AN ABSTRACT OF THE THESIS OF

KENNETH OAKLAND GAMON for the Ph. D.
(Name) (Degree)

in Mathematics presented on May 2, 1969
(Major) (Date)

Title: THE PROPAGATION OF WAVES AND PULSES IN THE PRESENCE OF CONICAL STRUCTURES

Abstract approved: Redacted for Privacy
(Major professor)

The electromagnetic field in a cone of arbitrary slant height with a symmetrically placed time harmonic ring source is studied. Through the use of the modified Helmholtz equation as an intermediate, we obtain the solution of the semi-infinite cone directly from the finite cone. To demonstrate the need for the modified Helmholtz equation a simple example is used in which the solution is known. The Green function is derived from a well known summation formula involving the eigenvalues and eigenfunctions, which are determined from the roots of certain Legendre and Bessel functions.

The results obtained here for the semi-infinite cone are compared with those obtained by Buchholz [4], and the special case $\theta_0 = \frac{\pi}{2}$ is compared with a double ring in free space. In both cases the results are in agreement.

Once the results have been obtained for the time harmonic case, they are generalized by Laplace transform theory to arbitrary time dependency. This is accomplished by finding the field for a "Dirac" pulse at time t and integrating the product of the pulse function and the "Dirac" pulse field with respect to time.

THE PROPAGATION OF WAVES AND PULSES IN THE PRESENCE OF CONICAL STRUCTURES

by

KENNETH OAKLAND GAMON

A THESIS

submitted to

OREGON STATE UNIVERSITY

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1969

APPROVED:

Redacted for Privacy

Professor of Mathematics

In Charge of Major

Redacted for Privacy

Chairman of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented May 2, 1969

Typed by Ramona Nestell

ACKNOWLEDGEMENT

I would like to express my sincere thanks to Professor Fritz Oberhettinger for his help and encouragement in writing this dissertation. I would also like to express my appreciation to both Dr. Oberhettinger and his wife, Joyce, for their hospitality during my numerous visits to discuss problems arising from this dissertation. My thanks also to Dr. D. S. Carter and Dr. A. T. Lonseth for their help and suggestions in the conduct of my Ph. D. program. Finally my thanks to my wife and family for their faith and confidence.

TABLE OF CONTENTS

THE PROPAGATION OF WAVES AND PULSES IN THE PRESENCE OF CONICAL STRUCTURES

CHAPTER 1

INTRODUCTION

Much work has been done with regard to the solution of the wave equation in a cone. One of the earlier works was by H. S. Carslaw [5].

In the paper by Carslaw the solution is found when the disturbance is a point source of sound and the obstacle is a semi-infinite right circular cone of any angle. His method was to use the ordinary Helmholtz equation and contour integration.

More directly related to the present dissertation is a paper by Buchholz [4]. In fact, as a special case of the solution in this dissertation, we get the results obtained earlier by Buchholz. The methods used by Buchholz are similar to those used by Carslaw.

In a more recent paper by Muki and Sternberg [16] the problem of heat conduction in a cone is solved using Mellin transforms and contour integration.

Shortly thereafter F. H. Northover [17] solves the problem of the diffraction of electromagnetic waves around a finite perfectly conducting cone in which the source is an axially pointing dipole outside the cone on the axis of symmetry nearest the vertex of the cone. The primary difficulty in this problem is satisfying the boundary conditions at two geometrically distinct boundaries simultaneously.

The present dissertation uses a very straightforward approach to extend the investigation to the case of a conical horn of a slant height a and terminated by a spherical cap of radius a. The excitation is due to a ring source inside the cone in a plane perpendicular to the axis of the cone and symmetric with the axis of the cone. The method used is a simple application of a well known expansion formula for Green's function. The computation of the solution and the extension of the solution to related problems is facilitated by using the modified Helmholtz equation as an intermediate. The advantage of the modified Helmholtz equation is that the solution is in terms of a series of monotonic functions rather than oscillatory functions.

The primary results obtained here are the solution of the finite cone problem with an axially symmetric time harmonic ring source and extending the theory to an arbitrary time dependent pulse. Also the energy flux over the far field is obtained. The method of solution was suggested by earlier work done by Professor F. Oberhettinger.

CHAPTER 2

A SIMPLE EXAMPLE DEMONSTRATING THE USE OF THE MODIFIED HELMHOLTZ EQUATION

The present dissertation obtains the solutions to the modified Helmholtz equation

$$\Delta u - \gamma^2 u = 0 \tag{1}$$

initially. These solutions are monotonic in nature and therefore converge more rapidly than an oscillatory solution. Upon obtaining the solution, γ is replaced by ik and the solution is obtained for

$$\Delta u + k^2 u = 0. \tag{2}$$

To more clearly demonstrate this, a simple example may be used in which the solution is known. Consider the sphere of radius a about the origin with a point source at the origin.

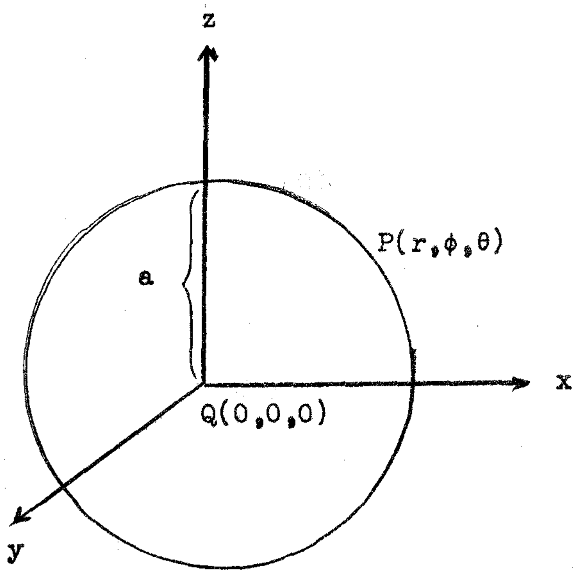

fig. 1

Then the first Green's function 22, p. 183 is

(3) $$G = -\sum \frac{u_n(P)u_n^*(Q)}{\gamma^2 + k_n^2}$$

where $\gamma = ik$ as stated earlier. But because of the symmetry, u is independent of θ and ϕ. Expressing equation (2) in polar coordinates we have

(4) $$\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} + k^2u = 0.$$

In this example since u is independent of and , (4) reduces to

$$\frac{1}{r}\frac{\partial^2(ru)}{\partial r^2} + k^2u = 0,$$

which has a solution

$$u = \frac{\sin kr}{r}.$$

The boundary condition u = 0 at r = a leads to the solutions for the eigenvalues

$$k = k_n = \frac{n\pi}{a}.$$

The condition of finiteness of the eigenfunctions is satisfied since

$$\lim_{r\to 0}\frac{\sin kr}{r} = k.$$

Let $u = k$ at $r = 0$. The normalization factor is obtained by integrating u^2 over the sphere.

$$N^2 = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{a} \left[\frac{\sin kr}{r}\right]^2 r^2 dr \sin\theta \, d\theta \, d\phi$$

$$= \int_0^a \sin^2(kr) \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi$$

$$= 2\pi \int_0^a \sin^2\left(\frac{n\pi r}{a}\right) dr \int_0^{\pi} \sin\theta \, d\theta$$

$$= 2\pi\left(\frac{a}{2}\right)(2) = 2\pi a.$$

Thus

$$N = \sqrt{2\pi a} \text{ and } \frac{u}{N} = \frac{1}{\sqrt{2\pi a}} \cdot \frac{\sin\left(\frac{n\pi r}{a}\right)}{r} = u_n.$$

