol "*" denotes complex conjugation. If a zero is located at $z = z_1$ then another zero must be located at $z = z_1^*$. Also for a sequence of length N there will be N-1 zeroes on the z-plane.

Definition of a Filter:

The term filter will be used to describe a system whose primary function is to alter the z-transform of a sequence evaluated on the unit circle. A filter is generally used to set values of the z-transform equal to zero.

Definition of an Averaging Filter:

An averaging filter is a finite impulse response system described by the impulse response,

$$h(n) = \begin{cases} 1 & 0 \leq n \leq N-1 \quad N \text{ an integer constant} \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

The output of this system is the sum of the last N input values. If the output were divided by N, then we would have the arithmetical average of the last N input values, hence the name averaging filter.

The z-transform of the impulse response of this system is,

$$H(z) = \sum_{n=0}^{N-1} z^{-n}$$

When evaluated on the unit circle H(z) becomes,

$$H(\exp(jw)) = \sum_{n=0}^{N-1} \exp(-jwn)$$

Averaging Filter Theorem:

Theorem 5:

Given the averaging filter described by eq. (19) then the z-transform of h(n) can be factored as

$$H(z) = \prod_{m-1}^{N-1} (z^{-1} - \exp(-j2\pi m/N))$$

Proof:

From the theory of polynomials , H(z) must have N-1 zeroes.

H(z) evaluated on the unit circle is

$$H(\exp(jw)) = \sum_{n=0}^{N-1} \exp(-jwn) \ .$$

Using the summation identity

$$\sum_{n=0}^{N-1} \exp(-jwn) = \frac{1 - (\exp(-jw))^{-(n+1)}}{1 - \exp(-jw)^{-1}}$$

gives

$$H(\exp(jw)) = \frac{\sin(wN/2)}{\sin(w/2)} \exp(-jw(N-1)/2) \ .$$

The magnitude of H(exp(jw)) is

$$| H(\exp(jw)) | = \left| \frac{\sin(wN/2)}{\sin(w/2)} \right|$$

This function is zero for the angles $w = 2\pi m / N$ for $m = 1,2,\ldots,N-1$.

This exhaust all possible zeroes for the z-transform H(z) and so

$$H(z) = \prod_{m=1}^{N-1} (z^{-1} - \exp(-2\pi m/N)) \qquad \text{Q.E.D.}$$

Fig. (3) graphically demonstrates the above theorem for an averaging filter where $N = 7$.

Definition of Complex Demodulation:

Multiplication of a sequence by $\exp(-j2\pi kn/N)$, where k and N are any integer constants and n is the index value of the input sequence, is called complex demodulation.

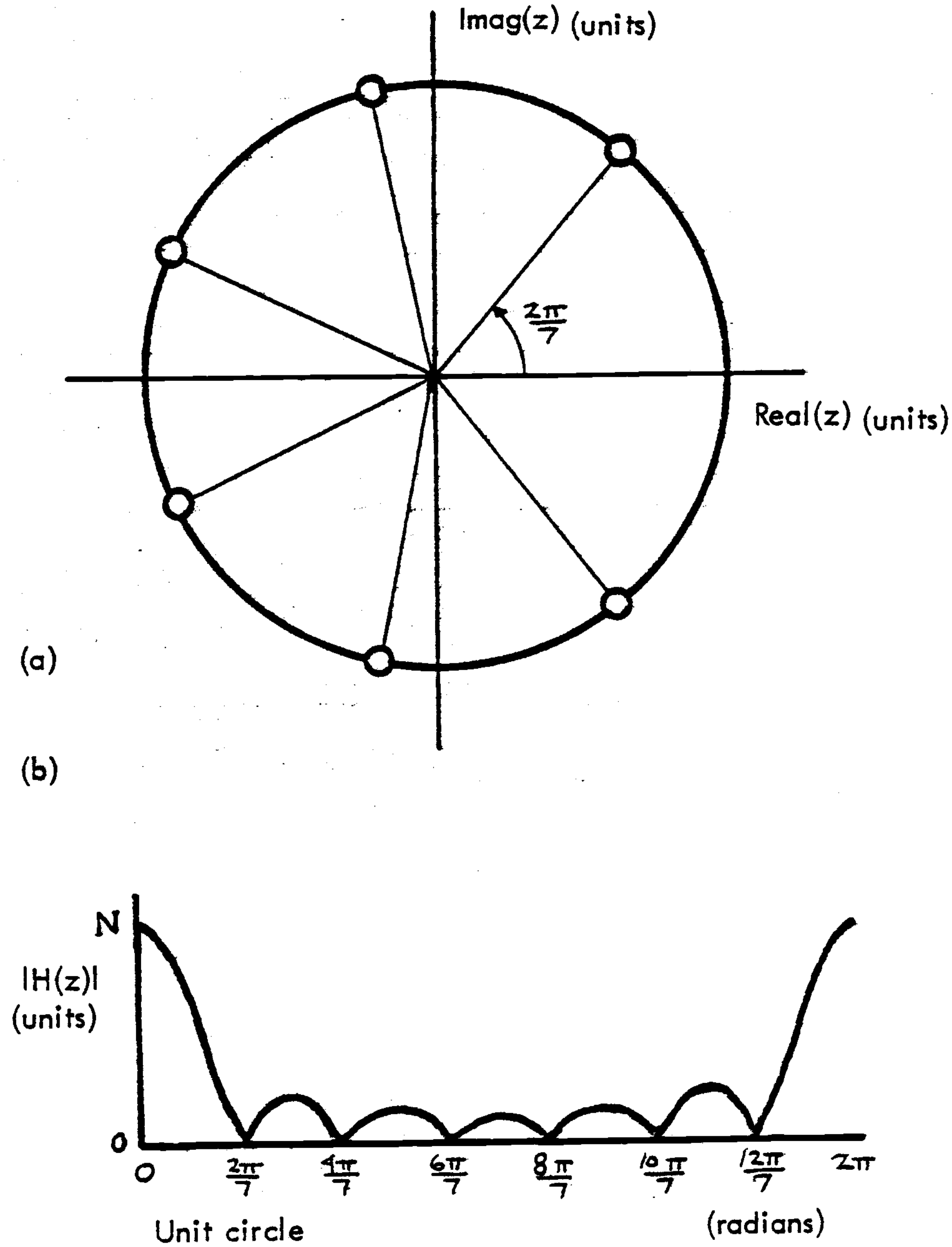

Fig. 3 (a) Unit circle plot and (b) Unit circle magnitude plot of the z-transform of the impulse response of an averaging filter where $h(n) = 1$ for $n = 0, 1, 2, 3, 4, 5,$ and 6 and $h(n) = 0$ for all other values of n. Unit circle magnitude plot is the function $|\sin(w7/2)/\sin(w/2)|$.

The Complex Demodulation Theorem:

Theorem 6:

Given a sequence h(n) and its z-transform H(z). If a new sequence is defined as $\hat{h}(n) = h(n) \exp(-j\, 2\pi kn/N)$ where k and N are integer constants, then the z-transform of $\hat{h}(n)$ is

$$\hat{H}(z) = H(z)\Big|_{z = z\exp(-j2\pi k/N)} \tag{20}$$

Proof:

$$H(z) = \sum_{n=0}^{\infty} h(n)\, z^{-n}$$

$$H(z \exp(-j2\pi k/N)) = \sum_{n=0}^{\infty} h(n)\, z^{-n} \exp(-j2\pi kn/N)$$

$$= \sum_{n=0}^{\infty} (h(n) \exp(-j2\pi kn/N))\, z^{-n}$$

Thus $H(z) = H(z \exp(-j2\pi k/N))$ Q.E.D.

The evaluation of H(z) at $z = z \exp(-j2\pi k/N)$ can be viewed as a rotation of the z-plane by the angle $-2\pi k/N$. This is graphically demonstrated in Fig. 4.

A Complex Demodulating Averaging Filter:

The previous results can be combined to give the general complex demodulating averaging filter defined by the impulse response,

$$h(n) = \begin{cases} \exp(\,j2\pi kn/N) & 0 \leq n \leq N-1 \\ 0 & \text{otherwise} \end{cases} \tag{21}$$

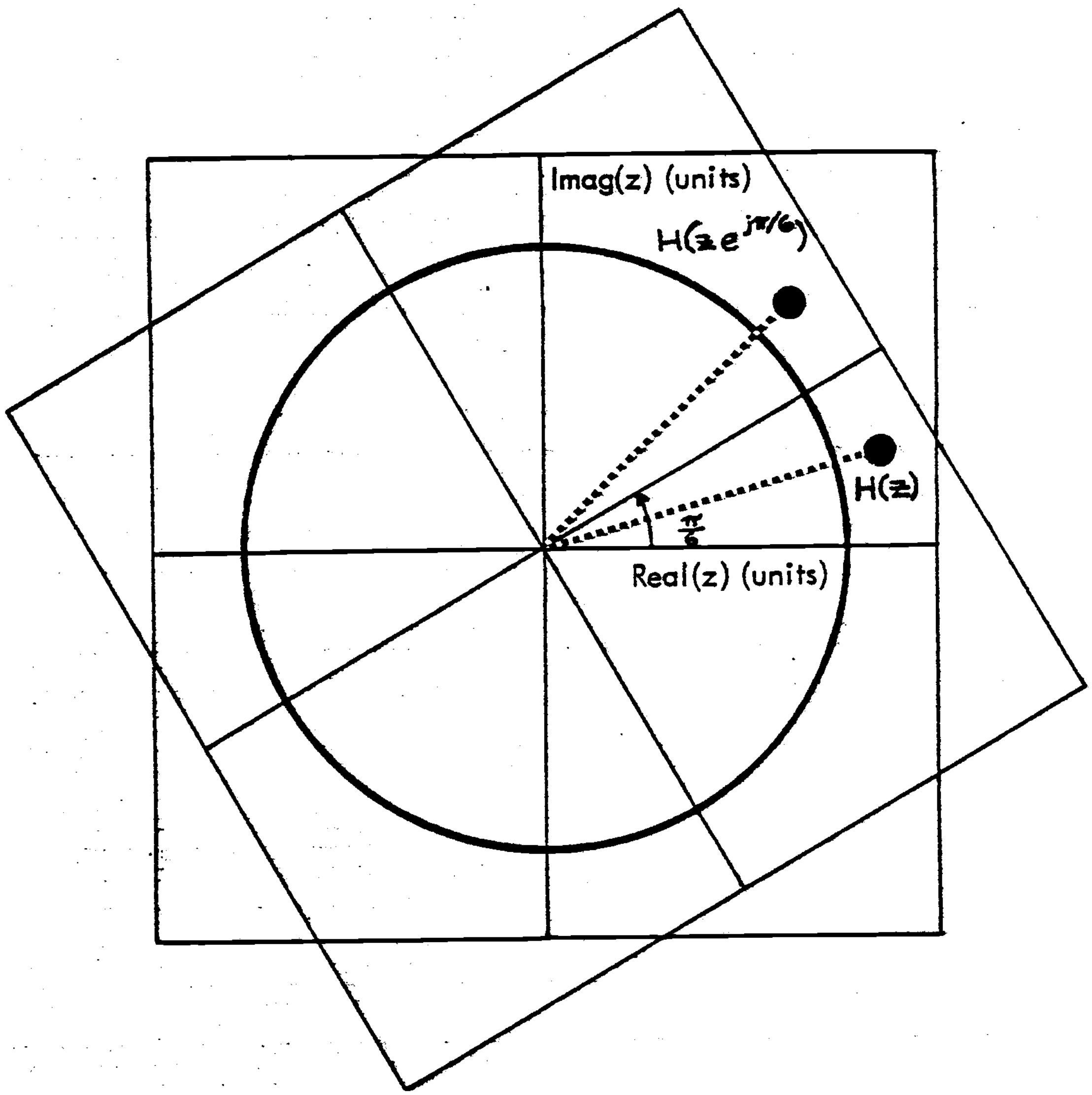

Fig. 4 Complex demodulation viewed as a rotation of the z-plane.

where k and N are finite integers. For k = 0 this expression reduces to the simple averaging filter given by eq. (19). The z-transform of this expression can be factored into the form,

$$H(z) = \prod_{\substack{m=0 \\ m \neq k \bmod(N)}}^{N-1} (z^{-1} - \exp(\ j2\pi m/N)) \tag{22}$$

The zeroes of this filter are evenly spaced around the unit circle every $2\pi/N$ radians except at the point $z = \exp(\ j2\pi k/N)$. A picture of this can be obtained by rotating Fig. (3a) by $2\pi k/N$ radians.

