AN ABSTRACT OF THE THESIS OF

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Abstract approved: ______________________
David S. Carter

D. S. Carter has described the following model for security prices. Let $z(t) = (z_i(t))$, $(t \geq 0)$, $(i=1,\ldots,n)$ be a continuous stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t, t \geq 0)$ be a nondecreasing family of sub σ-fields of $\mathcal{F}$ to which z is adapted. Here $z(t)$ is a vector of security prices and $\mathcal{F}_t$ contains the information known up to and including time t. Suppose there are stochastic processes $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t))$, $(t \geq 0)$, $(i,j=1,\ldots,n)$ such that the following moment rate conditions hold:

(1) $$\lim_{h\downarrow 0} \frac{1}{h} E[z_i(t+h)-z_i(t) \mid \mathcal{F}_t] = a_i(t) \quad \text{a.s.}$$

(2) $$\lim_{h\downarrow 0} \frac{1}{h} E[(z_i(t+h)-z_i(t))(z_j(t+h)-z_j(t)) \mid \mathcal{F}_t] = B_{ij}(t) \quad \text{a.s.}$$

(3) $$\lim_{h\downarrow 0} \frac{1}{h} E[\prod_{k=1}^{m} | z_{i_k}(t+h)-z_{i_k}(t) | \mid \mathcal{F}_t] = 0 \quad \text{a.s.}$$

for some $m \geq 3$.

The question arose as to how this compares with models in which prices are given by stochastic integrals. This thesis discusses the relationship between stochastic integrals and processes satisfying conditions (1), (2) and (3) for various values of m.

It is shown that for each value of m a broad class of Ito stochastic integrals satisfy (1), (2) and (3). Furthermore, under reasonable restrictions on the processes involved, a process z that satisfies (1), (2) and (3) with $m = 3$ is representable as an Ito stochastic integral.

A preliminary result, of interest in itself, is that sufficient conditions are obtained for the conditional expectations of a continuous stochastic process to have continuous versions.

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I INTRODUCTION

Motivated by a search for a model of security prices, D. S. Carter suggested a stochastic model as described below. Carter was interested in generalizing the standard models in which security prices are represented by diffusion processes, in such a way as to avoid the Markov and independence assumptions of diffusion processes while retaining a characterization in terms of drift and covariance functions.

To describe Carter's model let $z(t)=(z_i(t)),(t\geqslant 0),(i=1,\dots,n)$ be a continuous stochastic process on the probability space $(\Omega,\mathcal{F},P)$. Let $(\mathcal{F}_t,\ t\geqslant 0)$ be an increasing family of sub σ-fields of $\mathcal{F}$ to which z is adapted. Here $z(t)$ is a vector of the security prices at time t and $\mathcal{F}_t$ contains the information known up to and including time t. Then there are stochastic processes $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t)),(t\geqslant 0),(i,j=1,\dots,n)$ such that

(1) $$\lim_{h\downarrow 0} \frac{1}{h} E[z_i(t+h)-z_i(t) \mid \mathcal{F}_t] = a_i(t) \quad \text{a.s. } (i=1,\dots,n),$$

(2) $$\lim_{h\downarrow 0} \frac{1}{h} E[(z_i(t+h)-z_i(t))(z_j(t+h)-z_j(t)) \mid \mathcal{F}_t] = B_{ij}(t) \quad \text{a.s.}$$

$(i,j=1,\dots n)$ and

$$(3)\quad \lim_{h\downarrow 0} \frac{1}{h} E\left[\prod_{k=1}^{m} |z_{i_k}(t+h)-z_{i_k}(t)| \;\middle|\; \mathcal{F}_t\right] = 0 \quad \text{a.s.}$$

for some $m \geq 3$, where $(i_k : k=1,\ldots,m)$ is any sequence of m indices from $\{1,\ldots,n\}$. Here a is the drift vector and B is the covariance matrix.

The defining equations for a diffusion process are similar to (1), (2) and (3). Specifically, a stochastic process $\xi(t) = (\xi_i(t))$, $(t \geq 0)$, $(i=1,\ldots,n)$ is a <u>diffusion process</u> if ξ is a continuous Markov process and there exist functions $\mu_i, v_{ij} : R^n \times [0,\infty) \to R$ such that for all $t \geq 0$ and $(i,j=1,\ldots,n)$

$$(4)\quad \lim_{h\downarrow 0} \frac{1}{h} E[\xi_i(t+h)-\xi_i(t) \mid \xi(t)=x] = \mu_i(x,t)$$

$$(5)\quad \lim_{h\downarrow 0} \frac{1}{h} E[(\xi_i(t+h)-\xi_i(t))(\xi_j(t+h)-\xi_j(t)) \mid \xi(t)=x] = V_{ij}(x,t)$$

$$(6)\quad \lim_{h\downarrow 0} \frac{1}{h} E[\| \xi(t+h)-\xi(t)\|^{2+\delta} \mid \xi(t)=x] = 0$$

for some $\delta > 0$, where $\| x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$.

There is a close relationship between diffusion processes and stochastic integrals. Let $\mu = (\mu_i)$ be a vector, and $\sigma = (\sigma_{ij})$ a matrix, of functions from $R^n \times [0,\infty) \to R$. Consider the stochastic integral equation

$$(7) \quad \xi(t) = \xi(0) + \int_0^t \mu(\xi(s),s)\,ds + \int_0^t \sigma(\xi(s),s)\,dw(s)$$

where w is an n-dimensional Brownian motion. Ito (1951) has shown, under suitable continuity and growth conditions on μ and σ, that there is a unique solution to (7) satisfying

$$E\left[\int_0^T \xi_i^2(t)\,dt \right] < \infty$$

for all $T \geqslant 0$ and $i=1,\dots,n$. Furthermore, this solution is a diffusion process with

$$V_{ij} = \sum_{k=1}^{n} \sigma_{ik}\,\sigma_{jk} \qquad ,$$

in correspondence with the notation in (5), [see Friedman (1975, Ch. 5) for an account more in line with our notation].

With this in mind, the question arose as to how Carter's model compares with models in which prices are represented by stochastic integrals. The main result of this thesis is to show, under reasonably broad restrictions on the stochastic processes involved, that these two approaches are mathematically equivalent. In section 3.2 we give conditions under which an Ito stochastic integral

$$(8) \quad z(t) = z(0) + \int_0^t a(s)\,ds + \int_0^t b(s)\,dw(s)$$

satisfies equations (1), (2) and (3) for fixed values of m with

$$B_{ij} = \sum_{k=1}^{n} b_{ik}\, b_{jk} \quad .$$

Section 3.3 is devoted to showing under suitable hypotheses -- including the assumptions that $a = (a_i)$ and $B = (B_{ij})$, $(i,j=1,\ldots,n)$ are bounded and that B is strictly positive definite -- that a process satisfying (1), (2) and (3) with $m = 3$ can be represented by a stochastic integral. Roughly speaking, we use (1), (2) and (3) to integrate

$$\lim_{h \downarrow 0} \frac{1}{h} E[f(z(t+h)) - f(z(t)) \mid \mathcal{F}_t]$$

with respect to t for sufficiently smooth functions $f : R^n \to R$, to obtain a form of Ito's formula. This allows us to apply a result of Stroock and Varadhan (1979), which states under certain restrictions that if a stochastic process z satisfies Ito's formula with a and B bounded, then there exists an n-dimensional Brownian motion w such that (8) holds with $bb^T = B$. When B is only nonnegative definite, we show that there is a stochastic integral (possibly on a different probability space) that has the same finite dimensional probability distributions as z.

In Chapter II the necessary background material in probability theory, continuous parameter stochastic processes and Ito integrals is briefly sketched with references given in lieu of proofs. Little effort has been made to reference the original sources. It is assumed that the reader has a basic knowledge of measure theory.

Chapter II contains one noteworthy result which we were unable to find in the literature. In section 2.6 the theory of Banach space-valued random variables is used to establish a sufficient condition for the conditional expectations of a continuous stochastic process to have continuous versions.

II MATHEMATICAL PRELIMINARIES

2.1 Probability Spaces and Conditional Expectation

Let $(\Omega, \mathcal{F}, P)$ be a measure space where Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω and P is a positive σ-additive measure defined on $\mathcal{F}$. If $P(\Omega) = 1$ then $(\Omega, \mathcal{F}, P)$ is a probability space. When dealing with probability spaces, the term "almost surely (a.s.)" is used interchangeably with "almost everywhere (a.e.)". On a probability space convergence in measure is called convergence in probability.

A real valued measurable function on a probability space is called a random variable. If a random variable f is integrable then we write $E[f]$ for $\int_\Omega f\,dP$ and we call this integral the expectation of f. An n-dimensional random variable is an R^n- valued measurable function on a probability space. Most of the definitions and results for random variables can be extended to n-dimensional random variables by dealing with their components individually.

Let $\mathcal{E}$ be a sub σ-field of $\mathcal{F}$ and f an integrable random variable. The conditional expectation of f with respect to $\mathcal{E}$ is defined to be any $\mathcal{E}$-measurable random variable, $E[f \mid \mathcal{E}]$, such that for all $C \in \mathcal{E}$

$$\int_C f\,dP = \int_C E[f \mid \mathcal{E}]\,dP.$$

The Radon-Nikodym Theorem gives existence and uniqueness a.s. of $E[f \mid \mathcal{E}]$.

Following is a list of properties of conditional expectation. In this list f and g are integrable random variables on a probability space $(\Omega, \mathcal{F}, P)$, a and b are real numbers, $\mathcal{E}$ and $\mathcal{G}$ are sub σ-fields of $\mathcal{F}$.

<u>2.1.1</u> [Ash (1972, Ch. 6)] (a) If f is equal to a constant k a.s. then $E[f \mid \mathcal{E}] = k$ a.s.

(b) If $f \leqslant g$ a.s. then $E[f \mid \mathcal{E}] \leqslant E[g \mid \mathcal{E}]$ a.s.

(c) $|E[f \mid \mathcal{E}]| \leqslant E[|f| \mid \mathcal{E}]$ a.s.

(d) $E[af + bg \mid \mathcal{E}] = a\,E[f \mid \mathcal{E}] + b\,E[g \mid \mathcal{E}]$ a.s.

(e) $E[f \mid \mathcal{F}] = f$ a.s.

(f) $E[f \mid \{\emptyset, \Omega\}] = E[f]$ a.s.

(g) If $\mathcal{E} \subset \mathcal{G}$ then

$$E[E[f \mid \mathcal{G}] \mid \mathcal{E}] = E[f \mid \mathcal{E}] \quad \text{a.s.}$$

and

$$E[E[f \mid \mathcal{E}] \quad \mathcal{G}] = E[f \mid \mathcal{E}] \quad \text{a.s.}$$

In particular

$$E[E[f \mid \mathcal{E}]] = E[f] \quad \text{a.s.}$$

(h) If g is $\mathcal{E}$-measurable and fg is integrable then $E[fg \mid \mathcal{E}] = g\, E[f \mid \mathcal{E}]$ a.s.

We also have the Dominated Convergence Theorem for conditional expectations:

2.1.2 [Ash (1972, p. 257)] Let f_m be a sequence of random variables such that

$$\lim_{m\to\infty} f_m = f \quad \text{a.s.}$$

If there is an integrable random variable g such that

$$| f_m | \leqslant g ,$$

then $E[f \mid \mathcal{E}]$ exists and

$$\lim_{m\to\infty} E[f_m \mid \mathcal{E}] = E[f \mid \mathcal{E}] \quad \text{a.s.}$$

Since conditional expectation is order preserving (2.1.1 (b)), it is a simple exercise to show that Hölder's inequality is true for conditional expectations.

2.1.3 Let p and q be real numbers greater than 1 such that $1/p + 1/q = 1$. If $|f|^p$ and $|g|^q$ are integrable then $|fg|$ is integrable and

$$E[|fg| \mid \mathcal{G}\,] \leqslant (E[|f|^p \mid \mathcal{G}\,])^{1/p}(E[|g|^q| \,\mathcal{G}\,])^{1/q} \quad \text{a.s.}$$

A useful consequence of Hölder's inequality occurs when $g = 1$:

<u>2.1.4</u> If $p \geqslant 1$ and $|f|^p$ is integrable then

$$E[|f| \mid \mathcal{G}\,]^p \leqslant E[|f|^p \mid \mathcal{G}\,] \quad \text{a.s.}$$

2.2 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A finite family $\mathcal{C}_i$, $(i = 1,\dots,m)$ of sub σ-fields of $\mathcal{F}$ is independent if for all $C_i \in \mathcal{C}_i$, $(i = 1,\dots,m)$

$$P(C_1 \cap \cdots \cap C_m) = P(C_1) \cdots P(C_m) \quad .$$

A family $\mathcal{G}_i$, $(i \in I)$ of sub σ-fields of $\mathcal{F}$ is independent if every finite subcollection is independent.