Since $u_n^*(Q) = u_n^*(0) = \frac{n\pi}{a} \cdot \frac{1}{\sqrt{2\pi a}}$, from (3) the first Green's function is

$$\overline{G}_1 = -\frac{1}{2\pi a} \frac{\pi}{ar} \sum_{n=1}^{\infty} \frac{n \sin\left(\frac{n\pi r}{a}\right)}{\gamma^2 + \left(\frac{n\pi}{a}\right)^2}$$

$$= -\frac{1}{4\pi r} \cdot (2\pi) \sum_{n=1}^{\infty} \frac{n \sin(\frac{n\pi r}{a})}{\gamma^2 a^2 + n^2\pi^2}$$

$$= \frac{1}{4\pi r} \frac{\sinh(\gamma(r-a))}{\sinh(\gamma a)} .$$

This last step follows from the well known Fourier series:

$$\frac{\sinh(\gamma(r-a))}{\sinh(\gamma a)} = -2\pi \sum_{n=1}^{\infty} \frac{n \sin(\frac{n\pi r}{a})}{\gamma^2 a^2 + n^2\pi^2} .$$

Now for large values of a, we have the following asymptotic relations

$$\sinh(a) \sim \cosh(a) \sim e^{a}$$

and so the limit as $a \to \infty$ of $\overline{G}_1$ exists and

$$\lim_{a\to\infty} \overline{G}_1 = \frac{1}{4\pi r}(\sinh(\gamma r) - \cosh(\gamma r)) = -\frac{1}{4\pi r} \cdot e^{-\gamma r}.$$

Putting $\gamma = ik$ we get

$$G_1 = -\frac{1}{4\pi r} e^{-ikr},$$

the free space value for u.

On the other hand, had we not used the modified Helmholtz equation, we would have obtained

$$G_1 = \frac{1}{4\pi r} \frac{\sin(k(r-a))}{\sin(ka)} ,$$

the solution inside the sphere of radius a. But now if we try to get the free space solution by letting $a \to \infty$, we see that the limit as $a \to \infty$ for G_1 does not exist. The problem comes from the oscillatory nature of the sine function, whereas in the case of the modified Helmholtz equation, we were dealing with the hyperbolic functions which are monotonic.

CHAPTER 3

EXCITATION OF AN ELECTROMAGNETIC FIELD BY A RING SOURCE INSIDE A PERFECTLY CONDUCTING FINITE CONE

Consider a perfectly conducting cone with a slant height a and a spherical cap of radius a. Let the semi-opening angle be θ_0. The field is excited by an alternating current of frequency ω flowing along a circular path whose plane is perpendicular to the axis of the cone and whose center is on the axis of the cone.

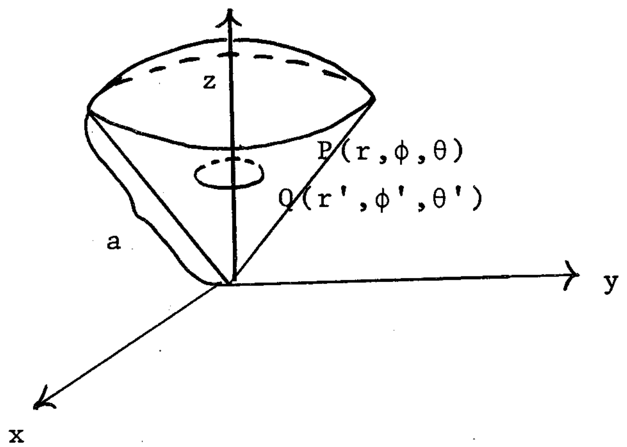

fig. 2

$P(r, \phi, \theta)$ is a point of observation inside the cavity and $Q(r', \phi', \theta')$ is a current element tangent to the circle. The coordinates are spherical.

$$x = r \cos \phi \sin \theta \qquad y = r \sin \phi \sin \theta \qquad z = r \cos \theta$$

The vector potential of the field at P due to the ring has only a ϕ component because of the symmetry. Since the direction of the vector potential for each current element is the same as that of the current element, we can show geometrically that the ϕ component is the only component by superimposing the point of observation P onto the plane of the ring. For sake of simplicity and without loss of generality we can take P above the x axis.

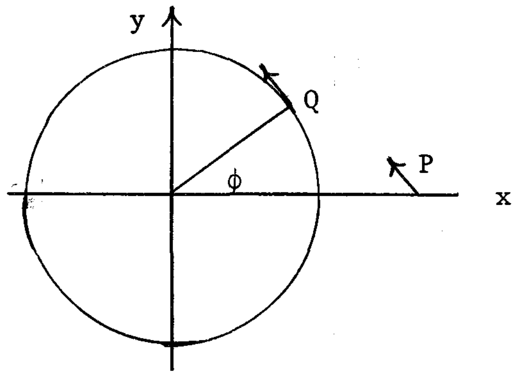

fig. 3

Let A be the strength of the source point Q at P, which is a function of the distance from P to Q.

$$A_x = - A \sin \phi',$$

$$A_y = A \cos \phi',$$

$$A_z = 0.$$

Clearly for each $Q(r', \phi', \theta')$ on the ring there is a corresponding point $Q_1(r', 2\pi - \phi', \theta')$ such that the distance from P to each of the two points is the same. Therefore the x components will be equal in magnitude but opposite in sign and

$$\int_{\phi'=0}^{2\pi} A_x \, d\phi' = 0.$$

The remaining contribution over the ring is A_y. But with P located as it is, $A_y = A_\phi$. Clearly the vector potential is a function of $(\phi - \phi')$ for fixed r and θ.

Therefore $A = (A_r, A_\phi, A_\theta)$ with $A_r = A_\theta = 0$ and $A_\phi \neq 0$.

Since our vector potential has only one non-zero component A_ϕ, with $u = A_\phi$, u satisfies Helmholtz's equation (2) with the frequency of the current and c the velocity of light. Then the electric and magnetic components of the field are

(5) $$E = \text{grad div } A + k^2 A, \quad H = \text{curl } A.$$

Because of the symmetry of the field, A_ϕ is independent of ϕ and

$$\text{div } A = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}(r^2 \sin\theta\, A_r) + \frac{\partial}{\partial\phi}(rA_\phi) + \frac{\partial}{\partial\theta}(r \sin\theta\, A_\theta)\right] = 0$$

Therefore $E = k^2 A$ or

(6) $$E_r = E_\theta \equiv 0, \quad E_\phi = k^2 A_\phi = k^2 u.$$

Curl A is $\nabla \times A$ and in spherical coordinates

$\nabla \times A =$

$$\left(\frac{1}{r^2 \sin\theta}(r \cos\theta\, A_\phi + r \sin\theta \frac{\partial A_\phi}{\partial\theta}),\ 0,\ -\frac{\cos\theta}{\sin\theta}A_\phi - \frac{\partial A_\phi}{\partial r}\right),$$

or

$$(7) \qquad H_r = \frac{1}{r \sin\theta}\left(A_\phi \cos\theta + \sin\theta \frac{\partial A_\phi}{\partial\theta}\right),$$

$$H_\phi = 0,$$

$$H_\theta = - A_\phi \frac{\cos\theta}{\sin\theta} - \frac{\partial A_\phi}{\partial r} .$$

The condition $u = 0$ on the boundary leads to the boundary conditions:

$$E_r = E_\phi = 0 \text{ for } \theta = \theta_0,$$

$$E_\phi = E_\theta = 0 \text{ for } r = a.$$

A_ϕ can be determined by determining the influence at P of a point source at Q, and then integrating over all such point sources. We start by finding the first Green's function using [22, p. 183]:

$$(8) \qquad G(P, Q) = \sum \frac{u_n(P)u_n^*(Q)}{k^2 - k_n^2} .$$

The u_n are the normalized eigenfunctions, with $u_n(P)$ the value of u_n at the point P. The $u_n^*(Q)$ are the complex conjugates at the source point Q. The k_n are the eigenvalues corresponding to the eigenfunctions u_n.

