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In this dissertation, we use Fourier-analytic and spectral theory methods to analyze the behavior of solutions of the incompressible Navier-Stokes equations in 2D and 3D (with an eye towards better understanding turbulence). In particular, we investigate the possible existence of so-called *ghost solutions* to the Navier-Stokes Equations. Such solutions, if they exist, would be dynamic in time and yet have constant energy and enstrophy profiles. First, we completely analyze the case of ghost solutions in the simpler Stokes system, and use results from that to construct nonstationary constant-energy solutions (as well as solutions with constant higher order norms) when the spatial domain is the 3D torus. We then explore the properties of finite-mode solutions on the 2D torus, providing a constraining relationship between the spectral structure of a finite-mode solution and that of any force that might generate it. As a consequence of the main result regarding the spectral structure of finite-mode solutions we disprove the existence of so-called *chained ghost solutions*, which have been investigated in several recent papers in this area.

On the Existence of Solutions to the Incompressible Navier-Stokes Equations with Constant Energy and Enstrophy

by
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APPROVED:

Major Professor, representing Mathematics

Head of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Sarah Hagen, Author

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1 Introduction

This dissertation explores the equations that describe the evolution of the motion of fluids—a system of partial differential equations known as the Navier-Stokes equations (NSE). These equations are derived from the physical principles of the conservation of momentum and the conservation of mass. These equations are famous because of a remarkable disconnect between practice and theory: the numerical methods and models are applied with success in areas ranging from blood flow to weather prediction, and yet not only is there no explicit solution for these equations in general, it is not even known whether the problem is well-posed in three dimensions (at least not at the time of this printing). This significant gap between theory and practice is why the Clay Institute lists the resolution of the existence and uniqueness of physically realistic solutions as one of its seven Millennium Prizes. However, independent of the resolution of the millennium problem, there remain large areas of open problems of interest to mathematicians and scientists alike concerning the general behavior of the solutions that do exist. Indeed, in the case of 2 dimensions the existence/uniqueness problem has been solved, and yet little is known about the qualitative properties of the solutions, especially in connection with applications.

In particular, turbulent fluid flow (a phenomenon familiar to anyone who has seen a crashing waterfall or experienced a choppy airplane flight) is poorly understood both physically and mathematically. The existing empirical theory of turbulent flow, while being appropriately descriptive, remains disconnected from the equations that govern the flow, and thus is inadequate for making predictions. Mathematically, we would like to ground the physical theory in the Navier-Stokes equations themselves in a way that is both explanatory and predictive. We would like to develop rigorous conditions that give rise to or preclude turbulence and to understand statistical behavior of turbulent phenomena. The empirical theory points towards the importance of two properties for understanding turbulence: kinetic energy (a measure of total fluid motion) and enstrophy (a measure of the "swirliness" of the fluid). As a result, much ongoing work is dedicated to understanding the evolution of these quantities in actual solutions. It is in this area that the research in this dissertation lies.

1.1 Outline of this Dissertation

The goal of this chapter is to provide the reader with an introduction to the basic theory of the Navier-Stokes equations. There are various ways of stating the equations and the well-established theorems. Different subtleties arise as a result of one's choice of domain and boundary conditions. Thus, another goal of this introductory chapter is to precisely state the functional setting and results related to our choice of domain and boundary conditions.

In Chapter 2 we look at the Navier-Stokes equations from a dynamical systems perspective and connect this perspective to the more general theory of turbulence. It is in this chapter where we introduce and motivate the central problem of this dissertation: Do there exist nonstationary solutions to the Navier-Stokes equations whose energy and enstrophy remain constant in time? Chapter 3 will be a brief survey of results in the literature related to this problem.

Chapters 4 and 5 contain the two main theorems of this dissertation. The first theorem, established in Chapter 4, gives the construction of a nonstationary constant-energy solution to the Navier-Stokes equations in three dimensions. This theorem is then generalized to extend this construction to higher order norms. The second main theorem, established in Chapter 5, explores the spectral structure of finite-mode solutions to the Navier-Stokes equations in two dimensions. A consequence of this theorem is that if the force is an eigenvector of the Stokes operator, then any nonstationary solution with constant energy and enstrophy must be supported on an infinite number of eigenvectors of the Stokes operator. This precludes the existence of so-called *chained ghost solutions* introduced by Tian and Zhang in [25].

1.2 The Navier-Stokes Equations

The incompressible Navier-Stokes equations (NSE) are a set of partial differential equations that describe the motion of a Newtonian fluid. Derived from the principles of conservation of momentum and mass, they are the following:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = F \\ \nabla \cdot u = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

over some spatial domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ with appropriate boundary or decay conditions. Here $u = \mathbf{u}(\mathbf{x}, t)$ is the fluid velocity vector field, ν is the kinematic viscosity, $p = p(\mathbf{x}, t)$ is the pressure (per density), $F = \mathbf{F}(\mathbf{x})$ is the body force vector field (per density), and $u_0 = \mathbf{u}_0(\mathbf{x})$ is the initial condition. The unknowns in these equations are the vector u and the scalar p.

We make several simplifications in our analysis of the Navier-Stokes equations. First, we consider these equations with periodic boundary conditions. That is, the domain Ω is the torus (in 2 or 3 dimensions) given by $\Omega = [0, L]^n_{per}$, $n = 2, 3$. This assumption is useful because it allows us to in essence ignore boundary conditions while maintaining a compact domain. Though this assumption is not physically realistic, it is useful for considering idealized cases or situations where the boundary is far enough removed from the region of concern that its effects can be neglected [13] (p.30). It is also often the preferred setting for numerical simulations (see, for example, [1]). This is because, concretely, the assumption of periodic boundary conditions allows us to explicitly write the functions under consideration in terms of their Fourier expansions (see equation (1.21)), which thus allows us to perform explicit calculations.

Next, we assume that the velocity and body force have zero space average over their domain. That is, we assume

$$\int_\Omega u \, \mathrm{d}x = 0, \qquad \int_\Omega F \, \mathrm{d}x = 0. \tag{1.2}$$

This assumption simplifies certain calculations and can be done without loss of generality since all cases can be reduced to such a scenario. Mathematically, the justification amounts to performing an appropriate change of variables (see [13] p.31). Physically, the justification for centering the velocity at the average stems from the equivalence of Newton's laws between any two inertial frames (known as Galilean invariance).

Thus, in this dissertation we consider the following specific formulation of the Navier-Stokes equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = F \\ \nabla \cdot u = 0 \\ \int_\Omega u dx = 0, \quad \int_\Omega F dx = 0 \\ u(x,0) = u_0 \end{cases} \tag{1.3}$$

on a periodic spatial domain $\Omega = [0, L]^n_{per}$, $n = 2, 3$, with u_0, F at least in $[L^2(\Omega)]^n$.

Two physical quantities that we will deal with extensively are the energy and enstrophy (each per unit mass) of the fluid. [1] These are given by $e(u)$ and $E(u)$ respectively and are defined as follows:

$$e(u) = \frac{1}{2L^n} \int_\Omega |u(x)|^2 dx \tag{1.4}$$

[1]The enstrophy is often defined as $\int_\Omega |\omega(x)|^2 dx$ where $\omega = \nabla \times u$ is the curl of u. On a periodic domain these two definitions are equivalent. This can be shown using integration by parts and the divergence-free condition.

$$E(u) = \frac{1}{2L^n} \sum_{i=1}^{n} \int_\Omega |\nabla u_i(x)|^2 dx. \tag{1.5}$$

We associate with the system (1.3) two natural Hilbert spaces. The first is the space of divergence-free zero-space-average functions with bounded energy, referred to here as H.[2] The inner product on H is the usual $[L^2(\Omega)]^n$ inner product defined by

$$(u, v) = \int_\Omega u(x) \cdot v(x) dx. \tag{1.6}$$

For the norm on H we use the notation

$$|u| = (u, u)^{1/2} = \left(\int_\Omega |u(x)|^2 dx \right)^{1/2} \tag{1.7}$$

where the difference between whether $|\cdot|$ refers to the norm in H or the modulus of a vector is hopefully clear from context. Thus, we define H as follows:

$$H := \left\{ u \in [L^2(\Omega)]^n : \nabla \cdot u = 0, \int_\Omega u(x) dx = 0 \right\}. \tag{1.8}$$

Another natural space that we associate with this system is the subspace of H consisting of divergence-free zero-space-average functions with finite enstrophy, which we refer to as V. We write the natural norm on V, which in our case is equivalent to the $[H^1(\Omega)]^n$ norm[3], as

$$\|u\| = ((u, u))^{1/2} = \left(\int_\Omega \left| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u(x) \right|^2 dx \right)^{1/2}. \tag{1.9}$$

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[2] The following definitions of H and V can be found in, say, [13]. There (and in other places) the zero-space average and periodic boundary conditions are indicated in the notation itself. For example, our definitions of H and V are the same as the definitions of $\dot{H}_{\text{per}}$ and $\dot{V}_{\text{per}}$ given in equations (5.17) and (5.18) in [13]. These spaces are often alternatively (and equivalently) defined in the literature as the closure in $[L^2(\Omega)]^n$ and, respectively, in $[L^2(\Omega)]^n$ of the space of infinitely differentiable divergence-free functions on Ω with zero-space average.

[3] On a periodic domain with zero-space-average the Poincaré inequality (eq. (1.36)) holds, and thus the H^1 norm, defined as $\left(\int_\Omega |u(x)|^2 dx + \int_\Omega \left| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} u(x) \right|^2 dx \right)^{1/2}$, and the norm below are equivalent

Thus we define V as follows:

$$V := \left\{ u \in [H^1(\Omega)]^n : \nabla \cdot u = 0, \int_\Omega u(x)dx = 0 \right\}. \tag{1.10}$$

An often-used variant of the well-known Helmholtz decomposition (sometimes called the Helmholtz-Weyl decomposition) tells us that any vector in $[L^2(\Omega)]^n$ can be orthogonally decomposed into the sum of a gradient and a divergence-free vector field, and this decomposition is unique. That is, for any $u \in \Omega$, we have $u = h + \nabla g$ where $h \in H$ (and thus is such that $\nabla \cdot h = 0$), g is a scalar function such that $\nabla g \in H$, and $\int_\Omega h \cdot \nabla g dx = 0$. The Leray Projector $\mathbb{P}_L : [L^2(\Omega)]^n \to H$ is defined as follows: given $u = h + \nabla g$ (where h, g are from the Helmholtz-Weyl decomposition), $\mathbb{P}_L(u) = h$. That is, the Leray projector is an orthogonal projection of $[L^2(\Omega)]^n$ onto H. Notice that in system 1.3, since u is divergence-free by hypothesis, we have that $\mathbb{P}_L(u) = u$. Also notice that since ∇p is a gradient, we have that $\mathbb{P}_L(\nabla p) = 0$.

Taking the Leray Projection onto H of (1.3) we have the following functional formulation of the Navier-Stokes equations:

$$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = f \\ u(0) = u_0 \end{cases} \tag{1.11}$$

where $A(u) := \mathbb{P}_L(-\Delta u)$ is the Stokes operator, $B(u,v) := \mathbb{P}_L((u \cdot \nabla)v)$ is a bilinear operator, and $f := \mathbb{P}_L(F)$. We note that $\mathbb{P}_L$ commutes with $\frac{d}{dt}$, and since we impose periodic boundary conditions on our spatial domain, we have $Au = -\Delta u$.

Without loss of generality, we may consider this functional formulation of the Navier-Stokes equations given in (1.11). This is due to the fact that since u is already divergence-free, it is not changed by Leray projection. In addition, the zero-space-average and divergence-free conditions on u guarantee that the pressure term ∇p is completely determined by the velocity term u and the forcing data F. Taking the divergence of both sides and exploiting the fact that $\nabla \cdot u = 0$ we have $\Delta p = -\sum_{i,j=1}^n \frac{d^2}{dx_i dx_j}(u_i(x)u_j(x))$. This equation for p has a unique solution in the space of functions with zero-space average and periodic boundary conditions.[4]

The advantage of using the functional formulation of (1.11) is that it allows us to view the NSE as an evolution equation in H. This provides the foundation for taking a dynamical systems approach to understanding the behavior of solutions to the NSE. This will be discussed more in Chapter 2.

The Stokes operator $A : D(A) \subset H \to H$ is well-known to be self-adjoint with compact inverse. Thus, as a consequence of Hilbert's spectral theorem we know there exists an orthonormal basis of vectors $\{\omega_j\}_{j\in\mathbb{N}} \in H$ such that for each ω_j we have $A\omega_j = \lambda_j \omega_j$ for some real number λ_j (that is, each ω_j is an eigenvector of A). Thus we may write any vector field $u(x) \in H$ as follows:

$$u(x) = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} u_j \omega_j(x) \tag{1.12}$$

with $u_j \in \mathbb{R}$. We may define the powers A^s of the Stokes operator A (for $s \in \mathbb{R}$) as follows:

$$A^s u = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \lambda_j^s u_j \omega_j(x). \tag{1.13}$$

The domain of A^s is given by

$$\left\{ u : \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \lambda_j^{2s} (u_j)^2 < \infty \right\}. \tag{1.14}$$

For $s \geq 0$ we have $A^s \subseteq H$. For $s < 0$ we have $H \subset A^s$.

In the space-periodic case we have an explicit formulation of ω_j and λ_j. Indeed, we may explicitly write each eigenvalue as

$$\lambda_j = \left(\frac{2\pi}{L}\right)^2 j \cdot j; \qquad j \in \mathbb{Z}^n \setminus \{0\} \tag{1.15}$$

[4]Indeed, the operator $(-\Delta)^{-1}$ has a natural definition in Fourier space. Just as $-\Delta$ is given by multiplying the j^{th} Fourier mode by λ_j for each $j \in \mathbb{Z}^n \setminus \{0\}$, the operator $(-\Delta)^{-1}$ is given by dividing the j^{th} Fourier mode by λ_j.

and the eigenvectors of A may be represented by trigonometric polynomials. Thus, any function in H may alternatively be written as a Fourier series

$$u(x) = \sum_{j \in \mathbb{Z}^n \setminus \{0\}} \hat{u}_j e^{i(2\pi/L)j\cdot x} \tag{1.16}$$

where the $\hat{u}_j \in \mathbb{C}^n$ are such that $\sum_{j\in\mathbb{Z}^n\setminus\{0\}} |\hat{u}_j|^2 < \infty$ (to ensure finite energy), $\hat{u}_j = \overline{\hat{u}}_{-j}$ (to assure u is real), and $\hat{u}_j \cdot j = 0$ (to assure that u is divergence-free) for all $j \in \mathbb{Z}^n \setminus \{0\}$. By Parseval's identity we have

$$|u|^2 = \sum_{j \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}_j|^2, \tag{1.17}$$

$$\|u\|^2 = \sum_{j \in \mathbb{Z}^n \setminus \{0\}} \lambda_j |\hat{u}_j|^2. \tag{1.18}$$

Thus we may rewrite our spaces H and V as follows:

$$H = \left\{ u(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{u}_k e^{i(2\pi/L)k\cdot x} : \hat{u}_{-k} = \overline{\hat{u}}_k, \hat{u}\cdot k = 0, \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |\hat{u}|^2 < \infty \right\}, \tag{1.19}$$

$$V = \left\{ u(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{u}_k e^{i(2\pi/L)k\cdot x} : \hat{u}_{-k} = \overline{\hat{u}}_k, k \cdot \hat{u} = 0, \sum_{k \in \mathbb{Z}^n \setminus \{0\}} |k|^2 |\hat{u}|^2 < \infty \right\}. \tag{1.20}$$

Note that we may write the norm on V in terms of the Stokes operator as follows: $\|u\|^2 = (A^{1/2}u, A^{1/2}u)$.

We may now represent (1.11) as an infinite dimensional system of coupled ordinary differential equations by rewriting it in terms of its Fourier series:

$$\hat{u}_j'(t) + \nu\lambda_j \hat{u}_j(t) + \widehat{B(u,u)}_j(t) = \hat{f}_j; \qquad j \in \mathbb{Z}^n \setminus \{0\}. \tag{1.21}$$

Two dimensionless quantities that are useful for understanding fluid flow are the generalized Grashof number (introduced in [14]) and the well-known Reynolds number. The definition of the generalized Grashof number depends on the number of spatial dimensions of the flow. It is defined as

$$G = \frac{|f|}{\nu^2 \lambda_0} \quad \text{in 2D}; \qquad G = \frac{|f|}{\nu^2 \lambda_0^{3/4}} \quad \text{in 3D}. \tag{1.22}$$

where $\lambda_0 = \left(\frac{2\pi}{L}\right)^2$ is the smallest eigenvalue of the Stokes operator and f is the Leray projection of the force. In order to define the Reynolds number one must first define some appropriate (time-independent) "average" of the fluid velocity. Several reasonable choices are possible, so let $\langle |u| \rangle$ refer to an appropriate average of the fluid velocity. Then we define the Reynolds number as

$$Re = \frac{\langle |u| \rangle}{\nu \lambda_0^{1/2}}. \tag{1.23}$$

1.3 Properties of the Nonlinear Term

In this section we list some important properties of the nonlinear term $B(u, u)$.

Proposition 1.3.1. *For $u, v, w \in V$ we have $(B(u, v), w) = -(B(u, w), v)$.*

The proof, which may be found in [3] (p.53), uses integration by parts and exploits the fact that u is divergence-free and the fact that the spatial domain is periodic.

Corollary 1.3.2. *For $u \in V$ we have $(B(u, u), u) = 0$.*

Proposition 1.3.3. *For $u \in D(A)$ we have $(B(u, u), Au) = 0$ in 2D.*

The proof similarly uses integration by parts, the periodicity of the domain, and the divergence-free condition on u (see [23] p.404).

There are many ways to bound the trilinear term $(B(u, v), w)$ for $u, v, w \in V$, all of which start with an application of the generalization of Hölder's inequality (1.35). In particular, we will see the following bound (in 2D) in Chapter 3.

Proposition 1.3.4. *Let $\Omega = [0, L]^2_{per}$ and $u, v, w \in H$. Then*

$$|(B(u, v), w)| \le c_L^2 |u|^{1/2} \|u\|^{1/2} \|v\| |w|^{1/2} \|w\|^{1/2},$$

where c_L is the constant from the Ladyzhenskaya inequality (1.39).

To obtain this bound first apply the generalization of Hölder's inequality (1.35) where $f_1 = u, f_2 = \nabla v, f_3 = w$, $r = 1$, and $p_1 = 4, p_2 = 2, p_3 = 4$. Then apply the Ladyzhenskaya inequality (1.39) twice (once on $\|u\|_4$ and once on $\|w\|_4$).

The following representation of the Fourier coefficients of the nonlinear term is well-known, but we provide the calculations here:

Proposition 1.3.5. *Given $u \in H$ and $v \in V$ we may explicitly write the Fourier coefficients of $B(u,v)$ as follows:*

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \left[(\hat{u}_{j-k} \cdot k)\hat{v}_k - \frac{(\hat{u}_{j-k} \cdot k)(\hat{v}_k \cdot j)}{|j|^2} j \right]. \tag{1.24}$$

In 2D we may alternatively write the Fourier coefficients of $B(u,v)$ as follows:

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L} \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{(\hat{u}_{j-k} \cdot k)(\hat{v}_k \cdot j^\perp)}{|j|^2} j^\perp \tag{1.25}$$

where for $j = (j_1, j_2)$ we define $j^\perp = (-j_2, j_1)$.