If f_i, $(i \in J)$ is a collection of random variables then $\sigma(f_i, i \in J)$ will denote the smallest σ-field with respect to which each f_i is measurable. A collection of random variables f_i, $(i \in J)$ is independent if the collection of σ-fields $\sigma(f_i)$, $(i \in J)$ is independent. Often, for convenience, we will say that two σ-fields, or two random variables, are independent when the collection consisting of the two objects is independent. When we say that a random variable g is independent of a σ-field $\mathcal{C}$, we mean that $\sigma(g)$ and $\mathcal{C}$ are independent.

2.2.1 [Loève (1978, p. 15)] If the integrable random variable f and the σ-field $\mathcal{C}$ are independent, then

$$E[f \mid \mathcal{C}] = E[f] \quad \text{a.s.}$$

2.2.2 [Ash (1972, p. 227)] If the integrable random variables f_i, $(i = 1,\ldots,m)$ are independent, then

$$E[f_1 \cdots f_m] = E[f_1] \cdots E[f_m] .$$

2.3. Stochastic Processes

An n-dimensional stochastic process on a probability space $(\Omega, \mathcal{F}, P)$ is an R^n-valued function of two variables $f : I\times\Omega \to R^n$ where I, the index set, is any set such that for each $t \in I$ the function $f(t, \cdot) : \Omega \to R^n$ is an n-dimensional random variable. We will denote the random variable $f(t, \cdot)$ by $f(t)$. We will use the notation $f(t) = (f_i(t))$, $(t \in I)$, $(i = 1,\ldots,n)$ to denote an n-dimensional stochastic process where $f_i(t)$ is the i^{th} component. We will refer to this process as f. In our applications, I will usually be an interval of real numbers. For a fixed $\omega \in \Omega$ the function $f(\cdot,\omega) : I \to R^n$ is called a sample function or sample path of f.

A collection f_i, $(i \in J)$ of stochastic processes, with a common index set I, is independent if the collection of σ-fields $\sigma(f_i(t), t \in I)$, $(i \in J)$ is independent.

Given a one-dimensional stochastic process $f(t)$, $(t\in I)$, let $D = (t_1,\ldots,t_r)$ be a finite sequence of distinct elements of I. Then let $f_D = (f(t_1),\ldots,f(t_r))$ and notice that f_D is an r-dimensional random variable. We use f_D to induce a probability measure Φ_D on $(R^r, \mathcal{B}^r)$ by the formula

$$\Phi_D(E) = P(f_D^{-1}(E)) \quad \text{for} \quad E \in \mathcal{B}^r .$$

Let $\mathcal{D}$ be the collection of all finite sequences of distinct elements of I. Then $\{\Phi_D : D \in \mathcal{D}\}$ is called the system of finite dimensional probability distributions determined by f.

If $f(t)$ and $g(t)$, $(t \in I)$ are two stochastic processes on a probability space, we say that f and g are stochastically equivalent if for every $t \in I$, $f(t) = g(t)$ a.s. We then say that g is a version of f. Often, given a process f we will need to find another process which is a version of f and satisfies certain conditions. When the original process is no longer referred to we will use the same symbol for the version of f as for f. The next result shows that two versions of the same process are equivalent for empirical purposes.

2.3.1 [Yeh (1973, p. 3)] Two versions of the same stochastic process have the same system of finite dimensional probability distributions.

2.4 Separability

Let $f(t)$, $(t \in I)$ be a one-dimensional stochastic process where I is an interval of real numbers. Unfortunately functions such as $h = \lim_{t \to c} f(t)$ or $g = \sup (f(t) : t \in I)$ may or may not be random variables. As an example consider a set of the form

$$A = \{\omega : g(\omega) > d\} = \bigcup_{t \in I} \{\omega : f(t,\omega) > d\} \quad .$$

The union on the right involves an uncountable number of sets. Even in the cases where A is measurable, $P(A)$ is not in general determined by the finite dimensional probability distributions of f (for an example, see Ash and Gardner (1975, p. 161)). The notion of separable stochastic processes was introduced to avoid these difficulties.

Suppose that I is an interval. A stochastic process $f(t)$, $(t \in I)$ on a probability space $(\Omega, \mathcal{F}, P)$ is <u>separable</u> if there is a countable dense subset I_0 of I, called the <u>separating set</u>, and a set $N \in \mathcal{F}$ with $P(N) = 0$, called the <u>negligible set</u>, such that the following condition holds for $\omega \notin N$:

If A is a closed interval and J is an open interval and $f(t,\omega) \in A$ for all $t \in I_0 \cap J$, then $f(t,\omega) \in A$ for all $t \in I \cap J$.

<u>2.4.1</u> [Doob (1953, p. 55)] Let $f(t)$, $(t \in I)$ be a separable stochastic process with separating set I_0 and negligible set N.

Suppose $\omega \notin N$, $t_0 \in I$, and

$$\lim_{t \to t_0, t \in I_0} f(t,\omega)$$

exists (and is thus a random variable due to the countability of I_0). Then

$$\lim_{t \to t_0, t \in I} f(t,\omega)$$

exists and the two limits are equal.

An <u>extended random variable</u> is a measurable function from a probability space to the extended real line. An <u>extended stochastic process</u> is a function $f : I \times \Omega \to \overline{R}$ such that for each $t \in I$, $f(t,\cdot)$ is an extended random variable. The concepts and results discussed so far for random variables and stochastic processes are also valid in the extended case.

<u>2.4.2</u> [Doob (1953, p. 53)] If $f(t)$, $(t \in I)$ is a separable stochastic process, then $\sup (f(t) : t \in I)$ is an extended random variable.

Statements (2.4.1) and (2.4.2) will allow us to study some sample path properties of separable processes. We still need the existence of separable versions.

2.4.3 (Separability Theorem), [Doob (1953, p. 57)] Let $f(t)$, $(t \in I)$ be a stochastic process where I is an interval. Then there exists a separable extended real valued stochastic process g that is a version of f.

Notice that by (2.3.1), replacing a process by any of its separable versions does not change the finite dimensional probability distributions.

2.5 Integration of Sample Paths

A stochastic process $f(t)$, $(t \in I)$ on a probability space $(\Omega, \mathcal{F}, P)$ is <u>measurable</u> if I is a Borel measurable subset of R and $f : I\times\Omega \to R$ is a measurable function on the product space $I\times\Omega$.

The following result, which is an application of Fubini's Theorem, uses measurability to justify the integration of sample functions.

<u>2.5.1</u> [Doob (1953, p. 62)] Let $f(t)$, $(t \in I)$ be a measurable stochastic process. Then

(i) almost all sample functions of f are measurable functions of t.

(ii) if $E[f(t)]$ exists for all $t \in I$ then $E[f(t)]$ is a measurable function of t.

(iii) if $D \subset I$ is Borel measurable and $E[|f(t)|]$ is Lebesgue integrable over D, then almost all sample functions of f are Lebesgue integrable over D and $\int_D f(t)\,dt$ is a random variable.

To establish conditions under which we can interchange an integral and a conditional expectation, we need the following result:

<u>2.5.2</u> <u>Lemma</u> If h and k are $\mathcal{E}$-measurable random variables on the probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{E}$ is a sub σ-field of

$\mathcal{F}$ and $E[\eta h] = E[\eta k]$ for every bounded $\mathcal{G}$-measurable random variable η then $h = k$ a.s.

Proof: Let

$$\eta(\omega) = \begin{cases} 1 & \text{if } h(\omega) - k(\omega) \geq 0 \\ -1 & \text{if } h(\omega) - k(\omega) < 0 \end{cases} .$$

Since η is a bounded $\mathcal{G}$-measurable random variable, we have $E[\eta(h-k)] = 0$. Hence $E[\,|h-k|\,] = 0$, and $h = k$ a.s. □

2.5.3 Theorem Let $f(t)$, $(t \in I)$ be a measurable stochastic process on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ a sub σ-field of $\mathcal{F}$. Suppose that $E[\,|f(t)|\,]$ is Lebesgue measurable and $\int_I E[|f(t)|]\,dt < \infty$. If there exists a measurable version of the process $E[f(t)|\mathcal{G}]$, $(t \in I)$, then for any such version

$$\int_I E[f(t)|\mathcal{G}]\,dt = E\left[\int_I f(t)\,dt \,\middle|\, \mathcal{G}\right]$$

Proof: Let η be any bounded $\mathcal{G}$-measurable random variable. In view of the properties of conditional expectations (2.1.1) we see that,

$$\int_I E[\,|\eta\, E[f(t) \mid \mathcal{G}]\,|\,]\,dt$$

$$\leq \int_I E[\,E[\,|\eta\, f(t)| \mid \mathcal{G}\,]\,]\,dt$$

$$= \int_I E[\,|\eta f(t)|\,]\,dt$$

$$< \infty \quad \text{since } \eta \text{ is bounded.}$$

Using this and the hypothesis that $E[|f(t)|]$ is Lebesgue integrable, we may apply Fubini's Theorem twice to obtain

$$E[\eta \int_I E[f(t)|\mathcal{E}]\,dt\,]$$
$$= \int_I E[\eta\, E[f(t)\,|\,\mathcal{E}\,]\,]\,dt$$
$$= \int_I E[\eta\, f(t)]\,dt$$
$$= E[\eta \int_I f(t)\,dt\,] \quad .$$

Also by (2.1.1),

$$E[\eta\, E[\int_I f(t)\,dt\,|\,\mathcal{E}\,]\,]$$
$$= E[\eta \int_I f(t)\,dt\,] \quad .$$

This establishes the formula

$$E[\eta \int_I E[f(t)|\mathcal{E}\,]\,dt\,]$$
$$= E[\eta\, E[\int_I f(t)\,dt\,|\,\mathcal{E}\,]\,] \,,$$

for all bounded $\mathcal{E}$-measurable random variables η. The theorem now follows by Lemma 2.5.2. □

Let $f(t)$, $(t \in I)$ be a stochastic process on $(\Omega, \mathcal{F}, P)$, where I is an interval, and $(\mathcal{F}_t, t \in I)$ a nondecreasing family of sub σ-fields of $\mathcal{F}$; i.e., if $s, t \in I$ and $s < t$ then $\mathcal{F}_s \subset \mathcal{F}_t$. If $f(t)$ is $\mathcal{F}_t$-measurable for each $t \in I$ then f is <u>adapted</u> to $(\mathcal{F}_t, t \in I)$.

We will generally need a condition stronger than adapted. For each $t \in I$ let $I_t = [-\infty, t] \cap I$ and let $\mathcal{B}(I_t)$ be the Borel σ-field on I_t. Then f is <u>progressively measurable</u> with respect to $(\mathcal{F}_t, t \in I)$ if for all $t \in I$ the stochastic process given by f restricted to I_t is $\mathcal{B}(I_t) \times \mathcal{F}_t$-measurable. When it is clear which family of sub σ-fields we are referring to, we simply say that f is progressively measurable. In the next section we will state a result which gives sufficient conditions for the existence of a progressively measurable version of a process.

2.6 Sample Path Continuity

Given a stochastic process $f(t)$, $(t \in I)$ on a probability space $(\Omega, \mathcal{F}, P)$ and a real interval I, we say that f is <u>right continuous</u> if almost every sample function of f is right continuous on I. (Although "a.s. right continuity" would be appropriate here, it is common practice to omit "a.s." in this context.) If almost every sample path of f is continuous then we say that f is <u>continuous</u>.

The following is a list of results relating continuity with separability and progressive measurability.

<u>2.6.1</u> [Ash and Gardner (1975, p. 163)] If $f(t)$, $(t \in I)$ is a continuous stochastic process, then f is separable and the separating set I_0 can be taken as any countable dense subset of I.

<u>2.6.2</u> [Ash and Gardner (1975, p. 163)] If $f(t)$, $(t \in I)$ is right continuous, then f is separable. Further if I has a right endpoint y, then the separating set I_0 can be taken as any countable dense subset of I containing y.

<u>2.6.3</u> [Yeh (1973, p. 57)] If $f(t)$, $(t \in I)$ is separable and is stochastically equivalent to a continuous stochastic process then f is itself continuous.

