AN ABSTRACT OF THE THESIS OF

SHYR TUNG LUO for the Degree of Doctor of Philosophy in Nuclear Engineering presented on August 23, 1985.

Title: Stability Analysis of Dynamic Heat Transfer

Abstract Approved: Redacted for Privacy

Stephen E. Binney

The stability analysis for a heated tube system is an important safety feature of a nuclear power plant. Although the system theory is well established for linear systems described by ordinary differential equations, there is still a shortage of theory dealing with a distributed system, which is described by partial differential equations. By use of the finite difference technique, several important conclusions for distributed systems are obtained systematically. These results can be applied directly to the heated tube system as shown in this thesis. All these results can be considered as a starting point for the stability analysis of a distributed system. Further expansion of these results to a feedback system and closed loop system are desirable, but beyond the scope of this thesis.

Stability Analysis of Dynamic Heat Transfer

by

Shyr Tung Luo

A THESIS

Submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed August 23, 1985

Commencement June, 1986

APPROVED:

Redacted for Privacy

Professor of Nuclear Engineering in charge of major

Redacted for Privacy

Head of Department of Nuclear Engineering

Redacted for Privacy

Dean of Graduate School

Date thesis is presented August 23, 1985

Typed by Shyr Tung Luo

ACKNOWLEDGMENTS

I especially appreciate the assistance, encouragement and advice of my major professor and thesis advisor, Dr. Stephen E. Binney. His helpful suggestions made the completion of this thesis possible.

I would also like to thank my thesis committee members Dr. James Rawers, Dr. Brian Dodd, Dr. Wojciech Kolodziej and Dr. Walter Loveland, for helping me set up the proper study program.

A special thanks goes to the late Dr. Karl Hornyik, my former advisor, for introducing me into this field.

I would like to thank my parents for their continuing faith and encouragement during my studies abroad. This dissertation is dedicated to my wife, Pu-Chin Wang, for her love and understanding during my graduate studies at Oregon State University.

TABLE OF CONTENTS

CHAPTER PAGE

LIST OF FIGURES

TABLE OF RULES

STABILITY ANALYSIS OF DYNAMIC HEAT TRANSFER

1. INTRODUCTION

Beside the reactor itself, the steam generator is one of the principal components in a pressurized water reactor (PWR). The steam generator transfers the heat from the primary loop to the secondary loop as shown in Figure 1-1. Heat is generated by fission in the reactor core. It is convected from there by the primary coolant and passed on to the secondary loop by way of the steam generator[1]. Eventually heat is transformed to electric power by the steam which drives a turbine-generator. As shown in this simple scheme, the steam generator therefore is the coupling link between the primary and the secondary loop in the PWR plant. This indicates that it is essential to have a good understanding of the stability problem of the steam generator for predicting PWR response under normal and accidental conditions. It is well understood that any empirical designs developed without careful analysis are often high in cost and poor in performance.

In recent years the problem of stability of two-phase flow systems has become of interest to engineers. Experiments have been performed and results are available in the literature[2-4], but there is still a shortage of adequate theoretical analysis. Most mathematical analyses of flow instabilities performed until now are based on linearization of the conservation equation[5,6]. Most of these, in turn, make use of some frequency domain stability criteria. Typically a transfer function is calculated numerically to determine stability in terms of a Nyquist plot[7]. Another approach involves the use of Lagrangian coordinates instead of Euler coordinates for the fluid governing equations[8]. Both approaches introduce simplifying assumptions and employ a considerable amount of numerical calculations. Because of a lack of stability criteria for a distributed system (i.e., the system governing equations are partial differential equations and the variables are functions of space and time), the development of improved analytical tools to study and predict this kind of system becomes necessary.

In this thesis a simple stability criterion for linear partial differential equations has been investigated. These results are applied to the stability analysis of a heated tube system. As shown later the stability for a distributed system can be easily and systematically described by a distributed spatial eigenvalue. The main objectives of the work reported in this thesis are to derive the stability criteria for a distributed system and to apply this analysis to a heated tube system.

When a point of theory is intimately related to practical application, its conclusion may strengthen not only our theoretical insight, but our intuitive understanding as well. The background of this development involves small perturbation theory and stability criteria for a system controlled by an ordinary differential equation (ODE). Throughout this thesis, only the conclusion of the background material is used; no attempt is made to repeat the development of its theory.

In the analysis of many physical systems, a complete description of the behavior of the system is unnecessary and all that is required is a knowledge of whether the system is stable, i.e., whether its response to a bounded excitation remains bounded or become infinite as time approach infinity. Often, the onset of oscillations represents the limit of operating conditions for a given system. Several rules of thumb are available for precautions which should be taken in the design of such needed is a logical theoretical basis for the prediction of system performance in advance and for systematic optimization of the operating condition.

This analysis was originally derived in the course of consulting work to model the once through steam generator for Gesellschaft fur Reaktorsicherheit Co. in West Germany and has no claim to complete generality, although it is felt that a similar procedure could be applied to different systems. A computer code was developed to simulate the steam generator. It was found that the inner iteration loop of the computer program had convergence problems. A stability study was needed to solve this problem. The heated tube model, which corresponds to the inner iteration loop, is discussed in Chapter 4. The results showed that the mathematical model was stable. The convergence problem was actually caused by the iteration technique, as shown in Appendix B.

In this paper a discrete mathematical model for the stability analysis of partial differential equations of a vertical heated tube is presented. The governing equations for fluids are nonlinear partial differential equations. A small perturbation technique is used to linearize the governing equations[9-11]. These equations are then discretized for the space parameter to get a typical set of difference differential equations. This procedure is similar to a "lumping" procedure or engineering approximation. Finally, stability criteria are applied to this model[12]. This technique is explained in greater detail in Chapter 2 and its implementation in Appendix A. The expansion of this procedure for simultaneous partial differential equations is also presented in Chapter 3. Because of the use of trapezoidal integration, the difference differential equations have some special characteristics which can be used for eigenvalue searching in stability analysis[13-15]. These characteristics are analyzed and some rules are established and presented.

In Chapter 4, the implementation of this approach in a heated tube is presented. The heat transfer mechanism may be divided into three regions:

1. Subcooled region: in which heat transfer to the fluid occurs by subcooled forced convection and nucleate boiling. The quality is zero throughout this region.
2. Two-phase flow region: in which the quality of steam changes from zero to unity. The major heat transfer mechanisms on the secondary side are

steam changes from zero to unity. The major heat transfer mechanisms on the secondary side are nucleate boiling and forced convection vaporization.

3. Superheated steam region: in which the heat transfer mechanism is single-phase forced convection. Throughout this region the quality is unity.

All these three regions are analyzed by the procedure discussed in Chapters 2 and 3.

It is assumed that pressure drops in the elements of the loop are so low compared with absolute pressure that compressibility effects and changes in fluid properties can be ignored[5,6]. For a high-pressure system local changes in pressure will have a negligible effect on the fluid properties. Boiling two-phase flow in a channel is inherently hydrodynamically unstable. Transient flow excursions or flow oscillations are apt to develop by means of buoyancy or compressibility effects. Such hydrodynamic instability of a two-phase flow is not desired, since it usually precipitates a high-temperature excursion of the heating element, thus causing a so-called premature burnout[16]. That is to say, when flow instability sets in, component failure due to thermal fatigue can occur at a much lower heat flux than under steady state conditions as reported in the literature. The result of this analysis is presented in Chapter 4. Chapter 5 contains concluding remarks for the stability of a heated tube. Throughout this thesis, it is intended to systematize as much as possible the procedures for stability analysis of a dynamic system.

This analysis is closely related to and dependent on numerical analysis, which is the basic tool in solving nonlinear partial differential equations in this computer era. A complete and general understanding of convergence for numerical analysis is difficult, especially for a complicated nonlinear system such as a heat transfer problem. With the help of stability analysis as shown in this thesis, it may be determined whether a numerical instability or a system instability causes the convergence problem of a computer program. This analysis will be useful in the actual computer program debugging process.

1. Reactor
2. Core
3. Turbine
4. Condenser
5. Steam line
6. Primary coolant
7. Steam generator
8. Feedwater pump
9. Primary coolant pump
10. Secondary condensate line

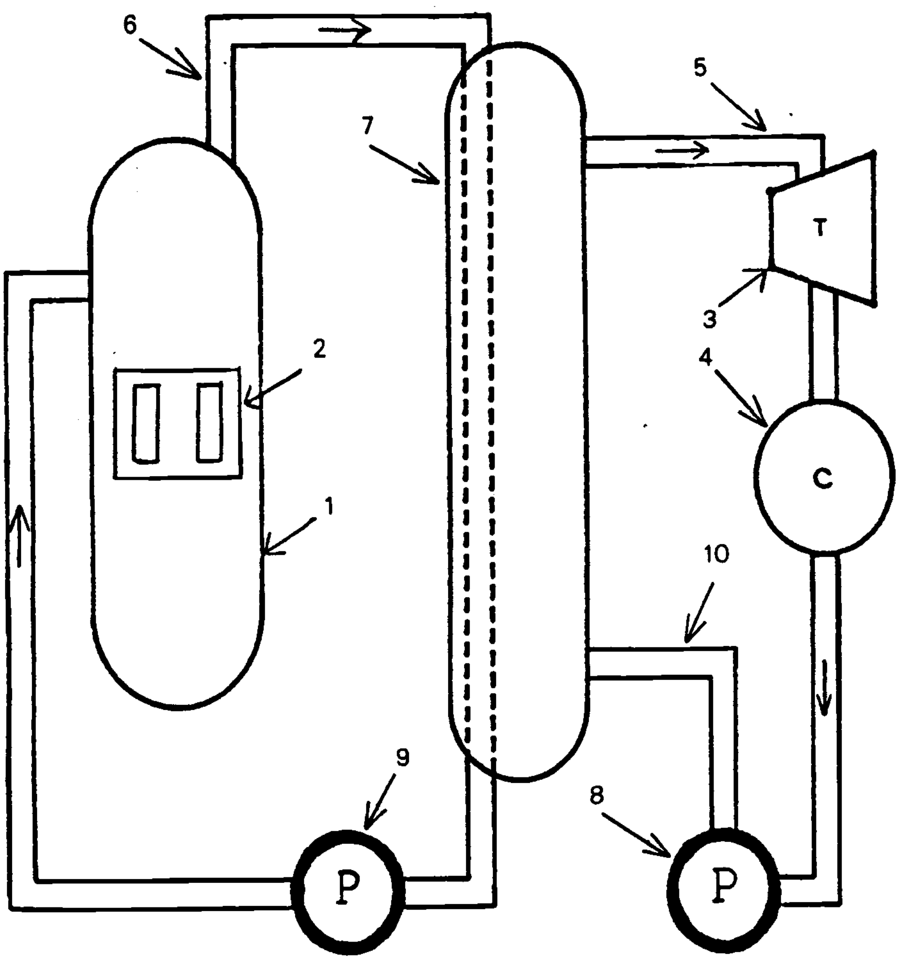

Figure 1-1 Simplified PWR Schematic

2. BACKGROUND

2.1. PERTURBATION METHOD FOR NONLINEAR EQUATIONS

Many of the problems facing physicists, engineers, and applied mathematicians involve such difficulties as nonlinear governing equations, variable coefficients, and nonlinear boundary conditions at complex known or unknown boundaries that preclude solving these equations exactly. Consequently, solutions are usually approximated using numerical techniques, analytic techniques, and combinations of both. Foremost among the analytic techniques are the systematic methods of perturbations (asymptotic expansions) in terms of a small or a large parameter or coordinate[10,11]. This section is concerned only with these perturbation techniques. The use of linearization is based theoretically on the Liapunov Theorem which states that the stability of the linearized system corresponds to the stability of the non-linear system operating under quasi-equilibrium conditions[17]. Linearized models can only predict the system stability boundary and appraise the stability margin. They cannot predict transient unstable system behavior.

The perturbation method can be summarized by the following procedure:

1. Determine the governing equation with boundary conditions.
2. Find the equilibrium solution.
3. Assume a small perturbation variable.
4. Change the governing equation variable by the small perturbation variable.
5. Eliminate the equilibrium part from the governing equation.
6. Neglect higher order terms.
7. Obtain the resultant linearized equation.

