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Two numerical methods are presented that can be used to solve second order nonlinear ordinary differential equations with periodic boundary conditions. One of these methods is a shooting method developed solely for the periodic problem. The other, "quasilinearization," is a method applicable to a wide variety of problems. It is presented in a quite general setting; and then is used to solve the periodic problem. Under suitable hypotheses both methods are shown to converge. Numerical results are given.

Secondly, we prove bifurcation of solutions of nonlinear Sturm-Liouville problems as well as some related global results. The approach used does not use degree theory, Liaupunov-Schmidt theory, or functional analysis; but instead, elementary facts about continuous functions and differential equations.

BOUNDARY VALUE PROBLEMS AND BIFURCATION THEORY FOR ORDINARY DIFFERENTIAL EQUATIONS

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David Malloy

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APPROVED:

Redacted for privacy

John Walter Lee, Professor of Mathematics
(in charge of major)

Redacted for privacy

Richard Schori, Department of Mathematics

Redacted for privacy

Robert W. Newburgh, Dean of Graduate School

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PREFACE

In 1976, A. Granas, R.B. Guenther, and J.W. Lee wrote a paper titled "On a Theorem of S. Bernstein" [6] in which they give conditions for existence and uniqueness of solutions of certain ordinary differential equations. The problems considered formed a class of nonlinear equations with Dirichlet, Neumann, Sturm-Liouville, or periodic boundary conditions. In a subsequent paper [10] they present numerical methods ("shooting methods") for solving the Dirichlet, Neumann and Sturm-Liouville problems; and gave proof that the methods converge. The case of periodic boundary conditions was not considered in [10] although this case is of considerable practical and theoretical importance.

Chapter Two of this thesis is concerned with the numerical aspects of the periodic problem. Two numerical schemes are presented. The first ("shooting") might be considered the most natural extension of the methods used in [10]. The second, the technique of "quasilinearization," is a scheme applicable to a wide variety of problems, although it has not, to my knowledge, been applied to periodic boundary value problems. In both cases, we prove that the methods converge under suitable hypotheses; among them the hypotheses used in [6] to establish existence and uniqueness of solutions. Finally, we do some numerical experiments.

Chapter Three, "Bifurcation Theory," was motivated in part by an informal seminar held in the spring of 1978 by John Lee, Ron Guenther, and myself. There we considered several topics, among them the Euler buckling beam problem and the degree theory of Krasnoselskii. We were interested in showing basic facts about solutions of second order boundary value problems involving a parameter.

Degree theory yields the results quickly, once the machinery is set up. However, it seemed desirable to prove the results in an elementary fashion, relying only, if possible, on the basic theory of ordinary differential equations.

Such a program is undertaken here. The approach is modeled, in part, after a paper by Macki and Waltman [17]; but there are some major differences. For one, the Macki and Waltman paper doesn't assume uniqueness of initial value problems. We make this assumption, which simplifies the proof greatly, and enables us to draw stronger conclusions. In particular, we show that each bifurcating branch of solutions forms a continuous curve, and we are able to make more definitive statements about the initial "shape" of the curves. The exposition here is indeed elementary: the major tools used are properties of continuous functions, and some comparison theorems from ordinary differential equations.

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CHAPTER I. PRELIMINARIES

A. COMMENTS AND NOTATION

This paper will concern itself with finding the numerical solution of second order nonlinear boundary value problems of the form:

$u'' = f(t, u, u')$ on the finite closed interval $[\alpha, \beta]$ where $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

A solution to such a problem is a real valued function u which is twice differentiable on $[\alpha, \beta]$ and satisfies the equation:

$$u''(t) = f(t, u(t), u'(t)) \text{ for each } t \in [\alpha, \beta].$$

Also, u may be required to satisfy the additional (boundary) conditions:

$$au(\alpha) + a'u'(\alpha) + bu(\beta) + b'u'(\beta) = A$$

$$cu(\alpha) + c'u'(\alpha) + du(\beta) + d'u'(\beta) = B$$

$$\text{where rank} \begin{bmatrix} a & a' & b & b' \\ c & c' & d & d' \end{bmatrix} = 2.$$

Special cases of the boundary conditions include:

initial conditions	:	$u(\alpha) = A \quad u'(\alpha) = B$
Dirichlet boundary conditions	:	$u(\alpha) = A \quad u(\beta) = B$
Neumann boundary conditions	:	$u'(\alpha) = A \quad u'(\beta) = B$

Sturm-Liouville boundary conditions: $au(\alpha) + a'u'(\alpha) = 0$

$du(\beta) + d'u'(\beta) = 0$

and periodic boundary conditions : $u(\alpha) = u(\beta) \quad u'(\alpha) = u'(\beta)$.

There are many excellent numerical schemes that provide approximate solutions to problems with initial conditions. These methods have the following characteristics: the interval $[\alpha, \beta]$ is partitioned by a finite set of "grid" points $\alpha = t_0 < t_1 < \ldots < t_n = \beta$, and $t_i - t_{i-1}$, is called the i-th stepsize. Given the initial data $u(\alpha)$ and $u'(\alpha)$, approximations to the solution u and its derivative u' are generated at each of the grid points. The approximations y_i and y'_i to $u(t_i)$ and $u'(t_i)$ may depend only on the immediately preceding approximations y_{i-1} and y'_{i-1} (as in the Runge-Kutta method we used), or they may depend on some or all of the preceding values $(y_j, y'_j)_{0\leq j\leq i-1}$, as in the "multi-step" methods.