Definition of the Fourier Transform:

The Fourier transform of a sequence h(n) is the z-transform of the sequence evaluated on the unit circle.

$$H(\exp(jw)) = \sum_{n=0}^{\infty} h(n) \exp(-jwn) \tag{23}$$

where w is a real variable.

This function is periodic since $H(\exp(jw)) = H(\exp(j(w+2\pi m))$, m any integer.

Sampling in the z-plane:

Theorem 7:

Given a z-transform X(z). Define the function

$$\hat{X}(k) = X(z)\Big|_{z = \exp(-j2\pi k/N)} \quad , \ k = 0, 1, \ldots, N-1 \ . \tag{24}$$

Then on the unit circle (z = exp(jw))

$$X(\exp(jw)) = \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \hat{X}(k) \left[\frac{\sin\left((w - \frac{2\pi k}{N}) \frac{N}{2} \right)}{\sin\left((w - \frac{2\pi k}{N}) \frac{1}{2} \right)} \right] \exp\left(-j(w - \frac{2\pi k}{N})(\frac{N-1}{2}) \right) \right\} \tag{25}$$

Proof:

The proof of this theorem is contained in reference [22] .

Definition of the Discrete Fourier Transform:

The definition of the Fourier transform eq. (23) results in a periodic function. When this function is evaluated at N equally spaced points on the unit circle (eq. (24)) then a closed form expression is available (eq. (25)) to determine the value of the z-transform at all other points on the unit circle. For this reason the discrete Fourier transform is defined as

$$\hat{X}(k) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi nk/N), \quad k = 0, 1, 2, \ldots, N-1 \ . \tag{26}$$

Comments About the Discrete Fourier Transform:

The discrete Fourier transform may be viewed as N parallel complex averaging filters in the following manner. If an input sequence, $x(p)$, is convolved with the impulse response of a complex demodulating averaging filter (eq. (21)) then one term of the discrete Fourier transform is computed. To see this define k sequences as,

$$g_k(m) = \sum_{n=0}^{N-1} \hat{x}(n) \exp(-j2\pi kn/N) , \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ k = 0, 1, 2, \ldots \end{array}$$

where $\hat{x}(n) = x(n+m)$. The above relation can be related to the relationship for a complex demodulating averaging filter. Also $g_k(0)$ is the same as $\hat{X}(k)$ defined by eq. (26). The above process is commonly called a sliding discrete Fourier transform as m is allowed to vary. Frequently, to reduce the amount of computation, $g_k(m)$ is computed only for $m = 0, 2M, 3M, \ldots$ where M is any positive integer. This process is commonly refered to as a hopping discrete Fourier transform. It will be seen in the next section that this process is actually a decimation of the sequence $g_k(m)$.

Definition of a Decimator:

A decimator is a system with an input sequence x(n) and an output sequence y(n) defined as

$$y(n) = x(nM) \tag{27}$$

where M is a positive integer constant and is defined as the decimation ratio.

The Decimation Theorem:

Theorem 8:

Given the decimator described by eq. (27). The z-transform of the decimator output sequence in terms of the z-transform of the input sequence x(n) is given by,

$$Y(z) = \frac{1}{M} \sum_{m=0}^{M-1} \left\{ X(z) \Big|_{z = \exp(-j2\pi m/M)\, z^{1/M}} \right\} \tag{28}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} X(\exp(-j2\pi m/M)\, z^{1/M})$$

where X(z) is the z-transform of the input sequence.

Proof:

Define the sequence

$$w(n) = \begin{cases} x(n) & n = 0, M, 2M, \ldots \\ 0 & \text{otherwise} \end{cases}$$

w(n) can be written as

$$w(n) = x(n) \left\{ \frac{1}{M} \sum_{m=0}^{M-1} \exp(j2\pi m/M) \right\}$$

since the term in brackets equals 1 for n = 0, M, 2M,...and 0 otherwise.

Now let

$$y(n) = w(nM)$$

hence $$Y(z) = \sum_{n=0}^{\infty} w(nM)\, z^{-n}$$

But since w(n) = 0 except at integer multiples of M we can rewrite Y(z) as

$$Y(z) = \sum_{n=0}^{\infty} w(n)\, z^{-n/M}$$

$$= \sum_{n=0}^{\infty} \left\{ x(n) \left[\frac{1}{M} \sum_{m=0}^{M-1} \exp(j2\pi mn/M) \right] \right\} z^{-n/M}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} \sum_{n=0}^{\infty} \left\{ x(n) \exp(j2\pi mn/M)\, z^{-n/M} \right\}$$

$$= \frac{1}{M} \sum_{m=0}^{M-1} \left[X(z) \Big|_{z = \exp(-j2\pi m/M)\, z^{1/M}} \right] \qquad \text{Q.E.D.}$$

It should be noted that $z^{1/M}$ is to be evaluated for the princple value. For example, if z = 1 and M = 2 then $z^{1/2} = 1$ but not -1.

Definition of an Interpolator:

An interpolator is a system with the input sequence x(n) and the output sequence y(n) defined as

$$y(n) = \begin{cases} x(n/L) & n = 0, L, 2L, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

where L is a positive integer constant defined as the interpolation ratio.

The Interpolation Theorem:

Theorem 9:

Given the interpolator described by equation (29), then the z-transform of the output sequence y(n) in terms of the z-transform of the input sequence x(n) is given by

$$Y(z) = X(z)\Big|_{z = z^L} = X(z^L) \tag{30}$$

Proof:

$$Y(z) = \sum_{n=0}^{\infty} x(n/L)\, z^{-n}$$

But since $x(n/L) = 0$ for $n \neq 0, 2L, \ldots$ we can rewrite Y(z) as

$$Y(z) = \sum_{n=0}^{\infty} x(n)\, z^{-Ln} = X(z^L) \qquad \text{Q.E.D.}$$

The processes of interpolation and decimation are graphically shown in Fig. 5. Notice that L-1 zero values are inserted between each sample for interpolation and every Mth point is retained for decimation.

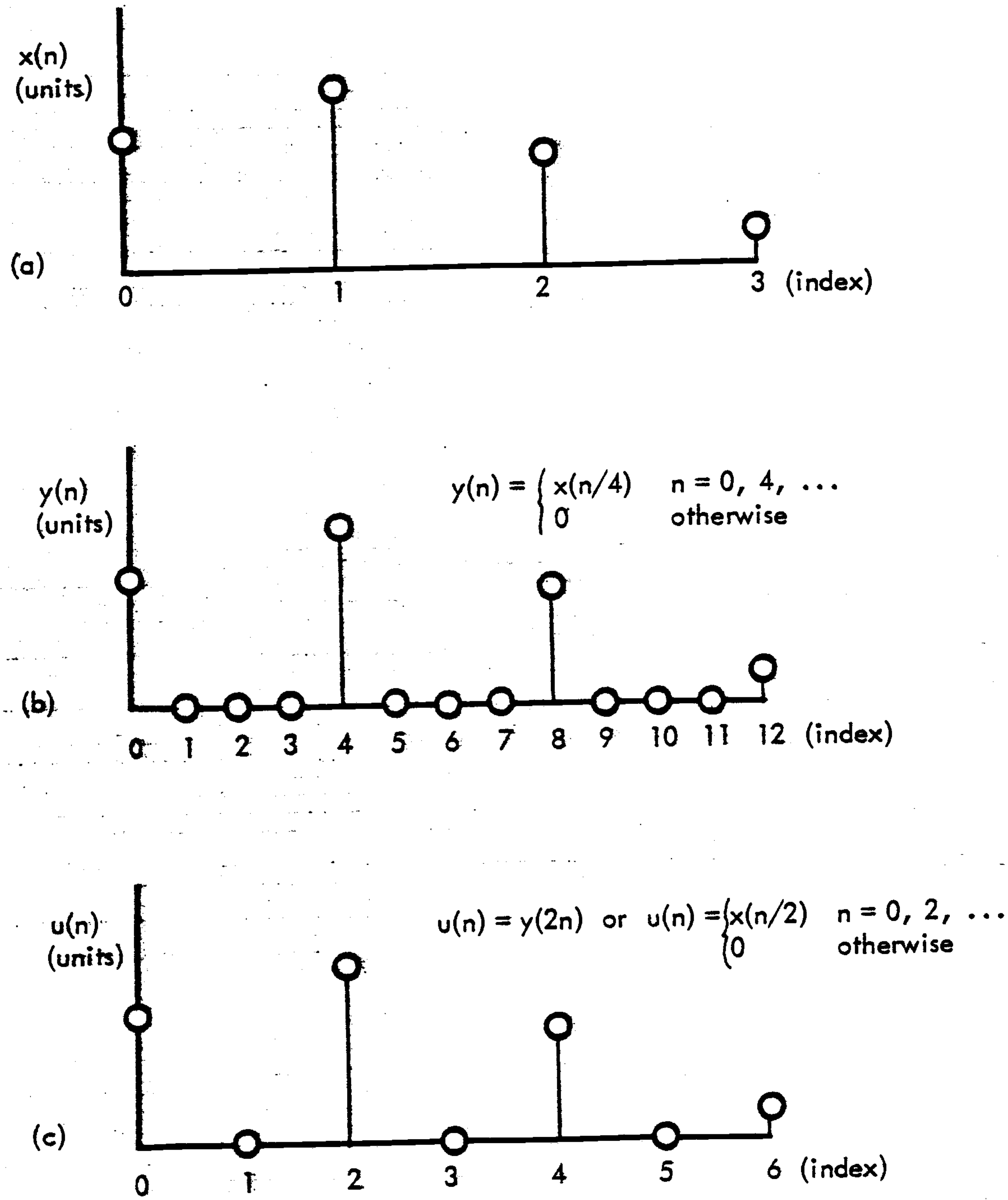

Fig. 5 (a) Original sequence x(n). (b) x(n) interpolated by 4. Three zero values are inserted between each sample. (c) y(n) decimated by 2. Notice that this is equivalent to interpolating x(n) by 2.

Evaluation of z-transform on the unit circle resulting from interpolation and decimation:

The effect of interpolation and decimation as defined above can be graphically demonstrated on the unit circle as shown in Fig. (6). If a sequence x(n) is interpolated according to eq. (29) giving the sequence y(n), then on the unit circle, evaluating eq. (30) with $z = \exp(jw)$

$$Y(\exp(jw)) = X(\exp(jwL)) , \qquad -\pi < w < \pi . \tag{31}$$

As w varies from $-\pi$ to π, wL will vary from $-\pi L$ to πL.