Proof. First we represent $(u \cdot \nabla)v$. Consider the following calculations:

$$\begin{aligned}
(u \cdot \nabla)v &= (u \cdot \nabla) \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{v}_k e^{ik \cdot x} \\
&= \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{v}_k \sum_{j=1}^{n} u_j \frac{\partial}{\partial x_j} e^{ik \cdot x} \\
&= i \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{v}_k (u \cdot k) e^{ik \cdot x} \\
&= i \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \hat{v}_k \left(\sum_{j \in \mathbb{Z}^n \setminus \{0\}} \hat{u}_j e^{ij \cdot x} \cdot k \right) e^{ik \cdot x} \\
&= i \sum_{j,k \in \mathbb{Z}^n \setminus \{0\}} (\hat{u}_j \cdot k)\, \hat{v}_k e^{i(k+j) \cdot x} \\
&= i \sum_{j,k \in \mathbb{Z}^n \setminus \{0\}} (\hat{u}_j \cdot k)\, \hat{v}_k e^{i(k+j) \cdot x}
\end{aligned}$$

Now we reindex with $m = k + j$ to get

$$(u \cdot \nabla)v = i \sum_{m \in Z^n \setminus \{0\}} \sum_{k \in \mathbb{Z}^n \setminus \{0\}} (\hat{u}_{m-k} \cdot k)\, \hat{v}_k e^{im \cdot x}$$

To get $B(u,v)$ we need to project $(u\cdot\nabla)v$ onto the divergence-free vector fields. This means we need $\widehat{B(u,v)}_m \cdot m = 0$ for each $m \in \mathbb{Z}^n \setminus \{0\}$. Since L^2 can be orthogonally decomposed into gradients and divergence-free vector fields, we can calculate $B(u,v)$ in two ways. First, we may subtract off the projection of $(u\cdot\nabla)v$ onto the space of gradients. In this case we project $[\widehat{(u\cdot\nabla)v}]_m$ onto m and then subtract. Thus we have

$$\widehat{B(u,v)}_m = i \sum_{k\in\mathbb{Z}^n\setminus\{0\}} (\hat{u}_{m-k}\cdot k)\hat{v}_k - \frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m)}{|m|^2} m.$$

Alternatively, in 2D we can project $[\widehat{(u\cdot\nabla)v}]_m$ onto the direction of $m^\perp = (-m_2, m_1)$ as follows:

$$\widehat{B(u,v)}_m = i \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m^\perp)}{|m^\perp|^2} m^\perp.$$

Since $|m^\perp| = |m|$ we may rewrite this as

$$\widehat{B(u,v)}_m = i \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{(\hat{u}_{m-k}\cdot k)(\hat{v}_k\cdot m^\perp)}{|m|^2} m^\perp.$$

□

1.4 Standard Results in 2 and 3 Dimensions

In this section we briefly mention some standard results of existence and uniqueness theory for the Navier-Stokes equations. A necessary step in this process is to introduce various notions of so-called "solutions" to the system (1.3). We begin with the definition of a classical solution.

Definition 1.4.1. *A* **classical solution** *to the Navier-Stokes equations (system (1.3)) for $t \geq 0$, with $u_0, f \in H$, is a function u such that*

(i) u and its classical derivatives $\frac{\partial u}{\partial t}, \nabla u, \Delta u$ exist and are in $[L^2(\Omega)]^n$ for all $t > 0$.

(ii) u *and* $p = (-\Delta)^{-1}\left[\sum_{i,j=1}^{n} \frac{d^2}{dx_i dx_j}(u_i(x)u_j(x))\right]$ *(where* p *is periodic with zero-space average and* $(-\Delta)^{-1}$ *is defined as in footnote 4) solve the equations of (1.11) for* $t > 0$.

(iii) $\lim_{t\to 0} |u(t) - u_0| = 0$.

A common strategy for proving existence and uniqueness of solutions is to find functions that satisfy some related but weaker conditions than the ones of the initial problem. The hope is to then show that such "easier-to-find" functions (known as *weak* solutions) are unique and must actually satisfy the original conditions as well. This strategy motivates the additional types of solutions that we will mention.

First we introduce the space of test functions $\mathcal{D}_\sigma$ (following the notation and treatment in [21]), defined as

$$\mathcal{D}_\sigma := \{\phi \in C_0^\infty(\Omega \times [0,\infty)) : \nabla \cdot \phi(t) = 0 \ \forall t \in [0,\infty)\}. \tag{1.26}$$

If we multiply the momentum equation in (1.3) by a test function ϕ and then integrate over the whole space and over the time interval $[0, s)$ we have the following condition that is easier to satisfy than the momentum equation:

$$\int_0^s \left[-(u, \frac{\partial \phi}{\partial t}) + (\nabla u, \nabla \phi) + ((u\cdot\nabla)u, \phi) - (F, \phi)\right] dt = (u_0, \phi(0)) - (u(s), \phi(s)), \tag{1.27}$$

(where we use integration by parts, the periodic boundary conditions in space, and the fact that ϕ is divergence-free). Clearly, any classical solution will satisfy equation (1.27). However, the converse is not true (see [21] Example 3.1). Indeed, recent work by Buckmaster and Vicol ([2]) has shown that even under certain restrictions on the class of solutions to equation (1.27) the problem is not well posed due to non-uniqueness. Thus, we want to impose additional, but natural, restrictions on any supposed 'weak' solution to assure that such a solution is desirable. There are various choices one can make here. Leray-Hopf weak solutions are perhaps the most important type of weak solution considered in the literature. We follow [21] in defining Leray-Hopf weak solutions as follows:[5]

Definition 1.4.2. *A* **Leray-Hopf weak solution** *to the Navier-Stokes equations (system (1.3)) with* $u_0, f \in H$ *is a function* u *such that*

(i) $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ *for all* $T > 0$*,*

(ii) u *is such that*

$$\frac{1}{2}|u|^2 + \int_s^t \nu \|u\|^2 \leq \frac{1}{2}|u(s)|^2 \text{ for all } t > s$$

for $s = 0$ *and for almost every* $s > 0$*.*

(iii) u *is such that*

$$\int_0^s \left[-(u, \frac{\partial \phi}{\partial t}) + (\nabla u, \nabla \phi) + ((u \cdot \nabla)u, \phi) - (F, \phi) \right] dt = (u_0, \phi(0)) - (u(s), \phi(s))$$

for all $\phi \in D_\sigma$ *and for almost every* $s > 0$*.*

The first condition in the definition of a weak solution (*i.e.* the regularity condition) comes from the energy balance equation that holds for classical solutions (see equation (1.30)). Integrating the energy balance equation in time and applying the Cauchy-Schwarz and Young inequalities (equations (1.33), (1.32)) to the (f, u) term shows that classical solutions satisfy that first condition in the definition of a weak solution. The second condition is known as the strong energy inequality and is a natural condition that emerges from the construction of such weak solutions using what is known as the Galerkin method.

As we will see, global existence in time of Leray-Hopf weak solutions is known, but uniqueness is not. This motivates the following definition of a so-called *strong* solution (again following [21]).

[5] The Leray-Hopf weak solutions considered here are stronger than the weak solutions considered in [2]. Thus the non-uniqueness result of that paper does not apply.

Definition 1.4.3. *A* **strong solution** *to the Navier-Stokes equations (system (1.3)) on the interval* $[0,T]$ *is a weak solution with the additional property that*

$$u \in L^{\infty}(0,T;H^1) \cap L^2(0,T;H^2). \tag{1.28}$$

The appeal of strong solutions lies in the fact that if they exist they are unique. They also guarantee that all the terms in the Navier-Stokes equation are square integrable in the space-time domain. That is, $u, \frac{\partial u}{\partial t}, \Delta u, (u\cdot\nabla u)u \in [L^2(\Omega\times(0,T))]^n$. Thus, in existence/uniqueness theory one seeks strong solutions to the Navier-Stokes equation.

Finally, we define what are known as steady-state (alternatively, stationary or equilibrium) solutions.

Definition 1.4.4. *A* **steady-state** *(or* **stationary**, *or* **equilibrium***) solution to the Navier-Stokes equations is a time-independent solution,* $u \in V$*, to the system*

$$\nu Au + B(u,u) = f \tag{1.29}$$

for a given $f \in H$.

We provide the following classical theorems regarding existence and uniqueness of solutions to the NSE (proofs of which may be found in, say, [3], among others).

Theorem 1.4.5 (Global in Time Existence of Weak Solutions, see [21])**.** *Let* $\Omega = [0,L]^n_{per}$*,* $n = 2,3$*. Given* $u_0, f \in H$*, there exists at least one Leray-Hopf weak solution* u *to the Navier-Stokes equations (system (1.3)) that is defined for all* $t > 0$.

Theorem 1.4.6 (Local in Time Existence of Strong Solutions, see [13], [21])**.** *Let* $\Omega = [0,L]^n_{per}$*,* $n = 2,3$*. Given* $u_0 \in V$ *and* $f \in H$*, there exists time* $T > 0$ *such that there exists a (unique) strong solution* u *to the Navier-Stokes equations (system (1.3)) at least on the interval* $[0,T]$.

Theorem 1.4.7 (Global in Time Existence of Strong Solutions in 2D, see [13])**.** *Let* $\Omega = [0,L]^2_{per}$*. Given* $u_0 \in V$ *and* $f \in H$*, there exists a (unique) strong solution* u *to the Navier-Stokes equations (system (1.3)) that is defined for all* $t > 0$.

Theorem 1.4.8 (Existence of Steady-State Solutions (see [3], or [22]). *For any* $f \in H$ *there exists a steady-state solution, often denoted* u^**, to the Navier-Stokes equations (in 2D or 3D) such that* u^* *is in the domain of* A*.*

In addition to existence in time, we have the following theorems regarding analyticity in time of strong solutions:

Theorem 1.4.9 (Time Analyticity in 3D (see [3])). *Let* $\Omega = [0, L]^3_{per}$*. Given* $u_0 \in V$ *and* $f \in H$*, there exists a* $T > 0$ *and an open neighborhood* D_T *in the complex plane including* $(0, T)$ *such that the solution* $u(t)$ *of (1.11) is analytic as a function from* D_T *to the domain of* A*.*

In addition, given the spectral representation of u *as given in (1.12), we also have that the coefficients* $u_j(t)$*,* $j \in \mathbb{Z}^3 \setminus \{0\}$*, are each analytic on* D_T*.*

Theorem 1.4.10 (Time Analyticity in 2D (see [3])). *Let* $\Omega = [0, L]^2_{per}$*. Given* $u_0 \in H$ *and* $f \in H$*, there exists an open neighborhood* D *in the complex plane including the positive real axis* $(0, \infty)$ *such that the solution* $u(t)$ *of (1.11) is analytic as a function from* D *to the domain of* A*.*

In addition, given the spectral representation of u *as given in (1.12), we also have that the coefficients* $u_j(t)$*,* $j \in \mathbb{Z}^2 \setminus \{0\}$*, are each analytic on* D*.*

We also have the following results on the real-analytic nature of functional norms of strong solutions:

Theorem 1.4.11. *Let* $\Omega = [0, L]^3_{per}$*,* $u_0 \in V$ *and* $f \in H$*. Then* $|u(t)|^2, \|u(t)\|^2$*, and* $|Au(t)|^2$ *are analytic on* $(0, T)$ *where* T *is the same as in Theorem 1.4.9.*
Let $\Omega = [0, L]^2_{per}$ *and* $u_0, f \in H$*. Then* $|u(t)|^2, \|u(t)\|^2$*, and* $|Au(t)|^2$ *are analytic on* $(0, \infty)$*.*

For completeness' sake, we collect here the energy and enstrophy balance equations for the NSE. For strong solutions to the NSE we have the following energy balance equation:

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \nu\|u\|^2 = (f, u). \tag{1.30}$$

This is obtained by taking the L^2 inner product of the the Navier-Stokes equations with the solution u. In two dimensions only, we also have the enstrophy balance equation:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \nu|Au|^2 = (f, Au), \tag{1.31}$$

which is obtained by taking the inner product of the NSE with Au.

1.5 Useful Inequalities

In this section we collect some inequalities that are common in the literature and which we make use of in this dissertation.

Young's Inequality *Let $a, b \geq 0$ and $\varepsilon > 0$. Then*

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{b^q}{\varepsilon^{q/p} q}, \quad \textit{where } p, q > 1 \textit{ and } \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{1.32}$$

Cauchy-Schwarz Inequality *For vectors u, v in an inner product space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$ we have*

$$|(u, v)| \leq \|u\|\|v\|, \tag{1.33}$$

with equality when u is a multiple of v.

Hölder's Inequality *Let $\|\cdot\|_p$ refer to the L_p norm defined by $\left(\int_\Omega |\cdot|^p\right)^{1/p}$. Then for measurable functions f, g we have*

$$\int_\Omega |fg| \leq \|f\|_p\|g\|_q \tag{1.34}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. This also holds for $p = \infty, q = 1$.

This inequality may be generalized as follows:

Let $r, p_1, \dots, p_n \in (0, \infty]$ be such that $\sum_{k=1}^n \frac{1}{p_k} = \frac{1}{r}$, and let $f_1, \dots, f_n$ be measurable functions. Then

$$\left\|\prod_{k=1}^n f_k\right\|_r \leq \prod_{k=1}^n \|f_k\|_{p_k}. \tag{1.35}$$

While the Poincaré inequality can be stated much more generally, the following version suffices for our purposes:

Poincaré Inequality *There exists a constant C, independent of u, such that*

$$|u| \leq C\|u\| \tag{1.36}$$

for all $u \in H$.

Indeed, in our case we may calculate C explicitly:

$$\|u\|^2 = \sum_{j\in\mathbb{Z}^n\setminus\{0\}} \lambda_j u_j^2 \geq \lambda_0 \sum_{j\in\mathbb{Z}^n\setminus\{0\}} u_j^2 = \lambda_0|u|^2.$$

Thus we have $C = \lambda_0^{-1/2}$ where $\lambda_0 = \left(\frac{2\pi}{L}\right)^2$ is the smallest eigenvalue of the Stokes operator. A similar inequality holds between the terms $\|u\|$ and $|Au|$:

$$\|u\|^2 \leq (\lambda_0)^{-1/2}|Au|^2 \tag{1.37}$$

Grönwall's Inequality *Let $u(t), a(t), b(t)$ be real continuous functions defined on an open interval I. If $\frac{d}{dt}u(t) \leq a(t)u(t) + b(t)$ then*

$$u(t) \leq u(t_0)e^{\int_{t_0}^t a(s)ds} + e^{\int_{t_0}^t a(s)ds}\int_{t_0}^t e^{-\int_{t_0}^s a(x)dx}b(s)ds \tag{1.38}$$

for $t_0, t \in I$.

Ladyzhenskaya Inequality (in 2D and 3D) *Let $\Omega = [0, L]^2_{per}$. For any $u \in V$ there exists a constant c_L (which depends only on Ω) such that*

$$\|u\|_{L^4(\Omega)} \leq c_L|u|^{1/2}\|u\|^{1/2}. \tag{1.39}$$

Let $\Omega = [0, L]^3_{per}$. For any $u \in V$ there exists a constant C_L (which depends only on Ω) such that

$$\|u\|_{L^4(\Omega)} \leq C_L|u|^{1/4}\|u\|^{3/4}. \tag{1.40}$$

Agmon's Inequality (in 2D and 3D) *Let $\Omega = [0, L]^2_{per}$. For any $u \in D(A)$ there exists a constant c_A such that*

$$\|u\|_{L^\infty(\Omega)} \leq c_A|u|^{1/2}|Au|^{1/2}. \tag{1.41}$$

Let $\Omega = [0, L]^3_{per}$. *For any* $u \in D(A)$ *there exists a constant* C_A *such that*

$$\|u\|_{L^\infty(\Omega)} \leq C_A \|u\|^{1/2} |Au|^{1/2}. \tag{1.42}$$

2 Attractors and Turbulence Theory

In this chapter, we look at the Navier-Stokes equations from a dynamical systems perspective. The goal of this chapter is to properly motivate the central problem of this dissertation: Do there exist non-stationary solutions to the Navier-Stokes equations whose energy and enstrophy remain constant over time? There are two somewhat independent motivations for considering this problem. One comes from a dynamical systems perspective and the other comes from the theory of turbulence. We introduce each in turn.

2.1 Attractors of the Navier-Stokes Equations

The advantage of looking at the NSE in the following form:

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u,u) = f \\ u(x,0) = u_0 \end{cases}$$

is that the system may be viewed as an ODE in the space H. Thinking of the momentum equation of the NSE as an evolution equation in H, we may view the NSE as describing a dynamical system with parameter f. The attractor of a dynamical system (a definition we will make more precise momentarily) in some sense captures the long-term dynamics of the system, since every solution becomes arbitrarily close to the attractor in finite time (see Figure 2.1). The attractor is the place to look for understanding persistent phenomena of the system.

Suppose you have a dynamical system in some appropriate function space, (suggestively called) H, with unique solutions corresponding to each possible initial condition in H. Denote by $S(t)u_0$ the unique solution $u(t)$ corresponding to initial condition

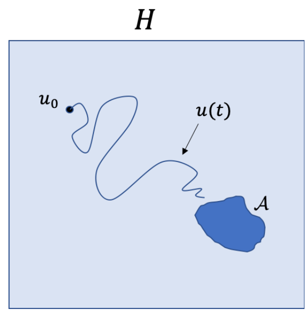


FIGURE 2.1: Visualization of arbitrary solution and the global attractor in H

u_0. Then the operators $S(t) : u_0 \in H \to u(t) \in H$ form a semi-group

$$\begin{cases} S(t+s) = S(t) \circ S(s) & \forall s, t \geq 0 \\ S(0) = I & (Identity). \end{cases}$$

Following the treatment in [23] we define an attractor of such a dynamical system as follows:

Definition 2.1.1. *An* **attractor***, $\mathcal{A}$, is a subset of H with the following properties:*

1. *$S(t)\mathcal{A} = \mathcal{A}$. That is, $\mathcal{A}$ is an invariant set under $S(t)$.*
2. *$\mathcal{A}$ belongs to an open set U such that, for every $u_0 \in U$ we have $\operatorname{dist}(S(t)u_0, \mathcal{A}) \to 0$ as $t \to \infty$.*

We say that $\mathcal{A}$ is a **global (or universal) attractor** *if, in addition, $\mathcal{A}$ is compact, and for any bounded set $B \subset H$ we have that $\mathcal{A}$ attracts B uniformly (that is, $\sup_{x \in S(t)B} \inf_{y \in \mathcal{A}} d(x,y) \to 0$ as $t \to \infty$).*

Many deep results regarding the attractor of the Navier-Stokes equations in two-dimensions are known. Indeed, these results motivate the attempt to understand better the relationship between the energy and enstrophy of a solution and its long-term behavior. Since long-term existence and uniqueness in three dimensions is not guaranteed (as far as we know), the notion of an attractor does not exist in the same sense. Thus we consider only the two dimensional case in what follows.

