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Counting problems are rich in opportunities for students to make meaningful mathematical connections and develop non-algorithmic thinking; their accessible nature and applications to computer science make counting problems a valuable part of mathematics curricula. However, students struggle in various ways with counting, and while previous studies have indicated that listing may be a useful way to address student difficulties, little work has been done toward understanding exactly how students may connect lists of outcomes to their solutions to counting problems. To begin to address this, I conducted twenty task-based interviews with undergraduate students to probe the ways in which students conceptualize the relationship between sets of outcomes and counting processes. In this thesis, I describe the ways that students listed outcomes using an elaboration of English's (1991) solution strategies, and I frame my findings about their understanding using Lockwood's (2013) model of students' combinatorial reasoning. I discover that students reason about the relationship between lists of outcomes and counting processes with varying levels of sophistication, and I suggest that teachers could help students by making connections between sets of outcomes and counting processes more explicit.

Listing as a Potential Connection between Sets of Outcomes and Counting Processes

by
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Chair of the Department of Mathematics

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I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Sarah A. Erickson, Author

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Chapter 1: Introduction

In our increasingly digital world, the mathematical area of combinatorics has become more important than ever due to its applications in computer science, electrical engineering, probability, and statistics. Apart from having many uses inside and outside of mathematics, researchers have also argued also that combinatorics is a worthwhile topic for students to learn in and of itself as well. Kapur (1970) believed it should play a role in school mathematics, due to the accessible nature of combinatorial problems (since they do not depend on calculus), their ability to provide motivation for working with computers, their effectiveness in teaching students systematic reasoning, and the context they provide for helping students develop difficult mathematical concepts, such as mapping, relations, equivalence classes, and isomorphisms (p. 114). Combinatorics is rich in opportunities for students to make mathematical connections, make conjectures, rigorously justify claims, and in turn construct proofs for mathematical propositions.

One key component of combinatorics is enumerative combinatorics, the solving of counting problems. Counting problems often show up in mathematics curriculum, since their solutions tend to require non-algorithmic, creative mathematical thinking. However, while crucial to combinatorics and valuable for developing creative mathematical thinking, it is well established that students struggle when solving counting problems (Eizenberg & Zaslavsky, 2004; Melusova & Vidermanova, 2015). This is seen in both low levels of success for obtaining correct solutions (Batanero et al., 1997), and in tendencies for students to defend combinatorial solutions with surface features of a problem or empirical patterns, rather than with rigorous mathematical justifications (Lockwood, Swinyard, & Caughman, 2015). Eizenberg and Zaslavsky (2004) said about counting problems,

> Most problems do not have readily available solution methods, and create much uncertainty regarding how to approach them and what method to employ. There are numerous examples in which two different solutions yielding different answers to the same problem may seem equally convincing. (p. 16)

In other words, students often find counting difficult, because for many counting problems it is possible to think of two ways to approach the problem, and these two ways may be equally convincing, but these two ways actually give different numerical answers. For this reason, it can be hard to know when one is solving a counting problem correctly. Equally troubling can be the fact that many counting problems have solution sets whose cardinalities are quite large, making it unfeasible to verify that a solution to a counting problem is correct by explicitly enumerating each outcome.

While there remains much to learn about how to help students overcome difficulties solving counting problems, some work has been done in an attempt to address these difficulties. These efforts have included characterizing student errors (Batanero et al., 1997), studying combinatorial solution strategies of young children (English, 1991; Maher, at al., 2011), looking at combinatorial solution strategies of secondary-school students (Melusova, 2015), creating models of students' combinatorial reasoning (Lockwood, 2013), and determining student ways of thinking about combinatorial solution sets (Halani, 2013). One key result that has emerged from the combinatorics education community is the importance of attending to sets of outcomes for students' correct solving of combinatorial problems. Lockwood (2013) argued that robust combinatorial understanding is rooted in sets of outcomes, and suggested that emphasis on relating the outcomes of a counting problem and the solution for solving the problem may be beneficial for helping students count successfully. This suggestion was confirmed in a later study in which statistical analysis was done on the effects of systematic listing on student performance solving counting problems (Lockwood & Gibson, in press). The positive correlation between listing and success at solving counting problems merits some consideration, and more work needs be done to investigate the relationship between student-generated lists of outcomes and correct, rigorously justified solutions to counting problems.

In this study, I attempt to expand the knowledge we have about students' combinatorial listing by focusing on the listing behaviors of undergraduate students and how it affects their success in solving counting problems involving arrangement. In Chapter 2, I summarize previous empirical research that has been done towards understanding student listing behaviors, as well as student reasoning about lists of outcomes. Key in the literature is a study conducted by English (1991) on the combinatorial solution strategies of young children. I present in detail the framework she used to categorize the combinatorial solution (listing) strategies of young children, providing examples to help clarify this framework to the reader. In Chapter 2 I also outline the theoretical perspectives I used for my research. In particular, I made use of Lockwood's (2013) model of students' combinatorial reasoning for the collection and analysis of my data, which I describe in detail in Chapter 3 of this thesis.

In Chapter 4, I show and explicate the results of this study. An important assumption I make in explaining my results is the notion that, for students, the activity of creating a list (or partial list) of outcomes is valuable, but rote production of lists not always enough to guarantee that the correct solution will be achieved (Batanero, 1997), or that the student will be able to provide justification as to why a particular solution is correct (Lockwood, Swinyard, & Caughman, 2015). Listing is a useful activity for students to engage in, and I do not wish to diminish its importance, but rather I assert that equally important is intentionality in the strategy used to list and productively connecting the list to the solution of the combinatorial task being solved. Keeping this in mind, I address in this thesis the following research questions:

1. How can undergraduate students' combinatorial lists be usefully categorized? In particular, can the combinatorial solution strategies identified by English (1991) be usefully applied to characterize combinatorial solution strategies utilized by undergraduate students?
2. Does the data in this study of undergraduate students and their listing activity corroborate with existing evidence that explicitly writing outcomes is helpful for students?
3. When solving counting problems involving arrangements, what are the ways in which students explicitly make meaningful connections between lists of outcomes and solutions to counting problems?

Before describing my review of relevant literature that addresses components of these research questions, I note that this study was conducted within the context of a larger study aimed primarily at probing student understanding of the concept of factorials in a combinatorial setting. Data for this broader study was collected and analyzed data with another researcher from a large university in the western United States. It is appropriate to situate the study that this thesis explicates in the context of this broader study, because within the study on student reasoning about factorials, I was afforded the opportunity to study students who were attuned to the factorial formula and what counting processes may be associated with it. This allowed me to narrow my focus and make a more detailed analysis of student reasoning about lists of outcomes as they relate to counting processes for problems that involve arrangement.

Chapter 2: Literature Review

In this chapter, I synthesize work that previous researchers have done relevant to my study of student reasoning about how lists of outcomes relate to counting processes. In Section 2.1, I summarize prior empirical studies that have been conducted and how they informed my investigation. I expound upon previous studies that have addressed specific difficulties that students have while solving counting problems, ways that visualization and specifically writing outcomes can help remedy some of these difficulties, and issues that students can encounter with using lists of outcomes to connect to a coherent solution to a counting problem. An important subsection, 2.1.3, expands on English's (1991) solution strategies that young children exhibit while solving counting problems. In Section 2.2, I present theoretical perspectives I used to formulate my research questions, collect data, and analyze the data to answer my research questions. These include in particular Lockwood's (2013) model of student's combinatorial reasoning.

2.1 Themes among Previous Empirical Studies

2.1.1 Difficulties Students Face While Counting

It is widely acknowledged in the mathematics-education community that students struggle to correctly solve counting problems (Eizenberg & Zavlavsky, 2004). Hadar and Hadass (1981) in particular said that, "Combinatorics is a field which most of the students find very complicated" (p. 435). To address this problem, researchers have categorized student errors while solving counting problems (Batanero et al., 1997), analyzed students' initial intuitions while counting (Fischbein & Grossman, 1997), created models of students' combinatorial reasoning (Lockwood, 2013), and articulated students' ways of thinking and ways of understanding in the context of solving counting problems (Halani, 2013; Harel, 2008).

Batanero et al. (1997) in particular studied the nature of student mistakes and found that regardless of whether the students in their study had received combinatorial instruction or not, they struggled to solve counting problems correctly. They found that, when solving counting problems, errors were made for a variety of reasons, including confusion over the type of objects being counted, misunderstanding of whether given elements should be considered distinguishable or indistinguishable, excluding some elements from the configurations, and giving a mistaken intuitive answer without any justification. Fischbein and Grossman (1997) also looked at how student intuition played a role in their counting, hypothesizing that those intuitions were governed by certain intellectual schema that students have. They found by studying various age groups of students that "With a few exceptions, the intuitive estimations expressed, in fact, a particular intellectual schema. The particular procedure was, generally, inadequate and consequently…the guesses expressed incorrect solutions" (Fischbein & Grossman, 1997, p. 35). Thus, they found that students' intuition tended to lead them to using inappropriate procedures and wrong estimations while trying to solve counting problems.

Another difficulty addressed in the literature is the challenges many students have with evaluating the reasonableness of their solutions. The cardinality of a set of outcomes being counted can be quite large, making it impossible for students to check their answers by writing all of the outcomes. Because of this, students need more efficient ways of checking the solution to a counting problem, and Eisenberg and Zaslavsky (2004) showed that many students do not have efficient verification strategies. In their study on verification strategies of undergraduate combinatorics students, they note, "many of the students who made attempts to verify their incorrect solutions…were not able to come up with efficient verification strategies and were thus neither able to detect an error nor to correct their solution" (p. 32). In other words, they discovered that when undergraduate students struggled to verify their answers to counting problems, this unsurprisingly resulted in their inability to determine if answers were unreasonable, much less help them see how to fix their erros. They also discovered that many students did not even try to verify their solutions to counting problems.

To summarize, students face a variety of challenges while attempting to solve basic enumerative combinatorics problems. The reasons for student errors are copious, and even attempts to verify solutions may not be helpful for students to determine the reasonableness of their solutions. In the proceeding section, we will see one way that researchers have tried to address this difficulty: using visualization while solving counting problems, and specifically using concrete lists as a way of understanding the solution to a particular counting problem.

2.1.2 Visualization and Listing Outcomes as Possible Remedies for Student Difficulties

While student difficulties in counting persist, there is evidence that students' use of visualization while solving counting problems might be helpful. In this paper, I use the term visualization in the same way that Maher and Speiser used it in a 1997 study to describe an explicit or mental model that students construct to make sense of a mathematical problem or idea (p. 128). In their study they discovered some of the usefulness of visualization in the context of counting when they studied young children solving basic counting problems with the visualization of block towers. They said, "We find that children's working theories empower very striking and effective ways of working with mathematical ideas, often using concrete objects, in very particular ways, first as evidence for specific arguments, then as anchors for quite abstract constructions" (Maher & Speiser, 1997, p. 126). In another study, Maher and Martino (1996) gave an example of a 10-year-old girl who was able to give sophisticated justifications to solutions to counting problems when given block towers to visualize her outcomes. Finally, Halani (2013) also found undergraduate students were able to use Venn diagrams to represent each of the ways of thinking about counting problems that she identified.

In the domain of counting, there is evidence that students might be able to visualize the set of outcomes they are counting by making a physical list, and that this may be useful when solving counting problems. Lockwood (2013) explained the importance of attending to the set of outcomes while solving counting problems, and more recently Lockwood and Gibson (in press) found evidence for a potential link between student success and the activity of writing outcomes. In their study of the effects of student listing, Lockwood and Gibson reported that students who listed for some problems and did not list for others answered significantly more questions correctly when they listed versus when they did not. They also found statistically significant evidence that students were more likely to have listed on questions they answered correctly versus questions they answered wrong. While these results are not enough to show causation (it may be the case the stronger students are more naturally inclined to write a list of outcomes while solving counting problems), they argued that the correlation was promising (p. 15).

In conclusion, having students think about concrete lists of outcomes as a way to visualize the counting problem they are trying to solve has been shown to be potentially useful. Other researchers have noticed this as a potential way to help students count, and there have been a variety of ways that researchers have tried to achieve a greater understanding of how students list. This has included analyzing ways of thinking that students have about combinatorial solution sets (Halani, 2013), and making careful observations about the particular strategies that students use while they list outcomes (English 1991). The latter is explored more in depth in the following section.

2.1.3 English's Categories for Children's Listing Strategies

Critical to my review of previous literature on student listing behavior and the ways students tie their lists to a solution to a counting problem is the framework that English (1991) developed to categorize combinatorial solution strategies. This particular subsection explains her categories and illustrates them with concrete examples. I will draw on her categories of listing in presenting my analysis and results in Chapters 4 and 5.

To help gain greater insight into how students think about combinatorial solution sets and use them to solve counting problems, English (1991) conducted a study to document the strategies that young children used to solve basic enumerative combinatorics problems. The problems involved using only one operation (multiplication), and the children were given attractive manipulatives to help them keep track of what they were counting as they solved problems. In particular, the problems asked about the number of ways that bears could be dressed in different outfits, with each bear able to wear a top and a pair of pants. The children were given small toy bears and differently colored outfits to "dress the bears" to aid them in their problem-solving. More bears and outfits were given to the children than were needed to solve the counting problems asked, so English could see how attuned the children were to the possibility of over counting. The children's ages ranged between 4 years 6 months and 9 years 10 months, and data about their generation of outcomes were collected in task-based interviews.

