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Title: Smooth Nonparametric Conditional Quantile Profit Function Estimation

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Recently, in an attempt to produce robust production frontier estimators, Aragon et al. [2005, Nonparametric frontier estimation: a conditional quantile-based approach. Econometric Theory 21, 358-389] and Martins-Filho and Yao [2008, A smooth nonparametric conditional quantile frontier estimator. Journal of Econometrics 143, 317-333] considered the estimation of nonparametric $\alpha-$ frontier models based on conditional quantiles with $\alpha \in (0,1)$. There exist, however, a large and growing literature in economics devoted to the estimation of profit functions. In this paper, we first define an $\alpha-$profit function based on the quantile of the suitably defined conditional distribution for profits. Second we propose a smooth nonparametric conditional quantile estimator for the $\alpha-$profit function model. Our estimator is computationally simple, resistant to outliers and extreme values, and smooth. In addition, the estimator is shown to be consistent and asymptotically normal under mild regularity conditions. A small simulation study provides evidence of the finite sample properties for the estimator.

Smooth Nonparametric Conditional Quantile Profit Function Estimation

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Anton Piskunov, Author

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Academic

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Personal

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1. INTRODUCTION

Investigation of the efficiency (or productivity) of a firm, an industry or other objects of interest and specification of a frontier, corresponding to efficient units have been the subject of research in economics since the fundamental papers of Farrell (1957), Shephard (1953, 1970) and Debreu (1951). These papers introduced the basic ideas for construction of measures of productive efficiency of the firm based on what has become known as Farrel and Shephard distance functions, which in turn depend on production function (frontier). A number of subsequent papers were devoted to the development of different procedures for production function estimation. There exist, in addition, a large and growing literature in economics devoted to the estimation of profit functions (or profit frontier). The goal of this paper is to propose a robust profit frontier estimator and investigate it's statistical properties.

1.1. Literature and methodology review

It is convenient to start with a small literature review which will shed some light on the existing methods of frontier estimation.

Consider a pair $(x, y) \in \Re_+^D \times \Re_+$, we can describe a technology where firms use inputs x in order to produce output y as

$$\Psi = \{(x, y) \in \Re_+^D \times \Re_+ : x \textit{ can produce } y\}.$$

A production function and Farrel's output distance function for some pair (x_0, y_0) are defined respectively as

$$g(x) = \sup\{y \in \Re_+ : (x, y) \in \Psi, x \in \Re_+^D\}, \quad F_o(x_0, y_0) = \sup\{\beta : (x_0, \beta y_0) \in \Psi, (x_0, y_0) \in \Psi, \beta \geq 1\}.$$

Then we can define a Farrel's output efficiency measure for (x_0, y_0) as

$$0 \leq \; D_o(x_0, y_0) \;=\; \frac{y_0}{g(x_0)} \;=\; \frac{1}{F_o(x_0, y_0)} \;\leq\; 1,$$

where $D_o(x_0, y_0)$ is also known as Shephard's output distance function. Alternatively, we define

$$D_o(x_0, y_0) \;\equiv\; \inf\{\theta : (x_0, \frac{y_0}{\theta}) \in \Psi,\; \theta \in (0, 1]\}.$$

The main objective in production and efficiency analysis is, given a random sample of production pairs $\{X_i, Y_i\}_{i=1}^{n} \in \Psi$, to obtain an estimator $\hat{g}(\cdot)$ for $g(\cdot)$ and use it to obtain estimated efficiencies $\hat{D}_o(X_i, Y_i)$ for $i = 1, 2, ..., n$.

Estimators for production frontiers can be divided broadly into those which are based on: parametric or nonparametric models which, in turn, can be subdivided into stochastic and non-stochastic.

Popular among the parametric stochastic frontiers are the models and estimators proposed by Aigner, Lovell and Schmidt (1977), and Kumbhakar and Lovell (2000). These models impose some known parametric structure on the production technology and assume (in)efficiency can be modeled as a random shock to output with some specific distribution. In addition, output is subject to random shocks which may result in production plans that are inside or outside the set Ψ. This model can in general be easily estimated using Maximum Likelihood based estimation, which is known to be asymptotically normal and efficient. However, it has important disadvantages: (i) by assuming a parametric form for the technology, restrictions on the shape of the frontier can lead to potential misspecification; (ii) a large number of distributional assumptions could be made on the efficiency and random shocks with impact on estimation. Results are sensitive to these assumptions and there is in general lack of robustness of the results to these distributional assumptions. Even if one could analyze properties of the model with several different distributional assumptions (Greene (1990)), it is not possible to analyze impacts of all kinds of distributions.

In deterministic frontier models, there exists no random shocks to output that are related to (in)efficiency.The most frequently used deterministic frontier estimation procedures are Data Envelopment Analysis (DEA) first introduced by Charnes, Cooper and Rhodes (1978), and Free Disposal Hull (FDH) proposed by Deprins et al. (1984). Both procedures use similar nonparametric, enveloping techniques based on free disposability of the technology assumption and the assumption that all observation lie in the set Ψ, but FDH is more general since it does not require convexity of the technology. Both, DEA and FDH are very easy to implement using linear programming algorithms and have known asymptotic properties given by Gijbels et al. (1999) and Park et al. (2000). However, these estimators have a number of important disadvantages. First, because of the enveloping nature of DEA and FDH, they are very sensitive to extreme values and are inherently biased. In other words, DEA and FDH provide frontier estimators which based on the best production plans in the sample and do not give us much information on how close an estimated frontier is to the true one. Second, even if true frontier is smooth, FDH and DEA provide us with discontinuous or piecewise linear functions. Even though there were some attempts to resolve some of these deficiencies (Simar and Wilson (2004)), various alternative procedures have emerged based on different approaches to frontier modeling and estimation. Among these are partial frontiers of order-m introduced by Cazals et al. (2002) and order-α conditional quantile frontiers proposed by Aragon et al. (2005).

Aragon et al. (2005) define the production function as,

$$g(x) \;=\; \sup\{y \in \Re_+ : F(y|x) < 1\} \;=\; \inf\{y \in \Re_+ : F(y|x) = 1\},$$

where $F(y|x) = \frac{F(x,y)}{F_X(x)}$, $F(x,y) = P(\{X \le x, Y \le y\})$ and $F_X(x) \equiv P(\{X \le x\})$ is the associated marginal distribution of X. Since $F_X(x) > 0$, they focus attention on a set

$$\Psi^* \;=\; \{(x,y) \in \Psi : F_X(x) > 0\}.$$

Aragon et al. (2005) observe that the last definition of the production function (which coincides with a conditional quantile of the order one) suggests the possibility to constructing a production function of continuous order $\alpha \in [0,1]$. They propose the following,

$$q_\alpha(x) \;=\; \inf\{y \in \Re_+ : F(y|x) \ge \alpha\}.$$

as an $\alpha-$frontier. The interpretation for $q_\alpha(x)$ is that it is the level of production exceeded by $(1-\alpha) * 100\%$ of producion plans that use inputs less than or equal to x. If $F(y|x)$ is strictly increasing on the support $[0, g(x)]$, then $q_\alpha(x) = F^{-1}(\alpha|x)$, where $F^{-1}(y|x)$ is the inverse of $F(y|x)$[1]. In this context any allocation (x,y) belongs to some quantile curve of order-α. Hence it allows us to dermine the level of productivity of every production plan relative to the other

[1]See Proposition 2.1 in Aragon et al. (2005).

plans using the same or less amount of inputs. Another advantage of this approach is that by construction, it does not require any assumptions on the shape of the technology and as $\alpha \to 1$, $q_\alpha(x) \to g(x)$[2]. Aragon et al. (2005) proposed an estimator of q_α based on empirical conditional distribution $\hat{F}(y|x)$, given by

$$\hat{q}_{\alpha,n}(x) \;=\; \hat{F}^{-1}(\alpha|x) \;=\; \inf\{y : \hat{F}(y|x) \geq \alpha)\}$$

and show that $\sqrt{n}(\hat{q}_{\alpha,n}(x) - q_\alpha(x)) \xrightarrow{d} N(0, \sigma^2(x,\alpha))$, where $\sigma^2(x,\alpha) = \alpha(1-\alpha)/(f^2(q_\alpha(x|x))F_X(x))$.

Recently Martins-Filho and Yao (2008) proposed a smooth conditional quantile frontier estimator based on Azzalini (1981), which is based on the integration of a Rosenblatt density estimator. They show that, under suitable assumptions, their estimator has some advantages comparable to Aragon et al. For finite samples, their estimator has a smaller variance, which is good property, but smoothness introduces a bias term, which Aragon's estimator does not have. Monte-Carlo simulation conducted by Martins-Filho and Yao reveals, that when the data is such that firms are mostly relatively efficient, then the bias term is dominated over the advantage of smaller variance and it is probably better to use Aragon's estimator, but for the data where we have a small number of efficient firms their estimator is preferable. Asymptotically, both estimators are equivalent since the bias term disappears and variances become the same. We now turn to the main topic of this thesis.

1.2. Statement of the problem and organization of the Thesis

In the previous section we discussed different methods for production frontier estimation. Previous attempts to obtain profit efficiency e.g., Fare and Grosskopf (2005), Briec et al.(2006), Maruyama E. and Torero M. (2007), used parametric stochastic frontier estimators or DEA/FDH estimators, which have some of the disadvantages discussed above. Here we build upon the estimation proposed by Martins-Filho and Yao (2008) to construct a robust and smooth estimator for a profit function (profit frontier). First, we define an $\alpha-$profit function based on the quantile of the suitably defined conditional distribution for profits. Second, we propose a smooth nonparametric conditional quantile estimator for the $\alpha-$profit function model. Because of the nature of the

[2]See Theorem 2.4 in Aragon et al. (2005).

methodology we use, our estimator is computationally simple, resistant to outliers and extreme values, and smooth. In addition, we establish that the estimator is consistent and asymptotically normal under mild regularity conditions.

This thesis has the following structure. In Chapter 2 we define $\alpha-$profit functions and investigate some of it's basic properties. Then, in Chapter 3, we introduce our smooth conditional quantile estimator for $\alpha-$profit function and impose some assumptions, necessary for further analysis. In Chapter 4, we provide the main results, which shed light on statistical properties of the estimator. Finally, in Chapter 5, we present a small Monte-Carlo study in order to check finite sample properties of our estimator. Proofs of all results and some interesting properties are listed in the Appendices.

2. CONDITIONAL QUANTILE REPRESENTATION OF THE PROFIT FUNCTION

2.1. Definition of the conditional quantile profit function

Because of the nature of the conditional quantile representation of the production frontier we could consider only the case of a single output and multiple inputs[3]. One of the advantages of efficiency estimation using conditional quantile profit function is that we can easily consider the case of multiple inputs and multiple outputs, which in many instances more realistic.

Let $x \in \Re_+^D$ and $y \in R_+^M$ be input and output vectors, and $w = (w_1, w_2, \ldots, w_D) \in \Re_{++}^D$, $p = (p_1, p_2, \ldots, p_M) \in \Re_+^M$ be vectors of input and output prices respectively.

Given a pair of prices $(p, w) \in \Re_+^M \times \Re_{++}^D$, a level of output $y \in \Re_+^M$ and a technology Ψ, profit can be defined as

$$\pi(p, w, y) = py - c(w, y),$$

where $c(w, y) = \inf\{wx : (x, y) \in \Psi,\ w \in \Re_{++}^D\}$ is a cost function. Assume, that for all prices $(p, w) \in \Re_+^M \times \Re_{++}^D$, profit $\pi(p, w, y)$ is bounded above as a function of y, i.e. $\exists\ 0 < B_\pi < \infty$ such that $0 \leq \pi(p, w, y) \leq B_\pi$ for all y.

Thus, the profit function can be defined as

$$\begin{aligned} \pi(p, w) &= \sup\{\pi(p, w, y) : y \in \Re_+^M\} \\ &= \sup\{py - c(w, y) : y \in \Re_+^M\} \\ &= \sup\{py - wx : (x, y) \in \Psi\}. \end{aligned} \tag{2.1}$$

Profit function, when it exists, has the following properties:

(i) Nondecreasing in p: if $p^0 \geq p^1$ then $\pi(p^0, w) \geq \pi(p^1, w),\ \forall\ w$.

(ii) Non increasing in w: if $w^0 \geq w^1$ then $\pi(p, w^0) \leq \pi(p, w^1),\ \forall\ p$.

(iii) Homogeneity in p, w: $\pi(kp, kw) = k\pi(p, w)$ for $k > 0$.

(iv) Convex in p. Let $p = \lambda p^0 + (1 - \lambda)p^1$ for $0 \leq \lambda \leq 1$. Then $\pi(p, w) \leq \lambda\pi(p^0, w) + (1 - \lambda)\pi(p^1, w)$.

[3] See (Simar and Daouia 2007) for an attempt to deal with multiple outputs. Their proposed estimator suffers from the curse of dimensionality.

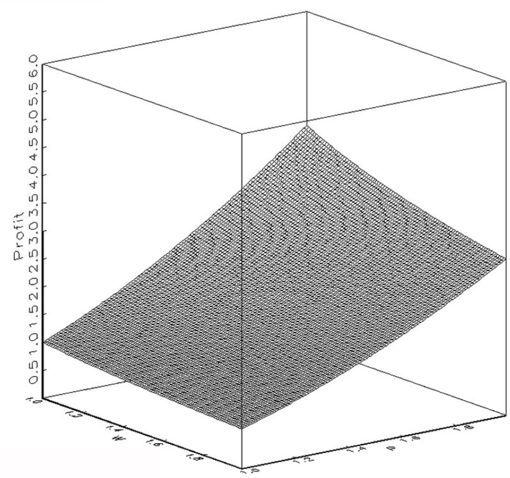

FIGURE 2.1: 3D graph of the profit function

Figure 2.1 illustrates sample profit function (generated as described later in the Monte Carlo section) and provides some intuition for properties (i), (ii) and (iv).

The set of all attainable profits, given the technology Ψ, can be defined by

$$\Psi_\pi = \{(p, w, \pi) : \pi \leq \pi(p, w), p \in \Re_+^M, w \in \Re_+^D\}.$$

Let $F(p, w, \pi)$ be an absolutely continuous distribution function with associated density $f(p, w, \pi)$, and F_{PW}, F_P, $F_{P\Pi}$ and f_{PW}, f_P, $f_{P\Pi}$ be, respectively, marginal distributions and densities for the corresponding variables appearing as subscripts.

Before we proceed with a definition of the conditional quantile profit function, notice that when we deal with quantile of distribution for a random variable X, based on a conditioning set C we can define

$$q(\alpha|C) = \inf\{x : F(x|C) \geq \alpha\},$$

where $F(x|C) = \frac{Prob(X \leq x, C)}{Prob(C)}$ is a conditional distribution of X given a conditioning set C, such that $Prob(C) > 0$.

Thus, we can define the profit function as:

$$\pi(p, w) = \sup\{\pi \in [0, B_\pi] : F(\pi|p, w) < 1\} = \inf\{\pi \in [0, B_\pi] : F(\pi|p, w) = 1\},$$

where $F(\pi|p, w) = \frac{Prob(\Pi \leq \pi, C)}{Prob(C)}$ is a conditional distribution of π given conditioning set $C =$

$\{P \leq p, W \geq w\}$ with $Prob(C) > 0$, and B_π is an upper bound for the profits given prices $(p, w) \in \Re_+^M \times \Re_+^D$.

The concept can be extended to a profit function of continuous order α by

$$\pi_\alpha(p, w) = \inf\{\pi \in [0, B_\pi] : F(\pi|p, w) \geq \alpha\},$$

Note that denominator in the expression for $F(\pi|p, w)$ can be written as,

$$\begin{aligned} Prob(P \leq p, W \geq w) &= \int_0^p \int_w^\infty \int_0^{B_\pi} f(P, W, \Pi) d\Pi dW dP \\ &= \int_0^p \int_w^\infty f_{PW}(P, W) dW dP \end{aligned}$$

Since $\int_0^\infty f_{PW}(p, W) dW = f_P(p)$,

$$\int_w^\infty f_{PW}(p, W) dW = f_P(p) - \int_0^w f_{PW}(p, W) dW.$$

Hence, we write

$$\int_0^p \int_w^\infty f_{PW}(P, W) dW dP = F_P(p) - F_{PW}(p, w).$$

Similarly we can write,

$$\int_0^p \int_w^\infty \int_0^\pi f(P, W, \Pi) d\Pi dW dP = F_{P\Pi}(p, \pi) - F(p, w, \pi).$$

Consequently, the conditional distribution would be,

$$F(\pi|p, w) = \frac{F_{P\Pi}(p, \pi) - F(p, w, \pi)}{F_P(p) - F_{PW}(p, w)},$$

where $F_P(p) - F_{PW}(p, w) > 0$. Thus, we can focus our attention on the set

$$\Psi_\pi^* = \{(p, w, \pi) \in \Psi_\pi : F_P(p) - F_{PW}(p, w) > 0\}.$$

Figure 2.2 represents an α-profit function for one output and a fixed input price $\bar{w}$. Figure 2.3 shows an $\alpha-$profit function for the case of one input and a fixed output price $\bar{p}$. Here the profit function is convex to the origin with respect to input prices. The area of interest in the Figure 2.2, i.e. a set of efficient points for given (p, w), are all points on the left hand side of $\bar{p}$ and above the $\pi_\alpha(\bar{p}, \bar{w})$. In Figure 2.3 all points on the right hand side of the $\bar{w}$ and above $\pi_\alpha(\bar{p}, \bar{w})$ are the plans efficient relative to $(\bar{p}, \bar{w})$ under π_α.

A three dimensional picture with shaded area of interest for given (p, w) is shown in the Figure 2.4.

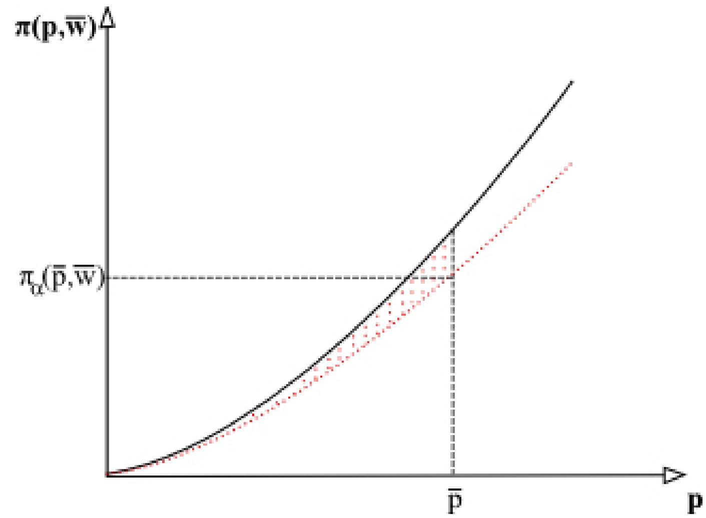

FIGURE 2.2: Profit with fixed input price

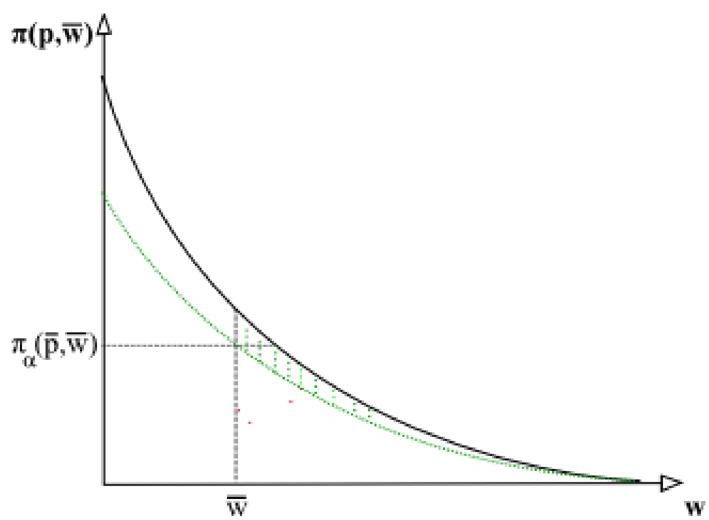

FIGURE 2.3: Profit with fixed output price

2.2. General properties of the conditional quantile profit function

Similarly to the case of production function we can think of $\pi_\alpha(p, w)$ as the profit level exceeded by $(1 - \alpha) * 100\%$ of firms that face output prices less than p and input prices greater than w. We now show that, the set of quantile curves $\{(p, w, \pi_\alpha(p, w)) : F_P(p) - F_{PW}(p, w) > 0,\ \alpha \in [0, 1]\}$ fills the space Ψ^*_π, i.e. we show that for every firm represented by triplet a (p, w, π) there exists a corresponding α quantile curve, which gives us the level of profit efficiency of the observed firm.