Notice only step 6 is an approximation. A detailed development of this procedure is shown in the following. The mathematical model for heat transfer problems is a nonlinear partial differential equation. A general one dimensional nonlinear first order partial differential equation has the following form:

$$\dot{X}(Z,t) = A(X,Z)\frac{\partial}{\partial Z}X(Z,t) + B(X,Z)X(Z,t) + U(X,Z) \tag{2-1}$$

where

$$\dot{X}(Z,t) = \frac{\partial}{\partial t}X(Z,t)$$

and A(X,Z), B(X,Z) and U(X,Z) are continuous functions. Assume the equilibrium solution of Equation (2-1) is

$$X_e = \dot{X}(Z,0) \tag{2-2}$$

Then, for a small perturbation, the governing equation can be linearized as follows:

Let

$$X(Z,t) = X_e + x(Z,t) \tag{2-3}$$

Equation (2-3) is substituted into Equation (2-1), and the equilibrium part is eliminating by virtue of the equilibrium reference solution.

$$
\begin{aligned}
\dot{x} = \; & A(X_e,Z) + \left(\frac{\partial}{\partial X}A\right)\Big|_{X_e}\cdot x \quad \frac{\partial}{\partial Z}(X_e + x) \\
& + B(X_e,Z) + \left(\frac{\partial}{\partial X}B\right)\Big|_{X_e}\cdot x \quad (X_e + x) \\
& + U(X_e,Z) + \left(\frac{\partial}{\partial X}U\right)\Big|_{X_e}\cdot x
\end{aligned}
\tag{2-4}
$$

Equation (2-4) can be rearranged to give

$$
\begin{aligned}
\dot{x} = \; & A(X_e,Z)\frac{\partial}{\partial Z}X_e + B(X_e,Z)\cdot X_e + U(X_e,Z) \\
& + A(X_e,Z)\frac{\partial}{\partial Z}x \\
& + \left(\frac{\partial}{\partial X}A\right)\Big|_{X_e} \left(\frac{\partial}{\partial Z}X_e\right) + B(X_e,Z) + \left(\frac{\partial}{\partial X}B\right)\Big|_{X_e}\cdot X_e \\
& \qquad + \left(\frac{\partial}{\partial X}U\right)\Big|_{X_e}\cdot x \\
& + \left(\frac{\partial}{\partial X}B\right)\Big|_{X_e}\cdot x^2 + \left(\frac{\partial}{\partial X}A\right)\Big|_{X_e}\cdot\frac{\partial}{\partial Z}X\cdot x
\end{aligned}
\tag{2-5}
$$

The first term in brackets is just the equilibrium state solution, which has a value of zero. Thus

$$\dot{x} = A(X_e,Z)\cdot(\frac{\partial}{\partial Z}x)$$

$$+ (\frac{\partial}{\partial X}A)\Big|_{X_e}\cdot(\frac{\partial}{\partial Z}X_e) + B(X_e,Z) + (\frac{\partial}{\partial X}B)\Big|_{X_e}\cdot X_e$$

$$+ (\frac{\partial}{\partial X}U)\Big|_{X_e}\cdot x$$

$$+ (\frac{\partial}{\partial X}B)\Big|_{X_e}\cdot x^2 + (\frac{\partial}{\partial X}A)\Big|_{X_e}\cdot(\frac{\partial}{\partial Z}x)\cdot x \qquad (2\text{-}6)$$

In order to simplify the system of equations, those terms which have higher than second order are neglected, which results in

$$\dot{x} = A(X_e,Z)\cdot(\frac{\partial}{\partial Z}x)$$

$$+ (\frac{\partial}{\partial X}A)\Big|_{X_e}\cdot(\frac{\partial}{\partial Z}X_e) + X_e\cdot(\frac{\partial}{\partial X}B)\Big|_{X_e} + (\frac{\partial}{\partial X}U)\Big|_{X_e}$$

$$+ B(X_e,Z)\cdot x \qquad (2\text{-}7)$$

In the general form of linearized partial differential equations, Equation (2-7) can be written as

$$\dot{x} = f(Z)\cdot(\frac{\partial}{\partial Z}x) + g(Z)\cdot x \tag{2-8}$$

Appendix A contains an example to develop a better understanding of the perturbation method.

2.2 DISCRETIZED METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

The reduction of a differential equation to a set of finite difference equations is a very powerful method of numerical analysis, and the trapezoidal method is well known in integration problems. Applying the same technique to a partial differential equation (PDE) will change it to a set of ordinary differential equations (ODE). The technique used here to discretize a PDE is extensively described in many textbooks on numerical methods or system simulation[18,19]. Therefore in this section the trapezoidal integration technique is directly applied to the general first order partial differential equation

$$\dot{X} = f(Z)\frac{\partial}{\partial Z}X + g(Z)X \tag{2-9}$$

with initial condition and boundary condition

$$X(Z,0) = X_i(Z) \tag{2-10}$$

$$X(0,t) = X_b(t) \tag{2-11}$$

Usually the time domain is infinite but the Z domain is finite (e.g., $0 < Z < L$). The Z domain is divided into n equally spaced sections as shown in Figure 2-1. Consider the following definitions:

$$X(Z_0,t) = X_e(t) = X_b(t) \quad \text{since } Z_0 = 0$$

$$X(Z_1,t) = X_1(t)$$

$$\vdots$$

$$X(Z_n,t) = X_n(t). \tag{2-12}$$

Integration of Equation (2-9) from Z_i to Z_{i+1} yields

$$\int (\dot{X})\ dZ = \int \left[f(Z)\frac{\partial}{\partial Z}X + g(Z)X\right] dZ \tag{2-13}$$

The integral sign represents the integration from Z_i to Z_{i+1} throughout this thesis. The left hand side (LHS) can be expressed by the trapezoidal rule as

$$LHS = (\Delta Z/2)(\dot{X}_{i+1} + \dot{X}_i) \qquad (2\text{-}14)$$

where $\Delta Z = Z_{i+1} - Z_i = L/n$. Similarly the right hand side (RHS) can be expressed by the two terms:

$$RHS_1 = \int f(Z)(\frac{\partial}{\partial Z}X)\, dZ$$

$$= f(Z_{i+.5}) \int (\frac{\partial}{\partial Z}X)\, dZ$$

$$= f(Z_{i+.5})(X_{i+1} - X_i) \qquad (2\text{-}15)$$

where it was assumed that f(Z) does not vary significantly over the region of integration and can thus be evaluated at the midpoint. Similarly,

$$RHS_2 = \int g(Z)(X)\, dZ$$

$$= g(Z_{i+.5}) \int X\, dZ$$

$$= g(Z_{i+.5})(X_{i+1} + X_i)(\Delta Z/2) \qquad (2\text{-}16)$$

Substituting Equations (2-14), (2-15) and (2-16) into Equation (2-13) yields

$$\dot{X}_{i+1} + \dot{X}_i = (2/\Delta Z) f(Z_{i+.5})(X_{i+1} - X_i)$$

$$+ g(Z_{i+.5})(X_{i+1} + X_i)$$

$$= (2/\Delta Z) f(Z_{i+.5}) + g(Z_{i+.5}) \ (X_{i+1})$$

$$+ \ -(2/\Delta Z) f(Z_{i+.5}) + g(Z_{i+.5}) \ (X_i) \tag{2-17}$$

or

$$\dot{X}_{i+1} + \dot{X}_i = A_{i+1} X_{i+1} + B_{i+1} X_i \tag{2-18}$$

where

$$A_{i+1} = g(Z_{i+.5}) + (2/\Delta Z) f(Z_{i+.5})$$

$$B_{i+1} = g(Z_{i+.5}) - (2/\Delta Z) f(Z_{i+.5}) \tag{2-19}$$

For i=0 to n-1, there are n equations which are

$$\dot{X}_1 = A_1X_1 + B_1X_e - \dot{X}_e$$

$$\dot{X}_2 + \dot{X}_1 = A_2X_2 + B_2X_1$$

$$\vdots$$

$$\dot{X}_n + \dot{X}_{n-1} = A_nX_n + B_nX_{n-1} \qquad (2\text{-}20)$$

These equations can be written in matrix form by the following definition

$$(X) = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}_{n \times 1} \qquad (M) = \begin{bmatrix} 1 & & & 0 \\ 1 & 1 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 1 \end{bmatrix}_{n \times n}$$

$$(U) = \begin{bmatrix} B_1 X_e - \dot{X}_e \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \qquad (N) = \begin{bmatrix} A_1 & & & 0 \\ B_2 & A_2 & & \\ & \ddots & \ddots & \\ 0 & & B_n & A_n \end{bmatrix}_{n \times n} \tag{2-21}$$

Then Equations (2-20) can be written as

$$(M)(\dot{X}) = (N)(X) + (U) \tag{2-22}$$

It is noticed that by this method a PDE with finite Z domain can be reduced to a set of ODE's which can be written in a matrix form with lower triangular matrices as coefficients. It is also interesting to point out that the lower triangular matrix has many marvelous characteristics which are extremely useful in stability analysis. A detailed discussion of these properties appears in Chapter 3.

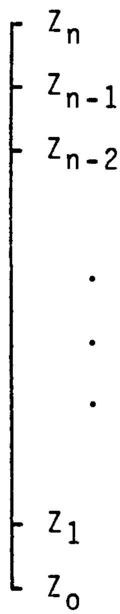

Figure 2-1 Discretized Mesh Points

2.3 STABILITY THEORY

System theory has developed into a scientific and engineering discipline which seems destined to have an impact upon all aspects of modern society. A general equation can be written for a system, that is,

$$(\dot{X}) = (A)(X) + (B)(U) \tag{2-23}$$

where (A) and (B) are constant matrix coefficients. (X) is referred to as a state function. The stability of a system described by Equation (2-23) can be summarized by considering the following definitions and rules:[12,13]

Definition: The equilibrium solution (if it exists) of Equation (2-23) is called the equilibrium state (X_e).

Definition: X_e is the asymptotically stable equilibrium state if there exists some $\delta > 0$ such that $|X(0) - X_e| < \delta$ implies $|X(\infty) - X_e| > 0$.

Definition: X_e is a weakly stable equilibrium state if for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|X(0) - X_e| < \delta \text{ implies } |X(\infty) - X_e| < \epsilon.$$

Definition: X_e is an unstable equilibrium state if for any $\delta > 0$ there exists a state X(0) such that

$$\left| X(0) - X_e \right| < \delta, \text{ yet } X(\infty) - X_e \rightarrow k,$$

where $\delta < k < \infty$.

Rule 2.1: The stability for a system described by Equation (2-23) is guaranteed if and only if all the eigenvalues of matrix (A) have a negative real part.

Rule 2.2: The system is weakly stable if the matrix (A) has only one zero as an eigenvalue, but no eigenvalues with positive real part.

Rule 2.3: The system is unstable if the matrix (A) has any eigenvalues with positive real part or multiple zero eigenvalues.

Therefore instead of solving Equation (2-23), the stability of the system can be easily obtained by determining the eigenvalues of matrix (A) and applying the above mentioned rules.

3. EXTENSION OF STABILITY THEORY

3.1 EIGENVALUE SEARCHING

There is an easy way to find eigenvalues for a lower triangular matrix[14,15,19].

Rule 3.1: The eigenvalues for a lower (or upper) triangular matrix are the diagonal elements of this matrix.

As was pointed out in Section 2.2, the matrix form for a discretized partial differential equation always has lower triangular matrices as coefficients. So eigenvalues are obvious once the matrices are known. However, most practical problems involve a set of partial differential equations. After the discretization procedure, the set of partial differential equations will become a set of simultaneous ordinary differential equations with lower triangular matrices as coefficients. For example, consider

$$\dot{X}_1 = A_{11}X_1 + A_{12}X_2 + U_1$$

$$\dot{X}_2 = A_{21}X_1 + A_{22}X_2 + U_2 \qquad (3\text{-}1)$$

where A_{11}, A_{12}, A_{21} and A_{22} are lower triangular matrices. If Equation (3-1) is written in matrix form, the result is

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \qquad (3\text{-}2)$$

A simple procedure is developed here for the eigenvalue search of this kind of matrix. Let $A_{mn}(a_{ij})$ represent the element in column j, row i of matrix A_{mn}. Then for a matrix F defined as

$$F = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{bmatrix} (mn \times mn) \qquad (3\text{-}3)$$

where A_{ij} are n x n lower triangular matrices, m corresponds to the number of simultaneous partial differential equations, and n corresponds to the number of discretized mesh points, the following rule can be established:

Rule 3.2: The eigenvalues for the matrix defined in Equation (3-3) are the solution of

$$\prod_{i=1}^{n} \begin{vmatrix} S-A_{11}(a_{ii}) & A_{12}(a_{ii}) & \cdots & A_{1m}(a_{ii}) \\ A_{21}(a_{ii}) & S-A_{22}(a_{ii}) & \cdots & A_{2m}(a_{ii}) \\ \vdots & & & \vdots \\ A_{m1}(a_{ii}) & A_{m2}(a_{ii}) & \cdots & S-A_{mm}(a_{ii}) \end{vmatrix}_{(m \times m)} = 0 \qquad (3\text{-}4)$$

By this rule, the search for eigenvalues becomes much easier. For example, let m=2 and n=50. Instead of solving eigenvalues for a 100x100 matrix, it is only necessary to solve for the eigenvalues of a 2x2 matrix 50 times. Besides the computation saving, this theory can be applied to the searching for the largest real part of eigenvalue, which is important in stability analysis.