If the sequence x(n) is decimated then the relationship between Y(z) and X(z) is not simple. Evaluating eq. (28) on the unit circle gives

$$Y(\exp(jw)) = \frac{1}{M} \sum_{m=0}^{M-1} X(\exp(j(w-2\pi m)/M)) , \qquad -\pi < w < \pi . \tag{32}$$

In this case as w varies from $-\pi$ to π, Y(exp(jw)) will be the average of X(exp(jw)) at M equally spaced points around the unit circle. Fig. (6) illustrates the evaluation of these z-transforms for interpolation by 4 and decimation by 3.

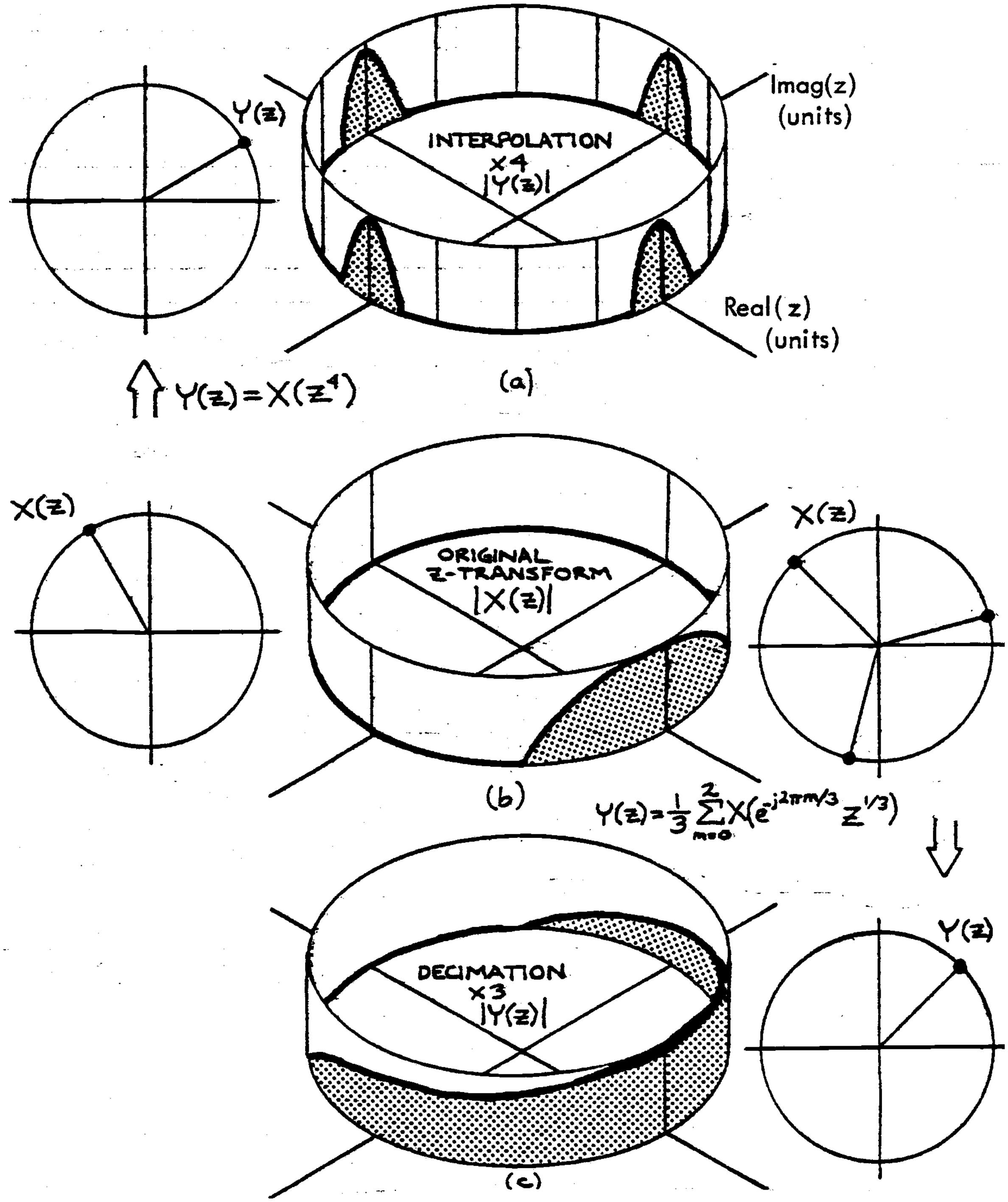

Fig. 6 (a) Sketch of the z-transform resulting from interpolating a sequence x(n) by 4. (b) Z-transform of the original sequence x(n). (c) Sketch of the z-transform resulting from decimating x(n) by 3.

Comments About Aliasing:

A commonly used term in the literature is "aliasing". Y(exp(jw)) is defined in this thesis to have aliasing distortion if two or more terms in eq. (32) are non-zero for some value of w.

For some arbritary sequence g(n), it is theoretically possible to filter this sequence to produce a new sequence x(n) such that at most only one term in the summation of eq. (32) is non-zero for $-\pi < w < \pi$. In general the desired filter can only be approximated when it is physically implemented. Methods for this approximation are discussed in Appendix A.

The Aliasing Theorem:

Conceptually, a sequence g(n) can be filtered, resulting in a sequence x(n), such that the z-transform of x(n) is zero for some regions on the unit circle. This concept leads to the following theorem.

Theorem 10:

Given the z-transform X(z) such that X(exp(jw)) = 0 for all

$$|w| < \frac{2\pi}{M}\left(\frac{m}{2}\right) = w_1 \qquad -\pi < w < \pi$$

and

$$|w| > \frac{2\pi}{M}\left(\frac{m+1}{2}\right) = w_2$$

where m is an integer, $0 \leq m \leq M-1$. If the sequence y(n) is the sequence x(n) decimated by M, where x(n) is the sequence corresponding to X(z), then

$$Y(\exp(jw)) = \frac{1}{M} X(\exp(j(w_1 + \frac{w}{M}))) \qquad \text{for m even and } 0 < w < \pi \text{ or m odd and } -\pi < w < 0$$

$$\frac{1}{M} X(\exp(j(-w_1 + \frac{w}{M}))) \qquad \text{for m even and } -\pi < w < 0 \text{ or m odd and } 0 < w < \pi$$

Proof:

See Appendix C. Fig. (7) illustrates this for M = 8 and m = 5 and 6.

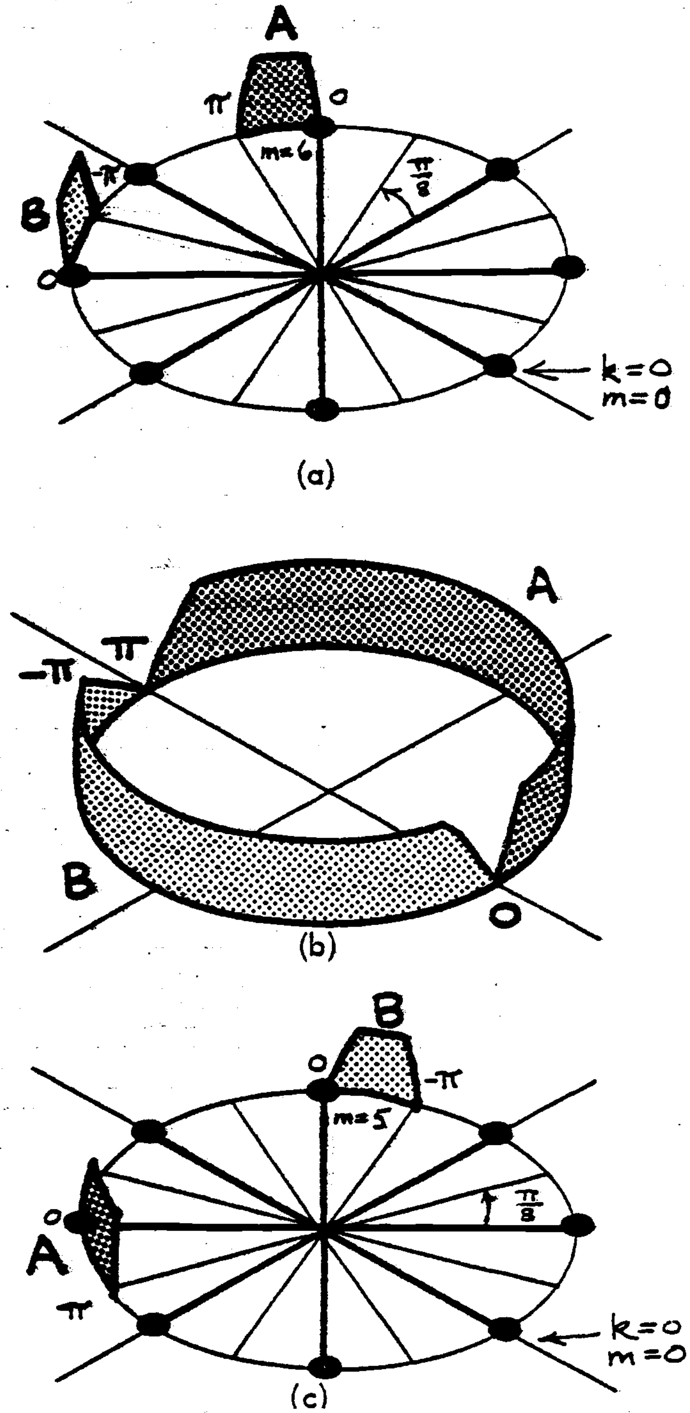

Fig. 7 Illustration of the Aliasing Theorem. (a) Sketch of z-transform of a sequence where $w_1 = 6\pi/8$ and $w_2 = 7\pi/8$. (b) Z-transform sketch resulting after decimating the sequence in (a) by 8. (c) Sketch of z-transform of a sequence where $w_1 = 5\pi/8$ and $w_2 = 6\pi/8$. When this sequence is decimated by 8 the resulting z-transform will look similar to Fig. (7b). The letters A and B indicate where each region is mapped to.

An Example:

Fig. (8) shows typical examples of the processes described in this chapter. The sequence in Fig. (8a) is interpolated by 4 resulting in the sequence shown in Fig. (8b). The sequence in Fig. (8c) is a filtered version of the sequence in Fig. (8b). Finally, the sequence in Fig. (8d) is the sequence in Fig.(8c) decimated by 3. Because of the choice of filter cut off points the sequence in Fig. (8d) looks very similar to the sequence in Fig. (8a).

Summary:

Closed-form expressions can be derived which relate an interpolated or decimation sequence to the original sequence by using their z-transforms. These expressions provide a convenient picture for many aspects of signal processing.