As English analyzed the data from her interviews with the children, there were six hierarchical strategies that emerged, each increasingly sophisticated. She denoted these strategies as Solution Strategies A through F, with A being the least sophisticated, and F being the most sophisticated. English's characterization of each of the categories is summarized in Table 1.

Solution Strategy	**Description**
Solution Strategy A	"Random selection of items with no rejection of inappropriate items" (p. 458).
Solution Strategy B	"Trial-and-error procedure with random item selection and rejection of inappropriate items" (p. 458).
Solution Strategy C	"Emerging pattern in item selection, with rejection of inappropriate items" (p. 458).
Solution Strategy D	"Consistent and complete cyclical pattern in item selection, with rejection of inappropriate items" (p. 459).
Solution Strategy E	"Emergence of an 'odometer' pattern in item selection, with possible item rejection" (p. 460).
Solution Strategy F	"Complete odometer pattern in item selection, with no rejection of items" (p. 461).

Table 1. English's (1991) characterization of young children's listing strategies.

Going into more detail, children transitioned from Solution Strategies A-C to D-F by exhibiting a complete, cyclical pattern used to generate each of the outcomes being counted. English explained about Solution Strategy D,

> In contrast to the previous strategy, the present one is characterized by a consistent and complete cyclical pattern in item selection, with the pattern having the potential to generate all possible combinations. When children use a cyclical pattern in the selection of one item type only, they frequently do not follow any particular order in selecting items of the other type; they simply select any item which will produce a different outfit and reject those which are inappropriate. (English, 1991, p. 459)

To further differentiate between Solution Strategies E and F, English clarified that the odometer pattern in Solution Strategy E is incomplete, and this could be due to an "over-exhaustion" or duplication of outcomes produced by holding a given item constant, a failure to exhaust all possible outcomes from a given item held constant, or an otherwise failure to determine when generation of all of the outcomes being counted is complete (English, 1991, p. 460-461). From here on out in this thesis, the phrases "complete odometer strategy" and "Solution Strategy F" will be considered synonymous in meaning. Solution Strategy E will be referred to as an "incomplete odometer strategy."

Later in 2012, Halani used English's categories in her study of students' ways of thinking about combinatorics solution sets. In her study, she observed two ways of thinking that undergraduate students engaged in while using the complete odometer strategy to answer counting problems, *Standard Odometer Thinking* and *Wacky Odometer Thinking*. Students who engaged in Standard Odometer Thinking generated outcomes by picking an item to hold constant in a fixed position while cycling through the items that can be placed in the other positions. Then, a new item would be chosen to hold constant in the same fixed position to begin the process of cycling through items to be placed in the other positions. Alternatively, if a student engaged in Wacky Odometer Thinking, outcomes would be still be generated by picking an item to be held constant in a fixed position while cycling through other items, but new outcomes would then be created by changing the *position* of the first item being held constant, and then cycling next through the positions that the other items can occupy (Halani, 2012). In this way, Halani extended English's Solution Strategy F to encompass two ways of thinking, one in which items are held constant in a fixed position (like an odometer in a car), and another in which an item is held constant in different fixed positions.

In conclusion, English's (1991) study on the solution strategies of young children, including the extension of the complete odometer strategy contributed by Halani (2012), was critical to my study on the listing behaviors of undergraduate students and the way in which students use these lists to solve counting problems. In this thesis, I used and elaborated on English's categories to characterize listing strategies of undergraduate students, enabling me to talk about their lists in a more organized way.

2.1.4 Student Difficulties with Effective Listing

While much has been learned about the methods students use to list outcomes and the ways in which students think about lists of outcomes, there is still work to be done to learn how to support students' effective use of listing to find and justify solutions to counting problem. Lockwood and Gibson (in press) reported that organized, appropriate notation and articulation of what constituted a desirable outcome is helpful for students as they use their lists to solve the problems. However, Hadar and Hadass (1981) noted that students often find it difficult to recognize what outcomes of a given counting problem may look like.

Additionally, since a complete list of outcomes is often impossible for students to create, due to a large cardinality of the solution set, students often can only list a subset of the outcomes to be enumerated. Lockwood and Gibson found that even a partial list of outcomes can help students understand underlying structures of outcomes and solve counting problems (Lockwood & Gibson, in press), but making an illuminative partial list is not always a trivial task for students. Hadar and Hadass found that choosing an appropriate subset can be challenging for students as well. They explained,

> The breaking of a combinatorial problem…into subproblems, possibly through transforming its formulation into a more explicit one, is a major breakthrough in the process of seeking a solution. However, achieving it is a big obstacle for students…Many times students form subsets of possibilities to be counted, which are not mutually exclusive. Also, very often, the union of the partial sets counted does not coincide with the whole set under discussion. (Hadar & Hadass, 1981, p. 438)

Finally, there is evidence that intentionally writing outcomes in an organized manner is an important component in using lists of outcomes to find a solution to a counting problem (Lockwood & Gibson, in press). However, students may fail to find a systematic way in which to write outcomes. Lockwood and Gibson found that an incomplete implementation of the odometer strategy could result in students not getting all of the outcomes, or making incorrect assumptions about how their lists would generalize. Batanero also identified non-systematic listing as an important cause for student errors while solving counting problems (Batanero et al., 1997).

In summary, students trying to solve counting problems may have difficulties articulating what constitutes a desirable outcome, finding appropriate subsets that can shed light on the underlying structure of the larger set of outcomes, and utilizing a systematic, intentional strategy for writing outcomes. Because of this, there is more work that needs to be done to learn about how students can overcome these difficulties to successfully connect outcomes to solutions to counting problems. Just because students can write down outcomes, and even if they can write down outcomes in a systematic yet rote way, this does not necessarily mean that these outcomes are immediately illuminative for students trying to solve a given counting problem. Along similar lines, Cooper and Alibali claimed, "Rather than asking simply which types of illustrations serve learners better, it is important to identify how learners with different backgrounds and skill levels utilize visual representations when solving problems" (Cooper & Alibali, 2012, p. 287) In the context of counting, I feel it may be important to not only think about how lists are beneficial for students, but think about helping students use those lists to connect outcomes to their process for enumerating those outcomes to obtain a solution. As Cooper and Alibali found out, just because a student may have a visualization of what is happening in a mathematics problem, it does not mean that he or she necessarily knows how to connect that to a solution (Cooper & Alibali, 2012). This leads us to the primary goal of this paper: to gain insight into how students think about the relationship between lists of outcomes and their solutions for solving counting problems.

2.2 Theoretical Perspectives: Lockwood's (2013) Model of Students' Combinatorial Thinking

In the following subsections, I will describe the theoretical perspectives that were used throughout data collection and analysis. These perspectives include use of Lockwood's (2013) model of students' combinatorial thinking. When a student approaches a counting problem, Lockwood (2013) contends that students' thinking about that the problem may have three components: *formulas/expressions*, *counting processes*, and *sets of outcomes*. This is illustrated in Figure 1.

Listing as a Potential Connection between Sets of Outcomes and Counting Processes

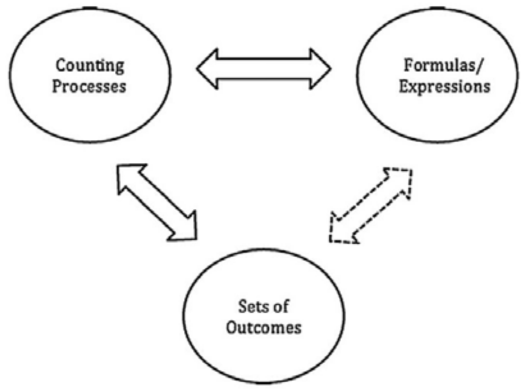

Figure 1. Lockwood's (2013) model of students' combinatorial thinking[1].

For a given counting problem, sets of outcomes refer to collections of objects that could be enumerated by a student counting – the set of outcomes is the set whose cardinality is the solution to the counting problem. Counting processes are defined to be procedures that a student could engage in to find the cardinality of the set of outcomes for the given counting problem. A counting process can be thought of as an intentional, strategic way of organizing or generating outcomes to be counted. For many counting problems, multiple counting processes could be conceived to obtain a solution to the problem. The results of counting processes expressed using mathematical symbols are the formulas/expressions components of a counting problem, the numerical value of which is the solution to the counting problem.

In addition, between each of the components are key bi-directional relationships that a student can use to understand a solution to a counting problem. For instance, a particular formula or expression can represent a counting process used to solve a problem, and conversely, counting processes when expressed symbolically create formulas and expressions that compute the answer to a counting problem. The relationships between formulas/expressions and sets of outcomes are less clear, although Lockwood et al. (2015) suggested that empirical patterning may be an appropriate characterization of some of these relationships. Students moving from the

[1] The bidirectional arrow between Formulas/Expressions and Sets of Outcomes is dotted in Lockwood's model, because she found no evidence in her data of what exactly that relationship might look like for students (Lockwood, 2013, p. 255).

sets of outcomes component to the formulas/expressions component might be able to notice a numerical pattern by exhaustively listing the outcomes for counting problems with numerically small answers. Then without appealing to an argument as to why the cardinalities of the sets of outcomes may follow a certain pattern, students are able to produce an accurate formula. Conversely, a student might move from the formulas/expressions component to sets of outcomes component by thinking of a formula or expression conceptually as a set of objects with cardinality equal to the numerical value of the formula or expression.

Of particular interest in this paper is the bi-directional relationship that Lockwood conjectures between sets of outcomes and counting processes for a given counting problem. A student moving from sets of outcomes to counting processes can articulate how a particular set of outcomes could be organized or enumerated by an intentional, strategic counting processes. Conversely, a student could be given a counting process, and describe an organization or structure that particular counting process would impose on a collection of objects to be counted. While researchers have established in the existing literature that considering sets of outcomes is potentially beneficial to students' successful counting (Lockwood, 2014; Lockwood & Gibson, in press), little research has been done on how students might conceptualize the bi-directional relationship between their procedures for solving counting problems, and the corresponding organization of outcomes that those procedures afford. This conceptualization is central to the research questions addressed by the study described in this thesis.

To explain clearly each component and conjectured relationship in the model, I provide a concrete example using Lockwood's (2013) model to discuss solutions to a problem I gave to students in the interviews conducted for this study. Examining this problem will allow for an in-depth look at the details of this problem, hopefully clarifying the mathematics in subsequent sections of the paper. It is called the Cattle Problem, and its statement is the following: "How many ways are there to rearrange the letters in the word CATTLE if the two T's must appear together at the beginning or end of the word?" To find a solution for this problem, we may think of each arrangement as a particular way to fill six slots, or positions, with the letters C, A, T, T, L, and E. A particular counting process imposing the organization of outcomes in Table 2 could entail first recognizing that the set of outcomes can be partitioned into two subsets, the first subset containing each arrangement of the letters with the two T's at the beginning, and the second subset containing the arrangements with the two T's at the end. Within each subset, there are 4 letters which can fill the first unoccupied slot in an arrangement. (For instance, if the two T's are first, we could choose C to fill the third slot.) For each way that the first unoccupied slot is filled, there are 3 letters that would fill the second unoccupied slot in an arrangement. (For example, if the two T's are first and C is chosen to fill the third slot, we can choose to fill the fourth slot with the letters A, L, or E.) For each way the second unoccupied slot is filled, there are two ways to fill the third unoccupied slot, creating a forced final choice of letter to be inserted into the final slot. (Continuing with the previous example, if the two T's are chosen to be first, C is chosen to fill the third slot, and A is chosen to fill the fourth slot, we may choose L or E to fill the fifth slot in an arrangement. Depending on our choice, we will then be forced to fill the sixth slot with E or L, respectively.) By this counting process, we have that there are 4! ways to arrange the remaining letters once the two T's are fixed, giving us the answer conveyed using the following expressions:

$$2 \times 4 \times 3 \times 2 \times 1 = 2 \times 4! = 48.$$

The counting process described in the preceding paragraph is reflected in the above formula via an application of the Multiplication Principle.[2] The counting process also creates a particular organization of the set of outcomes by position, and listing outcomes according to this organization is an example of utilizing English's (1991) Solution Strategy F (the odometer strategy) for generating outcomes. Table 2 shows how one subset of these outcomes are organized and is a representative example of how a student engaged in Standard Odometer Thinking (Halani, 2012) might visualize outcomes for the Cattle Problem.

[2] The Multiplication Principle states that the number of ways to carry out a sequence of independent tasks is the product of the number of ways to complete each task individually (see Tucker, 2002 for more information).

TTCALE	TTACLE	TTLCAE	TTECAL
TTCAEL	TTACEL	TTLCEA	TTECLA
TTCLAE	TTALCE	TTLACE	TTEACL
TTCLEA	TTALEC	TTLAEC	TTEALC
TTCEAL	TTAECL	TTLECA	TTELCA
TTCELA	TTAELC	TTLEAC	TTELAC

Table 2. Outcomes in the Cattle Problem beginning with TT arranged by position.

