

AN ABSTRACT OF THE THESIS OF

PETER JOHN MURRAY for the M. S. in Mathematics
(Name) (Degree) (Major)

Date thesis is presented August 14, 1964

Title THE GREATEST INTEGER PART FUNCTION

Abstract approved Redacted for Privacy
(Major professor)

This thesis contains a collection of properties of the greatest integer part function which were obtained by an extensive literature search. A few original properties are stated and proved and some of the properties which were found unproved in the literature are proved.

THE GREATEST INTEGER PART FUNCTION

by

PETER JOHN MURRAY

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

MASTER OF SCIENCE

August 1964

APPROVED:

Redacted for Privacy

Associate Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Mathematics Department

Redacted for Privacy

Dean of Graduate School

Date thesis is presented August 14, 1964

Typed by Carol Baker

ACKNOWLEDGMENT

The author wishes to express his appreciation to Professor Stalley and to the library staff for their invaluable assistance.

TABLE OF CONTENTS

CHAPTER	PAGE
I INTRODUCTION	1
II PROPERTIES	5
Elementary Properties	5
Representations of $[x]$ in Terms of Other Functions	7
Formulas Relating $[x]$ to the Values of Other Functions of Number Theory	9
Summation Formulas	15
Miscellaneous Properties	31
Unsolved Problems	44
III COMMENTS AND PROOFS	46
IV INCORRECT RESULTS	60
BIBLIOGRAPHY	61

THE GREATEST INTEGER PART FUNCTION

CHAPTER I. INTRODUCTION

This thesis is a collection of properties of the greatest integer part function, defined as the function whose domain is the set of real numbers and whose value corresponding to a given real number is the largest integer less than or equal to the given number.

Applications of this function include the distribution of primes, divisibility properties of the integers, quadratic reciprocity, and game theory.

We answer some general questions about the nature of this function. The greatest integer part function

1. has as its domain the set of real numbers and has as its range the set of integers;
2. is discontinuous for integral values of its argument and continuous elsewhere;
3. has no inverse;
4. is monotonically increasing; and
5. possesses a zero derivative for non-integral values of its argument and has no derivative for integral values of its argument.

It is difficult to establish the origin of this function. Gauss (1777-1855) used it (4), and it is found quite commonly in the literature by the 1880's. The notation for the value of this function has

developed along two different lines. Most French and a few German mathematicians have used $E(x)$, a notation having its origin in the French word "entier", meaning integer, for example P. G. Lejeune-Dirichlet (24; 31), A. Legendre (30; 66), R. Lipschitz (25), and J. J. Sylvester (57, p. 738-739). Most others have used $[x]$, including L. Gegenbauer (11; 12; 13), J. Hacks (20; 21; 22), L. Kronecker (30), and K. Gauss (4; 30). The latter notation is in almost universal use at the present time, and for this reason it has been employed in this paper. However, in view of the present trend toward functional notation and because of the representation of other functions of number theory, it would have seemed equally fitting, at least to this writer, to have used $E(x)$.

The greatest integer part function has been of interest chiefly as a tool in the study of other concepts and for this reason a search of the literature is difficult. The sources of information which I have used include the following:

Mathematical Reviews, 1940 to the present;

Jahrbuch über die Fortschritte der Mathematik, 1868
to the present;

Zentralblatt für Mathematik, 1930 to the present;

L. E. Dickson. History of the Theory of Numbers,
volume 1, published in 1919 (10); and

The Royal Society of London Catalogue of Scientific
Literature, 1800 to 1900.

This thesis is organized in the following way. Following the introduction is a list of all of the properties which were found, including some unsolved problems. This chapter is divided into six sections. In each section similar properties are grouped together and special cases follow general results. Proofs of many of the properties are not available. Properties 1.16, 4.4, 5.22, and 5.23 are original, at least in that they were not found explicitly in the literature. Chapter III contains proofs of these original properties, comments about several other properties, and proofs of still other properties which are from among those given in the literature without proof. Chapter IV contains four incorrect results which were found in the literature.

Below each property in Chapter II we give further information. First we give the source. Then there appears a statement as to whether or not the property is proved in that source. If the proof is not reasonably complete in the source, we say it is not proved. Finally if there is a proof of the property or a comment in Chapter III, we so indicate. It is possible that some of the items called comments may be interpreted by others as trivial proofs. There are two exceptions to the pattern stated above. If the property is original with this author, we so state and indicate that the proof is in Chapter III. In the section containing the unsolved problems, only the source is given since there is no proof.

The following functions from number theory will be used:

The greatest common divisor of the integers m and n , whose value is denoted by (m, n) .

τ where $\tau(n)$ is the number of positive integers which divide the positive integer n .

ϕ where $\phi(n)$ is the number of positive integers less than or equal to and relatively prime to the integer n .

σ where $\sigma(n)$ is the sum of the positive integral divisors of the positive integer n , including 1 and n .

μ where $\mu(n)$ is equal to 1 if $n = 1$, 0 if n is a positive integer which contains a square factor greater than 1, and $(-1)^k$ if n is a product of k distinct positive prime factors.

It will be assumed that all variables represent real numbers unless otherwise stated. The symbol γ will denote Euler's constant,

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^n \frac{1}{i} - \log n \right).$$

CHAPTER II. PROPERTIES

Elementary Properties

$$1.1 \quad x - 1 < [x] \leq x < [x] + 1.$$

(38, p. 79), proof given, comment in Chapter III.

$$1.2 \quad \text{If } n \text{ is an integer and } x - 1 < n \leq x \text{ or } n \leq x < n + 1, \text{ then } n = [x].$$

(38, p. 79), proof given, comment in Chapter III.

$$1.3 \quad \text{If } n \text{ is an integer, then } [x + n] = [x] + n.$$

(38, p. 79), proof given, comment in Chapter III.

$$1.4 \quad [x] + [y] \leq [x + y] \leq [x] + [y] + 1.$$

(38, p. 79), proof given, comment in Chapter III.

$$1.5 \quad [x] + [-x] = \begin{cases} 0 & \text{if } x \text{ is an integer,} \\ -1 & \text{otherwise,} \end{cases}$$

(38, p. 79), proof given, comment in Chapter III.

$$1.6 \quad \text{If } n \text{ is an integer, then } \left[\frac{[x]}{n} \right] = \left[\frac{x}{n} \right].$$

(38, p. 79), proof given, comment in Chapter III.

$$1.7 \quad -[-x] \text{ is the least integer greater than or equal to } x.$$

(38, p. 79), proof given, comment in Chapter III.

$$1.8 \quad \left[x + \frac{1}{2} \right] \text{ is the integer nearest to } x. \text{ If two integers are} \\ \text{equally near to } x, \text{ this expression gives the larger of} \\ \text{the two.}$$

(38, p. 79), proof given, comment in Chapter III.

1.9 $-\left[-x + \frac{1}{2}\right]$ is the integer nearest to x . If two integers are equally near to x , this expression gives the smaller of the two.

(38, p. 79), proof given, comment in Chapter III.

$$1.10 \quad [2x] + [2y] \geq [x] + [y] + [x + y].$$

(17, p. 45-46), proof given.

$$1.11 \quad [2x] - 2[x] = \begin{cases} 0 & \text{if } 0 \leq x - [x] < \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} \leq x - [x] < 1. \end{cases}$$

(59, p. 97), proof not given.

$$1.12 \quad \left[x + \frac{1}{2}\right] - [x] = [2x] - 2[x].$$

(17, p. 173-174), proof not given, proof in Chapter III.

1.13 If $0 < a < 1$, we have

$$[x] - [x - a] = \begin{cases} 0 & \text{if } a \leq x - [x] < 1, \\ 1 & \text{if } 0 \leq x - [x] < a. \end{cases}$$

(59, p. 97), proof not given.

1.14 Let k and m be positive integers, then

$$\left[\frac{k}{m}\right] - \left[\frac{k-1}{m}\right] = \begin{cases} 1 & \text{if } m \mid k \\ 0 & \text{if } m \nmid k. \end{cases}$$

(38, p. 80), proof not given, proof in Chapter III.

1.15 If m , n and k are positive integers and $(m, n) = 1$, then

$$\left[-k \frac{n}{m}\right] = -\left[\frac{kn}{m}\right] - 1 + \left[\frac{k}{m}\right] - \left[\frac{k-1}{m}\right].$$

(26, p. 155), proof not given, proof in Chapter III.

1.16 If n is a positive integer, then

$$[\sqrt{n}] - [\sqrt{n-1}] = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

Original result, proof in Chapter III.

Representations of $[x]$ in Terms of Other Functions

2.1 Let $F(x) = \frac{1}{\pi} \text{Arcsin}(|\sin \pi x|)$, and

$$G(x) = \lim_{n \rightarrow \infty} \{1 + |\sin \pi(x - F(x))|\}^n.$$

Then

$$[x] = x - |F(x) + 2^{1-G(x)} - 1|.$$

(43, p. 706-707), proof given.

2.2 Let m and n be positive integers such that $n \nmid m$. Then

$$\left[\frac{m}{n}\right] = \frac{m}{n} - \frac{1}{2} + \frac{1}{2n} \sum_{i=1}^{n-1} \left\{ \frac{\sin 2\pi mi}{n} \cot\left(i \frac{\pi}{n}\right) \right\}.$$

(61, p. 51), proof not given.

2.3 Let m and n be positive integers such that $n \nmid m$, let

$i = \sqrt{-1}$, and let e be the base of the natural logarithms.

Then

$$\begin{aligned} \left[\frac{m}{n}\right] &= \frac{m}{n} - \frac{n-1}{2n} - \frac{1}{n} \sum_{k=1}^{n-1} \left\{ e^{\frac{2\pi i m k}{n}} \cdot \frac{e^{\frac{2\pi i k}{n}}}{1 - e^{\frac{2\pi i k}{n}}} \right\} \\ &= \frac{m}{n} - \frac{n-1}{2n} + \frac{1}{2n} \sum_{k=1}^{n-1} \cos\left(\frac{2\pi k m}{n}\right) \\ &\quad + \frac{1}{2n} \sum_{k=1}^{n-1} \left\{ \sin\left(\frac{2\pi k}{m}\right) \cot\left(\frac{\pi k}{m}\right) \right\}. \end{aligned}$$

(62, p. 95-96), proof given.

2.4 Define

$$G(x) = \begin{cases} 1 & \text{if } x \text{ is a non-zero integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$[x] = x - \frac{1}{2} + \frac{1}{\pi} \sum_{i=1}^{\infty} \frac{\sin 2i\pi x}{i} - \frac{1}{2} G(x).$$

(44, p. 194), proof not given.

$$2.5 \quad [x] = x + \frac{1}{4} \sum_{i=1}^{\infty} \frac{\sin 2i\pi x}{i} + \frac{4}{\pi} \sum_{i=1}^{\infty} \left\{ \sum_{j=1}^i \frac{\cos 2\pi i x}{(2j-1)i} - \sum_{j=1}^i \frac{\cos 2(1+i-2j)\pi x}{(2j-1)(2i+1-2j)} \right\}.$$

(44, p. 194), proof given.

2.6 If r is an integer greater than x , then

$$[x] = \frac{1}{2} \sum_{i=1}^{r-1} \{1 + \operatorname{sgn}(x-i)\}.$$

(30, p. 346-348), proof not given, comment in Chapter III.