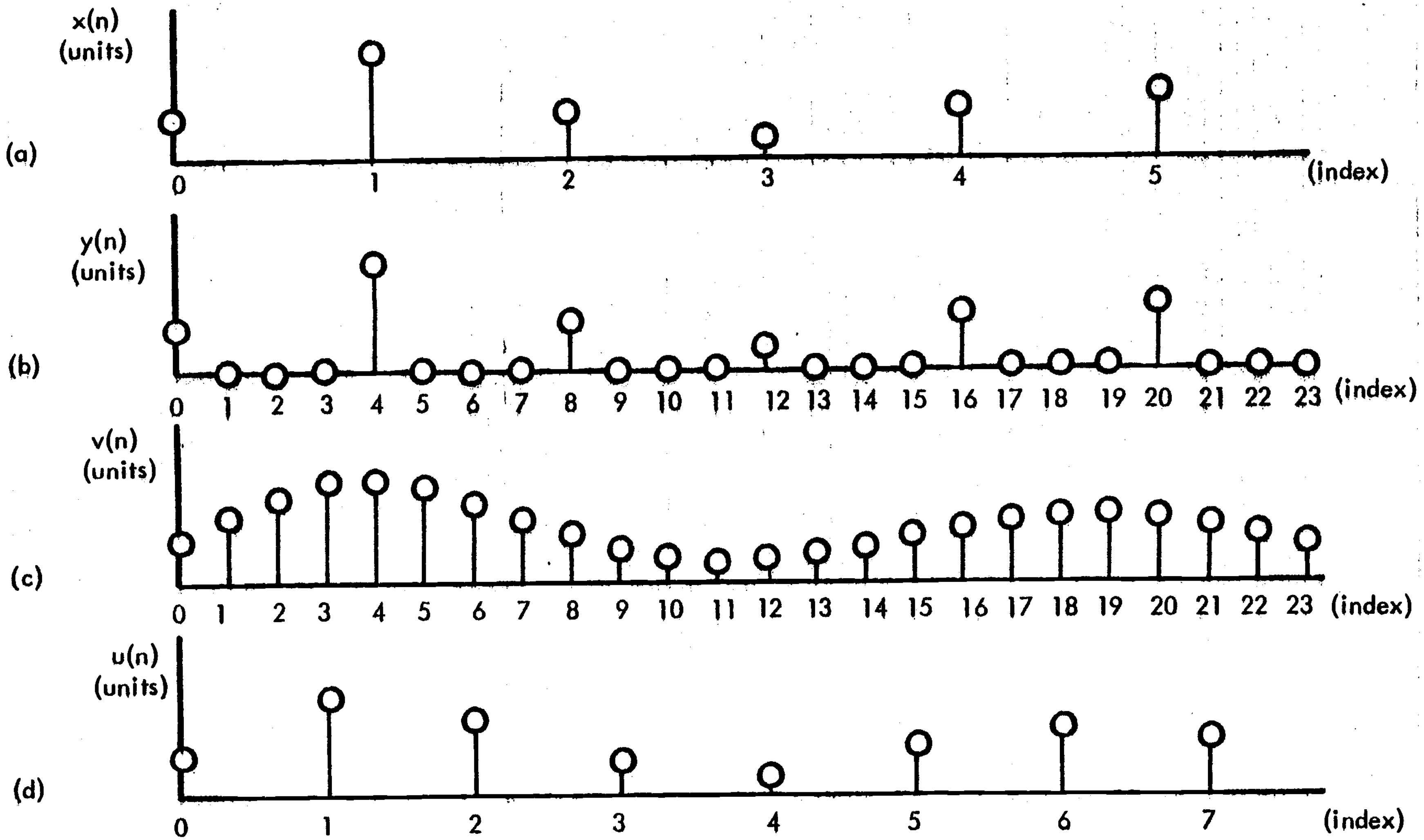

Fig. 8 (a) Original sequence. (b) x(n) interpolated by 4. (c) typical response when y(n) is low pass filtered. (d) v(n) decimated by 3.

III. Decimation Applied to a Sonar Signal

A sequence of real data was acquired by sampling a sonar signal at regular intervals in time. A plot of 8192 points in this sequence is shown in Fig.(9a). For plotting purposes each point in the sequence is connected to the next point with a straight line segment. The magnitude of this sequence's z-transform evaluated on the unit circle is shown in Fig. (9b). The largest magnitude is represented by 0 dB. Only a small region as indicated in Fig. (9b) is of interest.

Aliasing is graphically demonstrated in Figs. (10a) and (10b). Fig. (10a) shows the 4096 point sequence resulting from decimating the sequence in Fig.(9a) by 2. When this sequences z-transform is evaluated on the unit circle, aliasing becomes evident. Prominent peaks of the first sequence's z-transform are labeled with letters in Fig. (9b). When these peaks are summed as predicted by eq.(32) in chapter 2 they appear as images in Fig.(10b).

This is an unsatisfactory way of reducing the number of points in the 8192 point sequence due to aliasing. To make it satisfactory, all of the unwanted components on the unit circle must be filtered so that they will not sum into the region of interest.

A 128 point FIR filter was designed using the Remez exchange algorithm (Appendix A) to band pass filter the original sequence so that only the region of interest was significant. The filter's z-transform evaluated on the unit circle is shown in Fig. (11). From Theorem 10 in chapter 2 the cut off frequencies were chosen as $w_1 = 6\pi/8$ and $w_2 = 7\pi/8$. This filter will pass the region of interest and allow the filtered sequence to be decimated by 8. Since the filter has reasonably good attenuation, in this case -60 dB, Theorem 10 will apply. This is justifiable since the apparatus used to collect and convert the data has limited accuracy. Fig. (12) shows the resulting decimated sequence and its z-transform evaluated on the unit circle.

This process represents a large reduction in the number of data points which must be stored. Instead of storing 8192 sample points, only 1024 samples points need to be stored. From these samples, the z-transform for the region of interest can be computed. Due to the reduced number of sample points, additional signal processing will require less computation.

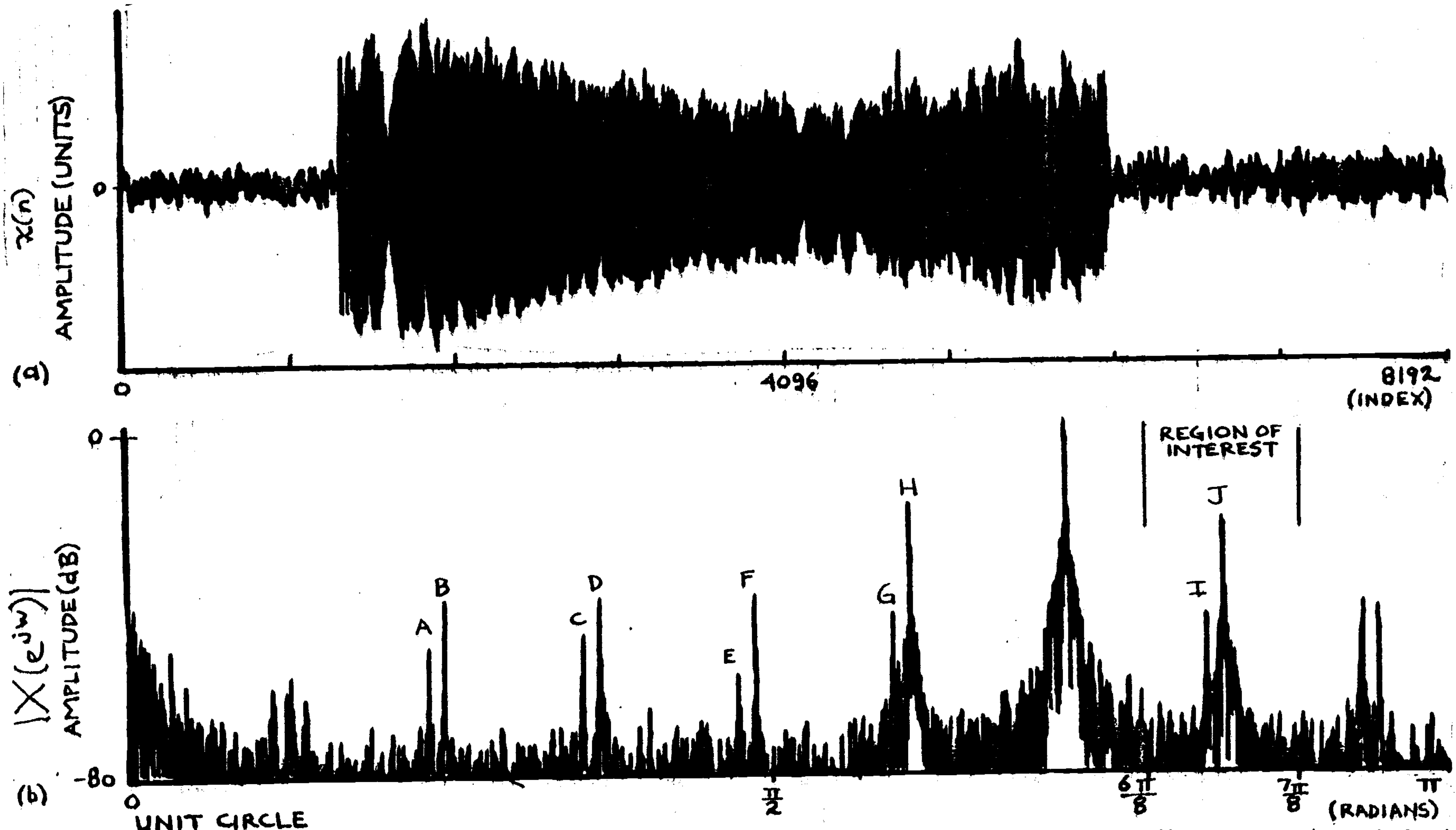

Fig. 9 (a) Original 8192 point sequence. (b) Z-transform on the unit circle. Only a small region contains desired information. All other regions are noise.

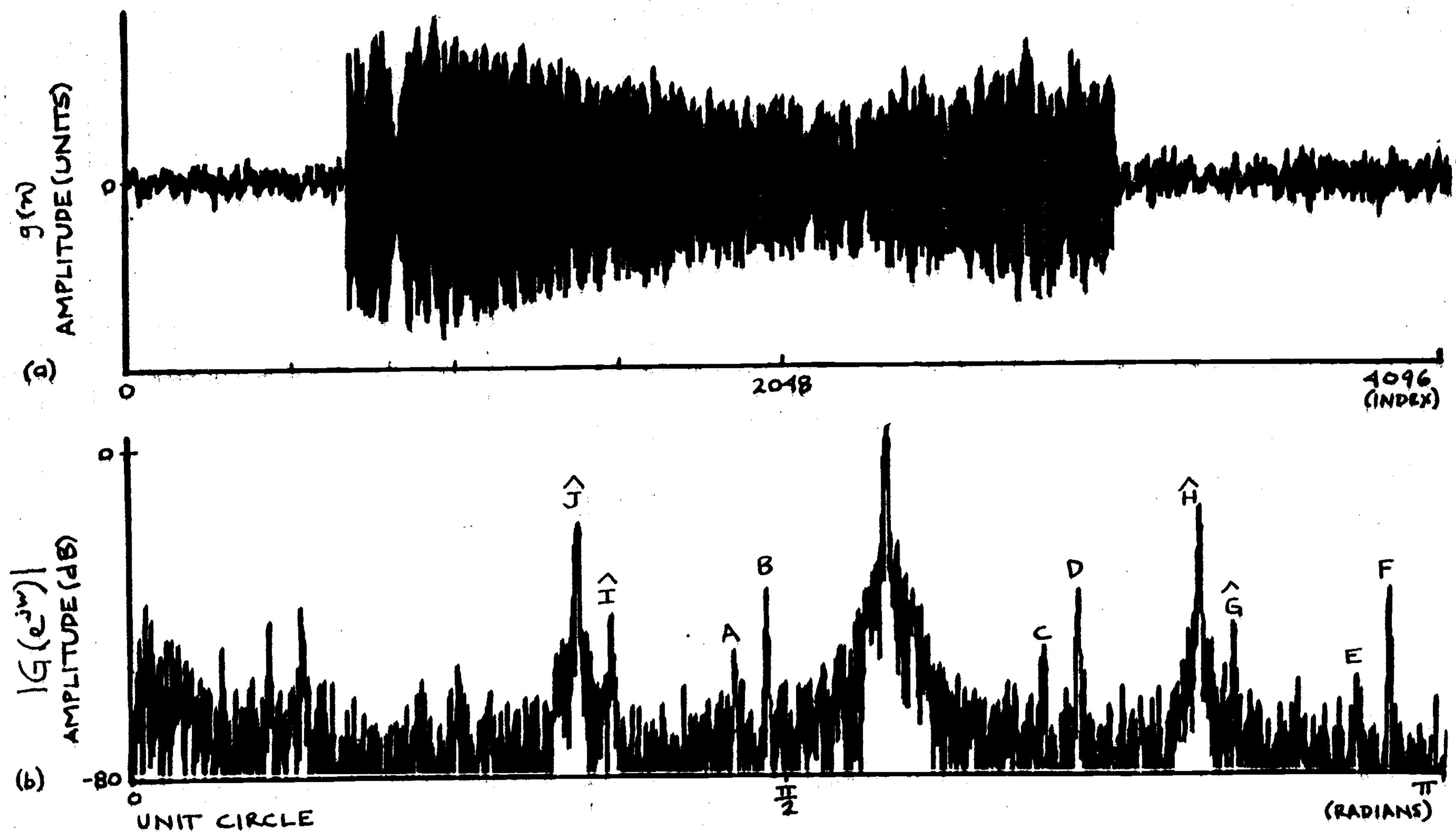

Fig. 10 (a) Original sequence decimated by 2. (b) Z-transform of decimated sequence on the unit circle. The components which summed due to aliasing are labeled with letters. Letters with hats are the same magnitude as the corresponding letter in Fig. 9.