In practice, the accuracy of the approximations depends on many factors. Error is inherent: computing machines are unable to do real number arithmetic with perfect precision; and since the number of grid points is finite, the differential equation is only sampled at a finite number of points. A common feature of useful algorithms is that if the precision of computation is perfect, decreasing the stepsizes results in better approximations to the true solution.

For a more thorough discussion, see [2].

Since we desire approximate numerical solutions to boundary value problems, our philosophy is to somehow reduce such a problem to one or more initial value problems, and then to use an initial value method to provide us with a solution.

The following notation will be used: $C^n[\alpha, \beta]$ denotes the space of n-times continuously differentiable functions on $[\alpha, \beta]$; $C[\alpha, \beta]$, the continuous functions on $[\alpha, \beta]$. A lower subscript on a function will indicate a differentiation.

For example, if $f = f(t, u, u')$, $f_u(t, u, u') = \frac{\partial}{\partial u} f(t, u, u')$ and $f_{u'}(t, u, u') = \frac{\partial}{\partial u'} f(t, u, u')$. The norms to be used are the "usual" ones:

On $C^n[\alpha, \beta]$ we use

$$||u|| = \max_{t \in [\alpha, \beta]} \{|u(t)|, |u'(t)|, \cdots, |u^{(n)}(t)|\}$$

On R^n we use the Euclidean norm:

$$||(x_1, \cdots, x_n)|| = \sqrt{x_1^2 + \cdots + x_n^2}$$

The norm on a product AxB, where A and B are Banach spaces, will be:

$$||(a, b)||_{AxB} = \max \{||a||_A, ||b||_B\}.$$

B. THEOREMS FROM ORDINARY DIFFERENTIAL EQUATIONS

The following definitions and theorems from the theory of ordinary differential equations will be essential. They are stated in the language of second order differential equations; in the terms that they will be needed. (The results are usually stated in terms of a first order system of differential equations. The transformation from a second order equation to a first order system is straightforward [7, 2b]). In what follows, $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

DEFINITION: f satisfies a <u>Lipschitz condition in a region</u> $R \subseteq [\alpha, \beta] \times \mathbb{R} \times \mathbb{R}$ if there is a constant k so that if (t, x_1, y_1), $(t, x_2, y_2) \in R$, then $|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq k|x_1 - x_2| + k|y_1 - y_2|$.

DEFINITION: f is <u>locally Lipschitz</u> if f satisfies a Lipschitz condition in some open neighborhood of each point in its domain.

REMARK: If f is continuously differentiable, then f is locally Lipschitz [7, 2.3, Lemma 1].

THEOREM A (Uniqueness): If f is locally Lipschitz, then there is at most one solution to the problem

$$u'' = f(t, u, u') \qquad u(t_0) = x_0 \qquad u'(t_0) = \alpha_0$$

PROOF: [7, 2.5, Theorem 3].

THEOREM B (Local Existence): Let f satisfy a Lipschitz condition in $R = [t_0, t_0 + \varepsilon] \times [x_0 - \eta, x_0 + \eta] \times [\alpha_0 - \eta, \alpha_0 + \eta]$.

Let $M = \max_{(t, x, \alpha)\in R}\{|f(t, x,\alpha)|, |\alpha|\}$. Then there exists a solution u of

$$u'' = f(t, u, u') \qquad u(t_0) = x_0 \qquad u'(t_0) = \alpha_0$$

for $t\in[t_0, t_0 + h]$ where $h = \min\{\varepsilon, \frac{\eta}{M}\}$.

PROOF: [7, 2.5, Theorem 4].

THEOREM C (Existence of Maximal Solutions): Let f satisfy a local Lipschitz condition in the region $R\subseteq[\alpha, \beta] \times \mathbb{R} \times \mathbb{R}$. If $(t_0, x_0, \alpha_0)\in R$ then the initial value problem

$$u'' = f(t, u, u') \qquad u(t_0) = x_0 \qquad u'(t_0) = \alpha_0$$

has a unique solution u with maximal domain of definition (that is, no other solution to the problem has a larger domain). Further, suppose that the maximal solution u is defined on an interval with endpoints a and b. Define the function $\rho(t)$ to be the distance of the point $(t, u(t), u'(t))$ to the boundary of R. Then either $\lim_{t\to c}\rho(t) = 0$ or $\lim_{t\to c}||(u(t), u'(t))|| = \infty$ for c = a and c = b.

PROOF: [7, 2.5, Theorem 11].

THEOREM D (Differentiable Dependence on Initial Conditions): Let f be locally Lipschitz in the region $R\subseteq[\alpha, \beta] \times \mathbb{R} \times \mathbb{R}$, and let u be the uniquely defined solution with maximal domain to the initial value problem

$$u'' = f(t, u, u') \qquad u(t_0) = x_0 \qquad u'(t_0) = \alpha_0$$

$$(t_0, x_0, \alpha_0)\in R.$$

Let the domain of u be the interval J and let [a, b] be any closed subinterval of J containing t_0. Then there is an $\varepsilon > 0$ such that for every (t, x, α) such that $||(t, x, \alpha) - (t_0, x_0, \alpha_0)|| < \varepsilon$ there is a unique solution y of

$$u'' = f(t, u, u') \qquad u(t) = x \qquad u'(t) = \alpha,$$

whose domain contains [a, b]. Further, if f is continuously differentiable, the function y will depend continuously and differentiably on the initial condition (t, x, α).

PROOF: [7, 2.5, Theorem 9].