2.7 If m and n are odd positive integers, then

$$\left[\frac{kn}{m}\right] = \frac{1}{2} \sum_{i=1}^{m-1} \left\{ 1 + \operatorname{sgn} \left(\frac{k}{n} - \frac{i}{m} \right) \right\}, \quad (k=1, 2, 3, \dots, n-1).$$

(30, p. 346-348), proof not given, comment in Chapter III.

2.8 If m and n are odd positive integers, then

$$\left[\frac{km}{n} + \frac{1}{2}\right] = \frac{1}{2} \sum_{i=1}^{\frac{m-1}{2}} \left\{ 1 + \operatorname{sgn} \left(\frac{i}{m} + \frac{k}{n} - \frac{1}{2} \right) \right\}, \quad (k=1, 2, 3, \dots, n-1).$$

(30, p. 346-348), proof not given, comment in Chapter III.

Formulas Relating $[x]$ to the Values of Other Functions of Number Theory

3.1 If n is a positive integer and $\binom{n}{2}$ is a binomial coefficient, then

$$\binom{n}{2} = \sum_{i=2}^n \phi(i) \left[\frac{n}{i} \right].$$

(39, p. 37-39), proof given

3.2 If a and b are positive integers, then

$$\frac{a^{\phi(b)} + b^{\phi(a)} - 1}{ab} = \left[\frac{a^{\phi(b)-1}}{b} \right] + \left[\frac{b^{\phi(a)-1}}{a} \right] + 1.$$

(8, p. 148), proof given.

3.3 If n is a positive integer, then

$$\sum_{i=1}^n \mu(i) \left[\frac{n}{i} \right] = 1.$$

(32, p. 301), proof given.

3.4

$$\sum_{i=1}^{[x]} \{ \mu(i) ([\frac{x}{i}] - 1) \} + \sum_{i=1}^{[x]} \mu(i) \sin^2 \left\{ \frac{\pi}{2} ([\frac{x}{i}] - 1) \right\} = 2.$$

(48, p. 45-53), proof given, comment in Chapter III.

3.5

$$\sum_{i=1}^{[x]} \mu(i) \sin^2 \left(\frac{\pi}{2} \left[\frac{x}{i} \right] \right) = -1.$$

(48, p. 45-53), proof given, comment in Chapter III.

$$3.6 \quad \sum_{i=1}^{[x]} \mu(i) \tau\left[\frac{x}{i}\right] = 2 \sum_{i=1}^{[x]} \sum_{j=1}^{[\sqrt{x/i}]} \mu(i) \left[\frac{x}{ij}\right] - \sum_{i=1}^{[x]} \mu(i) [\sqrt{x/i}]^2.$$

(48, p. 45-53), proof given, comment in Chapter III.

3.7 If n is a positive integer, then,

$$\tau(n) = 1 + \sum_{i=1}^{n-1} \left(\left[\frac{n}{i} \right] - \left[\frac{n-1}{i} \right] \right).$$

(55, p. 55), proof given.

3.8 If n is a positive integer, then

$$\sum_{i=1}^n [n/i] \tau(i^2) = \sum_{j=1}^n \tau^2(j).$$

(10, p. 298), proof not given, comment in Chapter III.

3.9 If $N(2, x)$ denotes the number of integers n less than or equal to x for which $\tau(n)$ is an even integer, then

$$N(2, x) = [x] - [\sqrt{x}].$$

(41, p. 118-119), proof not given.

3.10 Let $J(n) = \sum_{i=1}^n \tau(i)$, where n is a positive integer. Then

$$\sum_{i=1}^{\infty} J([j/i]) = \frac{1}{2} ([j]^2 + [j]).$$

(57, p. 738-739), proof given.

3.11 If n is a positive integer, then

$$\sum_{i=1}^n \tau(i) = 2 \sum_{i=1}^{[\sqrt{n}]} [n/i] - [\sqrt{n}]^2.$$

(59, p. 99), proof given.

3.12 If n is a positive integer, then

$$\tau(n) = [\sqrt{n}] = [\sqrt{n-1}] + 2 \sum_{i=1}^{[\sqrt{n-1}]} \left\{ \left[\frac{n}{i} \right] - \left[\frac{n-1}{i} \right] \right\}.$$

(55, p. 55), proof not given.

3.13 Define $\tau_a(n)$ to be the number of a^{th} power divisors of n .

Then

$$\sum_{i=1}^n \tau_a(i) = \sum_{i=1}^{[\sqrt[a]{n}]} \left[\frac{n}{i^a} \right].$$

(22, p. 1-52), proof given.

3.14 If n is a positive integer, then

$$\sum_{i=1}^n \tau(i) = \sum_{i=1}^n \left[\frac{n}{i} \right].$$

(22, p. 1-52), proof given, comment in Chapter III.

- 3.15 If $\tau_2(n)$ is the number of quadratic divisors of the positive integer n , then

$$\sum_{i=1}^n \tau_2(i) = \sum_{i=1}^{[\sqrt{n}]} \left[\frac{n}{i^2} \right] .$$

(22, p. 1-52), proof given, comment in Chapter III.

- 3.16 If $\sigma_a(n)$ is the sum of the a^{th} power divisors of n , then

$$\sum_{i=1}^n \sigma_a(i) = \sum_{i=1}^{[\sqrt[a]{n}]} i^a \left[\frac{n}{i^a} \right] .$$

(22, p. 1-52), proof given.

- 3.17 If n is a positive integer, then

$$\sum_{i=1}^n \sigma(i) = \sum_{i=1}^n i \left[\frac{n}{i} \right] .$$

(22, p. 1-52), proof given, comment in Chapter III.

- 3.18 If $\sigma_2(n)$ is the sum of the quadratic divisors of the positive integer n , then

$$\sum_{i=1}^n \sigma_2(i) = \sum_{i=1}^n i^2 \left[\frac{n}{i^2} \right] .$$

(22, p. 1-52), proof given, comment in Chapter III.

3.19 If m is an odd positive integer, then

$$\tau(1) + \tau(3) + \dots + \tau(m) = \sum_{i=1}^{\frac{m+1}{2}} \left[\frac{1}{2} \cdot \frac{m+2i-1}{2i-1} \right] .$$

(22, p. 1-52), proof given.

3.20 If m is an odd positive integer, then

$$\sigma(1) + \sigma(3) + \dots + \sigma(m) = \sum_{i=1}^{\frac{m+1}{2}} (2i-1) \left[\frac{1}{2} \cdot \frac{m+2i-1}{2i-1} \right] .$$

(22, p. 1-52), proof given.

3.21 Let m be a positive integer and define $k(n)$ to be the sum of the odd divisors and the halves of the even divisors of the positive integer n . Then

$$\sum_{i=1}^m k(i) = \sum_{i=1}^m i \left[\frac{m}{i} \right] - \sum_{i=1}^{\left[\frac{m}{2} \right]} i \left[\frac{m}{2i} \right] .$$

(22, p. 1-52), proof given.

3.22 Let m be a positive integer and define $h(n)$ to be the difference of the odd divisors and the halves of the even divisors of the positive integer n in that order. Then

$$\sum_{i=1}^m h(i) = \sum_{i=1}^m (-1)^i i \left[\frac{m}{i} \right] .$$

(22, p. 1-52), proof given.

- 3.23 Let $J_k(n)$ be the number of different sets of k positive integers less than or equal to n whose greatest common divisor is relatively prime to n . Then

$$\sum_{i=1}^n \left[\frac{n}{i} \right] J_k(i) = \sum_{i=1}^n i^k .$$

(31, p. 78), proof given.

- 3.24

$$[x] = \sum_{i=1}^{[x]} \mu(i) \tau \left[\frac{x}{i} \right] .$$

(3, p. 313), proof given.

- 3.25 Define

$$U(x) = \begin{cases} 1 & \text{if } x \geq 1 . \\ 0 & \text{if } 0 < x < 1 . \end{cases}$$

Then

$$[x] = \sum_{i=1}^{\infty} U(x/i),$$

if $x > 0$.

(45, p. 221), proof given.

- 3.26 If n is a positive even integer, m is a positive rational number, p is a prime number, B_i is the i^{th} Bernoulli number, and $(m, p) = 1$, then

$$\frac{m^n - 1}{nm^{n-1}} B_n \equiv \sum_{i=1}^{p-1} \{ i^{n-1} [\frac{i m}{p}] \} \pmod{p}.$$

(18, p. 111-112), proof not given.

3. 27 With the same hypotheses as for Property 3.26, we have

$$\frac{m^n - 1}{nm^{n-1}} \cdot \frac{1}{p} \sum_{i=1}^{p-1} i^{n-1} \equiv \sum_{i=1}^{p-1} \{ i^{n-1} [\frac{i m}{p}] \} \pmod{p}.$$

(18, p. 111-112), proof not given.

Summation Formulas

4. 1 If n is a positive integer, then

$$\sum_{i=1}^{\infty} [\frac{n}{2^i} + \frac{1}{2}] = n.$$

(33, p. 49-50), proof given, comment in Chapter III.

4. 2 If n is an integer, then

$$\sum_{i=1}^n (2i-1)[n/i] = \sum_{i=1}^n [n/i]^2.$$

(59, p. 98), proof not given.

4. 3

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i} \left[\frac{\log i}{\log 2} \right] = \gamma, \text{ where } \gamma \text{ is Euler's constant.}$$

(47, p. 116-117), proof given

4. 4 If p is a positive integer, $a > 1$, and $q = [a^p]$, then

$$\sum_{i=1}^p [a^i] + \sum_{i=1}^q [\log i] = \begin{cases} p(q+1) & \text{if } a \text{ is an integer,} \\ pq & \text{otherwise.} \end{cases}$$

Original result, proof in Chapter III.

4. 5 If p is a positive integer, e is the base of the natural logarithms, and $q = [e^p]$, then

$$\sum_{i=1}^p [e^i] + \sum_{i=1}^q [\log_e i] = pq.$$

(22, p. 23), proof not given, proof in Chapter III.

4. 6 If p is a positive integer and $q = 10^p$, then

$$\sum_{i=1}^p 10^i + \sum_{i=1}^q [\log_{10} i] = p(q+1).$$

(22, p. 23), proof not given, proof in Chapter III.

4. 7 If m is a positive integer, then

$$\sum_{i=0}^{[m/3]} \left[\frac{m-3i}{2} \right] = \left[\frac{m^2 + 2m + 4}{12} \right].$$

(29, p. 185), proof given.

4.8 If n is a positive integer, then

$$\sum_{i=1}^n \left[\sqrt{i} + \frac{1}{2} \right] = \left[\sqrt{n} + \frac{1}{2} \right] \left(\frac{3n + 1 - \left[\sqrt{n} + \frac{1}{2} \right]^2}{3} \right) .$$

(36, p. 86), proof not given.

4.9 If n is a positive integer and γ is Euler's constant, then

$$\sum_{i=1}^n [n/i] = n(\ln n + 2\gamma - 1) + O(\sqrt{n}).$$

(23, p. 262), proof given.

4.10 If a , d , n , and p are integers and $h = \left[\frac{an+d}{p} \right]$, then

$$\sum_{i=1}^n \left[\frac{ai+d}{p} \right] + \sum_{i=1}^h \left[\frac{pi-(d+i)}{a} \right] = nh.$$

(66, p. 245), proof given.

4.11 If a , d , n , p , and q are integers and h is defined by the equation $(i+1)a + d = (h+1)p + q$, then

$$\sum_{i=1}^n \left[\frac{ai+d}{p} \right] = \sum_{i=0}^h \left[\frac{pi+q}{a} \right] .$$

(66, p. 252), proof given.