2.6.4 [Ash and Gardner (1975, p. 170)] If the continuous stochastic process $f(t)$, $(t \in I)$ is adpated to a nondecreasing family of σ-fields $(\mathcal{F}_t, t \in I)$ then f has a version which is both progressively measurable with respect to $(\mathcal{F}_t, t \in I)$ and separable.

Next we state a theorem due to Kolmogorov that gives a sufficient condition for a process to be continuous.

2.6.5 [Ash and Gardner (1975, p. 164)] Let $f(t)$, $(t \in I)$ be a separable stochastic process. If there are positive constants, a, b, and C such that

$$E[|f(t)-f(s)|^b] \leq C\,|t-s|^{1+a}$$

for all $s,t \in I$, then f is continuous.

2.6.6 Corollary If $f(t)$, $(t \in I)$ is a stochastic process on $(\Omega, \mathcal{F}, P)$ satisfying the hypotheses in (2.6.5) with $b \geq 1$, and $\mathcal{E}$ is a sub σ-field of $\mathcal{F}$, then there is a version of the process $E[f(t) \mid \mathcal{E}]$, $(t \in I)$ which is continuous.

Proof: We will show that every version of $E[f(t) \mid \mathcal{E}]$ satisfies the hypotheses of (2.6.5) with the same a, b, and C as for f. Hence, any separable version of $E[f(t) \mid \mathcal{E}]$ is continuous. Making use of (2.1.1) and (2.1.5) we see that

$$E[\,|\,E[f(t)\,|\,\mathcal{G}\,] - E[f(s)\,|\,\mathcal{G}\,]\,|^{b}]$$
$$\leq E[(E[\,|\,f(t) - f(s)\,|\ \,|\,\mathcal{G}\,])^{b}]$$
$$\leq E[E[\,|f(t) - f(s)|^{b}\,|\,\mathcal{G}\,]]$$
$$= E[\,|\,f(t) - f(s)\,|^{b}]$$
$$\leq C\,|\,t - s\,|^{1+a}$$

□

Next we wish to give another condition under which continuity of a process f implies that the process $E[f(t)\,|\,\mathcal{G}\,]$ has a continuous version. To do this we make use of the theory of Banach space valued random variables and Bochner integration. The source used for this material was Diestel and Uhl (1977, pp. 41-50, 121-123).

Let $(\Lambda, \mathcal{S}, \mu)$ be a finite measure space, X a Banach space with norm $\|\ \|$, and $\mathcal{A}$ the Borel σ-field of X. A function $f : \Lambda \to X$ is a simple function if there exists $x_i \in X$ and $E_i \in \mathcal{S}$, $(i = 1,\ldots,n)$ such that

$$f = \sum_{i=1}^{n} x_i\, \chi_{E_i}$$

where χ_{E_i} is the characteristic function of the set E_i. A function $f: \Lambda \to X$ is

(i) strongly measurable if there is a sequence of simple functions f_n with

$$\lim_{n\to\infty} \| f_n - f \| = 0 \quad \text{a.e.}$$

(ii) <u>measurable</u> if the inverse image of a measurable set is measurable

(iii) <u>weakly measurable</u> if, for each bounded linear functional $x^* : X \to R$, the real valued composite function $x^* \circ f$ is measurable.

First we will establish the relationships between these different concepts of measurability. (Diestel and Uhl (1977) only define strong and weak measurability.)

<u>2.6.7</u> (Pettis' Theorem) A function $f : \Lambda \to X$ is strongly measurable if and only if

(i) there exists $N \in \mathcal{S}$ with $\mu(N) = 0$ and such that $f(\Lambda \setminus N)$ is a separable subset of X, and

(ii) f is weakly measurable.

If f is measurable then $x^* \circ f$ is measurable for each linear functional x^*. Thus, measurability implies weak measurability. This plus Pettis' Theorem gives us the next result:

<u>2.6.8</u> Let X be a separable Banach space. Then a function $f : \Lambda \to X$ is strongly measurable if and only if f is measurable.

<u>2.6.9</u> <u>Lemma</u> If $f : \Lambda \to X$ is strongly measurable then $\| f \| : \Lambda \to R$ is measurable.

Proof: Let f_n be a sequence of simple functions such that

$$\lim_{n\to\infty} \| f_n - f \| = 0 \quad \text{a.s.}$$

Then $\| f_n \|$ is a sequence of simple functions from Λ to R and since

$$\big| \| f_n \| - \| f \| \big| \leqslant \| f_n - f \| ,$$

we have

$$\lim_{n\to\infty} \big| \| f_n \| - \| f \| \big| = 0 \quad \text{a.e.}$$

Thus $\| f \|$ is strongly measurable. Since R is separable, $\| f \|$ is measurable by (2.6.8). □

A function $f : \Lambda \to X$ is Bochner integrable if f is strongly measurable and if there is a sequence of simple functions f_n such that

$$\lim_{n\to\infty} \int_\Lambda \| f_n - f \| \, d\mu = 0 .$$

If f is Bochner integrable, then for each $E \in \mathcal{G}$ we define

$$\int_E f \, d\mu = \lim_{n\to\infty} \int_E f_n \, d\mu ,$$

where

$$\int_E f_n \, d\mu = \sum_{i=1}^{n} x_i \, \mu(E_i)$$

for

$$f_n = \sum_{i=1}^{n} x_i \, \chi_{E_i} \quad .$$

Standard techniques are used to show that this definition is independent of the sequence of simple functions chosen to approximate f.

<u>2.6.10</u> (Bochner) A strongly measurable function $f : \Lambda \to X$ is Bochner integrable if and only if

$$\int_\Lambda \| f \| \, d\mu < \infty \, .$$

<u>2.6.11</u> Let T be a bounded linear operator on a Banach space X into a Banach space Y. If $f : \Lambda \to X$ is Bochner integrable then $Tf : \Lambda \to Y$ is Bochner integrable and for each $G \in \mathcal{G}$

$$T\left(\int_G f \, d\mu\right) = \int_G Tf \, d\mu \quad .$$

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A strongly measurable function $f : \Omega \to X$ is called an <u>X-valued random variable</u>. Let $\mathcal{C}$ be a sub σ-field of $\mathcal{F}$ and f a Bochner integrable X-valued random variable. We define the <u>conditional</u> <u>expectation</u> of f with respect to $\mathcal{C}$ to be any $\mathcal{C}$-strongly measurable X-valued random variable $\mathcal{E}[f \mid \mathcal{C}]$ such that for all $C \in \mathcal{C}$

$$\int_C \mathcal{E}[f \mid \mathcal{C}]\, d\mu = \int_C f\, d\mu \,.$$

Now if $\mathcal{E}[f \mid \mathcal{C}]$ exists it is uniquely defined a.s. The usual existence proof for conditional expectations does not apply since the Radon-Nikodym Theorem does not generally hold for Banach spaces. However, by defining $\mathcal{E}[f \mid \mathcal{C}]$ for simple functions and using an extension argument, the existence of $\mathcal{E}[f \mid \mathcal{C}]$ for Bochner integrable f can be established (cf. Diestel and Uhl (1977, p. 123)).

We will use as our Banach space the space $C(I)$ of real valued continuous function on a finite closed interval I, with the sup norm. Let $\mathcal{B}_C$ be the σ-field generated by the open sets of $C(I)$. Since $(C(I), \|\ \|)$ is separable, our three concepts of measurability coincide by (2.6.7) and (2.6.8).

Let $f(t)$, $(t \in I)$ be a continuous stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. Let $\Omega_0 \in \mathcal{F}$ be such that $P(\Omega_0) = 1$ and if $\omega \in \Omega_0$ then the sample function $f(\cdot, \omega)$ is in $C(I)$.

<u>2.6.12</u> [Freedman (1971, p. 50)] The function $F : \Omega \to C(I)$ given by

$$F(\omega) = \begin{cases} f(\cdot,\omega) & \text{if } \omega \in \Omega_0 \\ 0 & \text{otherwise} \end{cases}$$

is $\mathcal{B}_c$-measurable.

<u>2.6.13</u> <u>Theorem</u> Let $f(t)$, $(t \in I)$ be a continuous stochastic process on the probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{C}$ a sub σ-field of $\mathcal{F}$. If

$$E\left[\sup_{t \in I} |f(t)|\right] < \infty ,$$

then any separable version of the process $E[f(t) \mid \mathcal{C}]$, $(t \in I)$ is continuous.

<u>Proof</u>: Let F be as in (2.6.12). Since $C(I)$ is separable, F is strongly measurable by (2.6.9). Next for a.e. $\omega \in \Omega$

$$\| F(\omega) \| = \sup_{t \in I} | f(t,\omega)|$$

so $E[\| F \|] < \infty$. Thus by (2.6.10) F is Bochner integrable, so $\mathcal{E}[F \mid \mathcal{C}]$ exists. For $t \in I$ define the function $\xi_t : C(I) \to R$ by formula $\xi_t(g) = g(t)$, and notice that ξ_t is a bounded linear operator.

We will apply (2.6.11) and the definitions of conditional expectation to establish the identity

$$\xi_t(\mathcal{E}[F \mid \mathcal{C}]) = E[f(t) \mid \mathcal{C}] \text{ a.s.}$$

Since $\mathcal{E}[F \mid \mathcal{C}]$ is in $C(I)$ this gives us a continuous version of $E[f(t) \mid \mathcal{C}]$. Then (2.6.3) shows that any separable version of $E[f(t) \mid \mathcal{C}]$ is continuous.

Let $C \in \mathcal{C}$ and make the calculation

$$\int_C E[f(t) \mid \mathcal{C}]\, dP = \int_C f(t)\, dP$$

$$= \int_C \xi_t\,(F)\, dP$$

$$= \xi_t \left(\int_C F\, dP\right)$$

$$= \xi_t \left(\int_C \mathcal{E}[F \mid \mathcal{C}]\, dP\right)$$

$$= \int_C \xi_t\left(\mathcal{E}[F \mid \mathcal{C}]\right) dP\ ,$$

which gives the desired result. □

2.7 Brownian Motion and Martingales

A Brownian motion is a stochastic process $w(t)$, $(t \geq 0)$ satisfying

(i) $w(0) = 0$ a.s.

(ii) for any $0 \leq t_0 < t_1 < \cdots < t_m$ the random variables $w(t_k) - w(t_{k-1})$, $(k = 1,\dots,m)$ are independent.

(iii) if $0 \leq s < t$ then for every Borel set A

$$P(w(t) - w(s) \in A) = \int_A \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{x^2}{2(t-s)^2}\right) dx$$

i.e., the random variable $w(t) - w(s)$ is normally distributed with mean 0 and variance $t-s$.

2.7.1 [Friedman (1975, Ch. 3)] If w is a Brownian motion, then

(i) w has a version which is continuous .

(ii) almost all the sample paths of w are nowhere differentiable.

(iii) almost all the sample paths of w have infinite variation on any finite interval.

Further, if the Brownian motion w is adapted to a nondecreasing family of σ-fields $(\mathcal{F}_t, t \geq 0)$ then by (2.6.4) and (2.6.3) we see that w has a version that is continuous and progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$. From now on, in discussing Brownian motion, we will assume that such a version has been chosen. Next we give a characterization of Brownian motion due to Doob (1953, p. 384).

<u>2.7.2</u> Let $f(t)$, $(t \geqslant 0)$ be a continuous stochastic process adapted to a nondecreasing family of σ-fields $(\mathcal{F}_t, t \geqslant 0)$. If for all $0 \leqslant s < t$

(i) $f(0) = 0$ a.s. ,

(ii) $E[f(t) - f(s) \mid \mathcal{F}_s] = 0$ a.s. and

(iii) $E[(f(t)-f(s))^2 \mid \mathcal{F}_s] = t - s$ a.s.

then f is a Brownian motion.

An n-dimensional stochastic process $w(t) = (w_i(t))$, $(t \geqslant 0)$, $(i=1,\ldots,n)$ is an n-dimensional Brownian motion if each w_i is a Brownian motion and the collection of processes w_i, $(i=1,\ldots,n)$ is independent.

The proof for the next result, which is an extension of (2.7.2), may be found in Friedman (1975, p. 51).