THEOREM E (Differentiable Dependence on Initial Conditions and a Parameter): Let $g:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \times A \rightarrow \mathbb{R}$ be continuously differentiable where $A \subseteq \mathbb{R}^n$ is an open set. Let u be the maximal solution to

$$u'' = g(t, u, u', \lambda_0) \qquad u(t_0) = x_0 \qquad u'(t_0) = \alpha_0$$
$$\lambda_0 \in A.$$

Let the domain of u be the interval J and let [a, b] be a closed subinterval of J containing t_0. Then there is an $\varepsilon > 0$ so that if $||(t, x, \alpha, \lambda) - (t_0, x_0, \alpha_0, \lambda_0)|| < \varepsilon$ there is a solution y of

$$u'' = g(t, u, u', \lambda) \qquad u(t) = x \qquad u'(t) = \alpha \qquad \lambda \in A,$$

whose domain contains [a, b]. Further, y depends continuously and differentiably on the initial conditions and parameter (t, x, α, λ).

PROOF: [7, 2.5, Theorem 10].

THEOREM F (Comparison Theorem): Let u and v be solutions to the first order differential equations $u' = g(t, u)$ $v' = h(t, v)$, respectively, where $g(t, u) \leq h(t, u)$ for $t \in [\alpha, \beta]$ and g or h satisfies a Lipschitz condition. Let $u(\alpha) = v(\alpha)$. Then $u(t) \leq v(t)$ for all $t \in [\alpha, \beta]$.

PROOF: [4, 1.12].

COROLLARY F.1: In the previous theorem, if $t_1 > \alpha$, then either $u(t_1) \leq v(t_1)$ or $u(t) \equiv v(t)$ for $\alpha \leq t \leq t_1$.

PROOF: [4, 1.12].

COROLLARY F.2: In the previous theorem, assume g and h satisfy a Lipschitz condition. If $u(\alpha) < v(\alpha)$ then $u(t) < v(t)$ for all $t \in [\alpha, \beta]$.

PROOF: [4, 1.12].

The main existence and uniqueness result for boundary value problems to be used is an extension of a classical theorem by S. Bernstein:

THEOREM G: Suppose $f(t, u, u')$ is continuous in $[\alpha, \beta] \times \mathbb{R} \times \mathbb{R}$ and there is a constant $M > 0$ so that $uf(t, u, 0) \geq 0$ for $|u| > M$. Suppose further that $|f(t, u, u')| \leq A(t, u)u'^2 + B(t, u)$ where $A, B \geq 0$ are functions bounded for (t, u) in $[\alpha, \beta] \times [-M, M]$. Then the problems

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) = 0 \qquad \text{(Dirichlet)}$$

$$u'' = f(t, u, u') \qquad u'(\alpha) = u'(\beta) = 0 \qquad \text{(Neumann)}$$

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta) \text{ (periodic)}$$

all have at least one solution. If in addition, f_u and $f_{u'}$ are bounded and $f_u \geq 0$ then the solution of the Dirichlet problem is unique, and any two solutions of the Neumann or the periodic problem differ by a constant. Finally, if in addition to this, $f_u(t_0, u, u') > 0$ for a fixed $t_0 \in [\alpha, \beta]$, then solutions to the Neumann and the periodic problems are unique.

PROOF: [6].

C. NEWTON'S METHOD

In this section we present one of our most important tools: Newton's method--a scheme for finding the roots of a nonlinear equation. We present it and its proof in a Banach space context, as we will need it in this generality for some of the applications.

Let $(A, \|\ \|)$ and $(B, \|\ \|)$ be Banach spaces, $D \subseteq A$ an open set and $F:D \to B$ continuously Frechét differentiable. Newton's method is an iterative procedure used to solve the equation $F(x) = 0$. It works as follows: an initial vector, x_1, an approximation to the solution, is guessed and a sequence is generated by solving the recursion formula

$$0 = F(x_n) + F'(x_n)(x_{n+1}-x_n) \tag{1}$$

for x_{n+1} in D. If $\{x_n\}$ converges to some vector x_0 in D, then (1) reduces to $F(x_0) = 0$, as desired.

The following convergence result will be used.

THEOREM H (Newton's Method): Let $F:D \subseteq A \to B$, as above. Assume that $F(x_0) = 0$, that $F'(x_0)^{-1}$ exists and is continuous, and that $F'(x)$ is continuous at x_0. Then there is a $\delta > 0$ so that if $\|x_0 - x_1\| < \delta$, the Newton sequence $x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)$ is well defined, and converges to x_0.

PROOF: Without loss of generality, we can assume $x = 0$ in the proof. The main idea of the proof is to show that the function $G(x) = x - F'(x)^{-1}F(x)$ is contractive in a certain ball centered at the origin. First we show that the function

$$k(x) = \quad x - F'(x)^{-1}F(x) \quad \text{is } o(x)\text{, that is,}$$

that $\lim_{||x||\to 0} \frac{k(x)}{||x||} = 0.$

$$\begin{aligned} \text{Write } k(x) &= ||x - F'(x)^{-1}F(x)|| \\ &= ||x - F'(0)^{-1}F(x) + (F'(0)^{-1} - F'(x)^{-1})F(x)|| \\ &\leq ||x - F'(0)^{-1}F(x)|| + \\ &\quad ||(F'(0)^{-1} - F'(x)^{-1})F(x)||, \end{aligned}$$

and handle each piece separately.