2.1.1 The Global Attractor in Two Dimensions

Recall that in two dimensions we have global (forward in time) existence and uniqueness of solutions to the system (1.11). Thus, for every initial condition u_0 in H we have a well-defined solution operator $S(t)u_0 = u(t)$. We also have that both the energy and enstrophy balance equations, (1.30) and (1.31), hold in this case. Applying the Cauchy-Schwarz, Young, Poincaré, and Grönwall inequalities (in order) to the energy and enstrophy balance equations yields the following relations:

$$|u|^2 \leq e^{-\nu(2\pi/L)^2 t}|u_0|^2 + (1 - e^{-\nu(2\pi/L)^2 t})\frac{|f|^2}{\nu^2(2\pi/L)^4},$$
$$\|u\|^2 \leq e^{-\nu(2\pi/L)^2 t}\|u_0\|^2 + (1 - e^{-\nu(2\pi/L)^2 t})\frac{|f|^2}{\nu^2(2\pi/L)^2}.$$

These imply that any bounded set in H eventually becomes a subset of the closed ball $B = \{u \in V : \|u\| \leq 2\nu\frac{2\pi}{L}G\}$ (where G is the Grashof number (1.22)). Indeed, any bounded set is attracted to this ball uniformly. The set

$$\mathcal{A} = \bigcap_{t\geq 0} S(t)B \tag{2.1}$$

is invariant under $S(t)$ and is such that $\text{dist}(S(t)u_0, \mathcal{A}) \to 0$ as $t \to \infty$ for all $u_0 \in H$. Thus we have that $\mathcal{A}$ is a global attractor for the NSE in two dimensions. In fact, the following stronger properties of $\mathcal{A}$ are well known:

Theorem 2.1.2 (See [3], Chapter 14)**.** *Let $\Omega = [0, L]^2_{per}$. The global attractor $\mathcal{A}$ of the dynamical system associated with the Navier-Stokes equations (1.11) is compact, connected, and has finite fractal and Hausdorff dimension.*

It also follows from the above discussion that for all $u \in \mathcal{A}$ we have

$$|u|^2 \leq \nu^2 G^2 \tag{2.2}$$
$$\|u\|^2 \leq \nu^2(2\pi/L)^2 G^2 \tag{2.3}$$

In order to understand the relationships between the energy and enstrophy that exist independent of the force f, in [7] they consider notions of energy and enstrophy that are dimensionless and normalized by the energy of the force. These normalized and dimensionless energy and enstrophy are defined (respectively) as follows:

$$\mathbf{e} = \frac{|u|^2}{\nu^2 G^2} \tag{2.4}$$

$$\mathbf{E} = \frac{\|u\|^2}{\nu^2 (2\pi/L)^2 G^2} \tag{2.5}$$

An important relationship between the energy and enstrophy of solutions in the global attractor (in the 2D periodic case) was proved in [7]:

Theorem 2.1.3. *[See Theorem 5.2 in [7]] Let $f \in H$. For all $u \in \mathcal{A}$*

$$\|u\|^2 \leq \frac{|f|}{\nu}|u|. \tag{2.6}$$

This implies that for solutions in the attractor we have

$$\mathbf{E} \leq \sqrt{\mathbf{e}}. \tag{2.7}$$

When restated in terms of the normalized dimensionless energy and enstrophy, the Poincaré inequality becomes

$$\mathbf{E} \geq \mathbf{e}. \tag{2.8}$$

Thus, Theorem 2.1.3 together with the Poincaré inequality give a bounded region in the $\mathbf{e}-\mathbf{E}$ plane where the attractor must "project". That is, geometrically speaking, the attractor exists in the $\mathbf{e}-\mathbf{E}$ plane somewhere in the shaded region in Figure 2.2.

Further refinements to this bound on the attractor in the $\mathbf{e}-\mathbf{E}$ plane are given in [7] and elsewhere (see, for example [6], [11]). This line of investigation leads to several natural questions that we explore in this dissertation.

2.1.2 Natural Questions and a First Look at our Problem

A question that arises naturally in this area is how much we can determine regarding the dynamics of the system by simply considering the energy and enstrophy

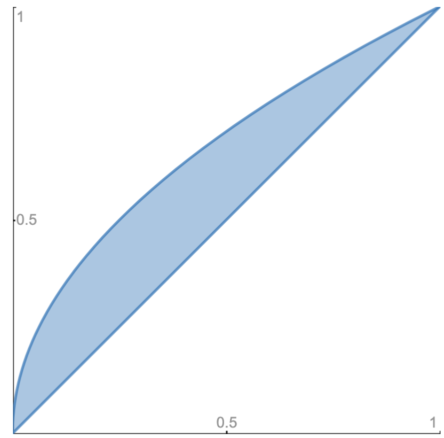


FIGURE 2.2: Bounds on $\mathcal{A}$ in the $\mathbf{e}-\mathbf{E}$ plane

of solutions in the attractor. In light of Theorem 2.1.3, this question might be asked in the following way: what can we say about the general dynamics of a solution simply by considering its trajectory in the $\mathbf{e}-\mathbf{E}$ plane? While it is likely too much to hope that an entire solution might be determined by its behavior in the $\mathbf{e}-\mathbf{E}$ plane, there is reason to believe that these quantities may still tell us quite a bit.

As a first test regarding the usefulness of considering trajectories in the $\mathbf{e}-\mathbf{E}$ plane, it would be nice to know whether a solution with a stationary trajectory in the $\mathbf{e}-\mathbf{E}$ plane is necessarily a stationary solution. If a solution that evolves dynamically in time can project to a single point in the $\mathbf{e}-\mathbf{E}$ plane, then that would indicate that the energy and enstrophy profiles alone miss something fundamental about the dynamics of the underlying solution. Thus we have the following research question:

Question 2.1.4. *Can there exist a non-stationary solution $u(t)$ to the Navier-Stokes equations (1.11) whose energy and enstrophy remains constant in time? That is, can we have $\frac{d}{dt}u(t) \neq 0$ while having $\frac{d}{dt}|u(t)| = 0$ and $\frac{d}{dt}\|u(t)\| = 0$?*

This is the question that directly motivates the research in this dissertation. In particular, the main theorem of Chapter 5 implies that, in two spatial dimensions, when the force is an eigenvector of the Stokes operator, no solution with a finite Fourier series representation can be such that it is nonstationary with constant energy and enstrophy.

We also consider in this dissertation two less restrictive, but related questions:

Question 2.1.5. *Can there exist a non-stationary solution* $u(t)$ *to the Navier-Stokes equations (1.11) whose energy remains constant in time? That is, can we have* $\frac{d}{dt}u(t) \neq 0$ *while having* $\frac{d}{dt}|u(t)| = 0$*?*

Question 2.1.6. *Can there exist a non-stationary solution* $u(t)$ *to the Navier-Stokes equations (1.11) whose enstrophy remains constant in time? That is, can we have* $\frac{d}{dt}u(t) \neq 0$ *while having* $\frac{d}{dt}\|u(t)\| = 0$*?*

In Chapter 4 we answer these last two questions in the affirmative in the case of three spatial dimensions. We do this by a direct construction of such solutions.

Independent motivation for considering these questions comes from the theory of turbulence—a topic we turn to now.

2.2 Turbulence

Turbulence is commonly used to refer to the chaotic motion that is often observed when, for example, a fluid moving at a high velocity encounters an obstacle. A key feature of turbulent motion is that the velocity field describing the motion becomes erratic and unpredictable (despite the deterministic nature of the system). Interestingly, as turbulence in a fluid increases (say, by increasing the velocity or the Reynolds number of the fluid) there comes a point when some amount of predictability (in a statistical sense) is restored. That is, experimental results concerning certain averages (such as average kinetic energy or average enstrophy) once again become predictable. This situation where the turbulence becomes uniform enough to restore statistical predictability is known as "fully developed turbulence."

The physical theory of fluid turbulence in three dimensions was pioneered in the

1920s by L. F. Richardson [20][6] and formalized by A. Kolmogorov in the 1940s ([17], [16]). This classical theory of 3D turbulence proceeds along the following lines. Energy is introduced into the system at relatively large length scales (small wavenumber) due to the forcing term in the NSE (it is generally assumed that the forcing term is supported on a finite set of wave numbers relatively far removed from the scales where viscous dissipation dominates). Whether or not we are in a turbulent regime, it is thought that this energy is transferred via the non-linear term to smaller and smaller length scales until it reaches scales where viscous dissipation dominates and the energy leaves the system. Turbulent regimes are characterized by what is called an "inertial range" of scales where the rate of transfer of energy through each wavenumber (via the non-linear term) in this range is nearly the same as the overall energy dissipation rate of the system. This transfer of energy through the inertial range from small wave numbers to large wave numbers is known as a "direct energy cascade". The inertial range is thought to occur for wave numbers between the largest wave number of the force and the wave numbers where viscous dissipation begins to be felt.

The theory in two dimensions is necessarily different than in three dimensions due to the fact that in two dimensions we have the important orthogonality property $(B(u,u),Au) = 0$, yielding an enstrophy balance equation in 2D that mirrors the energy balance equations in 2D and 3D. The two dimensional theory, due to Kraichnan [18], is characterized by a direct *enstrophy* cascade through an inertial range towards large wave numbers and an inverse cascade of energy to smaller wave numbers.

These theories of turbulence as a self-similar flux of energy (or enstrophy) through an inertial range were developed on physical grounds somewhat divorced from the Navier-Stokes equations themselves. Even the rigorous results regarding the bounds on the inertial range are derived from the energy/enstrophy balance equations and dimensional analysis alone. More recent work has focused on deriving aspects of

[6]It is in this work (p. 66) where we get Richardson's famous little poem about turbulence: "Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity."

the physical theory of turbulence from the mathematical theory of the Navier-Stokes equations (see for example [13] and [10]). It is in this spirit that we find another motivation for considering solutions to the Navier-Stokes equations with constant energy and enstrophy.

2.2.1 A Sufficient Condition for 2D Turbulence

The central goal of the paper "Statistical Estimates for the Navier-Stokes Equations and the Kraichnan Theory of Fully Developed Turbulence" [10] is to reproduce aspects of Kraichnan's physical theory of 2D turbulence from the Navier-Stokes equations alone and to provide rigorous support for the physical theory. We discuss elements of their strategy and several results below.

In the case where there is no force (and thus no energy introduced to the system) the energy and enstrophy balance equations (eq.s 1.30, 1.31) are simply

$$\frac{1}{2}\frac{d}{dt}|u|^2 = -\nu\|u\|^2, \tag{2.9}$$

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\nu|Au|^2. \tag{2.10}$$

Thus, the terms $\nu\|u(t)\|^2$ and $\nu|Au(t)|^2$ represent, respectively, the rate of energy dissipation and enstrophy dissipation at time t. Since turbulence is a statistical phenomenon, we want to consider the *average* energy and enstrophy dissipation rates. However, there is no guarantee that the naive averages $\lim_{T\to\infty}\frac{1}{T}\int_0^T \nu\|u(t)\|^2 dt$ and $\lim_{T\to\infty}\frac{1}{T}\int_0^T \nu|Au(t)|^2 dt$ exist. One issue addressed in [10] is the more or less explicit assumptions in the physical theory that not only do such limits exist, but that turbulent solutions satisfy an ergodic property insofar as it is assumed that over time turbulent solutions sample almost all of the attractor (and so there is a direct correspondence between time averages and ensemble averages). By appealing to a generalized notion of a limit and certain properties of invariant measures, the authors of [10] are able to rigorously support the thrust of these physical assumptions showing that some version holds even in the case where the classical limits do not exist (see also [13] for a prior presentation of these ideas). Such generalized limits are denoted by angle brackets in the literature, and thus we write $\langle \nu \|u\|^2 \rangle$ and $\langle \nu |Au|^2 \rangle$ to refer to the average energy and enstrophy dissipation rates respectively.

If we let $\langle \mathfrak{e}_k(u) \rangle$ represent the average rate of nonlinear energy flux through wavenumber k (from smaller wavenumbers to larger wavenumbers), then theory states that an energy cascade occurs when, for an appreciable range of k, we have $\langle \mathfrak{e}_k(u) \rangle \approx \langle \nu \|u\|^2 \rangle$ (or, rather, when $\frac{\langle \mathfrak{e}_k(u) \rangle}{\langle \nu \|u\|^2 \rangle} \approx 1$). If we let $\langle \mathfrak{E}_k(u) \rangle$ represent the rate of nonlinear enstrophy flux through wavenumber k, then theory states that an enstrophy cascade occurs when, for an appreciable range of k, we have $\langle \mathfrak{E}_k(u) \rangle \approx \langle \nu |Au|^2 \rangle$ (or, rather, when $\frac{\langle \mathfrak{E}_k(u) \rangle}{\langle \nu |Au|^2 \rangle} \approx 1$). The quantities $\mathfrak{e}_k(u)$ and $\mathfrak{E}_k(u)$ may be reasonably (and rigorously) defined by considering the spectral projections (sometimes referred to as low-pass and high-pass filters) $P_k(u) = \sum_{|j| \leq |k|} u_j \omega_j$ and $Q_k = \sum_{|j| > |k|} u_j \omega_j$ (where u_j and ω_j are as in equation (1.12)). Indeed, we have

$$\mathfrak{e}_k(u) = (B(P_k(u), P_k(u)), Q_k(u)) - (B(Q_k(u), Q_k(u)), P_k(u)), \tag{2.11}$$

$$\mathfrak{E}_k(u) = (B(P_k(u), P_k(u)), AQ_k(u)) - (B(Q_k(u), Q_k(u)), AP_k(u)). \tag{2.12}$$

Taking averages using generalized limits quickly leads to the following theorem:

Theorem 2.2.1 (See Proposition 3.1 and Corollary 3.2 in [10]). *Let $\Omega = [0, L]^2_{per}$ and let f be such that it is supported on a finite number of eigenvectors of the Stokes operator A. That is, let $f = \sum_{j:|j|<|\bar{j}|}$ where $\bar{j}$ is the largest wavenumber in the support of f. Let k be such that $|k| > |\bar{j}|$. Then for any invariant probability measure μ on $D(A)$ we have*

$$\langle \mathfrak{e}_k(u) \rangle = \frac{\nu}{L^2} \langle \nu \|Q_k(u)\|^2 \rangle, \tag{2.13}$$

$$\langle \mathfrak{E}_k(u) \rangle = \frac{\nu}{L^2} \langle \nu |AQ_k(u)|^2 \rangle. \tag{2.14}$$

Moreover, this implies that

$$0 \leq k^2 \langle \mathfrak{e}_k(u) \rangle \leq \langle \mathfrak{E}_k(u) \rangle. \tag{2.15}$$

This theorem shows that indeed, beyond the spectrum of the force, there is a net flux of energy and enstrophy to higher wavenumbers (smaller scales). Moreover, equation (2.15) shows that for large wavenumbers the flux of energy is completely dominated by the flux of enstrophy.

Straighforward computations using Theorem 2.2.1 yields the following results

Theorem 2.2.2 (See Lemma 5.7, Theorem 5.8, and Remark 5.9 in [10])**.** *Let* $\Omega = [0,L]^2_{per}$ *and let* f *be such that* $f = \sum_{j:|j|<|\bar{j}|}$*, where* $\bar{j}$ *is the largest wavenumber in the support of* f*. For* k *such that* $|k| > |\bar{j}|$ *we have*

$$1 - \frac{k^2}{\langle \|u\|^2\rangle / \langle |u|^2\rangle} \le \frac{\langle \mathfrak{e}_k(u)\rangle}{\langle \nu \|u\|^2\rangle} \le 1, \tag{2.16}$$

$$1 - \frac{k^2}{\langle |Au|^2\rangle / \langle \|u\|^2\rangle} \le \frac{\langle \mathfrak{E}_k(u)\rangle}{\langle \nu |Au|^2\rangle} \le 1. \tag{2.17}$$

Note that the equations in Theorem 2.2.2 yield sufficient conditions for energy and enstrophy cascades. Equation (2.16) tells us that, for values of k such that the ratio of k^2 to $\langle \|u\|^2\rangle / \langle |u|^2\rangle$ is small, the nonlinear flux of energy is comparable to the global energy dissipation rate, and thus we have an energy cascade through those values of k. Equation (2.17) tells us that, for values of k such that the ratio of k^2 to $\langle |Au|^2\rangle / \langle \|u\|^2\rangle$ is small, the nonlinear flux of enstrophy is comparable to the global enstrophy dissipation rate, and thus we have an enstrophy cascade through those values of k.

2.2.2 A Second Look at our Problem

Given the fact that two-dimensional systems are impossible to physically realize, direct empirical validation of the physical theory is necessarily limited. (Experiments typically involve soap film flowing under the force of gravity or thin horizontal layers of fluid on the order of 1mm thick driven by an electric current). Even in numerical simulations, observations of two-dimensional turbulence remain somewhat elusive without modification (by the introduction of an *ad hoc* "friction" term) of the Navier-Stokes equations (see, for example, [1]). Theorem 2.2.1 provides a sufficient condition for a direct energy cascade (in 2D or 3D) and a sufficient condition for a direct enstrophy cascade (in 2D only). The key is finding solutions to the NSE where the ratio of the average enstrophy to average energy, or average palinstrophy ($|Au|^2$) to average energy, are large.

One place to look for such solutions is in the space of solutions whose energy, enstrophy, and palinstrophy remain constant over time. For the enstrophy cascade characteristic of two-dimensional turbulence the requirement is that the ratio of average palinstrophy to average enstrophy is large. Thus we are led to several new research questions:

Question 2.2.3. *Can there exist a non-stationary solution $u(t)$ to the Navier-Stokes equations (1.11) whose palinstrophy remains constant in time? That is, can we have $\frac{d}{dt}u(t) \neq 0$ while having $\frac{d}{dt}|Au(t)| = 0$?*

Question 2.2.4. *Can there exist a non-stationary solution $u(t)$ to the Navier-Stokes equations (1.11) whose enstrophy and palinstrophy remain constant in time? That is, can we have $\frac{d}{dt}u(t) \neq 0$ while having $\frac{d}{dt}\|u(t)\| = 0$ and $\frac{d}{dt}|Au(t)| = 0$?*

In Chapter 4 we answer Question 2.2.3 in the affirmative in the case of three spatial dimensions. A consequence of the main theorem of Chapter 5 is that, in two spatial dimensions, when the force is an eigenvector of the Stokes operator, no solution with a finite Fourier series representation can be such that it is nonstationary with constant enstrophy and palinstrophy. Thus we have a partial result concerning Question 2.2.4.

As evidenced by equation (2.15), even an extremely weak direct "cascade" of energy beyond the spectrum of the force would imply a strong direct enstrophy cascade in two dimensions. Thus, even solutions where the ratio of average enstrophy to average energy is small but comparable to the size of k^2 for a range of k would satisfy the sufficient condition for turbulence in 2D. Thus we are led once again to Questions 2.1.4, 2.1.5, and 2.1.6, concerning the existence of nonstationary solutions with constant energy, enstrophy, or both.

3 Recent Work on the Problem of Ghost Solutions

A central question of this dissertation concerns the existence of so-called ghost solutions to the Navier-Stokes equations. The concept of ghost solutions was introduced only very recently ([11]). In this section we review the recent work on the problem of ghost solutions. While the concept of ghost solutions is quite natural, as we will see, satisfying results in the literature are extremely limited.

3.1 Ghost Solutions

The term *ghost solution* was first introduced in [11] to refer to non-stationary solutions to the NSE whose energy and enstrophy remain constant for all time. In the original context the spatial domain of the fluid flow is the 2D torus, the force is an eigenvector of the Stokes operator, and the solution lies in the global attractor. However, the concept of a ghost solution makes sense even outside of these specifications. Indeed, given the motivations discussed in Section 2.2.2, it is desirable to consider a more general setting since in the case with a periodic domain and eigenvector force turbulence is not possible (see [4]). Thus, we define ghost solutions as follows:

Definition 3.1.1. *A* **ghost solution** *is a non-stationary solution, $u(x,t)$, to (1.11) (in 2D or 3D) such that $|u(t)|$ and $\|u(t)\|$ are constant in time for all t where u is defined.*

Note that this definition allows the possibility of ghost solutions that exist outside of the attractor. The only restriction on the forcing function f is that it is in H. While in this paper we work on a periodic domain, that is not an essential feature of the definition.