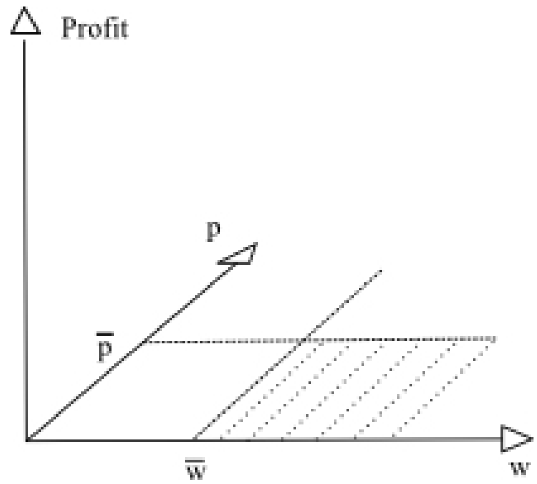

FIGURE 2.4: Area corresponding to the conditional quantile profit function for given $(\bar{p}, \bar{w})$.

Assumption A2.1 *$\forall$ $(p, w) \in \Re_+^M \times \Re_+^D$, s.t. $F_P(p) - F_{PW}(p, w) > 0$, we have $F(\pi_1|p, w) > F(\pi_2|p, w)$ whenever $\pi_1 > \pi_2$ for $\pi_1, \pi_2 \in [0, B_\pi]$.*

Theorem 2.1 *Suppose Assumption A2.1 holds. Then,*

$$\forall (p, w, \pi) \in \Psi_\pi^*, \text{ we have } \pi = \pi_\alpha(p, w) \text{ where } \alpha = F(\pi|p, w)^4.$$

This theorem tells us that, as in the case of production α-frontier, we can use $\alpha = F(\pi|p, w)$ as a measure of the efficiency of the observed firm.

Another property of the profit α-frontier that we need to check, is does it satisfy monotonicity properties of the profit function for the case of fixed input or output prices which we described before? To show this we need to impose more monotonicity assumptions on the $F(\pi|p, w)$.

Assumption A2.2 *Let $A_p(\bar{w}) = \{p \in \Re_+^M : F_P(p) - F_{PW}(p, \bar{w}) > 0\}$, where $\bar{w} \in \Re_+^D$ is any fixed input price vector. Then $\forall$ $\pi \in [0, B_\pi]$ and $\forall$ $p_1, p_2 \in A_p(\bar{w})$ s.t. $p_1 \leq p_2$, we have $F(\pi \mid p_1, \bar{w}) \geq F(\pi \mid p_2, \bar{w})$, i.e $F(\pi|p, \bar{w})$ is non increasing in output prices p for any fixed input price $\bar{w}$ and any profit π.*

Assumption A2.3 *Let $A_w(\bar{p}) = \{w \in \Re_+^D : F_P(\bar{p}) - F_{PW}(\bar{p}, w) > 0\}$, where $\bar{p} \in \Re_+^M$ is any fixed output price vector. Then $\forall$ $\pi \in [0, B_\pi]$ and $\forall$ $w_1, w_2 \in A_w(\bar{p})$ s.t. $w_1 \leq w_2$, we have $F(\pi \mid \bar{p}, w_1) \leq F(\pi \mid \bar{p}, w_2)$, i.e $F(\pi|\bar{p}, w)$ is nondecreasing in input prices w for any fixed output*

[4] All proofs are in Appendix A.

price $\bar{p}$ and any profit π.

Assumption A2.2 says that the probability of profit being less than π decreases with output prices for fixed input prices. Conversely, assumption A2.3 says that the probability of profit being less than π increases with input prices for fixed output prices. Now we state:

Theorem 2.2 *(i) The quantile function $(p, \bar{w}) \mapsto \pi_\alpha(p, \bar{w})$ is monotone nondecreasing on the set $A_p(\bar{w})$ for every order $\alpha \in [0,1]$ and fixed $\bar{w}$ if and only if the function $(p, \bar{w}) \mapsto F(\pi|p, \bar{w})$ is monotone non increasing on the set $A_p(\bar{w})$ for any profit $\pi \in [0, B_\pi]$, i.e. satisfies Assumption A2.2.*
(ii) The quantile function $(\bar{p}, w) \mapsto \pi_\alpha(\bar{p}, w)$ is monotone non increasing on the set $A_w(\bar{p})$ for every order $\alpha \in [0,1]$ if and only if the function $(\bar{p}, w) \mapsto F(\pi|\bar{p}, w)$ is monotone nondecreasing on the set $A_w(\bar{p})$ for any profit $\pi \in [0, B_\pi]$, i.e. satisfies Assumption A2.3.

Theorem 2.2 shows that under our assumptions on the conditional distribution function, the conditional quantile profit function satisfies the monotonicity properties associated with the function.

3. STOCHASTIC MODEL AND ESTIMATION

3.1. Smooth conditional quantile estimator

In this section we introduce empirical and smooth estimators for the conditional α-quantile of the profit function, i.e. what we call an α-profit frontier.

Let $S_n = \{(P_i, W_i, \Pi_i)\}_{i=1}^n$ be a sequence of independent identically distributed random vectors taking values in Ψ_π^*. Here n is the number of firms observed, Π_i is an profit and (P_i, W_i) are output and input prices faced by the i$^{\text{th}}$ firm. We first define an estimator $\hat{F}(\pi|p,w)$ for $F(\pi|p,w)$ as

$$\hat{F}(\pi|p,w) = \begin{cases} 0 & \text{if } \pi = 0, \\ \frac{\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)}{\hat{F}_P(p) - \hat{F}_{PW}(p,w)} & \text{if } \pi > 0, \end{cases} \tag{3.1}$$

where $\hat{F}_{P\Pi}(p,\pi)$, $\hat{F}(p,w,\pi)$, $\hat{F}_P(p)$, $\hat{F}_{PW}(p,w)$ are defined using two different estimators: (i) as empirical distributions following Aragon et al. (2005) and (ii) as integrals over Π of suitable defined Rosenblatt density estimators following Martins-Filho and Yao (2008). Both techniques we have already discussed briefly in the Introduction. For the case (i) we define,

$$\begin{aligned} \hat{F}_{P\Pi}(p,\pi) &= n^{-1} \sum_{i=1}^{n} I(\Pi_i \leq \pi, P_i \leq p), \\ \hat{F}(p,w,\pi) &= n^{-1} \sum_{i=1}^{n} I(\Pi_i \leq \pi, P_i \leq p, W_i \leq w), \\ \hat{F}_P(p) &= n^{-1} \sum_{i=1}^{n} I(P_i \leq p), \\ \hat{F}_{PW}(p,w) &= n^{-1} \sum_{i=1}^{n} I(P_i \leq p, W_i \leq w), \end{aligned}$$

and for (ii),

$$\begin{aligned}\hat{F}_{P\Pi}(p,\pi) &= (nh_n)^{-1}\sum_{i=1}^{n}\left(\int_0^{\pi} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right) I(P_i\leq p),\\ \hat{F}(p,w,\pi) &= (nh_n)^{-1}\sum_{i=1}^{n}\left(\int_0^{\pi} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right) I(P_i\leq p, W_i\leq w),\\ \hat{F}_P(p) &= n^{-1}\sum_{i=1}^{n} I(P_i\leq p),\\ \hat{F}_{PW}(p,w) &= n^{-1}\sum_{i=1}^{n} I(P_i\leq p, W_i\leq w),\end{aligned}$$

where $I(A)$ is the indicator function for the set A, $K(\cdot)$ is a suitably defined kernel function and h_n is a non stochastic sequence of bandwidths such that $h_n > 0,\ h_n \to 0$ as $n \to \infty$. The difference between estimators in (i) and (ii) is that we use smooth nonparametric estimator of the distribution function (Azzalini, 1981) in the direction of the profit π instead of using an empirical one, but we still have empirical distributions in the direction of p and w. It was shown by Martins-Filho and Yao (2008) for the case of production functions, that smooth kernel based estimator implemented in the direction of univariate output has $\sqrt{n}$ rate of convergence. This is the rate of convergence of an asymptotically efficient parametric estimator and thus is the best rate of convergence one could obtain. As we show in later chapters, our estimator also achieves a $\sqrt{n}$ rate of convergence. It is possible to smooth estimators in the direction of prices as well, but as a result the estimator will suffer the curse of dimensionality problem.

It can be shown that, $\hat{F}(\pi|p,w)$ is asymptotically a distribution function, i.e. for suitably defined kernels: (f.i) $\hat{F}(\pi|p,w)$ is nondecreasing in π; (f.ii) $\hat{F}(\pi|p,w)$ is right continuous in $\Re_+$; (f.iii) $\lim_{\pi\to 0}\hat{F}(\pi|p,w) = 0$; (f.iv) there exists some $N(p,w)$ such that for all $n > N(p,w)$ we have $\lim_{\pi\to\infty}\hat{F}(\pi|p,w) = 1$[5].

Now, assuming that $\pi_\alpha(p,w)$ is the unique order α quantile for the conditional distribution $F(\pi|p,w)$, we define the estimator $\hat{\pi}_{\alpha,n}(p,w)$ as the root of

$$\hat{F}(\hat{\pi}_{\alpha,n}(p,w)|p,w) = \alpha,\ \text{ for } \alpha \in (\mathit{0},\mathit{1}] \text{ and } (p,w) \in \Re_+^M \times \Re_+^D, \tag{3.2}$$

and thus

$$\hat{\pi}_{\alpha,n}(p,w) = \hat{F}^{-1}(\alpha|p,w), \tag{3.3}$$

[5]See Appendix C.

which constitute an estimator for conditional α-quantile (frontier) of the profit function. Now, using the mean value theorem, absolute continuity of F and smoothness of the kernel function we can write

$$\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) = \frac{F(\pi_\alpha(p,w)|p,w) - \hat{F}(\pi_\alpha(p,w)|p,w)}{\hat{f}(\bar{\pi}_{\alpha,n}(p,w)|p,w)},$$

where $\hat{f}(\bar{\pi}_{\alpha,n}(p,w)|p,w) = \frac{\partial \hat{F}(\pi|p,w)}{\partial \pi} = \frac{(nh_n)^{-1}\sum_{i=1}^{n} K(\frac{\Pi_i-\pi}{h_n})[I(P_i \le p) - I(P_i \le p, W_i \le w)]}{\hat{F}_P(p) - \hat{F}_{PW}(p,w)}$ for $\pi \ge 0$ (notice that for $\pi < 0$, $\hat{f}(\pi|p,w) = 0$) and $\bar{\pi}_{\alpha,n}(p,w) = t\hat{\pi}_{\alpha,n}(p,w) + (1-t)\pi_\alpha(p,w)$ for $t \in (0,1)$.

3.2. Assumptions

The stochastic properties of the estimator defined in previous the section are obtained under the following regularity conditions:

Assumption A3.1 *(a) $S_n = \{(P_i, W_i, \Pi_i)\}_{i=1}^n$ is a sequence of independent random vectors taking values in Ψ_π^* and having the same distribution F as the vector (P, W, Π) with support in Ψ_π^*;*

(b) Ψ_π^* is compact and $0 < f(p,w,\pi) < B_f$ for all $(p,w,\pi) \in \Psi_\pi^*$;

(c) $\exists\ 0 < w^{min} \in \Re_+^D$ s.t. $\forall\ (p,w,\pi) \in \Psi_\pi^*$, $w_k^{min} \le w_k\ \forall\ k = 1,\ldots,D$, i.e. we assume that input prices are bounded away from zero.

The assumption that S_n is an independent and identically distributed sequence, and the existence of the density f as a bounded function in Ψ_π is standard in deterministic frontier literature (Aragon et al., 2005; Cazals et al., 2002; Park et al., 2000; Martins-Filho and Yao, 2007a, 2008). Notice, that assumption A3.1(c) is needed, since if we allow w to be zero, we get infinite profits $\pi(p,w,y) = py - c(w,y)$ and thus profit function (maximum) does not exist.

Assumption A3.2 *(a) $K(\gamma) : S_K \to \Re$ is a symmetric bounded function with compact support $S_K = [-B_K, B_K]$ such that:*

(b) $\int_{-B_K}^{B_K} K(\gamma)d\gamma = 1$;

(c) $\int_{-B_K}^{B_K} \gamma K(\gamma)d\gamma = 0$, $\int_{-B_K}^{B_K} \gamma^2 K(\gamma)d\gamma = \sigma_K^2$;

(d) for all $\gamma, \gamma' \in S_K$ we have $|K(\gamma) - K(\gamma')| \le m_K|\gamma - \gamma'|$ for some $0 < m_K < \infty$;

(e) for all $\gamma, \gamma' \in \Re$ we have $|\kappa(\gamma) - \kappa(\gamma')| \le m_K|\gamma - \gamma'|$ for some $0 < m_K < \infty$, where $\kappa(\lambda) = \int_{-B_K}^{\lambda} K(\gamma)d\gamma$.

Assumption A3.2 is standard in nonparametric estimation and is satisfied by commonly used kernels such as Epanechnikov, Biweight and others.

Assumption A3.3 *(a) f is continuous in Ψ_π^*;*

(b) for all (p,w) such that $F_P(p) - F_{PW}(p,w) > 0$ and for all $\alpha \in (0,1]$, $f(\pi_\alpha(p,w)|p,w) > 0$, where $f(\cdot|p,w)$ is the derivative of $F(\cdot|p,w)$;

(c) for all $(p,w,\pi),(p,w,\pi') \in \Psi_\pi^*$, $|f(p,w,\pi') - f(p,w,\pi)| \leq m_f|\pi' - \pi|$ for some $0 < m_f < \infty$;

(d) F is twice continuously differentiable in the interior of Ψ_π^*.

Before we state the next assumption, we adopt the following notations:

(i) Let $(p,w) \in \Re_+^M \times \Re_+^D$, then for any three subsets $A_p \subseteq C_p = \times_{i=1}^M[0,p_i]$, $A_w \subseteq C_w = \times_{i=1}^D(w_i^{min}, w_i]$ and $B \subseteq [0,\pi(p,w)]$, $\pi(A_p \times A_w) = \{\pi(p,w) : p \in A_p, w \in A_w\}$ and $\pi^{-1}(B) = \{(p,w) : p \in C_p, w \in C_w, \pi(p,w) \in B\} = (A_p \times A_w) \subseteq (C_p \times C_w)$, (but note that $\exists\ (p_0,w_0) \in A_p \times A_w \ni \pi(p_0,w_0) > \pi(p,w)$);

(ii) Let $\bar{w} \in \Re_+^D$ be fixed and $\pi = \pi(p,w) \leq \pi' = \pi(p',w')$, for some $(p,w),(p',w') \in \Re_+^M \times \Re_+^D$. Then $\pi_{\bar{w}}^{-1}([\pi,\pi']) = \{p : \pi \leq \pi(p,\bar{w}) \leq \pi'\}$.

(iii) Note that for $(p,w) \in \Re_+^M \times \Re_+^D$, $\pi(p,w)$ is not the maximal value in the region $C_w \times C_p$, since profit function increases when w decreases. Since we assumed that profit is bounded, i.e. for given $(y,p,w) \in \Re_+^M \times \Re_+^D$, $\exists\ B_\pi > 0$, *s.t.* $\pi(y,p,w) \leq B_\pi$, and given that every element of w id bounded below by w_i^{min}, the maximal value of the profit function in the region $C_w \times C_p$ must belong to $\Re_+$ and is given by $\pi_{max(p,w)} = \pi(p,w^{min})$, where $w^{min} \in \Re_+^D$ is provided by Assumption A3.1(c).

(iv) Denote $\Re_w^D = \{w \in \Re_+^D : \ w_k^{min} < w_k \ \forall\ k = 1,\ldots,D\}$.

Assumption A3.4 *For all $\pi,\pi' \in G$, where G is compact subset of $(0,\infty)$, we have $|\int_{\pi^{-1}([\pi,\pi'])} d(P,W)| \leq m_{\pi^{-1}}|\pi' - \pi|$, for some $0 < m_{\pi^{-1}} < \infty$.*

Assumption A3.4 imposes Lipschitz type condition on the integral on the left hand side of the last inequality.

4. ASYMPTOTIC CHARACTERIZATION OF THE ESTIMATOR

4.1. Asymptotic properties

In order to establish asymptotic properties of $\hat{\pi}_{\alpha,n}(p,w)$ we need to proof some intermediate results provided by Lemma 4.1 and Lemma 4.2. Lemma 4.1 is an extension of Lemma 1 in Martins-Filho and Yao (2008), with a difference in the object of interest: we are interested in properties of the difference $\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)$ because of the nature of our estimator. Notice that, asymptotically the difference between $\hat{F}_{p\Pi}(p,\pi) - \hat{F}(p,w,\pi)$ and the difference between corresponding empirical distributions is the order at which our bias and variance converge to zero. Lemma 4.2 is used in the Theorem 4.2 in order to obtain the asymptotic normality of our estimator.

Lemma 4.1 *For all $(p,w) \in \Re_+^M \times \Re_+^D$ and $\pi \in \Re_+$ and under assumptions A3.1, A3.2(a), A3.2, A3.3, we have:*

$$(a)\ E(\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) = \begin{cases} F_{P\Pi}(p,\pi) - F(p,w,\pi) + B_{E,n} + o(h_n^2) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n^2) & \text{if } \pi > \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n) & \text{if } \pi = \pi_{max(p,w)}, \end{cases}$$

where

$$B_{E,n} = \frac{h_n^2}{2}\sigma_K^2 \left[\int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) dPdW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) d(P,W) \right];$$

(b)

$$V\quad (\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) =$$
$$\begin{cases} \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + B_{V,n} + o(h_n/n) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + o(h_n/n) & \text{if } \pi \geq \pi_{max(p,w)}, \end{cases}$$

where

$$B_{V,n} = 2n^{-1}h_n\sigma_\kappa \left[\int_{\pi^{-1}([\pi,\pi_{max,(p,w)}])} f(P,W,\pi) d(P,W) - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max,(p,w)}])} f(P,W,\pi) dPdW \right],$$

and $\kappa(\lambda) = \int_{-B_K}^{\lambda} K(\gamma)d\gamma$, $\sigma_\kappa = \int_{-B_K}^{B_K} \gamma\kappa(\gamma)K(\gamma)d\gamma$, $f^{(1)}(P,W,\pi)$ denotes the first derivative of f with respect to Π, and $0 < h_n \to 0$ is a nonstochastic sequence of bandwidths.

Lemma 4.2 *Let $0 < h_n \to 0$ as $n \to \infty$ be a nonstochastic sequence of bandwidths with $nh_n^2 \to \infty$.*

Assume A3.1, A3.2, A3.3 and A3.4, then for a compact subset $G \subset (0, \pi(p,w))$ we have:

$$\sup_{\pi \in G} \left| \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right) [I(P_i \le p) - I(P_i \le p, W_i \le w)] \right.$$
$$\left. - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi, \pi_{max(p,w)}])} f(P,W,\pi) dP dW + \int_{\pi^{-1}([\pi, \pi_{max(p,w)}])} f(P,W,\pi) d(P,W) \right| = o_p(1).$$

Theorems 4.1 and 4.2 establish consistency and asymptotic normality of $\hat{\pi}_{\alpha,n}(p,w)$. Notice that in addition to our assumptions imposed in Chapter 3, Theorem 4.2 uses the following assumption $\min_{\{i:P_i \le p,\}} \Pi_i \ge h_n B_k$, which implies that as the number of observations that satisfy $\{i : P_i \le p\}$ goes to infinity, then corresponding level of profit Π_i is bounded away from zero. The Theorem 4.1, following below, establishes consistency of $\hat{\pi}_{\alpha,n}(p,w)$:

Theorem 4.1 *Let $0 < h_n \to 0$ be a nonstochastic sequence of bandwidths with $nh_n \to \infty$ as $n \to \infty$. Assume that for a given $(p,w) \in \Re_+^M \times \Re_+^D$ and some n we have $\min_{\{i:P_i \le p\}} \Pi_i \ge h_n B_k$ and A3.1, A3.2, A3.3, A3.4. Then we have that,*

$$\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) = o_p(1) \tag{4.1}$$

Asymptotic normality of $\hat{\pi}_{\alpha,n}(p,w)$ is obtained in Theorem 4.2.