An example is the easiest way to explain Rule 3.2. Consider the following 3x3 matrix:

$$\begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 2 & 6 & 1 \end{bmatrix}$$

Since this is a lower triangular matrix, the eigenvalues will be 3, 4 and 1 by Rule 3.1. Now consider a 6x6 matrix, such as:

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 5 & 1 & 0 \\ 1 & 0 & 9 & 0 & 4 & -3 \\ \hdashline 2 & 0 & 0 & 8 & 0 & 0 \\ 2 & 3 & 0 & 6 & 5 & 0 \\ 5 & 1 & 4 & 2 & 9 & 1 \end{array}\right] \qquad (3\text{-}4a)$$

Notice this 6 x 6 matrix can be partitioned into a 2 x 2 matrix, where each partitioned submatrix is a 3 x 3 lower triangular matrix. Therefore instead of solving for the eigenvalues of the 6x6 matrix directly, the eigenvalues can be found for the three 2x2 matrices

$$\begin{bmatrix} 2 & 0 \\ 2 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 \\ 3 & 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 9 & -3 \\ 4 & 1 \end{bmatrix}.$$

The eigenvalues will be 2, 8, 2, 6, 3 and 7 by Rule 3.2. The time consumed for eigenvalue searching is proportional to n^4, so the time savings for an (mn x mn) matrix will be n^3. For this example the time savings will be 27 times. For a discretized problem, where n is much larger than m, for example for n=50 and m=2, the time savings will be 125,000 times.

Before developing the next rule, some notation needs to be defined to simplify the problem.

Definition: Diagonal matrix D(A): Given an n x n matrix A, the diagonal matrix is an n x n matrix with all elements equal to zero except the diagonal elements which are equal to the diagonal elements of matrix A, correspondingly, that is

$$D(A) = \begin{bmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}$$

Rule 3.3: The eigenvalues for a lower triangular matrix are the diagonal elements of its diagonal matrix.

Some useful rules for matrix manipulation follow:

Rule 3.4: The inverse matrix of a lower triangular matrix is a lower triangular matrix.

Rule 3.5: Let B represent the inverse matrix of a lower triangular matrix A. Then each element of matrix D(B) is the reciprocal of the corresponding element of matrix D(A).

For instance

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 1 & 7 & 4 \end{bmatrix}$$

$$D(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$D(A^{-1}) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

Let matrix $C = A \cdot B$, where matrices A and B are lower triangular matrices. Then the following rules apply:

Rule 3.6: The product matrix of two lower triangular matrices is a lower triangular matrix. That is to say, matrix C is also a lower triangular matrix.

Rule 3.7: The elements of diagonal matrix D(C) are the products of the elements of diagonal matrices D(A) and D(B), respectively.

That is,

$$D(A \cdot B) = D(C) = \begin{bmatrix} c_{11} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & c_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \cdot b_{11} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large 0} & & a_{nn} \cdot b_{nn} \end{bmatrix}$$

For example,

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 5 & 0 \\ 6 & 4 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 7 & 0 \\ 3 & 8 & 9 \end{bmatrix}$$

Then

$$C = A \cdot B = \begin{bmatrix} 6 & 0 & 0 \\ 9 & 35 & 0 \\ 19 & 36 & 9 \end{bmatrix}$$

$$D(C) = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 35 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 3x2 & 0 & 0 \\ 0 & 5x7 & 0 \\ 0 & 0 & 1x9 \end{bmatrix}$$

The definition of a matrix in no way rules out the possibility that the elements of a matrix are themselves matrices[19]. In fact, it is often convenient to subdivide, or partition, a matrix into submatrices and then regard the original matrix as a new matrix having these submatrices as elements.

Definition: A matrix is known as a block lower triangular matrix if can be partitioned to form a lower triangular matrix.

An interesting property of a block matrix is shown below. In particular, it is helpful to regard an mn by mn matrix $A = \left[a_{ij}\right]$ as an n by n square matrix whose elements are the respective m by m submatrices of A. For instance

$$A = \begin{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} & \begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 9 \\ 8 & 7 \end{bmatrix} & \begin{bmatrix} 4 & 7 \\ 3 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 8 \\ 5 & 1 \end{bmatrix} \end{bmatrix}_{6 \times 6} \qquad (3\text{-}5)$$

can be written as

$$A = \begin{bmatrix} (a)_{11} & (a)_{12} & (a)_{13} \\ (a)_{21} & (a)_{22} & (a)_{23} \\ (a)_{31} & (a)_{32} & (a)_{33} \end{bmatrix} \tag{3-6}$$

where

$$(a)_{11} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \qquad (a)_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$(a)_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (a)_{21} = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \text{, etc.} \tag{3-7}$$

The following rule then applies:

Rule 3.8: The product of two matrices is equivalent to the matrix product of their partitioned submatrices.

For example

$$A \cdot B = \begin{bmatrix} (a)_{11}(b)_{11} + (a)_{12}(b)_{21} + (a)_{13}(b)_{31}, & \cdots \\ (a)_{21}(b)_{11} + (a)_{22}(b)_{21} + (a)_{23}(b)_{31}, & \cdots \\ (a)_{31}(b)_{11} + (a)_{32}(b)_{21} + (a)_{33}(b)_{31}, & \cdots \end{bmatrix}$$

Rule 3.9: The determinant, $\det[A]$, of a block lower triangular matrix A is equivalent to the determinant of its partitioned matrices whose elements are the determinants of each submatrix, i.e.,

$$\det[A] = \det \begin{bmatrix} \det(a)_{11} & \det(a)_{12} & \det(a)_{13} \\ \det(a)_{21} & \det(a)_{22} & \det(a)_{23} \\ \det(a)_{31} & \det(a)_{32} & \det(a)_{33} \end{bmatrix}$$

For example the 6 x 6 matrix $[A]$ defined as Equation (3-5) has as its determinant

$$\det \lfloor \bar{A} \rfloor = \det \begin{bmatrix} 3 & 0 & 0 \\ 5 & -3 & 0 \\ -65 & -17 & -38 \end{bmatrix} = 342$$

Rule 3.10: The product of two block lower triangular matrices is also a block lower triangular matrix.

Rule 3.11: If a matrix can be partitioned to become a block lower triangular matrix, then the eigenvalues of the original matrix are the eigenvalues of each diagonal submatrix.

For example, the eigenvalue of A defined in Equation (3-5) is equal to the eigenvalues of $(a)_{11}$, $(a)_{22}$ and $(a)_{33}$ as defined in Equation (3-7).

Rule 3.12: The eigenvalues of a block lower triangular matrix are the eigenvalues of the elements (i.e., submatrices) of its block diagonal matrices.

Let $C = A \cdot B$ where A and B are block lower triangular matrices.

Rule 3.13: The matrix C is also a block lower triangular matrix and its block diagonal submatrix is the product of the block diagonal submatrix of A and B.

$$D(C) = D(A \cdot B) = \begin{bmatrix} (a)_{11} \cdot (b)_{11} & & 0 \\ & \ddots & \\ 0 & & (a)_{nn} \cdot (b)_{nn} \end{bmatrix}$$

For example, if

$$B = \begin{bmatrix} \begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 3 & 4 \\ 7 & 9 \end{bmatrix} & \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 6 \\ 8 & 5 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \end{bmatrix}$$

and A is defined in Equation (3-5), then

$$D(C) = D(A \cdot B) = D(A) \cdot D(B)$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 6 & 4 \end{bmatrix} & & 0 \\ & \begin{bmatrix} 3 & 6 \\ 4 & 7 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 7 & 5 \end{bmatrix} & \\ 0 & & \begin{bmatrix} 2 & 8 \\ 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \end{bmatrix}$$

$$D(C) = \begin{bmatrix} \begin{bmatrix}36 & 22\\ 27 & 17\end{bmatrix} & & 0 \\ & \begin{bmatrix}48 & 33\\ 57 & 39\end{bmatrix} & \\ 0 & & \begin{bmatrix}18 & 26\\ 7 & 8\end{bmatrix} \end{bmatrix}$$

3.2 STABILITY FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Stability theory is widely used in linear systems described by linear ordinary differential equations. However, in many engineering problems, partial differential equations are involved, for which stability criteria are needed. If the discretized method discussed in Section 2.2 is applied, a general linear partial differential equation can be changed to a set of ordinary difference differential equations. Then all the stability criteria for ordinary differential equations can be directly applied. Some interesting results are obtained in the following.

From Equations (2-20) and (2-21) and Rules 3.3 and 3.4, it is noted that the inverse matrix of M is a triangular matrix and its diagonal matrix has unity for all elements. Multiplying both sides of Equation (2-13) by the inverse matrix of M yields

$$\dot{X} = (M^{-1}N)X + M^{-1}U \tag{3-8}$$

Rules 3.5 and 3.6 indicate that the product of the inverse of matrix M and matrix N is a lower triangular matrix whose diagonal matrix is the diagonal matrix of N (i.e., M has unity diagonal elements).

$$D(M^{-1}N) = D(N) \tag{3-9}$$

Since the eigenvalues for a lower triangular matrix are just the diagonals of that matrix, the eigenvalues for matrix $(M^{-1}N)$ are from Equation (2-19):

$$A = g(Z) + (2/\Delta Z)f(Z) \tag{3-10}$$

where ΔZ can be selected as small as possible. Let

$$H(Z) = \lim_{\Delta Z \to 0} (\Delta Z/2)\cdot A = f(Z)$$

Definition: A function f(Z) is defined as positive (negative) in the interval $0 < Z < L$ if, for all the points Z in this interval, the function f has only positive (negative) values.

For a general linear first order partial differential equation such as Equation (2-9), the criteria of stability for $0 < Z < L$ are derived as following:

1. If f(Z) is always negative, then the system is equilibrium stable.
2. If f(Z) has a positive value at any point Z, then the system is unstable.
3. If f(Z)=0 for any Z in $0 < Z < L$, then the criterion depends on g(Z). If g(Z) is always negative, then the system is equilibrium stable. If g(Z) is positive at any point Z, the system is unstable.
4. If f(Z) and g(Z) are both zero at any point, and for all other points f(Z) has negative values, then the system is weakly stable.

The same criteria are here applied to numerical method analysis. As shown below, criteria exist for selection of the number of mesh points (n) in order that the numerical analysis of the differential equations will not cause divergence for a stable system. The criteria for stability are:

1. If both g(Z) and f(Z) are negative for all Z in $0 < Z < L$, then there is no limitation for selection of number of mesh points.
2. If f(Z) is negative and g(Z) is positive for a given Z, then the choice should satisfy

   $$g(Z) + 2 \cdot f(Z)/\Delta Z < 0$$

3. If f is positive and g is negative, then the choice should satisfy

   $$g(Z) + 2 \cdot f(Z)/\Delta Z < 0$$

   In this case, the system is unstable.

The convergence and the accuracy of the solution are also dependent on the selection of ΔZ. A more specific discussion is shown in Reference 19 .