3.2 Immediate Results

It is worth mentioning that at the time that ghost solutions were introduced, several existing theorems in the literature could be immediately applied to show that ghost solutions do not exist under certain circumstances. We discuss a few of these theorems here.

The first results are due to Marchioro ([19]). These results, presented as a single theorem, give two conditions for when the global attractor of the 2D NSE reduces to a single point. The first condition may be stated as follows:

Theorem 3.2.1 (First Condition in Main Theorem of [19]). *Let $\Omega = [0, L]^2_{per}$, and let f be such that $Af = \lambda_1 f$ where λ_1 is the smallest eigenvalue of the Stokes operator A. Then the steady-state solution u^* guaranteed by Theorem 1.4.8 is unique, and the global attractor $\mathcal{A}$ is such that $\mathcal{A} = \{u^*\}$.*

This theorem implies that when the force is an eigenvector of the Stokes operator associated with the smallest eigenvalue, there cannot exist ghost solutions in the 2D attractor since, in this case, it consists of a single stationary solution. This fact was proven again in [7] (see Theorem 5.2 and Remark 5.1). The second part of the theorem shows that for small perturbations on such a force, the attractor remains a single stationary point.

Theorem 3.2.2 (Second Condition in Main Theorem of [19]). *Let $\Omega = [0, L]^2_{per}$, and let f be such that $Af = \lambda_1 f$ where λ_1 is the smallest eigenvalue of the Stokes operator A. Then there exists $\varepsilon_1, \varepsilon_2 > 0$ (dependent upon $|f|$ and ν) such that for any $\tilde{f}$ where $|f - \tilde{f}| < \varepsilon_1$ and $\|f - \tilde{f}\| < \varepsilon_2$, the steady-state solution u^* guaranteed by Theorem 1.4.8 (for the NSE with force $\tilde{f}$) is unique, and the global attractor $\mathcal{A}$ is such that $\mathcal{A} = \{u^*\}$.*

A well-known theorem states that when the energy of the force is small enough the attractor consists of a single stationary solution.

Theorem 3.2.3. *Let* $\Omega = [0, L]^2_{per}$ *for* $n = 2, 3$. *Let* f *be such that*

$$G < \frac{1}{c_L^2},$$

where G *is the Grashof number (1.22) and* c_L *is the constant from the Ladyzhenskaya inequality (1.39). In this case the steady-state solution* u^* *guaranteed by Theorem 1.4.8 is unique, and the global attractor* $\mathcal{A}$ *is such that* $\mathcal{A} = \{u^*\}$.

Proof. Let u^* be the steady-state solution and let v be any other solution that exists for all $t > 0$. Subtracting the NSE with solution v from the NSE with solution u^* we have

$$\frac{d}{dt}(u^* - v) + \nu A(u^* - v) + B(u^*, u^*) - B(v, v) = 0. \tag{3.1}$$

We may rewrite $B(u^*, u^*) - B(v, v)$ as $B(u^* - v, u^*) - B(v, u^* - v)$. Then taking the inner product with $u^* - v$ we get

$$\frac{d}{dt}|u^* - v|^2 + \nu\|u^* - v\|^2 = -(B(u^* - v, u^*), u^* - v). \tag{3.2}$$

The right-hand side of the equation is bounded by $c_L^2 G\nu\|u^* - v\|^2$. This bound is obtained by first applying the bound on the trilinear term given by (1.3.4) and combining terms to get $|(B(u^* - v, u^*), u^* - v)| \leq c_L^2|u^* - v|\|u^* - v\|\|u^*\|$. Applying the Poincaré inequality (1.36) to $|u^* - v|$ and applying the attractor bound (2.3) to $\|u^*\|$ implies $|(B(u^* - v, u^*), u^* - v)| \leq c_L^2 G\nu\|u^* - v\|^2$. Combining this with equation (3.2) yields that $\frac{d}{dt}|u^* - v|^2 \leq c_L^2 G\nu\|u^* - v\|^2$ is negative when $G < \frac{1}{c_L^2}$. Thus, for $G < \frac{1}{c_L^2}$, by Poincaré we have

$$\frac{1}{2}\frac{d}{dt}|u_0 - v|^2 \leq \nu\lambda_1|u_0 - v|^2(c_L^2 G - 1). \tag{3.3}$$

Then using Gronwall we have

$$|u_0 - v|^2 \leq |u_0 - v(0)|^2 e^{\nu\lambda_1(c_L^2 G - 1)t} \tag{3.4}$$

which goes to 0 as $t \to \infty$ so long as $G < \frac{1}{c_L^2}$. □

A similar theorem can be found more recently in [7]. In that paper they prove that if $|A^{-1/2}f|$ is small enough (relative to the viscosity and domain) then again the attractor consists of a single stationary point.

3.3 The Initial Paper of Foias, Jolly, and Yang

In this section we review the results of the seminal paper by Foias, Jolly, and Yang [11] where ghost solutions were introduced. As mentioned at the beginning of this chapter, the authors of this paper were working on a periodic domain in two dimensions, and an assumption throughout their paper is that the force is an eigenvector of the Stokes operator. An advantage of working in this framework is that we have the following equivalence:

$$\frac{d}{dt}|u|^2 = \frac{d}{dt}\|u\|^2 = 0 \quad \Longleftrightarrow \quad \nu\|u\|^2 = (f, u) = \frac{\nu}{\lambda_f}|Au|^2, \tag{3.5}$$

where λ_f is the eigenvalue associated with f (*i.e.* $Af = \lambda_f f$). This follows immediately from the energy and enstrophy balance equations. Note that this equivalence depends on both the fact that the domain is 2D periodic and the fact f is an eigenvector of A. Note also that (3.5) implies that $\frac{d}{dt}|u|^2 = \frac{d}{dt}\|u\|^2 = 0 \Longrightarrow \frac{d}{dt}|Au|^2 = 0$. That is, constant energy and enstrophy implies constant palinstrophy in this regime. Thus, one has a powerful tool for analyzing ghost solutions in this framework.

The authors of [11] quickly put equivalence (3.5) to work to prove that a ghost solution cannot be an eigenvector of the Stokes operator (this is their Proposition 6.2). We pause to note that this result is an immediate corollary of the main theorem of Chapter 5 in this dissertation.

Several other interesting relations follow quickly from equivalence (3.5):

Theorem 3.3.1 (see Proposition 6.3 in [11])**.** *Let $\Omega = [0, L]^2_{per}$ and let $Af = \lambda_f f$. Let u be a ghost solution. Then the following relationships hold:*

$$|Au - f/2| = |f/2| \tag{3.6}$$

$$\left(\frac{d}{dt}u, f\right) = \left(\frac{d}{dt}u, Au\right) = \left(\frac{d}{dt}u, u\right) = \left(\frac{d}{dt}u, Au - f\right) = 0 \tag{3.7}$$

$$|B(u,u)|^2 + |Au|^2 = \left|\frac{d}{dt}u\right|^2 + |f|^2 \tag{3.8}$$

$$(B(u,u), f) = |B(u,u)|^2 - \left|\frac{d}{dt}u\right|^2 = |f|^2 - |Au|^2 = |Au - f|^2 \tag{3.9}$$

$$|B(u,u) - f|^2 = \left| \frac{d}{dt} u \pm f/2 \right|^2 \tag{3.10}$$

$$\frac{d}{dt}(|B(u,u)|^2) = \frac{d}{dt}\left(\left| \frac{d}{dt} u \right|^2 \right). \tag{3.11}$$

When f is such that $Af = \lambda_f f$ it follows that $u = f/(\nu\lambda_f)$ is a stationary solution to the NSE. This solution is dubbed u_* in [11] and is referred to as the *primary fixed point*. In [7] it was shown that if a solution in the attractor exists on the parabola in the dimensionless $\mathbf{e} - \mathbf{E}$ plane (as mentioned in 2.1.3), then the force f must be an eigenvector of the Stokes operator and the solution is the primary fixed point u_* (see Theorem 5.2 and Remarks 5.1 and 5.2 in [7]). This was restated in [11], where the force is assumed to be an eigenvector of the Stokes operator, as follows:

Theorem 3.3.2 (See Proposition 3.1 in [11]). *Let* $\Omega = [0, L]^2_{per}$*, let* f *be such that* $Af = \lambda_f f$ $(\lambda_f \in \mathbb{R})$*, and let* $u_* = f/(\nu\lambda_f)$*. Then we have*

$$\mathcal{A} \cap \{u \in D(A) : \mathbf{E} = \sqrt{\mathbf{e}}\} = \{u_*\}. \tag{3.12}$$

The authors of [11] prove several results that relate the behavior of solutions relative to the primary fixed point u_*. Many of these have direct implications for ghost solutions. For example, they show that if u is a solution such that $\|u\| > \|u_*\|$, then $\frac{d}{dt}|u|^2 < 0$ and thus u is not a ghost solution (see Lemma 4.7 in [11]). Geometrically, this rules out the possibility of ghost solutions above the line $\mathbf{E} = \mathbf{E}_*$ in the dimensionless energy-enstrophy plane (where $\mathbf{E}_*$ is the dimensionless enstrophy of u_*). Further results demarcating regions of the $\mathbf{e} - \mathbf{E}$ plane where either the energy or enstrophy changes in time (and thus there are no ghost solutions in these regions) can be found in Sections 5 and 6 of [11]. In particular, it is shown that ghost solutions can only exist within a certain horizontal band of the $\mathbf{e} - \mathbf{E}$ plane:

Theorem 3.3.3 (See Theorems 6.4 and 6.5 in [11]). *Let* $\Omega = [0, L]^2_{per}$ *and let* f *be such that* $Af = \lambda_f f$ $(\lambda_f \in \mathbb{R})$*. If* u *is a ghost solution with dimensionless enstrophy*

E *then*

$$\mathbf{E}_* > \mathbf{E} \geq \frac{G^2}{\lambda_f + c_A G (\ln e\lambda_f)^{1/2}} \tag{3.13}$$

where c_A *is the constant from the Agmon inequality.*

The firmest result regarding ghost solutions states that no ghost solution exists if the energy of the force is small relative to λ_f:

Theorem 3.3.4 (See Proposition 6.6 in [11])**.** *Let* $\Omega = [0, L]^2_{per}$ *and let* f *be such that* $Af = \lambda_f f$ $(\lambda_f \in \mathbb{R})$*. In the case where*

$$\frac{\lambda_f}{G(\ln(2e\lambda_f))^{1/2}} > \left(c_A^2 e + \frac{c_L^4}{2e}\right)^{1/2} \tag{3.14}$$

(where c_A *and* c_L *are the constants from the Agmon and Ladyzhenskaya inequalities), there cannot exist any ghost solutions.*

3.4 Chained Ghost Solutions

Following the initial paper by Foias, Jolly, and Yang, the researchers Tian and Zhang investigated the existence of ghost solutions in [25]. In this paper, continuing in the framework of a 2D periodic spatial domain with eigenvector force, they provide additional results regarding properties of ghost solutions. They also introduce a "degenerate" subtype of ghost solutions, which they dubbed *chained ghost solutions*, that are particularly amenable to analysis. In [24] Tian and You continued the study of chained ghost solutions for specific values of λ_f. We discuss the results of these subsequent papers ([25],[24]) below.

The central project of [25] was to investigate geometric structures common to ghost solutions in the framework of a 2D periodic spatial domain with eigenvector force. For a given ghost solution u the authors consider the (time-dependent) subspace of H given by

$$\tilde{H}(t) = \text{span}\{f, u(t), Au(t), A^2u(t)\}, \tag{3.15}$$

since for any $v \in \tilde{H}(t)$ the product (v, u) is constant. They show that for any ghost solution u it must be the case that $\text{span}\{f\} \subsetneq \text{span}\{f, u\} \subsetneq \text{span}\{f, u, Au\}$, but their work allows for the possibility that $A^2 u \in \text{span}\{f, u, Au\}$. In considering this potential degeneracy, they investigate ghost solutions that satisfy the following relation:

$$A^2 u = \gamma(t) f + \beta(t) u + \alpha(t) Au. \tag{3.16}$$

In their paper they explicitly calculate the values of $\gamma(t), \beta(t)$, and $\alpha(t)$, showing that γ is actually time-independent and that $\beta(t)$ and $\alpha(t)$ are only time-dependent if $|A^{3/2}u|$ is time-dependent. These considerations led the authors to introduce the following subtype of ghost solution:

Definition 3.4.1 (See [25] Definition 6.1)**.** *Consider the system (1.11) with* $\Omega = [0, L]^2_{per}$ *and suppose that* f *is an eigenvector of the Stokes operator. A* **chained ghost solution** *is a ghost solution in the global attractor satisfying the following relation:*

$$A^2 u(t) = \gamma f + \beta u(t) + \alpha A u(t), \quad \forall t \in \mathbb{R}, \tag{3.17}$$

for real scalars α, β, *and* γ.

Unlike the original definition of ghost solutions, the motivation for considering chained ghost solutions necessarily relies on the domain being the 2D torus and the force being an eigenvector of the Stokes operator.

The authors of [25] go on to prove several results concerning chained ghost solutions. The most relevant result for this dissertation is the fact that chained ghost solutions may be decomposed as a sum of three eigenvectors of the Stokes operator. This theorem was restated nicely in [24]. We state a simplified version of the result here:

Theorem 3.4.2 (See Theorem 6.3 in [25] or Theorem 2.1 in [24])**.** *A chained ghost solution* $u(t)$ *may be written in the following form:*

$$u(t) = u_+(t) + u_-(t) + \frac{\|u\|^2}{|f|^2} f,$$

where u_+ *and* u_- *are eigenvectors of the Stokes operator* A *corresponding to eigenvalues* μ_+ *and* μ_-*, respectively, which themselves stand in the relation* $\mu_- < \lambda_f < \mu_+$.

Since $\frac{\|u\|^2}{|f|^2}$ is a scalar and f is assumed to be an eigenvector of the Stokes operator, this theorem implies that any chained ghost solution may be written as a sum of three eigenvectors of A. The authors of [25] conclude their paper with a proof that if f is such that $\lambda_f = 2\left(\frac{2\pi}{L}\right)^2$ then there do not exist any chained ghost solutions. We note that their proof relies heavily on the spectral structure of chained ghost solutions described in Theorem 3.4.2.

The project of investigating chained ghost solutions continued in [24]. This paper has two main results, both pertaining to situations where chained ghost solutions do not exist. The first is that in the case where $\lambda_f = 2\left(\frac{2\pi}{L}\right)^2$ there do not exist chained ghost solutions. The other, more general result, is that if λ_f and μ_- (from Theorem 3.4.2) are of the form $k^2\left(\frac{2\pi}{L}\right)^2$ for some $k \in \mathbb{R}$, then there do not exist chained ghost solutions. The authors of [24] conjecture at the end of their paper that in fact chained ghost solutions are impossible. We prove this conjecture as a corollary to the main theorem in Chapter 5.

4 The Construction Theorem

In this chapter we construct a nonstationary solution to (1.11) in 3D that has constant energy everywhere it is defined. We begin by constructing a nonstationary solution to the Stokes system (*i.e.*, the Navier-Stokes equations but without the non-linear term) that is defined and has constant energy for all $t \geq 0$. We then show that this construction can be modified to create a nonstationary constant-energy solution to the full Navier-Stokes equations in 3D. These results are generalized in several ways. The first is that, given an $s \in \mathbb{R}$, we can construct a nonstationary solution on the 3D torus such that the norm $|A^{s/2}u|$ is constant. We also generalize this result to dimensions greater than three. That is, for any $s \in \mathbb{R}$ we can construct a nonstationary solution to the NSE with domain $[0, L]^n_{per}$ for $n \geq 3$ with norm $|A^{s/2}u|$ constant. We also show that this construction method does not work in two dimensions.

4.1 Stokes System

By the incompressible *Stokes system* we refer to the following set of partial differential equations:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + \nabla p = F \\ \nabla \cdot u = 0 \\ \int_\Omega u dx = 0, \quad \int_\Omega F dx = 0 \\ u(x,0) = u_0 \end{cases} \tag{4.1}$$

on a periodic spatial domain $\Omega = [0, L]^n_{per}$, $n = 2, 3$. This system is simply the NSE without the non-linear term. The functional formulation of the Stokes system in H is as follows:

$$\begin{cases} \frac{du}{dt} + \nu A u = f \\ u(0) = u_0. \end{cases} \tag{4.2}$$

Recall that we may write any function in H as follows:

$$u(x) = \sum_{j \in \mathbb{Z}^n \setminus \{0\}} u_j \omega_j(x) \tag{4.3}$$

where $\{\omega_j\}_{j \in \mathbb{Z}^2 \setminus \{0\}}$ is a set of orthonormal eigenvectors of A spanning H with explicit eigenvalues $\lambda_j = \left(\frac{2\pi}{L}\right)^2 |j|^2$. Considering this eigenvector expansion, we see that (4.2) is equivalent to the following (possibly infinite) system of linear ordinary differential equations:

$$\frac{d}{dt} u_j(t) + \nu \lambda_j u_j(t) = f_j, \qquad j \in \mathbb{Z}^n \setminus \{0\}. \tag{4.4}$$

Each of these equations can be solved explicitly for all $t \geq 0$:

$$u_j(t) = \left(u_j(0) - \frac{f_j}{\nu \lambda_j} \right) e^{-\nu \lambda_j t} + \frac{f_j}{\nu \lambda_j}. \tag{4.5}$$

From here it can be seen that the global attractor for this system consists of the steady-state solution: $u^* = \frac{1}{\nu} A^{-1} f$ (*i.e.* $u_j^* = \frac{f_j}{\nu \lambda_j}$ for each $j \in \mathbb{Z}^2 \setminus \{0\}$) since solutions in the attractor must be bounded for all time.

In what follows we will construct a nonstationary constant-energy solution to the Stokes system. For specificity, we work on the 2D torus with $\Omega = [0, L]^2_{per}$. We begin with a lemma regarding a necessary and sufficient condition for the existence of nonstationary constant-energy solutions to the Stokes system.

Lemma 4.1.1. *Any nonstationary constant-energy solution to the Stokes system (4.2) requires that the force and initial condition satisfy the following condition:*

$$\sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \left(u_j(0) - \frac{f_j}{\nu \lambda_j} \right)^2 e^{-2\nu \lambda_j t} + \frac{2 f_j}{\nu \lambda_j} \left(u_j(0) - \frac{f_j}{\nu \lambda_j} \right) e^{-\nu \lambda_j t} = 0 \tag{4.6}$$

where at least one u_j is such that $u_j(t) \neq \frac{f_j}{\nu \lambda_j}$. The converse also holds.

Proof. The energy of the solution to (4.2) is given by

$$\frac{1}{2} |u(t)|^2 = \frac{1}{2} \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} u_j(t)^2$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \left[\left(u_j(0) - \frac{f_j}{\nu\lambda_j} \right) e^{-\nu\lambda_j t} + \frac{f_j}{\nu\lambda_j} \right]^2$$

$$= \frac{1}{2} \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \left(u_j(0) - \frac{f_j}{\nu\lambda_j} \right)^2 e^{-2\nu\lambda_j t} + \frac{2f_j}{\nu\lambda_j} \left(u_j(0) - \frac{f_j}{\nu\lambda_j} \right) e^{-\nu\lambda_j t} + \left(\frac{f_j}{\nu\lambda_j} \right)^2.$$

If the energy is to remain constant then it must have the same energy as the solution in the global attractor. That is, a constant-energy solution must have energy equal to $\frac{1}{2}\sum_{j\in\mathbb{Z}^2}\left(\frac{f_j}{\nu\lambda_j}\right)^2$. Such a solution would then require that

$$\sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \left(u_j(0) - \frac{f_j}{\nu\lambda_j} \right)^2 e^{-2\nu\lambda_j t} + \frac{2f_j}{\nu\lambda_j} \left(u_j(0) - \frac{f_j}{\nu\lambda_j} \right) e^{-\nu\lambda_j t} = 0. \tag{4.7}$$

Note that if we try to make the sum zero by making all the terms zero, this requires that we set $u_j(0) = \frac{f_j}{\nu\lambda_j}$ for each j. However, if this is the case then we have that each $u_j(t)$ is constant in time, and in fact that $u(t)$ is the steady-state solution. Thus, in order for $u(t)$ to be nonstationary, we need for $u_j(t) \neq \frac{f_j}{\nu\lambda_j}$ for at least one j.