Theorem 4.2 *Let $0 < h_n \to 0$ be a nonstochastic sequence of bandwidths with $nh_n^2 \to \infty$ and $nh_n^4 = O(1)$ as $n \to \infty$. Assume that for a given $(p,w) \in \Re_+^M \times \Re_+^D$ and some n we have that $\min_{\{i:P_i \le p\}} \Pi_i \ge h_n B_k$ and A3.1, A3.2, A3.3, A3.4. Then for all $\alpha \in (0,1)$ we have*

$$v_n(p,w)^{-1} \sqrt{n}(\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) - B_{E,n}) \xrightarrow{d} N(0,1), \tag{4.2}$$

where

$$B_{E,n} = -\frac{h_n^2}{2}\sigma_K^2 \frac{\left[\int_{\Re_w^D} \int_{\pi_W^{-1}([\pi_\alpha(p,w), \pi_{max(p,w)}])} f^{(1)}(P,W,\pi) dP dW - \int_{\pi^{-1}([\pi_\alpha(p,w), \pi_{max(p,w)}])} f^{(1)}(P,W,\pi) d(P,W)\right]}{(F_P(p) - F_{PW}(p,w)) f(\pi_\alpha(p,w)|p,w)} + o(h_n^2)$$

and

$$\begin{aligned} v_n^2(p,w) &= \frac{1}{(F_P(p) - F_{PW}(p,w)) f(\pi_\alpha(p,w)|p,w))^2} \Bigg[F_{P\Pi}(p, \pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) \\ &+ \frac{(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)))^2}{F_P(p) - F_{PW}(p,w)} + 2h_n \sigma_\kappa B_{v,n} \Bigg] + o(h_n) \end{aligned}$$

where

$$B_{v,n} = \int_{\pi^{-1}([\pi_\alpha(p,w), \pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) d(P,W) - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi_\alpha(p,w), \pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) dP dW.$$

It could be easily obtained that if we use solely empirical estimator instead of smooth, we would get similar asymptotic results with the only difference, that in that case we would not have bias $B_{E,n}(p,w)$ and extra term in variance $B_{v,n}$ (it is similar to the conditional quantile production function estimator provided by Aragon et al (2005)). Notice that we obtained that our smooth estimator has $\sqrt{n}$ rate of convergence, which is very good result. Another appeal of the smooth estimator is it's finite sample properties suggested by the Theorem 4.2. Since we subtract a nonnegative term $B_{v,n}$ in the variance of our smooth estimator, for finite samples, the smooth estimator has less variance than the empirical one. When $n \to \infty$, $B_{v,n}$ disappears, which means that smooth and empirical asymptotically have the same variance. At the same time we have bias term $B_{E,n} = O(h_n^2)$, which does not exist for empirical estimator, but it is also disappears asymptotically when $nh_n^4 = o(1)$. Notice, that drawing a parallel with the case of the estimation of the production function, we get similar results as in Martins-Filho and Yao (2008), just as one would expect.

In the next theorem we turn our attention to the estimation of the true frontier $\pi_1(p,w)$.

Theorem 4.3 *Assume that* $\min_{\{i:P_i \leq p, W_i \geq w\}} \Pi_i \geq h_n B_K$ *and that A3.1, A3.2 holds. In addition, assume that density* f *is strictly positive on the frontier* $\{(p,w,\pi(p,w)) : F_P(p) - F_{PW}(p,w) > 0\}$ *and that* $\pi(p,w)$ *is continuously differentiable. Then for all* (p,w) *in the support of* (P,W) *we have*
(a) there exists $N(p,w) > 0$ *such that* $\forall\ n > N(p,w)$ $\pi_{1,n}(p,w) = \max_{\{i:P_i \leq p, W_i \geq w\}} \Pi_i + h_n B_K$*;*
(b) $n^{1/(d+1)}(\pi_1(p,w) - \pi_{1,n}(p,w) + h_n B_K) \xrightarrow{d} Weibull(\mu_x^{d+1}, d+1)$.

Park et al. (2000) provide an expression for a constant μ_x and it's consistent estimator. Notice, that we get exactly the same result as in Martins-Filho and Yao (2008) in corresponding theorem for true production frontier. Thus, we can just make the same conclusion (which follows from the results obtained in Park et al. (2000)), that this result suggests, that the bias term associated with the estimation of the true frontier $\pi_1(p,w)$ using $\hat{\pi}_{1,n}(p,w)$ could be smaller than that associated with the FDH estimator.

4.2. Bandwidth selection

Implementation of the profit $\alpha-$frontier estimator requires the selection of bandwidth. Following standard practice in nonparametric methods, similarly to Martins-Filho and Yao (2008), we select our bandwidth by minimizing an asymptotic approximation of the estimator's mean integrated squared error (AMISE) over all α. Disregarding terms of order $o(h_n^4)$ and $o(h_n/n)$, we get $AMISE(\hat{\pi}_{\alpha,n}(p,w);h_n)$ as a function of h_n:

$$\begin{aligned} MISE(\hat{\pi}_{\alpha,n}(p,w);h_n) &= E\left(\int_0^1 (\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w))^2 d\alpha\right) \\ &= \int_0^1 V(\hat{\pi}_{\alpha,n}(p,w)) + Bias^2(\hat{\pi}_{\alpha,n}(p,w))d\alpha \\ &= A_1(p,w) + A_2(p,w) + A_3(p,w), \end{aligned}$$

where

$$\begin{aligned} A_1(p,w) &= \frac{h_n^4 \sigma_K^4}{4(F_P(p) - F_{PW}(p,w))^2} \int_0^1 \frac{I_1^2(p,w,\alpha)}{f^2(\pi_\alpha(p,w)|p,w)} d\alpha, \\ A_2(p,w) &= \frac{1}{n} \frac{1}{F_P(p) - F_{PW}(p,w)} \int_0^1 \frac{\alpha(1-\alpha)}{f^2(\pi_\alpha(p,w)|p,w)} d\alpha, \\ A_3(p,w) &= \frac{1}{n} \frac{2h_n\sigma_\kappa}{(F_P(p) - F_{PW}(p,w))^2} \int_0^1 \frac{I_2(p,w,\alpha)}{f^2(\pi_\alpha(p,w)|p,w)} d\alpha, \end{aligned}$$

with

$$\begin{aligned} I_1(p,w,\alpha) &= \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) dPdW \\ &- \int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) d(P,W) \end{aligned}$$

and

$$\begin{aligned} I_2(p,w,\alpha) &= \int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) d(P,W) \\ &- \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) dPdW. \end{aligned}$$

Then, using standard technique for minimization problems, the bandwidth that minimizes $AMISE(\hat{\pi}_{\alpha,n}(p,w);h_n)$ is given by

$$h_n^* = \left(\frac{2\sigma_\kappa \int_0^1 \frac{I_2(p,w,\alpha)}{f^2(\pi_\alpha(p,w)|p,w)} d\alpha}{\sigma_K^4 \int_0^1 \frac{I_1^2(p,w,\alpha)}{f^2(\pi_\alpha(p,w)|p,w)} d\alpha} \right)^{1/3} n^{-1/3} = Cn^{-1/3}.$$

Since our expression for AMISE accounts for all possible values of α, h_n^* can be interpreted as a global optimal bandwidth with respect to α for given output and input prices (p,w). Note that, since the estimator for profit $\alpha-$frontier is constructed as a quantile estimator with univariate smoothing in the profit direction only for the underlying conditional distribution, the order $O(n^{1/3})$

was expected. It is not surprising, similar result was obtained in Martins-Filho and Yao (2008), Azzalini (1981) etc. where a kernel estimator is used to estimate an unconditional distribution. However, the constant C is different from theirs, this is also expected because of the difference in conditional sets. Recall that in our case we have $\{P \leq p, W \geq w\}$ instead of, for example, $\{X \leq x\}$ in Martins-Filho and Yao (2008).

The practical use of h_n^* requires the estimation of the unknowns $f(\cdot), f^{(1)}(\cdot)$ appearing in its equation, as in the traditional plug-in bandwidth selection methods. In the next section we provide an estimation procedure for these unknowns and check finite sample performance of our estimator via a small Monte Carlo study.

5. MONTE CARLO STUDY

In this section, we perform a Monte Carlo study, which implements our smooth estimator for profit $\alpha-$frontier (S) and compares it's finite sample performance with the empirical estimator (E) constructed using empirical distribution only, similarly to Aragon et al (2005).

The data in the experiment are simulated according to the model $\Pi_i = \pi(P_i, W_i)R_i,\ i = 1, 2, ..., n$ where Π_i represents profit; the univariate output prices P_i and input prices W_i are pseudo random variables, generated from a uniform distribution with support given by $[b_l, b_u]$. $R_i = exp(-U_i)$, where U_i are pseudo random variables representing efficiency shocks, generated independently from exponential distribution with parameter $\theta = \frac{1}{3}$. In order to satisfy monotonicity properties of the profit function we chose the following functional form: $\pi(P_i, W_i) = \frac{P_i^2}{\sqrt{W_i}}$, which is convex in p and nondecreasing in p for fixed w; and is non increasing in w for fixed p. Similar DGP has been considered in Martins-Filho and Yao (2008, 2007a,b), Aragon et al. (2005), Park et al. (2000).

In our study, we estimate profit $\alpha-$frontiers for $\alpha = 0.25, 0.5, 0.75, 0.99$. For our specification of $\pi(p, w)$, we consider three sample sizes $n = 200, 600$ and 1000 and perform 1000 repetitions for each experiment. Using 10 equally spaced points in each direction P and W (total 100 points) of the support of $\pi(p, w)$, we obtain the averaged bias, standard deviation and root mean squared error of estimator for each α.

5.1. Estimator implementation

The empirical profit α-frontier is implemented similarly to one described in Aragon et al. (2005) with the only difference in conditioning set. For convenience, we provide below a description of the algorithm.

Let N_{pw} be the number of observations (P_i, W_i) such that $P_i \leq p$ and $W_i \geq w$, i.e. $N_{pw} = \sum_{i=1}^{n} I(\{P_i \leq p, W_i \geq w\})$, and, for $j = 1, \ldots, N_{pw}$, denote $\Pi_{(i_j)}$ the j$^{\text{th}}$ order statistic of the observations Π_i such that $\{P_i \leq p, W_i \geq w\} : \Pi_{i_1} \leq \Pi_{i_2} \leq \ldots \leq \Pi_{i_{N_{pw}}}$. Supposing that $N_{pw} > 0$, we have,

$$\hat{F}(\pi|p,w) = \begin{cases} 0 & \text{if } \pi < \mathit{\Pi}_{i_1}, \\ k/N_{pw} & \text{if } \mathit{\Pi}_{i_k} \le \pi < \mathit{\Pi}_{i_{k+1}},\ 1 \le k \le N_{pw}-1, \\ 1 & \text{if } \pi \ge \mathit{\Pi}_{i_{N_{pw}}}. \end{cases}$$

Therefore, we obtain for every $\alpha > 0$,

$$\hat{\pi}_{\alpha,n}(p,w) = \begin{cases} \Pi_{i_{\alpha N_{pw}}} & \text{if } \alpha \mathrm{N_{pw}} \in \mathrm{N}^*, \\ \Pi_{i_{[\alpha N_{pw}]+1}} & \text{otherwise,} \end{cases}$$

where $[\alpha N_{pw}]$ denotes integral part of αN_{pw}: the largest integer less than or equal to αN_{pw} and N^* stands for the set of all nonnegative integers.

Our smooth estimator is implemented using the Epanechnikov kernel and the following plug-in bandwidth

$$\hat{h}_\pi = \left(\frac{2\sigma_\kappa \int_0^1 \frac{\hat{I}_2(p,w,\alpha)}{\hat{f}^2(\pi_\alpha(p,w)|p,w)} d\alpha}{\sigma_K^4 \int_0^1 \frac{\hat{I}_1^2(p,w,\alpha)}{\hat{f}^2(\pi_\alpha(p,w)|p,w)} d\alpha} \right)^{1/3} n^{-1/3},$$

where $\hat{I}_1(p,w,\alpha)$, $\hat{I}_2(p,w,\alpha)$, $\hat{f}(\pi_\alpha(p,w)|p,w)$ are estimators for $I_1(p,w,\alpha)$, $I_2(p,w,\alpha)$ and $f(\pi_\alpha(p,w)|p,w)$ appearing in h_n^*. Notice, that

$$\hat{f}(\hat{\pi}_{\alpha,n}(p,w)|p,w) = \frac{(ng_n)^{-1}\sum_{i=1}^n K(\frac{\Pi_i - \pi}{g_n})[I(P_i \le p) - I(P_i \le p, W_i \le w)]}{\hat{F}(p) - \hat{F}(p,w)},$$

where $\hat{F}(p)$ and $\hat{F}(p,w)$ are empirical distribution functions. Since $\hat{f}(\hat{\pi}_{\alpha,n}(p,w)|p,w)$ is suitably defined Rosenblatt density estimator, we utilize the *rule-or-thumb* bandwidth of Silverman (1986) for g_n. In $I_1(p,w,\alpha)$, $I_2(p,w,\alpha)$ the area of integration $\pi_W^{-1}(\pi_\alpha(p,w),\pi_{max(p,w)})$, $\forall\ W$ and $\pi^{-1}(\pi_\alpha(p,w),\pi_{max(p,w)})$ need to be estimated. Then to estimate $I_1(p,w,\alpha)$ and $I_2(p,w,\alpha)$ consider

$$\begin{aligned} \int_{a1}^{a2}\int_{b1}^{b2} f^{(1)}(p,w,\pi)dPdW &= \left(\int_0^{a2}\int_0^{b2} f^{(1)}(P,W,\pi)dPdW - \int_0^{a2}\int_0^{b1} f^{(1)}(P,W,\pi)dPdW\right) \\ &- \left(\int_0^{a1}\int_0^{b2} f^{(1)}(P,W,\pi)dPdW - \int_0^{a1}\int_0^{b1} f^{(1)}(P,W,\pi)dPdW\right), \end{aligned}$$

for some positive bounds $a1$, $a2$, $b1$ and $b2$. Given our estimator for conditional distribution and any two bounds $a > 0$ and $b > 0$, a natural estimator for $I(\pi) = \int_0^a \int_0^b f^{(1)}(P,W,\pi)dPdW$ is given by

$$\hat{I}(\pi) = \frac{1}{ng_{n1}} \sum_{i=1}^n K^{(1)}\left(\frac{\pi - \Pi_i}{g_{n1}}\right) I(P_i \le b, W_i \le a),$$

where $K^{(1)}(\psi) = \frac{dK(\psi)}{d\psi}$ for a bandwidth g_{n1}. The optimal bandwidth g^*_{n1} can be obtained in a similar manner as it was obtained in Martins-Filho and Yao (2008), using their Lemma 3 with a minor changes in areas of integration. For $I_2(p, w, \alpha)$ we consider an estimator for $H(\pi) = \int_0^a \int_0^b f(P, W, \pi) dP dW$ width some constants $a > 0,\ b > 0$. The estimator can be defined as

$$\hat{H}(\pi) = \frac{1}{n g_{n2}} \sum_{i=1}^{n} K\left(\frac{\pi - \Pi_i}{g_{n2}}\right) I(P_i \leq b, W_i \leq a),$$

for a bandwidth g_{n2}. Since $\hat{H}(\pi)$ is suitably defined Rosenblatt density estimator, we can use the *rule-of-thumb* bandwidth of Silverman (1986) for g_{n2}. Finally, we can use FDH estimator for $\hat{\pi}^{-1}(\cdot)$ and get initial estimator for $\hat{\pi}_{\alpha,n}(p, w)$ using true optimal bandwidth h^*_n based on some known simple function for $\pi(p, w)$ and joint standard normal distribution for (p, w, π).

Given the results of the Theorem 2 we can construct asymptotic confidence intervals for the smooth α-profit function estimator. Similarly to Martins-Filho and Yao, given that the asymptotic bias is $O(h_n^2)$ and $h^* \propto n^{-1/3}$, we have that $O(\sqrt{n} h_n^2) = O(n^{-1/6}) = o(1)$. Hence, the normalized bias vanishes asymptotically and for 97.5% quantile $Z_{0.975}$ of standard normal distribution, we obtain

$$\lim_{n\to\infty} P(\hat{\pi}_{\alpha,n}(p, w) - n^{-1/2}(\hat{S}_2^2)^{1/2} Z_{0.975} \leq \pi_\alpha(p, w) \leq \hat{\pi}_{\alpha,n}(p, w) + n^{-1/2}(\hat{S}_2^2)^{1/2} Z_{0.975}) = 0.95,$$

where $\hat{S}_2^2 = \frac{\alpha(1-\alpha)}{(\hat{F}_P(p) - \hat{F}_{PW}(p,w))\hat{f}(\hat{\pi}_{\alpha,n}(p,w)|p,w)^2}$. $\hat{F}_P(p)$, $\hat{F}_{PW}(p, w)$ and $\hat{f}(\hat{\pi}_{\alpha,n}(p, w)|p, w)$ are estimated as described above. The asymptotic confidence interval for the empirical α-profit frontier is constructed in a similar manner.

5.2. Results and analysis

In previous section we described a procedure for the choice of the optimal bandwidth, but even the *rule-of-thumb* bandwidth supports our theoretical results. All results in this section are obtained with the *rule-of-thumb* bandwidth and it is most likely that with the optimal one, performance of the smooth estimator would be much better.

Figure 5.1 depicts the true $\alpha-$profit frontier with estimated smooth and empirical frontiers for α ranging over 0.01, 0.02, ..., 1 for simulated data set of size $n = 50$ and $\pi(p, w) = \pi(1.9, 1.1)$. As it was expected, our α-profit frontier is a smooth function of α and the empirical α-profit frontier

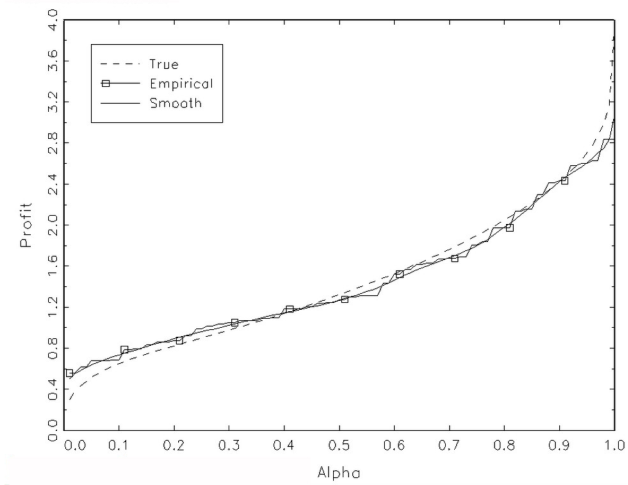

FIGURE 5.1: Plot of true $\alpha-$profit frontier with estimated smooth and empirical $\alpha-$frontiers.

is not. Table 5.1 provides biases, standard deviations and root mean squared errors for smooth and empirical $\alpha-$profit frontier estimators for $\alpha = 0.25,\ 0.5,\ 0.75$, and 0.99. As we can see, biases of the smooth estimator are larger than empirical one for small α and vise-verse when α is high. But in terms of standard deviations and root mean squared errors our smooth estimator perform better than empirical estimator for every α, which suggests that smaller variance of the smooth estimator compensates for an additional term appearing in the bias. Notice also, that when n growing root mean squared errors of both estimators become closer, which supports our asymptotic results and suggests that for large n, two estimators become equivalent. We also obtained that root mean squared errors for both estimator get larger for α close to 1, which is not surprising since there are relatively less representative data available for high α.

The empirical coverage probability (the frequency that the estimated confidence interval contains the true α-profit frontier in 1000 repetitions) is shown in the Table 5.2 for $(p_{10}, w_{10}) = (1.33, 1.67)$, $(p_{20}, w_{20}) = (1.56, 1.44)$ and $(p_{30}, w_{30}) = (1.78, 1.22)$. For most experiments we obtain that our smooth α-profit function estimator is superior to the empirical estimator, although both estimators behave very good for high values of profit function at (p_{30}, w_{30}); and not really good for small profits at (p_{30}, w_{30}).