We start by finding G(P, Q) for Q(r', 0, θ'). A solution to (2) must satisfy

$$(9)\quad \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\,\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial f}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\,\frac{\partial f}{\partial\phi}\right)\right) + k^2 f = 0$$

in spherical coordinates. Such a solution periodic in ϕ is

$$(10)\quad f = \cos(m\phi)\, r^{-\frac{1}{2}}\, J_\nu(kr)\, P^m_{\nu-\frac{1}{2}}(\cos\theta),$$

$$m = 0, 1, 2, \ldots;\ \nu > \tfrac{1}{2}\,.$$

In order to establish that f is a solution to (9), consider first using [7, vol. 1, p. 148]

$$(11)\quad \frac{d}{d\theta} P^m_{\nu-\frac{1}{2}}(\cos\theta) = P^{m+1}_{\nu-\frac{1}{2}}(\cos\theta) + m\cot\theta\, P^m_{\nu-\frac{1}{2}}(\cos\theta).$$

From [7, vol. 1, p. 161]

$$\frac{d}{d\theta}\left(\sin\theta\,\frac{d}{d\theta} P^m_{\nu-\frac{1}{2}}(\cos\theta)\right) = \sin\theta\, P^{m+2}_{\nu-\frac{1}{2}}(\cos\theta) +$$

$$+ 2(m+1)\cos\theta\, P^{m+1}_{\nu-\frac{1}{2}}(\cos\theta) +$$

$$+ \left(\frac{m^2\cos^2\theta}{\sin\theta} - m\sin\theta\right) P^m_{\nu-\frac{1}{2}}(\cos\theta)$$

$$= \left[\sin\theta\left(-\nu^2 + \tfrac{1}{4}\right) + \frac{m^2}{\sin\theta}\right] P^m_{\nu-\frac{1}{2}}(\cos\theta).$$

Now we have

$$(12)\quad \frac{\partial}{\partial\theta}(\sin\theta\,\frac{\partial}{\partial\theta}f) = [(-\nu^2+\tfrac{1}{4})\sin\theta + \frac{m^2}{\sin\theta}]\,f.$$

For the term involving r, we use [7, vol. 2, p. 11]

$$\frac{d}{dr}(r^{-\frac{1}{2}}J_\nu(kr)) = -(\nu+\tfrac{1}{2})\,r^{-\frac{3}{2}}J_\nu(kr) + \frac{k}{\sqrt{r}}J_{\nu-1}(kr)$$

and

$$\frac{d}{dr}(r^2(\frac{k}{\sqrt{r}}J_{\nu-1}(kr) - (\nu+\tfrac{1}{2})\,r^{-\frac{3}{2}}J_\nu(kr)))$$

$$= [(\nu-\tfrac{1}{2})(\nu+\tfrac{1}{2})\frac{1}{\sqrt{r}} - k^2r^{\frac{3}{2}}]\cdot J_\nu(kr)$$

$$= [\nu^2 - k^2r^2 - \tfrac{1}{4}]\frac{1}{\sqrt{r}}J_\nu(kr)\ .$$

From this it follows that

$$(13)\quad \frac{\partial}{\partial r}(r^2\sin\theta\,\frac{\partial f}{\partial r}) = \sin\theta\,[\nu^2 - k^2r^2 - \tfrac{1}{4}]\,f\ .$$

For the partial derivatives with respect to ϕ,

$$\frac{d^2}{d\phi^2}\cos(m\phi) = -m^2\cos(m\phi),$$

and

$$(14)\quad \frac{\partial^2}{\partial\phi^2}f = -m^2 f.$$

Substituting (12), (13), and (14) into (9), we obtain

$$\left[\frac{1}{r^2}\left(-\nu^2+\tfrac{1}{4}\right)+\frac{m^2}{r^2\sin^2\theta}+\frac{1}{r^2}\left(\nu^2-r^2k^2-\tfrac{1}{4}\right)-\frac{m^2}{r^2\sin^2\theta}+k^2\right]f$$

$$= 0,$$

which verify that f is a solution of (9). The fact that f is periodic in ϕ with period $\frac{2\pi}{m}$ is obvious.

From our boundary conditions, f = 0 for r = a and $\theta = \theta_0$, we must have

(15) $$J_\nu\,(ka) = 0,$$

and

(16) $$P^m_{\nu-\frac{1}{2}}\,(\cos\,\theta_0) = 0.$$

Since a and θ_0 are fixed, we must choose ν as a root of (16) and k as the corresponding root of (15). We accomplish this by choosing

$$k = k_{\nu,n} = \tau_{\nu,n}/a,$$

where $\tau_{\nu,n}$ is the n-th positive root of $J_\nu\,(x) = 0$. But since all eigenvalues $k_{\nu,n}$ must be real [22, p. 170] it follows that all $\tau_{\nu,n}$ are real. However, according to a theorem by Lommel [23, ch. 15] this is only possible when ν is real. Let $\tau_{\nu,n}$ denote the n-th positive root of $J_\nu(x) = 0$ ($J_{-\nu}(x)$ has the same roots). The roots of (16) must then be real. But by [7, vol. 1, p. 144]

$$P^m_{-\nu-\frac{1}{2}}(x) = P^m_{\nu-\frac{1}{2}}(x).$$

Denote by $\alpha_{m,\ell}$ the ℓ-th positive root of (16):

$$\nu = \alpha_{m,\ell}, \quad k = k_{\nu,n} = \frac{\tau_{\nu,n}}{a}$$

Using [7, vol. 1, p. 159] we can write

$$(17)\quad P^m_{\nu-\frac{1}{2}}(\cos\theta_0) = (-1)^m \frac{\Gamma(\nu + m + \frac{1}{2})\sin^{-m}(\theta_0)}{\Gamma(\nu - m + \frac{1}{2})\Gamma(m + \frac{1}{2})} \cdot$$

$$\cdot\, \theta_0 \int_0^1 (\cos(\theta_0 x) - \cos\theta_0)^{m-\frac{1}{2}} \cos(\nu\theta_0 x)\, dx,$$

where $m = 0, 1, 2, \ldots$ and

$$(\nu + \tfrac{1}{2} - m) \neq 0, -1, -2, \ldots \quad .$$

If $(\nu + \frac{1}{2} - m)$ is a non-positive integer, then $P^m_{\nu-\frac{1}{2}}(\cos\theta_0)$ vanishes identically. From the integral we can see that the integrand is positive for $0 \leq x < \frac{\pi}{2\nu\theta_0}$. But for the integral to be zero, the integrand must be negative for some $0 < x < 1$, so

$$\frac{\pi}{2\nu\theta_0} < 1 \quad \text{and} \quad \nu > \frac{\pi}{2\theta_0} .$$

So $\nu > \frac{1}{2}$ if $\theta_0 \leq \pi$, which must be true in order to have a cone. Hence the finiteness condition for f is satisfied:

$$r^{-\frac{1}{2}} J_\nu(kr) = r^{-\frac{1}{2}} \sum_0^\infty \frac{(-1)^n (\frac{kr}{2})^{\nu+2n}}{n!\Gamma(\nu + n + 1)}$$

which is finite as $r \to 0$ for $\nu > \frac{1}{2}$.