4.12 If a , n , and p are positive integers, then

$$\sum_{i=1}^n \left[\frac{i^a}{p} \right] + \sum_{i=1}^{\left[\frac{n^a}{p} \right]} \left[\sqrt[i]{ip} \right] = n \left[\frac{n^a}{p} \right] .$$

(66, p. 253), proof given.

4.13 If d , n , and p are positive integers, then

$$\sum_{i=1}^n \left[\frac{i^2+d}{p} \right] + \sum_{i=1}^{\left[\frac{n^2+d}{p} \right]} \left[\sqrt[ip]{ip - (d+n)} \right] = n \left[\frac{n^2+d}{p} \right] .$$

(66, p. 253), proof given.

4.14 If n is a positive integer, then

$$\sum_{i=1}^{\left[\sqrt{n} \right]} \left[n/i \right] - \sum_{i=\left[\sqrt{n} \right]+1}^n \left[n/i \right] = \left[\sqrt{n} \right]^2 .$$

(66, p. 253), proof given.

4.15 Let p_1, p_2, \dots, p_r be all of the primes such that $p_i^2 \leq n$, where n is a given positive integer. Then the number of positive primes less than or equal to n is

$$n + r - 1 - \sum_{i=1}^r \left[\frac{n}{p_i} \right] + \sum_{\substack{i < j \\ i, j=1}}^r \left[\frac{n}{p_i p_j} \right] \dots + (-1)^r \sum_{\substack{i < j < \dots < r \\ i, j, \dots, r=1}}^r \left[\frac{n}{p_i p_j \dots p_r} \right] .$$

(17, p. 47-49). proof given.

- 4.16 Let a positive integer d be represented as a product of powers of distinct primes in the form

$$d = p_1^{a_1} p_2^{a_2} \dots p_s^{a_s} .$$

Let n and r be positive integers, define

$$g_r(d) = p_1^{[a_1/r]} p_2^{[a_2/r]} \dots p_s^{[a_s/r]} ,$$

and let $\phi_k(n)$ be the number of k^{th} powers less than n and relatively prime to n . Then

$$\sum_{i=1}^n \{g_r(i)\}^k = \sum_{i=1}^{[\sqrt[r]{n}]} \left[\frac{n}{i^r} \right] \phi_k(i) .$$

(11, p. 219-224), proof given.

- 4.17 (Bougaief's formula). If $g_2(i)$ is defined as in Property

4.17, where we set $r = 2$, and n is a positive integer, then

$$\sum_{i=1}^n g_2(i) = \sum_{i=1}^{[\sqrt{n}]} [n/i^2] \phi(i) .$$

(11, p. 219-224), proof given, proof in Chapter III.

- 4.18 Let $g_r(d)$ be defined as in Property 4.17, $S_{k-1}(d) = \sum_{i=1}^d i^{k-1}$, and let $\phi^{(k-1)}(d)$ be the sum of the $(k-1)$ st powers of the integers which are less than or equal to d and relatively prime to d . Then

$$\sum_{i=1}^n S_{k-1} \{ g_r(i) \} = \sum_{i=1}^{\lceil \sqrt[n]{n} \rceil} \left\{ \left\lfloor \frac{n}{i^2} \right\rfloor \phi^{(k-1)}(i) \right\} .$$

(11, p. 219-224), proof given.

4. 19 If n is a positive integer, then

$$\sum_{i=0}^{n-1} \left[x + \frac{i}{n} \right] = [nx] .$$

(38, p. 82), proof not given, proof in Chapter III.

4. 20 If m is an even integer, then

$$\sum_{i=1}^{m/2} \left[x + \frac{2i-1}{m} \right] = \begin{cases} \frac{[mx]}{2} & \text{if } 2 \mid [mx] , \\ \frac{[mx] + 1}{2} & \text{otherwise .} \end{cases}$$

(53, p. 93), proof not given.

4. 21 If m is an even integer, then

$$\sum_{i=1}^{\frac{m}{2}-1} \left[x + \frac{2i}{m} \right] = \begin{cases} \frac{[mx]}{2} - [x] & \text{if } 2 \mid [mx] , \\ \frac{[mx] - 1}{2} - [x] & \text{otherwise .} \end{cases}$$

(53, p. 94), proof not given.

4.22 If m is an odd integer, then

$$\sum_{i=1}^{\frac{m-1}{2}} \left[x + \frac{2i-1}{m} \right] = \begin{cases} \frac{[mx] - [x]}{2} & \text{if } 2 \mid [mx] - [x] , \\ \frac{[mx] - [x] - 1}{2} & \text{otherwise} . \end{cases}$$

(53, p. 95), proof not given.

4.23 If m is an odd integer, then

$$\sum_{i=1}^{\frac{m-1}{2}} \left[x + \frac{2i}{m} \right] = \begin{cases} \frac{[mx] - [x]}{2} & \text{if } 2 \mid [mx] - [x] , \\ \frac{[mx] - [x] + 1}{2} & \text{otherwise.} \end{cases}$$

(53, p. 95), proof not given.

4.24 If m and n are integers, then

$$\sum_{i \geq 1} \left[\left(\frac{x}{i^m} \right)^{1/n} \right] = \sum_{j \geq 1} \left[\left(\frac{x}{j^n} \right)^{1/m} \right] .$$

(19, p. 703-704), proof given.

4.25 Define $\psi(p, q)$ to be $\sum_{i=1}^{\frac{q-1}{2}} \left[\frac{ip}{q} \right]$, where p and q are

positive integers and q is odd (Gauss' ψ - function). Then

$$(a) \quad \psi(-p, q) = -\frac{q-1}{2} - \psi(p, q),$$

$$(b) \quad \psi(p, -q) = -\psi(p, q),$$

and

$$(c) \quad \psi(-p, -q) = -\psi(-p, q) = \frac{q-1}{2} + \psi(p, q).$$

(4, p. 67), proof given.

4.26 If n is an integer and

$$S(x) = \sum_{i=1}^{\infty} \frac{\sin 2\pi i x}{i\pi},$$

Then

$$\begin{aligned} \sum_{i=1}^x [x/i] &= \sum_{i=1}^{2x} \left[\frac{x}{i} - i \right] + \frac{1}{2} \sum_{i=1}^{2x} \left(\left[\frac{x}{i} \right] - \left[\frac{x-1}{i} \right] \right) \\ &\quad + \sum_{i=1}^{2x} S(x/i). \end{aligned}$$

(63, p. 167-168), proof given.

4.27 If m and n are positive odd integers and $(m, n) = 1$, then

$$\sum_{i=1}^{\frac{m-1}{2}} \left[i \frac{n}{m} + \frac{1}{2} \right] = \sum_{j=1}^{\frac{n-1}{2}} \left[j \frac{m}{n} + \frac{1}{2} \right].$$

(30, p. 347), proof not given.

4.28 If h and k are positive integers, then

$$\sum_{i=1}^{hk-1} [i/h] [i/k] = \frac{1}{12} (h-1)(k-1) (4hk + h + k + 1).$$

(35, p. 593), proof not given.

4. 29 If $(p, q) = d$, then

$$\begin{aligned} & q \sum_{i=1}^{p-1} i \left[\frac{iq}{p} \right] + p \sum_{i=1}^{q-1} i \left[\frac{ip}{q} \right] \\ &= \frac{1}{12} (p-1)(q-1)(8pq-p-q-1) + (d-1)(6pq-d-1) . \end{aligned}$$

(45, p. 221-222), proof given.

4. 30 If $(p, q) = 2$, then

$$\begin{aligned} & \sum_{i=1}^{p-1} i \left[\frac{iq}{p} \right] + p \sum_{i=1}^{q-1} i \left[\frac{ip}{q} \right] \\ &= \frac{1}{12} (p-1)(q-1)(8pq-p-q-1) + 6pq-3 . \end{aligned}$$

(35, p. 593), proof given, comment in Chapter III.

4. 31 For a rectangular array of numbers where $A_{j,k}$ is the element in the j^{th} row and k^{th} column, we have

$$\sum_{i=1}^n \sum_{j=1}^{[{}^a\sqrt{i}]} A_{j,k} = \sum_{j=1}^{[{}^a\sqrt{n}]} \sum_{i=j^a}^n A_{j,k} .$$

(16, p. 276), proof not given.

4. 32 If a is an integer greater than or equal to 2, then

$$\sum_{i=1}^n [{}^a\sqrt{i}] + \sum_{i=1}^{[{}^a\sqrt{n}]} i^a = [{}^a\sqrt{n}] (n+1) .$$

(16, p. 276), proof not given, comment in Chapter III.

4.33 If n is a positive integer, then

$$\sum_{i=1}^n [\sqrt{i}] + \sum_{i=1}^{[\sqrt{n}]} i^2 = [\sqrt{n}] (n+1) .$$

(22, p. 49), proof not given, comment in Chapter III.

4.34 If n is a positive integer and $q = [\sqrt[3]{n}]$, then

$$\sum_{i=1}^n [\sqrt[3]{i}] = q(n+1) - \frac{q^2(q+1)^2}{4} .$$

(16, p. 275) proof not given, comment in Chapter III.

4.35 Let $m = [\sqrt{n}]$, where n is a positive integer. Then

$$\sum_{i=1}^n [\sqrt{i}] = m \frac{6n-2m^2-3m+5}{6} = m(n+1) - \frac{m(m+1)(2m+1)}{6} .$$

(16, p. 275), proof not given, comment in Chapter III.

4.36 If a and n are positive integers and a is greater than or equal to 2, then

$$\sum_{i=1}^n [a\sqrt{i}]^2 = \sum_{j=1}^{[a\sqrt{n}]} (2j-1) (n-j^a + 1) .$$

(16, p. 277), proof not given, comment in Chapter III.

- 4.37 If a and n are positive integers and a is greater than or equal to 2, then

$$\sum_{i=1}^n [\sqrt[i]{a}]^2 = (n+1) [\sqrt[n]{a}]^2 - 2 \sum_{j=1}^{[\sqrt[n]{a}]} j^{a+1} \sum_{j=1}^{[\sqrt[n]{a}]} j^a.$$

(16, p. 277), proof not given, comment in Chapter III.

- 4.38 If $[\sqrt{n}] = m$, where n is a positive integer, then

$$\sum_{i=1}^n [\sqrt{i}]^2 = m^2(n+1) - \frac{m(m+1)(3m^2+5m+1)}{6}$$

(16, p. 277), proof not given, comment in Chapter III.

- 4.39 If $A_{j,k}$ is defined as in Property 4.31, a , r , and n are integers with $r \leq n$, and $a \leq n$, then

$$\sum_{i=a}^n \sum_{j=a}^{[i/r]} A_{j,k} = \sum_{j=a}^{[n/r]} \sum_{i=rj}^n A_{j,k}.$$

(16, p. 277), proof not given.

- 4.40 If $A_{j,k}$ is defined as in Property 4.31 and n is a positive integer, then

$$\sum_{i=1}^n \sum_{j=1}^{2^i-1} A_{j,k} = \sum_{j=1}^{2^n-1} \sum_{i=1+\lceil \log_2 j \rceil}^n A_{j,k}.$$

(16, p. 277), proof not given.