First, $||x - F'(0)^{-1}F(x)||$ is $o(x)$:
since $F(x) = F(0) + F'(0)x + o(x) = F'(0)x + o(x)$, it follows that

$$\begin{aligned} ||x - F'(0)^{-1}F(x)|| &= ||x - F'(0)^{-1}(F'(0)x + o(x))|| \\ &= ||x - x + F'(0)^{-1}o(x)|| \\ &= o(x), \end{aligned}$$

since $F'(0)^{-1}$ is continuous.

Next, $||(F'(0)^{-1} - F'(x)^{-1})F(x)||$ is $o(x)$. Because $F'(0)^{-1}$ exists, $\frac{||F(x)||}{||x||}$ is bounded in a neighborhood of 0, and

$\lim_{||x||\to 0} (F'(0)^{-1} - F'(x)^{-1}) = 0$, by the continuity of $F'(x)$ at 0.

Thus $k(x) = o(x)$ as asserted. It follows that there is a $\delta_1 \geq 0$ so that $||x - F'(x)^{-1}F(x)|| \leq \frac{1}{2}||x||$ for $||x|| < \delta_1$. Let $\varepsilon > 0$ be so that $\{x : ||x|| < \varepsilon\} \subset D$. (Recall that D, the domain of F, is open). Let δ_2 be such that $F'(x)^{-1}$ exists for $||x|| \leq \delta_2$. Now let $\delta = \min\{\varepsilon, \delta_1, \delta_2\}$. For all x_1 such that $||x_1|| < \delta$, the Newton sequence will be well defined. Further, x_n converges to 0 because $||x_n|| \leq \frac{1}{2^n}\, ||x_1||$. Hence the proof is complete.

DISCUSSION: To conclude then, if F is continuously differentiable, the Newton sequence is well defined and will converge to a solution of $F(x) = 0$ if

i) a solution y_0 of $F(x) = 0$ exists,

ii) $F'(y_0)^{-1}$ exists and is continuous at y_0,

iii) $||y_0 - y_1||$ is sufficiently small, where y_1 is the initial guess.

We will use Newton's method often. In each application, in order to check that the method converges, we must address each of these conditions.

To handle the first condition, that roots exist, we will need existence (and uniqueness) results, such as THEOREM G. Finding good first guesses--condition (iii) will be a problem, as we will see. Estimates exist on "how close" is "close enough" but they are quite unwieldy [9], [11].

Our "convergence" proof, then, will run along the following lines: first, an existence theorem is used to conclude there are solutions; second, we show that under suitable hypotheses condition (ii) is satisfied, that is, that the relevant derivative is invertible and finally, conclude that if our initial guess is close enough to a root of F, the method will converge.

For a more complete discussion of Newton's method, as well as many refinements, see [11].

D. THE DIRICHLET PROBLEM

In this section we illustrate how Newton's method can be used to help solve the kinds of problems we are interested in.

Suppose $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that we want to numerically solve the Dirichlet problem:

$$u'' = f(t, u, u') \qquad u(\alpha) = 0 \qquad u(\beta) = 0$$

Suppose that a unique solution, $u(t)$, to this problem exists. We could, for example, assume the hypotheses of THEOREM G.

If we knew $u'(\alpha)$ (or $u'(\beta)$), then we could use an initial value method to compute $u(t)$. The plan then, is to somehow compute $u'(\alpha)$, the missing initial condition.

First, let $y(t, a)$ be the unique maximal solution (THEOREM C) to the initial value problem

$$u'' = f(t, u, u') \qquad u(\alpha) = 0 \qquad u'(\alpha) = a$$

Now let $h(a) = y(\beta, a)$, if it exists, i.e., if the solution $y(t, a)$ is defined on the entire interval $[\alpha, \beta]$. By THEOREM D it follows that h is a differentiable function with an open domain. Further, since the Dirichlet problem has a unique solution, h has a unique root, namely, $u'(\alpha)$. To determine this root we apply Newton's method to the function h and form the sequence

$$a_{n+1} = a_n - \frac{h(a_n)}{h'(a_n)} ,$$

where a_1 is our initial guess, or "shooting slope." We then iterate until $|h(a_n)|$ is less than a prescribed tolerance.

Note that $h'(a_n) = y_a(\beta, a_n)$, where $y_a(t, a_n)$ solves the differential equation:

$$y_a''(t, a_n) = f_u(t, y(t, a_n), y'(t, a_n))y_a(t, a_n) + f_{u'}(t, y(t, a_n), y'(t, a_n))y_a'(t, a_n)$$

$$y_a(\alpha, a_n) = 0 \qquad y_a'(\alpha, a_n) = 1$$

So for each iteration, we are required to solve two initial value problems. A linear one-to determine $y_a(\beta, a_n) = h'(a_n)$, and a non-linear one to determine $h(a_n) = y(\beta, a_n)$.

Now we must consider the problem of whether the sequence is well defined, and if it converges to a solution. Note that THEOREM G gives us conditions on f so that a unique solution to the Dirichlet problem exists. In [10], it is shown that under the same hypotheses, condition (ii) of Section C is fulfilled: that the relevant derivative is invertible. Here we require that $h'(a) \neq 0$, where a is the correct initial slope. So, we are assured convergence as long as the initial guess is close enough.

If the initial guess is not close enough the Newton sequence may not converge and may not even be defined. To carry out the iterative step, recall that a_n must be in the domain of h and $h'(a_n) \neq 0$. Here, a_n not in the domain of h means that the solution of the initial value problem

$$u'' = f(t, u, u') \qquad u(\alpha) = 0 \qquad u'(\alpha) = a_n$$

doesn't extend across the interval $[\alpha, \beta]$. See [10] for an example.