3.3 STABILITY FOR SIMULTANEOUS PARTIAL DIFFERENTIAL EQUATIONS

A further expansion of stability theory to simultaneous partial differential equations is useful in fluid mechanics and heat transfer problems, since the governing equations simultaneously involve the conservation of mass, momentum and energy. A general first order simultaneous set of linear partial differential equations can be represented as

$$\dot{X}_1 = \left[F_{11}\frac{\partial}{\partial Z}X_1 + \quad \cdots \quad + F_{1m}\frac{\partial}{\partial Z}X_m\right] + \left[G_{11}X_1 + \quad \cdots \quad + G_{1m}X_m\right] + U_1$$

$$\vdots$$

$$\dot{X}_m = \left[F_{m1}\frac{\partial}{\partial Z}X_1 + \quad \cdots \quad + F_{mm}\frac{\partial}{\partial Z}X_m\right] + \left[G_{m1}X_1 + \quad \cdots \quad + G_{mm}X_m\right] + U_m \qquad (3\text{-}11)$$

where F, G and U are all functions of Z and m is the number of equations. Equation (3-11) can be rewritten in matrix form as

$$(\dot{X}) = (F)\frac{\partial}{\partial Z}(X) + (G)(X) + (U) \qquad (3\text{-}12)$$

where

$$(X) = \begin{bmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{bmatrix}_{m \times 1} \qquad (F) = \begin{bmatrix} F_{11} & \cdots & F_{1m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ F_{m1} & \cdots & F_{mm} \end{bmatrix}_{m \times m}$$

$$(U) = \begin{bmatrix} U_1 \\ \cdot \\ \cdot \\ \cdot \\ U_m \end{bmatrix} \qquad (G) = \begin{bmatrix} G_{11} & \cdots & G_{1m} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ G_{m1} & \cdots & G_{mm} \end{bmatrix}_{m \times m}$$

(3-13)

If Equation (3-12) is integrated from Z_i to Z_{i+1}, the result is

$$(\dot{X})\ dZ = \int (F)\frac{\partial}{\partial Z}(X)\ dZ$$

$$+ \int (G)(X)\ dZ$$

$$+ \int (U)\ dZ$$

By the trapezoidal rule, the left hand side is

$$LHS = (\Delta Z/2)\left[(\dot{X})_{i+1} + (\dot{X})_i\right] \qquad (3\text{-}13a)$$

where $(X)_i = (X(Z_i))$. The first term on the right hand side is

$$RHS_1 = \int (F)\frac{\partial}{\partial Z}(X)\ dZ$$

$$= (F)_{i+.5} \int \frac{\partial}{\partial Z}(X)\ dZ$$

where (F) is considered constant over the small interval Z_i to Z_{i+1}. Then

$$RHS_1 = (F)_{i+.5}\left[(X)_{i+1} - (X)_i\right] \tag{3-13b}$$

The second term on the right hand side is

$$RHS_2 = \int (G)(X)\, dZ$$

$$= (G)_{i+.5} \int (X)\, dZ$$

$$= (G)_{i+.5}\left[(X)_{i+1} + (X)_i\right](\Delta Z/2) \tag{3-13c}$$

by an analysis similar to the RHS_1 term. The third term on the right hand side is similarly given by

$$RHS_3 = \int (U)\, dZ$$

$$= (\Delta Z/2)\left[(U)_{i+1} + (U)_i\right] \tag{3-13d}$$

Combination of Equations (3-13a), (3-13b), (3-13c) and (3-13d) yields

$$(\dot{X})_{i+1} + (\dot{X})_i = (2/\Delta Z)(F)_{i+.5}\left[(X)_{i+1} - (X)_i\right]$$

$$+ (G)_{i+.5}\left[(X)_{i+1} + (X)_i\right]$$

$$+ (U)_{i+1} + (U)_i$$

or

$$(\dot{X})_{i+1} + (\dot{X})_i = (A)_{i+1}(X)_{i+1} + (B)_{i+1}(X)_i + (C)_{i+1} \tag{3-14}$$

where

$$(A)_{i+1} = (G)_{i+.5} + (2/\Delta Z)(F)_{i+.5}$$

$$(B)_{i+1} = (G)_{i+.5} - (2/\Delta Z)(F)_{i+.5}$$

$$(C)_{i+1} = (U)_{i+1} + (U)_i \tag{3-15}$$

For i=0 to n-1, there are n differential difference matrix equations, represented as

$$(\dot{X})_1 = (A)_1(X)_1 + (B)_1(X)_0 + (C)_1 - (\dot{X})_0$$

$$(\dot{X})_2 + (\dot{X})_1 = (A)_2(X)_2 + (B)_2(X)_1 + (C)_2$$

$$\vdots$$

$$(\dot{X})_n + (\dot{X})_{n-1} = (A)_n(X)_n + (B)_n(X)_{n-1} + (C)_n \tag{3-16}$$

If $[X]$, $[M]$, $[N]$, $[U]$ and (I) are defined as

$$[X] = \begin{bmatrix} (X)_1 \\ \vdots \\ (X)_n \end{bmatrix}_{mn \times 1}$$

$$[M] = \begin{bmatrix} (I) & & & & \\ (I) & (I) & & 0 & \\ & & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & (I) & (I) \end{bmatrix}_{mn \times mn}$$

$$[N] = \begin{bmatrix} (A)_1 & & & & \\ (B)_2 & (A)_2 & & 0 & \\ & & \ddots & & \\ & & \ddots & \ddots & \\ & 0 & & (B)_n & (A)_n \end{bmatrix}_{mn \times mn}$$

$$[U] = \begin{bmatrix} (B)_1(X)_0 + (C)_1 - (\dot{X})_0 \\ (C)_2 \\ \vdots \\ (C)_n \end{bmatrix}_{mn \times 1}$$

and

$$(I) = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{m \times m} \qquad (3\text{-}17)$$

then Equation (3-16) can be written in matrix form as

$$[M]\,[\dot{X}] = [N]\,[X] + [U] \qquad (3\text{-}18)$$

Notice that $[M]$ and $[N]$ are block lower triangular matrices and the inverse matrix of M is

$$[M]^{-1} = \begin{bmatrix} (I) & & (0) \\ & \ddots & \\ (I) & & (I) \end{bmatrix}_{mn \times mn} \qquad (3\text{-}19)$$

Equation (3-19) indicates that the diagonal submatrices of M are block lower triangular matrices whose block diagonal matrices all have unit matrices as submatrices. If both sides of Equation (3-18) are multiplied by the inverse matrix of $[M]$, Equation (3-18) becomes

$$[\dot{X}] = [M]^{-1}[N][X] + [M]^{-1}[U] \qquad (3\text{-}20)$$

By Rules 3.10 and 3.13, the product of $[M]^{-1}[N]$ is a block lower triangular matrix with its diagonal submatrix equal to the diagonal submatrix of $[N]$, namely

$$D([M]^{-1}[N]) = D[N] \qquad (3\text{-}21)$$

Since the eigenvalues for a lower block triangular matrix are just the eigenvalues of its submatrix, according to Rule 3.12, then the eigenvalues of matrix $[M]^{-1}[N]$ are the eigenvalues of matrix $(A)_{i+1}$ from the definition of $[N]$ in Equation (3-17). Then, according to Equation (3-15), the diagonal submatrices are

$$(A)_{i+1} = (G)_{i+.5} + (2/\Delta Z)(F)_{i+.5}$$

Since ΔZ can be selected as small as desired,

$$H(Z) = \lim_{\Delta Z \to 0} (\Delta Z/2) \cdot A = (F)$$

since as $\Delta Z \to 0$, the second term in Equation (3-21) is much greater than the first term.

For a general set of simultaneous linear first order partial differential equations having a form similar to Equation (3-11), the criteria of stability in the interval $0 < Z < L$ are derived as following:

1. If the eigenvalue of (F(Z)) is negative for all Z, then the system is equilibrium stable.
2. If (F) has a positive eigenvalue for any value of Z, then the system is unstable.
3. If (F) has a zero eigenvalue, then the criteria depends on (G). If (G) has a negative eigenvalue for all Z, then the system is stable. If (G) has any positive eigenvalues, then the system is unstable.
4. If (F) and (G) both have zero eigenvalues at some point, and at all other points Z have negative eigenvalues, then the system is weakly stable.

3.4 STABILITY FOR HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS

The same development as in Section 3.3 is used here for higher order partial differential equations. A general second order partial differential equation can be written in matrix form as

$$(\dot{X}) = (F_2)\frac{\partial^2}{\partial Z^2}(X) + (F_1)\frac{\partial}{\partial Z}(X) + (F_0)(X) + (U) \tag{3-22}$$

where $(\dot{X})$, (F_2), (F_1), (F_0) and (U) are m x m matrices. Integration of Equation (3-22) from Z_i to Z_{i+1} yields

$$\begin{aligned} \int (\dot{X})\, dZ = & \int (F_2)\frac{\partial^2}{\partial Z^2}(X)\, dZ \\ & + \int (F_1)\frac{\partial}{\partial Z}(X)\, dZ \\ & + \int (F_0)(X)\, dZ \\ & + \int (U)\, dZ \end{aligned} \tag{3-22a}$$

By the trapezoidal rule, the left hand side is

$$LHS = (\Delta Z/2)\left[(\dot{X})_{i+1} + (\dot{X})_i\right] \tag{3-22b}$$

The first term on the right hand side is

$$RHS_1 = \int (F_2)\frac{\partial^2}{\partial Z^2}(X)\ dZ$$

$$= (F_2)_{i+.5}\int \frac{\partial^2}{\partial Z^2}(X)\ dZ$$

$$= (F_2)_{i+.5}\left[\frac{\partial}{\partial Z}(X)_{i+1} - \frac{\partial}{\partial Z}(X)_i\right] \tag{3-22c}$$

where (F_2) is considered constant over the small interval Z_i to Z_{i+1}. The second term on the right hand side is

$$RHS_2 = \int (F_1)\frac{\partial}{\partial Z}(X)\ dZ$$

$$= (F_1)_{i+.5}\int \frac{\partial}{\partial Z}(X)\ dZ$$

$$= (F_1)_{i+.5}\left[(X)_{i+1} - (X)_i\right]$$

The third and fourth terms on the right hand side are similarly given by

$$RHS_3 = \int (F_0)(X)\, dZ$$

$$= (F_0)_{i+.5} \int (X)\, dZ$$

$$= (F_0)_{i+.5}(\Delta Z/2)\left[(X)_{i+1} + (X)_i\right] \tag{3-22d}$$

$$RHS_4 = \int (U)\, dZ$$

$$= (\Delta Z/2)\left[(U)_{i+1} + (U)_i\right] \tag{3-22e}$$

Combination of Equations (3-22a) through (3-22e) yields

$$(\dot{X})_{i+1} + (\dot{X})_i = (2/\Delta Z)(F_2)_{i+.5}\left[\frac{\partial}{\partial Z}(X)_{i+1} - \frac{\partial}{\partial Z}(X)_i\right]$$

$$+ (2/\Delta Z)(F_1)_{i+.5}\left[(X)_{i+1} - (X)_i\right]$$

$$+ (F_0)_{i+.5}\left[(X)_{i+1} + (X)_i\right]$$

$$+ \left[(U)_{i+1} + (U)_i\right]$$

or in a simple notation,

$$(\dot{X})_{i+1} + (\dot{X})_i = (A)_{i+1}\frac{\partial}{\partial Z}(X)_{i+1} + (B)_{i+1}\frac{\partial}{\partial Z}(X)_i$$

$$+ (C)_{i+1}(X)_{i+1} + (D)_{i+1}(X)_i$$

$$+ (E)_{i+1}$$

where

$$(A)_{i+1} = (2/\Delta Z)(F_2)_{i+.5}$$

$$(B)_{i+1} = -(A)_{i+1}$$

$$(C)_{i+1} = (2/\Delta Z)(F_1)_{i+.5} + (F_0)_{i+.5}$$

$$(D)_{i+1} = -(2/\Delta Z)(F_1)_{i+.5} + (F_0)_{i+.5}$$

$$(E)_{i+1} = (U)_{i+1} + (U)_i$$

For i = 0 to n-1, there are n equations, namely

$$(\dot{X})_1 = (A)_1\frac{\partial}{\partial Z}(X)_1 + (B)_1\frac{\partial}{\partial Z}(X)_0$$

$$+ (C)_1(X)_1 + (D)_1(X)_0 + (E)_1 - (\dot{X})_0$$

$$(\dot{X})_2 + (\dot{X})_1 = (A)_2\frac{\partial}{\partial Z}(X)_2 + (B)_2\frac{\partial}{\partial Z}(X)_1$$

$$+ (C)_2(X)_2 + (D)_2(X)_1 + (E)_2$$

$$\vdots$$

$$(\dot{X})_n + (\dot{X})_{n-1} = (A)_n\frac{\partial}{\partial Z}(X)_n + (B)_n\frac{\partial}{\partial Z}(X)_{n-1}$$

$$+ (C)_n(X)_n + (D)_n(X)_{n-1} + (E)_n$$

These equations can be presented in a block matrix form as follows:

$$\left[M\right]\left[\dot{X}\right] = \left[N\right]\frac{\partial}{\partial Z}\left[X\right] + \left[P\right]\left[X\right] + \left[E\right] \qquad (3\text{-}23)$$

where $[X]$, $[M]$, $[N]$, $[P]$ and $[E]$ are defined as

$$X = \begin{bmatrix} (X)_1 \\ \vdots \\ (X)_n \end{bmatrix}_{mn \times 1}$$

$$M = \begin{bmatrix} (I) & & & & (0) \\ (I) & (I) & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ (0) & & & (I) & (I) \end{bmatrix}_{mn \times mn}$$

$$\left[N\right] = \begin{bmatrix} (A)_1 & & & (0) \\ (B)_2 & (A)_2 & & \\ & \ddots & \ddots & \\ (0) & & (B)_n & (A)_n \end{bmatrix}_{mn \times mn}$$

$$\left[P\right] = \begin{bmatrix} (C)_1 & & & (0) \\ (D)_2 & (C)_2 & & \\ & \ddots & \ddots & \\ (0) & & (D)_n & (C)_n \end{bmatrix}_{mn \times mn}$$

$$\left[E\right] = \begin{bmatrix} (B)_1 \frac{\partial}{\partial Z}(X)_0 + (D)_1 (X)_0 + (E)_1 - (\dot{X})_0 \\ (E)_2 \\ \vdots \\ (E)_n \end{bmatrix}_{mn \times 1}$$