4. 41 If n is a positive integer and x is positive, then

$$\sum_{i=1}^{[x]} \left[\frac{x}{i} \right] = \sum_{i=1}^n \left[\frac{x}{i} \right] + \sum_{i=1}^{\left[\frac{x}{n} \right]} \left[\frac{x}{i} \right] - n \left[\frac{x}{n} \right] .$$

(40, p. 46), proof given.

4. 42 If x is a positive irrational number and n is a positive integer, then

$$\sum_{i=0}^n [ix] + \sum_{i=0}^{[nx]} [i/x] = n[nx] .$$

(26, p. 154), proof given, comment in Chapter III.

4. 43 If x is a positive irrational number and n is a positive integer, then

$$\sum_{i=0}^n [ix] = \sum_{i=0}^{[nx]} [-i/x] + [nx] + n[nx] .$$

(26, p. 154), proof given, comment in Chapter III.

4. 44 If $(m, n) = 1$ and r is a positive integer, then

$$\sum_{i=0}^r \left[i \frac{m}{n} \right] + \sum_{i=0}^{\left[r \frac{m}{n} \right]} \left[i \frac{n}{m} \right] = r \left[r \frac{m}{n} \right] + \left[\frac{r}{n} \right] .$$

(26, p. 155), proof given, comment in Chapter III.

4. 45 If $(m, n) = 1$ and r is a positive integer, then

$$\sum_{i=0}^{\lfloor n/r \rfloor} \left[i \frac{m}{n} \right] + \sum_{i=0}^{\lfloor m/r \rfloor} \left[i \frac{n}{m} \right] = \left[\frac{m}{r} \right] \left[\frac{n}{r} \right] + \left[\frac{1}{r} \right] .$$

(26, p. 156), proof given.

4. 46 If $(m, n) = 1$, then

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \left[i \frac{m}{n} \right] + \sum_{i=1}^{\lfloor m/2 \rfloor} \left[i \frac{n}{m} \right] = \left[\frac{m}{2} \right] \left[\frac{n}{2} \right] .$$

(22, p. 50), proof given, comment in Chapter III.

4. 47 If p is a prime number, then

$$\sum_{i=1}^{p-1} \left[\frac{i^2}{p} \right] + \sum_{i=1}^{p-1} [\sqrt{ip}] = (p-1)^2 .$$

(22, p. 38), proof given.

4. 48 If p is a prime number, then

$$\sum_{j=1}^{p-1} \sum_{i=1}^{p-1} \left[\frac{ij}{p} \right] = \left(\frac{p-1}{2} \right)^2 (p-2) .$$

(21, p. 206-207), proof given.

4. 49 If p is a prime number, $p > 2$, then

$$\sum_{j=1}^{p-1} \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{ij}{p} \right] + \sum_{j=1}^{p-1} \sum_{i=1}^{\lfloor j/2 \rfloor} \left[p \frac{i}{j} \right] = \left(\frac{p-1}{2} \right)^3 .$$

(21, p. 206-207), proof given.

4. 50 Let x , y , and z be odd integers such that $(x, y, z) = 1$ and

$$x = \frac{py}{q} = \frac{pz}{r}, \text{ then}$$

$$\begin{aligned} & \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{i q}{p} \right] \left[\frac{i r}{p} \right] + \sum_{j=1}^{\frac{q-1}{2}} \left[\frac{j r}{q} \right] \left[\frac{j p}{q} \right] + \sum_{k=1}^{\frac{r-1}{2}} \left[\frac{k p}{r} \right] \left[\frac{k q}{r} \right] \\ &= \frac{p-1}{2} \cdot \frac{q-1}{2} \cdot \frac{r-1}{2} . \end{aligned}$$

(5, p. 124), proof given.

4. 51 Assuming the hypotheses of Property 4. 50, we have

$$\sum_{j=1}^y \left[\frac{j r}{q} \right] + \sum_{k=1}^z \left[\frac{k q}{r} \right] = yz .$$

(5, p. 124), proof given.

4. 52 Assuming the hypotheses of Property 4. 50, we have

$$\begin{aligned} & \sum_{i=1}^{\frac{p-1}{2}} \left\{ \sum_{j=1}^{\frac{q-1}{2}} \left[\frac{j r}{q} \right] = \sum_{k=1}^{\frac{q-1}{2}} \left[\frac{k q}{r} \right] \right\} + \sum_{j=1}^{\frac{q-1}{2}} \left\{ \sum_{k=1}^{\frac{r-1}{2}} \left[\frac{k p}{r} \right] + \sum_{i=1}^{\frac{r-1}{2}} \left[\frac{i r}{p} \right] \right\} \\ & + \sum_{k=1}^{\frac{r-1}{2}} \left\{ \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{i q}{p} \right] + \sum_{j=1}^{\frac{q-1}{2}} \left[\frac{j p}{q} \right] \right\} = \frac{p-1}{2} \cdot \frac{q-1}{2} \cdot \frac{r-1}{2} . \end{aligned}$$

(5, p. 124), proof given.

4. 53 If m and n are positive odd integers and $(m, n) = 1$, then

$$\sum_{i=0}^{\frac{n-1}{2}} \left[i \frac{m}{n} \right] + \sum_{i=0}^{\frac{m-1}{2}} \left[i \frac{n}{m} \right] = \frac{m-1}{2} \cdot \frac{n-1}{2} .$$

(38, p. 68), proof given, comment in Chapter III.

4. 54 If m and n are positive integers and $(m, n) = d$, then

$$\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-1)}{2} + \frac{d-1}{2} .$$

(21, p. 205), proof given.

4. 55 If $(m, n) = 1$ and r is an integer such that $1 \leq r < \frac{n}{2}$, then

$$\sum_{i=r}^{n-r} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-2r+1)}{2} .$$

(26, p. 154), proof not given, proof in Chapter III.

4. 56 If $(m, n) = 1$, then

$$\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-1)}{2} .$$

(59, p. 97), proof given, proof in Chapter III.

4. 57 If $(m, n) = 1$, then

$$\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-1)}{2} + \frac{1}{2} \sum_{i,j} \operatorname{sgn} \left(\frac{i}{n} - \frac{j}{m} \right) ,$$

($i=1, 2, \dots, n-1$; $j=1, 2, \dots, n-1$) .

(30, p. 347), proof not given.

4. 58 If $(m, n) = 1$, then

$$\sum_{i=1}^{n-1} \left[2i \frac{m}{n} \right] + \sum_{j=1}^{m-1} \left[j \frac{n}{2m} \right] = \frac{(m-1)(n-1)}{2} .$$

(30, p. 347), proof not given.

4. 59 If m and n are odd integers with $(m, n)=1$, then

$$\frac{n-1}{2} \sum_{i=1} \left[\left(i + \frac{n-1}{2} \right) \frac{m}{n} \right] = \frac{(m-1)(n-1)}{4} + \sum_{i=1}^{\frac{n-1}{2}} \left[i \frac{m}{n} + \frac{1}{2} - \frac{m}{2n} \right] .$$

(54, p. 337-342), proof given

4. 60 If m and n are odd integers with $(m, n) = 1$, then

$$\frac{n-1}{2} \sum_{i=1} \left[2i \frac{m}{n} \right] = \sum_{i=1}^{m-1} \left[i \frac{n}{2m} \right] .$$

(54, p. 337-342), proof given

4. 61 If m and n are odd integers with $(m, n)=1$, then

$$\frac{n-1}{2} \sum_{i=1} \left[i \frac{m}{n} + \frac{1}{2} \right] + \sum_{i=1}^{\frac{n-1}{2}} \left[i \frac{m}{n} - \frac{m}{2n} \right] = \frac{(m-1)(n-1)}{4} .$$

(54, p. 337-342), proof given.

4.62 If $(m, n) = d$, then

$$d = 2 \sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] - mn + m + n = 2 \sum_{i=1}^{\lfloor n/2 \rfloor} \left[i \frac{m}{n} \right] + 2 \sum_{i=1}^{\lfloor m/2 \rfloor} \left[i \frac{n}{m} \right] - 2 \left[\frac{m}{2} \right] \left[\frac{n}{2} \right] + \begin{cases} 0 & \text{if both } m \text{ and } n \text{ are even,} \\ 1 & \text{otherwise.} \end{cases}$$

(21, p. 205-208), proof not given

Miscellaneous Properties

5.1 If $(m, n) = 1$, $m < n$, n is an odd integer, and k is an integer less than or equal to $n-1$, then

$$\left[k \frac{m}{n} + \frac{1}{2} \right] + \left[(n-k) \frac{m}{n} + \frac{1}{2} \right] = m.$$

(54, p. 341), proof not given, proof in Chapter III.

5.2 If p is a prime number, the largest exponent e such that p^e divides $n!$ is

$$e = \sum_{i=1}^{\infty} \left[\frac{n}{p^i} \right].$$

(38, p. 79-80), proof given, comment in Chapter III.

5.3 Iff a and b are positive irrational numbers such that

$$a^{-1} + b^{-1} = 1, \text{ then the sequences } [an] \text{ and } [bn] \text{ for}$$

$n=1, 2, 3, \dots$ represent all of the positive integers without

repetition.

(55, p. 51-52), proof given, comment in Chapter III.

- 5.4 Iff a is a positive irrational number, then the sequences $[n + na]$ and $[n + na^{-1}]$ for $n = 1, 2, 3, \dots$ contain every positive integer exactly once.

(38, p. 83), proof not given, comment in Chapter III.

- 5.5 The n^{th} non-square integer is

$$n + \left[\sqrt{n} + \frac{1}{2} \right] .$$

(36, p. 85), proof not given.

- 5.6 The n^{th} non-triangular number is

$$n + \left[\sqrt{2n} + \frac{1}{2} \right] .$$

(36, p. 85), proof not given.

- 5.7 The n^{th} non- k^{th} power integer is

$$n + \left[\sqrt[k]{n + [k\sqrt{n}]} \right] .$$

(36, p. 85), proof not given.

- 5.8 Every integer of the form $[(1 + \sqrt{2})k]$ can also be written in the form $[\sqrt{2}m]$, where k and m are integers.

(52, p. 57-58), proof given.

- 5.9 If p is a prime number and $\binom{n}{p}$ denotes the binomial coefficient, then

$$\binom{n}{p} \equiv \left[\frac{n}{p} \right] \pmod{p}.$$

(42, p. 347-348), proof given, comment in Chapter III.

- 5.10 If p is a prime number, n is a positive integer, and v_p is the integer defined by $p^{v_p} \leq 2n < p^{1+v_p}$, then

$$v_p = \left[\frac{\log 2n}{\log p} \right].$$

(38, p. 167), proof not given.

- 5.11 If u is an integer, let p and q be odd integers such that $(p, q) = 1$, and let $A = \{ [p/q], [3p/q], \dots, [^{(q-2)p}_q] \}$.

Then

- (a). when p is of the form $(2x+1)u-1$, there are as many members of A of the form $2xu-1$ as of the form $(2x+1)u$;
- (b). when p is of the form $(2x+1)u+1$, there are as many members of A of the form $2xu$ as of the form $(2x+1)u-1$.

(13, p. 611-612), proof not given.

- 5.12 If u is an integer, let p and q be odd integers such that $(p, q)=1$, and let $B = \{ [2p/q], [4p/q], \dots, [^{(q-1)p}_q] \}$.

Then

- (a). when p is of the form $(2x+1)u-1$, there are as many members of B of the form $(2x+1)u-1$ as

of the form $2xu-2$;

- (b). when p is of the form $(2x+1)u+1$, there are as many members of B of the form $2xu+1$ as of the form $(2x+1)u$.