This method can be modified to handle other boundary value problems, such as Neumann or Sturm-Liouville. In each of these cases, we are missing a single initial condition--the root of an appropriately defined real valued function. We then use Newton's method to find it. For a discussion, see [10].

CHAPTER II. THE PERIODIC PROBLEM

A. INTRODUCTION

In this chapter we consider two methods to solve the periodic problem:

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta)$$

The first, which I call "shooting," is a natural extension of the method used in the last chapter to solve the Dirichlet problem. Next, a technique known as quasilinearization is introduced. It is a method applicable to a wide range of boundary value problems, but we use it only for the problem at hand, the periodic problem. We show that in each case, with the hypotheses of THEOREM G, the methods converge; given, as always, a sufficiently good first guess.

In the last section of this chapter we present some numerical results and draw some interesting conclusions.

B. THE SHOOTING METHOD FOR PERIODIC PROBLEMS

Here we present a method to solve the periodic boundary value problem

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta) \tag{1}$$

where $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. For now, assume there exists a unique solution to this problem. Let $u(t, x, a)$ be the unique maximal solution (THEOREM C) to the initial value problem

$$u'' = f(t, u, u') \qquad u(\alpha) = x \qquad u'(\alpha) = a \tag{2}$$

Now define the functions g and h by:

$$g(t, x, a) = u(t, x, a) - x$$
$$h(t, x, a) = u'(t, x, a) - a$$

$g(\beta, \cdot, \cdot)$ and $h(\beta, \cdot, \cdot)$ are continuously differentiable functions with a common open domain $D \subseteq \mathbb{R}^2$ (THEOREM D). D is non-empty because (1) is assumed solvable. Finding a solution to the periodic problem (1) is equivalent to finding a point $(x, a) \in D$ for which $g(\beta, x, a) = h(\beta, x, a) = 0$. Finally, define $F: D \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, a) = \begin{bmatrix} g(\beta, x, a) \\ h(\beta, x, a) \end{bmatrix}$$

F is continuously differentiable because g and h are.

Now apply Newton's method to the function F. The Newton sequence becomes:

$$[x_{n+1}, a_{n+1}] = [x_n, a_n] - F'(x_n, a_n)^{-1}F(x_n, a_n) \tag{3}$$

or, explicitly:

$$[x_{n+1}, a_{n+1}] = [x_n, a_n] - \frac{1}{g_x h_a - h_x g_a} \begin{bmatrix} h_a & -g_a \\ -h_x & g_x \end{bmatrix} \begin{bmatrix} g \\ h \end{bmatrix} \tag{4}$$

where each function is evaluated at (β, x_n, a_n).

Note that

$$g_x(t, x, a) = u_x(t, x, a) - 1$$
$$g_a(t, x, a) = u_a(t, x, a)$$
$$h_x(t, x, a) = u'_x(t, x, a)$$
$$h_a(t, x, a) = u'_a(t, x, a) - 1$$

and that $u_x(t, x, a)$ and $u_a(t, x, a)$ are both solutions of the differential equation:

$$v''(t) = f_u(t, u(t, x, a), u'(t, x, a))v(t) + f_{u'}(t, u(t, x, a), u'(t, x, a))v'(t) \tag{5}$$

These solutions statisfy the initial conditions:

$$u_x(\alpha, x, a) = 1 \qquad u'_x(\alpha, x, a) = 0 \tag{a}$$
$$u_a(\alpha, x, a) = 0 \qquad u'_a(\alpha, x, a) = 1 \tag{b}$$

respectively.

To solve the boundary value problem (1), we guess the initial conditions of a solution, (x_1, a_1). We then solve three initial value problems: (2), (5a) and (5b) by a suitable initial value method. Finally, we update our guess by the Newton method (3) or (4) to generate a sequence (x_n, a_n). We repeat this procedure until $g(\beta, x_n, a_n)$ and $h(\beta, x_n, a_n)$ are "sufficiently small." (In our numerical work, "sufficiently small" meant that $|g(\beta, x_n, a_n)| + |h(\beta_n, x_n, a_n)| < 10^{-7}$). We then take our most recent solution to (2), $u(t, x_n, a_n)$ to be our computed solution to (1).

REMARK: If f is linear (that is, if $f(t, u, u') = p(t)u + q(t)u' + r(t)$) then it is an easy but annoying calculation to show that Newton's method will converge in one step regardless of the initial data.

C. CONVERGENCE OF THE SHOOTING METHOD

If we assume the hypotheses of THEOREM G, we know that the periodic problem of the last section has a unique solution. In this section we prove that under the same hypotheses the relevant derivative in Newton's method is invertible near the "answer." That is, we show that $g_x(\beta, x_0, a_0)h_a(\beta, x_0, a_0) - h_x(\beta, x_0, a_0)g_a(\beta, x_0, a_0) \neq 0$, where (x_0, a_0) is the initial data satisifed by the unique solution to (1).

From these facts and the discussion in Chapter I, Section C, we conclude that if we can find a good first guess, the foregoing scheme will converge. In the proof, we will retain the notation of the previous section.

THEOREM 1: Let $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuously differentiable, $f_u(t, u, p) \geq 0$ and $f_u(t_0, u, p) > 0$ for a fixed $t_0 \in [\alpha, \beta]$. Then if $(x_0, a_0) \in D$ (the domain of F), $g_x(\beta, x_0, a_0)h_a(\beta, x_0, a_0) - g_a(\beta, x_0, a_0)h_x(\beta, x_0, a_0) < 0$.