The next series of graphs compares 3 dimensional graphs of the estimated smooth α-profit frontier and empirical one for different values of α. As we can see when α growing to 1 both estimated frontiers become closer to the true frontier (when $\alpha = 1$) and our smooth estimator indeed looks smoother than empirical one.

TABLE 5.1: Statistics for alpha profit function estimators

	Bias $\times 10^{-3}$		Std		RMSE	
	S	E	S	E	S	E
n=200						
0.25	1.97	0.91	0.0510	0.0535	0.0511	0.0535
0.5	5.06	4.35	0.0615	0.0638	0.0616	0.0638
0.75	2.59	-1.22	0.0764	0.0793	0.0765	0.0793
0.99	-14.57	-19.13	0.1038	0.1100	0.1055	0.1144
n=400						
0.25	1.53	1.13	0.0360	0.0374	0.0361	0.0374
0.5	2.56	0.15	0.0429	0.0443	0.0430	0.0443
0.75	0.88	-1.13	0.0534	0.0549	0.0533	0.0549
0.99	-6.19	-10.13	0.0761	0.0813	0.0764	0.0830
n=800						
0.25	0.150	0.02	0.0211	0.0220	0.0210	0.0220
0.5	1.69	0.42	0.0283	0.0293	0.0283	0.0294
0.75	0.91	-0.20	0.0380	0.0391	0.0381	0.0392
0.99	-2.90	-5.44	0.0549	0.0584	0.0550	0.0589

TABLE 5.2: Empirical coverage probability for $\alpha-$profit function estimators

	(p_{10}, w_{10})		(p_{20}, w_{20})		(p_{30}, w_{30})	
	S	E	S	E	S	E
n=200						
0.25	0.757	0.706	0.961	0.957	1	1
0.5	0.774	0.766	0.948	0.944	0.997	0.995
0.75	0.759	0.714	0.963	0.930	0.998	0.996
0.99	0.521	0.387	0.857	0.842	0.968	0.912
n=400						
0.25	0.779	0.772	0.978	0.968	1	1
0.5	0.818	0.813	0.957	0.955	0.999	0.998
0.75	0.806	0.776	0.954	0.946	0.999	0.998
0.99	0.656	0.618	0.908	0.826	0.984	0.968
n=800						
0.25	0.826	0.808	0.974	0.970	1	1
0.5	0.840	0.828	0.964	0.960	0.999	0.999
0.75	0.805	0.797	0.967	0.967	1	1
0.99	0.783	0.791	0.918	0.891	0.991	0.987

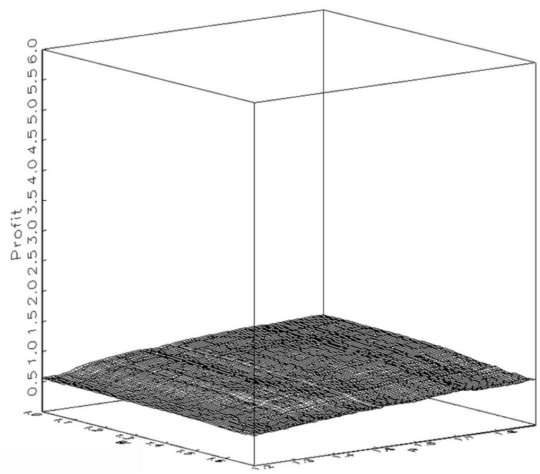

FIGURE 5.2: Smooth $\alpha-$profit frontier for $\alpha = 0.25$

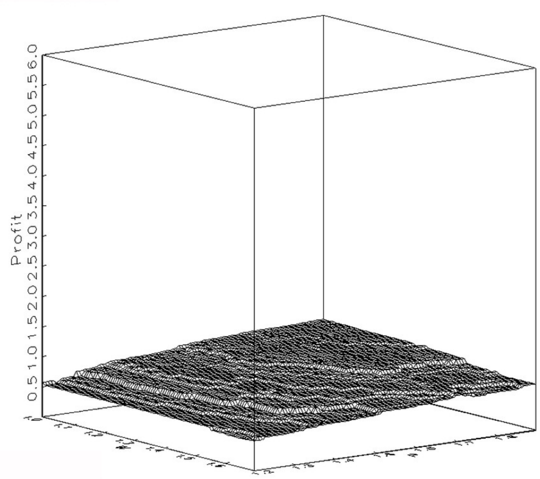

FIGURE 5.3: Empirical $\alpha-$profit frontier for $\alpha = 0.25$

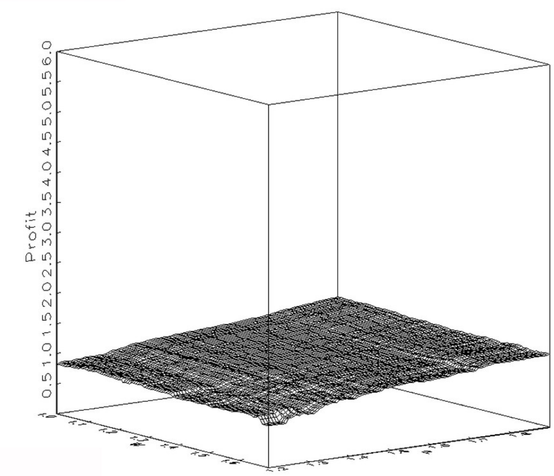

FIGURE 5.4: Smooth $\alpha-$profit frontier for $\alpha = 0.5$

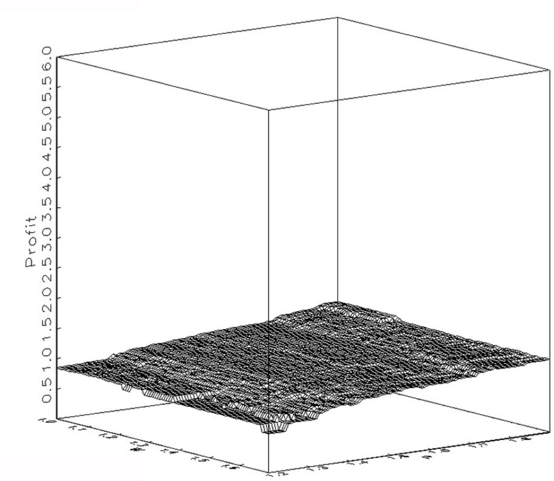

FIGURE 5.5: Empirical $\alpha-$profit frontier for $\alpha = 0.5$

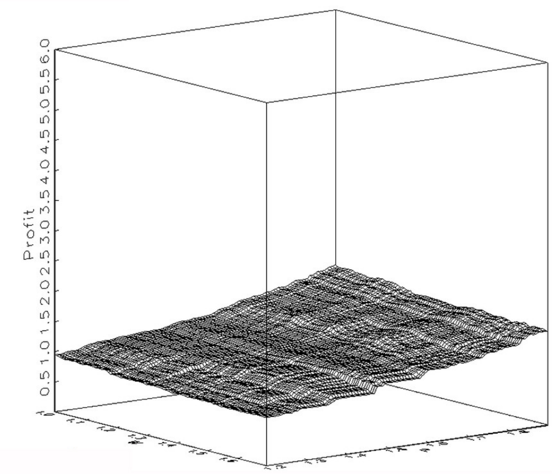

FIGURE 5.6: Smooth $\alpha-$profit frontier for $\alpha = 0.75$

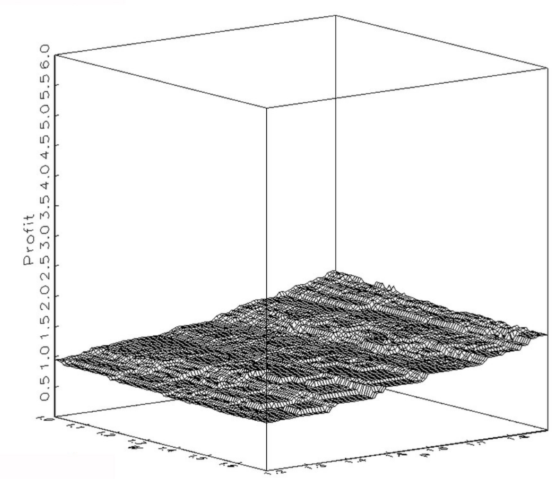

FIGURE 5.7: Empirical $\alpha-$profit frontier for $\alpha = 0.75$

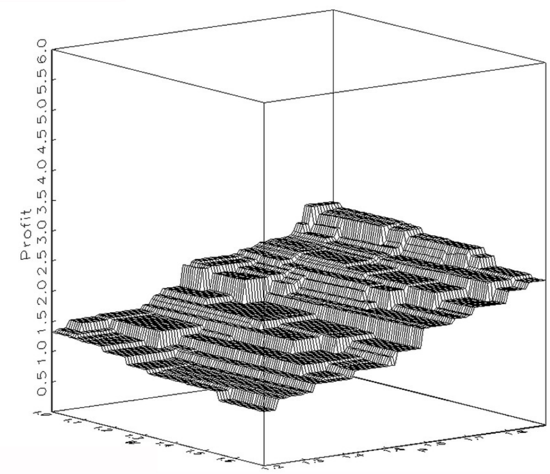

FIGURE 5.8: Smooth $\alpha-$profit frontier for $\alpha = 0.99$

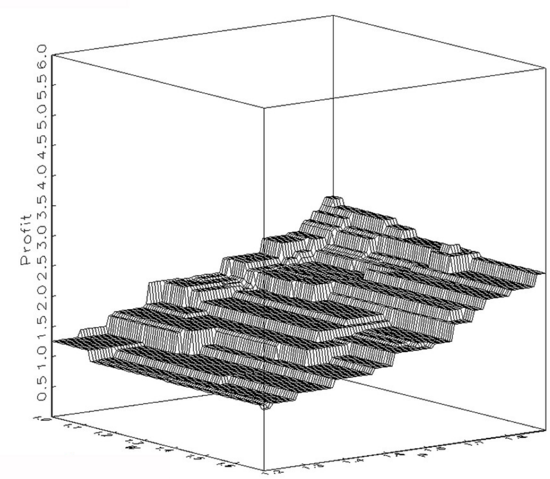

FIGURE 5.9: Empirical $\alpha-$profit frontier for $\alpha = 0.99$

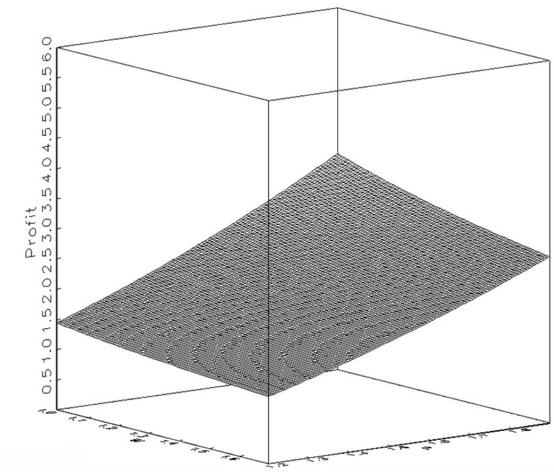

FIGURE 5.10: True profit frontier for $\alpha = 1$

6. CONCLUSIONS

In this thesis we proposed a smooth $\alpha-$quantile estimator of the profit function based on the suitably defined conditional distribution for profits. The estimator is pretty easy to implement and it is shown to be consistent and asymptotically normal with $\sqrt{n}$ rate of convergence. Comparing with the empirical version of the profit function, smoothness provides us with a smaller variance and thus better performance for finite samples. Although, we faced bias term, which does not appear for the empirical estimator, we show, that asymptotically bias and extra term in the variance disappear, making our smooth estimator and the empirical one equivalent for large n. Small Monte Carlo study, that we implemented, confirms the asymptotic theory predictions. One of the steps for future work is to introduce environmental variables into our model and investigate properties of the model and an impact of environmental variables on the estimator's performance.

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APPENDICES

A APPENDIX Proofs

Theorem 2.1 *Suppose Assumption A2.1 holds. Then,*

$$\forall (p, w, \pi) \in \Psi^*_\pi, \text{ we have } \pi = \pi_\alpha(p, w) \text{ where } \alpha = F(\pi|p, w).$$

Proof Let $(p_0, w_0, \pi_0) \in \Psi^*_\pi$ and $\alpha = F(\pi_0|p_0, w_0)$. Then having

$$\pi_\alpha(p_0, w_0) = \inf\{\pi : F(\pi|p_0, w_0) \geq \alpha\},$$

by strict monotonicity of $F(\pi|p, w)$ we have that the smallest π will be that one for which $F(\pi|p_0, w_0) = \alpha$. Thus, by A2.1 $\pi_\alpha(p_0, w_0) = \pi_0$.

Theorem 2.2 *(i) The quantile function $(p, \bar{w}) \mapsto \pi_\alpha(p, \bar{w})$ is monotone nondecreasing on the set $A_p(\bar{w})$ for every order $\alpha \in [0, 1]$ and fixed $\bar{w}$ if and only if the function $(p, \bar{w}) \mapsto F(\pi|p, \bar{w})$ is monotone nonincreasing on the set $A_p(\bar{w})$ for any profit $\pi \in [0, B_\pi]$, i.e., satisfies Assumption A2.2.*

(ii) The quantile function $(\bar{p}, w) \mapsto \pi_\alpha(\bar{p}, w)$ is monotone nonincreasing on the set $A_w(\bar{p})$ for every order $\alpha \in [0, 1]$ if and only if the function $(\bar{p}, w) \mapsto F(\pi|\bar{p}, w)$ is monotone nondecreasing on the set $A_w(\bar{p})$ for any profit $\pi \in [0, B_\pi]$, i.e. satisfies Assumption A2.3.

Proof (i) ($\Leftarrow$) Let $\bar{w} \in \Re^D_+$ be a fixed input price vector. Suppose that for any $\pi \in [0, B_\pi]$, the function $(p, \bar{w}) \mapsto F(\pi|p, \bar{w})$ is monotone nonincreasing on the set $A_p(\bar{w})$. Let $\alpha \in [0, 1]$ and $p_1, p_2 \in A_p(\bar{w})$ be such that $p_1 \leq p_2$. Then $F(\pi_\alpha(p_2, \bar{w})|p_1, \bar{w}) \geq \alpha$. Thus, since $F(\pi|p, w)$ satisfies Assumption A2.1, $\pi_\alpha(p_2, \bar{w}) \geq F^{-1}(\alpha|p_1, \bar{w})$ and therefore $\pi_\alpha(p_2, \bar{w}) \geq \inf\{\pi : F(\pi|p_1, \bar{w}) \geq \alpha\} = \pi_\alpha(p_1, \bar{w})$.

($\Rightarrow$) Let $\pi \in [0, B_\pi]$, $\bar{w} \in \Re^D_+$ and $p_1, p_2 \in A_p(\bar{w})$ such that $p_1 \leq p_2$. Suppose that the quantile function is monotone nondecreasing for every order α on $A_p(\bar{w})$, i.e., $\pi_\alpha(p_1, \bar{w}) \leq \pi_\alpha(p_2, \bar{w})$. Set $\alpha = F(\pi|p_2, \bar{w})$. Since we have $\pi_\alpha(p_2, \bar{w}) = \inf\{\bar{\pi} : F(\bar{\pi}|p_2, \bar{w}) \geq \alpha\}$, then $\pi \geq \pi_\alpha(p_2, \bar{w})$, and $\pi \geq \pi_\alpha(p_2, \bar{w}) \geq \pi_\alpha(p_1, \bar{w})$, which, given Assumption A2.1, implies that $F(\pi|p_1, \bar{w}) \geq F(\pi_\alpha(p_1, \bar{w})|p_1, \bar{w}) = \alpha = F(\pi|p_2, \bar{w})$.

Similarly for part two:

(ii) ($\Leftarrow$) Let $\bar{p} \in \Re^M_+$ be a fixed output price vector. Suppose that for any $\pi \in [0, B_\pi]$, the function $(\bar{p}, w) \mapsto F(\pi|\bar{p}, w)$ is monotone nondecreasing on the set $A_w(\bar{p})$. Let $\alpha \in [0, 1]$ and

$w_1, w_2 \in A_w(\bar{p})$ be such that $w_1 \leq w_2$. Then $F(\pi_\alpha(\bar{p}, w_1)|\bar{p}, w_2) \geq \alpha$. Thus, since $F(\pi|p, w)$ satisfies Assumption A2.1, $\pi_\alpha(\bar{p}, w_1) \geq F^{-1}(\alpha|\bar{p}, w_2)$ and therefore $\pi_\alpha(\bar{p}, w_1) \geq \inf\{\pi : F(\pi|\bar{p}, w_2) \geq \alpha\} = \pi_\alpha(\bar{p}, w_2)$.

($\Rightarrow$) Let $\pi \in [0, B_\pi]$, $\bar{p} \in \Re^M_+$ and $w_1, w_2 \in A_w(\bar{p})$ such that $w_1 \leq w_2$. Suppose that the quantile function is monotone non increasing for every order α on $A_w(\bar{p})$, i.e. $\pi_\alpha(\bar{p}, w_1) \geq \pi_\alpha(\bar{p}, w_2)$. Set $\alpha = F(\pi|\bar{p}, w_1)$. Since we have $\pi_\alpha(\bar{p}, w_1) = \inf\{\bar{\pi} : F(\bar{\pi}|\bar{p}, w_1) \geq \alpha\}$, then $\pi \geq \pi_\alpha(\bar{p}, w_1)$, and $\pi \geq \pi_\alpha(\bar{p}, w_2)$, which, given Assumption A2.1, implies that $F(\pi|\bar{p}, w_2) \geq F(\pi_\alpha(\bar{p}, w_2)|\bar{p}, w_2) = \alpha = F(\pi|\bar{p}, w_1)$.

Lemma 4.1 *For all $(p, w) \in \Re^M_+ \times \Re^D_+$ and $\pi \in \Re_+$ and under assumptions A3.1, A3.2, A3.3, we have:*

$$(a)\ E(\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) = \begin{cases} F_{P\Pi}(p,\pi) - F(p,w,\pi) + B_{E,n} + o(h_n^2) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n^2) & \text{if } \pi > \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n) & \text{if } \pi = \pi_{max(p,w)}, \end{cases}$$

where

$$B_{E,n} = \frac{h_n^2}{2}\sigma_K^2 \left[\int_{\Re^D_w} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) dP dW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi) d(P,W) \right];$$

(b)

$$V \quad (\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) =$$

$$\begin{cases} \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + B_{V,n} + o(h_n/n) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + o(h_n/n) & \text{if } \pi \geq \pi_{max(p,w)}, \end{cases}$$

where

$$B_{V,n} = 2n^{-1} h_n \sigma_\kappa \left[\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W) - \int_{\Re^D_w} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) dP dW \right],$$

and $\kappa(\lambda) = \int_{-B_K}^{\lambda} K(\gamma) d\gamma$, $\sigma_\kappa = \int_{-B_K}^{B_K} \gamma\kappa(\gamma) K(\gamma) d\gamma$, $f^{(1)}(P,W,\pi)$ denotes the first derivative of f with respect to Π, and $0 < h_n \to 0$ is a nonstochastic sequence of bandwidths.

Proof (a) We consider $E(\hat{F}_{P\Pi}(p,\pi))$ and $E(\hat{F}(p,w,\pi))$ separately.