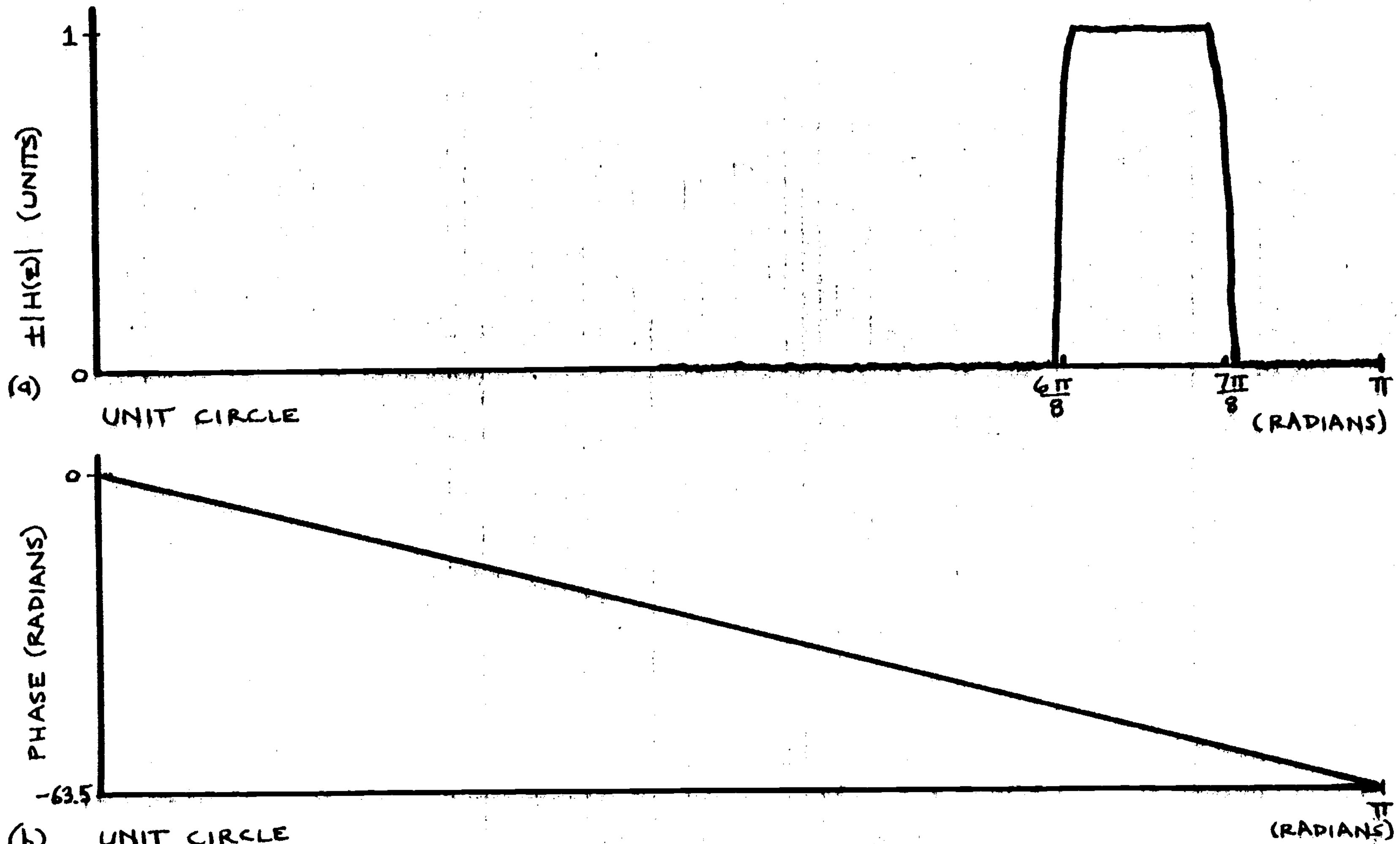

Fig. 11 (a) Magnitude response of 128 point FIR filter. (b) Linear phase response of filter. The magnitude function is plotted with both positive and negative values to achieve a linear phase plot. If the magnitude of H(z) were plotted, then the phase would change by π radians every time H(z) passes through a zero.

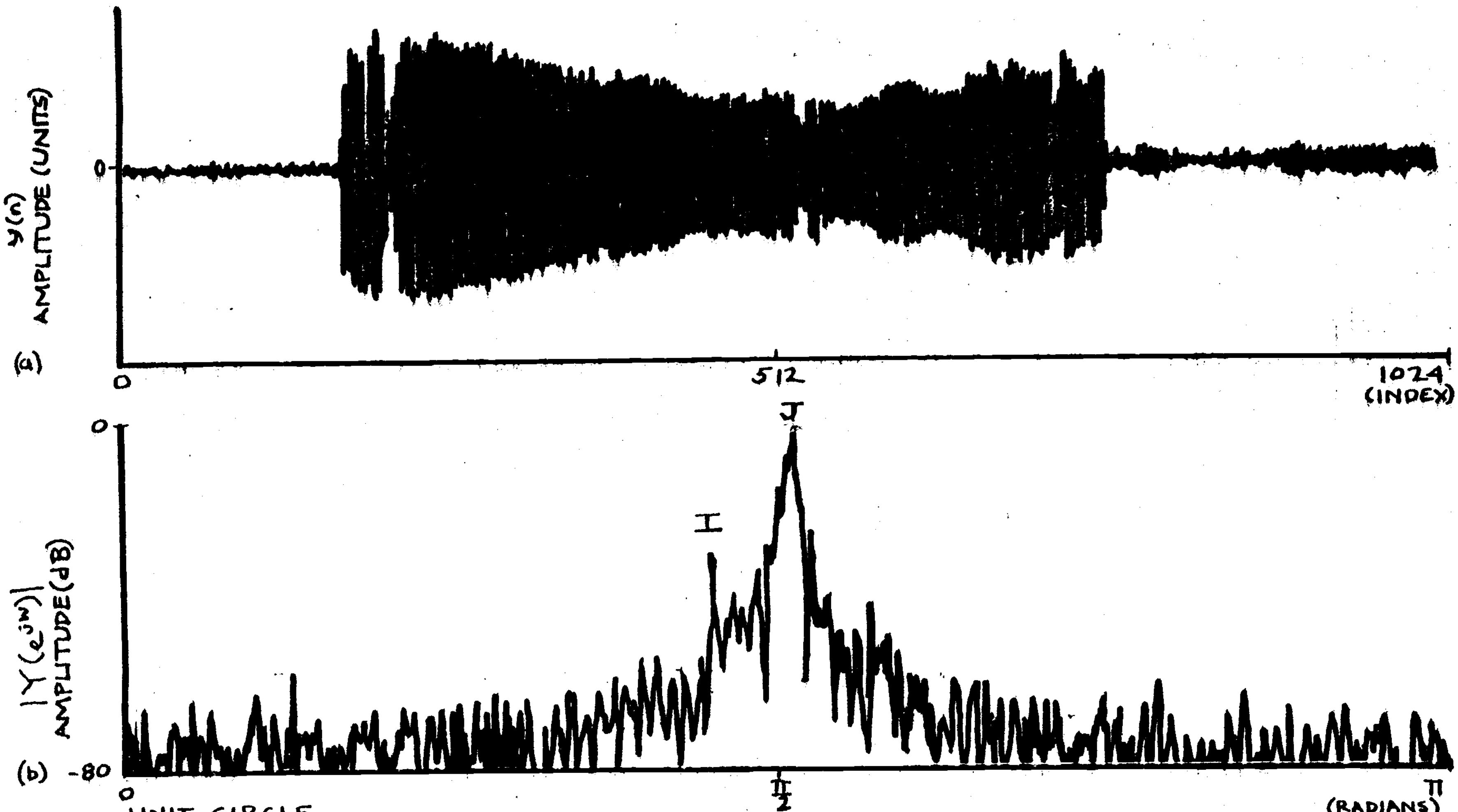

Fig. 12 (a) Sequence resulting after filtering the original sequence with a 128 point FIR filter and then decimating by 8. (b) Z-transform of sequence on the unit circle. Letters correspond to those in figure 9. The information in the region of interest has been preserved with a fewer number of samples.

Comments Concerning the Application of Interpolation and Decimation:

There is a serious question whether interpolation and decimation should be done with digital methods. If a sequence is to be generated by sampling an analog signal, as was the case in the previous example, then it may be better to under sample a bandpass filtered version of the signal rather than sampling the signal and then digitally filtering and decimating it. The cost and inflexibility of the analog filter along with its poor phase characteristics must be weighed against the complexity of digital filtering.

Comparing analog methods to digital methods is no trivial task and is very problem dependent. For example, in the laboratory where the data for the previous example was acquired, only an analog and digital computer are available. Since all processing of data can be done in non-real time it is easier to use digital methods for interpolation and decimation. Analog filters can be hard to implement on the analog computer whereas an FIR filter can typically be designed in less than one-half hour. If real time processing is required and as the amount of data to be processed increases, analog methods could provide the only solution since the digital hardware available may not be fast enough.

The literature contains many specialized techniques for efficient implementation of interpolation and decimation filters. Appendix B provides a summary of these methods, including linear interpolation, aliasing, octave band filtering, half-band filtering, and Hilbert Filtering. Some methods result in a complex valued sequence. If a general purpose computer is available which handles complex numbers, then these methods may not be unreasonable.

Summary:

Decimation can be applied to a sequence resulting from sampling a sonar signal to reduce the number of data points. The z-transform of the decimated sequence can be predicted from the Aliasing Theorem in Chapter 2.

IV. Conclusions and Recommendations

As indicated in chapters 2 and 3 the z-transform of a sequence evaluated on the unit circle is of prime importance due to the relationship between the unit circle and the Fourier transform domain of continuous signal processing. It may be concluded, therefore, if the region of interest for a given sequences z-transform is over only a small segment of the unit circle, then interpolation and decimation can be applied to produce a new sequence having a different number of samples without aliasing distortion. Interpolation results in an increase in the number of samples in the new sequence while decimation decreases the number of samples.

Equations are derived which describe the effects of interpolation and decimation. The hopping discrete Fourier transform is shown to be a decimation system.