(13, p. 611-612), proof not given

5.13 If u is an integer, let p and q be odd integers such that $(p, q)=1$, and let $C = \{ [p/q], [2p/q], [3p/q], \dots, [^{(q-1)}\frac{p}{q}] \}$.

Then

- (a). when p is of the form $2xu-2$, there are as many members of C of the form $2xu-2$ as of the form $(2x+1)u-1$;
- (b). when p is of the form $2xu+2$, there are as many members of C of the form $2xu+1$ as of the form $(2x+1)u$.

(13, p. 611-612), proof not given.

5.14 If u is an integer, let p and q be odd integers such that $(p, q)=1$, and let $D = \{ [^{(q+1)}\frac{p}{2q}], [^{(q+3)}\frac{p}{2q}], \dots, [^{(q-1)}\frac{p}{q}] \}$.

Then

- (a). when p is of the form $2xu-2$, there are as many members of D of the form $2xu-1$ as of the form $(2x+1)u-2$;

(b). when p is of the form $2xu+2$, there are as many members of D of the form $2xu$ as of the form $(2x+1)u-1$.

(13, p. 611-612), proof not given.

5.15 If u is an integer, let p and q be odd integers such that $(p, q)=1$, and let $B = \{ [\frac{2p}{q}], [\frac{4p}{q}], \dots, [\frac{(q-1)p}{q}] \}$. Then if n is an integer, B contains as many integers of the form $4n+1$ as of the form $4n+2$.

(13, p. 611-612), proof not given, comment in Chapter III.

5.16 If u is an integer, let p and q be odd integers such that $(p, q)=1$, and let $A = \{ [\frac{p}{q}], [\frac{3p}{q}], \dots, [\frac{(q-2)p}{q}] \}$. Then, for integral n ,

- (a). if p is of the form $4n+1$, A contains as many members of the form $4n+2$ as of the form $4n+3$.
- (b). if p is of the form $4n+3$, A contains as many members of the form $4n$ as of the form $4n+1$.

(13, p. 611-612), proof not given, comment in Chapter III.

5.17 If a, b, c, d, p , and q are positive integers, then a necessary and sufficient condition for

$$[\frac{an+b}{p}] = [\frac{cn+d}{q}],$$

for all non-negative integers n , is

$$(1). \quad \frac{a}{p} = \frac{c}{q},$$

and

$$(2). \quad \left[\frac{b}{(a, p)} \right] = \left[\frac{d}{(c, q)} \right] .$$

(2, p. 280), proof given.

5.18 If a, b, d , and p are positive integers with $b \neq d$ and the integers between b and d as well as $\max\{b, d\}$ are p -ic non-residues of a , then

$$\left[\frac{r+(an+b)^{1/p}}{s} \right] = \left[\frac{r+(an+d)^{1/p}}{s} \right]$$

for any integers r and s .

(2, p. 284-5), proof given.

5.19 If a, b, c, d , and p are positive integers with $b > d$, then a necessary and sufficient condition for

$$[(an+b)^{1/p}] = [(cn+d)^{1/p}] ,$$

for all non-negative integers n , is

$$(1). \quad a=c,$$

and

(2). the integers $d+1, d+2, \dots, b$ are p -ic non-residues of a .

(2, p. 284-5), proof given.

5.20 (Ramanujan). For all integers n ,

$$\left[\frac{1}{2} + \sqrt{n+\frac{1}{2}} \right] = \left[\frac{1}{2} + \sqrt{n+\frac{1}{4}} \right] .$$

(2, p. 285), proof given, comment in Chapter III.

5.21 If p and q are integers, then for all integers n ,

$$\left[\frac{1}{2} + \left(n + \frac{q}{2^p} \right)^{1/p} \right] = \left[\frac{1}{2} + \left(n + \frac{q+1}{2^p} \right)^{1/p} \right] .$$

(2, p. 285), proof given, comment in Chapter III.

5.22 If a, b, q , and r are integers, $b > 0$, and $a = bq + r$ where $0 \leq r < b$, then

$$r = a - b \left[\frac{a}{b} \right] .$$

Original result, proof in Chapter III.

5.23 The least non-negative remainder when an integer a is divided by a positive integer b is

$$a - b \left[a/b \right] .$$

Original result, proof in Chapter III.

5.24 If a and b are integers and b is positive, then the least absolute remainder of $a \pmod{b}$, or the negative remainder in case there are two remainders of the same absolute value, is given by

$$r = a - b \left[\frac{2a}{b} \right] + b \left[\frac{a}{b} \right] .$$

(36, p. 80), proof not given, proof in Chapter III.

5.25 If n and k are positive integers, let $S_k(n)$ be the sum of the digits of the number $S_{k-1}(n)$ in base b with $S_0(n) = n$.

Then

$$S_k(n) = S_{k-t}(n) - (b-1) \sum_{i=1}^t \sum_{j=1}^{S_{k-1}^{(i)}-1} \left[\frac{S_{k-1}^{(n)}}{b^j} \right].$$

(65, p. 260-262), proof given.

- 5.26 If m is a positive integer, then $[(1+\sqrt{3})^{2m+1}]$ is divisible by 2^{m+1} but not by 2^{m+2} .

(38, p. 83), proof not given.

- 5.27 If n is a positive integer, the integer $4n+1$ is prime if

$$\left[\frac{4n+1-y^2}{4y} \right] = \left[\frac{4n-3-y^2}{4y} \right]$$

for every odd integer y such that $1 < y \leq [\sqrt{4n+1}]$.

(12, p. 389), proof not given, comment in Chapter III.

- 5.28 An odd integer N , greater than or equal to 9, is prime iff $N+k^2$ is not a square for $k=0, 1, 2, \dots, [\frac{N-9}{6}]$.

(67, p. 128), proof not given.

- 5.29 The integers n and $n+2$ are simultaneously prime (twin primes) iff

$$\sum_{i \geq 1} ([\frac{n+2}{i}] + [\frac{n}{i}] - [\frac{n+1}{i}] - [\frac{n-1}{i}]) = 4.$$

(36, p. 86), proof not given, proof in Chapter III.

- 5.30 If p is a prime and $p \equiv 1 \pmod{4}$, then

$$\sum_{i=1}^{\frac{p-1}{4}} [\sqrt{ip}] = \frac{p^2-1}{12}.$$

(36, p. 88), proof not given.

- 5.31 Let P be the set of all positive integers. Then at least one of x and y is an integer if

$$[k(x+y)] = [kx] + [ky]$$

for all values of k in the set P .

(37, p. 600), proof given.

- 5.32 Let $U = \{1, -1\}$. Then at least one of x and y is an integer if

$$[k(x+y)] = [kx] + [ky]$$

for both values of k in the set U .

(38, p. 82), proof not given.

- 5.33 If a is an irrational number such that $0 < a < 1$, and

$$g_n = \begin{cases} 0 & \text{if } [na] = [(n-1)a], \\ 1 & \text{otherwise,} \end{cases}$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n g_i = a.$$

(38, p. 83), proof not given.

- 5.34 If p is an odd prime, then

$$p \text{ divides } [(2 + \sqrt{5})^p] - 2^{p+1}.$$

(50, p. 190), proof given.

- 5.35 2^n divides $[(3 + \sqrt{5})^n + \frac{1}{2}]$.

(36, p. 90), proof not given.

5.36 For all integers m ,

$$4\left[\frac{m}{4}\right]^2 + 5\left[\frac{m}{2}\right]^2 + m^2 - 4\left[\frac{m}{4}\right]\left[\frac{m}{2}\right] - 4m\left[\frac{m}{2}\right] + 2\left[\frac{m}{4}\right] + \left[\frac{m}{2}\right] - m = 0.$$

(62, p. 97), proof not given, comment in Chapter III.

5.37 If γ is Euler's constant, e is the base of the natural logarithms, and

$$x = \sum_{i=1}^n \frac{1}{i},$$

then

$$n = [e^{x-\gamma}].$$

(46, p. 341-42), proof given.

5.38 Let m be the number of quadratic non-residues of integer p which are less than or equal to $p/2$. Then

$$m \equiv \sum_{i=1}^{\frac{p-1}{2}} \left[\frac{i^2}{p} \right] \pmod{2}.$$

(61, p. 51), proof given.

5.39 The asymptotic density of integers n for which

$$(n, [\sqrt{n}]) = 1 \quad \text{is} \quad 6/\pi^2.$$

(36, p. 87), proof not given.

5.40 The expected value of $(n, [\sqrt{n}])$ is $\pi^2/6$.

(36, p. 87), proof not given.

5.41 If x is irrational and n is a positive integer, then

$$\lim_{n \rightarrow \infty} \left\{ \sum_{i=1}^n [ix] - \frac{xn(n+1)}{2} + \frac{n}{2} \right\} = 0 .$$

(51, p. 725), proof given.

5.42 If a and b are integers and m and r are positive integers, then

$$\sum_{i=0}^{r-1} \left\{ \left[\frac{a+b+i}{m} \right] - \left[\frac{a+i}{m} \right] - \left[\frac{b+i}{m} \right] + \left[\frac{i}{m} \right] \right\} \geq 0$$

(28, p. 35-41), proof given.

5.43 The cardinality of the solution set of the equation

$$ax+b = [x] ,$$

where a and b are real numbers, is given by

$$\left\{ \begin{array}{l} \left[-\frac{b}{a} \left| \frac{a}{1-a} \right| \right] - \left[-\frac{a+b}{a} \left| \frac{a}{1-a} \right| \right] \text{ if } a \neq 0, a \neq 1, \\ C \text{ if } a=0 \text{ and } b \text{ is an integer,} \\ 0 \text{ if } a=0 \text{ and } b \text{ is not an integer,} \\ D \text{ if } a=1 \text{ and } -1 < b \leq 0, \\ 0 \text{ if } a=1 \text{ and } b \leq -1 \text{ or } b > 0 , \end{array} \right.$$

where C denotes the cardinality of the continuum and D denotes the cardinality of a denumerable set.

(15, p. 6), proof given.

- 5.44 Define $N(a)$ to be the number of solutions to the equation $[x] = ax$. Then

$$N(a) = \frac{\left[-\left\lfloor \frac{a}{1-a} \right\rfloor \right]}{\left[\frac{-a^2}{1+a^2} \right]} .$$

(7, p. 439-440), proof given, comment in Chapter III.

- 5.45 There exists a number a such that if $g_0 = a$ and $g_{n+1} = 2^{g_n}$, then $[g_n]$ is prime for all integers n .

(64, p. 616-618), proof given.

- 5.46 If n is a positive integer, then

$$2 \sum_{i=1}^n [ix] \not\equiv 1 \pmod{(n+1)} .$$

(56, p. 6-7), proof given.

- 5.47 If k is a positive integer and x is a positive real number, then

$$0 \leq \sum_{i=1}^k \left\{ x \left[\frac{i}{x} \right] - (x+1) \left[\frac{i}{x+1} \right] \right\} \leq k .$$

(58, p. 308-310), proof given, comment in Chapter III.

- 5.48 Let x, y , and z be real numbers. Define

$$\Delta(x, y, z) = \Delta(y, x, z) = [zx + zy] - [zx] - [zy] .$$

Let $f(n) = \Delta(x, y, n)$, where n is an integer.