$E(\hat{F}(p,w,\pi))$: Since $h_n \to 0$ as $n \to \infty$, $\exists\ N(p,w) \in \Re_+$ such that $\forall\ n > N(p,w)$,

$$\begin{aligned} E(\hat{F}(p,w,\pi)) &= \frac{1}{h_n} \int_{C_w} \int_{C_p} \int_{[0,\pi(P,W)]} \int_0^\pi K\left(\frac{\Pi - \psi}{h_n}\right) d\psi f(P,W,\Pi) d\Pi dP dW \\ &= \int_{C_w} \int_{C_p} \int_{[0,\pi(P,W)]} \int_{-B_K}^{(\pi-\Pi)/h_n} K(\gamma) d\gamma f(P,W,\Pi) d\Pi dP dW \qquad \text{(A.1)} \end{aligned}$$

Let $F_f(p,w,\pi) = \int_{[0,\pi]} f(p,w,\gamma)d\gamma$, hence

$$E(\hat{F}(p,w,\pi)) = \int_{C_w}\int_{C_p}\int_{[0,\pi(P,W)]} \kappa\left(\frac{\pi-\Pi}{h_n}\right)\frac{\partial F_f(P,W,\Pi)}{\partial \Pi} d\Pi dP dW.$$

Using integration by parts,

$$\begin{aligned}\int_{[0,\pi(P,W)]} \kappa\left(\frac{\pi-\Pi}{h_n}\right) dF_f(P,W,\Pi) &= \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W)) \\ &+ \int_{\frac{\pi-\pi(P,W)}{h_n}}^{\pi/h_n} F_f(P,W,\pi-h_n\gamma)K(\gamma)d\gamma.\end{aligned}$$

By A3.3(d) and Taylor's theorem,

$$F_f(P,W,\pi-h_n\gamma) = F_f(P,W,\pi) - h_n\gamma f(P,W,\pi) + \frac{1}{2}h_n^2\gamma^2 f^{(1)}(P,W,\pi) + o(h_n^2),$$

where $f^{(1)}(P,W,\pi) = \frac{\partial f(P,W,\pi)}{\partial \Pi}$. Hence, we write $E(\hat{F}(p,w,\pi)) = E_{1n} + E_{2n} - E_{3n} + E_{4n} + o(h_n^2)$, where

$$\begin{aligned} E_{1n} &= \int_{C_w}\int_{C_p} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW \\ E_{2n} &= \int_{C_w}\int_{C_p} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW \\ E_{3n} &= h_n\int_{C_w}\int_{C_p} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \\ E_{4n} &= \frac{h_n^2}{2}\int_{C_w}\int_{C_p} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma dPdW. \end{aligned}$$

For $(p,w,\pi)\in\Psi_\pi^*$, if $\pi \le 0$ then $\hat{F}(p,w,\pi) = 0$. We now consider the limiting behavior of each term when: (1) $0 < \pi < \pi_{max(p,w)}$; (2) $\pi > \pi_{max(p,w)}$; (3) $\pi = \pi_{max(p,w)}$.

(1): Recalling our notations, adopted in the previous section, we can write:

$$\begin{aligned} E_{1n} &= \int_{\pi^{-1}([0,\pi])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))d(P,W) \\ &\quad + \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))d(P,W) \\ &= E_{11,n} + E_{12,n}. \end{aligned}$$

First, observe that $\left|\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right)\right| |F_f(P,W,\pi(P,W))| < B < \infty$ for some $0 < B < \infty$ given A3.1 and A3.2. Note that in the case of $E_{11,n}$, we have $(P,W)\in\pi^{-1}([0,\pi])$ and $\pi(P,W)\le\pi$, thus $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right)\to 1$. Hence by Lebesgue's dominated convergence (LDC) theorem

$$E_{11,n} \to \int_{\pi^{-1}([0,\pi])}\int_{[0,\pi(P,W)]} f(P,W,\Pi)d\Pi d(P,W).$$

For $E_{12,n}$, since $(P,W)\in\pi^{-1}([\pi,\pi_{max,(p,w)}])$ we have $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right)\to 0$, hence by LDC theorem, we have $E_{12,n}\to 0$.

For the case of E_{2n} we have:

$$\begin{aligned}
E_{2n} &= \int_{\pi^{-1}([0,\pi])} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma d(P,W)\\
&\quad+\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma d(P,W).
\end{aligned}$$

For $(P,W)\in\pi^{-1}([0,\pi])$ we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma\to 0$, and for $(P,W)\in\pi^{-1}([\pi,\pi_{max(p,w)}])$ we have $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma\to 1$. Therefore $E_{2n}\to\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])}\int_{[0,\pi]} f(P,W,\Pi)d\Pi d(P,W)$.

For the case of E_{3n} we have:

$$\begin{aligned}
E_{3n} &= h_n\int_{\pi^{-1}([0,\pi])} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma d(P,W)\\
&\quad+h_n\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma d(P,W)\\
&= E_{31,n}+E_{32,n}.
\end{aligned}$$

For $(P,W)\in\pi^{-1}([0,\pi])$ we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma K(\gamma)d\gamma\to 0$, hence $h_n^{-1}E_{31,n}\to 0$. Then, by A3.2(c), for $(P,W)\in\pi^{-1}([\pi,\pi_{max(p,w)}])$ we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma K(\gamma)d\gamma\to 0$, and therefore $h_n^{-1}E_{3n}\to 0$.

Now,

$$\begin{aligned}
E_{4n} &= \frac{h_n^2}{2}\int_{\pi^{-1}([0,\pi])} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma d(P,W)\\
&\quad+\frac{h_n^2}{2}\int_{\pi^{-1}([\pi,\pi_{max,(p,w)}])} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma d(P,W)\\
&= E_{41,n}+E_{42,n}.
\end{aligned}$$

For $(P,W)\in\pi^{-1}([0,\pi])$ we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma^2 K(\gamma)d\gamma\to 0$, hence $h_n^{-2}E_{41,n}\to 0$. For $(P,W)\in\pi^{-1}([\pi,\pi_{max(p,w)}])$ we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma^2 K(\gamma)d\gamma\to\sigma_K^2$ by A3.2(c), and

$$h_n^{-2}E_{4n}\to\frac{1}{2}\sigma_K^2\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)+o(h_n^2).$$

Hence, for $0<\pi<\pi(p,w)$ we have

$$E(\hat{F}(p,w,\pi))=F(p,w,\pi)+\frac{h_n^2}{2}\sigma_K^2\int_{\pi^{-1}([\pi,\pi_{max,(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)+o(h_n^2). \tag{A.2}$$

(2): If $\pi > \pi_{max(p,w)}$, $E_{1n} = \int_{\pi^{-1}([0,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))d(P,W)$ and since $(P,W) \in \pi^{-1}([0,\pi_{max(p,w)}])$ we have that $\pi(P,W) < \pi$ and $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 1$. Hence by LDC theorem $E_{1n} \to \int_{\pi^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))d(P,W) = F(p,w,\pi)$. Similarly, $E_{2n} \to 0$, $h_n^{-1}E_{3n} \to 0$, $h_n^{-2}E_{4n} \to 0$, since $\frac{\pi-\pi(P,W)}{h_n} \to \infty$ as $n \to \infty$. Consequently,

$$E(\hat{F}(p,w,\pi)) = F(p,w,\pi) + o(h_n^2). \tag{A.3}$$

(3): If $\pi = \pi_{max(p,w)}$, then $f(P,W,\pi)$ is not differentiable and $F_f(P,W,\pi - h_n\gamma) = F_f(P,W,\pi) - h_n\gamma f(P,W,\pi) + o(h_n)$. Hence we write $E(\hat{F}(p,w,\pi)) = E_{1n} + E_{2n} - E_{3n} + o(h_n)$. In this case,

$$E_{1n} = \int_{\pi^{-1}([0,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))d(P,W)$$

and since for $(P,W) \in \pi^{-1}([0,\pi_{max(p,w)}])$, we have that $\pi(P,W) < \pi$ and $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 1$ as $n \to \infty$. Hence by LDC theorem

$$E_{1n} \to \int_{\pi^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))d(P,W) = F(p,w,\pi).$$

Similarly,

$$E_{2n} = \int_{C_w}\int_{C_p} F_f(P,W,\pi) \int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW \to 0$$

and

$$E_{3n} = h_n \int_{C_w}\int_{C_p} f(P,W,\pi) \int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \to 0$$

as $n \to \infty$. Hence, when $\pi = \pi_{max(p,w)}$, we have

$$E(\hat{F}(p,w,\pi)) = F(p,w,\pi) + o(h_n). \tag{A.4}$$

Similarly we obtain $E(\hat{F}_{P\Pi}(p,\pi))$. Since $h_n \to 0$ as $n \to \infty$, $\exists\ N(p,w) \in \Re_+$ such that $\forall\ n > N(p,w)$,

$$\begin{aligned} E(\hat{F}_{P\Pi}(p,\pi)) &= \frac{1}{h_n}\int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]}\int_0^{\pi} K\left(\frac{\Pi-\psi}{h_n}\right)d\psi f(P,W,\Pi)d\Pi dPdW \\ &= \int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]}\int_{-B_K}^{(\pi-\Pi)/h_n} K(\gamma)d\gamma f(P,W,\Pi)d\Pi dPdW \end{aligned} \tag{A.5}$$

Let $F_f(p,w,\pi) = \int_{[0,\pi]} f(p,w,\gamma)d\gamma$, hence

$$E(\hat{F}_{P\Pi}(p,\pi)) = \int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]} \kappa\left(\frac{\pi-\Pi}{h_n}\right)\frac{\partial F_f(P,W,\Pi)}{\partial \Pi}d\Pi dPdW.$$

Using integration by parts,

$$\int_{[0,\pi(P,W)]} \kappa\left(\frac{\pi-\Pi}{h_n}\right) dF_f(P,W,\Pi) = \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W)) + \int_{\frac{\pi-\pi(P,W)}{h_n}}^{\pi/h_n} F_f(P,W,\pi-h_n\gamma)K(\gamma)d\gamma.$$

By A3.3(d) and Taylor's theorem,

$$F_f(P,W,\pi-h_n\gamma) = F_f(P,W,\pi) - h_n\gamma f(P,W,\pi) + \frac{1}{2}h_n^2\gamma^2 f^{(1)}(P,W,\pi) + o(h_n^2),$$

where $f^{(1)}(P,W,\pi) = \frac{\partial f(P,W,\pi)}{\partial \Pi}$. Hence, we write $E(\hat{F}(p,\pi)) = E_{1n} + E_{2n} - E_{3n} + E_{4n} + o(h_n^2)$, where

$$\begin{aligned}
E_{1n} &= \int_{\Re_w^D}\int_{C_p} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW \\
E_{2n} &= \int_{\Re_w^D}\int_{C_p} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW \\
E_{3n} &= h_n\int_{\Re_w^D}\int_{C_p} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \\
E_{4n} &= \frac{h_n^2}{2}\int_{\Re_w^D}\int_{C_p} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma dPdW.
\end{aligned}$$

We now consider the limiting behavior of each term when: (1) $0 < \pi < \pi_{max(p,w)}$; (2) $\pi > \pi_{max(p,w)}$; (3) $\pi = \pi_{max(p,w)}$.

(1): Recalling our notations, adopted in the previous section, we can write:

$$\begin{aligned}
E_{1n} &= \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW \\
&\quad + \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW \\
&= E_{11,n} + E_{12,n}.
\end{aligned}$$

First, observe that $\left|\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right)\right| |F_f(P,W,\pi(P,W))| < B < \infty$ for some $0 < B < \infty$ given A3.1. Note that in the case of $E_{11,n}$, we have $P \in \pi_W^{-1}([0,\pi])$ for given $W \in \Re_w^D$, and $\pi(P,W) \leq \pi$, thus $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 1$. Hence by Lebesgue's dominated convergence (LDC) theorem $E_{11,n} \to \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}\int_{[0,\pi(P,W)]} f(P,W,\Pi)d\Pi dPdW$. For $E_{12,n}$, since $P \in \pi_W^{-1}([\pi,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$, we have $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 0$, hence by LDC theorem, we have $E_{12,n} \to 0$.

For the case of E_{2n} we have:

$$\begin{aligned} E_{2n} &= \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW \\ &+ \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW. \end{aligned}$$

For $P \in \pi_W^{-1}([0,\pi])$ for given $W \in \Re_w^D$, we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma \to 0$, and for $P \in \pi_W^{-1}([\pi,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$, we have $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma \to 1$. Therefore $E_{2n} \to \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}\int_{[0,\pi]} f(P,W,\Pi)d\Pi dPdW$.

Now,

$$\begin{aligned} E_{3n} &= h_n\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \\ &+h_n\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \\ &= E_{31,n}+E_{32,n}. \end{aligned}$$

For $P \in \pi_W^{-1}([0,\pi])$ for given $W \in \Re_w^D$, we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma \to 0$, hence $h_n^{-1}E_{31,n} \to 0$. Then, by A3.2(c), for $P \in \pi_W^{-1}([\pi,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$, we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma \to 0$, and therefore $h_n^{-1}E_{3n} \to 0$.

Now,

$$\begin{aligned} E_{4n} &= \frac{h_n^2}{2}\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma dPdW \\ &+\frac{h_n^2}{2}\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma dPdW \\ &= E_{41,n}+E_{42,n}. \end{aligned}$$

For $P \in \pi_W^{-1}([0,\pi])$ for given $W \in \Re_w^D$, we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma \to 0$, hence $h_n^{-2}E_{41,n} \to 0$. For $P \in \pi_W^{-1}([\pi,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$, we have that $\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma^2 K(\gamma)d\gamma \to \sigma_K^2$ by A3.2(c), and

$$h_n^{-2}E_{4n} \to \frac{1}{2}\sigma_K^2\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW + o(h_n^2).$$

Hence, for $0<\pi<\pi_{max(p,w)}$ we have

$$E(\hat{F}_{P\Pi}(p,\pi)) = F_{P\Pi}(p,\pi) + \frac{h_n^2}{2}\sigma_K^2\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW + o(h_n^2). \tag{A.6}$$

(2): If $\pi > \pi_{max(p,w)}$, $E_{1n} = \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW$ and since $P \in \pi_W^{-1}([0,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$ we have that $\pi(P,W) < \pi$ and $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 1$. Hence by LDC theorem $E_{1n} \to \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))dPdW = F_{P\Pi}(p,\pi)$. Similarly, $E_{2n} \to 0$, $h_n^{-1}E_{3n} \to 0$, $h_n^{-2}E_{4n} \to 0$, since $\frac{\pi-\pi(P,W)}{h_n} \to \infty$ as $n \to \infty$. Consequently,

$$E(\hat{F}(p,\pi)) = F_{P\Pi}(p,\pi) + o(h_n^2). \tag{A.7}$$

(3): If $\pi = \pi_{max(p,w)}$, then $f(P,W,\pi)$ is not differentiable and $F_f(P,W,\pi - h_n\gamma) = F_f(P,W,\pi) - h_n\gamma f(P,W,\pi) + o(h_n)$. Hence we write $E(\hat{F}(p,\pi)) = E_{1n} + E_{2n} - E_{3n} + o(h_n)$. In this case,

$$E_{1n} = \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} \kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW$$

and since for $P \in \pi_W^{-1}([0,\pi_{max(p,w)}])$ for given $W \in \Re_w^D$, $\pi(P,W) < \pi$ we have that $\pi(P,W) < \pi$ and $\kappa\left(\frac{\pi-\pi(P,W)}{h_n}\right) \to 1$ as $n \to \infty$. Hence by LDC theorem

$$E_{1n} \to \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))dPdW = F_{P\Pi}(p,\pi).$$

Similarly,

$$E_{2n} = \int_{\Re_w^D}\int_{C_p} F_f(P,W,\pi(P,W)) \int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} K(\gamma)d\gamma dPdW \to 0$$

and

$$E_{3n} = h_n \int_{\Re_w^D}\int_{C_p} f(P,W,\pi(P,W)) \int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma K(\gamma)d\gamma dPdW \to 0$$

as $n \to \infty$. Hence, when $\pi = \pi_{max(p,w)}$, we have

$$E(\hat{F}_{P\Pi}(p,\pi)) = F_{P\Pi}(p,\pi) + o(h_n). \tag{A.8}$$

Finally combining terms, we get:

$$E(\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) = \begin{cases} F_{P\Pi}(p,\pi) - F(p,w,\pi) + B_{E,n} + o(h_n^2) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n^2) & \text{if } \pi > \pi_{max(p,w)}, \\ F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n) & \text{if } \pi = \pi_{max(p,w)}, \end{cases}$$

where

$$B_{E,n} = \frac{h_n^2}{2}\sigma_K^2 \left[\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)\right].$$

(b) Note, $V(\hat{F}_{P\Pi}(p,\pi)) - \hat{F}(p,w,\pi)) = \frac{1}{n}(V_{1n} - V_{2n})$, where

$$V_{1n} = E\left[\left(\frac{1}{h_n}\int_0^{\pi} K\left(\frac{\gamma-\Pi}{h_n}\right)d\gamma\right)^2 (I(P_i \le p) - I(P_i \le p, W_i \le w))\right]$$

and

$$V_{2n} = E\left[\left(\frac{1}{h_n}\int_0^{\pi} K\left(\frac{\gamma-\Pi}{h_n}\right)d\gamma\,(I(P_i \le p) - I(P_i \le p, W_i \le w))\right)\right]^2.$$

First let's examine the term V_{1n}. Given that $h_n \to 0$, $\exists\ N(p,w) \in \Re_+$ such that $\forall\ n > N(p,w)$ we have that:

$$\begin{aligned} V_{1n} &= \int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]} \kappa^2\left(\frac{\pi-\Pi}{h_n}\right)\frac{\partial F_f(P,W,\Pi))}{\partial \Pi} d\Pi dP dW \\ &\quad - \int_{C_w}\int_{C_p}\int_{[0,\pi(P,W)]} \kappa^2\left(\frac{\pi-\Pi}{h_n}\right)\frac{\partial F_f(P,W,\Pi))}{\partial \Pi} d\Pi dP dW \\ &= V_{11,n} - V_{12,n}. \end{aligned}$$

Now as in part (a), using integration by parts and the fact that $F_f(P,W,\pi - h_n\gamma) = F_f(P,W,\pi) - h_n\gamma f(P,W,\pi) + o(h_n)$, we obtain:

$$\begin{aligned} V_{11,n} &= \int_{\Re_w^D}\int_{C_p} \kappa^2\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W)) dP dW \\ &\quad +2\int_{\Re_w^D}\int_{C_p} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \kappa(\gamma)K(\gamma) d\gamma dP dW \\ &\quad -2h_n\int_{\Re_w^D}\int_{C_p} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma\kappa(\gamma)K(\gamma) d\gamma dP dW + o(h_n) \\ &= V_{111,n} + V_{112,n} + V_{113,n}, \end{aligned}$$

and

$$\begin{aligned} V_{12,n} &= \int_{C_w}\int_{C_p} \kappa^2\left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W)) dP dW \\ &\quad +2\int_{C_w}\int_{C_p} F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \kappa(\gamma)K(\gamma) d\gamma dP dW \\ &\quad -2h_n\int_{C_w}\int_{C_p} f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n} \gamma\kappa(\gamma)K(\gamma) d\gamma dP dW + o(h_n) \\ &= V_{121,n} + V_{122,n} + V_{123,n}. \end{aligned}$$

We consider the asymptotic behavior of each term for (1) $0 < \pi < \pi_{max(p,w)}$ and (2) $\pi \ge \pi_{max(p,w)}$.