Decimation is applied to a sequence resulting from sampling a typical sonar signal. Aliasing effects resulting from decimation without the use of a bandpass filter are shown. The effect on aliasing using a finite impulse response filter is demonstrated. The filter was designed in such a way that signals outside of the band of interest are attenuated below the original signal noise source level. It is shown that the band limited signals studied could be decimated by a factor of at least eight. The efficacy of the method used here depends critically on the effectiveness of the filter used to reduce the signal outside the band of interest below the source noise level.

Future studies should be concerned with the following:

1.) A means should be developed for measuring the error between a given signal and its reconstruction from filtered and decimated data.

2.) A study should be made of the minimization of the error criteria of 1.) by proper design of the filter decimator system.

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APPENDICES

Appendix A: Approximation of Bandlimited Systems (Ref. [11, 22])

A bandlimited system is defined as a system whose output z-transform is related to the input z-transform by the relation,

$$Y(\exp(jw)) = \begin{cases} X(\exp(jw)) & w_1 < |w| < w_2 \\ 0 & \text{otherwise} \end{cases}$$

where w_1 and w_2 are real constants and w is a real variable. This appendix discusses ways of approximating this system and explains the advantages of using a finite impulse response filter for the approximation.

A bandlimited system as defined above may be approximated with an infinite impulse response system (IIR) or a finite impulse response system (FIR). In physical implementations of an IIR system, past values of the output will affect the calculation of the present value of the output. FIR systems require only past values of the input to compute the present output.

The direct form implementation of an FIR system with output v(n) can be computed from the input sequence w(n) by the relationship

$$v(n) = \sum_{m=0}^{N-1} h(m)\, w(n-m)$$

where h(m), m = 0,1,..., N-1 are the filter coefficients and N is the duration of the impulse response. Since only the past values of the input are required to compute the output a computational savings is realized when the filter is used for interpolation and decimation in comparison to using IIR filters. In decimation only every M th output point must be computed where M is the decimation ratio. Furthermore, for an interpolated sequence only every L th point is nonzero where L is the interpolation ratio. Therefore the effective number of multiplications and additions discounting multiplications by zero and intermediate points is N/(LM) per output sample. With an IIR system these computational savings are not possible since every output value must be computed.

In general, IIR filters require less computation than a FIR filter to approximate a given characteristic on the unit circle. However for large values of decimation, FIR filters may require less computation since only every Mth point must be computed.

Another important feature of FIR filters is the ability to design them with exactly linear phase characteristics. This corresponds to a delay in the output sequence by a certain number of samples. IIR filters normally have non-linear phase characteristics which can distort the original signal.

For these reasons, FIR filters are attractive to use in interpolation and decimation systems. The following paragraphs describe some important features of FIR filters and discuss the various methods for designing FIR filters.

Linear Phase Condition for FIR Filters:

The impulse response for a FIR filter system with linear phase has the property that $h(n) = h(N-1-n)$, $n = 0, 1, \ldots, N-1$. This can be seen from the following argument.

The z-transform of the impulse response may be written as

$$H(z) = \sum_{n=0}^{(N/2)-1} h(n) z^{-n} + \sum_{n=N/2}^{N-1} h(n) z^{-n}$$

$$= \sum_{n=0}^{(N/2)-1} h(n) z^{-n} + \sum_{n=0}^{(N/2)-1} h(N-1-n) z^{-(N-1-n)}$$

assuming that N is even

$$H(z) = \sum_{n=0}^{(N/2)-1} h(n) \left[z^{-n} + z^{-(N-1-n)}\right]$$

assuming N is odd

$$H(z) = \left(\sum_{n=0}^{((N-1)/2)-1} h(n) \left[z^{-n} + z^{-(N-1-n)}\right] \right) + h\left(\frac{N-1}{2}\right) z^{-((N-1)/2)} .$$

Evaluating the above expression for $z = \exp(jw)$ for N even yields

$$H(\exp(jw)) = \exp(-jw((N-1)/2)) \left(\sum_{n=0}^{(N/2)-1} 2\,h(n) \cos\left(w\left(n - \frac{N-1}{2}\right)\right) \right)$$

and for N odd

$$H(\exp(jw)) = \exp(-jw((N-1)/2)) \cdot \left(h\left(\frac{N-1}{2}\right) + \sum_{n=0}^{\left(\frac{N-3}{2}\right)-1} 2\,h(n) \cos\left(w\left(n - \frac{N-1}{2}\right)\right) \right)$$

In both cases the terms in brackets are real implying a linear phase shift corresponding to a delay of $(N - 1)/2$ samples. Notice that for N even the delay is not an integer value.

Location of the Zeroes of an FIR Filter:

The symmetry condition discussed above for a linear phase FIR filter also implies certain things about the zeroes of the filter's z-transform. The z-transform of a finite impulse response filter is a polynomial of degree $N - 1$ where N is the duration of the impulse response. This polynomial may be factored to find its zeroes. There will be N-1 zeroes since the z-transform polynomial is of degree N-1. Since the impulse response is symmetrical, $H(z^{-1}) = z^{(N-1)}H(z)$. This means that H(z) and $H(z^{-1})$ are identical within a delay of N-1 samples. Therefore, the zeroes of H(z) are identical to the zeroes of $H(z^{-1})$ except at the points $z = 0$ and $z = \infty$.

It can be shown that if H(z) has a complex zero at $z = r\exp(jw)$ then it must also have a mirror image zero at $z = (1/r)\exp(-jw)$. Since the impulse response is real and hence the coefficients of the polynomial are real, every complex zero of H(z) must have a zero appearing at a complex conjugate location. Zeroes of an FIR filter not on the unit circle must appear in groups of four at $z = r\exp(jw)$, $z = r\exp(-jw)$, $z = (1/r)\exp(jw)$, and $z = (1/r)\exp(-jw)$. Zeroes on the unit circle appear in groups of two at $z = \exp(jw)$ and $z = \exp(-jw)$. A single zero can occur at $z = -1$ and zeroes on the real axis can appear in groups of two at $z = r$ and $z = 1/r$.

The location of the zeroes of an FIR filter become important when the filter has an even number of coefficients. (i.e. N even)
For N even a zero will always be located at $z = -1$ because of the symmetry of the filter's impulse response. A filter which must have a magnitude not equal to zero at $z = -1$ can not be adequately implemented when N is even. A digital high pass filter is an example since it has a non-zero magnitude at $z = -1$.

Thus differences exist between a linear phase FIR filter with an even number of coefficients and one with an odd number of coefficients. The even-number of coefficient filters result in a non-integer number of sample delay and are not suitable for implementing digital high pass filters.

The Remez Exchange Algorithm for FIR Filters:

The frequency response is normally specified in a FIR filter design problem. The Remez Exchange Algorithm is an efficient technique for computing the filter coefficient values. The input to the algorithm is a specification of the number of filter coefficients desired, the desired passband and stopband regions, the desired amplitudes in the pass and stopbands, and a relative weighting between ripple size in the passband and stopbands. The algorithm solves for a set of coefficients such that their z-transform evaluated on the unit circle has an equiripple characteristic. The algorithm allows the user to specify as many stopbands and passbands as desired.

The number of filter coefficients required for a given stopband and passband ripple for a low pass filter can be estimated by a formula given in [1] .

Other Design Methods:

Other methods are available to design FIR filter coefficients for some desired response on the unit circle. These methods are discussed in the literature.

Windowing -- A desired impulse response can be truncated by multiplying it by a window function. The disadvantage to this is the inability to specify the passband and stopbands precisely. As shown in [11] even though the window is optimum, the filter may not be optimum.

Frequency Sampling -- In the windowing method the impulse response is specified. Alternatively the z-transform of the impulse response evaluated on the unit circle can be specified. This specification can be sampled at evenly spaced intervals around the unit circle. If some regions are unspecified, then an algorithm can be developed which will design a filter to minimize the deviation of its response from the desired response. This method is normally implemented with linear programming.

Appendix B: Special Techniques for Implementing Interpolation and Decimation

Classical Method of Linear Interpolation: [19]

The process of linear interpolation can be viewed as a finite impulse response (FIR) filter. Given the interpolated sequence,

$$y(n) = \begin{cases} x(n/L) & n = 0, L, 2L, \ldots \\ 0 & \text{otherwise} \end{cases}$$

then interpolated values may be computed by using a linear approximation,

$$v(n+i) = x(n/L) + \left\{ \frac{[x(n/L+1) - x(n/L)]\, i}{L} \right\}$$

where n = 0, L, 2L, ... and i = 0, 1, ..., L-1.

Rearranging this equation yields,

$$v(n+i) = x(n/L)(1-i/L) + x(n/L+1)(i/L) .$$

To interpert this as a FIR filter an impulse response is required. To begin with the impulse response must be of length $N = 2L - 1$. If N is larger there would be more than two values of x(n) involved in the computation. If N is smaller then only one value of x(n) would be involved in the computation at some intermediate values. It is shown in [19] that the following non-causal impulse response when convolved with y(n) will linearly interpolate x(n) in the sense of determining v(n+i) for i = 0, 1, ..., L-1.

$$h(k) = \begin{cases} 1 - |k|/L & -L < k < L \\ 0 & \text{otherwise} \end{cases}$$

The impulse response is written in non-causal form for convenience. A causal impulse response can also be defined which has the same shape but delayed.

The z-transform of this filter evaluated on the unit circle is

$$| H(\exp(jw)) | = \frac{1}{L} \left(\frac{\sin^2(wL/2)}{\sin^2(w/2)} \right)$$

Other methods of interpolation such as Lagrange polynomials can be similarly treated. The characteristics of these methods on the unit circle discourage their use in signal processing applications. The real purpose of these filters is to approximate a bandlimited system. A better approximation can be obtained by using the Remez Exchange Algorithm discussed in Appendix A.