While Table 2 demonstrates one organization of the outcomes created by Standard Odometer Thinking (Halani, 2012), this is not the only organization possible. Another possible counting process that could be reflected in the product $2 \times 4!$ is the following: fix the two T's either at the beginning or end of an arrangement. Then, there are 4 remaining positions that could be filled with the letter C. (For example, we could fix the two T's to fill the first two slots and let C fill the third slot.) Once the positions of the two T's and the C are chosen, there are 3 remaining slots that the A could fill. (For example, if the two T's and the C occupy the first, second, and third slots, respectively, we could choose A to fill the fourth slot.) Finally, for each choice of placement for the two T's, C, and A, there are 2 remaining slots that the L could fill, creating a forced choice for the position of the E. (Continuing with the above example, if the two T's, C, and A occupy the first, second, third, and fourth positions, respectively, there are two choices for the position of L—the fifth or sixth position. For each choice, there is only one remaining position for the E—the sixth or fifth position, respectfully.) This counting process, while producing an identical formula for obtaining the solution to the counting problem, creates another distinct organization of the set of outcomes. A student listing in this manner would be also be categorized as using English's (1991) Solution Strategy F, because the student is holding an item constant (in this case, the position of the letter C), and cycling through all possible positions for the remaining letters. This is demonstrated in Table 3 and is an example of how a student engaging in Wacky Odometer Thinking (Halani, 2012) might conceptualize the set of outcomes for the Cattle Problem. Note the fixed positions of C's throughout each column of outcomes, suggesting that our first stage in the process is to decide into which of the four positions we place the C.

TTCALE	TTACLE	TTALCE	TTALEC
TTCAEL	TTACEL	TTAECL	TTAELC
TTCLAE	TTLCAE	TTLACE	TTLAEC
TTCEAL	TTECAL	TTEACL	TTEALC
TTCLEA	TTLCEA	TTLECA	TTLEAC
TTCELA	TTECLA	TTELCA	TTELAC

Table 3. Outcomes in the Cattle Problem beginning with TT arranged by letter placement.

Because of the various organizational schemes for the set of outcomes, looking at student thinking about this problem affords insight into how students articulate the relationship between sets of outcomes and the counting processes that organize them while solving counting problems. I draw upon these notions of counting processes, sets of outcomes, and the relationship between the two, throughout the remainder of the paper.

In summary, I assume in this study that students' reasoning about counting problems has multiple components consisting of the formulas/expressions, counting processes, and the set of outcomes that the process is intended to enumerate. These outcomes can be an explicit list, or more abstractly conceived by a student using an organizational scheme connecting outcomes to the process used for enumeration. It is particularly the connection between outcomes and counting processes that are explored in this thesis. While I recognize the importance of students being able to come up with correct answers to counting problems, and I acknowledge the usefulness of being able to systematically generate a list of outcomes, I assert that to have the most robust, connection-rich understanding of counting, students must be able to construct meaningful relationships between lists of outcomes and processes used to solve counting problems.

Chapter 3: Methods

In this chapter, I describe in detail the methodology in which I collected and analyzed data for this study. I discuss the participants involved in the data collection, how these participants were recruited, the procedures utilized to collect the data, and the techniques used to analyze the data.

3.1 Participants

The participants in this study were 20 undergraduate students at a large university in the western United States. Of these 20 students, 14 were taking a calculus class while data for this study was collected, and the remaining 6 were taking advanced mathematics classes (altogether, these 6 students were taking advanced calculus, topology, linear algebra, introductory probability, and introductory numerical analysis at the time of data collection). I sought participants at varying levels of collegiate mathematics, because I was interested in studying the reasoning of both students who had and had not previously seen discrete mathematics at the college level. I note also that the participants in this study encompassed the following STEM majors: four students in mathematics, one in biology, eleven in engineering, one in biochemistry and biophysics, one in physics and mathematics, one in zoology, and one in food science. Of the participants, fourteen were male students (ten in calculus, four in the advanced mathematics courses), and six were female (four in calculus, two in advanced courses).

3.2 Data Collection

The study reported in this thesis is situated within a larger study that targeted students' understanding of factorials. Thus, the overall design of the study and some of the interview tasks were developed in order to elicit responses that would shed light on student reasoning about factorials. Although this is not the explicit aim of this thesis, I felt it was relevant to frame the current study within this broader study about factorials. This enabled me to focus on student reasoning about listing outcomes in the context of arrangement problems, which in the case of the broader study could be solved using factorials.

The design of this research study was to conduct semi-structured, individual task-based clinical interviews with each participant (Hunting, 1997). Livescribe pens and notebooks were used by both the interviewer and participant during data collection. This technology was used because each Livescribe pen recorded audio and real-time pen strokes on the Livescribe paper, which facilitates an efficient analysis of their utterances and inscriptions that are captured in real time. Each of the 20 interviews began with questions aimed at gauging student reasoning about factorials, and then after these questions were asked, I proceeded to ask the students a sequence of counting problems. These problems are discussed in detail in Section 3.3.

To summarize, by narrowing my focus to arrangement problems that could be solved using one particular formula, I was able to give attention to the varied types of student reasoning about sets of outcomes and counting processes in this fixed context. Below, I discuss the specific interview tasks given to the research participants, providing justification for why the tasks were chosen and relating them to the components of Lockwood's model. Following the discussion of the tasks, I describe the data analysis.

3.3. Interview tasks

To study student connections between sets of outcomes and counting processes, I concentrated on arrangement problems that could be solved by manipulating one particular formula, $n!$. Even more specifically, I was interested to see whether and how students would be able to relate a counting process with the set of outcomes, and whether they could see the structure of a factorial in their sets of outcomes. Because the interviews were semi-structured, I chose which problems I wanted students to solve according to each individual student's ability, and according to which particular aspect of the student's reasoning I wanted to examine. As a result, the questions each student was asked to answer varied across the interviews. The following is a description of the counting problems I chose to ask the students, as well as justifications for using them in the interviews. While there were other tasks given to the students in the ambient study of students' reasoning about factorials, I focus in this thesis on the following three counting problems, since in each of these problems I either directly asked students to list outcomes, or I observed instances where in solving the problem, at least one student attempted to make a complete or partial list. This enabled me to use their lists to examine ways that students connected their list of outcomes to the process they eventually used to solve the counting problem.

3.2.1. The Cattle Problem

The Cattle Problem is stated as follows: “How many ways are there to rearrange the letters in the word CATTLE if the two T’s must appear together at the beginning or end of the word?” I previously discussed the problem at length in Section 2.2.1, including considering two possible ways of thinking about the solution that could generate lists of outcomes with different structures. Nineteen of the twenty students interviewed were asked to solve the Cattle Problem, and of those nineteen, eighteen of them were asked to list all of the outcomes beginning with the two T’s after finding a solution. Students were permitted to alter their pre-listing solution if they wanted to after writing the outcomes.

Because of the various organizational schemes for the set of outcomes, looking at student thinking about this problem afforded insight into how students articulate the relationship between sets of outcomes and the counting processes that organize them. Additionally, the problem provided insight into how students listed outcomes and whether that listing helped to solidify a relationship between counting processes and sets of outcomes for them.

3.2.2. The Horse Race Problem

The statement of the Horse Race Problem is as follows: “There are 10 horses in a race, how many ways are there for the horses to get 1st, second, and third place?” Out of the 20 students interviewed, 13 were asked to solve the Horse Race Problem. I did not explicitly ask any student who solved the problem to list any outcomes, but students were encouraged to list if they expressed in some way that they felt it would help them solve the problem.

A solution to this problem can be obtained by arguing that there are 10 possible horses that could win first place, and for every possible first-place horse, there are 9 remaining horses that could win second place, and after that 8 remaining horses that could win third place. This counting process yields the product $10 \times 9 \times 8 = 720$ total possible outcomes of the horse race. The counting process also gives rise to an organization of the outcomes of the race by placing. If each horse is denoted by a letter between A and J, Table 3 illustrates how some of the outcomes would be organized in relationship to this particular counting process. In Table 4, the outcomes depicted are those where horses A, B, and C win first place.

ABC	BAC	CAB
ABD	BAD	CAD
⋮	⋮	⋮
ABJ	BAJ	CAJ
ACB	BCA	CBA
ACD	BCD	CBD
⋮	⋮	⋮
ACJ	BCJ	CBJ
ADB	BDA	CDA
⋮	⋮	⋮

Table 4. Outcomes for the Horse Race Problem arranged by placing.

An alternative way to solve this problem might be to first make an unordered selection of three horses to place in the race. For any fixed selection of three horses, there are 3! ways to arrange those horses, in which each arrangement corresponds to a particular choice of one of those horses winning first, one earning second, and one earning third. Again, this counting process yields a formula producing the same value for the cardinality of the set of outcomes, $\binom{10}{3} \times 3! = 720$. But, it gives rise to a separate way of organizing the set of outcomes. Table 5 shows how this counting process yields organization by the choice of which three horses to place.

ABC	ABD	ABE	
ACB	ADB	AEB	
BAC	BAD	BAE	⋮
BCA	BDA	BEA	
CAB	DAB	EAB	
CBA	DBA	EBA	

Table 5. Outcomes for the Horse Race Problem organized by an unordered selection followed by an arrangement to determine placing.

Because so many different counting processes can be used to solve this problem, I was interested in seeing which counting process students would use, if they would use a listing technique to find the solution, and if they could articulate a connection between their counting process and a list of outcomes generated.

3.2.3 The Book Problem

Of the 20 students interviewed, I asked 4 of them the Book Problem, which is stated as follows: "How many ways are there to arrange the letters in the word BOOK?" Because of the semi-structured nature of the interviews, I only asked students to solve this problem if prior in the interview, they had indicated (through, for example, solving the Cattle Problem) that they thought 4! was the number of ways to arrange 4 objects, regardless if the objects were distinct or not. The few students who solved the Book Problem did so after I saw their problem-solving process on the Cattle Problem. None of the 4 students were explicitly asked to create a list of outcomes to solve the problem, but 3 of the 4 students nevertheless proceeded to make complete lists of outcomes for this problem. It is possible that these 3 lists were made, because the Book Problem was only asked after a student had already solved the Cattle Problem, and each student in solving the Cattle Problem was asked to write out a complete list of outcomes beginning with the letters TT. Thus, listing was already an activity that the students may have been thinking about when given the Book Problem, and therefore studying student solutions to the Book Problem provided for me another opportunity to see how students connect lists of outcomes to counting processes, and how this connection could be used to find a solution.

One solution to the problem involves recognizing that if the two O's are distinct (say, if the letters were B, O_1, O_2, and K), then there are 4! ways to arrange the four letters. However, since the two O's are not distinct, each arrangement of the four distinct letters has a "twin" that should not be counted toward the enumeration of arrangements of B, O, O, and K. (For instance, the outcomes BO_1O_2K and BO_2O_1K should be considered identical when solving the Book Problem.) Since the arrangements of B, O_1, O_2, and K can be partitioned into these equivalence classes of size 2, we find that the solution is $4!/2 = 12$.

The above counting process involves organizing the arrangements of B, O_1, O_2, and K, grouping the similar outcomes together, and counting each group of similar outcomes as 1 in the total enumeration of the arrangements of the letters in the word BOOK. Figure 2 illustrates how this counting process organizes the set of arrangements in stages.

BO_1O_2K	O_1BO_2K	O_2BO_1K	KBO_1O_2
BO_1KO_2	O_1BKO_2	O_2BKO_1	KBO_2O_1
BO_2O_1K	O_1O_2BK	O_2O_1BK	KO_1BO_2
BO_2KO_1	O_1O_2KB	O_2O_1KB	KO_1O_2B
BKO_1O_2	O_1KBO_2	O_2KBO_1	KO_2BO_1
BKO_2O_1	O_1KO_2B	O_2KO_1B	KO_2O_1B

BO_1O_2K BO_2O_1K	O_1BO_2K O_2BO_1K	O_1O_2KB O_2O_1KB	KBO_1O_2 KBO_2O_1
BO_1KO_2 BO_2KO_1	O_1BKO_2 O_2BKO_1	O_1KBO_2 O_2KBO_1	KO_1BO_2 KO_2BO_1
BKO_1O_2 BKO_2O_1	O_1O_2BK O_2O_1BK	O_1KO_2B O_2KO_1B	KO_1O_2B KO_2O_1B

BOOK	OBOK	OOKB	KBOO
BOKO	OBKO	OKBO	KOBO
BKOO	OOBK	OKOB	KOOB

Figure 2. Organization of the outcomes in the word BOOK.

Because of the potential for using outcomes and organization by equivalence to find a counting process, I used this problem to gain further insight into student reasoning about the relationship between sets of outcomes and counting processes.

3.3 Data Analysis

After conducting and recording the interviews, I listened to the interviews and watched the files outputted by the Livescribe pens. Content logs of each interview were created, providing a detailed, time-stamped description of what happened. Portions of interviews that were particularly relevant to understanding student reasoning were transcribed. For the counting problems that the students solved, I coded student responses and looked for particular factors of interest for each problem. I also recorded which problems each of the students solved by writing a full list of outcomes, writing a partial list of outcomes, or using another method not involving even a partial list of outcomes. For the coding, I discussed the categories that emerged with my academic adviser, and we worked together to clarify portions of students' problem-solving processes that were difficult to code.

For analysis of a counting problem on which students listed, I reported students' answers to the problem before they were asked list the outcomes, and their answers after listing the outcomes (if their answers changed). I also recorded and coded the students' justifications for their answers. I additionally documented a description of each student's list, and I coded the extent to which the students could identify a structure within their list of outcomes. Using this reported data, I analyzed the students' solutions to see which components of Lockwood's (2013) model students used to solve the problem and justify their answers.

Finally, for each list that every student made, I wrote a description of the strategy that the student used to generate the list of outcomes. In this description, I included details such as the order in which students wrote outcomes and utterances that students made indicating how they thought about their list generation. For each list, I compared my description to English's (1991) six hierarchical solution strategies and categorized the list based on which of English's categories best described the student's process for generating the list. When there were subtleties in a student's listing strategy preventing me from clearly using one of English's solution strategies to characterize the student's behavior, I made note of these subtleties explaining with evidence why they could not be categorized using one of English's solution strategies. Later, I returned to the subtleties I noted and looked for emergent themes in the ways undergraduate students listed that was not clearly characterized by one of English's solution strategies. These themes were discussed with my adviser to verify consistency in the coding of the undergraduate students' listing strategies and confirm correct interpretation of English's solution strategies.