(1):

$$\begin{aligned}
V_{111,n} &= \int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}\kappa^2\left(\frac{\pi-\pi(P,W)}{h_n}\right)F_f(P,W,\pi(P,W))dPdW \\
&\quad +\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}\kappa^2\left(\frac{\pi-\pi(P,W)}{h_n}\right)F_f(P,W,\pi(P,W))dPdW \\
&= v_{1n}+v_{2n}.
\end{aligned}$$

Given that $\pi<\pi_{max(p,w)}$, similarly to the part (a) we can use LDC theorem and obtain

$$v_{1n}\to\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}F_f(P,W,\pi(P,W))dPdW$$

and $v_{2n}\to 0$ as $n\to\infty$. Consequently,

$$V_{111,n}\to\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}F_f(P,W,\pi(P,W))dPdW,$$

and similarly,

$$V_{121,n}\to\int_{\pi^{-1}([0,\pi])}F_f(P,W,\pi(P,W))d(P,W).$$

Note that,

$$\begin{aligned}
V_{112,n} &= 2\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\kappa(\gamma)K(\gamma)d\gamma dPdW \\
&\quad +2\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}F_f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\kappa(\gamma)K(\gamma)d\gamma dPdW,
\end{aligned}$$

and by LDC theorem we have

$$\begin{aligned}
V_{112,n} &\to 2\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}F_f(P,W,\pi)\int_{-B_K}^{B_K}\kappa\left(\gamma\right)K(\gamma)d\gamma dPdW \\
&= \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}F_f(P,W,\pi)dPdW,
\end{aligned}$$

since $\int_{-B_K}^{B_K}\kappa\left(\gamma\right)K(\gamma)d\gamma=\frac{1}{2}$. Similarly,

$$V_{122}\to\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])}F_f(P,W,\pi)d(P,W).$$

Now,

$$\begin{aligned}
V_{113,n} &= -2h_n\int_{\Re_w^D}\int_{\pi_W^{-1}([0,\pi])}f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma\kappa(\gamma)K(\gamma)d\gamma dPdW \\
&\quad -\ 2h_n\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])}f(P,W,\pi)\int_{(\pi-\pi(P,W))/h_n}^{\pi/h_n}\gamma\kappa(\gamma)K(\gamma)d\gamma dPdW,
\end{aligned}$$

and by LDC theorem

$$\frac{V_{113,n}}{h_n} \to -2\sigma_\kappa \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)dPdW,$$

where $\sigma_\kappa = \int_{-B_K}^{B_K} \gamma\kappa(\gamma)K(\gamma)d\gamma$. Again,

$$\frac{V_{123,n}}{h_n} \to -2\sigma_\kappa \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)d(P,W).$$

Thus,

$$\begin{aligned} V_{1n} = F_{P\Pi}(p,\pi) \quad &- \quad F(p,w,\pi) \\ &+ \quad 2h_n\sigma_\kappa \left[\int \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)dPdW \right. \\ &- \quad \left. \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)dPdW \right] + o(h_n). \end{aligned}$$

(2) $V_{111,n} = \int_{\Re_w^D} \int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} \kappa^2 \left(\frac{\pi-\pi(P,W)}{h_n}\right) F_f(P,W,\pi(P,W))dPdW$ and by LDC theorem we have

$$V_{111,n} \to \int_{\Re_w^D} \int_{\pi_W^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))dPdW = F_{P\Pi}(p,\pi),$$

and

$$V_{121,n} \to \int_{\pi^{-1}([0,\pi_{max(p,w)}])} F_f(P,W,\pi(P,W))d(P,W) = F(p,w,\pi).$$

Similarly, $V_{112,n} \to 0$, $V_{122,n} \to 0$, $\frac{V_{113,n}}{h_n} \to 0$ and $\frac{V_{123,n}}{h_n} \to 0$ as $n \to \infty$. Consequently, $V_{1n} = F_{P\Pi}(p,\pi) - F(p,w,\pi) + o(h_n)$.

Now, since $V_{2n} = \left(E(\hat{F}(p,\pi) - \hat{F}(p,w,\pi))\right)^2$, we obtain directly from the results of part (a):

$$V_{2n} = \begin{cases} (F_{P\Pi}(p,\pi) - F(p,w,\pi))^2 + O(h_n^2) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ (F_{P\Pi}(p,\pi) - F(p,w,\pi))^2 + o(h_n^2) & \text{if } \pi > \pi_{max(p,w)}, \\ (F_{P\Pi}(p,\pi) - F(p,w,\pi))^2 + o(h_n) & \text{if } \pi = \pi_{max(p,w)}. \end{cases}$$

Thus, combining the results for V_{1n} and V_{2n} we have:

$$V \quad (\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)) =$$
$$\begin{cases} \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + B_{V,n} + o(h_n/n) & \text{if } 0 < \pi < \pi_{max(p,w)}, \\ \frac{1}{n}\left(F_{P\Pi}(p,\pi) - F(p,w,\pi)\right)\left(1 - F_{P\Pi}(p,\pi) + F(p,w,\pi)\right) + o(h_n/n) & \text{if } \pi \geq \pi_{max(p,w)}, \end{cases}$$

where

$$B_{V,n} = 2n^{-1}h_n\sigma_\kappa \left[\int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)d(P,W) - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)dPdW \right].$$

Lemma 4.2 *Let $0 < h_n \to 0$ as $n \to \infty$ be a nonstochastic sequence of bandwidths with $nh_n^2 \to \infty$. Assume A3.1, A3.2, A3.3 and A3.4, then for a compact subset $G \subset (0, \pi(p,w))$ we have:*

$$\sup_{\pi \in G} \left| \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right) [I(P_i \le p) - I(P_i \le p, W_i \le w)] \right.$$
$$\left. - \int_{\Re_+^D} \int_{\pi_W^{-1}([\pi, \pi_{max(p,w)}])} f(P,W,\pi) dPdW + \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W) \right| = o_p(1).$$

Proof Let

$$\begin{aligned} S_{0,pw}(\pi) &= \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right) [I(P_i \le p) - I(P_i \le p, W_i \le w)] \\ A_{0,pw}(\pi) &= \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi, \pi_{max(p,w)}])} f(P,W,\pi) dPdW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W). \end{aligned}$$

Then

$$\begin{aligned} \sup_{\pi \in G} |S_{0,pw}(\pi) - A_{0,pw}(\pi)| &\le \\ &\le \sup_{\pi \in G} |S_{0,pw}(\pi) - E(S_{0,pw}(\pi))| + \sup_{\pi \in G} |E(S_{0,pw}(\pi)) - A_{0,pw}(\pi)|. \end{aligned}$$

Note that,

$$\sup_{\pi \in G} |S_{0,pw}(\pi) - E(S_{0,pw}(\pi))| = O\left(\left(\frac{\ln(n)}{nh_n}\right)^{1/2}\right) = o_p(1),$$

when $nh_n \to \infty$ by Lemma 1 in Martins-Filho and Yao (2007a),

$$\begin{aligned} E(S_{0,pw}(\pi)) &= \int_{\Re_w^D} \int_{C_p} \int_{[0,\pi(P,W)]} \frac{1}{h_n} K\left(\frac{\Pi - \pi}{h_n}\right) f(P,W,\Pi) d\Pi dPdW \\ &- \int_{C_w} \int_{C_p} \int_{[0,\pi(P,W)]} \frac{1}{h_n} K\left(\frac{\Pi - \pi}{h_n}\right) f(P,W,\Pi) d\Pi dPdW \\ &= \int_{\Re_w^D} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} K(\gamma) f(P,W,\pi - h_n\gamma) d\gamma dPdW \\ &- \int_{C_w} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} K(\gamma) f(P,W,\pi - h_n\gamma) d\gamma dPdW \end{aligned}$$

Let

$$\begin{aligned} Q_{0,pw}(\pi) &= \int_{\Re_w^D} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} K(\gamma) f(P,W,\pi) d\gamma dPdW \\ &- \int_{C_w} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} K(\gamma) f(P,W,\pi) d\gamma dPdW \end{aligned}$$

and by A3.3(c)

$$\begin{aligned}
|E(S_{0,pw}(\pi)) - Q_{0,pw}(\pi)| &\leq \\
&\leq m_f h_n \left[\int_{\Re_w^D} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} |\gamma| K(\gamma) d\gamma dP dW \right. \\
&+ \left. \int_{C_w} \int_{C_p} \int_{-\pi/h_n}^{(\pi(P,W)-\pi)/h_n} |\gamma| K(\gamma) d\gamma dP dW \right] \\
&\leq m_f h_n \left[\int_{\Re_w^D} \int_{C_p} \int_{-B_K}^{B_K} |\gamma| K(\gamma) d\gamma dP dW \right. \\
&+ \left. \int_{C_w} \int_{C_p} \int_{-B_K}^{B_K} |\gamma| K(\gamma) d\gamma dP dW \right] = O(h_n) = o(1)
\end{aligned}$$

Given that $\pi \in G \subset (0, \pi_{max(p,w)})$, $\exists\ N(p,w)$ such that $\forall\ n > N(p,w)$ we have

$$\begin{aligned}
Q_{0,pw}(\pi) &= \int_{\Re_w^D} \int_{C_p} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) dP dW \\
&- \int_{C_w} \int_{C_p} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) dP dW \\
&= H_{1n}(p,w,\pi) - H_{2n}(p,w,\pi) + H_{3n}(p,w,\pi) - H_{4n}(p,w,\pi),
\end{aligned}$$

where

$$\begin{aligned}
H_{1n}(p,w,\pi) &= \int_{\Re_w^D} \int_{\pi_W^{-1}([0,\pi])} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) dP dW \\
H_{2n}(p,w,\pi) &= \int_{\pi^{-1}([0,\pi])} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) d(P,W) \\
H_{3n}(p,w,\pi) &= \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) dP dW \\
H_{4n}(p,w,\pi) &= \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} \kappa \left(\frac{\pi(P,W) - \pi}{h_n} \right) f(P,W,\pi) d(P,W).
\end{aligned}$$

For $H_{1n}(p,w,\pi)$ and $H_{2n}(p,w,\pi)$ we have that $(P,W) \in \pi^{-1}([0,\pi])$, which implies that $\pi(P,W) \leq \pi$ and consequently $\kappa\left(\frac{\pi(P,W)-\pi}{h_n}\right) \to 0$. Then since by our assumptions $\kappa(\cdot) \leq 1$,

$$\int_{\Re_w^D} \int_{\pi_W^{-1}([0,\pi])} f(P,W,\pi) dP dW < \infty$$

and

$$\int_{\pi^{-1}([0,\pi])} f(P,W,\pi) d(P,W) < \infty,$$

by LDC theorem $H_{1n}(p,w,\pi) \to 0$ and $H_{2n}(p,w,\pi) \to 0$. In addition, since $\kappa(\cdot)$ is a nondecreasing function we have that $H_{1n}(p,w,\pi) \geq H_{1(n+1)}(p,w,\pi)$ and $H_{2n}(p,w,\pi) \geq H_{2(n+1)}(p,w,\pi)$. Now given A3.2(e), A3.3(c), A3.4 and $f(p,w,\pi) < B_f$, we have that for fixed (p,w) and n, $H_{1n}(p,w,\pi)$ and $H_{2n}(p,w,\pi)$ are continuous in the argument π. Hence by theorem 7.13 in Rudin (1976) $\sup_{\pi \in G} |H_{1n}(p,w,\pi)| = o(1)$ and $\sup_{\pi \in G} |H_{2n}(p,w,\pi)| = o(1)$. A similar argument gives

$$\sup_{\pi \in G} \left| H_{3n}(p,w,\pi) - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max,(p,w)}])} f(P,W,\pi) dP dW \right| = o(1)$$

and

$$\sup_{\pi \in G} \left| H_{4n}(p,w,\pi) - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W) \right| = o(1).$$

Consequently we have,

$$\begin{aligned} \sup_{\pi \in G} \quad & \left| Q_{pw}(\pi) - \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max,(p,w)}])} f(P,W,\pi) dP dW \right. \\ + \quad & \left. \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W) \right| = o(1). \end{aligned}$$

Thus we obtain,

$$\begin{aligned} \sup_{\pi \in G} \quad & \left| \frac{1}{nh_n} \sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right) [I(P_i \leq p) - I(P_i \leq p, W_i \leq w)] \right. \\ - \quad & \left. \int_{\Re_w^D} \int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) dP dW + \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi) d(P,W) \right| = o_p(1). \end{aligned}$$

Theorem 4.1 *Let $0 < h_n \to 0$ be a nonstochastc sequence of bandwidths with $nh_n \to \infty$ as $n \to \infty$. Assume that for a given $(p,w) \in \Re_+^M \times \Re_w^D$ and some n we have $\min_{\{i: P_i \leq p, W_i \geq w\}} \Pi_i \geq h_n B_k$ and A3.1, A3.2, A3.3, A3.4. Then we have that,*

$$\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) = o_p(1) \tag{A.9}$$

Proof From Nadaraya (1964) given that $\pi_\alpha(p,w)$ is unique, $\forall \epsilon > 0 \ \exists \ 0 < \delta(\epsilon,p,w)$, where

$$\delta(\epsilon,p,w) = \min\{F(\pi_\alpha(p,w)+\epsilon \,|p,w)) - F(\pi_\alpha(p,w)|p,w), F(\pi_\alpha(p,w)|p,w)) - F(\pi_\alpha(p,w)-\epsilon \,|p,w)\}$$

and

$$P(|\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w)| > \epsilon) \leq P(|F(\hat{\pi}_{\alpha,n}(p,w)|p,w) - F(\pi_\alpha(p,w)|p,w)| > \delta(\epsilon,p,w)).$$

Then,

$$\begin{aligned}
|F(\hat{\pi}_{\alpha,n}(p,w)|p,w) \quad - \quad & F(\pi_\alpha(p,w)|p,w)| \\
&= |F(\hat{\pi}_{\alpha,n}(p,w)|p,w) - \hat{F}(\hat{\pi}_{\alpha,n}(p,w)|p,w)| \\
&\leq \sup_{\pi\in\Re_+} |F(\pi|p,w) - \hat{F}(\pi|p,w)| \\
&= \sup_{\pi\in\Re_+} \left| \frac{\hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)}{\hat{F}_P(p) - \hat{F}(p,w)} - \frac{F_{P\Pi}(p,\pi) - F(p,w,\pi)}{F_P(p) - F_{PW}(p,w)} \right| \\
&= \sup_{\pi\in\Re_+} \left| \frac{\hat{H}(p,w,\pi)}{\hat{H}_{PW}(p,w)} - \frac{H(p,w,\pi)}{H_{PW}(p,w)} \right| \\
&= \sup_{\pi\in\Re_+} \left| \frac{\hat{H}(p,w,\pi)}{\hat{H}_{PW}(p,w)} - \frac{H(p,w,\pi)}{\hat{H}_{PW}(p,w)} + \frac{H(p,w,\pi)}{\hat{H}_{PW}(p,w)} - \frac{H(p,w,\pi)}{H_{PW}(p,w)} \right| \\
&\leq \frac{1}{\hat{H}_{PW}(p,w)} \sup_{\pi\in\Re_+} \left| \hat{H}(p,w,\pi) - H(p,w,\pi) \right| \\
&+ \left| \frac{1}{\hat{H}_{PW}(p,w)} - \frac{1}{H_{PW}(p,w)} \right| \sup_{\pi\in\Re_+} |H(p,w,\pi)| \\
&\leq \frac{1}{\hat{H}_{PW}(p,w)} \sup_{\pi\in\Re_+} \left| \hat{H}(p,w,\pi) - H(p,w,\pi) \right| \\
&+ \left| \frac{1}{\hat{H}_{PW}(p,w)} - \frac{1}{H_{PW}(p,w)} \right| H_{PW}(p,w),
\end{aligned}$$

where $H_{PW}(p,w) = F_P(p) - F_{PW}(p,w)$, $H(p,w,\pi) = F_{P\Pi}(p,\pi) - F(p,w,\pi)$, $\hat{H}_{PW}(p,w) = \hat{F}_P(p) - \hat{F}_{PW}(p,w)$ and $\hat{H}(p,w,\pi) = \hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi)$. Now,

$$\begin{aligned}
\sup_{\pi\in\Re_+} \left| \hat{H}(p,w,\pi) - H(p,w,\pi) \right| &= \sup_{\pi\in\Re_+} \left| \hat{F}_{P\Pi}(p,\pi) - \hat{F}(p,w,\pi) - F_{P\Pi}(p,\pi) + F(p,w,\pi) \right| \\
&\leq \sup_{\pi\in\Re_+} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right| + \sup_{\pi\in\Re_+} \left| \hat{F}(p,w,\pi) - F(p,w,\pi) \right|.
\end{aligned}$$

(1): Consider the first sup:

$$\begin{aligned}
\sup_{\pi\in\Re_+} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right| &\leq \sup_{\pi\in[0,\pi_{max(p,w)}]} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right| \\
&+ \sup_{\pi\in(\pi_{max(p,w)},\infty)} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right|.
\end{aligned}$$

From Lemma 2 in Martins-Filho and Yao (2007b) we have that

$$\sup_{\pi\in[0,\pi_{max(p,w)}]} \left| \hat{F}_{P\Pi}(p,\pi) - E(\hat{F}_{P\Pi}(p,\pi)) \right| = o_p(1)$$

and

$$\sup_{\pi\in[0,\pi_{max(p,w)}]} \left| E(\hat{F}_{P\Pi}(p,\pi)) - F_{P\Pi}(p,\pi) \right| = o(1).$$

Thus, $\sup_{\pi \in [0, \pi_{max(p,w)}]} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right| = o_p(1)$.

Now, $\forall \pi \in (\pi_{max(p,w)}, \infty)$ we have that

$$F_{P\Pi}(p,\pi) = F_{P\Pi}(p,\pi_{max(p,w)}) = \int_{\Re_w^D} \int_{C_p} \int_{[0,\pi(P,W)]} f(P,W,\Pi) d\Pi dP dW = F_P(p).$$

In addition under assumption $\min_{\{i:P_i \leq p, W_i \geq w\}} \Pi_i \geq h_n B_k$ and given that $0 < \Pi_i < \pi_{max(p,w)}$ we have that $\forall\ \pi \in (\pi_{max,(p,w)}, \infty)$, $\Pi < \pi$. Hence $\exists\ N_p$ such that $\forall\ n > N_p$ we have:

$$\hat{F}_{P\Pi}(p,\pi) \quad = \quad \frac{1}{n} \sum_{i=1}^{n} \int_{-B_K}^{B_K} K(\gamma)\, d\gamma I(\{P_i : P_i \leq p\}) = \hat{F}(p),$$

since by our assumptions we have $\int_{-B_K}^{B_K} K(\gamma) d\gamma = 1$. Thus,

$$\begin{aligned} \sup_{\pi \in (\pi_{max(p,w)}, \infty)} \left| \hat{F}_{P\Pi}(p,\pi) - F_{P\Pi}(p,\pi) \right| &= \sup_{\pi \in (\pi_{max(p,w)}, \infty)} \left| \hat{F}_P(p) - F_P(p) \right| \\ &= \left| \hat{F}_P(p) - F_P(p) \right| = o_p(1), \end{aligned}$$

where the last equality follows from the fact that $E(\hat{F}_P(p)) = F_P(p)$ and $V(\hat{F}_P(p)) = \frac{1}{n} F_P(p)(1 - F_P(p)) \to 0$ as $n \to \infty$. Hence by Chebyshev's inequality $\hat{F}_P(p) - F_P(p) = o_p(1)$.

(2): Consider second supremum in $\sup_{\pi \in \Re_+} \left| \hat{H}(p,w,\pi) - H(p,w,\pi) \right|$:

$$\begin{aligned} \sup_{\pi \in \Re_+} \left| \hat{F}(p,w,\pi) - F(p,w,\pi) \right| &\leq \sup_{\pi \in [0,\pi_{max(p,w)}]} \left| \hat{F}(p,w,\pi) - F(p,w,\pi) \right| \\ &+ \sup_{\pi \in (\pi_{max(p,w)}, \infty)} \left| \hat{F}(p,w,\pi) - F(p,w,\pi) \right|. \end{aligned}$$

Then, again by Lemma 2 in Martins-Filho and Yao (2007b) we have that

$$\sup_{\pi \in [0,\pi_{max(p,w)}]} \left| \hat{F}(p,w,\pi) - F(p,w,\pi) \right| = o_p(1)$$

and $\forall\ \pi \in (\pi_{max(p,w)}, \infty)$ we have

$$F(p,w,\pi) = F(p,w,\pi_{max(p,w)}) = \int_{C_w} \int_{C_p} \int_{[0,\pi(P,W)]} f(P,W,\Pi) d\Pi dP dW = F_{PW}(p,w).$$

Then given assumption $\min_{\{i:P_i \leq p, W_i \geq w\}} \Pi_i \geq h_n B_k$ and given that $0 < \Pi_i < \pi_{max(p,w)}$ we have that $\forall\ \pi \in (\pi_{max(p,w)}, \infty)$, $\Pi_i < \pi$. Hence $\exists\ N_{pw}$ such that $\forall\ n > N_p$ we have:

$$\hat{F}(p,w,\pi) \quad = \quad \frac{1}{n} \sum_{i=1}^{n} \int_{-B_K}^{B_K} K(\gamma)\, d\gamma I(\{(P_i, W_i) : P_i \leq p, W_i \leq w\}) = \hat{F}_{PW}(p,w),$$

since by our assumptions we have $\int_{-B_K}^{B_K} K(\gamma)d\gamma = 1$. Thus,

$$\begin{aligned}
\sup_{\pi\in(\pi_{max(p,w)},\infty)} \left|\hat{F}(p,w,\pi) - F(p,w,\pi)\right| &= \sup_{\pi\in(\pi_{max(p,w)},\infty)} \left|\hat{F}_{PW}(p,w) - F_{PW}(p,w)\right| \\
&= \left|\hat{F}_{PW}(p,w) - F_{PW}(p,w)\right| = o_p(1),
\end{aligned}$$

where the last equality follows from the fact that $E(\hat{F}_{PW}(p,w)) = F_{PW}(p,w)$ and $V(\hat{F}_{PW}(p,w)) = \frac{1}{n}F_{PW}(p,w)(1 - F_{PW}(p,w)) \to 0$ as $n \to \infty$. Hence by Chebyshev's inequality $\hat{F}_{PW}(p,w) - F_{PW}(p,w) = o_p(1)$. Thus $\sup_{\pi\in\Re_+} \left|\hat{H}(p,w,\pi) - H(p,w,\pi)\right| = o_p(1)$. Now, to complete the proof notice that since $\hat{F}_P(p) \xrightarrow{p} F_P(p)$ and $\hat{F}_{PW}(p,w) \xrightarrow{p} F_{PW}(p,w)$, we have that $\hat{F}_P(p) = O_p(1)$ and $\hat{F}_{PW}(p,w) = O_p(1)$. Then provided that $F_P(p) - F_{PW}(p,w) > 0$ by Slutsky theorem we get that $\hat{F}_P^{-1}(p) - F_P^{-1}(p) = o_p(1)$ and $\hat{F}_{PW}^{-1}(p,w) - F_{PW}^{-1}(p,w) = o_p(1)$. Which implies that:

$$\begin{aligned}
\left|\hat{H}_{PW}(p,w) - H_{PW}(p,w)\right| &= \left|\hat{F}_P(p) - \hat{F}_{PW}(p,w) - F_P(p) + F_{PW}(p,w)\right| \\
&\leq \left|\hat{F}(p) - F_P(p)\right| + \left|\hat{F}_{PW}(p,w) - F_{PW}(p,w)\right| = o_p(1),
\end{aligned}$$

and thus $\frac{1}{\hat{H}(p,w)} - \frac{1}{H(p,w)} = o_p(1)$ which completes the proof since we get that $P(|\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w)| > \epsilon) \to 0$.