Multiplication of Equation (3-23) by M^{-1} yields

$$[\dot{X}] = [M]^{-1}[N]\frac{\partial}{\partial Z}[X] + [M]^{-1} P [X] + [M]^{-1}[E]$$

From Section 3.3, it has already been shown that the stability of this system is determined by the eigenvalues of $[M]^{-1}[N]$. Since both $[M]^{-1}$ and $[N]$ are lower triangular matrices, Equation (3-19) and Rules 3.10 and 3.13 indicate that the eigenvalues of $[M]^{-1}[N]$ are determined by the diagonal submatrix of $[N]$, that is, $(A)_i$, where

$$(A)_i = (2/\Delta Z)(F_2)_{i-.5}$$

In conclusion, if (H) and (G) are defined as:

$$(H) = \lim_{\Delta Z \rightarrow 0} (\Delta Z/2)(A) = (F_2)$$

and

$$(G) = \lim_{\Delta Z \rightarrow 0} (\Delta Z/2)(C) = (F_1)$$

the stability criteria for Equation (3-22) in the interval $0 < Z < L$ are:

1. If the eigenvalues of (H) are negative for all Z, then the system is equilibrium stable.
2. If (H) has positive eigenvalues for any point Z, then the system is unstable.
3. If (H) has a zero eigenvalue, then the criteria depend on (G). If (G) has a negative eigenvalue for all Z, then the system is stable. If (G) has any positive eigenvalues, then the system is unstable.
4. If (H) and (G) both have zero eigenvalues at the same point and at all other points have negative eigenvalues, the system is weakly stable.

Since (F_2) is a given function of Z, it is suggested to find the maximum eigenvalue by plotting the eigenvalues of (F_2) as an function of Z. For a finite number of mesh points, some important point affecting the results of the above conclusions may be missed otherwise.

3.5 STABILITY FOR SIMULTANEOUS PARTIAL DIFFERENTIAL EQUATIONS WITH A SIDE CONDITION

Another interesting extension of stability theory is presented here, namely the simultaneous linear partial differential equation with one side condition. Consider

$$\dot{X} = F_{11}(\frac{\partial}{\partial Z}X) + F_{12}(\frac{\partial}{\partial Z}Y) + F_{13}(\frac{\partial}{\partial Z}W)$$

$$+ G_{11}X + G_{12}Y + G_{13}W + U_1 \qquad (3\text{-}24)$$

$$\dot{Y} = F_{21}(\frac{\partial}{\partial Z}X) + F_{22}(\frac{\partial}{\partial Z}Y) + F_{23}(\frac{\partial}{\partial Z}W)$$

$$+ G_{21}X + G_{22}Y + G_{23}W + U_2 \qquad (3\text{-}25)$$

$$W = aX + bY \qquad (3\text{-}26)$$

Substituting Equation (3-26) into Equations (3-24) and (3-25) in order to eliminate terms containing W yields

$$\dot{X} = (F_{11} + aF_{13})(\frac{\partial}{\partial Z}X) + (F_{12} + bF_{13})(\frac{\partial}{\partial Z}Y)$$

$$+ (G_{11} + aG_{13})(X) + (G_{12} + bG_{13})(Y) + U_1 \qquad (3\text{-}27)$$

$$\dot{Y} = (F_{21} + aF_{23})(\frac{\partial}{\partial Z}X) + (F_{22} + bF_{23})(\frac{\partial}{\partial Z}Y)$$

$$+ (G_{21} + aG_{23})(X) + (G_{22} + bG_{23})(Y) + U_2 \qquad (3\text{-}28)$$

Define matrices (F) and (G) as

$$(F) = \begin{bmatrix} F_{11} + aF_{13} & F_{12} + bF_{13} \\ F_{21} + aF_{23} & F_{22} + bF_{23} \end{bmatrix}$$

$$(G) = \begin{bmatrix} G_{11} + aG_{13} & G_{12} + bG_{13} \\ G_{21} + aG_{23} & G_{22} + bG_{23} \end{bmatrix}$$

From the theory derived in Section 3.3, the stability criteria are determined as follows:

1. If all eigenvalues of matrix (F) are negative, then the system is stable.
2. If any eigenvalue of matrix (F) is positive, then the system is unstable.
3. If any eigenvalue of (F) is zero, the stability is determined by matrix (G).

As a special case, if the left hand side of Equation (3-26) is zero, then

$$Y = C(Z)\cdot X \tag{3-29}$$

and the situation reduces to a singular matrix problem. Substitution of Equation (3-29) into Equations (3-24) and (3-25) to eliminate terms containing Y yields

$$\dot{X} = F'_{11}(\frac{\partial}{\partial Z}X) + F_{13}(\frac{\partial}{\partial Z}W) + G'_{11}X + G'_{13}W + U_1 \tag{3-30a}$$

$$\dot{X} = F'_{21}(\frac{\partial}{\partial Z}X) + F'_{23}(\frac{\partial}{\partial Z}W) + G'_{21}X + G'_{23}W + U'_2 \tag{3-30b}$$

where

$$F'_{11} = F_{11} + C \cdot F_{12}$$

$$F'_{21} = F_{21}/C + F_{22}$$

$$F'_{23} = F_{23}/C$$

$$G'_{11} = G_{11} + C \cdot G_{12} + (\frac{\partial}{\partial Z} C) \cdot F_{12}$$

$$G'_{21} = G_{21}/C + G_{22} + (\frac{\partial}{\partial Z} C) \cdot F_{22}/C$$

$$G'_{23} = G_{23}/C$$

$$U'_2 = U_2/C \tag{3-31}$$

Notice that since both equations in (3-30) have left hand sides containing X only, all the rules derived before cannot be applied in this special case. Consider the discretized method again. Integration of Equation (3-30a) from Z_i to Z_{i+1} yields

$$\int (\dot{X})\, dZ = \int F'_{11}\left(\frac{\partial}{\partial Z}X\right) dZ$$

$$+ \int F_{13}\left(\frac{\partial}{\partial Z}W\right) dZ$$

$$+ \int G'_{11}X\, dZ$$

$$+ \int G'_{13}W\, dZ$$

$$+ \int U_1\, dZ$$

By the trapezoidal rule, the left hand side is

$$LHS = (\Delta Z/2)\left[(\dot{X})_{i+1} + (\dot{X})_i\right]$$

The terms on the right hand side are

$$RHS_1 = (F'_{11})_{i+.5}\left[(X)_{i+1} - (X)_i\right]$$

$$RHS_2 = (F_{13})_{i+.5}\left[(W)_{i+1} - (W)_i\right]$$

$$RHS_3 = (G'_{11})_{i+.5}\left[(X)_{i+1} + (X)_i\right](\Delta Z/2)$$

$$RHS_4 = (G'_{13})_{i+.5}\left[(W)_{i+1} + (W)_i\right](\Delta Z/2)$$

$$RHS_5 = \left[(U_1)_{i+1} + (U_1)_i\right](\Delta Z/2)$$

Combination of the right hand side and left hand side yields the following matrix form:

$$\left[M\right]\left[\dot{X}\right] = \left[N_{11}\right]\left[X\right] + \left[N_{12}\right]\left[W\right] + \left[U_1\right] \qquad (3\text{-}32)$$

where

$$\left[M\right] = \begin{bmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & 1 \end{bmatrix}$$

$$\left[N_{11}\right] = \begin{bmatrix} A_1 & & & \\ B_2 & A_2 & & \huge 0 \\ & \ddots & \ddots & \\ \huge 0 & & B_n & A_n \end{bmatrix} \tag{3-33}$$

and

$$A_i = (2/\Delta Z)(F'_{11})_{i+.5} + (G'_{11})_{i+.5}$$

$$B_i = -(2/\Delta Z)(F'_{11})_{i+.5} + (G'_{11})_{i+.5}$$

$$\left[N_{12}\right] = \begin{bmatrix} C_1 & & & \\ D_2 & C_2 & & \huge 0 \\ & \ddots & \ddots & \\ \huge 0 & & D_n & C_n \end{bmatrix} \tag{3-34}$$

and

$$C_i = (2/\Delta Z)(F_{13})_{i+.5} + (G'_{13})_{i+.5}$$

$$D_i = -(2/\Delta Z)(F_{13})_{i+.5} + (G'_{13})_{i+.5}$$

Integration of Equation (3-30b) in the same way yields

$$[M]\,[\dot{X}] = [N_{21}]\,[X] + [N_{22}]\,[W] + [U_2] \qquad (3\text{-}35)$$

where

$$[N_{21}] = \begin{bmatrix} A'_1 & & & \\ B'_2 & A'_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & B'_n & A'_n \end{bmatrix}$$

$$[N_{22}] = \begin{bmatrix} C'_1 & & & \\ D'_2 & C'_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & D'_n & C'_n \end{bmatrix}$$

and

$$A'_i = (2/\Delta Z)\,(F'_{21})_{i+.5} + (G'_{21})_{i+.5}$$

$$B_i' = -(2/\Delta Z)\,(F_{21}')_{i+.5} + (G_{21}')_{i+.5}$$

$$C_i' = (2/\Delta Z)\,(F_{23}')_{i+.5} + (G_{23}')_{i+.5}$$

$$D_i' = -(2/\Delta Z)\,(F_{23}')_{i+.5} + (G_{23}')_{i+.5}$$

If $[W]$ is eliminated from Equations (3-32) and (3-35), the result is

$$([N_{22}]^{-1} - [N_{12}]^{-1})\,[M]\,[X] = [N_{22}]^{-1}[U_2] - [N_{12}]^{-1}[U_1]$$

$$+ ([N_{22}]^{-1}[N_{21}] - [N_{12}]^{-1}[N_{11}])\ X$$

Then

$$[\dot{X}] = [G]\,[X] + [H]$$

where

$$[G] = [M]^{-1}([N_{22}]^{-1} - [N_{12}]^{-1})^{-1}\cdot$$

$$([N_{22}]^{-1}[N_{21}] - [N_{12}]^{-1}[N_{11}])$$

(3-36)

$$H = [M]^{-1}([N_{22}]^{-1} - [N_{12}]^{-1})^{-1}$$

$$([N_{22}]^{-1}[U_2] - [N_{12}]^{-1}[U_1])$$

Since $[M]$, $[N_{11}]$, $[N_{12}]$, $[N_{21}]$, and $[N_{22}]$ are all lower triangular matrices, the matrix $[G]$ is also a lower triangular matrix.

If rules 3.5, 3.6 and 3.7 are applied, the eigenvalues of $[G]$ are just the diagonal elements of matrix $[G]$ and

$$F(Z) = \lim_{\Delta Z \to 0} D([G])$$

$$= \lim_{\Delta Z \to 0} (C_i' - C_i)^{-1}(C_i'A_i - C_iA_i')$$

$$= (2/\Delta Z)^2(F_{23}' - F_{13})^{-1}(F_{23}F_{11}' - F_{13}F_{21}')$$

(3-37)

If the expressions for F_{11}' and F_{21}' in Equation (3-31) are used, the system stability is determined by a function

H(Z) which is defined as:

$$H(Z) = \frac{F_{23}(F_{11} + C\cdot F_{12})/C - F_{13}(F_{21}/C + F_{22})}{(F_{23}/C - F_{13})}$$

or

$$H(Z) = \frac{F_{23}(F_{11} + C\cdot F_{12}) - F_{13}(F_{21} + C\cdot F_{22})}{(F_{23} - C\cdot F_{13})}$$

(3-38)

4. IMPLEMENTATION

Excursion is a term used to describe a flow instability where a slight perturbation can cause a drastic transient in the flow conditions. A new equilibrium level is attained after the excursion. The change is irreversible and can also produce a temperature excursion. Flow oscillations, in which both flow rate and pressure undergo periodic oscillation about a mean level, occur under certain conditions of slight disturbance in two phase flow. They can be damped, neutral, or growing. In a growing oscillation, the amplitude of the flow and pressure fluctuations increases with time and many even cause a flow reversal. Such a situation can lead to the mechanical failure of a system. A damped oscillation generally seeks a stability point near the normal operating condition after a sufficient period of time. A neutral oscillation persists indefinitely and can be tolerated if the amplitude is not too large.