For the reverse implication note that if the condition in (4.6) is satisfied then we have that $\frac{1}{2}|u(t)|^2 = \frac{1}{2}\sum_{j\in\mathbb{Z}^2}\left(\frac{f_j}{\nu\lambda_j}\right)^2$ and the energy of u is constant. Again, the condition that some u_j is such that $u_j(t) \neq \frac{f_j}{\nu\lambda_j}$ guarantees that u is nonstationary. □

Suppose u is a nonstationary constant-energy solution to (4.2). Let j_0 be such that $u_{j_0}(0) \neq \frac{f_{j_0}}{\nu\lambda_{j_0}}$. Then we at least have two terms in equation (4.6) that need to be canceled:

$$\left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \right)^2 e^{-2\nu\lambda_{j_0} t}; \qquad \& \qquad \frac{2f_{j_0}}{\nu\lambda_{j_0}} \left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \right) e^{-\nu\lambda_{j_0} t}. \tag{4.8}$$

Since the terms $e^{-2\nu\lambda_{j_0}t}$ and $e^{-\nu\lambda_{j_0}t}$ are linearly independent for all $t \neq 0$, we know that

$$\left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \right)^2 e^{-2\nu\lambda_{j_0} t} + \frac{f_{j_0}}{\nu\lambda_{j_0}} \left(u_{j_0}(0) - \frac{2f_{j_0}}{\nu\lambda_{j_0}} \right) e^{-\nu\lambda_{j_0} t} \neq 0. \tag{4.9}$$

Notice that the term $\left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t}$ must be canceled by terms of the form $\frac{2f_k}{\nu\lambda_k}\left(u_k(0) - \frac{f_k}{\nu\lambda_k}\right) e^{-\nu\lambda_k t}$ where $\lambda_k = 2\lambda_{j_0}$. However, this creates new terms $\left(u_k(0) - \frac{f_k}{\nu\lambda_k}\right)^2 e^{-2\nu\lambda_k t}$, which themselves must be canceled by terms associated with an eigenvalue equal to $2\lambda_k = 4\lambda_{j_0}$. These considerations motivate the following lemma.

Lemma 4.1.2. *For any $j \in \mathbb{Z}^2 \setminus \{0\}$ and $n \in \mathbb{N}$ we have that $2^n\lambda_j$ is an eigenvalue of the Stokes operator.*

Proof. This follows almost directly from the Sum of Two Squares Theorem ([8]), which states that an integer m is the sum of two squares if and only if, in the prime factorization of m, all primes congruent to 3 modulo 4 are to an even power. Consider an arbitrary eigenvalue $\lambda_j = \left(\frac{2\pi}{L}\right)^2 j \cdot j$. Since $j \cdot j = j_1^2 + j_2^2$ is the sum of two squares, the prime factorization of $j \cdot j$ contains only even powers of primes congruent to 3 modulo 4. Note also that the prime factorization of $2^n j \cdot j$ then also has only even powers of primes congruent to 3 modulo 4 for any n. Thus $2^n j \cdot j = k_1^2 + k_2^2$ for some integers k_1, k_2. Thus, for $k = (k_1, k_2)$ we have $2^n\lambda_j = \lambda_k$. □

Remark 4.1.3. *For an explicit construction of a sequence of eigenvalues $\{2^n\lambda_j\}_{n\in\mathbb{N}}$, consider the following:*

For $j = (j_1, 0)$ with $j_1 \neq 0$ let

$$2^n\lambda_j = \begin{cases} \lambda_{(j_1 2^{(n-1)/2},\, j_1 2^{(n-1)/2})} & \text{if } n \text{ is odd} \\ \lambda_{(j_1 2^{n/2},\, 0)} & \text{if } n \text{ is even} \end{cases}$$

Before we move on to the next lemma, let us consider the following motivating calculations. Let j_0 be such that $f_{j_0} = 0$ and $u_{j_0}(0) \neq 0$. Then the j_0^{th} term in the sum (4.6) would simply be

$$u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0}t}. \tag{4.10}$$

Let λ_{j_1} be such that $\lambda_{j_1} = 2\lambda_{j_0}$. Then we have that the j_1^{st} term in sum (4.6) is

$$\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1}t} + \frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) e^{-\nu\lambda_{j_1}t}. \tag{4.11}$$

Since $\lambda_{j_1} = 2\lambda_{j_0}$ we have that $\frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) e^{-\nu\lambda_{j_1}t}$ from the j_1^{st} term cancels the j_0^{th} term $u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0}t}$ exactly when

$$\frac{2f_{j_1}}{\nu\lambda_{j_1}}\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right) = -u_{j_0}(0)^2, \tag{4.12}$$

or rather

$$u_{j_1}(0) = \frac{f_{j_1}}{2\nu\lambda_{j_0}} - \frac{\nu\lambda_{j_0}u_{j_0}(0)^2}{f_{j_1}}. \tag{4.13}$$

Of course this leaves us with the first part of the j_1^{st} term leftover: $\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1}t}$. If we let λ_{j_2} be such that $\lambda_{j_2} = 2\lambda_{j_1} = 4\lambda_{j_0}$ then the j_2^{nd} term of sum (4.6) is

$$\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right)^2 e^{-2\nu\lambda_{j_2}t} + \frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right)e^{-\nu\lambda_{j_2}t}. \tag{4.14}$$

This time we get that the second part of the j_2^{nd} term, $\frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right)e^{-\nu\lambda_{j_2}t}$, cancels the remainder from the j_1^{st} term, $\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 e^{-2\nu\lambda_{j_1}t}$, exactly when

$$\frac{2f_{j_2}}{\nu\lambda_{j_2}}\left(u_{j_2}(0) - \frac{f_{j_2}}{\nu\lambda_{j_2}}\right) = -\left(u_{j_1}(0) - \frac{f_{j_1}}{\nu\lambda_{j_1}}\right)^2 = -\frac{\nu^2\lambda_{j_0}^2u_{j_0}(0)^4}{f_{j_1}^2}, \tag{4.15}$$

or rather

$$u_{j_2}(0) = \frac{f_{j_2}}{4\nu\lambda_{j_0}} - \frac{2\nu^3\lambda_{j_0}^3u_{j_0}(0)^4}{f_{j_1}^2f_{j_2}}. \tag{4.16}$$

Continuing in this pattern, we get successive cancellation of terms in the j_n modes exactly when we define

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu2^n\lambda_{j_0}} - \frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^n(f_{j_k})^{2^{n-k}}}. \tag{4.17}$$

We now establish the following partial sum lemma:

Lemma 4.1.4. *Fix $j_0 \in \mathbb{Z}^2 \setminus \{0\}$. Choose a sequence $j_n \in \mathbb{Z}^2$, $n = 1, 2, 3\ldots$, such that $\lambda_{j_n} = 2^n\lambda_{j_0}$. Let $f_{j_0} = 0$ and let $f_{j_n} \neq 0$ for $n > 0$. Let $u_{j_0}(0) \neq 0$. If we define $u_{j_n}(0)$ for $n \geq 1$ recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu2^n\lambda_{j_0}} - \frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^n(f_{j_k})^{2^{n-k}}}, \tag{4.18}$$

then the formal sum

$$\sum_{n=0}^{\infty}\left(u_{j_n}(0) - \frac{f_{j_n}}{\nu\lambda_{j_n}}\right)^2 e^{-2\nu\lambda_{j_n}t} + \frac{2f_{j_n}}{\nu\lambda_{j_n}}\left(u_{j_n}(0) - \frac{f_{j_n}}{\nu\lambda_{j_n}}\right)e^{-\nu\lambda_{j_n}t} \tag{4.19}$$

has as its N^{th} partial sum

$$S_N = \left[\frac{\left(2\nu\lambda_{j_0}u_{j_0}(0)\right)^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^N(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t}. \tag{4.20}$$

Proof. We prove this by induction. As our base case we have

$$S_0 = \left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}}\right)^2 e^{-2\nu\lambda_{j_0}t} + \frac{2f_{j_0}}{\nu\lambda_{j_0}}\left(u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}}\right) e^{-\nu\lambda_{j_0}t} = u_{j_0}(0)^2 e^{-2\nu\lambda_{j_0}t}, \tag{4.21}$$

which is of the correct form. Now suppose we have that

$$S_N = \left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} \tag{4.22}$$

for $N \geq 0$. Distributing the square allows us to write this in a more convenient form for later:

$$S_N = \frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{2(N+1)}(\nu\lambda_{j_0})^2\prod_{k=1}^{N}(f_{j_k})^{2^{N+1-k}}} e^{-2^{N+1}\nu\lambda_{j_0}t} \tag{4.23}$$

Consider the following calculations:

$$\begin{aligned}
S_{N+1} &= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} + \frac{2f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} \\
&\qquad - \frac{2f_{j_{N+1}}}{\nu 2^{N+1}\lambda_{j_0}}\left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{N+2}\nu\lambda_{j_0}\prod_{k=1}^{N+1}(f_{j_k})^{2^{N+1-k}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= S_N + \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} - \left(\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{2N+2}(\nu\lambda_{j_0})^2\prod_{k=1}^{N}(f_{j_k})^{2^{N+1-k}}}\right) e^{-\nu\lambda_{j_{N+1}}t} \\
&= \left(u_{j_{N+1}}(0) - \frac{f_{j_{N+1}}}{\nu\lambda_{j_{N+1}}}\right)^2 e^{-2\nu\lambda_{j_{N+1}}t} \\
&= \left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^{N+1}}}{2^{(N+1)+1}\nu\lambda_{j_0}\prod_{k=1}^{N+1}(f_{j_k})^{2^{N+1-k}}}\right]^2 e^{-2^{(N+1)+1}\nu\lambda_{j_0}t}.
\end{aligned}$$

In the move to the second line we use the definition of $u_{j_{N+1}}$ as given in equation (4.18). The move to the third line is the result of simplification after using the fact that $\lambda_{j_{N+1}} = 2^{N+1}\lambda_{j_0}$. The move to the fourth line uses the formulation of S_N given in equation (4.23). The final line again uses the definition of $u_{j_{N+1}}$ as given in equation (4.18). Thus we have that S_{N+1} is of the correct form and the lemma is established. ☐

Remark 4.1.5. *Choose (j_n), f_{j_n}, and $u_{j_n}(0)$ as in Lemma 4.1.4. If we additionally choose $f_j = 0$ and $u_j(0) = 0$ for all j not in the sequence $\{j_n\}$, then we have the sum from (4.6) reduces to (4.19). For such choices of our parameters, we have that the identity (4.6) is satisfied exactly when*

$$\lim_{N\to\infty}\left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} = 0. \tag{4.24}$$

In one sense, it is easy to find appropriate values of $\lambda_{j_0}, \nu, u_j(0)$ and $\{f_{j_n}\}$ such that the limit in (4.24) is satisfied for all $t \geq 0$ (for example, set all parameters equal to 1). However, we should minimally require that our choices guarantee that the force and initial condition are at least in H and ideally in V. Our task now is to find values of $\lambda_{j_0}, \nu, u_{j_0}$ and $\{f_{j_n}\}$ such that the following criteria hold:

1. $u(0) \in V$
2. $f \in V$
3. $\lim_{N\to\infty}\left[\frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0}\prod_{k=1}^{N}(f_{j_k})^{2^{N-k}}}\right]^2 e^{-2^{N+1}\nu\lambda_{j_0}t} = 0.$

As a final preliminary result, we require the closed form of an important sum that arises in our calculations:

Lemma 4.1.6.

$$\sum_{k=1}^{N} k2^{N-k} = 2^{N+1} - N - 2 \tag{4.25}$$

Proof. This can be shown by a straighforward proof by induction. □

We now prove the following theorem which establishes the construction of a constant-energy solution to the Stokes system.

Theorem 4.1.7. *Fix $j_0 \in \mathbb{Z}^2 \setminus \{0\}$. Choose a sequence $j_n \in \mathbb{Z}^2$, $n = 1, 2, 3\ldots$, such that $\lambda_{j_n} = 2^n\lambda_{j_0}$. Let $f_{j_n} = \frac{1}{b^n}$ for $n > 0$ with $b > \sqrt{2}$, and let $f_j = 0$ for all other $j \in \mathbb{Z}^2 \setminus \{0\}$. Let $u_{j_0}(0) \neq 0$, and define $u_{j_n}(0)$ recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n\lambda_{j_0}} - \frac{(2\nu\lambda_{j_0}u_{j_0}(0))^{2^n}}{2^{n+1}\nu\lambda_{j_0}\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}. \tag{4.26}$$

Let $u_j(0) = 0$ for all other $j \in \mathbb{Z}^2 \setminus \{0\}$. Then f and $u(0)$ are in V and define a force and initial condition such that the solution to the Stokes system (4.2) is nonstationary with constant energy for all $t \geq 0$ provided $2\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$.

Proof. Let f, u be defined as in the hypothesis of the theorem. First we show that $f \in V$. We calculate the norm of f in V as follows:

$$\begin{aligned}\|f\|^2 &= \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \lambda_j f_j^2 \\ &= \sum_{n=1}^{\infty} \lambda_{j_n} f_{j_n}^2 \\ &= \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{1}{b^n}\right)^2 \\ &= \lambda_{j_0} \sum_{n=1}^{\infty} \left(\frac{2}{b^2}\right)^n.\end{aligned}$$

Thus, we have $f \in V$ for $b > \sqrt{2}$.

Next we calculate a closed form for the term $\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}$. We have

$$\begin{aligned}\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}} &= \prod_{k=1}^{n}(b^k)^{2^{n-k}} \\ &= \prod_{k=1}^{n} b^{k2^{n-k}} \\ &= b^{\sum_{k=1}^{n} k2^{n-k}} \\ &= b^{2^{n+1}-n-2}\end{aligned}$$

where the last identity uses Lemma 4.1.6.

Now we calculate the norm of $u(0)$ in V:

$$\begin{aligned}\|u(0)\|^2 &= \sum_{j \in \mathbb{Z}^2 \setminus \{0\}} \lambda_j u_j(0)^2 \\ &= \sum_{n=1}^{\infty} \lambda_{j_n} u_{j_n}(0)^2\end{aligned}$$

$$= \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{f_{j_n}}{\nu 2^n \lambda_{j_0}} - \frac{(2\nu\lambda_{j_0} u_{j_0}(0))^{2^n}}{2^{n+1}\nu\lambda_{j_0} \prod_{k=1}^{n} (f_{j_k})^{2^{n-k}}} \right)^2$$

$$= \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{1}{\nu 2^n \lambda_{j_0} b^n} - \frac{(2\nu\lambda_{j_0} u_{j_0}(0))^{2^n} b^{2^{n+1}-n-2}}{2^{n+1}\nu\lambda_{j_0}} \right)^2$$

$$= \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{1}{\nu 2^n \lambda_{j_0} b^n} - \frac{(2\nu\lambda_{j_0} u_{j_0}(0) b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0} b^2 b^n} \right)^2$$

$$= \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{1}{\nu 2^n \lambda_{j_0} b^n} \right)^2 - \sum_{n=1}^{\infty} 2^n \lambda_{j_0} 2 \left(\frac{1}{\nu 2^n \lambda_{j_0} b^n} \right) \left(\frac{(2\nu\lambda_{j_0} u_{j_0}(0) b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0} b^2 b^n} \right)$$

$$+ \sum_{n=1}^{\infty} 2^n \lambda_{j_0} \left(\frac{(2\nu\lambda_{j_0} u_{j_0}(0) b^2)^{2^n}}{2^{n+1}\nu\lambda_{j_0} b^2 b^n} \right)^2.$$

In order for $\|u(0)\|^2$ to be finite we need this final sum to converge. Given the dominance of the terms with exponent 2^n, we have convergence exactly when $2\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$. Thus, $u \in V$ provided $2\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$.

As mentioned in Remark 4.1.5, the necessary and sufficient condition for having a nonstationary constant-energy solution to the Stokes system is guaranteed in this case when the limit of the partial sums given by equation (4.20) in Lemma 4.1.4 is equal to 0.

Let us rewrite the N^{th} partial sum given by equation (4.20) given our specific choice of f.

$$S_N = \left[\frac{(2\nu\lambda_{j_0} u_{j_0}(0))^{2^N}}{2^{N+1}\nu\lambda_{j_0} \prod_{k=1}^{N} (f_{j_k})^{2^{N-k}}} \right]^2 e^{-2^{N+1}\nu\lambda_{j_0} t} = \left[\frac{(2\nu\lambda_{j_0} u_{j_0}(0) b^2)^{2^N}}{2^{N+1}\nu\lambda_{j_0} b^2 b^N} \right]^2 e^{-2^{N+1}\nu\lambda_{j_0} t}. \tag{4.27}$$

Notice that the requirement for $u \in V$, namely, $2\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$, is also sufficient to guarantee that $\lim_{N\to\infty} S_N = 0$ as desired.

□

Remark 4.1.8. *We may similarly construct a nonstationary solution to the Stokes system that has constant enstrophy. Define f as in Theorem 4.1.7. Let $u_{j_0}(0) \neq 0$,* *and define* $u_{j_n}(0)$ *recursively as*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n \lambda_{j_0}} - \frac{(\nu\lambda_{j_0} u_{j_0}(0))^{2^n}}{2^n \nu\lambda_{j_0} \prod_{k=1}^{n} (f_{j_k})^{2^{n-k}}}. \tag{4.28}$$

Let $u_j(0) = 0$ *for all other* $j \in \mathbb{Z}^2 \setminus \{0\}$*. Then* f *and* $u(0)$ *are in* V *and define a force and initial condition such that the solution to the Stokes system (4.2) is nonstationary with constant enstrophy for all* $t \geq 0$ *provided* $\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$*. The proof mirrors the proof of Theorem 4.1.7.*

Indeed, for any $s \in \mathbb{R}$ *we may construct a solution to the Stokes system such that* $|A^{s/2}u|$ *is constant. For a given* s *we define* $u_{j_n}(0)$ *recursively as follows:*

$$u_{j_n}(0) = \frac{f_{j_n}}{\nu 2^n \lambda_{j_0}} - \frac{(2^{1-s}\nu\lambda_{j_0} u_{j_0}(0))^{2^n}}{2^{n+1-s} \nu\lambda_{j_0} \prod_{k=1}^{n} (f_{j_k})^{2^{n-k}}}. \tag{4.29}$$

For $b^2 > 2^{(s/2)-1}$ *we have* $f \in D(A^{(s/2)-1})$ *and* $u(0) \in D(A^{s/2})$*. These data define a force and initial condition such that the solution to the Stokes system (4.2) is nonstationary with constant enstrophy for all* $t \geq 0$ *provided that* $2^{1-s}\nu\lambda_{j_0}|u_{j_0}(0)|b^2 \leq 1$*.*

Remark 4.1.9. *In our construction of a nonstationary constant-energy solution to the Stokes system there exists a time* $t < 0$ *such that the energy is no longer well-defined. Note that the energy of our solution is finite so long as the limit of the partial sums given by equation (4.20) converges. The limit diverges for values of* t *such that*

$$2\lambda_{j_0}\nu|u_{j_0}(0)|b^2 e^{-\nu\lambda_{j_0} t} > 1.$$

Remark 4.1.10. *In our definition of* $u(0)$*, the decay rate of the* f_{j_n} *coefficients competes against the convergence of the sum defining* $u(0)$*. Choosing* f_{j_n} *to be a geometric series caused the term* $\frac{1}{\prod_{k=1}^{n}(f_{j_k})^{2^{n-k}}}$ *to grow essentially at the rate of* $(b^2)^{2^n}$*. By chance, this matched the growth rate of the term* $(\lambda_{j_0}\nu u_{j_0}(0))^{2^n}$ *and allowed us to choose values of* $\lambda_{j_0}, \nu, u_{j_0}(0)$*, and* b *such that convergence of the sum defining* $u(0)$ *is guaranteed. However, if the* f_{j_n} *terms decay appreciably faster, then the sum defining* $u(0)$ *necessarily diverges. Thus, there is a limit to how smooth our choice of* f *can be.*

We end this subsection with a pair of theorems regarding nonstationary constant-energy and nonstationary constant-enstrophy solutions to the Stokes system. Note that the following theorems are independent of the nonstationary-constant energy and nonstationary constant-enstrophy constructions provided above.