<u>2.7.3</u> Let $f(t) = (f_i(t))$, $(t \geqslant 0)$, $(i=1,\ldots,n)$ be a continuous n-dimensional stochastic process adapted to the nondecreasing family of σ-fields $(\mathcal{F}_t, t \geqslant 0)$. If for all $0 \leqslant s < t$

(i) $f(0) = 0$ a.s. ,

(ii) $E[f(t) - f(s) \mid \mathcal{F}_s] = 0$ a.s. and

(iii) $E[(f_i(t)-f_i(s))(f_j(t)-f_j(s)) \mid \mathcal{F}_s] = \delta_{ij}(t-s)$ a.s. $(i,j=1,\ldots,n)$,

where δ_{ij} is the Kronecker delta, then f is an n-dimensional Brownian motion.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t, t \in I)$ be a nondecreasing family of sub σ-fields of $\mathcal{F}$, where I is an interval. A stochastic process $f(t)$, $(t \in I)$ is called a <u>martingale</u> with respect to $(\mathcal{F}_t, t \in I)$ if for all $s, t \in I$ with $t \geq s$

(i) $E[\,|f(s)|\,] < \infty$ and

(ii) $E[f(t)|\mathcal{F}_s] = f(s)$ a.s.

2.8 The Ito Integral

Let $f(t)$, $(t \in [\alpha,\beta])$, $(0 \leqslant \alpha < \beta)$ be a stochastic process and $w(t)$, $(t \geqslant 0)$ a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. In this section we will give Ito's definition (cf. Ito (1944)) of the integral

$$\int_{\alpha}^{\beta} f(t)\, dw(t)$$

and state those properties which will be used later in this dissertation. The proofs for the statements made in this section can be found in Friedman (1975, Ch. 4).

Since the sample functions of w are of unbounded variation (cf. 2.7.1) integration with respect to Brownian motion cannot be defined in the usual Lebesgue-Stieltjes sense. An integration by parts formula was used by Weiner; however, this restricts f to absolutely continuous processes.

Let $(\mathcal{F}_t, t \geqslant 0)$ be a nondecreasing family of sub σ-fields of $\mathcal{F}$ such that w is adapted to $(\mathcal{F}_t, t \geqslant 0)$ and that for each $t \geqslant 0$ $\sigma(w(\lambda+t) - w(t), \lambda \geqslant 0)$ is independent of $\mathcal{F}_t$. We assume that a version of w has been chosen so that w is progressively measurable with respect to $(\mathcal{F}_t, t \geqslant 0)$, (cf. 2.6.5). Next for $p \geqslant 1$, we define $K^p[\alpha,\beta]$ to be the set of all stochastic processes $f(t)$, $(t \in [\alpha,\beta])$ such that

(i) f is separable,

(ii) f is progressively measurable with respect to $(\mathcal{F}_t, t \geqslant 0)$ and

(iii) $\int_\alpha^\beta | f(t) |^p dt < \infty$ a.s.

Notice that if $f \in K^p[\alpha,\beta]$ and $1 \leqslant q \leqslant p$, then by Hölder's inequality

$$\int_\alpha^\beta | f(t) |^q dt \leqslant (\beta-\alpha)^{p/(p-q)} \left(\int_\alpha^\beta | f(t) |^p dt\right)^{q/p} ,$$

so that $f \in K^q [\alpha,\beta]$. Ito's integral will be defined for $f \in K^2[\alpha,\beta]$ and is thus defined for $f \in K^p[\alpha,\beta]$, $p \geqslant 2$.

We define $M^p[\alpha,\beta]$ to be the subset of $K^p[\alpha,\beta]$ such that

$$E[\int_\alpha^\beta |f(t)|^p dt] < \infty .$$

We say that a matrix of processes is in $K^p[\alpha,\beta]$ or $M^p[\alpha,\beta]$ if each element is in $K^p[\alpha,\beta]$ or $M^p[\alpha,\beta]$, respectively.

A stochastic process $f(t)$, $(t \in [\alpha,\beta])$ is a <u>step function</u> if there is a partition

$\alpha = t_0 < t_1 < \cdots < t_m = \beta$ of $[\alpha,\beta]$ such that for each $\omega \in \Omega$

$$f(t,\omega) = f(t_i,\omega) \quad \text{for} \quad t \in [t_i,t_{i+1}), \ (i = 0,\ldots,m-1).$$

2.8.1 If $f \in K^2[\alpha,\beta]$ then there is a sequence of step functions f_k in $K^2[\alpha,\beta]$ such that

$$\lim_{k\to\infty} \int_\alpha^\beta | f(t) - f_k(t) |^2 dt = 0 \quad \text{a.s.}$$

If g is a step function in $K^2[\alpha,\beta]$ with partition $\alpha = t_0 < t_1 < \cdots < t_m = \beta$, then the Ito integral of g with respect to the Brownian motion w is defined to be the random variable

$$\sum_{i=0}^{m-1} g(t_i) [w(t_{i+1}) - w(t_i)]$$

and is denoted by

$$\int_\alpha^\beta g(t) \, dw(t).$$

To define the Ito integral for any process f in $K^2[\alpha,\beta]$, let f_k be a sequence of step functions which approaches f in the sense of (2.8.1). It can be shown that the sequence

$$\int_\alpha^\beta f_k(t) \, dw(t)$$

is convergent in probability and that the limit is independent a.s. of the particular sequence f_k chosen. This limit is called the Ito integral of f with respect to the Brownian motion w and is written

$$\int_{\alpha}^{\beta} f(t)\ dw(t) \ .$$

We define $\int_{\alpha}^{\alpha} f(t)\ dw(t)$ to be zero.

2.8.2 Let f_1 and f_2 be processes in $K^2[\alpha,\beta]$ and λ_1 and λ_2 be real numbers. Then $\lambda_1 f_1 + \lambda_2 f_2$ is in $K^2[\alpha,\beta]$ and

$$\int_{\alpha}^{\beta} [\lambda_1 f_1(t) + \lambda_2 f_2(t)]\ dw(t)$$

$$= \lambda_1 \int_{\alpha}^{\beta} f_1(t)\ dw(t) + \lambda_2 \int_{\alpha}^{\beta} f_2(t)\ dw(t) \quad \text{a.s.}$$

For the next result notice that if $[\gamma,\delta]$ is a subinterval of $[\alpha,\beta]$ and $f \in K^2[\alpha,\beta]$ then the process given by f restricted to $[\gamma,\delta]$ is in $K^2[\gamma,\delta]$.

2.8.3 If $f \in K^2[\alpha,\beta]$ and $\gamma \in [\alpha,\beta]$, then

$$\int_{\alpha}^{\beta} f(t)\ dw(t) = \int_{\alpha}^{\gamma} f(t)\ dw(t) + \int_{\gamma}^{\beta} f(t)\ dw(t) \quad \text{a.s.}$$

2.8.4 If f is an element of $K^2[\alpha,\beta]$ or $M^2[\alpha,\beta]$ and ξ is a bounded $\mathcal{F}_{\alpha}$-measurable random variable then ξf is an element of $K^2[\alpha,\beta]$ or $M^2[\alpha,\beta]$, respectively, and

$$\int_{\alpha}^{\beta} \xi f(t)\ dw(t) = \xi \int_{\alpha}^{\beta} f(t)\ dw(t) \quad \text{a.s.}$$

<u>2.8.5</u> If $f \in M^2[\alpha,\beta]$, then

(i) $$E\left[\int_\alpha^\beta f(t)\, dw(t)\right] = 0$$

(ii) $$E\left[\int_\alpha^\beta f(t)\, dw(t) \mid \mathcal{F}_\alpha\right] = 0 \text{ a.s.}$$

(iii) $$E\left[\left(\int_\alpha^\beta f(t)\, dw(t)\right)^2\right] = E\left[\int_\alpha^\beta f^2(t)\, dt\right]$$

(iv) $$E\left[\left(\int_\alpha^\beta f(t)\, dw(t)\right)^2 \mid \mathcal{F}_\alpha\right] = E\left[\int_\alpha^\beta f^2(t)\, dt \mid \mathcal{F}_\alpha\right] \quad \text{a.s.}$$

<u>2.8.6</u> Let $f \in M^{2m}[\alpha,\beta]$ where m is a positive integer. Then

(i) $$E\left[\left(\int_\alpha^\beta f(t)\, dw(t)\right)^{2m}\right] \leqslant [m(2m-1)]^m (\beta-\alpha)^{m-1} E\left[\int_\alpha^\beta f^{2m}(t)\, dt\right]$$

(ii) $$E\left[\sup_{\tau \in [\alpha,\beta]} \left(\int_\alpha^\tau f(t)\, dw(t)\right)^{2m}\right] \leqslant [4m^3/(2m-1)]^m (\beta-\alpha)^{m-1} E\left[\int_\alpha^\beta f^{2m}(t)\, dt\right] .$$

Friedman gives a proof for (2.8.6) in the case where $\alpha = 0$. It is clear in the proof where to make the appropriate changes to obtain the result stated here. We will need the conditional expectation version of (2.8.6(i)). To obtain this we make use of the following lemma (notice its similarity to (2.5.2)):

2.8.7 Lemma If f and g are $\mathscr{C}$-measurable random variables with finite expectation and $E[\xi f] \leqslant E[\xi g]$ for every bounded, non-negative, $\mathscr{C}$-measurable random variable ξ then $f \leqslant g$ a.s.

Proof: Let

$$\xi(\omega) = \begin{cases} 1 & \text{if } (f-g)(\omega) > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Since ξ is bounded, nonnegative and $\mathscr{C}$-measurable $E[\xi(f-g)] \leqslant 0$. However, due to the way ξ was defined $\xi(f-g) \geqslant 0$. Thus $\xi(f-g) = 0$ a.s. The set on which $f - g = 0$ is included in the set on which $\xi = 0$; hence $\xi = 0$ a.s. Thus $g - f \geqslant 0$ a.s. □

2.8.8 Theorem If $f \in M^{2m}[\alpha,\beta]$ where m is a positive integer, then

$$E\left[\left(\int_\alpha^\beta f(t)\, dw(t)\right)^{2m} \Big| \mathscr{F}_\alpha\right]$$

$$\leqslant [m(2m-1)]^m (\beta-\alpha)^{m-1} E\left[\int_\alpha^\beta f^{2m}(t)\, dt \Big| \mathscr{F}_\alpha\right] \quad \text{a.s.}$$

Proof: Let ξ be a bounded, nonnegative, $\mathscr{F}_\alpha$-measurable random variable. Making use of properties (2.1.1), (2.8.4) and (2.8.6 (i)) we obtain

$$E\left[\xi^{2m}\ E\left[\left(\int_\alpha^\beta f(t)\ dw(t)\right)^{2m} \mid \mathcal{F}_\alpha\right]\right]$$

$$= E\left[E\left[\left(\int_\alpha^\beta \xi f(t)\ dw(t)\right)^{2m} \mid \mathcal{F}_\alpha\right]\right] \quad \text{a.s.}$$

$$= E\left[\left(\int_\alpha^\beta \xi f(t)\ dw(t)\right)^{2m}\right] \quad \text{a.s.}$$

$$\leq [m(2m-1)]^m\ (\beta-\alpha)^{m-1}\ E\left[\int_\alpha^\beta \xi^{2m}\ f^{2m}\ (t)\ dt\right] \quad \text{a.s.}$$

$$= E\left[\xi^{2m}\ [m(2m-1)]^m\ (\beta-\alpha)^{m-1}\ E\left[\int_\alpha^\beta f^{2m}\ (t)\ dt \mid \mathcal{F}_\alpha\right]\right] \text{a.s.}$$

Then by Lemma (2.8.7) we obtain the desired result. □

[A proof similar to that above can be used to prove the conditional expectation version of (2.8.6 (ii)).]

<u>2.8.9</u> If f is in $K^2[\alpha,\beta]$, then the process $z(t)$, $(t\in[\alpha,\beta])$ given by

$$z(t) = \int_\alpha^t f(s)\ dw(s)$$

has a continuous version.

The process z defined above is called the <u>indefinite Ito integral</u> of f. Henceforth, when we speak of an indefinite Ito integral, we shall always mean a continuous version.