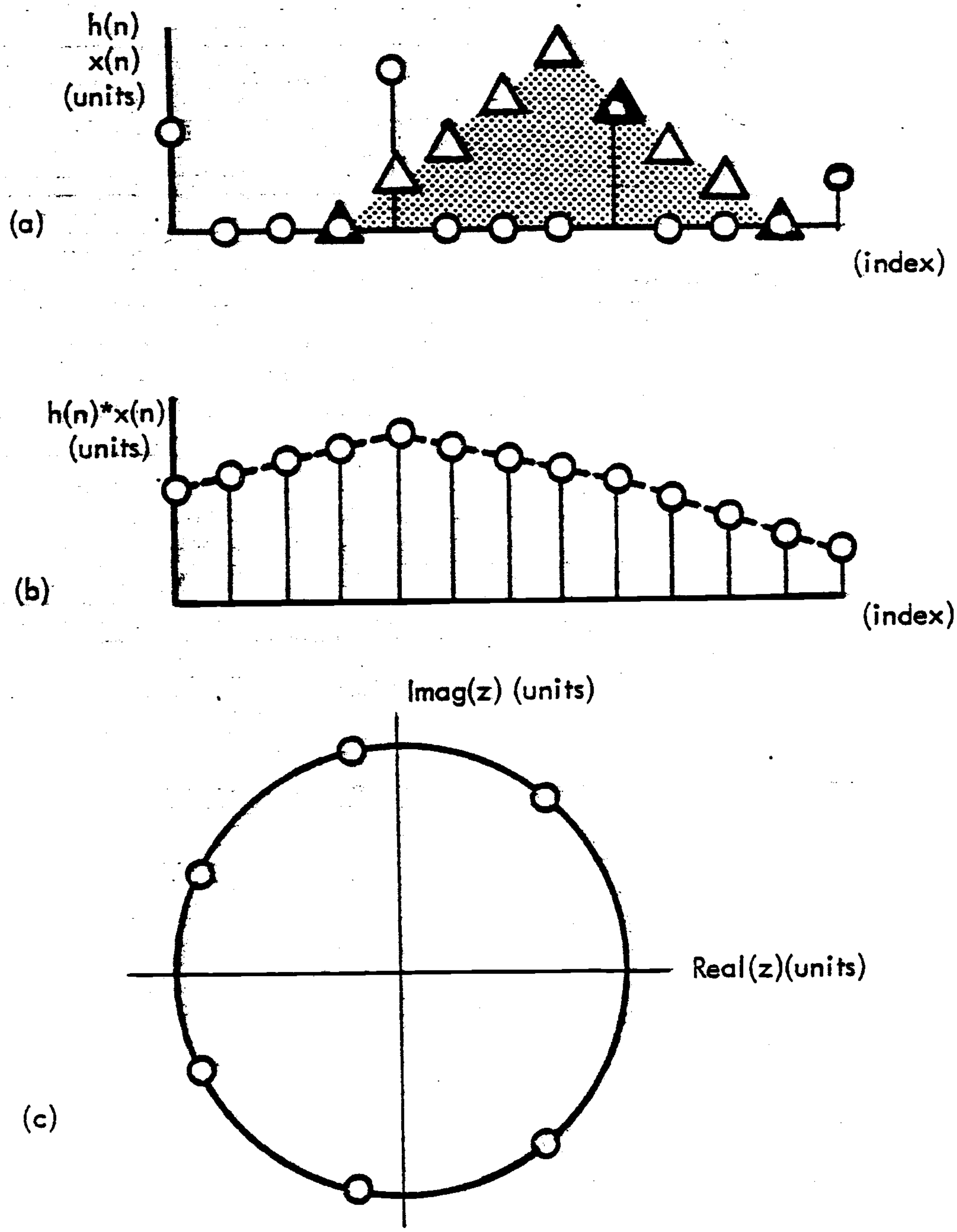

Fig. 13 (a) Interpolated sequence superimposed over a filter sequence. (b) Convolution of the interpolated sequence with the filter results in a linearly interpolated sequence. (c) Zero plot of the filter's z-transform.

The General Interpolation Filter Decimation Stage: [1]

A general interpolation decimation stage can be approximated by an interpolator in cascade with a FIR filter and decimator. If the number of filter coefficients is required to be an integer multiple of the interpolation ratio, L, then without loss of generality the output sequence may be represented as

$$y(k) = \sum_{i=0}^{Q-1} h(iL + (kM)\text{modulo}(L)) \cdot x(\text{integer}(kM/L) - i)$$

where y(k) is the output sequence, $k = 0, 1, 2, \ldots,$

h(m) is the mth filter coefficient, $m = 0, 1, \ldots, N-1,$

N is the number of filter coefficients,

L is the interpolation ratio,

M is the decimation ratio,

Q is an integer such that $QL = N$, and

x(p) is the input sequence, $p = 0, 1, 2, \ldots.$

The restriction on N is not very serious since zero value coefficients can always be added to the filter's impulse response. When $M = L = 1$ this equation reduces to the familiar convolution equation.

Design charts and procedures are available for interpolation and decimation filters where h(m) is assumed to be a low pass filter. In general h(m) may be a multi-bandpass filter. The effect of such a filter may be deduced from the aliasing equations derived in Chapter 2.

Bandpass Mapping: [20]

Given a sequence with a z-transform defined on the unit circle,

X(exp(jw))

a new transform may be defined as

$Y(\exp(j\tilde{w}))$

where $\tilde{w} = \pi \dfrac{w - w_1}{w_2 - w_1}$ $\qquad -\pi < \tilde{w} < \pi$

and w_1 and w_2 represent the endpoints of a segment of the unit circle.

This process takes a segment of the unit circle and maps it such that it covers the entire unit circle from $\tilde{w} = 0$ to $\pm\pi$, hence the name bandpass mapping. It can be implemented in the following way.

Define the auxiliary mappings

$$N(w) = \pi \frac{w}{w_2} \qquad 0 \leq N \leq n$$

$$O(N) = \pi - N \qquad 0 \leq O \leq n$$

$$Y(O) = \pi \frac{O}{O_1} \qquad 0 \leq Y \leq n$$

$$D(Y) = \pi - Y \qquad 0 \leq D \leq n$$

where $O_1 = O(N_1)$ and $N_1 = N(w_1)$. Combining the above mappings gives

$$\tilde{w}(w) = D(Y(O(N(w)))) \qquad -\pi < \tilde{w} < \pi$$

The functions O(N) and D(Y) are simple complex demodulator systems where the output is the input times $\exp(-j\,n\pi)$, where n is the input index. The functions N(w) and Y(O) can be implemented by the interpolator decimator stage described in the previous section. Assuming, without loss of generality, that

$$w_1 = w_2 (1 - L_2/M_2) \text{ , and}$$

$$w_2 = L_1 \pi / M_1$$

where L_1, L_2, M_1, and M_2 are the integer valued interpolation and decimation ratios of two different interpolator decimator stages then the function N(w) will be defined as an interpolator decimator stage with $L = L_1$, $M = M_1$ and a lowpass filter with cutoff frequency π / M_1. Likewise the function Y(O) is an interpolator decimator with $L = L_2$, $M = M_2$ and a lowpass filter with cutoff frequency π/M_2.

This method is totally general in the sense that it will work for any sequence and for any segment of the unit circle. Difficulties arise for many values of w_1 and w_2 when M_1 and M_2 are required to be large. For large values of M, very sharp lowpass filters must be designed which may require excessive amounts of computation. This problem can be partially solved by using more than two stages.

Octave Band Decomposition: [12]

An octave band filter approximates a bandlimiting system where

$w_1 = \pi/4$ and $w_2 = 3\pi/4$.

The advantage of this filter is that an FIR filter may be designed where every other coefficient is zero in addition to the first and last coefficients being zero. For example, a 26 point filter will require only seven non-zero values. This results in computational savings since multiplication by zero is unnecessary.

Using octave band filters a sequence may be decomposed into as many octave bands as desired. The procedure is as follows.

Assume that the sequence has been manipulated such that it is bandlimited to $\pi/2$. The sequence is passed through an octave band filter and the highest octave remains after filtering extending from $\pi/4$ to $\pi/2$. This sequence may be subtracted from the original sequence resulting in a new sequence which is bandlimited to $\pi/4$. If this new sequence is decimated by 2 it will be bandlimited to $\pi/2$ and the above process may be repeated to acquire the next octave band. (Fig. (14))

Octave band filters cannot be directly designed by the Remez Exchange Algorithm. However, the algorithm can be used to approximatean octave band filter. The design method available in the literature uses windowing techniques.

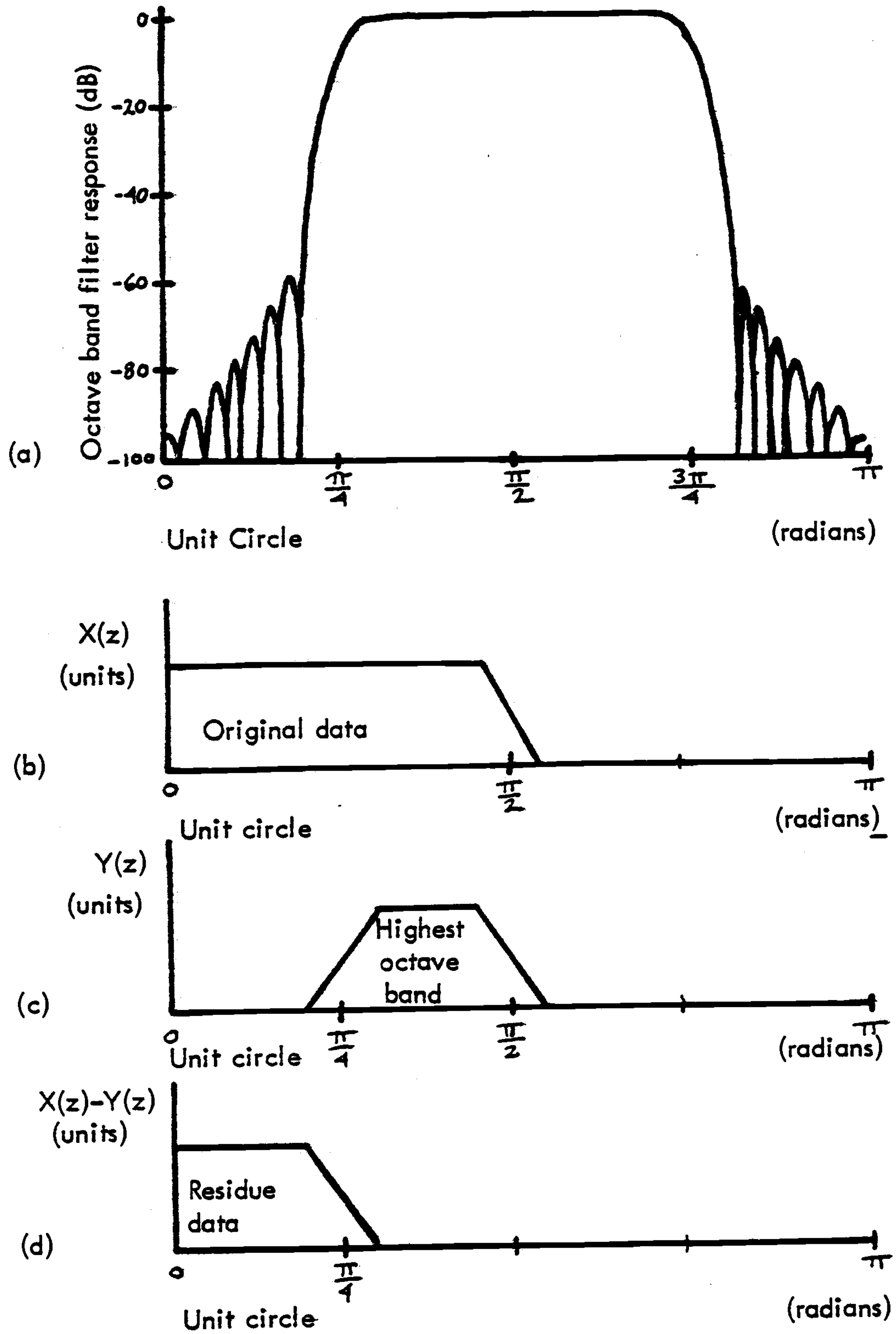

Fig. 14 (a) Typical response of an octave band filter. (b) Original data bandlimited to π/4. (c) Highest octave band after filtering X(z). (d) Residue after subtraction. Resulting signal may be decimated by 2 and the filtering process repeated to yield the next octave band.

Half-Band Filtering for Decimation: [3, 14, 16]

Another type of filter with a large number of zero valued coefficients is called a half-band filter. These filters approximate a bandlimited system where $w_1 = 0$ and $w_2 = \pi/2$.

If the impulse response of a half band filter has N coefficients, where N is an odd number, then the impulse response (Fig. (15))

$h(n) = 0$ for all

$$n = 0, 2, 4, \ldots, \frac{N-1}{2} - 2, \frac{N-1}{2} + 2, \ldots, N-5, N-3, N-1 .$$

Half-band filters may be designed with the Remez Exchange Algorithm by specifing a symmetrical pass band and stop band. Half-band filters are used in a manner similar to octave band filters. A sequence may be half-band filtered and decimated by two. The resulting sequence may be again half-band filtered and decimated by two.

Published designs are available where many of the non-zero filter coefficients are powers of two. This could facilitate a hardware implementation of the filter.