Theorem 4.1.11. *There does not exist a nonstationary solution to the Stokes system (4.2) with both constant energy and constant enstrophy, no matter what initial condition and time-independent force is chosen.*

Proof. Suppose $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$ for some $j_0 \in \mathbb{Z}^2 \setminus \{0\}$ (this is the condition for u to be nonstationary). Then in order to for u to have constant energy we require that the sum in (4.6) be 0. Thus, we must have (at least) the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = -\sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right). \tag{4.30}$$

In order to have constant enstrophy we have the following requirement (analogous to Equation 4.6):

$$\sum_{j\in\mathbb{Z}^2\setminus\{0\}} \lambda_j \left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right)^2 e^{-2\nu\lambda_j t} + \frac{2f_j}{\nu}\left(u_j(0) - \frac{f_j}{\nu\lambda_j}\right) e^{-\nu\lambda_j t} = 0. \tag{4.31}$$

Thus, if $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$, then in order to have constant enstrophy we must have the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \lambda_{j_0}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = -\sum_{|\lambda_n|=2\lambda_{j_0}} 2\lambda_{j_0}\frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right),$$

or rather

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = 2\left(-\sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n}\left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)\right). \tag{4.32}$$

Equations (4.30) and (4.32) can only be simultaneously satisfied if both sides of the equations are 0. $\square$

Theorem 4.1.12. *If the Stokes system (4.2) admits a nonstationary constant-energy solution then that solution must be supported on an infinite number of eigenvectors of the Stokes operator. In addition, the force must also be supported on an infinite number of eigenvectors of the Stokes operator.*

Proof. Suppose for contradiction that u is a nonstationary constant-energy solution to the Stokes system (4.2) that is supported on only a finite number of eigenvectors of the Stokes operator. As in Theorem 4.1.11, in order for u to be nonstationary we require $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ for some $j \in \mathbb{Z}^2 \setminus \{0\}$. Let j_0 be such that $u_{j_0}(0) - \frac{f_{j_0}}{\nu\lambda_{j_0}} \neq 0$ and for any other $j \in \mathbb{Z}^2 \setminus \{0\}$ such that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ we have $|j| \leq |j_0|$. In order to for u to have constant energy we require that the sum in (4.6) be 0, and so we must have (at least) the following cancellation:

$$\sum_{|\lambda_n|=\lambda_{j_0}} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right)^2 = - \sum_{|\lambda_n|=2\lambda_{j_0}} \frac{2f_n}{\nu\lambda_n f_n} \left(u_n(0) - \frac{f_n}{\nu\lambda_n}\right). \tag{4.33}$$

Since the left-hand side is not zero, the right-hand side must also be non-zero. However, this means that there must exist a $j_1 \in \mathbb{Z}^2 \setminus \{0\}$ such that $|j_1| > |j_0|$, $f_{j_1} \neq 0$, and $u_{j_1}(0) \neq \frac{f_j}{\nu\lambda_{j_1}}$. This contradicts our assumption on j_0. Thus u must be supported on an infinite number of eigenvectors of the Stokes operator. Not only that, we must have that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$ for an infinite number of j. As we just saw, for each j such that $u_j(0) - \frac{f_j}{\nu\lambda_j} \neq 0$, there must exist a j_1 such that $|j_1| > |j|$, $f_{j_1} \neq 0$, and $u_{j_1}(0) \neq \frac{f_j}{\nu\lambda_{j_1}}$. Thus f must also be supported on an infinite number of eigenvectors of the Stokes operator. □

Remark 4.1.13. *The analogous result holds for constant-enstrophy solutions. That is, if the Stokes system (4.2) admits a nonstationary constant-enstrophy solution then the solution and the force must be supported on an infinite number of eigenvectors of the Stokes operator. Indeed, this holds for any nonstationary solution u with $|A^{s/2}u|$ constant for some $s \in \mathbb{R}$.*

4.2 Extension to 3D Navier-Stokes

Recall that when working on the 3D torus, *i.e.* $\Omega = [0, L]^3_{per}$, we may write an element of H in terms of its Fourier expansion as follows:

$$u(x) = \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \hat{u}_j e^{i(2\pi/L) j \cdot x}$$

$\hat{u}_j \in \mathbb{C}^3$ such that $\overline{\hat{u}}_j = \hat{u}_{-j}$ and $j \cdot \hat{u}_j = 0$. As shown in Proposition 1.3.5, we may write the nonlinear term in (1.11) as

$$B(u,u) = \mathcal{P}_L[(u \cdot \nabla)u] = \mathcal{P}_L \left[\frac{i2\pi}{L} \sum_{j,k \in \mathbb{Z}^3 \setminus \{0\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} \right]. \tag{4.34}$$

We now construct our nonstationary constant-energy (for time $t \geq 0$) solution to the 3D Navier Stokes system on the torus using the results developed for the Stokes system. Recall that for any $j_0 \in \mathbb{Z}^2 \setminus \{0\}$ there exists a sequence of indices $j_n = (j_n(1), j_n(2))$ such that $\lambda_{j_n} = 2^n \lambda_{j_0}$ for all $n \geq 0$. Now consider the sequence $J_n \in \mathbb{Z}^3 \setminus \{0\}$ defined such that $J_n = (j_n(1), j_n(2), 0)$. Notice that we have

$$\lambda_{\pm J_n} = \left(\frac{2\pi}{L}\right)^2 J_n \cdot J_n = \left(\frac{2\pi}{L}\right)^2 \left((j_n(1))^2 + (j_n(2))^2 + 0^2\right) = \lambda_{j_n}. \tag{4.35}$$

Thus, we have that J_n so defined is such that $\lambda_{\pm J_n} = 2^n \lambda_{J_0}$ for $n \geq 0$.

First we define our forcing function $f(x) = \sum_{j \in \mathbb{Z}^3 \setminus \{0\}} \hat{f}_j e^{i(2\pi/L) j \cdot x}$ as follows. Let $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, 0, f_{j_n}) = (0, 0, 1/b^n)$ for $b > \sqrt{2}$, and $\hat{f}_j = 0$ for all other $j \in \mathbb{Z}^3 \setminus \{0\}$. By definition of the $\hat{f}_{J_n}$ terms we have that f satisfies the reality condition. Note also that f satisfies the divergence-free condition as well since $\hat{f}_{J_n} \cdot J_n = (0, 0, f_{j_n}) \cdot (j_n(1), j_n(2), 0) = 0$, and since $\hat{f}_j = 0$ (and thus $\hat{f}_j \cdot j = 0$) for all other j.

Let $u(x,t)$ be defined such that $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, 0, u_{j_n}(t))$, where $u_{j_n}(t)$ is from the nonstationary constant-energy solution of the Stokes system (for $t \geq 0$). Suppose further that $\hat{u}_j = 0$ for all other $j \in \mathbb{Z}^3 \setminus \{0\}$. Then, as with f, u satisfies the reality and divergence-free conditions.

We also have that $B(u,u) = 0$ for this choice of u. To see why, consider the sum

$$\sum_{j,k\in\mathbb{Z}^3\setminus\{0\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} \tag{4.36}$$

from equation (4.34). Notice that for $j \notin \{\pm J_n\}_{n=0}^{\infty}$ we have that $\hat{u}_j = 0$ and thus $(\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} = 0$ for all k. Similarly, if $k \notin \{\pm J_n\}_{n=0}^{\infty}$ we have that $\hat{u}_k = 0$ and thus $(\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} = 0$ for all j. Thus equation (4.34) reduces to

$$B(u,u) = \mathcal{P}_L \left[\frac{i2\pi}{L} \sum_{j,k\in\{\pm J_n\}} (\hat{u}_j \cdot k)\, \hat{u}_k e^{i(2\pi/L)(k+j)\cdot x} \right]. \tag{4.37}$$

However, if $j,k \in \{\pm J_n\}_{n=1}^{\infty}$, then $\hat{u}_j \cdot k = 0$ since $\hat{u}_j = (0,0,u_j)$ and $k = (k(1),k(2),0)$. Thus, we have that, for this definition of u, $B(u,u) = 0$.

Thus, this choice of u puts us back in the Stokes system. We now need to confirm that this choice of u satisfies $\frac{du}{dt} + \nu Au = f$ and that u has constant energy for $t \geq 0$. Consider the following calculations:

$$\begin{aligned}
\frac{du}{dt} + \nu Au &= \sum_{j\in\mathbb{Z}^3\setminus\{0\}} \left(\frac{d}{dt} + \nu A \right) \hat{u}_j e^{i(2\pi/L)j\cdot x} \\
&= \sum_{n=1}^{\infty} \left(\frac{d}{dt} + \nu A \right) \hat{u}_{J_n} e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \left(\frac{d}{dt} + \nu A \right) \hat{u}_{-J_n} e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} \left(0,0,u_{j_n}'(t) + \nu\lambda_{j_n} u(t)\right) e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \left(0,0,u_{j_n}'(t) + \nu\lambda_{j_n} u(t)\right) e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} (0,0,f_{j_n}) e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} (0,0,f_{j_n}) e^{i(2\pi/L)(-J_n)\cdot x} \\
&= \sum_{n=1}^{\infty} \hat{f}_{J_n} e^{i(2\pi/L)J_n\cdot x} + \sum_{n=1}^{\infty} \hat{f}_{J_n} e^{i(2\pi/L)(-J_n)\cdot x} \\
&= f
\end{aligned}$$

where the move to the fourth line is due to the fact that the u_{j_n} terms solve the Stokes system for those specific f_{j_n} terms. Thus we have that the u satisfies the 3D

Navier-Stokes system. Now we confirm that the energy of u is constant. We have:

$$\begin{aligned}
\frac{1}{2}|u(t)| &= \frac{L^3}{2}\sum_{j\in\mathbb{Z}^3\setminus\{0\}}|\hat{u}_j(t)|^2 \\
&= \frac{L^3}{2}\left(\sum_{n=1}^{\infty}\lambda_{J_n}|\hat{u}_{J_n}(t)|^2 + \sum_{n=1}^{\infty}|\hat{u}_{-J_n}(t)|^2\right) \\
&= \frac{L^3}{2}\left(\sum_{n=1}^{\infty}(u_{j_n}(t))^2 + \sum_{n=1}^{\infty}(u_{j_n}(t))^2\right) \\
&= L^3\sum_{n=1}^{\infty}\left(\frac{f_{j_n}}{\nu\lambda_{j_n}}\right)^2,
\end{aligned}$$

which is constant. The move to the last line is again justified by how the u_{j_n} and f_{j_n} terms were defined for the Stokes system. Straightforward calculations show that $u, f \in V$. Thus we have the following theorem:

Theorem 4.2.1. *Let $\{j_n\}_{n=0}^{\infty}$ and $f_j, u_j(0), u_j(t)$ for $j \in \mathbb{Z}^2 \setminus \{0\}$ be defined as in Theorem 4.1.7. Let $\Omega = [0, L]^3_{per}$ and define $J_n = (j_n(1), j_n(2), 0)$. Define $\hat{f}_j$ such that $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, 0, f_{j_n})$ and $\hat{f}_j = 0$ for all other $j \in \mathbb{Z}^3 \setminus \{0\}$. Define $\hat{u}_j(t)$ be such that $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, 0, u_{j_n}(t))$ and $\hat{u}_j(t) = 0$ for all other $j \in \mathbb{Z}^3 \setminus \{0\}$. Then the function $u(x, t) = \sum_{j\in\mathbb{Z}^3\setminus\{0\}} \hat{u}_j(t)e^{i(2\pi/L)j\cdot x}$ is a nonstationary constant-energy solution to the Navier-Stokes system with force $f(x) = \sum_{j\in\mathbb{Z}^3\setminus\{0\}} \hat{f}_j e^{i(2\pi/L)j\cdot x}$ for $t \geq 0$.*

Remark 4.2.2. *The above construction lends itself just as easily to creating nonstationary solutions with constant norm $|A^{s/2}u|$, for any $s \in \mathbb{R}$, on the 3D periodic domain. Simply define the parameters as in Remark 4.1.8.*

Remark 4.2.3. *The above construction may be modified to create nonstationary constant-energy (or constant higher norm) solutions in dimension $d > 3$. Simply define the indices by $J_n = (j_n^{(1)}, j_n^{(2)}, 0, \cdots, 0)$ and the non-zero Fourier coefficients of f and u by $\hat{f}_{J_n} = \hat{f}_{-J_n} = (0, \cdots, 0, f_{j_n})$ and $\hat{u}_{J_n}(t) = \hat{u}_{-J_n}(t) = (0, \cdots, 0, u_{j_n}(t))$.*

Remark 4.2.4. *This construction cannot be directly applied to the 2D Navier-Stokes system. In the 3D case we were able to construct the non-zero Fourier coefficients*

$\hat{u}_{J_n}(t)$ and the indices J_n such that they have disjoint support. This is what caused the nonlinear term to vanish and reduce the system to the Stokes system. This cannot be done in 2 dimensions since one cannot define the indices J_n such that they are 0 in the second component and maintain the requirement that $J_n = 2^n \lambda_{J_0}$. This is because if $J_n = (J_n(1), 0)$, then $\lambda_{J_n} = \left(\frac{2\pi}{L}\right)^2 (J_n(1))^2$ is always a perfect square. However, $2^n \lambda_{J_0}$ would only be a perfect square when n is even.

Remark 4.2.5. *We recall the Reynolds number and Grashof number defined as follows: $Re = \frac{\langle |u| \rangle}{\nu \lambda_1^{1/2}}$ and $Gr = \frac{|f|}{\nu^2 \lambda_1^{3/2}}$. We may also consider so-called "localized" Reynolds and Grashof numbers (localized to a specific mode) defined as $Re_j = \frac{|\hat{u}_j(0)|}{\nu \lambda_j^{1/2}}$ and $Gr_j = \frac{|\hat{f}_j|}{\nu^2 \lambda_j^{3/2}}$. Recall that the requirement on the parameters for maintaining constant energy is $2\nu \lambda_{j_0} u_{j_0}(0) b^2 \leq 1$. Thus, we may write this requirement in terms of localized Reynolds and Grashof numbers as follows:*

$$\frac{Re_{j_0}}{Gr_{j_2}} < 4.$$

5 Finite-Mode Solutions on the 2D Torus

In this chapter we explore the spectral structure of finite-mode solutions to the Navier-Stokes equations in two dimensions. The possibility of nonstationary finite-mode solutions to the NSE is unknown (at least in the attractor). The main theorem establishes the fact that (in two dimensions) if a solution is supported on only a finite number of modes, then nonlinear interactions necessarily excite modes beyond the spectrum of the solution. In order to avoid a contradiction, it must be the case that this excitement is exactly canceled by the force. Thus, as a corollary to our main theorem we have that, for any (nontrivial) finite-mode solution, it must be the case that the force is supported on wavenumbers beyond the spectrum of the solution. If the force is itself an eigenvector and the solution is a ghost solution, then it is necessarily the case that the force and the solution are supported on the same wavenumber. Thus, we conclude that there are no finite-mode ghost solutions when the force is an eigenvector of the Stokes operator. This incidentally rules out the possibility of chained ghost solutions.

5.1 Spectral Structure Theorem

We begin with a calculation that further simplifies the Fourier coefficient of the nonlinear term when Ω is the 2D torus.

Lemma 5.1.1. *The j^{th} Fourier mode of the nonlinear term of the NSE on the 2D torus may be written as*

$$\widehat{B(u,u)}_j = \frac{i\pi}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)(|k|^2 - |m|^2) \frac{u_m u_k}{|j|^2} j^\perp \tag{5.1}$$

where in general u_ℓ is a scalar in $\mathbb{C}$ defined such that $u_\ell \ell^\perp = \hat{u}_\ell$, with $\ell^\perp = (\ell_1, \ell_2)^\perp = (-\ell_2, \ell_1)$.

Proof. Recall the explicit representation of the j^{th} Fourier mode of the nonlinear term given in equation (1.25):

$$\widehat{B(u,v)}_j = \frac{2\pi i}{L} \sum_{k\in\mathbb{Z}^2\setminus\{0\}} \frac{(\hat{u}_{j-k}\cdot k)(\hat{v}_k \cdot j^\perp)}{|j|^2} j^\perp.$$

Recall that the divergence-free condition on u requires that $\hat{u}_j \cdot j = 0$ for all $j \in \mathbb{Z}^2\setminus\{0\}$. In 2D this means that we may write $\hat{u}_j = u_j j^\perp$ where u_j is now a scalar in $\mathbb{C}$ and $j^\perp = (-j_2, j_1)$. Writing u this way we write the j^{th} Fourier coefficient of the $B(u,u)$ term as follows:

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{k\in\mathbb{Z}^2} ((j-k)^\perp \cdot k)(k^\perp \cdot j^\perp) \frac{u_{j-k}u_k}{|j|^2} j^\perp. \tag{5.2}$$

Note that without loss of generality we relaxed the requirement on the index from $k \in \mathbb{Z}^2 \setminus \{0\}$ to $k \in \mathbb{Z}^2$ since an index of $k = (0,0)$ would not contribute to the sum anyway. By reindexing with $m = j - k$, the coefficient can be written as

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{\substack{k,m\in\mathbb{Z}^2\\ k+m=j}} (m^\perp \cdot k)(k^\perp \cdot (m+k)^\perp) \frac{u_m u_k}{|j|^2} j^\perp. \tag{5.3}$$

A simple calculation shows that for $a, b \in \mathbb{Z}^2$ we have $a^\perp \cdot b^\perp = a \cdot b$, given how we've chosen to define the perpendicular vectors. Thus we may also write the nonlinear term coefficient as

$$\widehat{B(u,u)}_j = \frac{2\pi i}{L} \sum_{\substack{k,m\in\mathbb{Z}^2\\ k+m=j}} (m^\perp \cdot k)(k \cdot (m+k)) \frac{u_m u_k}{|j|^2} j^\perp. \tag{5.4}$$

Consider the following calculation regarding the m^{th} coefficient of the nonlinear term:

$$\begin{aligned} 2\widehat{B(u,u)}_j &= \frac{2\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}} (m^\perp \cdot k)(k \cdot (m+k)) \frac{u_m u_k}{|j|^2} j^\perp + \frac{2\pi i}{L} \sum_{\substack{k,m\in\mathbb{Z}^2\\ k+m=j}} (k^\perp \cdot m)(m \cdot (k+m)) \frac{u_k u_m}{|j|^2} j^\perp \\ &= \frac{2\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2\\ m+k=j}} \left((m^\perp \cdot k)(k \cdot (m+k)) \frac{u_m u_k}{|j|^2} j^\perp + (k^\perp \cdot m)(m \cdot (k+m)) \frac{u_k u_m}{|j|^2} j^\perp \right) \end{aligned}$$

$$= \frac{2\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} \left((m^\perp \cdot k)(k \cdot (m+k)) \frac{u_m u_k}{|j|^2} j^\perp - (m^\perp \cdot k)(m \cdot (k+m)) \frac{u_k u_m}{|j|^2} j^\perp \right)$$

$$= \frac{2\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)((k-m) \cdot (m+k)) \frac{u_m u_k}{|j|^2} j^\perp$$

$$= \frac{2\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)(|k|^2 - |m|^2) \frac{u_m u_k}{|j|^2} j^\perp$$

Thus we may write

$$\widehat{B(u,u)}_j = \frac{\pi i}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)(|k|^2 - |m|^2) \frac{u_m u_k}{|j|^2} j^\perp. \tag{5.5}$$

□

Notice that if wavenumbers m, k are such that m is parallel to k then $m^\perp \cdot k = 0$, and so that term in the sum is 0. Notice also that if m, k are such that $|m| = |k|$ then that term in the sum is also 0. Thus, the only pairs of wave numbers that contribute to the nonlinear term in the j^{th} mode are pairs of different length, that are not parallel, and whose sum is equal to j. Finally note that each wavenumber in such a pair must correspond to a Fourier mode where u is supported in order to contribute to the nonlinear term.