Theorem 4.2 *Let $0 < h_n \to 0$ be a nonstochastic sequence of bandwidths with $nh_n^2 \to \infty$ and $nh_n^4 = O(1)$ as $n \to \infty$. Assume that for a given $(p,w) \in \Re_+^M \times \Re_w^D$ and some n we have that $\min_{\{i:P_i\leq p,W_i\geq w\}} \Pi_i \geq h_n B_k$ and A3.1, A3.2, A3.3, A3.4. Then for all $\alpha \in (0,1)$ we have*

$$v_n(p,w)^{-1}\sqrt{n}(\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) - B_{E,n}) \xrightarrow{d} N(0,1), \tag{A.10}$$

where

$$B_{E,n} = -\frac{h_n^2}{2}\sigma_K^2 \frac{\left[\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW - \int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)\right]}{(F_P(p)-F_{PW}(p,w))f(\pi_\alpha(p,w)|p,w)} + o(h_n^2)$$

and

$$\begin{aligned}
v_n^2(p,w) &= \frac{1}{(F_P(p) - F_{PW}(p,w))f(\pi_\alpha(p,w)|p,w))^2}\Bigg[F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) \\
&+ \frac{(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)))^2}{F_P(p) - F_{PW}(p,w)} + 2h_n\sigma_\kappa B_{v,n}\Bigg] + o(h_n)
\end{aligned}$$

where

$$B_{v,n} = \int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))d(P,W) - \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))dPdW.$$

Proof We write $\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) = (A_n + C_n)\left(\frac{1}{f(\pi_\alpha(p,w)|p,w)} + \beta_n\right)$, where

$$
\begin{aligned}
A_n &= F(\pi_\alpha(p,w)|p,w) - \frac{E\left[\hat{F}_{P\Pi}(\pi_\alpha(p,w),p) - \hat{F}(\pi_\alpha(p,w),p,w)\right]}{E\left[\hat{F}_P(p) - \hat{F}_{PW}(p,w)\right]} \\
C_n &= \frac{E\left[\hat{F}_{P\Pi}(\pi_\alpha(p,w),p) - \hat{F}(\pi_\alpha(p,w),p,w)\right]}{E\left[\hat{F}_P(p) - \hat{F}_{PW}(p,w)\right]} - \hat{F}(\pi_\alpha(p,w)|p,w) \\
\beta_n &= \hat{f}^{-1}\left(\bar{\pi}_{\alpha,n}(p,w)|p,w\right) - f^{-1}\left(\pi_{\alpha,n}(p,w)|p,w\right).
\end{aligned}
$$

Note that the theorem follows if we prove:

(a) $\beta_n = o_p(1)$,

(b) $A_n = -\frac{h_n^2}{2}\sigma_K^2 \frac{\left[\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW - \int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)\right]}{F_P(p)-F_{PW}(p,w)} +$
$o(h_n^2)$,

(c) $\left(\frac{s_n(p,w)}{\hat{F}_P(p)-\hat{F}_{PW}(p,w)}\right)^{-1}\sqrt{n}C_n \xrightarrow{d} N(0,1)$, where

$$
\begin{aligned}
s_n^2(p,w) &= F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) + \frac{\left(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right)^2}{F_P(p) - F_{PW}(p,w)} \\
&+ 2h_n\sigma_\kappa\left[\int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))d(P,W)\right. \\
&- \left.\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))dPdW\right] \\
&+ o(h_n).
\end{aligned}
$$

(a) It suffices to show that $\hat{f}\left(\bar{\pi}_{\alpha,n}(p,w)|p,w\right) - f\left(\pi_{\alpha,n}(p,w)|p,w\right) = o_p(1)$ for all $\alpha \in (0,1)$. Since we have already shown that $\hat{\pi}_{\alpha,n}(p,w) - \pi_\alpha(p,w) = o_p(1)$, by the Theorem 21.6 in Davidson (1994) it suffices to show that $B_{1,n} = \sup_{\pi\in G}\left|\hat{f}(\pi|p,w) - f(\pi|p,w)\right| = o_p(1)$, where $G \in (0, \pi_{max(p,w)})$

and G compact. Note that

$$\begin{aligned}
B_{1,n} &= \sup_{\pi \in G} \left| \frac{\frac{1}{nh_n}\sum_{i=1}^{n} K(\frac{\Pi_i - \pi}{h_n})\left[I(P_i \leq p) - I(P_i \leq p, W_i \leq w)\right]}{\hat{H}(p,w)} - \right. \\
&- \left. \frac{\int_{\Re_w^D}\int_{C_p} f(P,W,\pi)dPdW - \int_{C_w}\int_{C_p} f(P,W,\pi)dPdW}{H(p,w)} \right| \\
&\leq \sup_{\pi \in G} \left| \frac{\frac{1}{nh_n}\sum_{i=1}^{n} K(\frac{\Pi_i - \pi}{h_n})\left[I(P_i \leq p) - I(P_i \leq p, W_i \leq w)\right]}{\hat{H}_{PW}(p,w)} - \frac{H(p,w,\pi)}{H_{PW}(p,w)} \right| \\
&\leq \frac{1}{\hat{H}_{PW}(p,w)} \sup_{\pi \in G} \left| \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right)\left[I(P_i \leq p) - I(P_i \leq p, W_i \leq w)\right] - H(p,w,\pi) \right| + \\
&+ \left| \frac{1}{\hat{H}_{PW}(p,w)} - \frac{1}{H_{PW}(p,w)} \right| \sup_{\pi \in G} |H(p,w,\pi)|,
\end{aligned}$$

where $H_{PW}(p,w) = F_P(p) - F_{PW}(p,w)$, $\hat{H}_{PW}(p,w) = \hat{F}(p) - \hat{F}(p,w)$ and

$$H(p,w,\pi) = \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}]} f(P,W,\pi)dPdW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)d(P,W).$$

Now since by A3.1 $f(p,w,\pi) \leq B_f$ and Ψ_π^* is compact, we have

$$\sup_{\pi \in G} |H(p,w,\pi)| \leq B_f \left| \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}]} dPdW - \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} d(P,W) \right| = O(1).$$

In addition, given that $\left|\hat{F}_P(p) - \hat{F}_{PW}(p,w) - F_P(p) + F_{PW}(p,w)\right| = o_p(1)$ we have the second term on the right hand side of the inequality $\left| \frac{1}{\hat{F}(p) - \hat{F}(p,w)} - \frac{1}{F_P(p) - F_{PW}(p,w)} \right| = o_p(1)$. Then by Lemma 4.2,

$$\begin{aligned}
\sup_{\pi \in G} \quad & \left| \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{\Pi_i - \pi}{h_n}\right)\left[I(P_i \leq p) - I(P_i \leq p, W_i \leq w)\right] \right. \\
- & \left. \int_{\Re_w^D}\int_{\pi_W^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)dPdW + \int_{\pi^{-1}([\pi,\pi_{max(p,w)}])} f(P,W,\pi)d(P,W) \right| = o_p(1).
\end{aligned}$$

Hence $\beta_n = o_p(1)$.

(b)

$$\begin{aligned}
A_n &= F(\pi_\alpha(p,w)|p,w) - \frac{E\left[\hat{F}_{PW}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]}{E\left[\hat{F}_P(p) - \hat{F}_{PW}(p,w)\right]} \\
&= E\left(\left(\hat{F}_P(p) - \hat{F}_{PW}(p,w)\right)\right)^{-1}(A_{1n} - A_{2n}),
\end{aligned}$$

where,

$$\begin{aligned}
A_{1n} &= F(\pi_\alpha(p,w)|p,w)E\left[\hat{F}_P(p) - \hat{F}_{PW}(p,w)\right] - \left[F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right], \\
A_{2n} &= \left[F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right] - E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right].
\end{aligned}$$

Since $E\left[\hat{F}_P(p)-\hat{F}_{PW}(p,w)\right]=F_P(p)-F_{PW}(p,w)$, then $A_{1n}=0$. In addition, given that $0<\alpha<1$, we have $0<\pi_\alpha(p,w)<\pi(p,w)<\pi_{max(p,w)}$ and therefore from Lemma 4.1,

$$\begin{aligned}
A_{2n} &= -\frac{h_n^2}{2}\sigma_K^2\left[\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW\right. \\
&- \left.\int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)d(P,W)\right]+o(h_n^2).
\end{aligned}$$

Thus

$$\begin{aligned}
A_n &= -\frac{h_n^2}{2}\sigma_K^2\left[\frac{\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW}{F_P(p)-F_{PW}(p,w)}\right. \\
&- \left.\frac{F_P(p)-F_{PW}(p,w)}{\int\int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi)dPdW}F_P(p)-F_{PW}(p,w)\right]+o(h_n^2).
\end{aligned}$$

(c) Now note that,

$$\begin{aligned}
C_n &= \frac{E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w))-\hat{F}(p,w,\pi_\alpha(p,w))\right]}{E\left[\hat{F}_P(p)-\hat{F}_{PW}(p,w)\right]}-\hat{F}(\pi_\alpha(p,w)|p,w) \\
&= -\frac{1}{\hat{F}(p)-\hat{F}(p,w)}\left[\hat{F}(p,\pi_\alpha(p,w))-\hat{F}(p,w,\pi_\alpha(p,w))-\left(\hat{F}(p)-\hat{F}(p,w)\right)\times\right. \\
&\times \left.\frac{E\left[\hat{F}(p,\pi_\alpha(p,w))-\hat{F}(p,w,\pi_\alpha(p,w))\right]}{E\left[\hat{F}(p)-\hat{F}(p,w)\right]}\right] \\
&= -\frac{1}{\hat{F}_P(p)-\hat{F}_{PW}(p,w)}\frac{1}{n}\sum_{i=1}^n c_{in},
\end{aligned}$$

where

$$\begin{aligned}
c_{in}=\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} & K\quad(\frac{\Pi_i-\gamma}{h_n})\left[I(P_i\le p)-I(P_i\le p,W_i\le w)\right]d\gamma- \\
&- \quad [I(P_i\le p)-I(P_i\le p,W_i\le w)]\frac{E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w))-\hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p)-F_{PW}(p,w)}.
\end{aligned}$$

Hence we write $\sqrt{n}C_n=-\frac{1}{\hat{F}(p)-\hat{F}(p,w)}\sum_{i=1}^n Z_{in}$, where $Z_{in}=\frac{c_{in}}{\sqrt{n}}$, and note that $E(Z_{in})=0$,

$s_n^2 = \sum_{i=1}^n E(Z_{in}^2) = E(c_{in}^2)$ by A3.1. Let's write $E(c_{in}^2) = s_{1n} + s_{2n} + s_{3n}$, where

$$
\begin{aligned}
s_{1n} &= E\left[\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i - \gamma}{h_n})\left[I(P_i \le p) - I(P_i \le p, W_i \le w)\right] d\gamma\right]^2, \\
s_{2n} &= E\left(I(P_i \le p) - I(P_i \le p, W_i \le w)\right) \frac{\left(E\left[\hat{F}_{P\Pi}(p, \pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]\right)^2}{(F_P(p) - F_{PW}(p,w))^2}, \\
s_{3n} &= -2E\left[\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i - \gamma}{h_n})\left[I(P_i \le p) - I(P_i \le p, W_i \le w)\right] d\gamma\right] \times \\
&\times \quad \frac{E\left[\hat{F}_{P\Pi}(p, \pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p) - F_{PW}(p,w)}.
\end{aligned}
$$

Then from Lemma 4.1 (b) we have that

$$
\begin{aligned}
s_{1n} &= F_{P\Pi}(p, \pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) \\
&+ 2h_n\sigma_\kappa\left[\int_{\pi^{-1}([\pi_\alpha(p,w), \pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) d(P,W)\right. \\
&- \left.\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w)) dP dW\right] \\
&+ o(h_n),
\end{aligned}
$$

and from Lemma 4.1 (a)

$$
\begin{aligned}
s_{2n} &= (F_P(p) - F_{PW}(p,w))^{-1}\left(E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]\right)^2 \\
&= (F_P(p) - F_{PW}(p,w))^{-1} \times \\
&\times \Bigg(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) \\
&+ \frac{h_n^2}{2}\sigma_K^2\left[\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi_\alpha(p,w)) dP dW\right. \\
&- \left.\int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f^{(1)}(P,W,\pi_\alpha(p,w)) d(P,W)\right] + o(h_n^2)\Bigg)^2,
\end{aligned}
$$

and therefore we have $s_{2n} = (F_P(p) - F_{PW}(p,w))^{-1}\left(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right)^2 + o(h_n)$. Note that $s_{3n} = -2s_{2n} = -2(F_P(p) - F_{PW}(p,w))^{-1}\left(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right)^2 +$

$o(h_n)$. Combining these results we have that

$$\begin{aligned}
s_n^2(p,w) &= F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)) + \frac{\left(F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w))\right)^2}{F_P(p) - F_{PW}(p,w)} \\
&+ 2h_n\sigma_\kappa\left[\int_{\pi^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))d(P,W)\right. \\
&- \left.\int_{\Re_w^D}\int_{\pi_W^{-1}([\pi_\alpha(p,w),\pi_{max(p,w)}])} f(P,W,\pi_\alpha(p,w))dPdW\right] \\
&+ o(h_n).
\end{aligned}$$

Thus, provided that, $\lim_{n\to\infty}\sum_{i=1}^n E\left(\left|\frac{Z_{in}}{s_n(p,w)}\right|^{2+\delta}\right) = 0$ for some $\delta > 0$, by Liapunov's central limit theorem $\sum_{i=1}^n \frac{Z_{in}}{s_n(p,w)} \xrightarrow{d} N(0,1)$. Now note that

$$\begin{aligned}
\sum_{i=1}^n E\left(\left|\frac{Z_{in}}{s_n(p,w)}\right|^{2+\delta}\right) &= (s_n^2(p,w))^{-1-\frac{\delta}{2}}\sum_{i=1}^n E\left(|Z_{in}|^{2+\delta}\right) \\
&= (s_n^2(p,w))^{-1-\frac{\delta}{2}} n^{-\delta/2} E\left(\left|\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i-\gamma}{h_n})\left[I(P_i\le p) - I(P_i \le p, W_i \le w)\right]d\gamma\right.\right. - \\
&- \left.\left.[I(P_i\le p) - I(P_i\le p, W_i\le w)]\frac{E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p)-F_{PW}(p,w)}\right|^{2+\delta}\right) \\
&\le (s_n^2(p,w))^{-1-\frac{\delta}{2}}\frac{2^{1+\delta}}{n^{\delta/2}}\left[E\left(\left|\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i-\gamma}{h_n})\left[I(P_i\le p) - I(P_i\le p, W_i\le w)\right]d\gamma\right|^{2+\delta}\right)\right. + \\
&+ \left.E\left(\left|[I(P_i\le p) - I(P_i\le p, W_i\le w)]\frac{E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p)-F_{PW}(p,w)}\right|^{2+\delta}\right)\right]
\end{aligned}$$

where the last inequality follows form C_r-inequality. Hence given that $s_n^2(p,w) = O(1)$, to complete the proof it suffices to show that

$$\begin{aligned}
a_n &= n^{-\delta/2}E\left(\left|\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i-\gamma}{h_n})\left[I(P_i\le p) - I(P_i\le p, W_i\le w)\right]d\gamma\right|^{2+\delta}\right) = o(1) \\
b_n &= n^{-\delta/2}E\left(\left|[I(P_i\le p) - I(P_i\le p, W_i\le w)]\frac{E\left[\hat{F}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p)-F_{PW}(p,w)}\right|^{2+\delta}\right) = o(1).
\end{aligned}$$

First, note that

$$\begin{aligned}
a_n &= n^{-\delta/2}E\left(\left|\frac{1}{h_n}\int_0^{\pi_\alpha(p,w)} K(\frac{\Pi_i-\gamma}{h_n})\left[I(P_i\le p)-I(P_i\le p,W_i\le w)\right]d\gamma\right|^{2+\delta}\right)\\
&\le 2^{1-\delta}n^{-\delta/2}\left(\int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]}\left(\int_{-\Pi/h_n}^{(\pi_\alpha(p,w)-\Pi)/h_n}K(\gamma)d\gamma\right)^{2+\delta} f(P,W,\Pi)d\Pi dPdW+\right.\\
&+\left.\int_{C_w}\int_{C_p}\int_{[0,\pi(P,W)]}\left(\int_{-\Pi/h_n}^{(\pi_\alpha(p,w)-\Pi)/h_n}K(\gamma)d\gamma\right)^{2+\delta} f(P,W,\Pi)d\Pi dPdW\right)\\
&\le 2^{1-\delta}n^{-\delta/2}\left(\int_{\Re_w^D}\int_{C_p}\int_{[0,\pi(P,W)]} f(P,W,\Pi)d\Pi dPdW+\right.\\
&+\left.\int_{C_w}\int_{C_p}\int_{[0,\pi(P,W)]} f(P,W,\Pi)d\Pi dPdW\right)\to 0 \ as\ n\to\infty.
\end{aligned}$$

Now note that $b_n = n^{-\delta/2}E\left(I(P_i \le p) - I(P_i \le p, W_i \le w)\right)\left(\frac{E\left[\hat{F}_{P\Pi}(p,\pi_\alpha(p,w))-\hat{F}(p,w,\pi_\alpha(p,w))\right]}{F_P(p)-F_{PW}(p,w)}\right)^{2+\delta} \to$ 0, since $E\left(I(P_i \le p) - I(P_i \le p, W_i \le w)\right) = F_P(p) - F_{PW}(p,w) > 0$ and by Lemma 1

$$E\left(\hat{F}_{P\Pi}(p,\pi_\alpha(p,w)) - \hat{F}(p,w,\pi_\alpha(p,w))\right) \to F_{P\Pi}(p,\pi_\alpha(p,w)) - F(p,w,\pi_\alpha(p,w)).$$

Hence $\left(\frac{s_n(p,w)}{\hat{F}(p)-\hat{F}(p,w)}\right)^{-1}\sqrt{n}C_n \xrightarrow{d} N(0,1)$ since $\hat{F}(p) - \hat{F}(p,w) \xrightarrow{p} F_P(p) - F_{PW}(p,w)$.