Chapter 4: Results

In this chapter, I present the findings of this study as they addressed the research questions outlined in Chapter 2. I will present the frequencies of each listing strategy used by the undergraduate students by utilizing English's six hierarchical categories (1991), as well as discuss a particular application of her six solution strategies that emerged in the data. I will draw also on Halani's student ways of thinking about combinatorics solution sets, including emergence of her conjectured "Generalized Odometer Thinking" (Halani, 2012). Next, I will present evidence that emerged supporting Lockwood and Gibson's (in press) findings that listing is beneficial for students' productive solving of counting problems. Finally, I examine student lists and their connections to counting processes to see how students might think of the relationship between these critical components of Lockwood's (2013) model.

4.1 Undergraduate students' listing strategies

I begin by describing in detail the strategies that the undergraduate students used while listing outcomes. This will allow me in later sections to capture subtleties in a student's listing behavior and connect that to the counting process they used to solve a given counting problem. The listing strategies I use are a further exploration of the categories developed by English (1991).

In total, there were 31 complete or partial lists that the undergraduate students made in the interviews while solving the Cattle Problem, the Horse Race Problem, and the Book Problem. After collection of the interview data, I used English's (1991) six hierarchical solution strategies to categorize undergraduate students' listing strategies. It was discovered that her strategies, while originally created to characterize the combinatorial solution strategies of young children, were effective in characterizing the listing strategies of undergraduate students as well. A summary of the frequencies of each listing strategy used is presented in Table 6. Specifically, Table 6 is broken into two sections – one reporting on Single Solution Strategies (in which students used only one strategy in listing outcomes), and one reporting Multiple Solution Strategies (in which students employed more than one strategy on a particular list of outcomes). In sections 4.1.1 and 4.1.2 I highlight some key findings about each of these strategies.

4.1.1 Students' Use of a Single Solution Strategy: Prevalence of Complete Odometer Use in Generating Outcomes

I begin by discussing students' use of a single solution strategy. As can be observed in Table 6, undergraduate students used Solution Strategy F most frequently, with 19 of the 31 complete or partial lists used this strategy. A student list was coded as being generated via Solution Strategy F if there was consistent, clear use of holding an item constant as the student allowed the other items in the outcomes to vary. In order for a list to be categorized as being generated via Solution Strategy F, I examined the utterances the students made, along with their real-time listing activity, to see if they thought about their list generation as coming from holding a pivotal item constant.

Single Solution Strategies:			**Multiple Solution Strategies:**		**Total Lists Made:**
Total Single Strategies:	22		Total Multiple Strategies:	9	31
A	*	Switching Strategies:	E then D	2	
B	1		C then B	1	
C	1	Simultaneous Strategies:	F and B	4	
D	1		E and D	1	
E	*	Switching and Simultaneous Strategies:	B then (E and B)	1	
F	19				

Table 6. Listing Strategies Students Used.

For example, 13 of the 20 participants use Solution Strategy F to create a list of outcomes for the Cattle Problem. A nice example of this was the list created by Student 17. His list of outcomes generated for the Cattle Problem are shown in Figure 3. Examining his list, we see his list can be divided into four subsets, each of cardinality six, such that every outcome in each subset begins with the letters C, A, L, and E, respectively. Within each subset, the first two outcomes listed are those with a particular fixed second letter, with the last two letters interchanged. For instance, in the subset of outcomes beginning with C, we see that Student 17 first listed the two outcomes that had A as the second letter, then the outcomes with L and E as the second letter. This clear organization of the list of outcomes suggests a methodical process of generating the outcomes via holding a first letter, then second letter constant, while varying the last two. Then changing the second letter, varying the new last two letters, and so on, until all outcomes with a particular fixed first letter are created. Then this process begins again with a new first letter.

Indeed, when I asked Student 17 how he generated his outcomes, he replied as follows:

Figure 3. Student 17's list of outcomes for the Cattle Problem.

Student 17: "Ok, so I started with the first letter. I made the first letter the same for all the times I could. And then, I made the second letter the same for as many times as I could, and then I chose the third and fourth letter. Then I switched the third and fourth letter. And then, for the second letter, then I switched it with either the third or the fourth letter. And, um, and then the second letter became my new third letter. And, so that third and fourth letter I switched them again. And then I used

> the, the uh, whichever one of the second, third, or fourth letters I didn't use, and I made that the second letter, and then the third and fourth letter I chose from the remaining ones. And then I switched them. And then, I did that same process with a new first letter."

Looking at his response, we see that he described in detail a strategy composed of systematically holding different items constant in the first and second positions while cyclically varying the other items. This indicates that he was in fact generating his list of outcomes using a complete odometer strategy (English, 1991, p. 458).

Other examples of undergraduate students utilizing Solution Strategy F were seen in analysis of students' complete lists for the Book Problem. For example, Student 4 listed the outcomes in the word BOOK by systematically holding a first and second letter, varying the two remaining letters, then changing the letters being held first and second in a manner characteristic of the odometer strategy (English, 1991, p. 458). He did this carefully to ensure that he did not over count any outcomes, and arrived at the list shown in Figure 4.

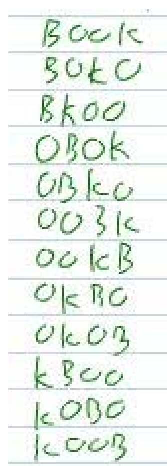

Figure 4. Student 4's list of outcomes for the Book Problem.

Finally, Solution Strategy F was also seen as students solved the Horse Race Problem. While no students made a complete while solving this problem, some partial lists that students created were still categorized using English's sixth and highest hierarchical solution strategy. Reasons for this were two-fold. First, the cardinality of the set of outcomes for the Horse Races problem (720) is too large for a student to feasibly write out every outcome. Second, I coded student lists as being generated by the complete odometer strategy based on student *ability* to list by holding a pivotal item constant while systematically varying the other items. A set of outcomes was coded as being created using Solution Strategy E if based on student utterances, if it was indicated that the student *could not* list all of the remaining outcomes. This consideration of whether a student indicated if they could meaningfully extrapolate a partial list of outcomes is an elaboration of English's (1991) categories, because for her an incomplete list was indication of a child being unable to carry a solution strategy to its conclusion. However, for undergraduate students solving more complex counting problems with large solution sets, the production of only partial list was not enough to say whether a student used Solution Strategy E or F. There was a need instead to look at what students said to decide if they could extrapolate their list (suggesting Solution Strategy F) or were not sure how to do this (suggesting Solution Strategy E). This distinction and subsequent further exploration of English's work is explored more fully in Section 5.2.1.

For now, I focus on illustrating how this distinction was used in categorizing undergraduate students' listing strategies. To do this, I present Student 12's partial list for the Horse Race Problem. As she reasoned through the problem, she coded the race outcomes as 3-letter sequences of the letters A, B, C, D, E, F, G, H, I, and J. Thus, the outcomes ABC indicated an outcome of the race in which horse A placed first, horse B placed second, and horse C placed third. She then listed all of the outcomes beginning AB, cycling systematically through all of the possibilities in which horses A and B placed first and second. Her list of outcomes can be seen in Figure 5.

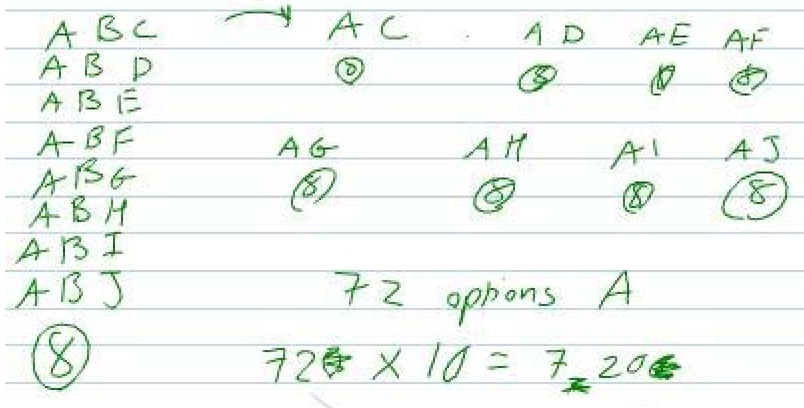

Figure 5. Student 12's list of outcomes for the Horse Race Problem.

After she listed all of the options beginning with AB, she described a clear extrapolation of her process to continue enumerating all of the possible outcomes for the Horse Race Problem. Her explanation is quoted as follows.

> Student 12: "So that's 8 possibilities, and then I could do AC, which will have 8, AD will have 8, AE will have 8, AF 8, AG 8, AH 8, AI 8, AJ will have 8. Ok. So then with A being first, I'm gonna have 1, 2, 3, 4, 5, 6, 7, 8, 9--72 options with A being the leader. And then I think they all get their chance—1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Yeah, so if they all get their chance to be first place, I think there should be that many [writes 72*10=720]."

Because she was able to clearly articulate a process by which she could continue listing all of the outcomes, her list was coded as being generated by use of the complete odometer strategy. Even though she did not write every outcome, it was clear to me that she could have if she chose to.

While Solution Strategy F was the most prevalent among the listing strategies used by undergraduate students in our interviews, there were a handful of lists that were created using another strategy that could be characterized by one of English's other solution strategies. For example, Student 15 created two complete lists of outcomes during her solving of the Cattle Problem, one list was generated using Solution Strategy D, and the second by an "F and B" strategy. (The latter will be explained in section 4.1.2.2.) For her first list, Student 15 wrote CALE as her first outcome, and then "shifted" the position of each letter in the outcome, putting the E first, the C second, the A third, and the L fourth. Thus, she then had the arrangement ECAL. She repeated this action again to obtain LECA, and finally ALEC. Exhausting the outcomes she could write using this shifting action, she began the process again with a new outcome, CLAE, and once again repeated this action to obtain three more outcomes. Student 15's list of outcomes can be seen in Figure 6.

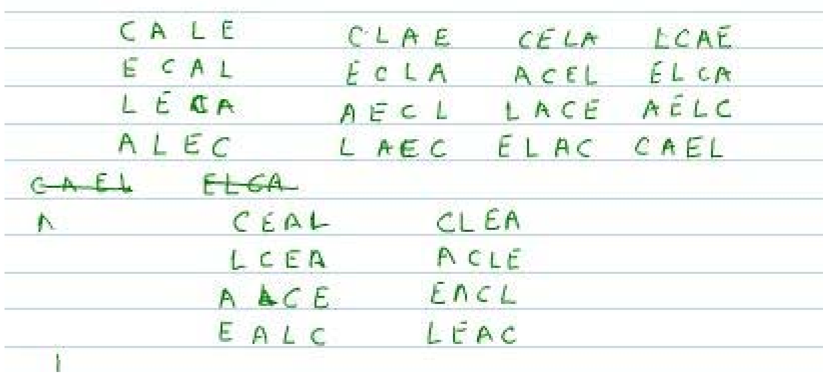

Figure 6. Student 15's first list of outcomes for the Cattle Problem.

This strategy was interesting, in that it was systematic and cyclical. However, it was based not on a fixed item in a particular position, but on a fixed relational ordering of the letters C, A, L, and E. Since an item was not being held constant as other items were systematically varied, I did not categorize the strategy as a complete odometer strategy. However, a clear, systematic pattern is being utilized to generate the outcomes. In fact, Student 15's strategy uses a concept that is much like problems asking the number of ways n people can sit in chairs around a circle. There are $(n-1)!$ ways for this to occur, because rather than counting each chair as being distinct, outcomes are distinguished via different relational orderings (where the only thing that matters is who sits next to whom, not exactly where each person sits). Likewise, there are 3! relational orderings of the letters C, A, L, E (or, there are 6 ways to place the letters C, A, L, and E in a circle), and so we obtain a solution of $3! \times 4 = 24$ for the number of arrangements in the Cattle Problem beginning with TT. While this list of outcomes was not generated using the odometer strategy, the cyclical, systematic pattern was clearly productive in generating outcomes to solve the Cattle Problem.

In short, when it came to students' clear use of one of English's (1991) solution strategies, Solution Strategy F, the complete odometer strategy, was the most prevalent. Of the 31 lists made by students in the interviews, 19 of them came from utilization of

Solution Strategy F. This is not a surprising result, because such a strategy fits in naturally with arrangement problems, and it is consistent with English's findings. The ages of children in her study ranged from 4-9 years old, and English found that the 8 and 9-year-old children were more likely to use the more sophisticated solution strategies, D and F (p. 464). Likewise, in our study with undergraduate students, all over the age of 18, we found that the majority of them used the most sophisticated listing strategy according to English's hierarchical categories. In addition, it is possible since we explicitly asked almost every student interviewed to write a list of outcomes for the Cattle Problem, it is possible that this prompted the undergraduate students to make an organized list when they otherwise might not have.

When just examining these results, it may seem that the undergraduate students all naturally and easily employed Solution Strategy F. However, I additionally point out that while many students used the complete odometer strategy to write a list of outcomes while solving a counting problem, other students utilized less sophisticated listing strategies, but this emerged when they used multiple listing strategies. As can be seen in Table 6, there were students who used some of English's less sophisticated strategies, such as Solution Strategy B. In the following section, I examine other less sophisticated strategies that undergraduate students utilized to list outcomes. As will be seen, these strategies often involve random listing that is combined in some way to another more sophisticated strategy.