The eigenfunctions which have yet to be normalized follow from (10):

$$(18) \qquad f_{m,\ell,n} = \cos(m\phi)\; r^{-\frac{1}{2}}\; J_\nu(kr)\; P^m_{\nu-\frac{1}{2}}(\cos\theta),$$

where $\nu = \alpha_{m,\ell}$ and $k = \frac{\tau_{\nu,n}}{a}$. Integrating f^2 over the cone, we obtain the square of the normalization factor N:

$$(19) \qquad N^2 = \int_0^{2\pi} \int_0^{\theta_0} \int_0^a [f_{m,\ell,n}]^2\; r^2 dr \sin\theta\; d\theta\; d\phi$$

$$= \int_0^{2\pi} \cos^2(m\phi) d\phi \int_0^a r J_\nu^2(kr)\; dr \int_0^{\theta_0} [P^m_{\nu-\frac{1}{2}}(\cos\theta)]^2 \sin\theta\, d\theta.$$

But from formula (10) of [7, vol. 2, p. 90] and (56) of [7, vol. 2, p. 12]

$$\int_0^a rJ_\nu^2\ (kr)\ dr$$

$$= \tfrac{1}{2}r^2\ [2J_\nu^2\ (kr) - 2J_{\nu+1}\ (kr)\ J_{\nu-1}\ (kr)]\ \Big|_0^a$$

$$= -\frac{a^2}{2}\ J_{\nu+1}\ (ka)\ J_{\nu-1}\ (ka)$$

$$= \frac{a^2}{2}\ [J_{\nu+1}\ (ka)]^2.$$

Now since $P^m_{\nu-\frac{1}{2}}$ (cos θ) as a function of ν satisfies the conditions of l'Hospital's rule we can use formula (1) of [7, vol. 1, p. 169] to evaluate

$$\int_0^{\theta_0} [P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta)]^2\ \sin\ \theta\ d\theta.$$

Writing one of the factors $P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta) = \lim_{\sigma\to\nu-\frac{1}{2}}\ P^m_\sigma\ (\cos\ \theta)$, we have

$$\int_0^{\theta_0} [P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta)]^2\ \sin\ \theta\ d\theta = \lim_{\sigma\to\nu-\frac{1}{2}}\ \frac{\sin\ \theta_0}{(\nu - \frac{1}{2} - \sigma)(\nu + \frac{1}{2} + \sigma)}\ \cdot$$

$$\cdot\ (P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta_0)\ \frac{d}{d\theta}\ P^m_\sigma\ (\cos\ \theta)\Big|_{\theta=\theta_0} - P^m_\sigma\ (\cos\ \theta_0)\ \frac{d}{d\theta}\ P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta)\Big|_{\theta=\theta_0})$$

$$= \lim_{\sigma\to\nu-\frac{1}{2}}\ [\frac{-\sin\ \theta_0}{(\nu - \frac{1}{2} - \sigma)(\nu + \frac{1}{2} + \sigma)}\ (P^m_\sigma\ (\cos\ \theta_0)\ \frac{d}{d\theta}\ P^m_{\nu-\frac{1}{2}}\ (\cos\ \theta)\Big|_{\theta=\theta_0})]$$

$$= \lim_{\sigma \to \nu - \frac{1}{2}} \left[\frac{-\sin \theta_0}{(\nu - \frac{1}{2} - \sigma)(\nu + \frac{1}{2} + \sigma)} P_\sigma^m (\cos \theta_0) P_{\nu-\frac{1}{2}}^{m+1} (\cos \theta_0) \right]$$

$$= \frac{-\sin \theta_0}{2\nu} P_{\nu-\frac{1}{2}}^{m+1} (\cos \theta_0) \lim_{\sigma \to \nu-\frac{1}{2}} \left[\frac{P_\sigma^m (\cos \theta_0)}{(\nu - \frac{1}{2} - \sigma)} \right]$$

$$= \frac{+\sin \theta_0}{2\nu} P_{\nu-\frac{1}{2}}^{m+1} (\cos \theta_0) \lim_{\sigma \to \nu-\frac{1}{2}} \left(\frac{d}{d\sigma} P_\sigma^m (\cos \theta_0) \right)$$

$$= \frac{\sin \theta_0}{2\nu} P_{\nu-\frac{1}{2}}^{m+1} (\cos \theta_0) \frac{d}{d\nu} \left(P_{\nu-\frac{1}{2}}^m (\cos \theta_0) \right),$$

where $\nu = \alpha_{m\ell}$. Next,

$$\int_0^{2\pi} \cos^2 (m\phi) \, d\phi = \begin{cases} 2\pi \text{ if } m = 0, \\ \pi \text{ if } m \neq 0 \end{cases}$$

$$= \frac{2\pi}{\varepsilon_m}$$

where ε_m is Neumann's number. Finally, we have

$$(20) \quad N^2 = \frac{\pi a^2}{2\nu\varepsilon_m} \sin \theta_0 \, P_{\nu-\frac{1}{2}}^{m+1} (\cos \theta_0) \frac{d}{d\nu} P_{\nu-\frac{1}{2}}^m (\cos \theta_0) \, [J_{\nu+1} (\tau_{\nu,n})]^2$$

where $\nu = \alpha_{m\ell}$. Hence the first Green's function is

$$G(P, Q) = (rr')^{-\frac{1}{2}} \frac{2}{\pi a^2 \sin \theta_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \varepsilon_m \nu \frac{\cos (m\phi)}{[k^2 - (\frac{\tau_{\nu,n}}{a})^2]} \cdot$$

$$\cdot \frac{J_\nu (\tau_{\nu,n} \frac{r}{a}) J_\nu (\tau_{\nu,n} \frac{r'}{a}) P^m_{\nu-\frac{1}{2}} (\cos \theta) P^m_{\nu-\frac{1}{2}} (\cos \theta')}{P^{m+1}_{\nu-\frac{1}{2}} (\cos \theta_0) \frac{d}{d\nu} P^m_{\nu-\frac{1}{2}} (\cos \theta_0) J^2_{\nu+1} (\tau_{\nu,n})}$$

where $\phi' = 0$, $\nu = \alpha_{m,\ell}$, and ε_m is Neumann's number.

If $\phi' \neq 0$, we need only replace ϕ by $(\phi - \phi')$ because of the symmetry of the cone. With this substitution we obtain

$$(21) \quad G(P, Q) = (rr')^{-\frac{1}{2}} \frac{2}{\pi a^2 \sin \theta_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \varepsilon_m \nu \frac{\cos (m(\phi - \phi'))}{k^2 - (\frac{\tau_{\nu,n}}{a})^2} \cdot$$

$$\cdot \frac{J_\nu (\tau_{\nu,n} \frac{r}{a}) J_\nu (\tau_{\nu,n} \frac{r'}{a}) P^m_{\nu-\frac{1}{2}} (\cos \theta) P^m_{\nu-\frac{1}{2}} (\cos \theta')}{P^{m+1}_{\nu-\frac{1}{2}} (\cos \theta_0) \frac{d}{d\nu} P^m_{\nu-\frac{1}{2}} (\cos \theta_0) J^2_{\nu+1} (\tau_{\nu,n})}$$

for general ϕ'. Summing over n, we have [7, vol. 2, p. 104], [7, vol. 2, p. 4]

$$(22) \quad \sum_{n=1}^{\infty} \frac{J_\nu (\tau_{\nu,n} \frac{r}{a}) J_\nu (\tau_{\nu,n} \frac{r'}{a})}{[k^2 - (\frac{\tau_{\nu,n}}{a})^2] J^2_{\nu+1} (\tau_{\nu,n})} = i\pi \frac{a^2}{4} \frac{J_\nu (kr)}{J_\nu (ka)} \cdot$$

$$\cdot [J_\nu (ka) H^{(2)}_\nu (kr') - J_\nu (kr') H^{(2)}_\nu (ka)], \quad r < r'.$$

This is a summation formula of the Fourier-Bessel type. If $r > r'$, we just interchange r and r'.