Boiling two phase flow is an extremely interesting subject because it involves the simultaneous transport of momentum, heat and mass between solids and fluids and also across the liquid-vapor interface.

As nuclear technology flourished, the study of boiling heat transfer received a greater impetus because boiling is an extremely efficient mode of transferring heat from both a nuclear reactor and a steam generator.

4.1 MATHEMATICAL MODEL FOR REACTOR HEAT TRANSFER

The nuclear reactor or the steam generator in Figure 1-1 can be simulated by a heated tube model.[20,21] The fluid is heated as it flows through a cylindrical tube of length L and diameter D as shown in Figure 4-1. The heat flux (i.e., q"(Z,t)) into the fluid is a function of time and position. A mathematic model is needed to describe the dynamic variation of the fluid in the tube.

Since the nuclear reactor is usually operated at high pressures (7 MPa on the secondary side of the steam generator and 15 MPa on the reactor side) and the pressure drop through the whole reactor or steam generator is about 0.1 MPa, it is suitable to assume that the compressibility effects and changes in fluid properties can be ignored, i.e., the pressure can be assumed to be a constant within the reactor or steam generator. The heat transfer of the fluid in the tube can be separated into three different regions: a subcooled region, a two phase region and a superheated region. The governing equations for the fluid can be written as:

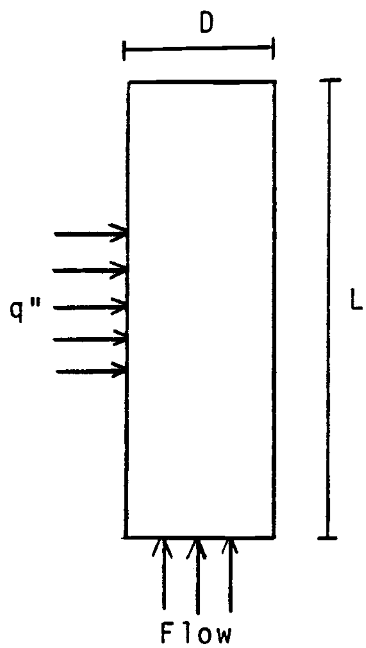

Figure 4-1. The Heated Tube Model

Mass conservation:

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial Z}(\rho V) \qquad (4\text{-}1)$$

Energy conservation:

$$\frac{\partial}{\partial t}(\rho h) = -\frac{\partial}{\partial Z}(\rho V h) + Pq''/A \qquad (4\text{-}2)$$

Physical property of fluid:

$$\rho = \rho(P, h) \tag{4-3}$$

where

$\rho(Z,t)$ is the fluid density (kg/m^3),

$V(Z,t)$ is the flow velocity (m/sec),

$h(Z,t)$ is the enthalpy of the fluid (kJ/kg),

$q''(Z,t)$ is the heat flux (kJ/m^2-s),

P is the heated perimeter of the tube (m), and

A is the cross sectional area of the tube (m^2)

The enthalpy is the physical property of the fluid that relates to the internal energy. That is, in the subcooled or superheated region, the temperature is used to represent the internal energy, but in the two-phase region the temperature is constant, so the enthalpy is used. With a proper initial condition and boundary conditions, Equations (4-1) through (4-3) will give the solution for the fluid properties ρ, V and h.

4.2 SUBCOOLED FLUID

For a subcooled fluid, the fluid is incompressible and the heat flux is constant. Then Equation (4-1) becomes:

$$\frac{\partial}{\partial t}\rho = -\frac{\partial}{\partial Z}(\rho V) = 0 \qquad (4\text{-}4)$$

so the product ρV is a function of time only. Thus Equation (4-2) becomes:

$$\rho\left(\frac{\partial}{\partial t}h\right) = -(\rho V)\left(\frac{\partial}{\partial Z}h\right) + Pq''/A \qquad (4\text{-}5)$$

Rearrangement of Equation (4-5) yields:

$$\frac{\partial}{\partial t}h = -(V)\left(\frac{\partial}{\partial Z}h\right) + Pq''/(A\rho) \qquad (4\text{-}6)$$

Because -(V) is always negative, the system is stable if V>0. If V=0, the system is equivalent to the heating of a closed pool, where eventually all the liquid will be evaporated.

4.3 TWO PHASE FLUID AND SUPERHEATED FLUID

For two phase fluid flow, the governing equation is the same as for a superheated fluid. Assume a constant pressure and time independent heat flux q" = q(z) and change variables such that $X = \rho$ and $Y = (\rho h)$. Then Equations (4-1) to (4-3) can be rewritten as:

$$\dot{X} = -\frac{\partial}{\partial z}(XV) \tag{4-7}$$

$$\dot{Y} = -\frac{\partial}{\partial z}(YV) + Pq''/A \tag{4-8}$$

$$X = f(Y) \tag{4-9}$$

There are three equations for the three unknowns X, Y and V. The method of stability analysis for this system is to (1) use the small perturbation method to linearize the governing equations, and (2) use the stability criteria derived in Chapter 3.

The first step of the perturbation method is finding the equilibrium solution. In equilibrium state, Equations (4-7) and (4-8) become

$$0 = -\frac{\partial}{\partial Z}(XV)$$

$$0 = -\frac{\partial}{\partial Z}(YV) + Pq''/A \qquad (4\text{-}10)$$

with boundary conditions

$$V(0,0) = V_e(0)$$

$$h(0,0) = h_e$$

and external heat flux $q''(Z)$. The equilibrium solutions are:

$$X = X_e(Z) \qquad (4\text{-}11)$$

$$Y = Y_e(Z) \qquad (4\text{-}12)$$

and

$$V = V_e(Z) \qquad (4\text{-}13)$$

For a small perturbation, the governing equations are then linearized as follows:

$$X(Z,t) = X_e(Z) + \overline{X}(Z,t) \qquad (4\text{-}14)$$

$$Y(Z,t) = Y_e(Z) + \overline{Y}(Z,t) \tag{4-15}$$

$$V(Z,t) = V_e(Z) + \overline{V}(Z,t) \tag{4-16}$$

where $\overline{X}$, $\overline{Y}$ and $\overline{V}$ are small perturbation variables. Substituting Equations (4-14), (4-15) and (4-16) into Equations (4-7) to (4-9) yields:

$$\dot{\overline{X}} = - \frac{\partial}{\partial Z}\left[(X_e + \overline{X})(V_e + \overline{V})\right]$$

$$\dot{\overline{Y}} = - \frac{\partial}{\partial Z}\left[(Y_e + \overline{Y})(V_e + \overline{V})\right] + Pq''/A$$

$$X_e + \overline{X} = f(Y_e + \overline{Y})$$

since $\dot{X}_e$ and $\dot{Y}_e$ are equal to zero by definition. Rearrangement yields:

$$\dot{\overline{X}} = - \frac{\partial}{\partial Z}\left[X_e V_e\right] - \frac{\partial}{\partial Z}\left[X_e \overline{V} + \overline{X} V_e + \overline{X}\overline{V}\right]$$

$$\dot{\overline{Y}} = - \left[\frac{\partial}{\partial Z} Y_e V_e\right] + Pq''/A - \frac{\partial}{\partial Z}\left[Y_e \overline{V} + \overline{Y} V_e + \overline{Y}\overline{V}\right]$$

$$X_e + \overline{X} = f(Y_e + \overline{Y})$$

Elimination of the equilibrium state part of these equations by the equilibrium solution and expansion of the third equation in a Taylor series yields:

$$\dot{X} = - \frac{\partial}{\partial Z}\left[X_e\overline{V} + \overline{X}V_e + \overline{X}\overline{V}\right]$$

$$\dot{Y} = - \frac{\partial}{\partial Z}\left[Y_e\overline{V} + \overline{Y}V_e + \overline{Y}\overline{V}\right]$$

$$\overline{X} = \left(\frac{\partial}{\partial Y}f\right)\Big|_{Y_e}\overline{Y} + 0.5\left(\frac{\partial^2}{\partial Y^2}f\right)\Big|_{Y_e}\overline{Y}^2 + \ldots$$

Simplification of these equations by neglecting higher order terms results in:

$$\dot{X} = - \frac{\partial}{\partial Z}\left[X_e\overline{V} + \overline{X}V_e\right]$$

$$\dot{Y} = -\left[\frac{\partial}{\partial Z}\; Y_e\overline{V} + \overline{Y}V_e\right]$$

$$\overline{X} = \left(\frac{\partial}{\partial Y}f\right)\Big|_{Y_e}\overline{Y}$$

Rearrangement of these equations yields:

$$\dot{X} = (-V_e)\frac{\partial}{\partial Z}\overline{X} + (0) + (-X_e)\frac{\partial}{\partial Z}\overline{V}$$

$$+ (-\frac{\partial}{\partial Z}V_e)\overline{X} + (0) + (-\frac{\partial}{\partial Z}X_e)\overline{V}$$

$$\dot{Y} = (0) + (-V_e)\frac{\partial}{\partial Z}\overline{Y} + (-Y_e)\frac{\partial}{\partial Z}\overline{Y}$$

$$+ (0) + (-\frac{\partial}{\partial Z}V_e)\overline{Y} + (-\frac{\partial}{\partial Z}Y_e)\overline{V}$$

$$\overline{X} = (\frac{\partial}{\partial Y}f)\Big|_{Y_e}\overline{Y} \qquad (4\text{-}17)$$

Equations (4-17) have the same form as Equations (3-24), (3-25) and (3-29) in Section 3.5. According to the results derived in Section 3.5, the stability criteria (see Equation (3-38) where $F_{11} = -V_e$, $F_{12} = 0$, $F_{13} = -X_e$, $F_{21} = 0$, $F_{22} = -V_e$ and $F_{23} = -Y_e$) depend on

$$H(Z) = \frac{(-Y_e)\ (-V_e) + (0)\ - (-X_e)\ (0) + C(-V_e)}{(-Y_e) - C(-X_e)}$$

$$= \frac{Y_e V_e - CX_e V_e}{CX_e - Y_e}$$

$$= -V_e$$

Thus the system is stable as long as $V_e > 0$, i.e., H(Z) is negative. This is the same conclusion as for subcooled heating.

4.4 EXCURSION INSTABILITY

As shown in Section 4.3, the heated tube system with a constant heat flux is a stable system. In terms of system analysis, this system can be represented by the following block diagram:

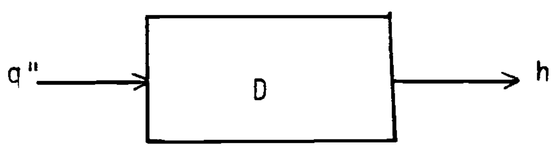

Figure 4-2 Block Diagram for Heated Tube System

where q" is the input, h is the output and D is an operator. If the output is the wall temperature of the tube instead of enthalpy, the diagram can be expanded as

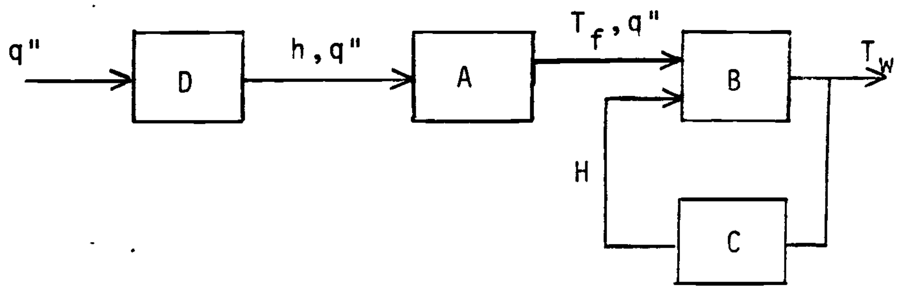

Figure 4-3. Block Diagram for Wall Temperature

where T_f is the temperature of the fluid, T_w is the temperature of the tube wall, and A, B, C are three different operators. Operator A corresponds to the steam tables

$$T_f = T(P,h),$$

operator B corresponds to Newton's law of heat convection, that is,

$$q'' = H(T_w - T_f)$$

which implies

$$T_w = T_f + q''/H,$$

and operator C corresponds to the heat transfer coefficient correlation

$$H = H(T_w, T_f, q'')$$

In terms of system analysis, the following diagram results:

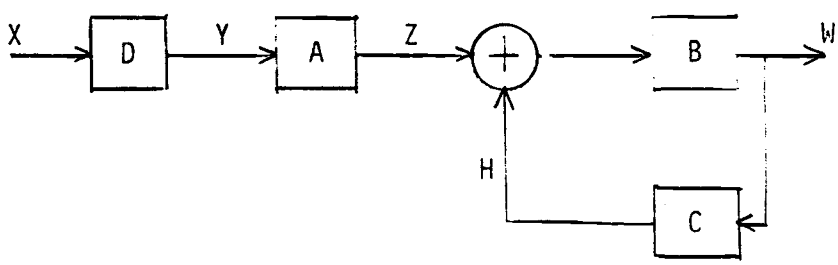

Figure 4-4. Block Diagram for Tube Wall Temperature

where X is the input q", Y is the combination of h and q", Z is T_f and q", and W is the output T_w. Notice that this is a feedback system.