4.1.2 Students' Use of Multiple Solution Strategies

Some children in English's (1991) study exhibited a listing behavior of switching between different strategies. She found that, for example, a child solving a basic counting problem with solution set having a cardinality of 9 might begin by employing a cyclical pattern to generate the first 6 outcomes, but then lose the pattern and generate the remaining 3 outcomes using a trial-and-error approach of creating new outcomes at random (p. 462). In my interviews with undergraduate students, I found some similar instances of students switching between different listing strategies, but I also encountered other ways in which students used multiple strategies. These additional ways that student incorporated multiple solution strategies sheds some light on (undergraduate) students' listing that was not addressed by English. In 4.1.2.1, I give examples that align with what English found, and in 4.1.2.2 I give examples from the data indicating a need to expand English's categories for students' combinatorial solution strategies.

4.1.2.1 Switching Between Strategies to Make a List of Outcomes. Through analysis of the interviews, there emerged a categorization of the students' lists that I denote *Multiple Strategies*. These consisted of two types: *switching strategies* and *simultaneous strategies*. A student that was listing via a switching strategy would list in a way that was characteristic of one of English's (1991) categories, but then change to a different solution strategy partway through creating a complete or partial list of outcomes. For example, a student who started listing using an odometer strategy of holding an item constant and cycling through the other items systematically, but then changed to a different strategy to complete the list, say using a cyclical pattern that is not holding an item constant while varying the others systematically, was coded "E then D." The E is used to indicate that the odometer strategy was used, but not completed throughout the listing task.

Throughout the interviews, there were only four instances where students switched their listing strategy partway through creating a list of outcome. As a clarification, if a student wrote down a partial list, then started a new list using a different strategy, this was *not* coded as a switching strategy. While the student is clearly changing his or her strategy, the significance of switching strategy according to my coding of the outcomes was changing solution strategy *within the generation of one list of outcomes*. Thus, if a student began a partial list by writing down random outcomes, and then started a new list using the complete odometer strategy, I would say this activity consisted of two separate lists, and the first would be coded as being generated by Solution Strategy B, and the second by Solution Strategy F.

Of students who switched between outcomes part-way through the generation of one list of outcomes, the strategy "E then D" was used twice, "C then B" was used once, and there was one instance of a student using a "B then (E and B)" strategy. This last strategy will be explained in more detail in section 4.1.2.2.

To illustrate, one student who used a strategy coded "E then D" was Student 7 when he wrote out a partial list of outcomes for the Cattle Problem. His list is shown in Figure 7.

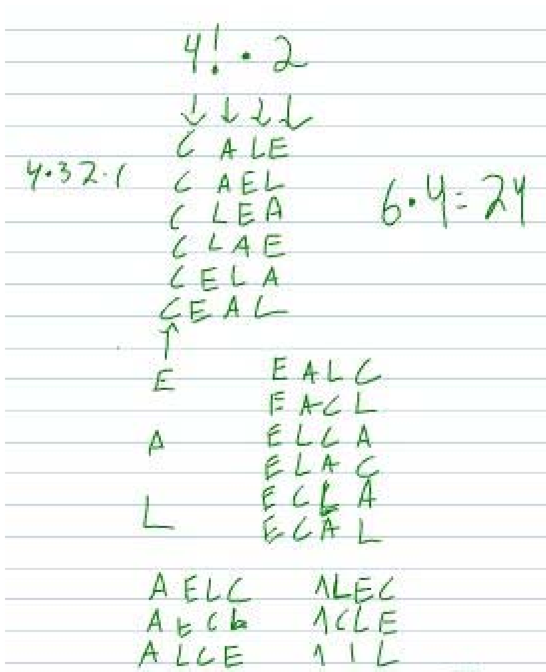

Figure 7. Student 7's partial list of outcomes for the Cattle Problem.

By looking at Student 7's list of outcomes, it appears that the list was generated by a use of the complete odometer strategy, because it can be seen in his list a pattern of a pivotal first letter, followed by a pivotal second letter. However, listening to Student 7 give a description of how he listed his outcomes, we see that is use of the odometer strategy is incomplete. His description is as follows.

> Student 7: "So, I'm pretty much, in order to do it more efficiently, I just kinda put all the same letter in the first [column]…and then I—that one I worked through slower, but then, with [the outcomes beginning with E], I just switched in C for E in all these. So, it's going to be the exact same here, but I just switched in C. And here I just switched in E for A, and so all the places where A was in here, E is now."

Seeing Student 7's description of his listing strategy, we observe that he did not think of his list generation as holding C, A, L, then E constant while systematically varying the other letters. Instead, he held C constant while systematically varying the other letters, but then followed a cyclical replacement pattern of swapping the C's and E's to create the outcomes beginning with E, and then followed the cyclical replacement pattern again to generate the outcomes beginning with A. While producing the same list as a list created with the complete odometer strategy, Student 7's way of thinking about his listing strategy was different enough that I felt I could not categorize it neatly as Solution Strategy F. Rather, there was an emergence of the odometer strategy in his generation of the outcomes beginning with C, and then a cyclical pattern that could be used to generate all of the outcomes. This cyclical pattern, however, cannot be seen as easily as holding an item constant while systematically varying the other letters. Rather, it is the six positions of letters in the outcomes beginning with C that is held constant, and the letters which fill those positions are cyclically changed. Thus, we see that Student 7 only showed an emergence of an odometer strategy, but then carried out Solution Strategy D to finish his partial list of outcomes.

4.1.2.2 Emergence of Solution Strategies Used Simultaneously. In addition to the switching strategies that occurred in the interviews, there were also instances of other solution strategies that students used that seemed to encompass more than one of English's (1991) categories. They neither fit neatly into one of her categories, nor could these strategies be characterized by switching between solution strategies. Instead, there was an emergence in the data of students using two of English's strategies simultaneously to generate a list of outcomes, and for this reason I categorized these as *simultaneous strategies*. Of the 31 total lists that students made, 6 of them were generated via use of two listing strategies simultaneously.

The most frequently seen simultaneous listing strategies were Solution Strategy F and B, which were coded as simply "F and B." This occurred when the student generated outcomes using a strategy that simultaneously both odometer and random-selection components. For example, Student 15 used Solution Strategy D to generate a complete list of outcomes for the Cattle Problem, and then created a new list using an "F and B" solution strategy. (The description of her first list of outcomes is explained in section 4.1.1.) To make her second list, she repeatedly held a fixed letter constant to be first, but then did not vary the other letters in a systematic way. Rather, they followed no clear pattern and appeared to have been varied randomly. Student 15's second list of outcomes for the Cattle Problem can be seen in Figure 8.

Listing as a Potential Connection between Sets of Outcomes and Counting Processes

Figure 8. Student 15’s second list of outcomes for the Cattle Problem.

To see if she was indeed varying the other items randomly, I asked her why she thought there might be six outcomes in each group with a fixed first letter. She responded as follows.

> Student 15: “Well, there’s three letters that don’t depend on the first letter. Um, but, I can’t think of, I don’t know. I can’t think of why there’s six. One thing I was noticing was, um, like, for these two, they both start with L and A, and then it’s just either EC or CE. So, yeah, maybe that’s it.”

Because she described observing two outcomes having a fixed first and second letter after she wrote those outcomes down, this suggests that she was not intentionally arranging the other items with a fixed first letter by next fixing a second letter. Thus, although she was using a pivotal first item to list outcomes, I could not classify her listing strategy as a complete odometer strategy due to the random component of her strategy.

Another example of two listing strategies being used simultaneously was in Student 18’s first partial list of outcomes for the Cattle Problem. This was the only instance in which a student used both switching *and* simultaneous solution strategies, which was coded “B then (E and B).” The resulting list can be seen in Figure 9.

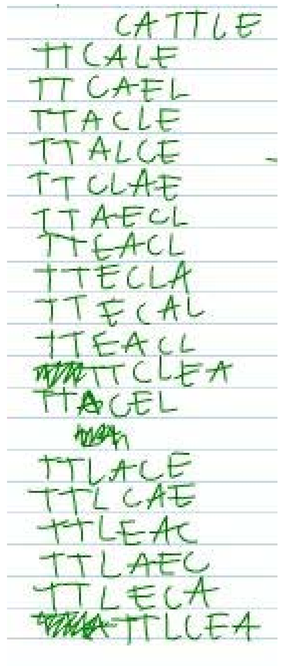

Figure 9. Student 18’s first partial list of outcomes for the Cattle Problem.

To begin her list, Student 18 wrote down 12 outcomes beginning with TT. To obtain those 12 outcomes, she said, “I’m just kinda mixing them around, I guess.” This utterance also suggests a lack of a systematic pattern, and this partial list was then categorized as being generated by Solution Strategy B. Then, however, after I asked her if she thought that she had listed all of the outcomes, she said no, and that she thought there may be more outcomes that she had not thought of. After saying this, I asked her if there were any other outcomes she could think of, and she wrote down five outcomes beginning with TTL: TTLACE, TTLCAE, TTLEAC, TTLAEC, and TTLECA. She explained that she realized she had missed outcomes beginning with TTL, and she then guessed that there were a total of seven outcomes beginning with TTL. She finally obtained the last outcome beginning with TTL by a trial-and-error thought process, and then realized that there was not another outcome that she could write down.

The above thought process, as well as the physical list itself, both suggest that while she was holding the L constant as the first letter in the list of outcomes beginning with TT, she did not vary the other items in a systematic manner. She also wrote down no more outcomes in this particular list, nor articulated a way to extrapolate her process to extend her partial list to a complete list. For this reason, the strategy she used for her last six outcomes was coded "E and B." In both Students 15 and 12, then, we gain some insight into listing by those students who did not immediately or naturally adopt a complete odometer strategy F. In particular, even as undergraduates these students were not always systematic and at times did not have mechanisms by which to keep track of their outcomes.

In conclusion, we find that English's (1991) strategies were effective in coding many of the strategies that undergraduate students used to write down lists of outcomes. However, there were also lists that students created which could not clearly be categorized using only one of English's six hierarchical solution strategies. Instead, there was a need to describe the listing strategy used with multiple solution strategies. These instances included examples of students switching between English's listing strategies, as well as utilizing strategies that simultaneously contained components characterized by different solution strategies. While the former was observed in English's studied with young children, I conjecture that the emergence of simultaneous solution strategies occurred because the problems I asked undergraduate students in our interviews were more complex than the tasks that English (1991) gave young children. In English's study, the tasks asked children involved only combinations of two items selected from discrete sets of items (p. 454). In studying the listing behavior of undergraduate students solving more complex counting problems, then, it will likely be beneficial to be attuned to simultaneous listing strategies.

4.2 Effects of Listing on Correctness of Counting Problem Solutions

In addition to looking closely at the lists of outcomes that students made and the strategies they used to generate those lists, I was also interested in seeing how listing affected the correctness of the students' answers to the counting problems given to them in the interviews. While there are numerous sources that document the difficulties that students have solving counting problems (Batanero et al., 1997; Eizenberg & Zaslavsky, 2004), there is some evidence in the literature that listing may be a productive activity for students to engage in while solving counting problems (Lockwood & Gibson, in press). Examining the interviews I conducted, I see some evidence that listing is in fact a worthwhile activity for obtaining correct solutions to counting problems.

4.2.1 Listing as a Way to Detect Counting Errors: Solutions for the Cattle Problem Before and After Listing

I focus this subsection of my results on solutions to the Cattle Problem. This is the only problem in which that I asked each student to write a complete list of outcomes, and thus allows us to study student answer to the problem before and after writing a complete list. First, Table 7 reports on the answers each student gave prior to listing all of the outcomes and after listing all of the outcomes. Student 11 did not answer the CATTLE problem and Student 16 did not give an answer prior to listing.

Student Answers to the CATTLE Problem Prior to Listing		
Type of Answer:	**Answer:**	**Frequency:**
Correct	2×4!	9
	4×6×2	1
	48	1
Incorrect	4×4	2
	(2×1×4×3×2×1) ×2	1
	16×2	1
	24	1
	(4+3+2+1+1+1) ×2	1
	2×4×9	1
None	*	1
Student Answers to the CATTLE Problem After Listing		
Type of Answer:	**Answer:**	**Frequency:**
Correct	2×4!	9
	4×6×2	6
	2×24	2
	2×12×2	1
	(4×3×2×1) ×2	1

Table 7. Summary of student responses to the CATTLE problem.

Looking at the students' responses to the Cattle Problem, we see that initially, only 11 out of the 19 students who solve the Cattle Problem came to a correct answer before making a list. However, after writing down a complete or partial list, every student arrived at a correct answer.

One example of a student who initially answered the Cattle Problem incorrectly was Student 13. Before writing a list of outcomes, his solution to the problem was (2×1×4×3×2×1) ×2. He explained,

> Student 13: "So, if the two T's are at the beginning, the first spot can be either of the T's, the second spot can be the other T, and then there's 4, 3, 2, 1. So, that would be if the T's are [at the beginning]. And then, you'd add that to if the T's are at the end. So it'd be 4, 3, 2, 1, and then either one of the T's."

Notice that this solution would be correct if the two T's are distinct. However, because the T's are identical, we see that Student 13's process over counts each arrangement once. Over counting is a common error in solving counting problems (Batanero et al., 1997; Halani, 2013), and it can often be a difficult error for students to detect. However, Student 13 was able to see his error after he wrote a complete list of outcomes beginning with TT using the complete odometer strategy. Once he finished his list, he remarked,

> Student 13: "Okay, so the reason I wrote, I got 48 here [from the original counting process], but only 24 [outcomes] here, is these T's are all the same, so TT—yeah, so I was saying, like, that the T's would be different. But, it's not….Well, [the number of outcomes beginning with TT] would be 24, and [the number of outcomes ending with TT] would be the same thing, so it would be 48."