Using (22) for the summation over n in (21) the first Green's function is

$$(23) \qquad G(P,Q) = \frac{i(rr')^{-\frac{1}{2}}}{2\sin\theta_0} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \nu\, \varepsilon_m \cos[m(\phi-\phi')] .$$

$$\cdot \frac{P^m_{\nu-\frac{1}{2}}(\cos\theta) P^m_{\nu-\frac{1}{2}}(\cos\theta')}{P^{m+1}_{\nu-\frac{1}{2}}(\cos\theta_0)\, \frac{d}{d\nu} P^m_{\nu-\frac{1}{2}}(\cos\theta_0)} F_\nu(kr, kr')$$

where $\nu = \alpha_{m,\ell}$ and

$$(24) \qquad F_\nu(kr,kr') = \frac{J_\nu(kr)}{J_\nu(ka)} [J_\nu(ka)\, H^{(2)}_\nu(kr') - J_\nu(kr') H^{(2)}_\nu(ka)]$$

for $r < r'$. This is still just the magnitude of the contribution to A due to the directed point source Q. Recalling figure 3,

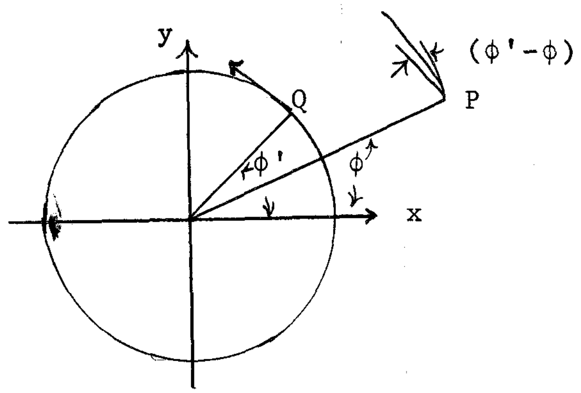

fig. 4

if we sum over all point sources, the contribution in the ϕ direction is the only one that remains. So

$$dA_\phi = G(P,Q)\cos(\phi - \phi')\,d\phi'$$

for an arbitrary point source Q, and

$$A_\phi = \int_0^{2\pi} G(P, Q)\cos(\phi - \phi')\,d\phi'$$

$$= \frac{i(rr')^{-\frac{1}{2}}}{2\sin\theta_0}\sum_{m=0}^{\infty}\sum_{\ell=0}^{\infty}\nu\,\varepsilon_m\,\frac{P^m_{\nu-\frac{1}{2}}(\cos\theta)P^m_{\nu-\frac{1}{2}}(\cos\theta')}{P^{m+1}_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}P^m_{\nu-\frac{1}{2}}(\cos\theta_0)}\;.$$

$$F_\nu(kr,kr')\int_0^{2\pi}\cos\,[m(\phi - \phi')]\cos(\phi - \phi')d\phi'.$$

But for $m \neq 1$ the integral is zero and for $m = 1$,

$$\int_0^{2\pi}\cos^2(\phi - \phi')\,d\phi' = \pi.$$

Therefore the vector potential $A = A_\phi$ due to the ring current is

$$(25)\quad A_\phi = \frac{i\pi(rr')^{-\frac{1}{2}}}{\sin\theta_0}\sum_{\ell=0}^{\infty}\nu\,\frac{P^1_{\nu-\frac{1}{2}}(\cos\theta)P^1_{\nu-\frac{1}{2}}(\cos\theta')}{P^2_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}P^1_{\nu-\frac{1}{2}}(\cos\theta_0)}F_\nu(kr,kr')$$

where $\nu = \beta_\ell$ are the positive roots of $P^1_{\nu-\frac{1}{2}}(\cos\theta_0) = 0$. From [7, vol. 1, p. 161],

$$(26)\qquad G_\nu(\theta, \theta') = \frac{P^1_{\nu-\frac{1}{2}}(\cos\theta)\, P^1_{\nu-\frac{1}{2}}(\cos\theta')}{P^2_{\nu-\frac{1}{2}}(\cos\theta_0)\, \frac{d}{d\nu} P^1_{\nu-\frac{1}{2}}(\cos\theta_0)}$$

$$= \frac{P^{-1}_{\nu-\frac{1}{2}}(\cos\theta)\, P^{-1}_{\nu-\frac{1}{2}}(\cos\theta')}{P_{\nu-\frac{1}{2}}(\cos\theta_0)\, \frac{d}{d\nu} P^{-1}_{\nu-\frac{1}{2}}(\cos\theta_0)}, \quad \nu = \beta_\ell .$$

We can now write

$$(27)\qquad A_\phi = \frac{i\pi(rr')^{-\frac{1}{2}}}{\sin\theta_0} \sum_{\ell=0}^{\infty} \beta_\ell\, G_{\beta_\ell}(\theta, \theta')\, F_{\beta_\ell}(kr, kr'),$$

or for the modified Helmholtz equation (1) we replace k by $-i\gamma$ in (27) to obtain

$$(28)\qquad \bar{A}_\phi = \frac{\pi i(rr')^{-\frac{1}{2}}}{\sin\theta_0} \sum_{\ell=0}^{\infty} \beta_\ell\, G_{\beta_\ell}(\theta, \theta')\, \bar{F}_{\beta_\ell}(\gamma r, \gamma r')$$

where

$$(29)\qquad \bar{F}_\nu(\gamma r, \gamma r') = F_\nu(-i\gamma r, -i\gamma r') =$$

$$\frac{2i}{\pi} \frac{I_\nu(\gamma r)}{I_\nu(\gamma a)} [I_\nu(\gamma a)\, K_\nu(\gamma r') - K_\nu(\gamma a)\, I_\nu(\gamma r')],$$

and $r < r'$. Here I_ν is the modified Bessel function and K_ν is the modified Hankel function.

As explained earlier, the transition from the Helmholtz equation to the modified Helmholtz equation can be considered an intermediate step to facilitate the analysis because of the monotonic behavior of the modified Bessel function. Once the solution is obtained, γ is replaced by ik and we have the solution to the original problem. If for instance $a \to \infty$ in (28) the semi-infinite cone is obtained.

Since for $\text{Re}\{\gamma\} > 0$, $K_\nu(\gamma a) \to 0$ and $I_\nu(\gamma a) \to \infty$ as $a \to \infty$, one obtains

$$(30) \qquad \overline{F}_\nu(\gamma r, \gamma r') = \frac{2i}{\pi} I_\nu(\gamma r) K_\nu(\gamma r'), \quad r < r'.$$

Now replacing γ by ik we have

$$(31) \qquad A_\phi = \frac{i\pi (rr')^{-\frac{1}{2}}}{\sin\theta_0} \sum_{\ell=0}^{\infty} \beta_\ell \, G_{\beta_\ell}(\theta, \theta') \, J_{\beta_\ell}(kr) H^{(2)}_{\beta_\ell}(kr')$$

which is the result obtained by Buchholz for the semi-infinite cone. Here through the use of the modified Helmholtz equation, we have obtained the semi-infinite cone solution as the limit of the solution of the finite cone as the slant height goes to infinity.

Now with (6), (7), and (27) we have the components of both the electric and magnetic fields for the ring source as described in the perfectly conducting cone, finite or semi-infinite.