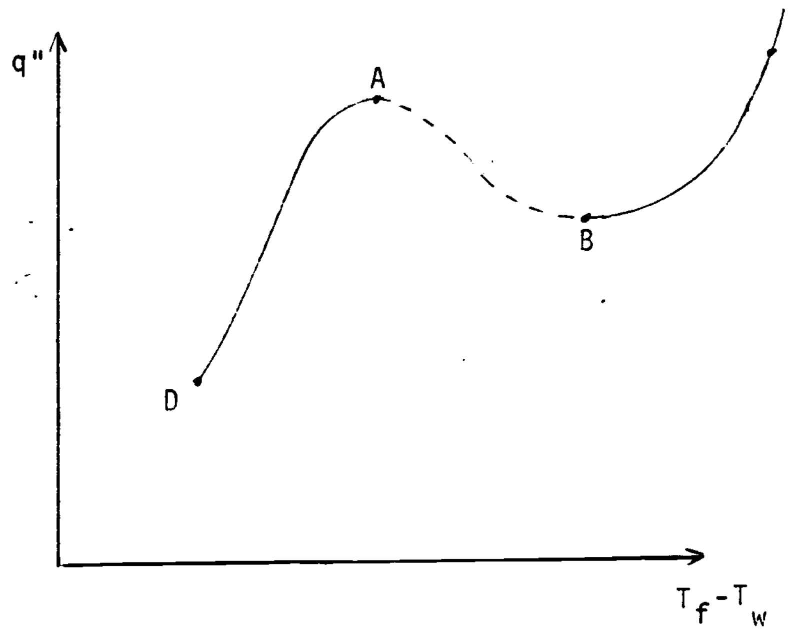

Figure 4-5. Heat Transfer Correlation

Many empirical data have been collected for this type of system. The feedback operator C can be represented in Figure 4.5. It is found that if the whole system is operating at point A of Figure 4-5, a small increase of q" will cause the operating point to shift to point C. This is the well known excursion phenomena in nuclear reactor safety analysis.[22,23] In terms of system analysis, in the range from A to B, the system has a positive feedback which corresponds to an excursion instability. In the range B to C or D to A, the system has a negative feedback, which corresponds to a stable system. Since there is a shortage of feedback analysis theory for a distributed system and since Figure 4-5 is in a graphical form instead of an analytical function, developing the theory for such a distributed system would be an interesting expansion beyond the scope of this thesis.

5. CONCLUSION

There is a certain difference between a physical problem and a mathematical model. This thesis discusses the stability analysis for a mathematical problem. For an actual physical problem, the empirical data should used to benchmark the mathematical results; otherwise some hazard may possibly exist.

Laplace transforms are usually used to solve the stability problem of a distributed system. This approach involves the numerical solution for the transformed equation and Nyquist plot. The disadvantage of this is it can not derive a general stability criterion for a general case. Each analysis must be performed independently with no idea about what is happening inside. This author is unaware of anyone trying to solve the stability problem of a distributed system directly from the eigenvalue approach as was done in this thesis. The reason this has not been previously done may be that this approach involves an eigenvalue search for a huge matrix that is beyond what a human or even a computer can comfortably and efficiently handle. By careful analysis of the matrix form, however, it was found that a general rule for eigenvalue searching for block triangular matrices exists to solve the huge matrix problem and the analytic solution was derived.

In this analysis, the stability criteria for a first order linear partial differential equation with distributed coefficients were derived first. Instead of discretized eigenvalues, distributed eigenvalues (i.e., eigenvalues as function of Z) were obtained. Based on these stability criteria, the general criteria for simultaneous partial differential equations were also determined. Furthermore, the set of simultaneous partial differential equations with a certain side condition was also solved. Special attention was paid to the singular matrix coefficients which have formats as in Equation (3-30). It can be predicted that for a singular matrix problem, the criteria matrix can be reduced to a lower order case, and the solution is achievable. The general case is worthy of further development. Based on those results, the system stability problem for a distributed parameter can be derived easily and systematically.

The heat transfer problem for a constant heat flux shows that the system is stable as long as the fluid velocity is positive. The heat transfer problem for a constant wall temperature can be simulated by a feedback system. With positive feedback, an excursion phenomena occurs, while with negative feedback a stable system exists.

In summary, the following were derived in this study:

1. The stability criteria for a linear partial differential equation.
2. The stability criteria for a set of simultaneous linear partial differential equations.
3. The stability criteria for a set of simultaneous linear partial differential equations with a side condition.
4. The stability criteria for a singular matrix coefficient.
5. The heat transfer stability of a heated tube for a time invariant heat flux without flow reversal.

It is important to point out that in order to make sense for the discretized method and limit calculations, all parameter functions in the governing equations should be smooth and continuous.

Whenever a linear mathematical model for a distributed system is established, the stability problem can be solved by the result derived in this paper. The trapezoidal approximation used in this paper can actually be replaced by the mean value theorem[24], which means no approximation has been made. A further study to include the feedback modeling for a distributed system is beyond the scope of this study and will be explored later. An extension of this study would be to establish a complete system theory for distributed systems including multi-dimensional systems, system feedback analysis, and closed loop system analysis.

BIBLIOGRAPHY

1. K. C. Lish, Nuclear Power Plant Systems and Equipment, Industrial Press Inc., New York (1972).

2. Y. A. Hassan and C. D. Morgen, "Transient Prediction of 19 Tube Once Through Steam Generator by RELAP5/MOD1," ASME, 71-HT-42.

3. C. J. Baroczy, "A Systematic Correlation for Two-Phase Pressure Drop," NAA-SR-MENO (1966).

4. G. B. Wallis and J. H. Heasley, "Oscillations in Two-Phase Flow Systems", Journal of Heat Transfer, A.S.M.E. Transactions, Series C, 83, 363 (1961).

5. K. C. Chan, "Thermal-Hydraulic Stability Analysis of Steam Generators," Ph.D. Thesis, University of California, Berkeley (1979).

6. P. Saha, "Thermally Induced Two-Phase Flow Instabilities, Including the Effect of Thermal Non-Equilibrium between the Phases" Ph.D. Thesis, Georgia Institute of Technology (1974).

7. C. T. Chen, Introduction to Linear System Theory, Holt, Rinehart and Winston, New York (1971).

8. W. M. Keys, Convection Heat and Mass Transfer, McGraw Hill Book Co., New York (1980).

9. J. J. Duderstadt and L. J. Hamilton., Nuclear Reactor Analysis, John Wiley and Sons, Inc., New York (1976).

10. A. H. Nayteh, Introduction to Perturbation Techniques, Blaisdell Publishing Co., Waltham, Massachusetts (1968)

11. J. Kevorkian and J. D. Cole, Perturbation Methods in Applied Mathematics, J. Wiley and Sons, Inc, New York (1981)

12. T. Kailath, Linear Systems, Prentice Hall, Inc., Englewood Cliffs, New York (1980).

13. L. Padulo and M. A. Arbib, System Theory, W. B. Saunders Co., Philadelphia (1974).

14. C. R. Wylie, Advanced Engineering Mathematics, McGraw Hill Book Co., New York (1975).

15. E. Kreyszig, Advanced Engineering Mathematics, J. Wiley and Sons Inc., New York (1972).

16. J. R. Lamarsh, Introduction to Nuclear Engineering, Addison-Wesley Publishing Co., Massachusetts (1983)

17. LaSalle and Lefshetz, Stability by Liapunov's Direct Method, Academic Press, New York (1961).

18. A. S. Jackson, Analog Computation, McGraw Hill Book Co., New York (1960).

19. B. Carnahan, H. A. Luther and J. O. Wilkes, Applied Numerical Methods, J. Wiley and Sons, Inc, New York (1969).

20. G. B. Wallis, One-Dimensional Two-Phase Flow, McGraw Hill Book Co., New York (1969).

21. J. R. Welty, C. E. Wicks and R. E. Wilson, Fundamentals of Momentum, Heat and Mass Transfer, J. Wiley and Sons, Inc., New York (1976).

22. R. T. Lahey, Jr. and F. J. Movely, The Thermal-Hydraulics of a Boiling Water Nuclear Reactor, American Nuclear Society (1979).

23. M. M. El-Wakil, Nuclear Heat Transport, International Textbook Co., New York (1971).

24. G. B. Thomas, Calculus and Analytic Geometry, Addison-Wesley Publishing Co., Massachusetts, (1972).

APPENDICES

APPENDIX A. IMPLEMENTATION OF PERTURBATION METHOD

It is frequently of interest to compute the small change or perturbation caused by a small change of the boundary condition of a system. Fortunately if this perturbation is small, one does not have to repeat the whole system calculation, but instead can use well-known techniques of perturbation theory to express the corresponding change in terms of the initial condition. Although the techniques are described here by means of an example that is an ordinary differential equation, the method is concise and applicable to partial differential equations as well. The perturbation method combined with a numerical technique such as a finite difference or a finite element technique, result in a very powerful and versatile technique.

In order to avoid the complexity of a physical problem, a purely mathematical problem is used as an illustration:

$$X' = -X^2 + 2X + 3 \quad \text{(A-1)}$$

where $X' = dX/dt$, with initial condition:

$$X(0) = 2 \quad \text{(A-2)}$$

The first step of the perturbation technique is to assume

$$X = X_e + x \quad \text{(A-3)}$$

where X_e is the equilibrium solution to Equation (A-1). Solving Equation (A-1) with the left hand side equal to zero (equilibrium state) yields

$$X_e = 3 \text{ or } -1 \quad \text{(A-4)}$$

Then substituting Equation (A-3) into Equations (A-1) and (A-2), respectively, yields

$$x' = (2 - 2X_e)x - x^2 \quad \text{(A-5)}$$

$$x(0) = X(0) - X_e \quad \text{(A-6)}$$

It should be pointed out that no approximation has been made in Equation (A-5), which has the same solution as Equation (A-1). The perturbation x represents the deviation from the equilibrium solution (i.e., $x = X - X_e$). If the perturbation is assumed to be small, the second order term in Equation (A-5) can be neglected, which yields

$$x' = (2 - 2X_e)x \qquad \text{(A-7)}$$

Instead of solving the nonlinear Equation (A-1), the analytical solution for the linear approximation Equation (A-7) is obtained:

$$x = \left[X(0) - X_e\right] \exp\left[(2 - 2X_e)t\right] \qquad \text{(A-8)}$$

$$X = X_e + x = X_e + \left[X(0) - X_e\right] \exp\left[(2 - 2X_e)t\right] \qquad \text{(A-9)}$$

Notice that the root $X_e = 3$ yields a stable solution, while $X_e = -1$ yields a divergent solution, so $X_e = 3$ is chosen. The exact solution of Equation (A-1) was solved by the predict-correct method. The comparison between the exact value and approximation is shown in Figure A-1. There are two points that should be noticed in this example. First, a nonlinear equation can be approximated by a linear equation in the neighborhood of the equilibrium state. Second, the approximate solution approaches the exact value asymptotically.