Here, Student 13 articulates that after having made a list of outcomes, he was able to see that the two T's were actually identical, and he was able to arrive to the correct answer. Thus, this illustrates one way in which writing a concrete list of outcomes can be beneficial for students in correctly solving counting problems. In particular, since he had to decide exactly what an outcome of the Cattle Problem looked like in order to list, he was able to detect over counting in his counting process.

Another example of a student using listing to correct a wrong solution to the Cattle Problem was seen in Student 19's solution. Initially, Student 19 argued that the solution to the problem would be (4+3+2+1+1+1) ×2. His reasoning is given as follows.

> Student 19: "Basically, when the two T's are at the front or back, there are only four other slots that create variation. So, it's like, those four other slots, there are four possibilities for, which you could equate to like a 4 factorial, because, you know, there are four possibilities for the first slot, three for the second and so on. So, 4 factorial plus there's one possibility for each location of the T's."

Following his line of reasoning, we see some understanding of the idea of a decreased number of options for each fixed position in an outcome, and the use of the 1's indicates some understanding that the two T's are identical. However, we also see use of an incorrect operation for expressing the number of outcomes for a task divided into distinct stages (in particular, Student 19 used addition rather than multiplication).

After Student 19 gave his solution, he expressed that he was unsure that his answer was correct, and he remarked that perhaps listing some outcomes might help him determine if he was on the right track. I encouraged him to do this, and like Student 13, he was able to see his error and utilize a correct counting process for solving the Cattle Problem. He listed the outcomes using the complete odometer strategy. After he listed, he observed first that the solution would be $6 \times 4 = 24$ total outcomes beginning with TT, because for each fixed first letter, there were six total outcomes. In addition, once he had written down a complete list of outcomes, he was able to articulate why there would be $4 \times 3 \times 2 \times 1$ total outcomes beginning with TT. This is explained in more detail in section 4.3.

In conclusion, the students' solutions to the Cattle Problem before and after writing down a complete or partial list supports the idea that systematic listing may be an effective way for students to catch errors in their counting processes. Eisenberg and Zaslavsky (2004) found that verifying combinatorial solutions to be a difficult task for students, and Lockwood (2013) proposed that students should learn to base their combinatorial arguments fundamentally in sets of outcomes (p. 258). These results corroborate with Lockwood's claims, suggesting the possible usefulness of more future studies on students' explicit use of outcomes to verify the correctness of counting processes.

4.2.2 Listing to Aid in Finding a Counting Process

Apart from using a list of outcomes to detect an error in a wrong solution to a counting problem, writing a list of outcomes can also be beneficial for helping students who are unsure how to go about solving a counting problem in coming up with a coherent counting process. Lockwood and Gibson's (2015) study provided some examples of students using a complete or partial list to come up with a solution to counting problems they appeared otherwise unable to solve, and in my interviews I was able to see an even clearer usefulness that listing seemed to have for stumped students as they verbalized their thought processes for me. This was seen both in instances where students made complete lists, and instances in which a partial list of outcomes was used.

One example where a student's complete list of outcomes led to a coherent, correct counting process was Student 20's solving of the Book Problem. When I first gave him the problem in my interview with him, he expressed that if the O's were distinct, he would know that the answer would be 4!. However, he observed that he would over count outcomes, because the positions of the two O's could in interchanged in any outcome to create an identical outcome. Therefore, he could see that the answer would have to be less than 4!, but he still was not sure how to answer the problem. After contemplating that the solution might be somehow related to 3!, he decided to try to write a complete list of outcomes. I encouraged him to do so, telling him he could take as much time as he needed to write out the outcomes. He then listed his outcomes using an "F and B" solution strategy, holding B, K, and O successively constant in the first position while randomly varying the other items for each fixed first letter. Student 20's complete list of outcomes for the Book Problem can be seen in Figure 10.

Figure 10. Student 20's complete list of outcomes for the Book Problem.

After Student 20 finished writing his list of outcomes, he observed that there were 12 total outcomes, and that $12 = 4!/2$. He said that this made sense, because if the two O's were different, there would be double the number of outcomes and you would have

4! total outcomes. I asked him why he would double the number of outcomes if the O's were different, and he said, "Because you can just switch them, essentially. In which case there would be 4 [distinct letters] and it would be 4!. But, because they're the same switched, it halves it. So, you would divide by 2." In this way, we see that after first being unsure how to solve the problem, creating a complete list helped Student 20 articulate a counting process of permuting 4 distinct letters, but then dividing by 2 to account for the fact that each outcome is equivalent to exactly one other outcome (namely, the outcome in which the two O's are swapped). This instance provides evidence that listing could help students solve counting problems involving division and aid in students articulate equivalence ways of thinking about outcomes (Halani, 2012).

In addition to seeing evidence that complete lists of outcomes may help students articulate coherent counting processes, I also encountered in the interviews instances where even a partial list was enough for a student to spot a pattern and use it to find a productive correct counting process. To illustrate, I provide Student 7's initial approach, partial list of outcomes, and subsequent solution for the Horse Race Problem. When I first gave Student 7 the problem, his initial reasoning was as follows.

> Student 7: "Mmm, I think it's the last seven—don't matter what order they're in….I'm thinking it might have to do with 'n choose r.' Would it be like 10 choose 3, or something like that? I might try going that direction with it, but, can't be too certain. Considering I don't even know the formula for it, I wouldn't know how to go about it….I don't know if that accounts for the order, so that's more of a blind guess than anything else."

These initial utterances suggest that he did not know how to approach the problem, and I asked him if he thought it might be useful to try writing out some outcomes. He agreed that might be helpful and said maybe if he wrote some outcomes, "something might spark for [him]." He began by writing the outcomes ABC and ABD, and said, "ooh! I'm starting to see a pattern! Here we go." He continued writing outcomes and utilized English's (1991) Solution Strategy F to write a partial list. Student 7's partial list of outcomes for the Horse Race problem may be seen in Figure 11.

Listing as a Potential Connection between Sets of Outcomes and Counting Processes

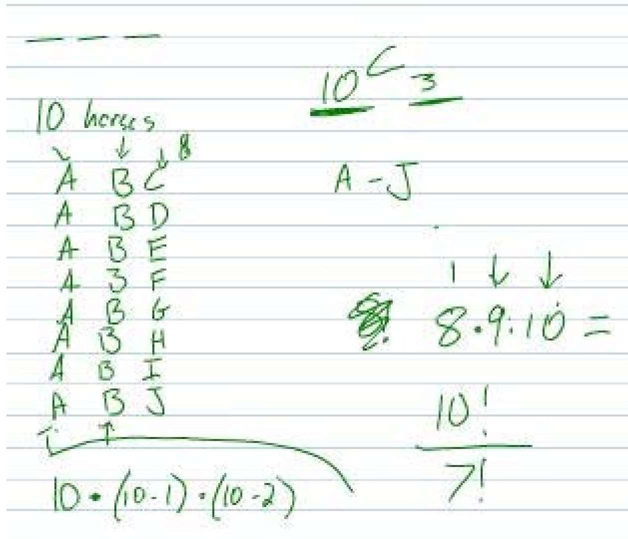

Figure 11. Student 7's partial list of outcomes for the Horse Race Problem.

After he finished his partial list he said that he thought the answer was $8 \times 9 \times 10$. I asked him then how he arrived at this answer, and his explanation is given as follows.

> Student 7: "I think it was when I started writing this out. I could be wrong, but this is where I was going with it. I was saying there were [8] options…in that [column], and then—this one there would be 9. There'd be 9: B through J. That's my guess. And then this one would be A through J, so there's 10 options. And so there's $8 \times 9 \times 10$."

While this response is hesitant, we still see that Student 7 was able to go from being entirely unsure how to solve the problem to arriving at a correct solution. The partial list of outcomes helped him to articulate which items could be used to fill each of the three positions in an outcome for the Horse Race Problem, leading him to a correct use of the Multiplication Principle to correctly solve the problem. This instance corroborates with Lockwood and Gibson's (in press) finding that even a partial list of outcomes can be beneficial for students arriving at productive counting processes for completing combinatorial tasks.

In conclusion, this study corroborates with existing literature on the usefulness of students making lists of outcomes as they carry out combinatorial tasks. In addition, since I was able to talk to the students in the interviews and ask about their reasoning, I was able to unpack some of the specific ways that listing appeared to help students solve counting problems. In my interviews with undergraduate students, I found evidence that listing is beneficial both to help detect errors and to help students find the way to a solution when they are stuck. These benefits occurred both when students wrote a complete list of outcomes, as well as in instances where only a partial list was produced.

4.3 Student Understanding of the Relationship between Lists of Outcomes and Counting Processes

The results of this study from sections 4.1 and 4.2 focus primarily on the sets of outcomes component of Lockwood's (2013) model of students' combinatorial reasoning. The former focused on how sets of outcomes written by undergraduate students could be usefully characterized, and the latter provided qualitative evidence that explicitly considering sets of outcomes may be beneficial to student success in carrying out combinatorial tasks. Lockwood and Gibson (in press) suggested the benefits of having more studies look closely into how listing affects student performance and thinking about counting problems (p. 34), and these results are a step toward greater understanding of this particular issue in mathematics education research.

In this final section of my results section, I turn my attention from the sets-of-outcomes component of Lockwood's (2013) model and turn to the crucial relationship in her model between sets of outcomes and counting processes. Lockwood (2013) contended that,

> [T]he link between counting processes and sets of outcomes can (and should be) a very flexible relationship, in which students fluidly move from one component to the other. If a student can easily coordinate a counting process and a set of outcomes, this affords them tractability in their counting. (p. 258)

In other words, to be a successful counter, Lockwood argued that it may not be enough for a student to list in an organized way, or even to use a list to conceive of a counting process. She believes that students may have the most success in solving counting problems and having a solid mathematical understanding of combinatorics when a clear relationship between sets of outcomes and counting processes can be articulated. In Chapter 2 of this thesis, I outlined how within a given counting problem, different organizational schemes of the set of outcomes can give rise to counting processes, and conversely any given counting process may be thought of as a way of imposing a certain structure on the set of outcomes for the counting problem. While there have been few studies explicitly addressing how this relationship affects robustness of student understanding about solutions to counting problems, other studies have demonstrated some of the consequences resulting from students failing to use this relationship (Lockwood et al., 2015).

Because of the importance of the relationship between sets of outcomes and counting processes, and the fact that relatively little literature that explicitly addresses this relationship exists, I was motivated to investigate student understanding of this relationship in our interviews with undergraduates. Earlier in this thesis, we have seen already some evidence of the usefulness of this relationship. As I discussed in section 4.2.1, comparing a counting process with a particular list of outcomes can be an effective way to verify the correctness of a counting process. The remainder of Chapter 4 will be devoted to sharing my results about how students articulated connections between sets of outcomes and counting processes. As we will see, the connectedness between these two critical components of Lockwood's model varied considerably, even within a fixed combinatorial task. To allow for clearer contrasts between different student articulations of this relationship, I will narrow my focus to student reasoning about the Cattle Problem. Therefore, my discussion will be in the context of counting arrangements of 4 distinct objects with a multiplicative counting process.

4.3.1 Strong Student Connections between Lists of Outcomes and Counting Processes: Evidence for Students Engaging in Halani's (2012) Conjectured Generalized Odometer Way of Thinking

Throughout the interviews, many students were able to explain clear relationships between their lists of outcomes for the Cattle Problem and the counting processes they used to solve it. One particularly nice example was Student 1's connection between his list, which was generated using Solution Strategy F, and his counting process for solving the Cattle Problem. When I first gave him the problem, Student 1 explained that his solution to the Cattle Problem would be expression $2 \times 4!$, and he corresponded this expression to the following counting process: he first considered the placement of the two T's, and then for each placement of the T's, there are 4 distinct letters to arrange. So, there are 4! possible way to arrange those four letters once the T's are fixed.

I then asked him to write out all of the outcomes for the Cattle Problem beginning with TT. In describing his strategy for listing the outcomes beginning with TT for the Cattle Problem, Student 1 said that he produced his list in the following way.

> Student 1: "First you set the first letter, and don't think about them. Don't think about it. And, you put the second letter and don't think about it. It's just, you have two [possibilities], EL and LE here, in this case AL and LA, and so—and, yeah."

Here, we see again a process in which a student is fixing a pivotal first letter, and then successively holding each possible second letter constant to interchange the last two letters, and in this way systematically vary the items in a way characteristic of the odometer strategy.

Finally, I asked Student 1 if he could see a structure of 4! in his set of outcomes beginning with TT. He answered,

> Student 1: "There are four groups [drawing a circle around each subset of outcomes with a fixed first letter], and in each group there are three chunks [drawing a circle around the pairs of outcomes beginning with CA, CE, and CL], and in each chunk, there are two twins. And, in each word, there's only one."