We next define the n-dimensional Ito integral. Let

$w(t) = (w_i(t)), (t \geq 0), (i=1,\ldots,n)$ be an n-dimensional Brownian motion and $(\mathcal{F}_t, t \geq 0)$ a nondecreasing family of σ-fields such that w is progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$ and that for each $t \geq 0$ $\sigma(w(\lambda+t) - w(t), \lambda \geq 0)$ is independent of $\mathcal{F}_t$. If $b(t) = (b_{ij}(t)), (t \in [\alpha,\beta])$ is an $m \times n$ matrix in $K^2[\alpha,\beta]$, then the Ito integral of b with respect to w is defined to be the m-dimensional random variable

$$\int_\alpha^\beta b(t)\ dw(t) = \left(\sum_{j=1}^{n} \int_\alpha^\beta b_{ij}(t)\ dw_j(t) \right), (i=1,\ldots,m) .$$

Next let $a(t) = (a_i(t))$ and $b(t) = (b_{ij}(t)), (t \in [\alpha,\beta]), (i=1,\ldots,m), (j=1,\ldots,n)$ be a stochastic processes satisfying

$$z(t) = z(0) + \int_\alpha^t a(s)\ ds + \int_\alpha^t b(s)\ dw(s) \qquad \text{a.s.}$$

for each $t \in [\alpha,\beta]$. By (2.8.9), (2.6.4) and (2.6.3) we can find a version of z that is continuous and progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$. If z is such a version then we say that z is a <u>stochastic integral</u>.

Our next result is the chain rule for the Ito calculus. It and several other identities derived from it are all known as Ito's formula. Examination of this equation highlights several of the major differences between Ito integrals and ordinary integrals.

2.8.10 Let $u(x,t)$, $(x \in R^m,\ t \in [\alpha,\beta])$ be a continuous real valued function in $R^m \times [\alpha,\beta]$ with continuous partial derivatives u_t, u_{x_i}, $u_{x_i x_j}$. Let $z(t) = (z_i(t))$, $(t \in [\alpha,\beta])$, $(i=1,\dots,m)$ be a stochastic integral

$$z(t) = z(\alpha) + \int_0^t a(t)\,dt + \int_0^t b(t)\,dw(t) \quad \text{a.s.,}$$

where $a = (a_i)$ and $b = (b_{ij})$, $(i=1,\dots,m)$, $(j=1,\dots,n)$ belong to $K^1[\alpha,\beta]$ and $K^2[\alpha,\beta]$, respectively. Then $u(z(t),\ t)$ is the stochastic integral

$$\begin{aligned} u(z(t),\ t) &= u(z(\alpha),\ \alpha) \\ &+ \int_\alpha^t [u_t\,(z(s),\ s) + \sum_{i=1}^m u_{x_i}\,(z(s),\ s)\,a_i(s) \\ &+ \frac{1}{2} \sum_{k=1}^n \sum_{i,j=1}^m u_{x_i x_j}\,(z(s),\ s)\,b_{ik}(s)\,b_{jk}(s)]\,ds \\ &+ \int_\alpha^t \sum_{k=1}^n \sum_{i=1}^m u_{x_i}\,(z(s),\ s)\,b_{ik}(s)\,dw_k(s) \quad \text{a.s.} \end{aligned}$$

The next result is obtained by applying (2.8.5) to the previous result. It is this that we will call Ito's formula:

2.8.11 Let $u : R^m \to R$ be a continuous function with continuous first and second derivatives. Assume further that z, a, and b satisfy the same hypotheses as in (2.8.10). Let

$$B_{ij}(t) = \sum_{k=1}^{n} b_{ik}(t)\, b_{jk}(t) \quad .$$

If the process

$$y(t) = \sum_{i=1}^{m} u_{x_i}(z(t))\, a_i(t) + \frac{1}{2} \sum_{i,j=1}^{m} u_{x_i x_j}(z(t)) B_{ij}(t), \quad (t \in [\alpha, \beta]).$$

is in $M^1[\alpha,\beta]$ and the process

$$\sum_{k=1}^{n} \sum_{i=1}^{m} u_{x_i}(z(t))\, b_{ik}(t) \ , \ (t \in [\alpha,\beta])$$

is in $M^2[\alpha,\beta]$, then for each $t \in [\alpha,\beta]$

$$E[u(z(t)) - u(z(\alpha)) \mid \mathcal{F}_\alpha]$$

$$= E\left[\int_\alpha^t y(s)\, ds \mid \mathcal{F}_\alpha \right] \qquad \text{a.s.}$$

2.9 Miscellaneous Results

First we quote a well-known inequality often used to prove Minkowski's inequality.

2.9.1 [Ash (1972, p. 85)] If $c,d \geqslant 0$ and $m \geqslant 1$, then

$$(c + d)^m \leqslant 2^{m-1} (c^m + d^m) \quad .$$

We need the following generalization of this result:

2.9.2 Lemma If c_i, $(i=1,\ldots,n)$ are nonnegative real numbers and m is a positive integer then

$$\left(\sum_{i=1}^{n} c_i\right)^m \leqslant 2^{(n-1)(m-1)} \sum_{i=1}^{n} c_i^{\,m} \quad .$$

Proof: The proof for $n \geqslant 2$ is by induction on n. For $n = 2$, (2.9.2) is (2.9.1). For the induction step from $n-1$ to n we use (2.10.1) to obtain:

$$\left(\sum_{i=1}^{n} c_i\right)^m = \left(c_n + \sum_{i=1}^{n-1} c_i\right)^m$$

$$\leqslant 2^{m-1} \left(c_n^{\,m} + \left[\sum_{i=1}^{n-1} c_i\right]^m\right)$$

$$\leqslant 2^{m-1}\left(c_n^{\ m} + 2^{(n-2)(m-1)} \sum_{i=1}^{n-1} c_i^{\ m}\right)$$

$$\leqslant 2^{m-1}\left(2^{(n-2)(m-1)} \sum_{i=1}^{n} c_i^{\ m}\right)$$

$$= 2^{(n-1)(m-1)} \sum_{i=1}^{n} c_i^{\ m} \quad . \qquad \square$$

<u>2.9.3</u> <u>Lemma</u> For $n = 2,3,4,\cdots$

(i) $\displaystyle\sum_{j=0}^{n} \binom{2n}{2j} = \sum_{j=0}^{n-1} \binom{2n}{2j+1}$

(ii) $\displaystyle\sum_{j=0}^{n} \binom{2n}{2j}(2n-2j) = \sum_{j=0}^{n-1} \binom{2n}{2j+1}(2n-2j-1)$

(iii) $\displaystyle\sum_{j=0}^{n} \binom{2n}{2j}(2n-2j)(2n-2j-1) = \sum_{j=0}^{n} \binom{2n}{2j+1}(2n-2j-1)(2n-2j-2)$

<u>Proof</u>: By the Binomial Theorem,

$$(x-1)^{2n} = \sum_{j=0}^{2n} \binom{2n}{j}(-1)^j x^{2n-j}$$

$$= \sum_{j=0}^{n} \binom{2n}{2j} x^{2n-2j} - \sum_{j=0}^{n-1} \binom{2n}{2j+1} x^{2n-2j-1}$$

Evaluation at $x = 1$ gives (i). Differentiation with respect to x, then evaluation at $x = 1$, gives (ii). Equation (iii) is obtained by a second differentiation with evaluation at $x = 1$. $\square$

2.9.4 Lemma If $f : [a,b) \to R$ is integrable and right continuous on $[a,b)$, then for each $t \in [a,b)$

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} f(s)\, ds = f(t) \quad .$$

Proof: Given $\varepsilon > 0$, use right continuity to choose $\delta > 0$ so that if $s \in [t, t+\delta)$, then $| f(s) - f(t) | < \varepsilon$. Then for $h \in (0,\delta)$

$$\left| \frac{1}{h} \int_t^{t+h} f(s)\, ds - f(t) \right|$$

$$= \left| \frac{1}{h} \int_t^{t+h} (f(s) - f(t))\, ds \right|$$

$$\leqslant \frac{1}{h} \int_t^{t+h} \left| f(s) - f(t) \right| ds$$

$$< \varepsilon \quad .$$

□

2.9.5 [Saks (1937, p. 204)] If $g : [a,b) \to R$ is continuous and right differentiable on $[a,b)$ and the right derivative of g is continuous at a point $t \in [a,b)$ then g is differentiable at t.

Combining this result with the Fundamental Theorem of Calculus we obtain

2.9.6 Lemma If $g : [a,b) \to \mathbb{R}$ is continuous on $[a,b)$ and has a continuous right derivative g_r' on $[a,b)$, then for each $c \in [a,b]$

$$\int_a^c g_r'(x)\, dx = g(c) - g(a) \quad .$$

III. THE MOMENT RATE CHARACTERIZATION

3.1 The Spaces M^p and Q^p

Given a probability space $(\Omega, \mathcal{F}, P)$, let $(\mathcal{F}_t, t \geq 0)$ be a nondecreasing family of sub σ-fields of $\mathcal{F}$. We define M^p to be the set of all stochastic processes $f(t)$, $(t \geq 0)$ on $(\Omega, \mathcal{F}, P)$ such that

(i) f is separable,

(ii) f is progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$ and

(iii) for all $T \geq 0$,

$$E\left[\int_0^T |f(t)|^p \, dt\right] < \infty .$$

Notice that f is in M^p iff for all $0 \leq \alpha < \beta < \infty$ the process given by f restricted to the interval $[\alpha,\beta]$ is in $M^p[\alpha,\beta]$.

We define Q^p to be the subset of M^p such that for all $T \geq 0$

$$E\left[\operatorname*{ess\,sup}_{t \in [0,T]} |f(t)|^p\right] < \infty .$$

We denote by DM^p the set of right continuous processes in M^p. The set CM^p consists of those continuous processes that are in M^p. The sets DQ^p and CQ^p are similarly defined.

The following results are consequences of Hölder's inequality and the fact that the essential supremum of a product of positive functions is less than or equal to the product of the essential suprema of these functions.

<u>3.1.1</u> If $1 \leqslant q \leqslant p$ then $M^p \subset M^q$ and $Q^p \subset Q^q$.

<u>3.1.2</u> If $f \in M^{2p}$ and $g \in M^2$ then $f^p g \in M^1$.

<u>3.1.3</u> If $f \in M^{4p}$ and $g \in M^4$ then $f^p g \in M^2$.

<u>3.1.4</u> If $f, g \in Q^4$ then $fg \in Q^2$.

<u>3.1.5</u> If $f \in Q^{2p}$ and $g \in Q^2$ then $f^p g \in Q^1$.

For notational convenience, we will say that a matrix of processes is in one of the spaces M^p, DM^p, CM^p, Q^p, DQ^p or CQ^p if all of its components are in that space.

<u>3.1.6</u> <u>Lemma</u> If $f \in DQ^p$, where p is a positive integer, then for any separable versions of the processes

$$E\left[\left(\int_t^{t+h} f(s)\,ds\right)^p \Big| \mathcal{F}_t\right] , \quad (h \geqslant 0)$$

and

$$E\left[\int_t^{t+h} f^p(s)\,ds \Big| \mathcal{F}_t\right] , \quad (h \geqslant 0) \quad ,$$

where t is fixed and h is the parameter, we have

(i) $\lim_{h\downarrow 0} E\left[\left(\frac{1}{h}\int_t^{t+h} f(s)\,ds\right)^p \Big| \mathcal{F}_t\right] = f^p(t)$ a.s.

(ii) $\lim_{h\downarrow 0} \frac{1}{h} E\left[\int_t^{t+h} f^p(s)\,ds \Big| \mathcal{F}_t\right] = f^p(t)$ a.s.

Proof: Let H_o be a separating set for $E\left[\left(\int_t^{t+h} f(s)\,ds\right)^p \Big| \mathcal{F}_t\right]$. To justify the use of the Dominated Convergence Theorem (2.1.2) notice that for $0 < h \leqslant 1$

$$\left|\frac{1}{h}\int_t^{t+h} f(s)\,ds\right|^p \leqslant \left(\frac{1}{h}\int_t^{t+h} \sup_{s\in[0,t+1]} |f(s)|\,ds\right)^p$$

$$= \sup_{s\in[0,t+1]} |f(s)|^p$$

and by hypothesis

$$E\left[\sup_{s\in[0,t+1]} |f(s)|^p\right] < \infty .$$

Then using (2.1.2) and (2.9.4) we obtain

$$\lim_{h\downarrow 0, h\in H_o} E\left[\left(\frac{1}{h}\int_t^{t+h} f(s)\,ds\right)^p \Big| \mathcal{F}_t\right]$$

$$= E\left[\lim_{h\downarrow 0, h\in H_o}\left(\frac{1}{h}\int_t^{t+h} f(s)\,ds\right)^p \Big| \mathcal{F}_t\right] \quad \text{a.s.}$$

$$= E\left[f^p(t) \mid \mathcal{F}_t\right] \quad \text{a.s.}$$

$$= f^p(t) \quad \text{a.s.}$$

Applying (2.4.1) establishes (i). Equation (ii) follows from (i) by noticing that if $f \in DQ^p$ then $f^p \in DQ^1$ and taking $p = 1$ in (i). □

3.2 Moment Rate Properties of Certain Stochastic Integrals

Let $w(t) = (w_i(t))$, $(t \geq 0)$, $(i=1,\ldots,n)$ be an n-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t, t \geq 0)$ a nondecreasing family of sub σ-fields of $\mathcal{F}$ such that w is adapted to $(\mathcal{F}_t, t \geq 0)$ and $(w(\lambda+t)-w(t), \lambda \geq 0)$ is independent of $\mathcal{F}_t$ for each $t \geq 0$. Given the stochastic processes $a(t) = (a_i(t))$ and $b(t) = (b_{ij}(t))$, $(t \geq 0)$, $(i,j = 1,\ldots,n)$, where $a \in M^1$ and $b \in M^2$, let $z = (z_i)$, $(i=1,\ldots,n)$ be a stochastic integral given by

$$z(t) = z(0) + \int_0^t a(s)\, ds + \int_0^t b(s)\, dw(s) \tag{3.2.1}$$

Recall that since z is a stochastic integral, it is continuous and progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$.