Define $F(x, y, N)$ as $\sum_{i=1}^N f(i)$ and define $A(x, y) = \lim_{N \rightarrow \infty} \frac{1}{N} F(x, y, N)$.

Denote $x - [x]$ by $r(x)$. Then

$$(a). \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N r(ix) = \begin{cases} \frac{1}{2} & \text{if } x \text{ is irrational,} \\ \frac{1}{2} - \frac{1}{2q} & \text{if } x = p/q \text{ where} \\ & (p, q) = 1 \text{ and } q \geq 1, \end{cases}$$

and

$$(b). \quad A(x, y) = \begin{cases} 0 & \text{if } x \text{ or } y \text{ is an integer,} \\ \geq \frac{1}{6} & \text{if neither } x \text{ nor } y \text{ is an integer,} \\ 1 & \text{if } x \text{ and } y \text{ are irrational and} \\ & x+y \text{ is integral.} \end{cases}$$

(27, p. 1-5), proof given.

5. 49 If $x \geq 1$, then

$$\int_1^x \left(\frac{1}{v} \sum_{i=1}^{[v]} \tau(i) \right) dv = \int_1^x \left(\frac{[v]}{v} \left[\frac{x}{v} \right] \right) dv.$$

(6, p. 90-91), proof given.

5. 50 (Wythoff's game) Two players, A and B, alternately remove counters from two piles according to the following rules. A player may, at his turn, remove any number of counters from either pile. If he wishes to remove counters from both piles, he must remove an equal number from each.

The player who takes the last counter wins. In order for A to win, he must leave one of the following combinations:

$$(1, 2), (3, 5), (4, 7), (6, 10), (8, 13), \dots$$

Then, no matter what B does, A can always convert to another "safe" combination.

The n^{th} pair of numbers giving a safe combination is

$$([nt], [nt^2]),$$

$$\text{where } t = \frac{1}{2}(\sqrt{5} + 1).$$

(9, p. 142-143), proof given.

Unsolved Problems

6.1 (Moser). Let e be the base of the natural logarithms.

Does the sequence $[e^n]$ for $n=1, 2, \dots$ contain infinitely many primes?

(36, p. 92)

6.2 (Erdős). Let e be the base of the natural logarithms. Does the sequence $[e^n]$ for $n = 1, 2, \dots$ contain infinitely many composite numbers?

(36, p. 92)

6.3 (Vijayaraghavan). Let a, b , and c be distinct real numbers and denote $x - [x]$ by $r(x)$. Does

$$r(x^a) = r(x^b) = r(x^c) \text{ imply that } r(x^a) = 0?$$

(1, p. 336)

6.4 (Mills) Is there a real number A such that $[A^n]$ is prime for every positive integer n ?

(36, p. 92), comment in Chapter III.

CHAPTER III. COMMENTS AND PROOFS

Properties are not restated before comments or trivial proofs but are restated before non-trivial proofs. Equations are numbered separately for each proof.

1.1 to 1.9. These properties may be found in many elementary number theory texts. One such source is (55, p. 51).

1.12. This property follows from Property 4.19 with $n=2$.

1.14. Let k and m be positive integers. Then

$$\left[\frac{k}{m} \right] - \left[\frac{k-1}{m} \right] = \begin{cases} 1 & \text{if } m \mid k \\ 0 & \text{if } m \nmid k. \end{cases}$$

Proof: If $m \mid k$, then $mp=k$, where p is an integer, and $\left[\frac{k}{m} \right] = \frac{k}{m} = p$. Hence

$$\begin{aligned} \left[\frac{k}{m} \right] - \left[\frac{k-1}{m} \right] &= p - \left[p - \frac{1}{m} \right] \\ &= p - p - \left[-\frac{1}{m} \right] \\ &= - \left[-\frac{1}{m} \right]. \end{aligned}$$

Since m is a positive integer, then $0 < \frac{1}{m} \leq 1$. Hence by Property 1.7, $- \left[-\frac{1}{m} \right] = 1$, and so

$$\left[\frac{k}{m} \right] - \left[\frac{k-1}{m} \right] = 1.$$

If $m \nmid k$, then $k = mq + r$ where q and r are integers and $1 \leq r \leq m-1$ (38, p. 3). Hence

$$\begin{aligned} \left[\frac{k}{m} \right] - \left[\frac{k-1}{m} \right] &= \left[q + \frac{r}{m} \right] - \left[q + \frac{r-1}{m} \right] \\ &= q - q - \left[\frac{r-1}{m} \right] \\ &= - \left[\frac{r-1}{m} \right] \\ &= 0, \end{aligned}$$

since $0 \leq r-1 \leq m-2$. This completes the proof.

1.15. This property follows immediately from Properties 1.5 and 1.14.

1.16. If n is a positive integer, then

$$[\sqrt{n}] - [\sqrt{n-1}] = \begin{cases} 1 & \text{if } n \text{ is a perfect square,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The difference between the square roots of two consecutive positive integers is less than 1, since

$$\sqrt{p+1} - \sqrt{p} = \frac{1}{\sqrt{p+1} + \sqrt{p}} < 1$$

if p is a positive integer.

If n is a perfect square, let $[\sqrt{n}] = \sqrt{n} = p$. Since $\sqrt{n-1} < \sqrt{n}$ it follows that $[\sqrt{n-1}] = p-1$. Hence

$$[\sqrt{n}] - [\sqrt{n-1}] = 1.$$

If n is not a perfect square, let $\sqrt{n-1} = p+h$ where p is an integer and $0 \leq h < 1$. Then $\sqrt{n} = p+k$ where

$0 < k < 1$, for if $k > 1$, then $n-1 < (p+1)^2 < n$, a contradiction.

Also, if $k=1$, then n is a perfect square, another contradiction.

Hence $[\sqrt{n}] = [\sqrt{n-1}] = p$ and $[\sqrt{n}] - [\sqrt{n-1}] = 0$. This

This completes the proof.

2. 6, 2. 7 and 2. 8. Here

$$\operatorname{sgn} x = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Several other formulas of this nature are given in the article cited.

3. 4, 3. 5, and 3. 6. Several other formulas of this nature are given in the article cited.

3. 8. Dickson's reference to the original source is not clear.

3. 14. Proof of this property is also given in (55, p. 53-54).

This is the case $a=1$ of Property 3. 13.

3. 15. This is the case $a=2$ of Property 3. 13.

3. 17 and 3. 18. These are the cases $a=1$ and $a=2$ respectively of Property 3. 16.

4. 1. This formula is proved in the source cited in the form

$$n=1 + \sum_{i=1}^{\infty} \left[\frac{n+2^{i-1}-1}{2^i} \right].$$

If we replace n by $n+1$, we obtain Property 4.1 .

4.4. If p is a positive integer, $a > 1$, and $q = [a^p]$, then

$$\sum_{i=1}^p [a^i] + \sum_{i=1}^q [\log_a i] = \begin{cases} p(q+1) & \text{if } a \text{ is an integer,} \\ pq & \text{otherwise.} \end{cases}$$

Proof by induction. If $p=1$, $[a^p] = [a] = a$ if a is an integer, and $[a] < a$ if a is not an integer. If $1 < a < 2$, $[a] = 1$, and hence

$$\sum_{i=1}^1 [a^i] + \sum_{i=1}^1 [\log_a i] = [a] + [\log_a 1] = [a] .$$

If $a \geq 2$, then

$$\begin{aligned} \sum_{i=1}^1 [a^i] + \sum_{i=1}^{[a]} [\log_a i] \\ &= \sum_{i=1}^1 [a^i] + \sum_{i=1}^{[a]-1} [\log_a i] + [\log_a [a]] \\ &= [a] + \begin{cases} 1 & \text{if } a \text{ is an integer,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

since $[\log_a b] = 0$ if $1 \leq b < a$.

Assume that

$$\sum_{i=1}^k [a^i] + \sum_{i=1}^{[a^k]} [\log_a i] = \begin{cases} k([a^k] + 1) & \text{if } a \text{ is an integer,} \\ k[a^k] & \text{otherwise.} \end{cases}$$

Then, if $1 < a < 2$, and $[a^{k+1}] > [a^k]$, we have

$$\begin{aligned}
 (1). \quad & \sum_{i=1}^{k+1} [a^i] + \sum_{i=1}^{[a^{k+1}]} [\log_a i] \\
 &= \sum_{i=1}^k [a^i] + [a^{k+1}] + \sum_{i=1}^{[a^k]} [\log_a i] + \sum_{j=[a^k]+1}^{[a^{k+1}]} [\log_a i] \\
 &= k[a^k] + [a^{k+1}] + k([a^{k+1}] - [a^k]) \\
 &= (k+1)[a^{k+1}],
 \end{aligned}$$

since $a^k < j < a^{k+1}$.

If $1 < a < 2$ and $[a^{k+1}] = [a^k]$, then

$$\begin{aligned}
 (2). \quad & \sum_{i=1}^{k+1} [a^i] + \sum_{i=1}^{[a^{k+1}]} [\log_a i] \\
 &= \sum_{i=1}^k [a^i] + [a^{k+1}] + \sum_{i=1}^{[a^k]} [\log_a i] \\
 &= k[a^k] + [a^{k+1}] \\
 &= k[a^{k+1}] + [a^{k+1}] \\
 &= (k+1)[a^{k+1}].
 \end{aligned}$$

If $a \geq 2$, then

$$\begin{aligned}
 (3). \quad & \sum_{i=1}^{k+1} [a^i] + \sum_{i=1}^{[a^{k+1}]} [\log_a i] \\
 &= \sum_{i=1}^k [a^i] + [a^{k+1}] + \sum_{i=1}^{[a^k]} [\log_a i] + \sum_{i=[a^k]+1}^{[a^{k+1}]-1} [\log_a i] + [\log_a [a^{k+1}]].
 \end{aligned}$$

If a is an integer, using the induction assumption, we may replace the right side of (3) by

$$(4) \quad k([a^k] + 1) + [a^{k+1}] + k([a^{k+1}] - 1 - [a^k]) + k + 1 \\ = (k+1)([a^{k+1}] + 1).$$

If a is not an integer, then $[a^n]$ is not an integer. Since $a \geq 2$, it follows that $a^n - a^{n-1} > 1$, so that $a^{n-1} < [a^n] < a^n$ and hence $n-1 \leq \log_a [a^n] < n$. The right side of (3) can be replaced by

$$(5) \quad k[a^k] + [a^{k+1}] + k([a^{k+1}] - 1 - [a^k]) + k \\ = (k+1)[a^{k+1}].$$

The property is proved by (1), (2), (4), and (5).

4. 5 and 4. 6. These are special cases of Property 4. 4 where $a = e$, the base of the natural logarithms, and $a=10$ respectively.

4. 17. This follows from Property 4. 16 where $r = 2$ and $k = 1$.