Thus the expression of the NSE in terms of Fourier modes becomes

$$\hat{u}_j' + \nu\lambda_j \hat{u}_j + \frac{i\pi}{L} \sum_{\substack{m,k\in\mathbb{Z}^2 \\ m+k=j}} (m^\perp \cdot k)(|k|^2 - |m|^2) \frac{u_m u_k}{|j|^2} j^\perp = \hat{f}_j; \qquad j \in \mathbb{Z}^2 \setminus \{0\}.$$

Theorem 5.1.2. *Let $u(x,t) = \sum_{\substack{j\in\mathbb{Z}^2\setminus\{0\} \\ |j|\le N}} \hat{u}_j(t) e^{ij\cdot x}$ be a finite-mode solution to the NSE on the 2D torus with nontrivial nonlinear term (i.e. $B(u,u) \neq 0$). Then there exist wavenumbers j, k such that the following criteria hold:*

1. $\hat{u}_{j+k} = 0$,

2. $\widehat{B(u,u)}_{j+k} = \frac{i\pi}{L}(j^\perp \cdot k)(|k|^2 - |j|^2)\frac{u_j u_k}{|j+k|^2}(j+k)^\perp$

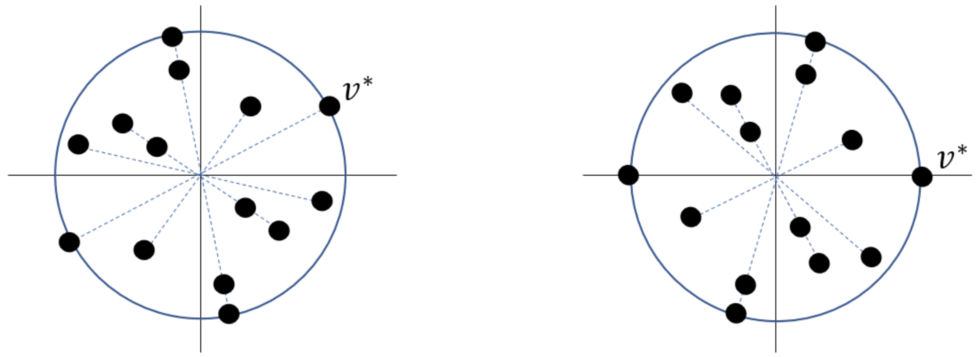
3. $\widehat{B(u,u)}_{j+k} \neq 0$

Proof. Suppose u is a finite-mode solution to the NSE with nontrivial nonlinear term. Let S be the finite set of vectors in $\mathbb{Z}^2 \setminus \{0\}$ associated with the Fourier modes where u is supported. That is, $j \in S$ if and only if $\hat{u}_j \neq 0$. We seek a pair of wave numbers $j, k \in \mathbb{Z}^2 \setminus \{0\}$ with the following properties:

1. $j, k \in S$
2. $j + k \notin S$
3. $j \nparallel k$
4. $|j| \neq |k|$
5. If $p, q \in S$ are such that $\{p, q\} \neq \{j, k\}$ and $p + q = j + k$ then either $p \parallel q$ or $|p| = |q|$

Property 2 is there to guarantee that the first criterion of our theorem is satisfied. Properties 1, 3, and 4 together imply that $\frac{i\pi}{L}(j^\perp \cdot k)(|k|^2 - |j|^2)\frac{u_j u_k}{|j+k|^2}(j+k)^\perp \neq 0$ and contributes to the sum for the $(j+k)^{\text{th}}$ mode of nonlinear term. (We recall that the Fourier coefficients of any solution to the NSE are analytic in time. This implies that if $u_j(t)$ and $u_k(t)$ are not identically zero, then the product $u_j(t)u_k(t)$ is not identically zero, since nonzero analytic functions may only take the value of 0 on a discrete set of points.) Property 5 implies that for any pair of vectors $p, q \in S$ such that $\{p, q\} \neq \{j, k\}$ we have $\frac{i\pi}{L}(p^\perp \cdot q)(|p|^2 - |q|^2)\frac{u_p u_q}{|p+q|^2}(p+q)^\perp = 0$ and does not contribute to the sum for the $(j+k)^{\text{th}}$ mode of nonlinear term. Properties 1, 3, 4, and 5 together establish the second and third criteria in the theorem.

We begin by choosing a vector $v^* \in S$ such that for any $v_i \in S$ we have $|v_i| \leq |v^*|$ (*i.e.* v^* has maximum length in S). We let $b_1 = v^*$ and $b_2 = (v^*)^\perp$ be a new basis for our vector space and we write the vectors in S in the coordinates of this new basis. Geometrically speaking, we reorient the plane so that v^* lies on the x-axis. [See Figure 5.1]



(a) Original orientation of vectors in S (b) Reorientation of vectors in S

FIGURE 5.1: Geometric Interpretation

We then arrange the elements of S in reverse-lexicographic order (according to this new basis). That is, j appears earlier in our list than k if and only if either the first component of j is greater than the first component of k (that is, j is farther along in the direction of v^* than k), or if the first components are equal then the second component of j is greater than the second component of k (that is, j is farther along in the direction 90° counterclockwise from v^* than k). Assuming S contains n vectors, let us relabel the vectors in S as $v_1, v_2, \ldots, v_n$ according to their reverse-lexicographic order. Thus, $v_1 = v^*$, and the rest of the vectors are ordered in terms of how close they are to v^* (prioritizing counter-clockwise rotation over clockwise rotation). Thinking in terms of Figure 5.1b, the vectors are ordered from right to left, and if a set of vectors are equally far right, these vectors are ordered from top to bottom.

Recall that the Fourier coefficients of u come in pairs. That is, if v_j is in S then so is $-v_j$. This implies that, given our ordering, the vectors $v_1, \ldots, v_{n/2}$ have nonnegative first components. Consider the vector $v_1 + v_2$. [See Figure 5.2]. We can see $v_1 + v_2 \notin S$ since if v_2 has positive first component then the first component of $v_1 + v_2$ is greater than the first component of v_1, which has the largest first component of any vector in S. If the first component of v_2 is 0 then the second component is positive and this implies that $v_1 + v_2$ has greater length than v_1, which already has length greater than or equal to any vector in S. Thus v_1 and v_2 are wavenumbers that satisfy Properties 1 and 2 of our list.

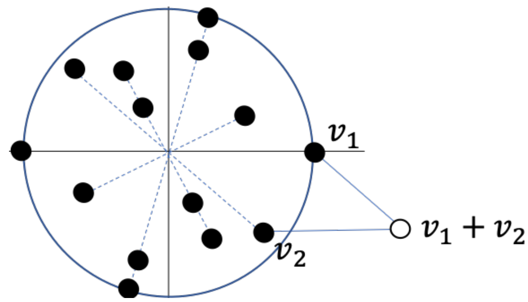


FIGURE 5.2: Vector $v_1 + v_2$

To establish Property 5 we show that if $v_p, v_q \in S$ are such that $v_p + v_q = v_1 + v_2$ then $\{v_p, v_q\} = \{v_1, v_2\}$. The proof of this straightforward. The geometric idea is that any other pair of vectors would have a sum that does not extend as far up and to the right as (*i.e.* has a smaller first or second component than) $v_1 + v_2$. Suppose $v_p, v_q \in S$ are such that $v_p + v_q = v_1 + v_2$. Note that if $v_p \in \{v_1, v_2\}$ then $v_p + v_q = v_1 + v_2$ implies that $\{v_p, v_q\} = \{v_1, v_2\}$. Similarly if $v_q \in \{v_1, v_2\}$ then $v_p + v_q = v_1 + v_2$ implies that $\{v_p, v_q\} = \{v_1, v_2\}$. So suppose $v_p, v_q \notin \{v_1, v_2\}$. Denote the i^{th} component of a vector v by $v(i)$. Note that, given the ordering of vectors in S under the basis $\{b_1, b_2\}$, for any $v \in S$ with $v \neq v_1$ we have that $v(1) \leq v_2(1) < v_1(1)$. Thus we have $v_p(1), v_q(1) \leq v_2(1) < v_1(1)$ and so $v_p(1) + v_q(1) < v_1(1) + v_2(1)$ and $v_p + v_q \neq v_1 + v_2$.

Thus v_1 and v_2 are wavenumbers that satisfy Property 5. If we also have that $v_1 \nparallel v_2$ and if $|v_1| \neq |v_2|$ then Properties 3 and 4 are satisfied and v_1 and v_2 are the required vectors for the theorem (This case is represented in Figure 5.2.)

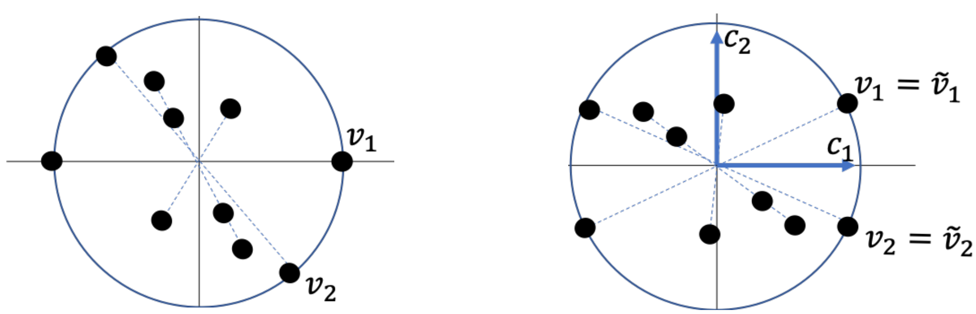
However, if $v_1 \parallel v_2$ or $|v_1| = |v_2|$ then v_1 and v_2 do not satisfy the theorem and we must seek a different pair of wavenumbers that satisfy the five properties listed above. We handle these cases in turn.

Case 1: $|v_1| = |v_2|$

For this case we again define a new basis for our vector space. Let $c_1 = \frac{v_1 + v_2}{2}$

and $c_2 = \frac{(v_1+v_2)^\perp}{2}$ be our new basis vectors (see Figure 5.3).[7]

In this case the vectors v_1 and v_2 have the same first coordinate in the basis $C = \{c_1, c_2\}$, and that first coordinate is strictly larger than the first coordinate (in C) of any other vector in S. (If any other vector had the same or larger first coordinate in C then that vector would have come between v_1 and v_2 in the ordering under the basis $\{b_1, b_2\}$, which is a contradiction.) Let $\tilde{v}_1 \dots, \tilde{v}_n$ be the reverse-lexicographic ordering of the vectors in S according to the basis C. Without loss of generality we may assume $\tilde{v}_1 = v_1$ and $\tilde{v}_2 = v_2$ as in Figure 5.3.



(a) Original orientation (b) Reoriented

FIGURE 5.3: $|v_1| = |v_2|$

Let $\tilde{v}_{j_0}$ be the first vector in this new ordering of S such that $|\tilde{v}_{j_0}| \neq |\tilde{v}_1|$ (see Figure 5.4 for example). If no such vector exists then $B(u,u) = 0$ and we are outside the scope of our theorem. We are now in a position to find pairs of vectors that satisfy the required five properties. Again, we must consider cases:

(1.a) $\tilde{v}_{j_0} \nparallel \tilde{v}_1$

In this case, the pair $\tilde{v}_1, \tilde{v}_{j_0}$ by definition satisfy Properties 1, 3, and 4. To see that $\tilde{v}_{j_0} + \tilde{v}_1 \notin S$ note that either $\tilde{v}_{j_0}$ has a positive first component, or $\tilde{v}_{j_0}$ has

[7] Why choose $c_1 = \frac{v_1+v_2}{2}$ and $c_2 = \frac{(v_1+v_2)^\perp}{2}$ as opposed to, say, $c_1 = v_1 + v_2$ and $c_2 = (v_1+v_2)^\perp$ or $c_1 = \frac{v_1+v_2}{|v_1+v_2|}$ and $c_2 = \frac{(v_1+v_{j_0})^\perp}{|v_1+v_2|}$? The choice is nothing deep. The problem with $v_1 + v_2$ is that such a vector will not fit compactly in our diagrams. The problem with $\frac{v_1+v_2}{|v_1+v_2|}$ is that it is more cumbersome to type.

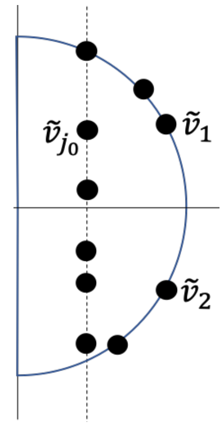


FIGURE 5.4: $\tilde{v}_{j_0} \nparallel \tilde{v}_1$

0 as its first component and has a positive second component. In the first case $\tilde{v}_{j_0} + \tilde{v}_1$ has a larger first component than $\tilde{v}_1$, and thus is not in S. In the second case $|\tilde{v}_{j_0} + \tilde{v}_1| > |\tilde{v}_1|$ and thus is not in S. Thus Property 2 is satisfied.

To see that Property 5 is satisfied we consider a pair of vectors $\tilde{v}_p, \tilde{v}_q \in S$ such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$. Without loss of generality, suppose $p < q$ (that is, $\tilde{v}_p$ comes before $\tilde{v}_q$ in our reverse-lexicographic ordering). We show that any such pair of vectors will be such that $1 < p < q < j_0$ and thus that $|\tilde{v}_p| = |\tilde{v}_q|$ (since $p, q < j_0$ implies that both $\tilde{v}_p$ and $\tilde{v}_q$ have the same length as $\tilde{v}_1$). Note that, given the ordering under the basis C, for any $\tilde{v} \in S$ such that $\tilde{v} \notin \{\tilde{v}_1, \tilde{v}_2\}$ we have that $\tilde{v}(1) < \tilde{v}_1(1) = \tilde{v}_2(1)$.

As we saw previously, if any other pair of vectors $\tilde{v}_p, \tilde{v}_q$ is such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ then neither $\tilde{v}_p$ nor $\tilde{v}_q$ is equal to $\tilde{v}_1$ or $\tilde{v}_{j_0}$.

Suppose $\tilde{v}_p = \tilde{v}_2$. Then $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ implies $\tilde{v}_q(1) = \tilde{v}_{j_0}(1)$ (since $\tilde{v}_p(1) = \tilde{v}_2(1) = \tilde{v}_1(1)$). This also implies $\tilde{v}_q(2) > \tilde{v}_{j_0}(2)$ (since $\tilde{v}_p(2) = \tilde{v}_2(2) < \tilde{v}_1(2)$). This implies that $q < j_0$ and thus that $1 < p < q < j_0$. This implies $|\tilde{v}_p| = |\tilde{v}_q| = |\tilde{v}_1|$ as desired.

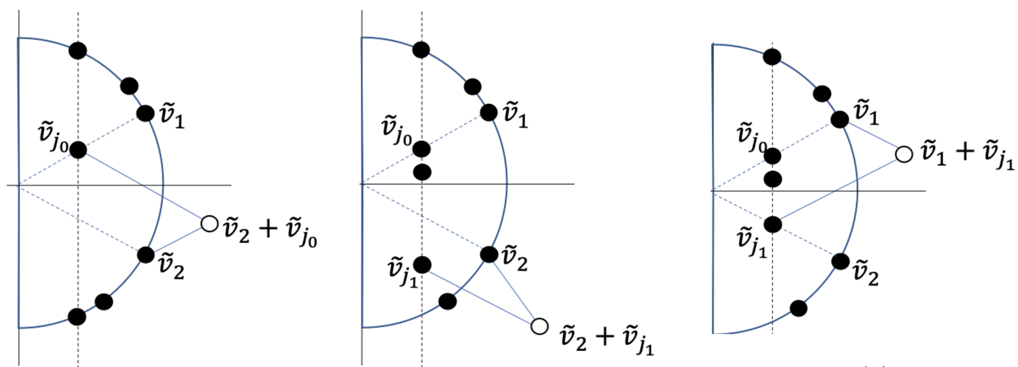
Suppose neither $\tilde{v}_p$ nor $\tilde{v}_q$ is equal to $\tilde{v}_2$. Then we have, in particular, that $\tilde{v}_p(1) < \tilde{v}_1(1)$. Thus $\tilde{v}_p + \tilde{v}_q = \tilde{v}_1 + \tilde{v}_{j_0}$ implies that $\tilde{v}_{j_0}(1) < \tilde{v}_q(1)$. This implies that $j_0 > q$ and thus that $1 < p < q < j_0$. Thus $|\tilde{v}_p| = |\tilde{v}_q| = |\tilde{v}_1|$ as desired. In this case the vectors $\tilde{v}_{j_0}$ and $\tilde{v}_1$ are vectors that satisfy the theorem.

(1.b) $\tilde{v}_{j_0} \parallel \tilde{v}_1$

In this case consider the set of vectors in S with the same first component (in the basis C) as $\tilde{v}_{j_0}$ and with size strictly less than $|\tilde{v}_1|$. Call this set of vectors S'. (S' is the set of vectors in Figure 5.4 that lie on the vertical dotted line, but not on the outer circle).