In particular,

$$E_r = E_\theta \equiv 0, \quad E_\phi = k^2 A_\phi,$$

$$H_r = \frac{1}{r \sin\theta}(A_\phi \cos\theta) + \frac{1}{r}\frac{\partial A_\phi}{\partial\theta}$$

$$= \frac{\cos\theta}{r\sin\theta} A_\phi$$

$$+ \frac{i\pi(rr')^{-\frac{1}{2}}}{r \sin\theta_0} \sum_{\ell=0}^{\infty} \nu \frac{P_{\nu-\frac{1}{2}}(\cos\theta) P^{-1}_{\nu-\frac{1}{2}}(\cos\theta') F_\nu(kr, kr')}{P_{\nu-\frac{1}{2}}(\cos\theta_0)\frac{d}{d\nu}(P^{-1}_{\nu-\frac{1}{2}}(\cos\theta_0))} ,$$

$$H_\phi = 0,$$

$$H_\theta = -A_\phi \tan - \frac{i\pi(r')^{-\frac{1}{2}}}{\sin\theta_0}\sum_{\ell=0}^{\infty} \beta_\ell G_{\beta_\ell}(\theta,\theta')\frac{d}{dr}[\frac{1}{\sqrt{r}}F_{\beta_\ell}(kr,kr')]$$

$$= -A_\phi \tan\theta + (\nu + \tfrac{1}{2})\frac{1}{r}A_\phi = \frac{ki\pi(rr')^{-\frac{1}{2}}}{\sin\theta_0}\sum_{\ell=0}^{\infty} \beta_\ell G_{\beta_\ell}(\theta, \theta').$$

$$\cdot \frac{J_{\nu-1}(kr)}{J_\nu(ka)}[J_\nu(ka)H^{(2)}_\nu(kr') - J_\nu(kr')H^{(2)}_\nu(ka)]$$

$$= (\nu + \tfrac{1}{2})\frac{1}{r}A_\phi - \frac{ki\pi(rr')^{-\frac{1}{2}}}{\sin\theta_0}\sum_{\ell=0}^{\infty} \beta_\ell G_{\beta_\ell}(\theta, \theta') .$$

$$\cdot \frac{J_{\nu-1}(kr)}{J_\nu(ka)}[J_\nu(ka)H^{(2)}_\nu(kr') - J_\nu(kr')H^{(2)}_\nu(ka)]$$

$$- A_\phi \tan\theta.$$

For the infinite horn ($a \to \infty$) it follows from (31) and from the above equation that $H_r = O(r^{-2})$, $H_\phi \equiv 0$, $H_\theta = O(r^{-1})$, $E_\phi = O(r^{-1})$, and $E_\theta = E_r \equiv 0$ as $r \to \infty$. This takes into account the fact that $H^{(2)}_{\beta_\ell}(kr) = O(r^{-\frac{1}{2}})$ for large r. Hence only H_θ and E_ϕ contribute to the far field and the Poynting vector $S = cE \times H$, (c is a constant) has only a radial component S_r. The character of the far field is therefore a spherical wave. Thus the radiated energy

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\theta_0} S_r^2 r^2 \sin\theta \, d\theta \, d\phi$$

with $r \to \infty$ can easily be obtained. However, this computation is not carried out here.

CHAPTER 4

THE SPECIAL CASE $\theta_0 = \frac{\pi}{2}$, AND $a \to \infty$

In the special case $\theta_0 = \frac{\pi}{2}$ and $a \to \infty$, the boundary becomes the x, y plane. This is equivalent to a double ring in free space where the second ring is the mirror image in the x, y plane of the original ring, only oppositely charged. In this way the condition u = 0 on the x, y plane is satisfied.

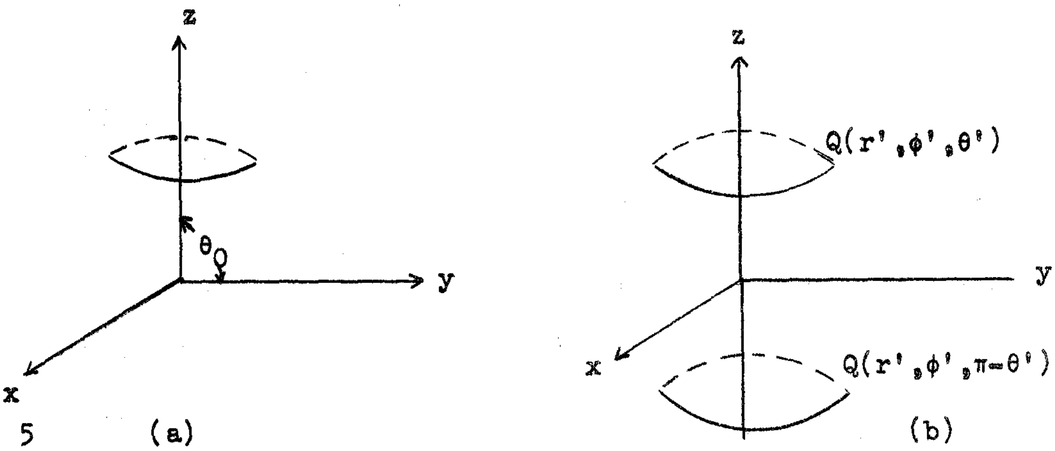

fig. 5 (a) (b)

In order to evaluate A_ϕ in the free space problem (fig. 5b) consider first an element $d\,A_\phi$ for $Q(r', \phi', \theta')$:

$$d\,\bar{A}_\phi = -\frac{1}{4\pi}\frac{e^{-\gamma\overrightarrow{PQ}}}{\overrightarrow{PQ}}\cos(\phi - \phi')$$

where $\overrightarrow{PQ} = \sqrt{r^2 + r'^2 - 2rr' \cos\xi}$,

$$\cos \xi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi').$$

$$d \bar{A}_\phi = - \frac{e^{-\gamma \overline{PQ}}}{4\pi \overline{PQ}} \cos (\phi - \phi') = - \frac{(rr')^{-\frac{1}{2}}}{4\pi} \cos (\phi - \phi').$$

$$\cdot \sum_{n=0}^{\infty} (2n + 1) I_{n+\frac{1}{2}} (\gamma r) K_{n+\frac{1}{2}} (\gamma r') P_n(\cos \xi), \quad r' > r.$$

Now using [7, vol. 1, p. 168]

$$P_n(\cos \xi) = \sum_{m=0}^{\infty} \varepsilon_m (-1)^m P_n^{-m}(\cos \theta) P_n^m(\cos \theta') \cos[m(\phi - \phi')],$$

and integrating over ϕ' from 0 to 2π, the only terms involving ϕ' are $\cos (\phi - \phi')$ and $\cos[m(\phi - \phi')]$:

$$\int_0^{2\pi} \cos (\phi - \phi') \cos [m(\phi - \phi')] \, d\phi' = \begin{cases} 0 \text{ if } m \neq 1 \\ \pi \text{ if } m = 1 \end{cases}.$$