PROOF: Let u be the solution of

$$u'' = f(t, u, u') \qquad u(\alpha) = x_0 \qquad u'(\alpha) = a_0.$$

Note that u is defined on $[\alpha, \beta]$ because $(x_0, a_0) \in D$.

Define $\Delta(t)$ by

$$\begin{aligned}\Delta(t) &= (u_x(t) - 1)(u_a'(t) - 1) - u_x'(t)u_a(t) \\ &= u_x(t)u_a'(t) - u_x'(t)u_a(t) - u_a'(t) - u_x(t) + 1\end{aligned}$$

That is, $\Delta(t) = g_x h_a - g_a h_x$ evaluated at (t, x_0, a_0); see (B.4) above.

Now define W(t) by

$$W(t) = u_x(t)u_a'(t) - u_x'(t)u_a(t).$$

Observe that W(t) is the Wronskian of the two solutions $u_a(t)$ and $u_x(t)$ to (5).

We have:

$$\Delta(t) = W(t) - u_a'(t) - u_x(t) + 1$$

I claim that $\Delta(t) < 0$ for $t > t_1$, where t_1 is defined by

$$t_1 = \inf_{t\in[\alpha,\ \beta]} \{t{:}f_u(t,\ u(t),\ u'(t)) > 0\},$$

and the infinium is well defined and less than β by the hypotheses of the theorem.

Once this is shown, $\Delta(\beta) < 0$, and the result follows. The claim follows from two facts:

i) $-u_x(t) + 1 \leq 0$ for $t\in[\alpha,\ \beta]$

ii) $W(t) - u_a'(t) < 0$ for $t > t_1$.

To verify (i) we must check that $u_x(t) \geq 1$ for $t\in[\alpha,\ \beta]$. Recall that u_x satisfies the initial value problem

$$u_x''(t) = f_u(t,\ u(t),\ u'(t))u_x(t) + f_{u'}(t,\ u(t),\ u'(t))u_x'(t)$$
$$u_x(\alpha) = 1 \qquad u_x'(\alpha) = 0$$

If u_x is not always greater than or equal to 1, by continuity there must be a point t_2 so that $u_x(t) > 0$ for $t\in[\alpha,\ t_2]$ and $u_x(t_2) < 1$.

Now define

$$z(t) = e^{\gamma(t-\alpha)} - 1 \text{ where } \gamma = \max_{t\in[\alpha,\ \beta]} \{ f_{u'}(t, u(t), u'(t)), 0\} + 1,$$

which is well defined since f is continuously differentiable.

Note that $z(t) > 0$ for $t > \alpha$, and $z'' - f_{u'} z' = \gamma^2 e^{\gamma(t-\alpha)} - f_{u'} \gamma e^{\gamma(t-\alpha)} > 0$.

Now let

$$y(t) = u_x(t) + \varepsilon z(t), \text{ where } 0 < \varepsilon < \frac{1-u_x(t_2)}{z(t_2)} .$$

y is positive on $[\alpha, t_2]$ since $z(t) \geq 0$ implies $y(t) \geq u_x(t) > 0$ for $t\in[\alpha, t_2]$. Also,

$$u''_x - f_{u'} u_{x'} = f_u u_x \geq 0 \text{ on } [\alpha, t_2], \text{ because}$$

$f_u \geq 0$ and $u_x > 0$ there; it follows that

$$y'' - f_{u'} y' > 0.$$

Hence, y cannot have an interior positive maximum (as $y'' > 0$ whenever $y' = 0$). Thus, y attains its maximum either at α or t_2, since y is positive on $[\alpha, t_2]$. But,

$$y(\alpha) = u_x(\alpha) = 1 \text{ and}$$

$$\begin{aligned} y(t_2) &= u_x(t_2) + \varepsilon z(t_2) \\ &< u_x(t_2) + \frac{1-u_x(t_2)}{z(t_2)} z(t_2) \\ &< 1 \end{aligned}$$

This shows that y achieves its maximum at $t = \alpha$ and so $y'(\alpha) \leq 0$. But $y'(\alpha) = u'_x(\alpha) + \varepsilon z'(\alpha) = 0 + \varepsilon\gamma > 0$, a contradiction, which establishes inequality (i).

Now to verify (ii). First of all, the Wronskian $W(t)$ satisfies the initial value problem

$$W'(t) = f_{u'}(t, u(t), u'(t))W(t) \qquad W(\alpha) = 1.$$

See [4, 2.3], or verify it directly.

Since u_a satisfies the differential equation:

$$u_a''(t) = f_u(t, u, u')u_a(t) + f_{u'}(t, u, u')u_a'(t)$$
$$u_a(\alpha) = 0 \qquad u_a'(\alpha) = 1,$$

u_a' will solve the initial value problem:

$$(u_a)' = f_{u'}(t, u(t), u'(t))u_a + k(t) \qquad u_a'(\alpha) = 1,$$

where $k(t) = f_u(t, u, u')u_a$.

I claim, first of all, that $u_a > 0$ for $t \in [\alpha, \beta]$, so that $k(t) \geq 0$. This is so because $u_a > 0$ in a deleted neighborhood of α by the initial conditions, and for u_a to return to 0 would imply the existence of a local positive maximum, a contradiction as before (as $u_a'' \geq 0$ whenever $u_a' = 0$). Hence $k(t) \geq 0$, and so by a standard comparison theorem (THEOREM F), $u_a'(t) \geq W(t)$. Furthermore $k(t) > 0$ for $t \in (t_1, t_1 + \delta)$ for some $\delta > 0$, and so by Corollary F.1, $u_a'(t) > W(t)$ for $t > t_1$. Hence the proof is complete.