Here, we see a clear connection between the Multiplication-Principle argument that Student 1 articulated in his counting process and the corresponding organization of the set of outcomes that this particular counting process affords. He showed not only a robust fluidity in his movement between sets of outcomes and counting processes, but also demonstrated an ability to engage in Generalized Odometer thinking (Halani, 2012), that is, holding an array of items constant (rather than just a single item, as was seen in the Solution Strategy "F and B"), systematically varying the remaining items constant, and relating it to a Multiplication-Principle counting process. In Halani's (2012) study, she said that her ideas about the Generalized Odometer Strategy were not rooted in her own empirical data, but were instead one of her own ways of thinking about the solution sets of many combinatorics problems (p. 1-241). Here in this study, we see evidence of students being able to engage in Generalized Odometer way of thinking as well.

Another interesting example of a student who clearly articulated the relationship between an organization of the set of outcomes and a counting process for solving the Cattle Problem was Student 6. Student 6 also arrived at a solution of $2 \times 4!$, but had a different organization of his set of outcomes, and thus also had a different counting process than Student 1. Student 6, like Student 1, also generated his outcomes using Solution Strategy F, but his strategy included first choosing not a fixed first letter, but a fixed position for the letter C. Thus, his first six outcomes that he wrote had C in the first position, and the next six had C in the second position, and so on. Interestingly, he did not follow a similar pattern for placing the other letters, but organized them according to the more standard odometer strategy of successively choosing a fixed letter to fill unoccupied positions. Student 6's complete list of outcomes for the Cattle Problem can be seen in Figure 12.

Listing as a Potential Connection between Sets of Outcomes and Counting Processes

Figure 12. Student 6's complete list of outcomes for the Cattle Problem.

I also asked Student 6 if he could see 4! reflected in the structure of his list of outcomes, and he replied, "Sure. Well, I mean a little bit abstractly. [Each group of six outcomes in my list] is 3 factorial, and like I said, I just copied it 4 times. I multiplied it by four, and I just deviated [the C] one position to the right each time." I asked him what he meant by each group being 3!, and he replied that he said that in each group where the position of the C was fixed, the arrangement of three distinct letters are identical. In this situation, Student 6 had condensed his understanding of 3! to simply be the arrangements of 3 distinct objects, in this case the letters A, L, and E, and systematically varied the position of C in relation to those letters. This is a fascinating application of the Multiplication Principle and also provides evidence of a student engaging in a Generalized Odometer way of thinking (Halani, 2012). In this particular example, Student 6 was holding the arrangements of 3 distinct items constant, and systematically varying the position of the last item in each outcome.

In summary, I saw in the interviews multiple instances of students using the sophisticated arguments to connect their lists of outcomes to multiplicative counting processes for the Cattle Problem. Even within the scope of examining one problem, students were able to come up with very different ways of structuring outcomes, and in turn very different ways of connecting those outcomes to counting processes. In addition, the emergence of students engaging in Generalized Odometer ways of thinking (Halani, 2012) is an interesting addition to the current body of literature in combinatorics education and may help motivate future studies on student reasoning about this way of thinking.

4.3.2 Student Difficulties Connecting Lists of Outcomes with Counting Processes

In this final section, I examine evidence in my data showing that, while several students excelled at articulating the relationship between sets of outcomes and counting processes, other students struggled to see this relationship. As a result, these students had a less robust understanding of the counting problems they were solving.

One example of an incomplete connection a student made between a counting process and an organization of outcomes for the Cattle Problem was in Student 2's reasoning. When I initially gave her the problem to solve, she explained, "So if I put the T's in the beginning, now I have 4 letters left, so I have 4 spaces. So we're gonna fix the T's, and now there are 4! ways to arrange the other three letters." Like other students I asked to solve the Cattle Problem, I then asked Student 2 to list the outcomes beginning with the two T's, which she did. Like many of the other students, she used Solution Strategy F to write out all of the outcomes. Student 2's complete list of outcomes for the Cattle Problem can be seen in Figure 13.

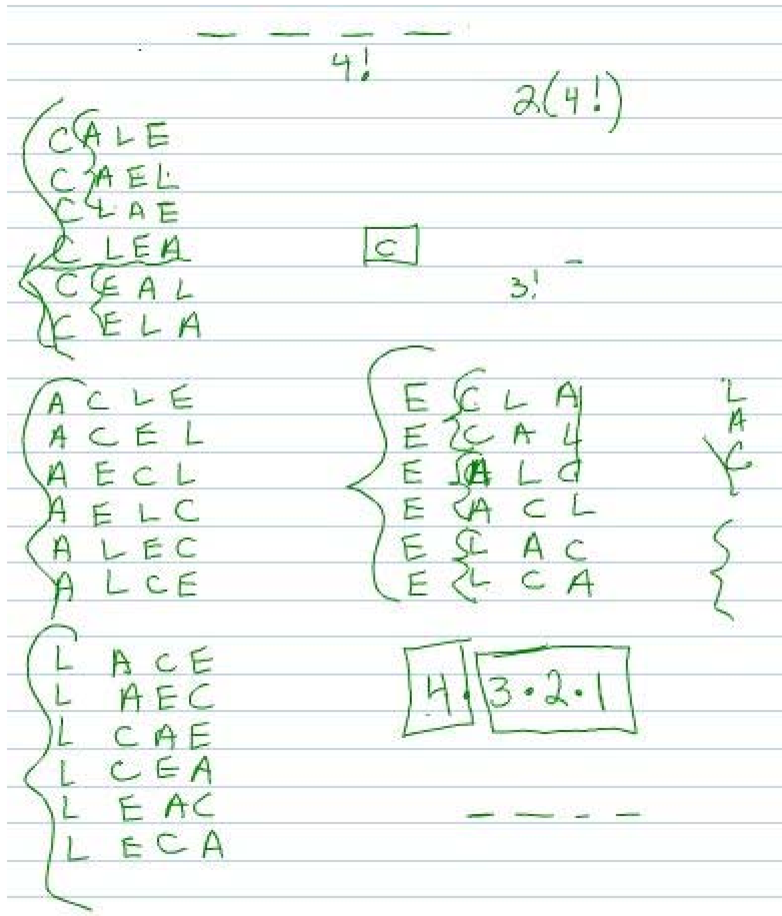

Figure 13. Student 2's complete list of outcomes for the Cattle Problem.

After she completed her list, I asked her if she if she could see 4! in the structure of her set of outcomes. She first answered, "I guess the structure is that I have four sets, and each set has six things in it." I asked if within each set if there was a $3 \times 2 \times 1$. Her response was the following.

> Student 2: "So, if we cover up the first letter, um, the second column has three different letters, oh but so does the third. That's not going to work. I mean there's kind of, like, when you get to this last column, and you read down, they don't repeat until you've gone through them all. Like, in the second column, where I have CC, AA, LL, first column I have ALC and then LCA. But I don't—that's not a coincidence, but I don't think that's really connected to the—yeah, I don't know."

In this excerpt, we see that Student 2 could see the structure of 4×6 in her list of outcomes, but was unable how her outcomes reflected the multiplication in 4!. She saw the $3{\times}2{\times}1$ as 6, but could not complete articulate how her list of outcomes were connected to the Multiplication Principle counting process she used to obtain a solution to the problem. This episode is noteworthy in our discussion about student reasoning, because even though she used the most sophisticated of English's (1991) Solution Strategies, she still was not able to see clearly how her outcomes connected to the multiplication $4 \times 3 \times 2 \times 1$. This suggests that while showing undergraduate students odometer listing strategies might be useful, and the odometer strategy is a productive way of listing outcomes in a clear, systematic way, they may still need additional support in making the connection between their list and the process they use to solve counting problems.

Even more evidence for the above point can be seen in Student 12's reasoning about her solution to the Cattle Problem. When I asked Student 12 to solve the problem, she was unsure how to go about solving it at first, and so she started by writing some outcomes where the two T's were at the beginning. She made a partial list, and her strategy was utilizing a cyclical pattern that she did not carry out to completion, and so her strategy was coded to be English's (1991) Solution Strategy C. However, after this she then created a new list using a more sophisticated listing strategy, and she wrote down a complete list of outcomes using the complete odometer strategy (English 1991). Student 12's partial list and complete list of outcomes for the Cattle Problem can be seen in Figure 14.

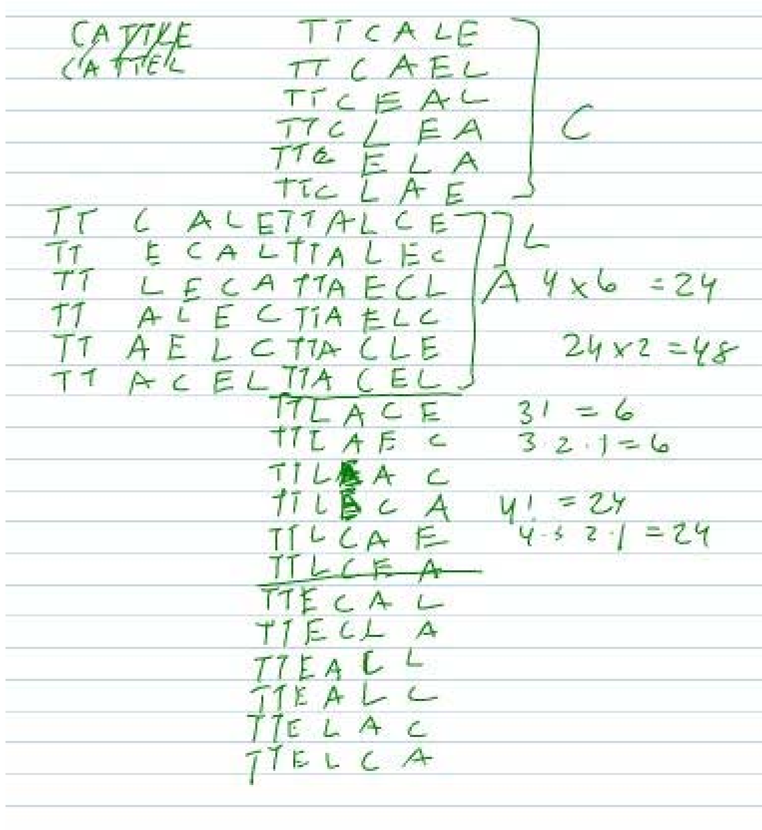

Figure 14. Student 12's complete and partial lists of outcomes for the Cattle Problem.

Halfway through making her complete list, she observed that she would get 4! outcomes beginning with the two T's. She then admitted that even though she was not originally thinking of using factorials to solve the problem, she said she could see now that "they may be a good way to go." After she finished her complete list of outcomes, I asked her if she could see the structure of $4 \times 3 \times 2 \times 1$ in her list of outcomes. She responded by first saying that she could see a 4 with all of the possible first letters, and then she noted, "And for each one then there's three possible second letters? And for the—after that there's two, but they are kinda rotating, so—um, to be honest there's not something jumping out at me that it's, like, oh this is why it's $4 \times 3 \times 2 \times 1$." In this particular example, I found that even though Student 12 utilize a sophisticated listing strategy and recognized that 4! would be the number of arrangements beginning with the two T's, she seemed unable to see any relationship between her outcomes and the counting process used to find the solution to the problem.

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Both Student 2's and Student 12's work on the Cattle Problem demonstrate that for students, making the connection between sets of outcomes and counting processes is not at all a trivial task, even though Lockwood argues that it is critical to use such a relationship in order to have a robust understanding of combinatorial tasks (Lockwood, 2013). These findings, that a student may have both a list of outcomes and a counting process, but little to no relationship between the two, is surprising and demonstrates a need for instructors to make such connections explicit for students who are learning to count. It also provides some motivation for possibly further research on student understanding this relationship. While my focus in this thesis was narrowed to student understanding of the relationship between lists of outcomes and counting processes in the context of the Cattle Problem, other contexts could be explored to gain more knowledge about how students use and think about this relationship in Lockwood's (2013) model of combinatorial thinking.

Chapter 5: Discussion and Conclusion

In this study, I focused on understanding undergraduate student listing behavior, and how it affects correctness of students' solutions to counting problems and the rigor to which students can justify their solutions. In this final chapter, I conclude by summarizing some of the key results from the study, exploring interesting points of discussion those results afforded, and providing suggestions for ways in which this study may be extended to continue to grow our knowledge of student understanding of lists of outcomes in the context of counting. In doing so, I hope to clarify important takeaways that this study provides for the mathematics education community, as well as give evidence for why I believe further research should be done on the listing aspect of students' solving of counting problems.

5.1 Summary of Key Results

In this study, there were three research goals:

1. Find a useful way to characterize undergraduate students' listing strategies while solving counting problems. In particular, my goal was to determine if the categories suggested by English (1991) for examining the combinatorial solution strategies of young children could be utilized or expanded to also categorize undergraduate student combinatorial solution strategies.
2. Examine the effects that listing had on undergraduate student success solving counting problems involving arrangement, and on their ability to rigorously justify their solutions to these counting problems.
3. Determine some ways that undergraduates explicitly make connections (or do not) between lists of outcomes they generate and counting processes they utilize to solve counting problems. In particular, I aimed at understanding student reasoning about the critical relationship between the sets of outcomes and counting processes components of Lockwood's (2013) model of students' combinatorial reasoning.

After conducting analysis of my data, interesting and applicable answers to each of the above research goals were found.

First, it was determined that although English's (1991) categories accurately described the listing behavior of many undergraduate students, there were several lists that undergraduates in my study made that could not completely be characterized using only one of English's solution strategies. What emerged was my characterization of student lists as being generated using two different types of multiple-strategy use: switching strategies and simultaneous strategies. While switching strategies was a phenomenon that English encountered with young children in her research (English, 1991, p. 462), simultaneous strategies was not addressed by English and is a new way to characterize student lists. While the simultaneous solution strategy that occurred most frequently in this study was "F and B," characterized by a student holding an item constant, but cycling through the other items randomly, I conjecture that other combinations of simultaneous solution strategies could possibly appear in students' generation of outcomes as well. As I described in section 4.1.1, undergraduates did not always use the most sophisticated solution strategy (the complete odometer strategy) for listing outcomes, and my additional elaboration of English's (1991) framework could give researchers and educators another tool for examining student lists and how they are used to solve counting problems.