4. 19. If n is a positive integer, then

$$\sum_{i=0}^{n-1} \left[x + \frac{i}{n} \right] = [nx].$$

Proof. If we let $g = [nx]$, then as a result of Properties 1. 6 and 1. 3, we have

$$\begin{aligned}
 (1). \quad [x + i/n] &= [\frac{nx+i}{n}] \\
 &= [\frac{[nx+i]}{n}] \\
 &= [\frac{[nx] + i}{n}] \\
 &= [\frac{g+i}{n}] .
 \end{aligned}$$

We can write

$$(2). \quad g+i = n [\frac{g+i}{n}] + r_i, \quad \text{where } 0 \leq r_i < n,$$

and solve for $[\frac{g+i}{n}]$ to obtain

$$(3). \quad [\frac{g+i}{n}] = \frac{g+i-r_i}{n}.$$

Using (1) and (3), we have

$$\begin{aligned}
 (4). \quad \sum_{i=0}^{n-1} [x + \frac{i}{n}] &= \sum_{i=0}^{n-1} [\frac{g+i}{n}] \\
 &= \sum_{i=0}^{n-1} \frac{g+i-r_i}{n} \\
 &= \sum_{i=0}^{n-1} \frac{g}{n} + \sum_{i=0}^{n-1} \frac{i}{n} - \sum_{i=0}^{n-1} \frac{r_i}{n} .
 \end{aligned}$$

As i ranges over the set $\{0, 1, 2, \dots, n-1\}$, then r_i ranges over the same set, but not necessarily in the same order. To prove this, assume that for some integers s and t , $s > t$, in the set $\{0, 1, 2, \dots, n-1\}$ $r_s = r_t$. Then, by (3), we have

$$\frac{g+s}{n} - [\frac{g+s}{n}] = \frac{g+t}{n} - [\frac{g+t}{n}] .$$

Hence

$$\frac{s-t}{n} = \left[\frac{g+s}{n} \right] - \left[\frac{g+t}{n} \right] .$$

But this is impossible since the right side is an integer and the left side is not. We have reached a contradiction and hence $r_s \neq r_t$. Since there are n distinct values of i , there must also be n distinct values of r_i , namely the members of the set $\{0, 1, 2, \dots, n-1\}$.

As a result of this,

$$\sum_{i=0}^{n-1} \frac{i}{n} - \sum_{i=0}^{n-1} \frac{r_i}{n} = 0,$$

so that (4) becomes

$$\sum_{i=0}^{n-1} \left[x + \frac{i}{n} \right] = \sum_{i=0}^{n-1} \frac{g}{n} = n \frac{g}{n} = g = [nx] .$$

This completes the proof.

- 4. 30. This is the case of $d=2$ of Property 4. 30.
- 4. 32. This is the case $A_{j,k}=1$ of Property 4. 31.
- 4. 33. This is the case $a=2$ of Property 4. 32. It can be proved by induction.
- 4. 34. This is the case $a=3$ of Property 4. 32.
- 4. 35. This follows from Property 4. 33.
- 4. 36. This is the case $A_{j,k}=2j-1$ of Property 4. 31.

4. 37. This follows from Property 4. 36.
4. 38. This is the case $a=2$ of Property 4. 37.
4. 42. This property is proved also in (22, p. 27). Zeller (66, p. 244) gives " x a non-integer" as the hypothesis but the other two sources cited give the property as stated here.
4. 43. This property follows from Property 4. 42.
4. 44. This is the analogue of Property 4. 42 when x is rational and equal to m/n .
4. 46. This is the case $r=2$ of Property 4. 45.
4. 53. This property is also proved in (26, p. 154).
4. 55. If $(m,n)=1$ and r is an integer such that $1 \leq r < n/2$, then

$$\sum_{i=r}^{n-r} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-2r+1)}{2} .$$

Proof by induction. If $r=1$, we have Property 4. 56, which is proved in (59, p. 97) and also proved independently following this proof.

Assume that for $k < \frac{n}{2} - 1$,

$$\sum_{i=k}^{n-k} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-2k+1)}{2} .$$

Then

$$\begin{aligned}
\sum_{i=k+1}^{n-(k+1)} \left[i \frac{m}{n} \right] &= \sum_{i=k}^{n-k} \left[i \frac{m}{n} \right] - \left[k \frac{m}{n} \right] - \left[(n-k) \frac{m}{n} \right] \\
&= \frac{(m-1)(n-2k+1)}{2} - \left[k \frac{m}{n} \right] - m - \left[-k \frac{m}{n} \right] \\
&= \frac{(m-1)(n-2k+1)}{2} - m - \left(\left[k \frac{m}{n} \right] + \left[-k \frac{m}{n} \right] \right) \\
&= \frac{(m-1)(n-2k+1)}{2} - m + 1 \\
&= \frac{(m-1)(n-2(k+1)+1)}{2},
\end{aligned}$$

by Property 1.5 since $n \nmid km$ by hypothesis. This proves the theorem.

4. 56. If $(m, n) = 1$, then

$$\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-1)}{2}.$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] &= \sum_{i=1}^{n-1} \left[(n-i) \frac{m}{n} \right] \\
&= \sum_{i=1}^{n-1} \left[m - i \frac{m}{n} \right] \\
&= \sum_{i=1}^{n-1} \{ m + \left[-i \frac{m}{n} \right] \} = \sum_{i=1}^{n-1} \{ m - 1 - \left[i \frac{m}{n} \right] \} \\
&= (m-1)(n-1) - \sum_{i=1}^{n-1} \left[i \frac{m}{n} \right],
\end{aligned}$$

by Properties 1. 3 and 1. 5. Hence

$$2 \sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = (m-1)(n-1)$$

and

$$\sum_{i=1}^{n-1} \left[i \frac{m}{n} \right] = \frac{(m-1)(n-1)}{2}.$$

Note that this is the case $r=1$ of Property 4. 55 and the case $d=1$ of Property 4. 54.

5. 1. If $(m, n) = 1$, $m < n$, n is an odd integer, and k is an integer less than or equal to $n-1$, then

$$\left[k \frac{m}{n} + \frac{1}{2} \right] + \left[(n-k) \frac{m}{n} + \frac{1}{2} \right] = m.$$

Proof.

$$\begin{aligned} (1). \quad & \left[k \frac{m}{n} + \frac{1}{2} \right] + \left[(n-k) \frac{m}{n} + \frac{1}{2} \right] \\ &= \left[k \frac{m}{n} + \frac{1}{2} \right] + \left[m - \left(k \frac{m}{n} + \frac{1}{2} \right) + 1 \right] \\ &= \left[k \frac{m}{n} + \frac{1}{2} \right] + m + 1 + \left[-\left(k \frac{m}{n} + \frac{1}{2} \right) \right], \end{aligned}$$

by Property 1. 3. Now $k \frac{m}{n} + \frac{1}{2}$ is not an integer, since

$$k \frac{m}{n} + \frac{1}{2} = \frac{2km+n}{2n},$$

and for this to be an integer, $2n$ must divide $2km+n$. But

if $2n$ divides $2km+n$, n must be even, a contradiction.

Hence $k \frac{m}{n} + \frac{1}{2}$ is not an integer, and by Property 1. 5,

the last member of (1) is equal to

$$m + 1 + (-1) = m$$

and the theorem is proved.

5. 2. This property is stated in two alternate ways in (55, p. 56) and (34, p. 200).

5. 3 and 5. 4. These properties are special cases of a general theorem about functions which represent all integers (see 14, p. 736-737). Another proof of Property 5. 3 can be found in (9, p. 135).

5. 9. Another proof is given in (50, p. 191).

5. 15. This is Stern's special case of Property 5. 12.

5. 16. This is Stern's special case of Property 5. 11.

5. 20. This is the case $r = d = 1$, $s = b = p = 2$, and $a = 4$ of Property 5. 18.

5. 21. This is the case $r = 1$, $s = 2$, $a = 2^p$, $b = q$, and $d = q + 1$ of Property 5. 18.

5. 22. If a, b, q , and r are integers, $b > 0$, and $a = bq + r$ where $u \leq r < u + b$, then

$$(1). \quad r = a - b \left[\frac{a-u}{b} \right] .$$

Proof. By hypothesis we have $u \leq a - bq < u + b$, which yields $u - a \leq -bq < u + b - a$, whence $a - u - b < bq < a - u$. Hence

$$\frac{a-u}{b} - 1 = \frac{a-u-b}{b} < q \leq \frac{a-u}{b} .$$

By Property 1. 2, $q = [\frac{a-u}{b}]$, and so (1) follows from $a=bq+r$.

5. 23. This is the case $u=0$ of Property 5. 22. Thus there exist unique integers q and r such that

$$a = bq + r, \quad 0 \leq r < b$$

by (38, p. 3). Since the next smaller remainder, $r-b$, is negative, r is the least non-negative remainder when a is divided by b .

5. 24. If a and b are integers and b is positive, then the least absolute remainder of $a \pmod{b}$, or the negative remainder in case there are two remainders of the same absolute value, is given by

$$(1) \quad r = a - b \left[\frac{2a}{b} \right] + b \left[\frac{a}{b} \right].$$

Proof. What must be shown is that r , defined by (1) lies in the range $-b/2 \leq r < b/2$. If we set $u = -b/2$ in Property 5. 22, the integer r of (1) of Property 5. 22 is the remainder of this theorem. Hence, using (1) of Property

5. 22, we have

$$(2) \quad r = a - b \left[\frac{a + \frac{b}{2}}{b} \right] = a - b \left[\frac{a}{b} + \frac{1}{2} \right].$$

If, in Property 4. 20, we set $n=2$, we have

$$[x] + \left[x + \frac{1}{2} \right] = [2x].$$

Hence, if $x = a/b$, we have

$$\left[\frac{a}{b} + \frac{1}{2} \right] = \left[\frac{2a}{b} \right] - \left[\frac{a}{b} \right].$$

Substituting this in (2), we have

$$\begin{aligned} r &= a - b \left(\left[\frac{2a}{b} \right] - \left[\frac{a}{b} \right] \right) \\ &= a - b \left[\frac{2a}{b} \right] + b \left[\frac{a}{b} \right], \end{aligned}$$

which was to be shown.

5. 27. Dickson (10, p. 427) asserts that there is a similar property for primes of the form $4n+3$ but he does not exhibit it.

5. 29. The integers n and $n+2$ are simultaneously prime if

$$(1) \sum_{i \geq 1} \left(\left[\frac{n+2}{i} \right] + \left[\frac{n}{i} \right] - \left[\frac{n+1}{i} \right] - \left[\frac{n-1}{i} \right] \right) = 4.$$

Proof. We rearrange the general term to obtain

$$\left(\left[\frac{n+2}{i} \right] - \left[\frac{n+1}{i} \right] \right) + \left(\left[\frac{n}{i} \right] - \left[\frac{n-1}{i} \right] \right),$$

and use Property 1. 14. The result (1) follows, since the only divisors of a prime p are 1 and p .

5. 36. A proof of this can be formed by considering the four cases $m \equiv 0 \pmod{4}$, $m \equiv 1 \pmod{4}$, $m \equiv 2 \pmod{4}$, and $m \equiv 3 \pmod{4}$.

5. 44. This property is a special case of Property 5. 42.

5. 47. The first part of the inequality is also proved in (49, p. 89-93).

6. 4. Note the similarity to Property 5. 44.

CHAPTER IV. INCORRECT RESULTS

The following results were found in the literature which are incorrect.

If n is a positive integer, then

$$\sum_{i=1}^n \phi(n) = \left[\frac{n+1}{2} \right]^2 .$$

When $n=2$, the left side of this equation is 2 and the right side is 1. This result appears in Dickson (10, p. 294). His reference to the original source is not clear.

Denote by $I_{a,b}(n)$ the number of fractions (in lowest terms) between a and b ($a > b$) with denominator n . Then

$$\sum_{i=1}^n [n/i] I_{a,b}(i) = \sum_{i=1}^n [(a-b)i] .$$

The word "between" is not defined in the source. When $n=1$, $a=5/4$, and $b=3/4$, then the left side is 1 and the right side is 0 regardless of the interpretation of the word "between". This result appears in (60, p. 126) .