Given an n-dimensional stochastic process $f(t) = f_i(t)$, $(t \geq 0)$, $(i=1,\ldots,n)$ adapted to $(\mathcal{F}_t, t \geq 0)$, a <u>moment rate function</u> for f is a process given by a limit of the form

$$\lim_{h \downarrow 0} \frac{1}{h} E\left[\prod_{k=1}^{m} (f_{i_k}(t+h) - f_{i_k}(t)) \,\middle|\, \mathcal{F}_t \right] ,$$

where $(i_k : k=1,\ldots,n)$ is a sequence of indices in $\{1,\ldots,n\}$ and a separable version has been chosen for the process

$$E\left[\prod_{k=1}^{m} (f_{i_k}(t+h) - f_{i_k}(t)) \,\middle|\, \mathcal{F}_t \right] , \quad (h \geq 0) .$$

For the remainder of this section the symbol z will denote a stochastic integral with a and b as in (3.2.1). Conditions will be placed on a and b that will guarantee the existence of moment rate functions for z and allow us to obtain explicit expressions for them.

<u>3.2.2</u> <u>Lemma</u> If $a \in M^1$ and $b \in M^2$ then

$$E[z(t+h) - z(t) | \mathcal{F}_t] = E\left[\int_t^{t+h} a(s)\, ds \,\Big|\, \mathcal{F}_t\right] \quad \text{a.s.}$$

<u>Proof</u>: We use (2.8.3) and (2.8.5) to see that

$$E[z(t+h) - z(t) | \mathcal{F}_t]$$

$$= E\left[\int_t^{t+h} a(s)\, ds + \int_t^{t+h} b(s)\, dw(s) \,\Big|\, \mathcal{F}_t\right] \quad \text{a.s.}$$

$$= E\left[\int_t^{t+h} a(s)\, ds \,\Big|\, \mathcal{F}_t\right] \quad \text{a.s.}$$ □

<u>3.2.3</u> <u>Theorem</u> If $a \in DQ^1$ and $b \in M^2$ then for any separable version of the process $E[z_i(t+h) - z_i(t) | \mathcal{F}_t]$, $(h \geq 0)$ we have

$$\lim_{h \downarrow 0} \frac{1}{h} E[z_i(t+h) - z_i(t) | \mathcal{F}_t] = a_i(t) \quad \text{a.s.}$$

<u>Proof</u>: This follows from Lemmas 3.2.2 and 3.1.6. □

For higher moments we require

3.2.4 Lemma If a and b are in M^{2p} and $E[z_i^{2p}(0)] < \infty$, $(i=1,\dots,n)$, where p is a positive integer, then z is in Q^{2p}.

Proof: By (2.9.2) there is a constant $C_{n,p}$ depending only on n and p such that

$$|z_i(t)|^{2p} \leq C_{n,p}\left[|z_i(0)|^{2p} + \left(\int_0^t |a_i(s)|\,ds\right)^{2p} + \sum_{j=0}^{n} \left|\int_0^t b_{ij}(s)\,dw_j(s)\right|^{2p}\right].$$

It suffices to show that the essential supremum of each term individually in the above equation has finite expectation. The first term is covered by hypothesis. For the second, we have for $T \geq 0$

$$\begin{aligned} E\left[\operatorname*{ess\,sup}_{t\in[0,T]} \left(\int_0^t |a_i(s)|\,ds\right)^{2p}\right] &= E\left[\left(\int_0^T |a_i(s)|\,ds\right)^{2p}\right] \\ &\leq E\left[T^{2p-1}\int_0^T a_i^{2p}(s)\,ds\right] \\ &< \infty, \end{aligned}$$

where Hölder's inequality was used on the integral $\int_0^T |a_i(s)|\,ds$.

To handle the Ito integrals we make use of (2.8.6) to see that for some constant K_p, depending only on p,

$$E\left[\operatorname*{ess\,sup}_{t\in[0,T]} \left| \int_0^t b_{ij}(s)\, dw_j(s) \right|\right]^{2p}$$

$$\leq K_p\, T^{p-1}\, E\left[\int_0^T b_{ij}^{2p}(s)\, ds\right]$$

$$< \infty$$

□

3.2.5 Theorem If $a, b \in DQ^2 \cap M^4$ then for any separable version of the process

$$E[(z_i(t+h) - z_i(t))(z_j(t+h) - z_j(t)) | \mathcal{F}_t], \ (h \geq 0)$$

we have

$$\lim_{h \downarrow 0} \frac{1}{h} E[(z_i(t+h) - z_i(t))(z_j(t+h) - z_j(t)) | \mathcal{F}_t]$$

$$= \sum_{k=1}^{n} b_{ik}(t)\, b_{jk}(t) \qquad \text{a.s.}$$

Proof: Since we are dealing with differences we can assume without loss of generality that $z(0) = 0$. First, since z is adapted to $(\mathcal{F}_t, t \geq 0)$,

$$\begin{aligned} &E[(z_i(t+h) - z_i(t))(z_j(t+h) - z_j(t)) | \mathcal{F}_t] \\ &\quad = E[z_i(t+h)z_j(t+h) - z_i(t)z_j(t) | \mathcal{F}_t] \\ &\quad - z_i(t)\, E[z_j(t+h) - z_j(t) | \mathcal{F}_t] \\ &\quad - z_j(t)\, E[z_i(t+h) - z_i(t) | \mathcal{F}_t] \quad \text{a.s.} \end{aligned} \qquad (3.2.6)$$

Next we will apply Ito's formula (2.8.11) with $u(x) = x_i x_j$ to the first term on the right side of the above equation to obtain

$$E[z_i(t+h)z_j(t+h) - z_i(t)z_j(t) \mid \mathcal{F}_t]$$

$$= E\left[\int_t^{t+h} [z_j(s)\ a_i(s) + z_i(s)\ a_j(s) + \sum_{k=1}^{n} b_{ik}(s)\ b_{jk}(s)]\ ds \,\middle|\, \mathcal{F}_t\right] \quad \text{a.s.} \qquad (3.2.7)$$

To show that the hypotheses of (2.8.11) are satisfied, first notice that by Lemma 3.2.4 z is in Q^4; thus by (3.1.2) $z_j b_{ik}$ and $z_j b_{jk}$ are in M^2. From (3.1.5) we observe that $z_j a_i$, $z_i a_j$ and $b_{ik} b_{jk}$ are in Q^1 and hence in M^1.

The above paragraph also shows that the integrand in (3.2.7) is in Q^1; so we can apply Lemma 3.1.6 to obtain

$$\lim_{h \downarrow 0} \frac{1}{h} E[z_i(t+h)z_j(t+h) - z_i(t)z_j(t) \mid \mathcal{F}_t]$$

$$= z_j(t)a_i(t) + z_i(t)a_j(t) + \sum_{k=1}^{n} b_{ik}(t)\ b_{jk}(t) \quad \text{a.s.}$$

Combining this with Theorem 3.2.3 and equation 3.2.6, we obtain the desired result. □

<u>3.2.8</u> <u>Lemma</u> If for an integer $m \geqslant 2$

$$\lim_{h \downarrow 0} \frac{1}{h} E[|z_j(t+h) - z_j(t)|^m | \mathcal{F}_t] = 0 \quad \text{a.s.} \quad (j=1,\ldots,n)$$

then

$$\lim_{h \downarrow 0} \frac{1}{h} E[\prod_{k=1}^{m} |z_{i_k}(t+h) - z_{i_k}(t)| \mid \mathcal{F}_t] = 0 \quad \text{a.s.}$$

for any sequence of indices $(i_k : k=1,\ldots,m)$ in $\{1,\ldots,n\}$.

<u>Proof</u>: This follows by applying Hölder's inequality (2.1.3) $m-1$ times in the following manner:

$$E [\prod_{k=1}^{m} |z_{i_k}(t+h) - z_{i_k}(t)| \mid \mathcal{F}_t]$$

$$\leqslant (E[|z_{i_1}(t+h) - z_{i_1}(t)|^m | \mathcal{F}_t])^{1/m}$$

$$\times (E[\prod_{k=2}^{m} | z_{i_k}(t+h) - z_{i_k}(t)|^{m/(m-1)} | \mathcal{F}_t])^{(m-1)/m} \quad \text{a.s.}$$

$$\leqslant \prod_{j=1}^{2} (E[| z_{i_j}(t+h) - z_{i_j}(t) |^m | \mathcal{F}_t])^{1/m}$$

$$\times (E[\prod_{k=3}^{m} |z_{i_k}(t+h) - z_{i_k}(t)|^{m/(m-2)} | \mathcal{F}_t])^{(m-2)/m} \quad \text{a.s.}$$

$$\vdots$$

$$\leqslant \prod_{k=1}^{m} (E[| z_{i_k}(t+h) - z_{i_k}(t)|^m | \mathcal{F}_t])^{1/m} \quad \text{a.s.} \quad \square$$

<u>3.2.9</u> <u>Theorem</u> Consider a process given by

$$E[\prod_{k=1}^{p} |z_{i_k}(t+h) - z_{i_k}(t)| \mid \mathcal{F}_t], \quad (h \geq 0)$$

where p is an integer greater than or equal to 3 and $(i_k : k=1,\dots,p)$ is a sequence of indices in $\{1,\dots,n\}$. If any one of the four conditions below holds, then every separable version of this process satisfies

$$\lim_{h\downarrow 0} \frac{1}{h} E[\prod_{k=1}^{p} |z_{i_k}(t+h) - z_{i_k}(t)| \mid \mathcal{F}_t] = 0 \quad \text{a.s.}$$

(1) p is even and $a,b \in DQ^p$;

(2) p is odd, $a \in DQ^p$ and $b \in DQ^{p+1}$;

(3) p is even, $a \in DQ^2 \cap M^{2p-4}$ and $b \in DQ^4 \cap M^{2p-4}$;

(4) p is odd, $a \in DQ^2 \cap M^{2p-2}$ and $b \in DQ^4 \cap M^{2p-2}$.