So

$$(32) \quad \bar{A}_{\phi,1} = \frac{(rr')^{-\frac{1}{2}}}{4\pi} \sum_{n=0}^{\infty} (2n + 1) I_{n+\frac{1}{2}} (\gamma r) \cdot$$

$$\cdot K_{n+\frac{1}{2}} (\gamma r') \varepsilon_1 P_n^{-1}(\cos \theta) P_n^1(\cos \theta')\pi$$

$$= \frac{1}{2}(rr')^{-\frac{1}{2}} \sum_{n=0}^{\infty} (2n + 1) I_{n+\frac{1}{2}}(\gamma r).$$

$$\cdot K_{n+\frac{1}{2}}(\gamma r') P_n^{-1}(\cos\theta)\; P_n^1(\cos\theta')$$

which is the effect of the upper ring in free space. If we now consider the second ring, then θ' is replaced by $\pi - \theta'$ in (31). The total field due to the two rings would be

$$\overline{A}_\phi = \frac{1}{2}(rr')^{-\frac{1}{2}} \sum_{n=0}^{\infty} (2n + 1) I_{n+\frac{1}{2}}(\gamma r) K_{n+\frac{1}{2}}(\gamma r') P_n^{-1}(\cos\theta).$$

$$\cdot [P_n^1(\cos\theta') - P_n^1(-\cos\theta')].$$

From [7, vol. 1, p. 144]

$$P_n^1(-x) = (-1)^{n+1}\; P_n^1(x),$$

so

$$\text{(33)} \qquad \overline{A}_\phi = \frac{1}{2}(rr')^{-\frac{1}{2}} \sum_{n=0}^{\infty} (2n + 1) I_{n+\frac{1}{2}}(\gamma r).$$

$$\cdot\; K_{n+\frac{1}{2}}(\gamma r') P_n^{-1}(\cos\theta) P_n^1(\cos\theta') [1 - (-1)^{n+1}]$$

$$= (rr')^{-\frac{1}{2}} \sum_{n=0}^{\infty} (4n + 1)\, I_{2n+\frac{1}{2}}(\gamma r) \cdot$$

$$\cdot\, K_{2n+\frac{1}{2}}(\gamma r')\, P_{2n}^{-1}(\cos\theta)\, P_{2n}^{1}(\cos\theta').$$

Equation (33) represents the free space potential due to the double ring, where $\gamma = ik$.

Now using (28) and (30) with $\theta_0 = \pi/2$ for our special case, β_ℓ is the ℓ-th positive root of

$$P^{1}_{\nu-\frac{1}{2}}(\cos\theta_0) = 0.$$

With $\theta_0 = \pi/2$ we have $\cos\theta_0 = 0$ and

$$P^{1}_{\nu-\frac{1}{2}}(0) = 2\pi^{-\frac{1}{2}} \cos\left[\pi/2(\nu + \tfrac{1}{2})\right] \frac{\Gamma(\frac{\nu}{2} + \frac{3}{4})}{\Gamma(\frac{\nu}{2} + \frac{1}{4})} = 0,$$

from which we get

$$\nu = \beta_\ell = 2\ell + \frac{1}{2};\ \ell = 0, 1, 2, \ldots .$$

Using (26) for $G_\nu(\theta, \theta')$ we have [15, p. 171]

$$(34)\quad P_{\nu-\frac{1}{2}}(\cos\theta_0) = P_{\beta_\ell-\frac{1}{2}}(0) = P_{2\ell}(0) = (-1)^{\ell}\, \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{1}{2} + \ell)}{\Gamma(1 + \ell)}$$

and

$$P^{-1}_{\nu-\frac{1}{2}}(0) = \frac{1}{2}\,\pi^{-\frac{1}{2}}\cos[\pi/2(\nu - \frac{3}{2})]\frac{\Gamma(\frac{\nu}{2} - \frac{1}{4})}{\Gamma(\frac{\nu}{2} + \frac{5}{4})} .$$

Using [7, vol. 1, p. 3]

$$\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z) = \frac{\pi}{\cos(\pi z)} ,$$

it follows that

$$P^{-1}_{\nu-\frac{1}{2}}(0) = \frac{1}{2}\pi^{\frac{1}{2}}\frac{1}{\Gamma(\frac{5}{4} + \frac{\nu}{2})\ (\frac{5}{4} - \frac{\nu}{2})} .$$

Now since

$$\frac{d}{dz}\Gamma(z) = \psi(z)\ \Gamma(z),$$

we have

$$\frac{d}{d\nu}P^{-1}_{\nu-\frac{1}{2}}(0) = \frac{\sqrt{\pi}}{4}\ \frac{\psi(\frac{5}{4} - \frac{\nu}{2}) - \psi(\frac{5}{4} + \frac{\nu}{2})}{\Gamma(\frac{5}{4} + \frac{\nu}{2}) \cdot \Gamma(\frac{5}{4} - \frac{\nu}{2})} ,$$

But from [7, vol. 1, pp. 46-47],

$$\lim_{\nu\to 2\ell+\frac{1}{2}} \frac{\psi(\frac{5}{4} - \frac{\nu}{2})}{\Gamma(\frac{5}{4} - \frac{\nu}{2})} = (-1)^{\ell}(\ell - 1)! ,$$

and since $\psi(\frac{5}{4} + \frac{\nu}{2})$ remains finite,

$$\frac{\psi(\frac{5}{4} + \frac{\nu}{2})}{\Gamma(\frac{5}{4} + \frac{\nu}{2})\ \Gamma(\frac{5}{4} - \frac{\nu}{2})} \to 0 \text{ as } \nu \to \beta_\ell = 2\ell + \tfrac{1}{2} .$$

So

$$(35) \qquad \frac{d}{d\nu} P^{-1}_{\nu-\frac{1}{2}}(0) \Big|_{\nu=\beta_\ell} = \frac{\sqrt{\pi}}{4} \frac{(-1)^\ell (\ell - 1)!}{\Gamma(\ell + \frac{3}{2})}$$

and

$$(36) \qquad P_{\nu-\frac{1}{2}}(0) \frac{d}{d\nu} P^{-1}_{\nu-\frac{1}{2}}(0) \Big|_{\nu=2\ell+\frac{1}{2}} = \frac{1}{4\ell(\ell + \frac{1}{2})} .$$

Using formula (11) of [7, vol. 1, p. 161] we have

$$(37) \qquad P^{-1}_{2\ell}(\cos \theta') = - \frac{1}{2\ell(2\ell + 1)} P^{1}_{2\ell} (\cos \theta').$$

Now substituting (36) and (37) into (28) we have

$$\overline{A}_\phi = (rr')^{-\frac{1}{2}} \sum_{\ell=0}^{\infty} (4\ell + 1)\, I_{2\ell+\frac{1}{2}} (\gamma r) \cdot$$

$$\cdot\, K_{2\ell+\frac{1}{2}} (\gamma r')\, P^{-1}_{2\ell}(\cos \theta)\, P^{1}_{2\ell}(\cos \theta')$$

which is the same as (33).

CHAPTER 5

TRANSIENT SOLUTIONS

So far this dissertation has been concerned with a time harmonic electrical ring source. Since many applications have other time dependencies, let us now consider a source that satisfies a "Dirac" pulse, and use the result to evaluate the field for an arbitrary time dependency g(t).

The method to be employed here was used by Fritz Oberhettinger [20] for diffraction of waves and pulses by wedges and corners. Laplace transform theory is the basic tool.