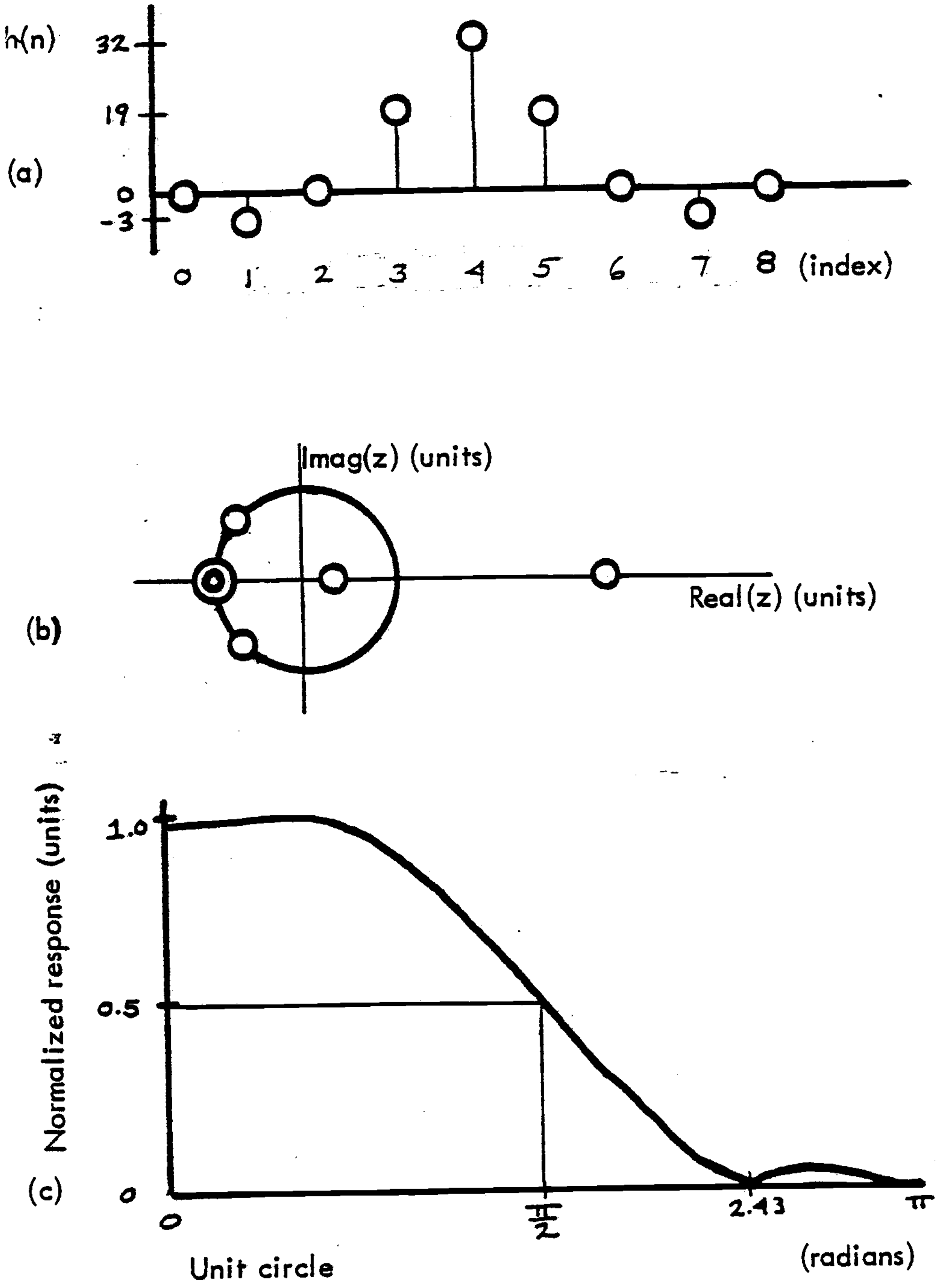

Fig. 15 (a) Nine-point half-band filter. (b) zero plot of half-band filter. (c) Unit circle magnitude plot of half-band filter.

Hilbert Filtering for Decimation: [11]

A single-side band sequence may be defined as one whose z-transform evaluated on the unit circle between π and 2π is exactly zero. That is for a sequence v(n) then

$$V(\exp(jw)) = 0 \qquad \pi \leq w \leq 2\pi$$

The sequence may also be bandlimited such that

$$V(\exp(jw)) = 0 \qquad 0 \leq w \leq w_1 \qquad w_2 \leq w \leq \pi$$

The sequence v(n) must be a complexed valued one since the z-transform of a sequence evaluated on the unit circle must satisfy the following condition if the sequence is real:

$$V^*(\exp(-jw)) = V(\exp(jw))$$ where * is complex conjugation.

Since v(n) is complex it may be represented as

$$v(n) = x(n) + j\,\hat{x}(n)$$

where x(n) and $\hat{x}(n)$ are real valued sequences.

For a single side band sequence

$$V(\exp(jw)) = X(\exp(jw)) + j\,\hat{X}(\exp(jw)) = 0 \qquad \pi \leq w < 2\pi$$

or

$$\hat{X}(\exp(jw)) = -j\,X(\exp(jw)) \qquad \pi \leq w < 2\pi$$

Since x(n) and $\hat{x}(n)$ are real sequences

$$\hat{X}(\exp(jw)) = -j\,X(\exp(jw)) \qquad 0 \leq w < \pi$$

Therefore the signal $\hat{x}(n)$ may be obtained from x(n) by filtering it with a filter with the frequency response H(exp(jw)) defined by

$$H(\exp(jw)) = \begin{cases} -j & 0 < w < \pi \\ j & \pi < w < 2\pi \end{cases} .$$

In this case V(exp(jw)) = 2X(exp(jw)) in the interval $0 < w < \pi$, and V(exp(jw)) = 0 in the interval $\pi < w < 2\pi$.

Like the bandlimited systems discussed previously, the Hilbert filter can be approximated by an FIR system. Hilbert filters can be designed with the same Remez Exchange Algorithm mentioned before. Unlike other linear phase FIR filters, a Hilbert filter is antisymmetric. For an impulse response h(n), n = 0, 1, ..., N-1, it is antisymmetric if h(n) = - h(N-n-1).

The aliasing equations derived in Chapter 2 will still apply to a single-sideband sequence. The only difference is that the resulting complex sequence may be decimated by twice as much without aliasing distortion. However, since a complex sequence requires two numbers to represent each element, no savings is realized in terms of the number of real values required to represent the sequence. An example of Hilbert filtering and decimation is shown in Fig. (16).

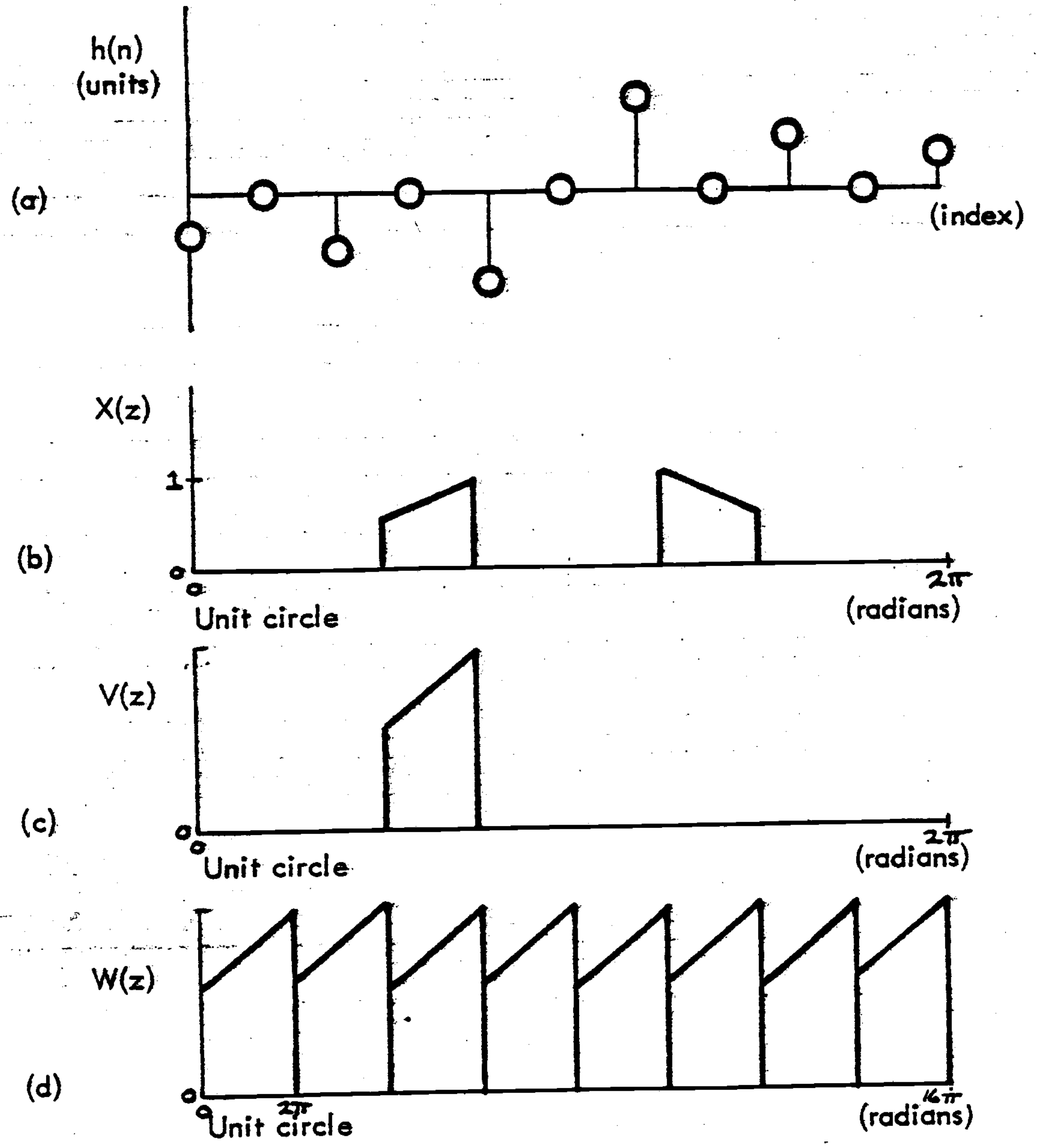

Fig. 16 (a) Typical Hilbert filter sequence. (b) Unit circle magnitude plot of a sequence. (c) Result of summing the sequence with its Hilbert filtered version. (d) Decimation of the sequence by 8.

Appendix C: Proof of Theorem 10

Proof:

From eq. (32) Chapter 2

$$Y(\exp(jw)) = (1/M) \sum_{k=0}^{M-1} X(\exp(j(w-2\pi k)/M)) \qquad -\pi < w < \pi$$

For the region $0 < w < \pi$ the above summation is zero except for the term $k = M - m/2$ when m is even and $k = m/2$ when m is odd, since $X(\exp(jw))$ for $|w| < \frac{2\pi}{M}\left(\frac{m}{2}\right)$ and $|w| > \frac{2\pi}{M}\left(\frac{m+1}{2}\right)$ is zero.

Substituting these values of k into eq.(32) and rearranging the exponent gives,

$$Y(\exp(jw)) = (1/M)X(\exp(j(w_1 + w/M))) \quad \text{for m even and } 0 < w < \pi \text{ or m odd and } -\pi < w < 0$$

$$= (1/M)X(\exp(j(-w_1 + w/M))) \quad \text{for m even and } -\pi < w < 0 \text{ or m odd and } 0 < w < \pi$$

Q.E.D.