We now present the main theorem of this section.

THEOREM 2: Assume

i) f is continuous in $[\alpha, \beta] \times \mathbb{R} \times \mathbb{R}$ and there is a constant $M > 0$ so that $uf(t, u, 0) \geq 0$ for $|u| > M$.

ii) $|f(t, u, u')| \leq A(t, u)u'^2 + B(t, u)$ where $A, B \geq 0$ are functions bounded for (t, u) in $[\alpha, \beta] \times [-M, M]$.

iii) f_u and $f_{u'}$ are bounded, $f_u \geq 0$ and $f_u(t_0, u, u') > 0$ for a fixed $t_0 \in [\alpha, \beta]$.

Then the problem

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta)$$

has a unique solution y. Further there is an $\varepsilon > 0$ such that if $|x_1 - y(\alpha)| + |a_1 - y'(\alpha)| < \varepsilon$ then the sequence defined by equation (B.4) is well defined and (x_n, a_n) will converge to $(y(\alpha), y'(\alpha))$.

PROOF: The hypotheses are the hypotheses of THEOREM G, hence the existence and uniqueness of a solution to the periodic problem is assured. By THEOREM 1, the relevant derivative in Newton's method is invertible. Finally, the existence of ε is guaranteed by THEOREM H.

D. QUASILINEARIZATION

In this section we present a numerical technique known as quasilinearization, a method applicable to a large variety of boundary value problems. The technique is actually a clever use of Newton's method, and we will see that it will converge in the case of periodic boundary conditions.

Suppose $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and that we wish to solve

$$u'' = f(t, u, u') \qquad u \in BC[\alpha, \beta] \cap C^2[\alpha, \beta] \tag{1}$$

where $BC[\alpha, \beta]$ denotes a space of functions on $[\alpha, \beta]$ that satisfy certain boundary conditions: Let $B_1, B_2: C^2[\alpha, \beta] \to \mathbb{R}$ be continuously differentiable functions and $N(B_i) = \{u \in C^2[\alpha, \beta] \mid B_i(u) = 0\}$. Then $BC[\alpha, \beta] = N(B_1) \cap N(B_2)$.

This framework contains all the boundary value problems mentioned above as well as many more. (For example, if we wish to consider a problem with Sturm-Liouville boundary conditions, we let $B_1(u) = au(\alpha) + a'u'(\alpha)$ and $B_2(u) = du(\beta) + d'u'(\beta)$). In particular, we include the possibility of nonlinear, inhomogeneous boundary conditions.

Now define the function

$$F: C^2[\alpha, \beta] \to C[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \text{ by}$$

$$F(u) = (u'' - f(t, u, u'), B_1(u), B_2(u)) \tag{2}$$

Since solutions of (1) are precisely the roots of F, Newton's method may apply. We start with a vector $u_1 \in C^2[\alpha, \beta]$ and try to solve

$$0 = F(u_n) + F'(u_n)(u_{n+1} - u_n) \tag{3}$$

for $u_{n+1} \in C^2[\alpha, \beta]$.

Since

$$F'(u)h = (h'' - f_u(t, u, u')h - f_{u'}(t, u, u')h', B_1'(u)h, B_2'(u)h),$$

solving (3) is equivalent to solving the differential equation:

$$\begin{aligned} 0 = {} & u_n'' - f(t, u_n, u_n') + (u_{n+1} - u_n)'' - \\ & f_u(t, u_n, u_n')(u_{n+1} - u_n) - f_{u'}(t, u_n, u_n')(u_{n+1} - u_n)' \end{aligned} \tag{4}$$

with the boundary conditions:

$$0 = B_1(u_n) + B_1'(u_n)(u_{n+1} - u_n) \tag{5}$$

$$0 = B_2(u_n) + B_2'(u_n)(u_{n+1} - u_n) \tag{6}$$

or, rewriting (4):

$$\begin{aligned} u_{n+1}'' = {} & f_u(t, u_n, u_n')u_{n+1} + f_{u'}(t, u_n, u_n')u_{n+1}' + \\ & f(t, u_n u_n') - f_u(t, u_n, u_n')u_n - f_{u'}(t, u_n, u_n')u_n' \end{aligned} \tag{7}$$

Note that the differential equation (7), taken together with the boundary conditions (5) and (6) form a linear problem for u_{n+1} for each n. Solution of linear problems is straightforward, see Section F. Also note that if u_n converges to some function u_0, equation (3) implies that $F(u_0) = 0$, so u_0 is a solution to (1), as desired.

REMARKS: If B_1 and B_2 are linear, as they will be in many of the problems we consider, equations (5) and (6) reduce to

$$B_1(u_{n+1}) = 0 \tag{8}$$

$$B_2(u_{n+1}) = 0 \tag{9}$$

Finally, if in addition, f is linear, the problem defined by equation (7) together with boundary conditions (8) and (9) is identical to the original problem (equation 1), and so the method will converge in a single step for any initial guess u_1.

E. CONVERGENCE OF QUASILINEARIZATION

To insure convergence of the preceding scheme we must consider the three conditions given in the discussion of Newton's method at the end of Chapter 1, Section C. Suppose that $F(u_0) = 0$, and consider the question of whether $F'(u_0)^{-1}$ exists. Since in our case $F'(u_0)$ is a linear differential operator, if $F'(u_0)$ is one-to-one then $F'(u_0)^{-1}$ will exist and be continuous. (In fact, $F'(u_0)^{-1}$ will be given by a Green's function [4, 10.15] and so will be a compact linear operator). Hence to verify the existence and continuity of $F'(u_0)^{-1}$ we must check that the only solution to the equation $F'(u_0)h = 0$ is the solution $h \equiv 0$.