In addition to providing more insight into the strategies that undergraduate students use to list outcomes for counting problems, this study is also consistent with previous work indicating that thinking explicitly about sets of outcomes and listing are a useful activities for students to engage in (Lockwood, 2013; Lockwood & Gibson, in press). In particular, in this study I encountered multiple instances where students expressed listing to be useful for correcting errors in their counting processes, as well as helping to give them ideas of where to go if there are stuck while attempting to solve a counting problem. While I make no broad, generalizable claims about how listing benefits all students, I contend that these results support the conclusion that listing is a useful activity, and they also shed some light on specific ways that writing outcomes might help some students. Previous research has shown that students struggle with finding productive counting processes for combinatorial enumeration (Batanero et al., 1997), and with verifying the accuracy of their answers to combinatorial tasks (Eizenberg & Zaslavsky, 2004). My findings provide evidence that listing may potentially help students having these particular difficulties with counting.

Finally, my third research question addressed the way in which students might not only use a list of outcomes they have generated to come up with a counting process, but also to have basis for a rigorous justification of why their counting process enumerates each outcome exactly once. This basis comes from an understanding of the key relationship between sets of outcomes and counting processes in Lockwood's

(2013) model. As the results of my study show, students can have a wide range of understandings about this relationship, ranging from a strong, well-articulated connection to an incomplete or even entirely nonexistent connection. Lockwood argued that an important feature of a robust understanding of counting is the ability to understand the structure a particular counting process imposes on a set of outcomes, as well as conversely how a particular organization of a set of outcomes yields a counting process that relies on that organization (Lockwood, 2013, p. 258). Even though understanding this relationship is so critical to robust combinatorial understanding, my results suggest that students do not always make this connection. Even if they can write down an organized list of outcomes and articulate a correct counting process, students are not always able to articulate a relationship between the two, and as a result can at times fail to see why a particular counting process may be appropriate for enumerating a particular set of outcomes. This is a noteworthy find and suggests that perhaps more time should be spent in combinatorics classrooms helping students explicitly make this connection, and engaging in systematic listing activities may be a productive activity to facilitate this.

In the next subsection, I bring up various points of discussion from the above results, and explain how they can be viewed and interpreted using my theoretical perspectives (outlined in Section 2.2).

5.2 Discussion

5.2.1 Partial Lists and English's Framework

While it has been acknowledged that the generation of partial lists is beneficial for students' successful solving of counting problems (Lockwood & Gibson, in press), English (1991) did not address ways in which to characterize partial lists of outcomes. This is not surprising, and nor do I think that English necessarily *should* have addressed it. The solutions to the counting problems that English gave to young children were numerically small in value, and the focus of her study was primarily on ways in which children generate all outcomes of a given problem.

However, when I studied the listing behaviors of undergraduate students, the need to categorize partial lists emerged, due both to the increased complexity and the larger sets of outcomes of the counting problems being asked to solve. The cardinality of the solution set for the Horse Race problem (720) was too large for students to feasibly write out all of the outcomes, and so every student who used listing to solve the problem made only a partial list. In addition, even though almost every student who answered the Cattle Problem was asked to list every outcome, there were still a few students who created a partial list as they began to solve the problem, but then stopped the list prematurely and started a new list that utilized a more (or less) sophisticated listing strategy. Since partial lists are so important, there is a need to categorize them so that researchers can have another tool for studying their effects on students' solving of counting problems.

In her study, English (1991) did account for a young child not carrying out a solution strategy to completion, but this was due to an inability of the child to continue producing outcomes, rather than an intentional choice to make only a partial list. As an example, when describing Solution Strategy E, "emergence of an 'odometer' pattern in item selection, with possible item rejection" (English, 1991, 460), English clarifies that, "The odometer pattern evident in this strategy is however, incomplete" (p. 460). She explained that this was due to one or more of the following:

1. "An 'over-exhaustion' or duplication of combinations with a given constant item,"
2. "A failure to exhaust all possible combinations with a given constant item," and
3. "A failure to identify task completion upon the exhaustion of all constant items" (p. 460-461).

Therefore, categorizing a student's listing strategy with Solution Strategy E requires indication of the students being unable to complete the odometer strategy, or unable to identify completion of the odometer. As I explained in Section 4.1.1, in order to modify English's (1991) categories to apply partial lists, I assert that it is not enough just to look at the outcomes a student made and see if every outcome was written. Instead, my modification requires consideration of students' utterances about *why* they made a partial list as well. As I explained in Section 4.1.1, there were instances in which students described a specific process in which they could meaningfully continue their listing strategy to extend their partial list to a full list. In terms of student reasoning about listing strategies, then, it is more appropriate to apply English's more sophisticated solution strategies, even though the student does not actually choose to make a full list of outcomes. In this way, we can capture subtle differences in the ways in which undergraduate students list.

In sum, the undergraduate students in the study at times created partial lists, and I adopted English's framework to account for that. In general, the fact that students use and extrapolate from partial lists is an important phenomenon that warrants further attention. Because counting problems typically have large sets of outcomes, and because we know that some listing activity is beneficial, a possible next step would be to more pointedly study how students think about the extrapolation of a partial list to find a correct answer. That is, it is worthwhile to study what mechanisms students employ as they extend to a complete list from a partial one, and how they can be sure they can do this correctly. This is related to broader issues of justification, and how students can be sure that extending a partial list does indeed capture all of the desirable outcomes of a problem.

5.2.2 Varying Robustness in Student Understanding about Lists of Outcomes and Counting Processes

As a final point of discussion, I highlight the variety that was seen in how undergraduate students thought about the relationship between lists of outcomes and counting processes. There were instances where students had different coherent counting processes and structures of the set of outcomes associated with the same formula, and instances where identical lists of outcomes were produced using fundamentally distinct solution strategies.

As an example of two students having different counting processes and lists of outcomes associated with the same formula, I compare Student 17's and Student 6's lists of outcomes and solutions to the Cattle Problem. Student 17's and Student 6's lists of outcomes can be compared in Figure 15. To answer the Cattle Problem, both students offered the solution $2 \times 4!$, but when asked to list outcomes beginning with TT, the resulting lists were structured very differently. When asked how the structure of the lists they wrote related to 4!, Student 17's response was the following:

> Student 17: "So, um, there's, uh, the four. That's the four, like, first letters. So, I start with C and then A, L, E. And then, times three. There's, um, three second letters that I have, um: A, L, E, in this first

example. And then, two, um, factorial, there's—I always have, um, I always have two letters that I switch, and then always the fourth letter's whatever one is left."

Student 6's outcomes for the Cattle Problem:

Student 17's outcomes for the Cattle Problem:

Figure 15. Side-by-side comparison of Student 6's and Student 17's lists of outcomes for the Cattle Problem.

As we see from Student 17's utterances, there was a strong connection between his list of outcomes and counting process—in particular he was able to articulate how the multiplication resulting in his answer 4! partitioned his outcomes based on fixed letters in the first, second, and third position of outcomes. In contrast, as we saw in Section 4.3.1, Student 6 thought of the 4! as structuring his list of outcomes based on a fixed position of the letter C, and fixed letters in the remaining positions. He was still able to articulate how the multiplication in 4! was reflected in a partitioning of his outcomes, but in a fundamentally different way. Using Halani's ways of student thinking about combinatorial solution sets, Student 17's list and connection to his counting process reflects Standard Odometer thinking, and Student 6's thinking reflects Wacky Odometer thinking (Halani, 2013, p. 124).

In this way, we can see that it is possible for students to think about a formula and how it connects to lists of outcomes in different, but equally useful ways. As a possible method for helping students understand the relationship between sets of outcomes and counting processes, one pedagogical strategy might be to alert students to different counting processes that can follow from the same formula, and how they structure a set of outcomes in different ways.

Finally, I also highlight the fact that students can create very similar lists of outcomes, but they might produce those outcomes using different strategies. This leads to varying degrees to which students can connect their list to their counting process for solving the problem. I again will use a comparison with Student 17, this time comparing his with the solution of Student 7 for the Cattle Problem. A side-by-side comparison of the complete lists of outcomes for both students can be seen in Figure 16. By looking at the figure and not hearing how students thought about their lists, it would be easy to assume that both lists were generated using the complete odometer strategy, since both lists appear to be organized by a common first and second letter in the outcomes. However, by examining the solution strategies both students used to write their outcomes, we see that the strategies were quite different, and in turn this resulted in different levels of understanding each student had about the relationship between their lists and the multiplication in 4!.

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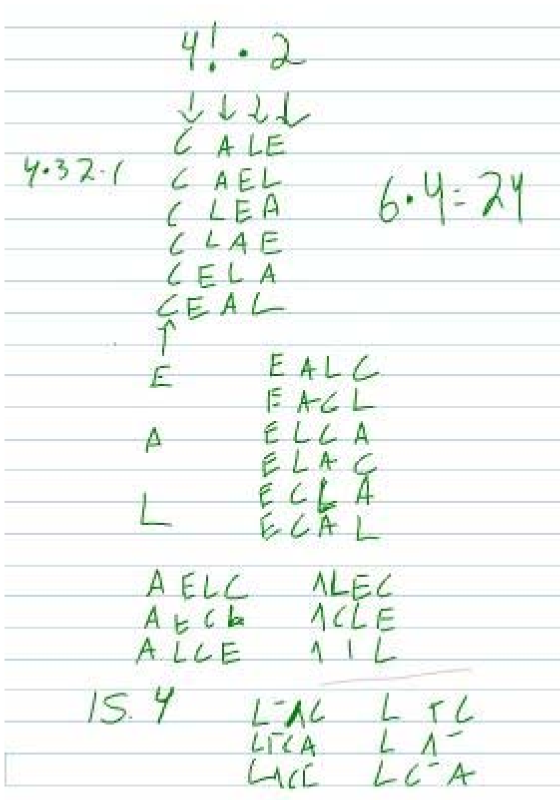
Student 7’s outcomes for the Cattle Problem:



Student 17’s outcomes for the Cattle Problem:

Figure 16. Side-by-side comparison of Student 7’s and Student 17’s lists of outcomes for the Cattle Problem.

Recall in Section 4.1.2.1, I explained that Student 7’s used solution strategy “E then D” for writing his outcomes, since he described a systematic switching pattern to generate his list, rather than repeatedly fixing letters in the first and second positions of an arrangement as Student 17 had done. Because a systematic switching process was used, rather than the more sophisticated complete odometer strategy that Student 17 utilized, there is a less clear connection between 4! and the structure of outcomes that Student 7 obtained. Indeed, when Student 7 was asked if 4! was reflected in the structure of his list, he replied, “I think the way I did it—it doesn’t really. I think there’s another way to do it that would show it a little better. $4 \times 3 \times 2 \times 1$, hmm.” After making these comments and then thinking for a while about the list he made, he was later able to describe a relationship between his list and 4! by looking at the number of options available in each “column” of the outcomes he wrote, but his initial response indicates that there was a less clear connection for him between the multiplication in 4! and the strategy he used to write and structure his list of outcomes. The explanation he gave about options available in each column of his outcomes, while correct and useful, was only articulated *after* he had completed his list and specifically looked for patterns in his list. This is evidence that he was not thinking about a connection to the multiplication in 4! while in the process of writing his outcomes, and so there was a less robust initial understanding of how his list of outcomes and counting process were related.

Since it is possible for students to find correct answers to counting problems and even write outcomes in a systematic way, but not connect the two in a meaningful way, this suggests according to Lockwood’s (2013) model that these students may have a gap in their combinatorial understanding. Since this gap in student understanding can exist even when systematic lists are generated and correct answers achieved, teachers should be aware of different solution strategies that students use and how they can give rise to more or less robust combinatorial reasoning.

5.3 Closing Comments and Further Directions

The purpose of this study was to achieve an improved understanding of undergraduate students listing strategies and how those listing strategies are connected to sets of outcomes. By using English’s (1991) categories with the modifications suggested in the results of my study, researchers can have a more precise tool to use for further study into how lists of outcomes affect student success solving counting problems. Researchers could use this study to begin to examine how English’s (1991) strategies may interact within student work on complicated counting problems, and apply her strategies to instances where students used a partial list to solve a counting problem. Teachers could also use my modified categories for student combinatorial solution strategies to gauge how their students are listing, and how these strategies could afford different levels of understanding about the relationship between sets of outcomes and counting processes. As I have shown, student reasoning about this relationship can vary tremendously, even within the context of one particular arrangement counting problem. Pedagogically, these findings mean that for teachers to help build robust combinatorial understanding in their students, they could intentionally point out to students ways that counting processes impose structures on a given set of outcomes, and in turn an organization of a set of outcomes can give rise to a process used to solve the counting problem that enumerates those outcomes.

In terms of possible further research, future studies could build upon this study toward continuing to unpack the ways students develop combinatorial conceptual knowledge about the relationship between sets of outcomes and counting processes. Specific ways that further research could be extended from this study might include exploring student understanding of the relationship between sets of outcomes and counting processes in other combinatorial contexts, such as in the context of problems involving combination or other combinatorial operations. Researchers could also test to see if my modified categorization of students' listing strategies applies to students at different stages in their learning, such as possibly at the high-school level.

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