<u>Proof</u>: By Lemma 3.2.8 we need only show that

$$\lim_{h\downarrow 0} \frac{1}{h} E[| z_i(t+h) - z_i(t) |^p \mid \mathcal{F}_t] = 0 \text{ a.s.}$$

<u>Case 1</u>: Suppose condition (1) holds. Write $p = 2m$. By (2.8.3) and Lemma 2.9.2 we have, for some constant $C_{m,n}$ depending only on m and n,

$$E[(z_i(t+h) - z_i(t))^{2m} | \mathcal{F}_t]$$

$$\leq C_{m,n} E\left[\left(\int_t^{t+h} a_i(s)\, ds \right)^{2m} + \sum_{j=1}^{n} \left(\int_t^{t+h} b_{ij}(s)\, dw_j(s) \right)^{2m} \Big| \mathcal{F}_t \right] .$$

Now we use Lemma 3.1.6 to see that

$$\lim_{h \downarrow 0} \frac{1}{h} E\left[\left(\int_t^{t+h} a_i(s)\, ds \right)^{2m} \Big| \mathcal{F}_t \right]$$

$$= \lim_{h \downarrow 0} h^{2m-1} E\left[\left(\frac{1}{h} \int_t^{t+h} a_i(s)\, ds \right)^{2m} \Big| \mathcal{F}_t \right]$$

$$= 0 \quad \text{a.s.}$$

For the remaining terms we apply (2.8.6) to obtain, for some constant K_m depending on m,

$$E\left[\left(\int_t^{t+h} b_{ij}(s)\, dw_j(s) \right)^{2m} \Big| \mathcal{F}_t \right]$$

$$\leq K_m h^{m-1} E\left[\int_t^{t+h} b_{ij}{}^{2m}(s)\, ds \Big| \mathcal{F}_t \right] \quad \text{a.s.} \tag{3.2.10}$$

Then noticing that $m \geq 2$ and applying Lemma 3.1.6, we find that

$$\lim_{h \downarrow 0} \frac{1}{h} K_m h^{m-1} E\left[\int_t^{t+h} b_{ij}{}^{2m}(s)\, ds \Big| \mathcal{F}_t \right] = 0 \quad \text{a.s.}$$

Case 2: Assume condition (2). Write $p = 2m+1$. By Hölder's inequality (2.1.3), we have

$$E[\,|z_i(t+h) - z_i(t)|^{2m+1} \mid \mathcal{F}_t]$$

$$\leq (E[(z_i(t+h) - z_i(t))^{2m} \mid \mathcal{F}_t])^{1/2}$$

$$\times (E[(z_i(t+h) - z_i(t))^{2m+2} \mid \mathcal{F}_t])^{1/2} \quad \text{a.s.}$$

This reduces Case 2 to Case 1.

Case 3: Suppose condition (3) hold and write $p = 2m$. By the Binomial Theorem

$$E[(z_i(t+h) - z_i(t))^{2m} \mid \mathcal{F}_t]$$

$$= E[\sum_{j=0}^{2m} \binom{2m}{j} (-1)^j z_i^{2m-j}(t+h)\, z_i^{j}(t) \mid \mathcal{F}_t] \quad \text{a.s.}$$

$$= E[\sum_{j=0}^{m} \binom{2m}{2j} z_i^{2m-2j}(t+h)\, z_i^{2j}(t) \mid \mathcal{F}_t]$$

$$- E[\sum_{j=0}^{m-1} \binom{2m}{2j+1} z_i^{2m-2j-1}(t+h)\, z_i^{2j+1}(t) \mid \mathcal{F}_t] \quad \text{a.s.}$$

Now making use of Lemma 2.9.3(i), we subtract

$$z_i^{2m}(t) \sum_{j=0}^{m} \binom{2m}{2j}$$

from the first term and add the equal expression

$$z_i^{2m}(t) \sum_{j=0}^{m-1} \binom{2m}{2j+1}$$

to the last term, to obtain

$$E[(z_i(t+h) - z_i(t)^{2m} \mid \mathcal{F}_t]$$

$$= \sum_{j=0}^{m} \binom{2m}{2j} \; z_i^{2j}(t) \; E[z_i^{2m-2j}(t+h) - z_i^{2m-2j}(t) \mid \mathcal{F}_t]$$

$$- \sum_{j=0}^{m-1} \binom{2m}{2j+1} z_i^{2j+1}(t) \; E[z_i^{2m-2j-1}(t+h) - z_i^{2m-2j-1}(t) \mid \mathcal{F}_t] \quad \text{a.s.} \qquad (3.2.11)$$

We will apply Ito's formula (2.8.11) with $u(x) = x_i^q$ for $q = 2,3,\ldots,2m$ to obtain

$$E[z_i^q(t+h) - z_i^q(t) \mid \mathcal{F}_t]$$

$$= E[\int_t^{t+h} \{ q \; a_i(s) \; z_i^{q-1}(s)$$

$$+ \frac{q(q-1)}{2} B_{ii}(s) \; z_i^{q-2}(s) \} \, ds \mid \mathcal{F}_t] \quad \text{a.s.} \qquad (3.2.12)$$

where

$$B_{ii}(s) = \sum_{k=1}^{n} b_{ik}^2(s) \quad .$$

First, however, the hypotheses in (2.8.11) must be verified. By

Lemma 3.2.4 we see that $z_i \in Q^{4m-4}$. From (3.1.4) $B_{ii} \in Q^2$, so by (3.1.5) $B_{ii}\, z_i^{q-2}$ and $a_i\, z^{q-1}$ are in Q^1. Finally by (3.1.2) $b_{ik}\, z_i^{q-1} \in M^2$.

By taking the last term to be 0, equation (3.2.12) also holds for $q = 1$ by Lemma 3.2.1. It holds trivially for $q = 0$, with all terms equal to 0.

The next step is to obtain the limits

$$\lim_{h \downarrow 0} \frac{1}{h} E[z_i^q(t+h) - z_i^q(t) \mid \mathcal{F}_t]$$

$$= q\, a_i(t)\, z_i^{q-1}(t) + \frac{q(q-1)}{2} B_{ii}(t)\, z_i^{q-2}(t) \quad \text{a.s.}$$

This is done by applying Lemma 3.1.6 to the right side of equation 3.2.12. The hypothesis that the integrand is in DQ^1 holds since a_i and b_{ij} are right continuous, z_i is continuous and, as shown above, $B_{ij}\, z^{q-2}$ and $a_i\, z_i^{q-1}$ are in Q^1.

Then applying this to equation 3.2.11, we obtain

$$\lim_{h \downarrow 0} \frac{1}{h} E[(z_i(t+h) - z_i(t)^{2m} \mid \mathcal{F}_t]$$

$$= \sum_{j=0}^{m} \binom{2m}{2j} (2m-2j)\, a_i(t)\, z_i^{2m-1}(t)$$

$$+ \frac{1}{2} \sum_{j=0}^{m} \binom{2m}{2j} (2m-2j)(2m-2j-1)\, B_{ii}(t)\, z_i^{2m-2}(t)$$

$$- \sum_{j=0}^{m-1} \binom{2m}{2j+1} (2m-2j-1)\, a_i(t)\, z_i^{2m-1}(t)$$

$$- \frac{1}{2} \sum_{j=0}^{m-1} \binom{2m}{2j+1} (2m-2j-1)(2m-2j-2)\ B_{ii}(t)\ z_i^{2m-2}(t) \qquad \text{a.s.}$$

Reference to Lemma 2.9.3 shows that the above expression is equal to 0 a.s.

Case 4: Assume condition (4) holds. The technique used to reduce Case 2 to Case 1 also serves to reduce Case 4 to Case 3. □

3.3 Representation Theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_t, t \geq 0)$ a nondecreasing family of sub σ-fields of $\mathcal{F}$. Suppose that $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t))$, $(t \geq 0)$, $(i,j=1,\ldots,n)$ are stochastic processes which are progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$. For any twice continuously differentiable function $f : R^n \to R$ define $L_t f$, $(t \geq 0)$ by

$$(L_t f)(x) = \frac{1}{2} \sum_{i,j=1}^{n} B_{ij}(t) f_{x_i x_j}(x) + \sum_{i=1}^{n} a_i(t) f_{x_i}(x) \qquad (3.3.1)$$

Notice that if $z(t)$, $(t \geq 0)$ is a progressively measurable stochastic process, then $(L_t f)(z(t))$, $(t \geq 0)$ defines another progressively measurable stochastic process.

We denote by $C_0^\infty (R^n)$ the space of real valued functions on R^n possessing continuous derivatives of all orders and having compact support. Following Stroock and Varadhan (1979) we say that a stochastic process $z(t)$, $(t \geq 0)$ is an Ito process with covariance B and drive a if for every $t \geq 0$ and $\omega \in \Omega$, $B(t,\omega)$ is symmetric nonnegative definite and z is continuous, progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$ and the process

$$f(z(t)) - \int_0^t (L_u f)(z(u))\, du, \ (t \geq 0)$$

is a martingale with respect to $(\mathcal{F}_t, t \geq 0)$ for all $f \in C_0^\infty (R^n)$.

(This generalizes Stroock and Varadhan's definition by omitting their requirement that a and B are bounded.)

A more convenient formulation of the martingale property for Ito processes is

$$E[f(z(t))-f(z(s))|\mathcal{F}_s]=E[\int_s^t (L_u f)(z(u))\, du|\mathcal{F}_s] \quad \text{a.s.,}$$

$$(0 \leqslant s \leqslant t) \qquad (3.3.2)$$

From this we see that Ito's formula (2.8.11), with $u \in C_0^\infty(R^n)$, holds for Ito processes.

Stroock and Varadhan (1979, p. 108) have proved the following theorem:

<u>3.3.3</u> Suppose that the stochastic processes $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t))$, $(t \geqslant 0)$, $(i,j=1,\ldots,n)$ are bounded and progressively measurable with respect to $(\mathcal{F}_t,\ t \geqslant 0)$. Suppose further that $B(t,\omega)$ is a strictly positive definite $n \times n$-symmetric matrix for each $t \geqslant 0$ and $\omega \in \Omega$. Let $b(t) = (b_{ij}(t))$, $(t \geqslant 0)$, $(i,j=1,\ldots,n)$ be a progressively measurable process such that $bb^T = B$. If $z(t)$, $(t \geqslant 0)$ is an Ito process with covariance B and drift a. Then there exists an n-dimensional Brownian motion w adapted to $(\mathcal{F}_t,\ t \geqslant 0)$ for which

$$z(t) = z(0) + \int_0^t a(s)\, ds + \int_0^t b(s)\, dw(s)$$

The above result needs the existence of a progressively measurable b such that $bb^T = B$. In our application B is a continuous process. Under this condition we can establish the existence of such a b. We will use the next result to accomplish this.

<u>3.3.4</u> [Friedman (1975, p. 128)] Let G be an open set in R. For each $x \in G$ let $D(x) = (D_{ij}(x))$ be a strictly positive definite $n \times n$-matrix. Let $d(x) = (d_{ij}(x))$ be the positive definite square root of $D(x)$. If $D_{ij} \in C^m$, i.e., has continuous derivatives up to order m, for all i and j, then for all i and j $d_{ij} \in C^m$.

To apply this result, notice that since B is symmetric its positive square root is symmetric. Thus we can use the square root of B for b. By (3.3.4), b is a continuous process which by (2.6.4) can be chosen to be progressively measurable.

In the following theorem we establish sufficient conditions on a process and its moment rates to insure that it is an Ito process.

<u>3.3.5</u> <u>Theorem</u> Let $z(t) = (z_i(t))$, $(t \geq 0)$, $(i=1,\ldots,n)$ be a continuous stochastic process that is progressively measurable with

respect to $(\mathcal{F}_t, t \geq 0)$. Let $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t))$, $(t \geq 0)$, $(i,j=1,\ldots,n)$ be stochastic processes in CQ^1 such that for each $\omega \in \Omega$ and $t \geq 0$, $B(t,\omega)$ is a symmetric nonnegative definite matrix. Suppose that whenever separable versions of the processes given by the conditional expectations below are taken, the following conditions hold for $u \geq 0$ and $i,j=1,\ldots,n$:

(i) there is a number $\delta_u > 0$ and a nonnegative random variable M_u with finite expectation such that for every h $(0,\delta_u)$

$$\frac{1}{h} \, E[z_i(u+h)-z_i(u) \mid \mathcal{F}_u]| \leq M_u \quad \text{a.s.,}$$

(ii) $\lim_{h\downarrow 0} \frac{1}{h} \, E[z_i(u+h)-z_i(u) \mid \mathcal{F}_u] = a_i(u)$ a.s.,

(iii) $\lim_{h\downarrow 0} \frac{1}{h} \, E[(z_i(u+h)-z_i(u))(z_j(u+h)-z_j(u)) \mid \mathcal{F}_u] = B_{ij}(u)$ a.s. and

(iv) $\lim_{h\downarrow 0} \frac{1}{h} \, E[|z_i(u+h)-z_i(u)|^3 \mid \mathcal{F}_u] = 0$ a.s.

Then z is an Ito process with covariance B and drift a.

Proof: We take separable versions of all processes occurring throughout the proof. All limits taken here are then justified in the sense of (2.4.1).