If n is a positive integer, then

$$1 \cdot \frac{1}{2^2} \cdot \frac{1}{3^3} \cdot \dots \cdot \frac{1}{n^n} < \left[\frac{2}{n+1} \right] \frac{[n(n+1)]}{2}$$

The right side of this inequality is one when $n = 1$ and zero for all other positive integers n . This result is from (50, p. 66).

If n is a positive integer, then

$$1 \cdot 2^2 \cdot 3^3 \dots n^n < \left[\frac{2n+1}{3} \right]^{\frac{n(n+1)}{2}}.$$

If we set $n = 5$, the left side of the inequality is 86,400,000 and the right side is 14,348,907. This result appears in (50, p. 66).

BIBLIOGRAPHY

1. American Mathematical Society. Report of the institute in the theory of numbers, University of Colorado, June 21-July 17, 1959.
2. Ayyangar, A. A. Krishnaswami. Some arithmetical identities. Journal of the Indian Mathematical Society 18: 278 - 285. 1930.
3. Bachmann, Paul. Zahlentheorie. vol. 2 Die Analytische Zahlentheorie. Leipzig, B. G. Teubner, 1921. 494 p.
4. Busche, E. Über die Function $\sum_{x=1}^{\frac{1}{2}(q-1)} \left[\frac{px}{q} \right]$. Journal für die Reine und Angewandte Mathematik 106: 65 - 80. 1890.
5. Busche, E. Über grosste Ganze. Journal für die Reine und Angewandte Mathematik 103: 118-125. 1888.
6. Buschman, R. G. Problem 4839: An integral identity. American Mathematical Monthly 67: 90 - 91. 1960.
7. Buschman, R. G. Problem E 1527. American Mathematical Monthly 70: 439-440. 1963.
8. Capelli, A. Ein Beitrage zum Fermatschen Satze. Verhandlungen des internationalen Mathematiker-Congressen, Leipzig 3: 148-150. 1904.
9. Coxeter, H. S. M. The golden section, phyllotaxis, and Wythoff's game. Scripta Mathematica 19: 135-143. 1953.
10. Dickson, Leonard Eugene. History of the theory of numbers. vol. 1. Washington, Carnegie Institute of Washington, 1919. 486 p.
11. Gegenbauer, Leopold. Über ein Theorem des Herrn Bugajef. Sitzungsberichte der Mathematisch - Naturwissenschaften Klasse der kaiserlichen Academie der Wissenschaften (Vienna), ser. 2, 95: 219-224. 1887.

12. Gegenbauer, Leopold. Ueber einen arithmetischen Satz des Herrn Charles Hermite. Sitzungsberichte der Mathematisch - Naturwissenschaften Klasse der kaiserlichen Akademie der Wissenschaften (Vienna), ser. 2a, 99: 387-403. 1891.
13. Gegenbauer, Leopold. Über grösste ganze Zahlen. Sitzungsberichte der Mathematisch - Naturwissenschaften Klasse der kaiserlichen Akademie der Wissenschaften (Vienna), ser 2, 94: 611-612. 1886.
14. Gilbert, E. N. Functions which represent all integers. American Mathematical Monthly 70: 736-737. 1963.
15. Golden, Robert Ray. The solution of certain equations in a variable and its greatest integer part. Masters thesis. Corvallis, Oregon State University, 1963. 37 numb. leaves.
16. Gould, H. W. Generalisation of a bracket function of L. Moser. Canadian Mathematical Bulletin 6: 275-277. 1963.
17. Griffin, Harriet. Elementary theory of numbers. New York, McGraw-Hill, 1954. 203 p.
18. Grün, Otto. Eine Kongruenz für Bernoullische Zahlen. Deutsche Mathematiker-Vereinigung, Jahresbericht 50: 111-112. 1940.
19. Gupta, Roshan Lal. Problem 4390: Sums of greatest integer functions. American Mathematical Monthly 58: 703-704. 1951.
20. Hacks, Jacob. Einige Sätze über Summen von Divisoren. Acta Mathematica 9: 177-181. 1886-7.
21. Hacks, Jacob. Über Einige für Primzahlen charakteristische Beziehungen. Acta Mathematica 17: 205-208. 1893.
22. Hacks, Jacob. Über Summen von grössten Ganzen. Acta Mathematica 10: 1-52. 1887.
23. Hardy, G. H. and E. M. Wright. Theory of numbers. Oxford, Clarendon Press, 1938. 403 p.

24. Hermite, Charles. Extract from a letter to Mr. Sylvester.
American Journal of Mathematics 6: 173-174. 1883-4.
25. Hermite, Charles. Sur Quelques points dans la théorie des
nombres. Acta Mathematica 2: 299. 1883.
26. Jacobsthal, Ernst. Beiträge zur Zahlentheorie IV. Kongelige
Norske Videnskabers Selskab. Forhandlinger 22: 154-156.
1949.
27. Jacobsthal, Ernst. Über die grösste ganze Zahl I. Kongelige
Norske Videnskabers Selskab. Forhandlinger 30: 1-5. 1957.
28. Jacobsthal, Ernst. Über eine zahlentheoretische Summe.
Norske Videnskabers Selskab. Forhandlinger 30: 35-41.
1957.
29. Kingston, J. M. Problem E978: A lemma used by Hansraj
Gupta. American Mathematical Monthly 59: 185. 1952.
30. Kronecker, Leopold. Bemerkung über die von Gauss mit $[x]$
bezeichnete arithmetische Function einer reellen Grösse
 x . Journal für die Reine und Angewandte Mathematik
106: 346-348. 1890.
31. Lejeune-Dirichlet, P. G. Ueber die Bestimmung der mittleren
Werthe in der Zahlentheorie. Abhandlung der Königlichen
Akademie der Wissenschaften zu Berlin, 1849. p. 78-81.
32. Meissel, E. $E(x)$ and reversion of series. Journal für die
Reine und Angewandte Mathematik 48: 301-306. 1854.
33. Mendelsohn, N. S. Problem 4346: Greatest integer function.
American Mathematical Monthly 58: 49-50. 1951.
34. Minkowski, Hermann. Zur Theorie der positiven quadratischen
Formen. Journal für die Reine und Angewandte Mathe-
matik 101: 200. 1887.
35. Mordell, L. J. The reciprocity formula for Dedekind sums.
The American Journal of Mathematics 78: 593-594. 1951.
36. Moser, Leo. University of Alberta, Edmonton, Alberta. Un-
published collection of problems in number theory.

37. Newman, D.J. Problem 4814: The greatest integer function. American Mathematical Monthly 66: 600. 1959.
38. Niven, Ivan and Herbert S. Zuckerman. An introduction to the theory of numbers. New York, Wiley, 1960. 250 p.
39. Perez-Cacho, L. La funcion $E(x)$ (Entero de x) en la teorie de numeros. Revista Mathematica Hispano-Americana, ser.4, 12: 36 - 40. 1952.
40. Pexider, J. V. Sur la fonction $E(x)$ representant l'entier contenu dans x . Rendiconti del circolo Matematico di Palermo 24: 46-63. 1907.
41. Pillai, S.S. On a congruence property of the divisor function. Journal of the Indian Mathematical Society 6: 118-119. 1942.
42. Piza, Pedro A. Problem 4322: Binomial coefficients and greatest integer function. American Mathematical Monthly 57: 347-348. 1950.
43. Porges, Arthur. An analytic expression for $[x]$. American Mathematical Monthly 66: 706-707. 1959.
44. Pringsheim, Alfred. Darstellung der zahlentheoretischen Function $E(x)$ durch eine unendliche Reihe. Mathematische Annalen 26: 193-196. 1886.
45. Rademacher, Hans. Eine arithmetische Summenformel. Monatshefte für Mathematik und Physik 39(2): 221-228. 1932.
46. Sandham, H.F. Problem E 938: Euler's constant. American Mathematical Monthly 58: 341-342. 1951.
47. Sandham, J.F. Problem 4353: Euler's constant. American Mathematical Monthly 58: 116-117. 1951.
48. Schröder, J. Zur Berechnung von Teilsummen der summatorischen Funktion der Möbius'schen Funktion $\mu(x)$. Norsk Matematisk Tidsskrift (Forening) 14: 45-53. 1932.

49. Selberg, Sigmund. On a conjecture by Ernst Jacobsthal. Det Kongelige Norske Videnskabers Selskabs 26: 89-93. 1953.
50. Shklarsky, D. O., N. N. Chentsov and I. M. Yaglom. The U.S. S.R. Olympiad problem book. San Francisco, Freeman, 1962. 452 p.
51. Sierpinski, W. Un théorème sur les nombres irrationnels. Bulletin International de l'Académie des Sciences de Cracovie, classe des Sciences mathématiques et naturelles, 1909, p. 725-727.
52. Skolem, Th. On certain distributions of integers in pairs with given differences. Mathematica Scandinavica 5: 57-58. 1957.
53. Stern, M. A. Sur un théorème de M. Hermite relative à la fonction $E(x)$. Acta Mathematica 8: 93-96. 1885-6.
54. Stern, M. A. Zur Theorie der Function Ex . Journal für die Reine und Angewandte Mathematik 106: 337-345. 1890.
55. Stewart, Bonnie Madison. Theory of numbers. New York, MacMillan, 1952. 261 p.
56. Stöhr, Alfred. Aufgabe 341. Jahresbericht der Deutschen Mathematiker-Verinigung 52: 6-7. April 1954.
57. Sylvester, James Joseph. Collected mathematical papers, vol. 4. London, Cambridge University Press, 1912. 756 p.
58. Tverberg, Helge. On two inequalities by S. Selberg. Mathematica Scandinavica 6: 308-310. 1958.
59. Uspensky, J. V. and M. A. Heaslett. Elementary number theory. New York, McGraw-Hill, 1939. 480 p.
60. Vahlen, K. Th. Über eine Verallgemeinerung der Euler'schen ϕ -Funktion. Zeitschrift für Mathematik und Physik 40: 126-127. 1895.
61. Vandiver, H. S. On some special arithmetic congruences. American Mathematical Monthly 11: 51. 1904.

62. Volosin, Ya. I. On the integral part of a rational number.
P. S. R. S. Augustākās Izglītības Ministrija. Pētera
Stučkas Latvijas Valsts Universitate. Zinatniskie Raksti.
Ministerstvo Vysshego Obrazovaniya. Latvišķi
Gosudarstvennyi Universitet im. Petra Stučki. Učēnye
Zapiski 28: 95-98. 1959.

63. Wigert, S. Recherches sur la representation analytique de la
fonction $\sum_{v=1}^q [q/v]$. Arkiv för Matematik, Astronomi,
och Fysik 1: 165-183. 1903-4.

64. Wright, E. M. A prime-representing function. American
Mathematical Monthly 58: 616-618. 1951.

65. Zaheln, Jean-Pierre. Sur les sommes des chiffres
successives d'un nombre. Euclides 8: 260-262. July 1948.

66. Zeller, Chr. Ueber Summen von grössten Ganzen bei
arithmetische Reihen. Nachrichten von der K. Gessel-
schaft der Wissenschaften und der Georg-Augusts
Universitat, May 14, 1879, p. 243-268.

67. Zsigmondy, K. Über eine allgemein giltige, additiv gebildete
Kriterien für Primzahlen. Monatshefte für Mathematik
und Physik 5: 123-128. 1894.