Theorem 4.3 *Assume that* $\min_{\{i:P_i\le p,W_i\ge w\}}\Pi_i \ge h_nB_K$ *and that A3.1, A3.2 hold. In addition, assume that density* f *is strictly positive on the frontier* $\{(p,w,\pi(p,w)) : F_P(p) - F_{PW}(p,w) > 0\}$ *and that* $\pi(p,w)$ *is continuously differentiable. Then for all* (p,w) *in the support of* (P,W) *we have*
(a) there exists $N(p,w) > 0$ *such that* $\forall\ n > N(p,w)$ $\hat{\pi}_{1,n}(p,w) = \max_{\{i:P_i\le p,W_i\ge w\}}\Pi_i + h_nB_K$*;*
(b) $n^{1/(d+1)}(\pi_1(p,w) - \hat{\pi}_{1,n}(p,w) + h_nB_K) \xrightarrow{d} Weibull(\mu_x^{d+1}, d+1)$.

Proof (a) Recall that by definition $\pi_{1,n}(p,w) = \inf\{\pi \in \Re_+ : \hat{F}(\pi|p,w) = 1\}$, i.e., $\hat{\pi}_{1,n}(p,w)$ is the greatest lower bound for the set under the constraint that

$$(nh_n)^{-1}\sum_{i=1}^n\int_0^\pi K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\left[I(P_i\le p)-I(P_i\le p,W_i\le w)\right] = n^{-1}\sum_{i=1}^n\left[I(P_i\le p)-I(P_i\le p,W_i\le w)\right].$$

Then under the assumption $\min_{\{i:P_i\le p,W_i\ge w\}}\Pi_i \ge h_nB_K$, there exists $N(p,w) \in \Re_+$ such that for all $n > N(p,w)$, we have that the equality holds for all $\pi \ge \max_{\{i:P_i\le p,W_i\ge w\}}\Pi_i + h_nB_K$, and it is false for all $\pi < \max_{\{i:P_i\le p,W_i\ge w\}}\Pi_i + h_nB_K$. Hence, $\hat{\pi}_{1,n}(p,w) = \max_{\{i:P_i\le p,W_i\ge w\}}\Pi_i + h_nB_K$ for all $n > N(p,w)$.

(b) Now, similarly to the Park et al. (2000) we have that FDH estimator is defined as $\theta_{FDH}(p,w) =$

$\max_{\{i:P_i \leq p, W_i \geq w\}} \Pi_i$ and under the assumptions in the theorem they show that $n^{1/(d+1)}(\pi_1(p,w) - \theta_{FDH}(p,w)) \xrightarrow{d} Weibull(\mu_x^{d+1}, d+1)$. Consequently, provided that $nh_n^{d+1} = O(1)$ we have that $n^{1/(d+1)}(\pi_1(p,w) - \hat{\pi}_{1,n}(p,w) + h_n B_K) \xrightarrow{d} Weibull(\mu_x^{d+1}, d+1)$.

B APPENDIX True quantile computation

One of the important parts of the Monte Carlo simulation is true quantile computation. Assume the functional form we had in Section 5, $\pi(p, w) = (p^2/\sqrt{w})R$. If (P, W) is fixed at (p, w), then $\pi = \pi(p, w)R$ is just a linear transformation of R. Since we assumed $R = exp(-Z)$ where Z is a random variable having exponential distribution with parameter β. Then,

$$f_R(x) = \frac{1}{\beta} e^{-\frac{1}{\beta}(-\log x)} \frac{1}{x} = \frac{1}{\beta} x^{\frac{1}{\beta}-1}.$$

Hence,

$$\begin{aligned} f_{\pi|P=p,W=w}(\pi) &= \frac{1}{\beta}\left(\frac{\pi}{\pi(p,w)}\right)^{\frac{1}{\beta}-1} \frac{1}{\pi(p,w)} \\ &= \frac{1}{\beta}\pi^{\frac{1}{\beta}-1} \frac{1}{\pi(p,w)^{1/\beta}}. \end{aligned}$$

Since we assumed p, w distributed uniformly from interval $[l, u]$, $f_{PW}(p, w) = \frac{1}{(u-l)^2}$ and

$$\begin{aligned} f(p, w, \pi) &= \frac{1}{\beta}\pi^{\frac{1}{\beta}-1} \frac{1}{\pi(p,w)^{1/\beta}} \frac{1}{(u-l)^2} \\ &= \frac{\pi^{\frac{1}{\beta}-1}}{\beta(u-l)^2}\left(\frac{\sqrt{w}}{p^2}\right)^{\frac{1}{\beta}}. \end{aligned}$$

Having the density explicitly, we can can integrate it and then obtain a conditional distribution function $F(\pi|p, w)$ inverse of which gives us a true quantile. A point (p, w, π) can lie in several different intervals and for every different case the area of integration is different. First consider different cases for p and w:

(1) $l < p < u$, $l < w < u$;

(2) $l < p < u$, $w \leq l$;

(3) $l < p < u$, $w \geq u$;

(4) $p \leq l$, $l < w < u$;

(5) $p \geq u$, $l < w < u$;

(6) $p \geq u$, $w \leq l$;

(7) $p \geq u$, $w \geq u$;

(8) $p \leq l$, $w \geq u$;

(9) $p \leq l$, $w \leq l$;

For the case (6) we have $F(\pi|p, w) = 1$, for cases (3), (4), (7), (8), (9) we have $F(\pi|p, w) = 0$

and for cases (1), (2), (5) we there several levels of π to check:

(i.1) $\pi \geq \pi(p, w)$;

(i.2) $\pi < \pi(l, u)$;

(i.3) $\pi(l, u) \leq \pi < \pi(l, w)$;

(i.4) $\pi(l, w) \leq \pi < \pi(p, u)$;

(i.5) $\pi(p, u) \leq \pi < \pi(p, w)$.

Those cases arise because for the functional form of the profit function we have chosen the following is true:

$$\pi(l, u) \leq \pi(l, w) \leq \pi(p, u) \leq \pi(p, w).$$

Notice, that we need to integrate over the shaded area in the Figure 2.3 and it is convenient to compute an integral over that area in the direction of (P, W, Π) directly instead of using differences of distribution functions we used in the theoretical part. Denote $F_q(p, w, \pi) = F_{P\Pi}(p, \pi) - F(p, w, \pi)$ and $F_{PW,q}(p, w) = F_P(p) - F_{PW}(p, wi)$, then $F(\pi|p, w) = \frac{F_q(p,w,\pi)}{F_{PW,q}(p,w)}$. Thus we get:

Case 1.1. $l < p < u$, $l < w < u$, $\pi \geq \pi(p, w)$.

$$\begin{aligned} F_q(p, w, \pi) &= \frac{1}{\beta(u-l)^2} \int_w^u w^{\frac{1}{2\beta}} \int_l^p p^{-\frac{2}{\beta}} \int_0^{p^2/\sqrt{w}} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\ &= \frac{(p-l)(u-w)}{(u-l)^2}, \\ F(\pi|p, w) &= \frac{(p-l)(u-w)}{(u-l)^2} / \frac{(p-l)(u-w)}{(u-l)^2} = 1. \end{aligned}$$

Case 1.2. $l < p < u$, $l < w < u$, $\pi \leq \pi(l, u)$.

$$\begin{aligned} F_q(p, w, \pi) &= \frac{1}{\beta(u-l)^2} \int_w^u w^{\frac{1}{2\beta}} \int_l^p p^{-\frac{2}{\beta}} \int_0^{\pi} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\ &= \frac{\pi^{\frac{1}{\beta}}}{(u-l)^2} \frac{2\beta^2}{(\beta-2)(2\beta+1)} \left(p^{1-\frac{2}{\beta}} - l^{1-\frac{2}{\beta}}\right) \left(u^{1+\frac{1}{2\beta}} - w^{1+\frac{1}{2\beta}}\right), \\ F(\pi|p, w) &= \frac{\pi^{\frac{1}{\beta}}}{(p-l)(u-w)} \frac{2\beta^2}{(\beta-2)(2\beta+1)} \left(p^{1-\frac{2}{\beta}} - l^{1-\frac{2}{\beta}}\right) \left(u^{1+\frac{1}{2\beta}} - w^{1+\frac{1}{2\beta}}\right). \end{aligned}$$

Case 1.3. $l < p < u$, $l < w < u$, $\pi(u,l) \le \pi < \pi(l,w)$.

$$\begin{aligned}
I_1 &= \frac{1}{\beta(u-l)^2}\int_{l^4/\pi^2}^{u} w^{\frac{1}{2\beta}} \int_{l}^{\pi^{1/2}w^{1/4}} p^{-\frac{2}{\beta}} \int_0^{p^2/\sqrt{w}} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{1}{(u-l)^2}\left[\frac{4\pi^{\frac{1}{2}}}{5}\left(u^{\frac{5}{4}} - \frac{l^5}{\pi^{\frac{5}{2}}}\right) + \frac{l^5}{\pi^2} - lu\right], \\
I_2 &= \frac{1}{\beta(u-l)^2}\int_{w}^{l^4/\pi^2} w^{\frac{1}{2\beta}} \int_l^p p^{-\frac{2}{\beta}} \int_0^{\pi} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{\pi^{\frac{1}{\beta}}}{(u-l)^2}\frac{2\beta^2}{(\beta-2)(2\beta+1)}\left(p^{1-\frac{2}{\beta}} - l^{1-\frac{2}{\beta}}\right)\left(\frac{l^{4+\frac{2}{\beta}}}{\pi^{2+\frac{1}{\beta}}} - w^{1+\frac{1}{2\beta}}\right) \\
I_3 &= \frac{1}{\beta(u-l)^2}\int_{l^4/\pi^2}^{u} w^{\frac{1}{2\beta}} \int_{\pi^{\frac{1}{2}}w^{\frac{1}{4}}}^{p} p^{-\frac{2}{\beta}} \int_0^{\pi} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{\pi^{\frac{1}{\beta}}}{(u-l)^2}\frac{\beta}{(\beta-2)}\left[p^{1-\frac{2}{\beta}}\frac{2\beta}{2\beta+1}\left(u^{1+\frac{1}{2\beta}} - \frac{l^{4+\frac{2}{\beta}}}{\pi^{2+\frac{1}{\beta}}}\right) - \frac{4\pi^{\frac{1}{2}-\frac{1}{\beta}}}{5}\left(u^{\frac{5}{4}} - \frac{l^5}{\pi^{5/2}}\right)\right], \\
F(\pi|p,w) &= (I_1 + I_2 + I_3)/\frac{(p-l)(u-w)}{(u-l)^2}.
\end{aligned}$$

Case 1.4. $l < p < u$, $l < w < u$, $\pi(l,w) \le \pi < \pi(p,u)$.

$$\begin{aligned}
I_1 &= \frac{1}{\beta(u-l)^2}\int_{w}^{u} w^{\frac{1}{2\beta}} \int_{l}^{\pi^{1/2}w^{1/4}} p^{-\frac{2}{\beta}} \int_0^{p^2/\sqrt{w}} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{1}{(u-l)^2}\left[\frac{4\pi^{\frac{1}{2}}}{5}\left(u^{\frac{5}{4}} - w^{\frac{5}{4}}\right) + lw - lu\right], \\
I_2 &= \frac{1}{\beta(u-l)^2}\int_{w}^{u} w^{\frac{1}{2\beta}} \int_{\pi^{\frac{1}{2}}w^{\frac{1}{4}}}^{p} p^{-\frac{2}{\beta}} \int_0^{\pi} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{\pi^{\frac{1}{\beta}}}{(u-l)^2}\frac{\beta}{(\beta-2)}\left[p^{1-\frac{2}{\beta}}\frac{2\beta}{2\beta+1}\left(u^{1+\frac{1}{2\beta}} - w^{1+\frac{1}{2\beta}}\right) - \frac{4\pi^{\frac{1}{2}-\frac{1}{\beta}}}{5}\left(u^{\frac{5}{4}} - w^{\frac{5}{4}}\right)\right], \\
F(\pi|p,w) &= (I_1 + I_2)/\frac{(p-l)(u-w)}{(u-l)^2}.
\end{aligned}$$

Case 1.5. $l < p < u$, $l < w < u$, $\pi(p,u) \leq \pi < \pi(p,w)$.

$$\begin{aligned}
I_1 &= \frac{1}{\beta(u-l)^2}\int_{p^4/\pi^2}^{u} w^{\frac{1}{2\beta}}\int_l^p p^{-\frac{2}{\beta}}\int_0^{p^2/\sqrt{w}} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{(p-l)}{(u-l)^2}\left(u-\frac{p^4}{\pi^2}\right), \\
I_2 &= \frac{1}{\beta(u-l)^2}\int_{w}^{p^4/\pi^2} w^{\frac{1}{2\beta}}\int_l^{\pi^{\frac{1}{2}}w^{\frac{1}{4}}} p^{-\frac{2}{\beta}}\int_0^{p^2/\sqrt{w}} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{1}{(u-l)^2}\left[\frac{4\sqrt{\pi}}{5}\left(\frac{p^5}{\pi^{5/2}}-w^{\frac{5}{4}}\right)-l\left(\frac{p^4}{\pi^2}-w\right)\right] \\
I_3 &= \frac{1}{\beta(u-l)^2}\int_{w}^{p^4/\pi^2} w^{\frac{1}{2\beta}}\int_{\pi^{\frac{1}{2}}w^{\frac{1}{4}}}^{p} p^{-\frac{2}{\beta}}\int_0^{\pi} \pi^{\frac{1}{\beta}-1} d\Pi dP dW \\
&= \frac{\pi^{\frac{1}{\beta}}}{(u-l)^2}\frac{\beta}{(\beta-2)}\left[p^{1-\frac{2}{\beta}}\frac{2\beta}{2\beta+1}\left(\frac{p^{4+\frac{2}{\beta}}}{\pi^{2+\frac{1}{\beta}}}-w^{1+\frac{1}{2\beta}}\right)-\frac{4\pi^{\frac{1}{2}-\frac{1}{\beta}}}{5}\left(\frac{p^5}{\pi^{5/2}}-w^{\frac{5}{4}}\right)\right], \\
F(\pi|p,w) &= (I_1+I_2+I_3)/\frac{(p-l)(u-w)}{(u-l)^2}.
\end{aligned}$$

Similarly we can obtain the result for the cases 2.1-2.5 if using w instead of l and for the cases 5.1-5.5 if using p instead of u in every integral above.

C APPENDIX Properties of $\hat{F}(\pi|p,w)$

In this Appendix we show that $\hat{F}(\pi|p,w)$ is asymptotically a distribution function, when constructed using the smooth estimators of Section 3.1.

(F.i) $\hat{F}(\pi|p,w)$ is nondecreasing in π.

Notice, that for the case when $\pi = 0$ we have $\hat{F}(\pi|p,w) = 0$ and the result is trivial. Let $0 < \pi_1 \leq \pi_2$, then notice that denominator of $\hat{F}(\pi|p,w)$ does not depend on π. Thus, examine the numerator. We must show that,

$$[\hat{F}_{P\Pi}(p,\pi_2) - \hat{F}(p,w,\pi_2)] - [\hat{F}_{P\Pi}(p,\pi_1) - \hat{F}(p,w,\pi_1)] \geq 0.$$

After simplifications we get:

$$\begin{aligned}
[\hat{F}_{P\Pi}(p,\pi_2) &- \hat{F}(p,w,\pi_2)] - [\hat{F}_{P\Pi}(p,\pi_1) - \hat{F}(p,w,\pi_1)] = \\
&= h_n^{-1}\sum_{i=1}^{n}\left[\int_0^{\pi_2} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma - \int_0^{\pi_1} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right] I(P_i \leq p)\left[1-I(W_i \leq w)\right].
\end{aligned}$$

Given assumption A3.2, we have

$$\int_0^{\pi_2} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma - \int_0^{\pi_1} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma \geq 0,$$

for all i. Since the other components of the summation are the indicator functions, the numerator is greater than or equal to zero.

(F.ii) $\hat{F}(\pi|p,w)$ *is right continuous in* $\Re_+$.

Definition A function $f(\cdot)$ is said to be right continuous at the point c if and only if the following holds: $\forall\ \epsilon > 0\ \exists\ \delta > 0\ s.t.\ \forall\ x$ in the domain with $c < x < c+\delta$ the value of $f(x)$ will satisfy $f(c)-\epsilon < f(x) < f(c)+\epsilon$ or $|f(x)-f(c)| < \epsilon$.

Let $\epsilon > 0$ be given, $\pi \in [0, B_\pi]$ and $\delta > 0$ be such that $\pi < x < \pi+\delta$. Then,

$$\begin{aligned}|(\hat{F}(\pi|p,w) - (\hat{F}(x|p,w)) \quad &| \\ &= \frac{h_n^{-1}\sum_{i=1}^n \left|\int_0^\pi K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma - \int_0^x K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right| I(P_i \leq p)\left[1-I(W_i\leq w)\right]}{\sum_{i=1}^n I(P_i\leq p)(1-I(W_i\leq w))}.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\int_0^\pi K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma - \int_0^x K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right| & \\ \leq & \left|\int_0^\pi K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma - \int_0^{\pi+\delta} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right| \\ = & \left|\int_\pi^{\pi+\delta} K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right| \\ = & \left|h_n\int_{\frac{\Pi_i-\pi-\delta}{h_n}}^{\frac{\Pi_i-\pi}{h_n}} K(\psi)\,d\psi\right| \leq m_K\delta.\end{aligned}$$

by assumptions A3.2. Now plugging this result back into the main equation we get

$$|(\hat{F}(\pi|p,w) - (\hat{F}(x|p,w))| \leq \frac{m_K}{h_n}\delta.$$

since $h_n > 0$, $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that $\frac{m_K}{h_n}\delta < \epsilon$, which establishes right continuity.

(F.iii) $\lim_{\pi\to-\infty}\hat{F}(\pi|p,w) = 0$.

Consider the numerator in $\hat{F}(\pi|p,w)$. Then,

$$\begin{aligned}\lim_{\pi\to-\infty}(\hat{F}_{P\Pi}(\pi,p)-\hat{F}(\pi,p,w)) &= \lim_{\pi\to-\infty}\frac{1}{nh_n}\sum_{i=1}^{n}\left[\int_0^{\pi}K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right]I(P_i\le p)(1-I(W_i\le w))\\ &= \frac{1}{nh_n}\sum_{i=1}^{n}\lim_{\pi\to-\infty}\left[\int_0^{\pi}K\left(\frac{\Pi_i-\gamma}{h_n}\right)d\gamma\right]I(P_i\le p)(1-I(W_i\le w))\\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\lim_{\pi\to-\infty}\int_{\frac{-\Pi_i}{h_n}}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi\right)I(P_i\le p)(1-I(W_i\le w)).\end{aligned}$$

Now, $\lim_{\pi\to-\infty}\int_{\frac{-\Pi_i}{h_n}}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi=0$, which gives the result.

(F.iv) There exists some $N(p,w)$ such that for all $n>N(p,w)$ we have $\lim_{\pi\to\infty}\hat{F}(\pi|p,w)=1$. Again we have,

$$\lim_{\pi\to\infty}\hat{F}(\pi|p,w)=\lim_{\pi\to\infty}\frac{\sum_{i=1}^{n}\left[h_n^{-1}\int_0^{\pi}K(\frac{\Pi_i-\gamma}{h_n})d\gamma\right]I(P_i\le p)\left[1-I(W_i\le w)\right]}{\sum_{i=1}^{n}I(P_i\le p)(1-I(W_i\le w))}.$$

Then using symmetry,

$$\begin{aligned}&\lim_{\pi\to\infty}h_n^{-1}\int_0^{\pi}K(\frac{\Pi_i-\gamma}{h_n})d\gamma\\ &\qquad= \lim_{\pi\to\infty}h_n^{-1}\int_0^{\pi}K(\frac{\gamma-\Pi_i}{h_n})d\gamma\\ &\qquad= \lim_{\pi\to\infty}\int_{-\frac{\Pi_i}{h_n}}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi.\end{aligned}$$

By assumption A3.2, for $n>N(p,w)$ we have that $-\frac{\Pi_i}{h_n}<-B_K$, therefore

$$\lim_{\pi\to\infty}\int_{-\frac{\Pi_i}{h_n}}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi=\lim_{\pi\to\infty}\int_{-B_K}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi=1,$$

since $\frac{\pi-\Pi_i}{h_n}\to\infty$ as $\pi\to\infty$ and $\lim_{\pi\to\infty}\int_{-B_K}^{\frac{\pi-\Pi_i}{h_n}}K(\psi)d\psi=1$.