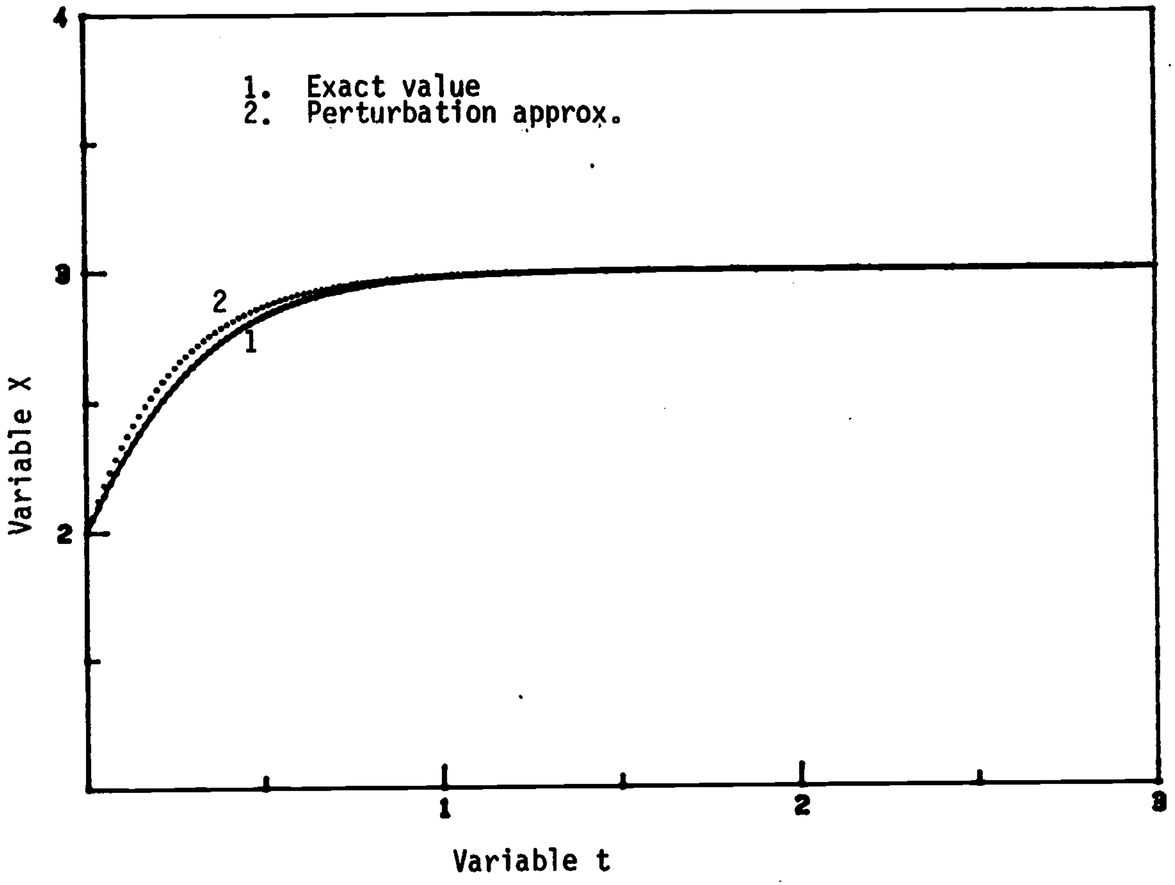

Figure A-1. Example Perturbation Method Solution

APPENDIX B. NUMERICAL CONVERGENCE PROBLEM AT A PHASE BOUNDARY

During the computation of two-phase flow, a common problem is the convergence at the phase boundary, i.e, the change from liquid to two-phase or from two-phase to vapor. The governing equations are the conservation of mass and energy equations, together with a side condition state equation which is actually the steam tables. The conservation equation gives enthalpy (h) as function of specific volume (v). The steam table gives (v) as a function of (h). Thus enthalpy and specific volume are related by two functions

$$h = F(v)$$

$$v = G(h)$$

By this logic, the problem is simplified to a root finding problem. A plot of F and G in the same figure is shown in

Figure B-1. Assume the iteration starts from point 1. The second point will be point 2 and then points 3, 4 and 5. However, the resulting iteration just repeats itself and never converges. This is a numerical convergence problem which comes from the singular point of the steam tables. Notice this is not a system instability. One way to solve this problem is to use the slope instead of the point for iteration (actually this can be considered as a modified Newton's method). Once the iteration loop is found, e.g., points 2, 3, 4 and 5), the slope method is begun. If line A is defined as the line passing through points 2 and 4 and line B is the line passing through point 3 and 5, then the formulas for lines A and B are:

$$A: v = (v_4 - v_2)/(h_4 - h_2)\ (h - h_2) + v_2$$

$$B: v = (v_3 - v_5)/(h_3 - h_5)\ (h - h_1) + v_5$$

The intersection will be

$$h = (h_3 + h_5)/2$$

since $h_3 = h_4$, $h_2 = h_5$, $v_2 = v_3$, and $v_4 = v_5$. In computer programming notation, this is written as:

$$h^{n+1} = (h^n + h^{n-1})/2$$

where h^n is the (n)th iteration value of enthalpy. This method was used successively to solve the convergence problem in the OTSG (once through steam generator) code. Results showed that the convergence was fast (usually no more than five iterations). It was found that during computer code development, it was hard to distinguish system instabilities from numerical instabilities, although once the numerical instability was identified, it was not hard to solve it.

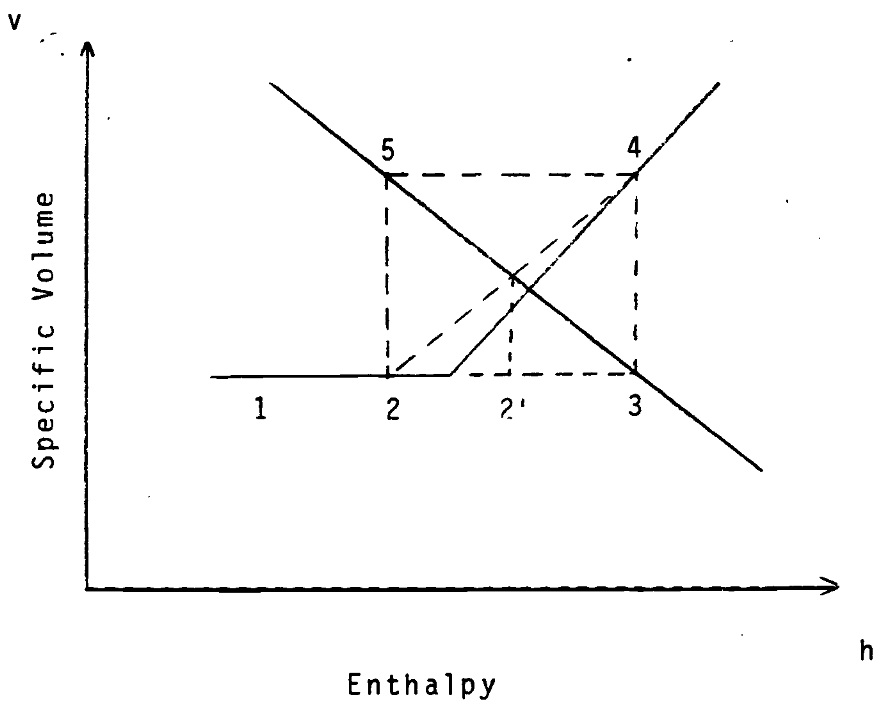

Figure B-1. The Convergence Problem at a Phase Boundary

APPENDIX C. PROOF OF MATRIX RULES

Rule 3.1 and Rules 3.3 through 3.7 for a lower triangular matrix are equivalent to the special case of a block lower triangular matrix (i.e., a 1x1 submatrix). Rules 3.8 and 3.9 are proved in reference 14. The proof for the rest of the rules involving block lower triangular matrices is shown below.

<u>Proof of Rule 3.10:</u>

Let C = A·B where A and B are mxm block lower triangular matrices with nxn submatrices as elements and $(A)_{ij}$, $(B)_{ij}$ and $(C)_{ij}$ are submatrices in column j and row i of matrices (A), (B) and (C), respectively. Prove that $(C)_{ij}$ equals (0) for $j > i$.

By Rule 3.8

$$(C)_{ij} = \sum_{k=1}^{n} (A)_{ik} \cdot (B)_{kj}$$

For k = i+1 to n, matrix $(A)_{ik}$ is (0), which yields

$$(C)_{ij} = \sum_{k=1}^{i} (A)_{ik} \cdot (B)_{kj}$$

Also for k = 1 to j-1 < i, matrix $(B)_{kj}$ is (0).

Thus

$$(C)_{ij} = \sum_{k=j}^{i} (A)_{ik} \cdot (B)_{kj}.$$

If j > i, the sum does not exist and

$$(C)_{ij} = (0)$$

<u>Proof of Rule 3.11:</u>

Let C = λI - A where A is an m by m block lower triangular matrices with n by n submatrices as elements. Prove that

$$|C| = \prod_{i=1}^{m} |\lambda I - A_{ii}|$$

where $|C|$ represents the determinant of matrix C. Since matrix C is also a block lower triangular matrix, by Rule 3.9 the determinant of matrix C is just the product of the determinants of its block diagonal submatrices. That is

$$|C| = \prod_{i=1}^{m} \left| \lambda I - A_{ii} \right|$$

Proof of Rule 3.12:

Since the diagonal matrix keeps the same diagonal submatrices as the original matrix, Rule 3.12 can be considered as a corollary of Rule 3.11.

Proof of Rule 3.13:

Use the same notation as the proof of Rule 3.10. Prove

$$(C)_{ii} = (A)_{ii} \cdot (B)_{ii}$$

By Rule 3.8

$$(C)_{ii} = \sum_{k=1}^{n} (A)_{ik} \cdot (B)_{ki}$$

For k = i+1 to n, matrix $(A)_{ik}$ is (0), and for k = 1 to i-1, matrix $(B)_{ki}$ is (0), which yields

$$(C)_{ii} = \sum_{k=i}^{i} (A)_{ik} \cdot (B)_{ik} \qquad = A_{ii} \cdot B_{ii}$$

<u>Proof of Rule 3.2:</u>

In order to avoid a complex mathematical notation, a simple example is given to shown the logic of this proof.

Let matrix A be defined as

$$A = \left[\begin{array}{ccc:ccc} A_{11} & 0 & 0 & B_{11} & 0 & 0 \\ X & A_{22} & 0 & X & B_{22} & 0 \\ X & X & A_{33} & X & X & B_{33} \\ \hdashline C_{11} & 0 & 0 & D_{11} & 0 & 0 \\ X & C_{22} & 0 & X & D_{22} & 0 \\ X & X & C_{33} & X & X & D_{33} \end{array}\right]$$

where X represents any number (i.e., X has no relationship to the eigenvalues of matrix A). Changing column 4 with with column 2 and then changing row 4 with row 2 yields

$$\left[\begin{array}{cc:cc:cc} A_{11} & B_{11} & 0 & 0 & 0 & 0 \\ C_{11} & D_{11} & 0 & 0 & 0 & 0 \\ \hdashline X & X & A_{33} & X & X & B_{33} \\ X & X & 0 & A_{22} & B_{22} & 0 \\ \hdashline X & X & 0 & C_{22} & D_{22} & 0 \\ X & X & C_{33} & X & X & D_{33} \end{array}\right]$$

Changing column 3 with column 4 and then changing row 3 with row 4 yields

$$\left[\begin{array}{cc:cc:cc} A_{11} & B_{11} & 0 & 0 & 0 & 0 \\ C_{11} & D_{11} & 0 & 0 & 0 & 0 \\ \hdashline X & X & A_{22} & 0 & B_{22} & 0 \\ X & X & X & A_{33} & X & B_{33} \\ \hdashline X & X & C_{22} & 0 & D_{22} & 0 \\ X & X & X & C_{33} & X & D_{33} \end{array}\right]$$

Changing column 4 with column 5 and then changing row 4 with row 5 yields

$$\left[\begin{array}{cc:cc:cc} A_{11} & B_{11} & 0 & 0 & 0 & 0 \\ C_{11} & D_{11} & 0 & 0 & 0 & 0 \\ \hdashline X & X & A_{22} & B_{22} & 0 & 0 \\ X & X & C_{22} & D_{22} & 0 & 0 \\ \hdashline X & X & X & X & A_{33} & B_{33} \\ X & X & X & X & C_{33} & D_{33} \end{array}\right]$$

The same logic can be applied to the matrix $C = \lambda I - A$. Notice this is a block lower triangular matrix, and by Rule 3.11 the eigenvalues are equivalent to the eigenvalues of the diagonal block submatrices. Although this proof involved a 6 by 6 matrix with 2 by 2 submatrices, a similar procedure can be followed for an m by m matrix with n by n submatrices.

APPENDIX D. APPLICATION OF THE HEATED TUBE MODEL

A complete steam generator simulation can be partitioned into the following three major parts: (1) the primary side heat transfer, (2) the secondary side heat transfer and (3) the wall heat transfer. In actual computer programming, these correspond to three subroutines, that is,

$$H_1 = H_1(q")$$

$$H_2 = H_2(q")$$

$$q" = Q(H_1, H_2)$$

where H_1 and H_2 are the enthalpy of the primary side and secondary side, respectively, and q" is the heat transferred from the primary side to the secondary side.

The heated tube model can be applied to both part (1) and part (2) of a steam generator simulation. The stability of a heated tube model which corresponds to the stability of the inner loop iteraion of a computer program is discussed in this thesis. The feedback model for distributed systems which corresponds to the outer loop of computer programs is desirable to be developed but is beyond the scope of this thesis.