- If $\tilde{v}_{j_0}$ is the only vector in this set (*i.e.* $S' = \{\tilde{v}_{j_0}\}$) then the pair $\tilde{v}_{j_0}, \tilde{v}_2$ satisfies all the conditions of the theorem. This is again because any other pair of vectors $\tilde{v}_p, \tilde{v}_q$ such that $\tilde{v}_p + \tilde{v}_q = \tilde{v}_2 + \tilde{v}_{j_0}$ are such that $|\tilde{v}_p| = |\tilde{v}_q|$, and the reasoning is similar to the case (1.a). See Figure 5.5a.
- If there are other vectors in S', then let $\tilde{v}_{j_1}$ be the vector in S' with the smallest second coordinate. If $\tilde{v}_{j_1}$ is not parallel to $\tilde{v}_2$ (see Figure 5.5b), then the pair $\tilde{v}_{j_1}, \tilde{v}_2$ satisfies all of the conditions of our theorem. Essentially, this is because any other pair of vectors whose sum matches the first coordinate of $\tilde{v}_{j_1} + \tilde{v}_2$ must either both have maximum length (which is what we want) or one is from S' and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$. However, since $\tilde{v}_{j_1}$ has the smallest second coordinate of any vector in S' and $\tilde{v}_2$ has a smaller second coordinate than $\tilde{v}_1$, no other sum of a vector from S' and a vector from $\{\tilde{v}_1, \tilde{v}_2\}$ will match the second coordinate of $\tilde{v}_{j_1} + \tilde{v}_2$.
- If both $\tilde{v}_{j_0} \parallel \tilde{v}_1$ and $\tilde{v}_{j_1} \parallel \tilde{v}_2$ (see Figure 5.5c), then the pair $\tilde{v}_{j_1}, \tilde{v}_1$ satisfies the theorem. As before, this is because any other pair of vectors whose sum matches the first coordinate of $\tilde{v}_{j_1} + \tilde{v}_1$ must either both have maximum length or one is from S' and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$. Note that in our current case, any vector from S' added to $\tilde{v}_1$ will have a positive second coordinate and any vector from S' added to $\tilde{v}_2$ will have a negative second component. This is due to the fact that $\tilde{v}_2(2) = -\tilde{v}_1(2)$ and for any $\tilde{v}_i \in S'$

we have $|\tilde{v}_i(2)| < |\tilde{v}_1(2)|$ (owing to the fact that the vectors from S' now are sandwiched between two vectors, $\tilde{v}_{j_0}$ and $\tilde{v}_{j_1}$, that are positive scalar multiples of $\tilde{v}_1$ and $\tilde{v}_2$ where that scalar is strictly less than 1). Thus, of all the sums of vectors where one is from S' and the other is from $\{\tilde{v}_1, \tilde{v}_2\}$, the sum $\tilde{v}_{j_1}+\tilde{v}_1$ is the only sum having the smallest positive second component. Thus, the pair $\tilde{v}_{j_1}, \tilde{v}_1$ satisfy the conditions of our theorem. Indeed, in this case, any vector from $S' \setminus \{\tilde{v}_{j_0}\}$ together with $\tilde{v}_1$ will meet the conditions of the theorem (as well as any vector from $S' \setminus \{\tilde{v}_{j_1}\}$ together with $\tilde{v}_2$).



(a) (b) (c)

FIGURE 5.5: $\tilde{v}_{j_0} \parallel \tilde{v}_1$

Thus we have shown that even in the case where $|v_1| = |v_2|$ we can find wavenumbers j, k that satisfy all the conditions of the theorem.

Case 2: $v_1 \parallel v_2$

If v_2 is parallel to v_1, let v_{k_0} be the first ordered vector from S such that v_{k_0} is not parallel to v_1. If no such vector exists then $B(u,u) = 0$ and we are outside the scope of our theorem. Thus we may assume that such a vector exists. Note that such a vector must have a nonnegative first component. (Otherwise we would have $-v_{k_0}$ precede v_{k_0} on the list. But we already said that any vector preceding v_{k_0} was parallel to v_1, implying that $-v_{k_0}$ and thus v_{k_0} was parallel to v_1. This is contrary to our assumption on k_0). We must consider two further subcases. The first is when

$|v_{k_0}| \neq |v_1|$. The second is when $|v_{k_0}| = |v_1|$.

(2.a) $|v_{k_0}| \neq |v_1|$:

In this case the pair v_1, v_{k_0} by definition satisfies Properties 1, 3, and 4. We consider the vector $v_1 + v_{k_0}$ (see Figure 5.6). To see that $v_1 + v_{k_0} \notin S$ note that either v_{k_0} has a positive first component, or v_{k_0} has 0 as its first component and has a positive second component. In the first case $v_{k_0} + v_1$ has a larger first component than v_1, and thus is not in S. In the second case $|v_{k_0} + v_1| > |v_1|$ and thus is not in S. Thus Property 2 is satisfied.

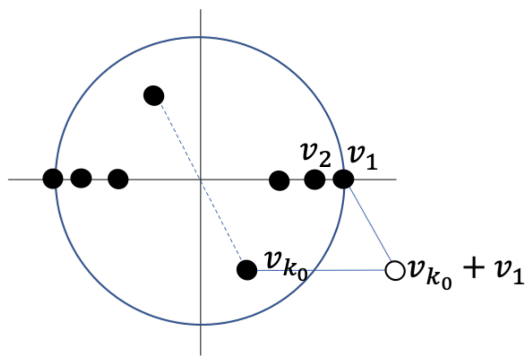


FIGURE 5.6: Vector $v_1 + v_{k_0}$

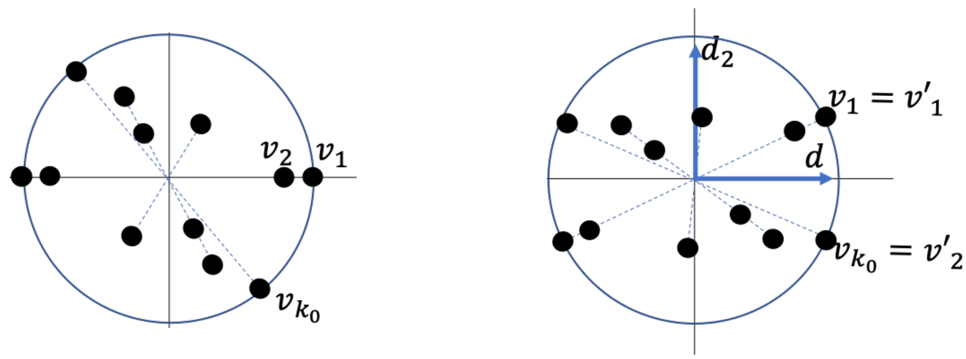
To see that Property 5 is satisfied we consider a pair of vectors $v_p, v_q \in S$ such that $v_p + v_q = v_1 + v_{k_0}$. We show that any such pair of vectors such that $\{v_p, v_q\} \neq \{v_1, v_{k_0}\}$ will also be such that $1 < p < q < k_0$ and thus that $v_p \parallel v_q$ (since $p, q < k_0$ implies that both v_p and v_q are parallel to v_1).

As we saw above, if one of $v_p, v_q \in \{v_1, v_{k_0}\}$ then $\{v_p, v_q\} = \{v_1, v_{k_0}\}$. So assume that neither v_p nor v_q is equal to v_1 or v_{k_0}. Note that since for any $v \in S \backslash \{v_1\}$ we have $v(1) < v_1(1)$, the identity $v_p + v_q = v_1 + v_{k_0}$ implies that $v_p(1), v_q(1) > v_{k_0}$. This implies that $1 < p, q < k_0$, and thus that $v_p \parallel v_q$.

Thus the conditions of the theorem are satisfied in this case.

(2.b) $|v_{k_0}| = |v_1|$:

For this case we redefine a basis for our vector space in a similar manner to what we did in Case 1. Let $d_1 = \frac{v_1+v_{k_0}}{2}$ and $d_2 = \frac{(v_1+v_{k_0})^{\perp}}{2}$ be our new basis vectors (see Figure 5.7).



(a) $|v_{k_0}| = |v_1|$

(b) Reoriented

FIGURE 5.7: $|v_{k_0}| = |v_1|$

In this case the vectors v_1 and v_{k_0} have the same first coordinate in the basis $D = \{d_1, d_2\}$, and that first coordinate is strictly larger than the first coordinate (in D) of any other vector in S. (If any other vector had the same or larger first coordinate in D then that vector would have come between v_1 and v_{k_0} in the ordering under the basis $\{b_1, b_2\}$. But by assumption the only such vectors are parallel to v_1. Thus each such vector is of the form $v_i = c_i v_1$ where $c_i < 1$, and so each such vector has smaller first coordinate than v_1 in any basis.) Let $v'_1, \ldots, v'_n$ be the reverse-lexicographic ordering of the vectors in S according to the basis D. Without loss of generality we may assume $v'_1 = v_1$ and $v'_2 = v_{k_0}$. (See Figure 5.7b)

Notice now that we are in the same position we were in at the beginning of Case 1. Thus, we may treat this situation in the same way as we treated Case 1.

□

5.2 Allowable Forces for Finite Mode Solutions

This section is dedicated to establishing limits on the types of forces that can admit of the possibility of a finite mode solution to the 2D NSE with non-trivial non-linear term. As a consequence, we will show that no nonstationary finite mode constant-energy solution (respectively, constant-enstrophy solution) is possible in the case when the force is an eigenvector of the Stokes operator. Incidentally, this proves the impossibility of so-called *chained ghost solutions* introduced in [25].

We begin with the following corollary to Theorem 5.1.2.

Corollary 5.2.1. *Let u be a finite mode solution to the NSE on the 2D torus with non-trivial non-linear term. Let S_f (resp. S_u) be the set of wave numbers associated with the Fourier modes where f (resp. u) is supported. For all pairs of wavenumbers $j, k \in S_u$ that satisfy the conditions of Theorem 5.1.2, f must be supported on the Fourier mode associated with $j + k$ (i.e. $j + k \in S_f$).*

In particular, let $k_f \in S_k$ be such that $|k_f| \geq k$ for all $k \in S_f$ and let $k_u \in S_u$ be such that $|k_u| \geq k$ for all $k \in S_u$. Then $|k_f| > |k_u|$.

Proof. Recall the following Fourier characterization of the NSE:

$$\hat{u}_j' + \nu\lambda_j\hat{u}_j + \widehat{B(u,u)}_j = \hat{f}_j; \qquad j \in \mathbb{Z}^2 \setminus \{0\}. \tag{5.6}$$

Suppose u is a finite mode solution to the NSE. Let j, k be a pair of wave numbers that satisfy the conditions of the theorem. Then the Fourier characterization of the NSE for the mode associated with wavenumber $j + k$ is as follows:

$$\widehat{B(u,u)}_{j+k} = \hat{f}_{j+k} \tag{5.7}$$

Since $\widehat{B(u,u)}_{j+k} \neq 0$ by Theorem 5.1.2, this implies that $\hat{f}_{j+k} \neq 0$. In the proof of Theorem 5.1.2 it is demonstrated that $|j + k| > k_u$ for any $k_u \in S_u$. □

Theorem 5.2.2. *Finite-mode constant-energy solutions to (1.11) in two dimensions are impossible when the force is an eigenvector of the Stokes operator. Similarly,* *finite-mode constant-enstrophy solutions are impossible when the force is an eigenvector of the Stokes operator.*

Proof. In the case where the solution is such that $B(u,u) = 0$ we are in the Stokes system (4.2). We proved in Theorem 4.1.12 that in this case no finite-mode constant-energy or finite-mode constant-enstrophy solutions are possible. Thus, we are only concerned with the case where the nonlinear term is nonzero.

Let us first consider constant-energy solutions to the NSE. The energy balance equation for the NSE is:

$$\frac{1}{2}\frac{d}{dt}|u|^2 = -\nu\|u\|^2 + (f,u) \tag{5.8}$$

If u is a solution with constant energy then $\frac{d}{dt}|u|^2 = 0$. In this case the balance equation becomes:

$$\nu\|u\|^2 = (f,u) \tag{5.9}$$

Suppose that f is an eigenvector of the Stokes operator associated with eigenvalue $|k_f|^2$. Then $f = \sum_{k:|k|=|k_f|} \hat{f}_k e^{ik\cdot x}$. Thus the term (f,u) becomes:

$$\sum_{k:|k|=|k_f|} \hat{f}_k \hat{u}_k(t). \tag{5.10}$$

Equation (5.9) requires that $(f,u) \neq 0$ (assuming of course that our solution is nonzero). Thus we require that $\sum_{k:|k|=|k_f|} \hat{f}_k \hat{u}_k(t) \neq 0$. In order for this to be the case there must exist a k_0 such that $\hat{f}_{k_0} \neq 0$ and $\hat{u}_{k_0} \neq 0$. However, given that f is an eigenvector, this means that $|k_0| = |k|$ for any $k \in S_f$ and $k_0 \in S_u$. This contradicts the final statement of Corollary 5.2.1.

Similarly, we consider constant-enstrophy solutions to the NSE. The enstrophy balance equation for the NSE is:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = -\nu|Au|^2 + (f,Au). \tag{5.11}$$

If u is a solution with constant enstrophy then $\frac{d}{dt}\|u\|^2 = 0$. In this case the enstophy balance equation becomes:

$$\nu|Au|^2 = (f,Au) \tag{5.12}$$

If f is an eigenvector of the Stokes operator associated with eigenvalue $|k_f|^2$ then the term $(f, Au) = (Af, u)$ becomes:

$$|k_f|^2 \sum_{k:|k|=|k_f|} \hat{f}_k \hat{u}_k(t) \tag{5.13}$$

Now equation (5.12) requires that $(f, Au) \neq 0$. As before, this means that u is supported on a vector associated with a wavenumber k_0 where $|k_0| = |k|$ for all $k \in S_f$, thus contradicting the final statement of Corollary 5.2.1. □

Corollary 5.2.3. *There are no finite mode ghost solutions to the NSE when the force is an eigenvector of the Stokes operator. In particular, there are no so-called chained ghost solutions.*

Proof. Ghost solutions require that both the energy and enstrophy remain constant over time. Theorem 5.2.2 demonstrates that each of these conditions is individually impossible for finite-mode solutions when the force is an eigenvector. Chained ghost solutions are a subset of the finite-mode ghost solutions when the force is an eigenvector. □

Naively, there is nothing to rule out the possibility of a finite-mode solution so long as the force is supported on all wavenumbers k where $\hat{u}_k = 0$ but $\widehat{B(u,u)}_k \neq 0$. However, for wavenumbers j and k where the conditions of Theorem 5.1.2 are satisfied, we have an interesting condition on the Fourier coefficients $\hat{u}_j$ and $\hat{u}_k$.

Proposition 5.2.4. *Let j, k be wavenumbers that satisfy the conditions of Theorem 5.1.2. Then the product $u_j u_k$ is constant, and both u_j and u_k are nowhere vanishing and never change sign.*

Proof. When j, k satisfy the condition of Theorem 5.1.2, the Fourier characterization of the $(j+k)^{\text{th}}$ mode of the NSE can be written as $c_{j,k} u_j(t) u_k(t) = f_{j+k}$, where $c_{j,k} = (j^\perp \cdot k)(k \cdot (j+k)) \frac{1}{|m|^2}$ is (a non-zero) constant. Since f_k is also (a nonzero) constant, this implies that the product $u_j(t)u_k(t)$ is constant. It is well-known that the functions $u_j(t)$ and $u_k(t)$ are analytic in time. Thus, in order for the product to be a non-zero constant, we must have that neither function is ever equal to 0. Since $u_j(t)$ and $u_k(t)$ are analytic and never 0, this means that $u_j(t)$ and $u_k(t)$ never change sign. $\square$

6 Conclusions and Future Work

A common theme uniting the main results of this dissertation is the essentially infinite nature of various solutions to the Navier-Stokes equations. The constructions of constant-normed solutions in Chapter 4 are necessarily infinite-dimensional. But beyond that, in the Stokes system, no finite-mode constant-norm solution is even possible. The question of whether there exist finite-mode constant-norm solutions to the Navier-Stokes system is partially answered in Chapter 5. By exploring the nature of the nonlinear term in two dimensions, we show that finite-mode constant-norm solutions are not possible when the force is an eigenvector of the Stokes operator and when the norm corresponds to the energy or enstrophy of the solution.

While the question of the existence of ghost solutions in general is still open, the strategies employed in this dissertation lend themselves to several further avenues of research. In particular, the geometric interpretation of the nonlinear interactions should bear more fruit. It seems reasonable to hope that in the case when the force is composed of two eigenvectors (a situation relevant for turbulence), a finite-mode constant-energy or constant-enstrophy solution is not possible. What is clear is that, in general, nonstationary finite-mode solutions (if they exist) exhibit a significant amount of symmetry. The exact nature of that symmetry is a topic for further investigation. In any event, the consequences of Theorem 5.1.2 have not been fully explored.

Bibliography

1. G. Boffetta and R. E. Ecke, *Two-dimensional turbulence*, Annual Review of Fluid Mechanics, 44 (2012), pp. 427–451.

2. T. Buckmaster and V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, Annals of Mathematics, 189 (2019), pp. 101–144.

3. P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, 1988.

4. P. Constantin, C. Foias, and O. P. Manley, *Effects of the forcing function spectrum on the energy spectrum in 2-D turbulence*, Physics of Fluids, 6 (1994), pp. 427–429.

5. R. Dascaliuc, C. Foias, and M. Jolly, *Some specific mathematical constraints on 2D turbulence*, Physica D: Nonlinear Phenomena, 237 (2008), pp. 3020–3029.

6. ——, *Estimates on enstrophy, palinstrophy, and invariant measures for 2-D turbulence*, Journal of Differential Equations, 248 (2010), pp. 792–819.

7. R. Dascaliuc, M. Jolly, et al., *Relations between energy and enstrophy on the global attractor of the 2-D Navier-Stokes equations*, Journal of Dynamics and Differential Equations, 17 (2005), pp. 643–736.

8. U. Dudley, *Elementary Number Theory: Second Edition*, Dover Books on Mathematics, Dover Publications, 2012.

9. L. Evans, *Partial Differential Equations*, American Mathematical Society, 2010.

10. C. Foias, M. Jolly, O. Manley, and R. Rosa, *Statistical estimates for the Navier–Stokes equations and the Kraichnan theory of 2-D fully developed turbulence*, Journal of Statistical Physics, 108 (2002), pp. 591–645.

11. C. Foias, M. Jolly, and M. Yang, *On single mode forcing of the 2D-NSE*, Journal of Dynamics and Differential Equations, 25 (2013), pp. 393–433.

12. C. Foias, M. S. Jolly, R. Lan, R. Rupam, Y. Yang, and B. Zhang, *Time analyticity with higher norm estimates for the 2D Navier–Stokes equations*, IMA Journal of Applied Mathematics, 80 (2015), pp. 766–810.

13. C. Foias, O. Manley, R. Rosa, and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, 2001.

14. C. Foias, O. P. Manley, R. Temam, and Y. M. Treve, *Asymptotic analysis of the Navier-Stokes equations*, Physica D: Nonlinear Phenomena, 9 (1983), pp. 157–188.

15. U. Frisch and A. N. Kolmogorov, *Turbulence: The Legacy of AN Kolmogorov*, Cambridge University Press, 1995.

16. A. N. Kolmogorov, *On degeneration (decay) of isotropic turbulence in an incompressible viscous liquid*, in Dokl. Akad. Nauk SSSR, vol. 31, 1941, pp. 538–540.

17. ——, *The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers*, Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences, 434 (1991), pp. 9–13.

18. R. H. Kraichnan, *Inertial ranges in two-dimensional turbulence*, Physics of Fluids, 10 (1967), p. 1417.

19. C. Marchioro, *An example of absence of turbulence for any Reynolds number*, Communications in Mathematical Physics, 105 (1986), pp. 99–106.

20. L. F. Richardson and P. Lynch, *Weather Prediction by Numerical Process*, Cambridge University Press, 2007.

21. J. C. Robinson, J. L. Rodrigo, and W. Sadowski, *The Three-Dimensional Navier-Stokes Equations*, Cambridge University Press, 2016.

22. R. Temam, *Navier–Stokes Equations and Nonlinear Functional Analysis*, Society for Industrial and Applied Mathematics, 1995.

23. ——, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer New York, 1997.

24. J. Tian and Y. You, *On a subclass of solutions of the 2D Navier–Stokes equations with constant energy and enstrophy*, Journal of Dynamics and Differential Equations, 31 (2019), pp. 1743–1775.

25. B. Zhang and J. Tian, *On solutions of the 2D Navier-Stokes equations with constant energy and enstrophy*, Indiana University Mathematics Journal, 64 (2015), pp. 1925–1958.