In considering the "Dirac" pulse, let $\Phi_D(t - t_1)$ denote the field at time t due to a "Dirac" pulse excitation imposed at time t_1. Then [20, p. 359]

$$\Phi_D(t - t_1) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \bar{G}(P, Q, \gamma)\, e^{\gamma(t-t_1)}\, d\gamma$$

where $\bar{G}$ is the Green's function for the modified Helmholtz equation. From (28)

(38) $\Phi_D(t - t_1) =$

$$\frac{(rr')^{-\frac{1}{2}}}{2\sin\theta_0} \sum_{\ell=0}^{\infty} \beta_\ell\, G_{\beta_\ell}(\theta, \theta') \int_{c-i\infty}^{c+i\infty} \bar{F}_{\beta_\ell}(\gamma r, \gamma r')\, e^{\gamma(t-t_1)}\, d\gamma$$

for the ring source in the cone. For the semi-infinite cone we use (30) for $\overline{F}_{\beta_\ell}(\gamma r, \gamma r')$ and so we need the inverse Laplace transform

$$\mathcal{L}_{\gamma}^{-1}\{I_\nu(\gamma r)\, K_\nu(\gamma r')\}\ .$$

From [7, vol. 2, p. 96]

$$(39)\qquad I_\nu(\gamma r)\, K_\nu(\gamma r') = \int_0^\infty \frac{x}{x^2+\gamma^2}\, J_\nu(xr)\, J_\nu(xr')\, dx.$$

But $\dfrac{x}{x^2+\gamma^2} = \displaystyle\int_0^\infty e^{-\gamma t}\sin(xt)\, dt$, from [8, p. 50]. With this

substituted in (39) we have

$$(40)\quad I_\nu(\gamma r)\, K_\nu(\gamma r') = \int_0^\infty \left\{\int_0^\infty e^{-\gamma t}\sin(xt)\, dt\right\} J_\nu(xr)\, J_\nu(xr')\, dx$$

$$= \int_0^\infty e^{-\gamma t}\left\{\int_0^\infty \sin(xt)\, J_\nu(xr)\, J_\nu(xr')\, dx\right\} dt.$$

Inverting (40) we have

$$(41)\quad \mathcal{L}_{\gamma}^{-1}\{I_\nu(\gamma r)\, K_\nu(\gamma r')\} = \int_0^\infty \sin(xt)\, J_\nu(xr)\, J_\nu(xr')\, dx.$$

But the right hand side of (41) is a Fourier sine transform [8, p. 102] and so:

$$(42) \quad \mathcal{L}_{\gamma}^{-1}\{I_{\nu}(\gamma r)\ K_{\nu}(\gamma r')\} = \begin{cases} 0 \text{ for } 0 < t < r' - r \\ \frac{1}{2}(rr')^{-\frac{1}{2}}\ P_{\nu-\frac{1}{2}}\left(\frac{r^2 + (r')^2 - t^2}{2rr'}\right) \\ \qquad \text{for } r' - r < t < r' + r \\ -\frac{\cos(\nu\pi)}{\pi(rr')^{\frac{1}{2}}}\ Q_{\nu-\frac{1}{2}}\left(\frac{t^2 - (r^2 + (r')^2)}{2rr'}\right) \\ \qquad \text{for } r + r' < t < \infty \end{cases}$$

where $Q_{\nu-\frac{1}{2}}$ is the ordinary Legendre function of the second kind. Now we have

$$(43) \quad \Phi_D(t - t_1) = \frac{(rr')^{-\frac{1}{2}}}{2\sin\theta_0} \sum_{\ell=0}^{\infty} \beta_\ell\ G_{\beta_\ell}(\theta, \theta')\ f_{\beta_\ell}(r, r', t - t_1)$$

with

$$(44) \quad f_{\beta_\ell}(r, r', t) = \mathcal{L}_{\gamma}^{-1}\{I_{\nu}(\gamma r)\ K_{\nu}(\gamma r')\}$$

where we have the semi-infinite cone. In the case of the finite cone we also need

$$\mathcal{L}_{\gamma}^{-1}\left\{\frac{I_{\nu}(\gamma r)}{I\nu(\gamma a)}\ K_{\nu}(\gamma a)\ I_{\nu}(\gamma r')\right\}$$

which is not known.

For the case of an arbitrary pulse, let $g(t)$ represent the pulse function. Define $g(t) \equiv 0$ for $t < 0$. Then the field is represented by [20, p. 359]

$$(45) \qquad \Phi(t) = \int_0^\infty g(\tau)\ \Phi_D(\tau - \tau)\, d\tau \ .$$

Concluding remark

The same analysis can be carried out for the case of a magnetic ring excitation. This would require the computation of the second Green's function. It may also be pointed out that the limiting case "diameter of the ring source very small" represents the excitation of the field by a dipole on and oriented along the axis. Specifically: a magnetic or an electric dipole as the limiting case for the electric or magnetic ring source respectively.

BIBLIOGRAPHY

1. Adachi, S. A theoretical analysis of semi-infinite conical antennas. IRE Transactions on Antennas and Propagation AP-8:534-547. 1960.

2. Banos, Jr., A., R.H. Huddlestone, D.C. Pridemore-Brown, and E.C. Taylor. The radiation field of an electric dipole antenna in a conical sheath. Journal of Mathematics and Physics 44:189-213. 1965.

3. Borgnis, F.E. and C.H. Papas. Electronmagnetic wave guides and resonators. In: Encyclopedia of physics, ed. by S. Flugge. Vol. 16. Berlin, Springer, 1958. p. 284-422.

4. Buchholz, H. Die Bewegung elektromagnetischer Wellen in einem kegelformigen Horn. Annalen der Physik, ser. 5, 37: 173-225. 1940.

5. Carslaw, H.S. The scattering of sound waves by a cone. Mathematische Annalen 75: 133-147. 1914.

6. Carslaw, H.S. and J.C. Jaeger. The determination of Green's function for the equation of conduction of heat in cylindrical coordinates by the Laplace transformation. Journal of London Mathematical Society 15: 273-281. 1940.

7. Erdelyi, A. et al. Higher transcendental functions. New York, McGraw-Hill. 1953. 3 vol.

8. Erdelyi, A. et al. Tables of integral transforms. Vol. 1, New York, McGraw-Hill. 1954. 391 p.

9. Felsen, F.B. Alternative field representations in regions bounded by spheres, cones, and planes. IRE Transactions on Antennas and Propagation AP-5:109-121. 1957.

10. ________ Radiation from ring sources in the presence of a semi-infinite cone. IRE Transactions on Antennas and Propagation AP-7:168-180. 1959.

11. ________ Electromagnetic properties of wedge and cone surfaces with linearly varying surface impedances. IRE Transactions on Antennas and Propagation AP-7:S231-S243. 1959.

12. Harrington, R. F. Time-harmonic electromagnetic fields. New York, McGraw-Hill. 1961. 480 p.

13. Keller, J. B. Backscattering from a finite cone. IRE Transactions on Antennas and Propagation AP-8:175-182. 1960.

14. Leitner, A. and C. P. Wells. Radiation by discs and conical structures. IRE Transactions on Antennas and Propagation AP-4:637-640. 1956.

15. Magnus, W., F. Oberhettinger and R. P. Soni. Formulas and theorems for the special functions of mathematical physics. New York, Springer. 1966. 508 p.

16. Muki, R. and E. Sternberg. Steady-state heat conduction in a circular cone. Zeitschrift für Angewandte Mathematik und Physik 11:306-315. 1960.

17. Northover, F. H. The diffraction of electromagnetic waves around a finite, perfectly conducting cone. Quarterly Journal of Mechanics and Applied Mathematics 15:1-9. 1962.

18. Oberhettinger, F. Diffraction of waves by a wedge. Communications on Pure and Applied Mathematics 7:551-563. 1954.

19. ____________ Tabellen zur Fourier Transformation. Berlin, Springer. 1957. 213 p.

20. ____________ On the diffraction and reflection of waves and pulses by wedges and corners. Journal of Research of the U. S. National Bureau of Standards 61:343-365. 1958.

21. Pridemore-Brown, D. C. The radiation field of a magnetic dipole antenna in a conical sheath. Journal of Mathematics and Physics 43:199-217. 1964.

22. Sommerfeld, A. Partial differential equations in physics. New York, Academic, 1964. 335 p.

23. Watson, G. N. Treatise on the theory of Bessel functions. Cambridge, Cambridge University, 1948. 804 p.