First we note that by Lemma 3.2.8 and condition (iv) we have, for $1 \leq i,j,k \leq n$,

$$\lim_{h \downarrow 0} \frac{1}{h} E[\prod_{m=i,j,k} | z_m(u+h)-z_m(u)| \quad |\mathcal{F}_u] = 0 \quad \text{a.s.} \tag{3.3.6}$$

Now let f be in $C_0^\infty(R^n)$. We will show that (3.3.2) holds for z. By Taylor's formula, we have for $h \geq 0$,

$$f(z(u+h))-f(z(u)) = \sum_{i=1}^{n} (z_i(u+h)-z_i(u)) f_{x_i}(z(u))$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} (z_i(u+h)-z_i(u))(z_j(u+h)-z_j(u)) f_{x_i x_j}(z(u))$$

$$+ \frac{1}{6} \sum_{i,j,k=1}^{n} [\prod_{m=i,j,k} (z_m(u+h)-z_m(u)) f_{x_i x_j x_k}(\zeta)]$$

where ζ is a point on the line connecting $z(u)$ and $z(u+h)$. Since $f_{x_i}(z(u))$ and $f_{x_i x_j}(z(u))$ are $\mathcal{F}_u$-measurable and $f_{x_i x_j x_k}$ are bounded functions we see that by (ii), (iii) and (3.3.6)

$$\lim_{h \downarrow 0} \frac{1}{h} E[f(z(u+h))-f(z(u)) \mid \mathcal{F}_u] = (L_u f)(z(u)) \quad \text{a.s.}$$

where $L_u f$ is as defined in (3.3.1). Thus for $0 \leq s \leq u$,

$$E[\lim_{h \downarrow 0} \frac{1}{h} E[f(z(u+h))-f(z(u)) | \mathcal{F}_u] | \mathcal{F}_s] = E[(L_u f)(z(u)) | \mathcal{F}_s] \quad \text{a.s.} \tag{3.3.7}$$

Let H_o be a separating set for the process $E[f(z(u+h)) - f(z(u))|\mathcal{F}_u]$, $(h \geq 0)$. By the Mean Value Theorem and the fact that f has compact support,

$$|f(z(u+h)) - f(z(u))| \leq C\ |z(u+h) - z(u)|$$

for some constant C. Using (i) with the above equation shows that we can apply the Dominated Convergence Theorem (2.1.2) and use (2.1.1g) and (2.4.1) to obtain

$$E[\lim_{h\downarrow 0} \frac{1}{h}\ E[f(z(u+h)) - f(z(u))\ |\mathcal{F}_u]\ |\mathcal{F}_s]$$

$$= E[\lim_{h\downarrow 0, h\in H_o} \frac{1}{h}\ E[f(z(u+h) - f(z(u))\ |\mathcal{F}_u]\ |\mathcal{F}_s] \quad \text{a.s.}$$

$$= \lim_{h\downarrow 0, h\in H_o} \frac{1}{h}\ E[f(z(u+h)) - f(z(u))\ |\mathcal{F}_s] \quad \text{a.s.}$$

$$= \lim_{h\downarrow 0} \frac{1}{h}\ E[f(z(u+h)) - f(z(u))\ |\mathcal{F}_s] \quad \text{a.s.}$$

Thus,

$$\lim_{h\downarrow 0} \frac{1}{h}\ E[f(z(u+h))-f(z(u))|\mathcal{F}_s]=E[(L_u\ f)(z(u))|\mathcal{F}_s] \quad \text{a.s.} \quad (3.3.8)$$

Since f_{x_i} $(z(t))$ and $f_{x_i x_j}$ $(z(t))$, $(t \geqslant 0)$ are continuous processes, f_{x_i} and $f_{x_i x_j}$ are bounded functions and a and B are in CQ^1, we see that the process $(L_u\ f)(z(u))$, $(u \geqslant 0)$ is in CQ^1. Then by Theorem 2.6.13 we may choose a version of the process $E[(L_u\ f)(z(u)) | \mathcal{F}_s]$, $(u \geqslant 0)$ which is continuous. Using (2.6.3) and (2.6.4) we make this choice so that the process is both continuous and progressively measurable with respect to $(\mathcal{F}_t,\ t \geqslant 0)$. In accordance with (3.3.8), this provides a continuous and progressively measurable version of

$$\lim_{h \downarrow 0} \frac{1}{h}\ E[f(z(u+h)) - f(z(u)) | \mathcal{F}_s],\ (u \geqslant 0).$$

Thus we can integrate (3.3.8) to obtain

$$\int_s^t \lim_{h \downarrow 0} \frac{1}{h}\ E[f(z(u+h)) - f(z(u))\ |\ \mathcal{F}_s]\ du$$

$$= \int_s^t E[(L_u\ f)(z(u))\ | \mathcal{F}_s]\ du \quad \text{a.s.}$$

We now apply Lemma 2.9.6 to the left side and Theorem 2.5.3 to the right side of the above equation to get (3.3.2). □

Combining (3.3.3) and Theorem 3.3.5, one obtains the following representation theorem, which characterizes certain Ito integrals in terms of moment rate properties:

<u>3.3.9</u> <u>Theorem</u> Suppose that z, a and B satisfy the hypothesis of Theorem 3.3.5. Suppose further that a and B are bounded and that for each $t \geqslant 0$ and $\omega \in \Omega$, $B(t,\omega)$ is strictly positive definite. Then for any progressively measurable process b such that $bb^T = B$, there is an n-dimensional Brownian motion w adapted to $(\mathcal{F}_t, t \geqslant 0)$ for which

$$z(t) = z(0) + \int_0^t a(s)ds + \int_0^t b(s)dw(s)$$

We next consider the case when B is only nonnegative definite. The difficulty is that we have no guarantee in this case that the probability space $(\Omega,\mathcal{F},P)$ is "large enough" to have a Brownian motion. To handle this type of situation, Doob (1953, p. 287) had the idea of adjoining a probability space, containing a Brownian motion independent of the process in question, to the original space. Quoting Stroock and Varadhan (1979, p. 108), "We then hold this new independent Brownian motion in reserve and only call on it to fill the gaps created by lapses in 'randomness' of the original Ito process." Using this technique, Stroock and Varadhan (1979, p. 108) obtained:

<u>3.3.10</u> Let $z(t) = (z_i(t))$, $(t \geqslant 0)$,$(i=1,\ldots,n)$ be an Ito process with covariance B and drift a, where a and B are bounded. Let $b(t) = (b_{ij}(t))$, $(t \geqslant 0)$, $(i,j=1,\ldots,n)$ be a

process which is progressively measurable with respect to $(\mathcal{F}_t, t \geq 0)$ such that $bb^T = B$. Then there is a probability space $(\Lambda, \mathcal{G}, Q)$, a nondecreasing family of sub σ-fields $(\mathcal{G}_t, t \geq 0)$ of $\mathcal{G}$, stochastic processes $\tilde{a}$, $\tilde{B}$, $\tilde{b}$ and $\tilde{z}$ and an n-dimensional Brownian motion $\tilde{w}$ on $(\Lambda, \mathcal{G}, Q)$ that are progressively measurable with respect to $(\mathcal{G}_t, t \geq 0)$, such that a, B, b and z have the same finite dimensional probability distributions as a, B, b and z, respectively,

and

$$\tilde{z}(t) = \tilde{z}(0) + \int_0^t \tilde{a}(s)ds + \int_0^t \tilde{b}(s)d\tilde{w}(s) \quad \text{a.s.}[Q] \ .$$

Now combining (3.3.10) and Theorem 3.3.5 we obtain

<u>3.3.11</u> <u>Theorem</u> Suppose that z, a and B satisfy the hypotheses in Theorem 3.3.5 and suppose further that a and B are bounded. Let b be a progressively measurable process such that $bb^T = B$. Then the conclusion of (3.3.10) holds.

We will end this section with a theorem which explains in what sense mathematical models described by a stochastic integral are mathematically equivalent to models described by Carter's conditions, as given in the introduction. First we need the following lemma.

<u>3.3.12</u> <u>Lemma</u> Let $a(t) = (a_i(t))$ and $b(t) = (b_{ij}(t))$, $(t \geq 0)$, $(i,j=1,\ldots n)$ be stochastic processes in $CQ^2 \cap M^4$ and CQ^4, respectively. If $z(t) = (z_i(t))$, $(t \geq 0)$, $(i=1,\ldots,n)$ is a stochastic integral given by

$$z(t) = z(0) + \int_0^t a(s)ds + \int_0^t b(s)dw(s) \quad \text{a.s.},$$

where w is an n-dimensional Brownian motion, and $B(t) = (B_{ij}(t))$, $(t \geq 0)$, $(i,j=1,\ldots,n)$ is a process given by $B = bb^T$, then z, a and B satisfy the hypotheses of Theorem 3.3.5

<u>Proof</u>: That for each $\omega \in \Omega$ and $t \geq 0$, $B(t,\omega)$ is a symmetric non-negative definite matrix follows from the form of B, i.e., bb^T. Since b is in CQ^4 we see by (3.1.5) that bb^T is in CQ^2 and hence that B is in CQ^1.

Conditions (ii), (iii) and (iv) in Theorem 3.3.5 follow from Theorems 3.2.3, 3.2.5 and 3.2.9(4), respectively.

To show that z satisfies condition (i), we have the following estimates, where the first step is justified by Lemma 3.2.2.

$$\frac{1}{h} |E[z_i(u+h)-z_i(u) | \mathcal{F}_u]| = \frac{1}{h}|E[\int_u^{u+h} a_i(s)ds | \mathcal{F}_u]| \quad \text{a.s.}$$

$$\leq \frac{1}{h} E[\int_u^{u+h} |a_i(s)| ds \, | \mathcal{F}_u] \quad \text{a.s.}$$

$$\leq E[\sup_{t \in [0,u+1]} | a_i(t) | \; | \mathcal{F}_u] \quad \text{a.s.}$$

Since $a \in Q^1$, we have

$$E[E[\sup_{t \in [0,u+1]} |a_i(t)| \ |\mathcal{F}_u]] = E[\sup_{t \in [0,u+1]} |a_i(t)|] < \infty .$$

This establishes (i) with $\delta_u = 1$ and

$$M_u = E[\sup_{t \in [0,u+1]} |a_i(t)| \ |\mathcal{F}_u] . \qquad \square$$

Making the observation that a bounded, progressively measurable stochastic process is in Q^p for all p, we combine Lemma 3.3.12, Theorem 3.3.9 and Theorem 3.2.9 to obtain

<u>3.3.13</u> <u>Theorem</u> Let $a(t) = (a_i(t))$ and $B(t) = (B_{ij}(t))$, $(t \geqslant 0)$, $(i,j=1,\dots,n)$ be bounded, progressively measurable, continuous stochastic processes, such that for each $t \geqslant 0$ and $\omega \in \Omega$, $B(t,\omega)$ is symmetric strictly positive definite. Let $z(t) = (z_i(t))$, $(t \geqslant 0)$, $(i=1,\dots,n)$ be a progressively measurable, continuous stochastic process. Then conditions 1 and 2 below are equivalent.

<u>Condition 1</u> Whenever separable versions of the processes given by the conditional expectations below are taken, then for all $u \geqslant 0$, $1 \leqslant i, j \leqslant n$, $m \geqslant 3$ and all finite sequences, $(i_k : k=1,\dots m)$ from $1,\dots,n$.

(i) there is a number $\delta_u > 0$ and a nonnegative random variable M_u with finite expectation such that for every $h \in (0,\delta_u)$

$$\left|\frac{1}{h} E[z_i(u+h) - z_i(u) \mid \mathcal{F}_u]\right| \leqslant M_u \quad \text{a.s.}$$

(ii) $$\lim_{h\downarrow 0} \frac{1}{h} E[(z_i(u+h) - z_i(u)) \mid \mathcal{F}_u] = a_i(u) \quad \text{a.s.}$$

(iii) $$\lim_{h\downarrow 0} \frac{1}{h} E[(z_i(u+h)-z_i(u))(z_j(u+h)-z_j(u)) \mid \mathcal{F}_u] = B_{ij}(u) \quad \text{a.s.}$$

(iv) $$\lim_{h\downarrow 0} \frac{1}{h} E[\prod_{k=1}^{m} |z_{i_k}(u+h)-z_{i_k}(u)| \quad \mid \mathcal{F}_u] = 0 \quad \text{a.s.}$$

Condition 2 For each progressively measurable $n \times n$ stochastic matrix $b(t)$, $(t \geqslant 0)$ that satisfies $bb^T = B$ there is an n-dimensional Brownian motion w adapted to $(\mathcal{F}_t, t \geqslant 0)$ such that

$$z(t) = z(0) + \int_0^t a(u)du + \int_0^t b(u)dw(u) \quad \text{a.s.}$$

A corresponding result is obtained for the case in which B is nonnegative definite by replacing Condition 2 above by the conclusion of (3.3.10).

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