Explicitly we require that the problem

$$h'' = f_u(t, u_0, u_0')h + f_{u'}(t, u_0, u_0')h' \tag{1}$$

with the boundary conditions:

$$B_1'(u_0)h = 0 \qquad B_2'(u_0)h = 0$$

has only the zero solution. We state below two important boundary value problems for which this is the case. Our first example is based on the following theorem from ordinary differential equations.

THEOREM I: Let $f:[\alpha, \beta] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the condition $|f(t, u_1, p_1) - f(t, u_2, p_2)| \leq K|u_1 - u_2| + L|p_1 - p_2|$ where $K, L > 0$ are constants so that $\frac{K(\beta-\alpha)^2}{8} + \frac{L(\beta-\alpha)}{2} < 1$. Then the Dirichlet problem

$$u'' = f(t, u, u') \qquad u(\alpha) = A \qquad u(\beta) = B$$

has a unique solution.

PROOF: [3, 1.1.1].

As a consequence, we deduce:

THEOREM 3: Let f be as in the preceding theorem. Then the Dirichlet problem $u'' = f(t, u, u')$ $u(\alpha) = A$ $u(\beta) = B$ has a unique solution u_0. Further, there is an $\varepsilon > 0$ so that if $||u_1 - u_0|| < \varepsilon$, the sequence u_n given by equation (D.3) is well defined and will converge to u_0.

PROOF: The proof is immediate in view of the remarks at the beginning of this section. We need only check that

$$\begin{aligned} h'' &= f_u(t, u_0, u_0')h + f_{u'}(t, u\ , u_0')h' \\ h(\alpha) &= h(\beta) = 0 \end{aligned} \qquad (2)$$

has only the zero solution.

But $|f_u(t, u_0, u_0')| \leq K$ and $|f_{u'}(t, u_0, u_0')| \leq L$ where K, L are as in THEOREM I, and so the solution to (2) is unique, by THEOREM I.

Finally, convergence for the periodic problem is assured by the following theorem:

THEOREM 4: Let f satisfy the hypotheses of THEOREM G. Then the Dirichlet, Neumann, and periodic problems:

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) = 0$$

$$u'' = f(t, u, u') \qquad u'(\alpha) = u'(\beta) = 0$$

$$u'' = f(t, u, u') \qquad u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta)$$

all have unique solutions. Let u_0 be the unique solution to any one of these problems. Then there is an $\varepsilon > 0$ so that if $||u_1 - u_0|| < \varepsilon$, the sequence generated by equation (3) is well defined, and u_n will converge to u_0.

PROOF: THEOREM G guarantees the existence and uniqueness of the solutions. We need only check that the only solution of

$$h'' = f_u(t, u_0, u_0')h + f_{u'}(t, u_0, u_0')h'$$

with the specified boundary conditions is $h \equiv 0$. It is shown in [10, 5.3] that this is the case.

F. LINEAR PROBLEMS WITH PERIODIC BOUNDARY CONDITIONS

To solve a nonlinear boundary value problem with periodic boundary conditions, the method of quasilinearization requires us to solve a sequence of linear differential equations with periodic boundary conditions. In this section we outline a method to accomplish this task. Other boundary conditions are handled similarly.

Suppose we wish to solve

$$h''(t) + p(t)u'(t) + q(t)u(t) = r(t)$$
$$u(\alpha) = u(\beta) \qquad u'(\alpha) = u'(\beta) \tag{1}$$

where p, q, r are continuous functions on $[\alpha, \beta]$.

Let:

v_1 solve the problem $u'' + p(t)u' + q(t)u = 0$

$u(\alpha) = 1 \qquad u'(\alpha) = 0$

v_2 solve the problem $u'' + p(t)u' + q(t)u = 0$

$u(\alpha) = 0 \qquad u'(\alpha) = 1$

w solve the problem $u'' + p(t)u' + q(t)u = r(t)$

$u(\alpha) = 0 \qquad u'(\alpha) = 0$

Taking advantage of the linearity, we see that the general solution of (1) is of the form $\psi(t) = w(t) + xv_1(t) + yv_2(t)$ $x, y \in R$.

We must choose x and y so that ψ satisfies the boundary conditions. Letting:

$$a = v_1(\beta) \qquad b = v_2(\beta) \qquad c = w(\beta)$$

$$a' = v_1'(\beta) \qquad b' = v_2'(\beta) \qquad c' = w'(\beta)$$

we want

$$\psi(\alpha) = x = c + xa + yb = \psi(\beta)$$

$$\psi'(\alpha) = y = c' + xa' + yb' = \psi'(\beta)$$

Solving for x and y, we get:

$$(a - 1)\, x + by = -\, c$$

$$a\, x + (b' - 1)y = -\, c' \text{ or}$$

$$\begin{bmatrix} a - 1 & b \\ a' & b' - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = - \begin{bmatrix} c \\ c' \end{bmatrix}$$

Now letting $\Delta = (a - 1)(b' - 1) - a'b$,

$$\begin{bmatrix} x \\ y \end{bmatrix} = - \frac{1}{\Delta} \begin{bmatrix} b' - 1 & - b \\ - a' & a - b \end{bmatrix} \begin{bmatrix} c \\ c' \end{bmatrix}$$

$$x = (c + bc' - cb')/\Delta$$

$$y = (c' + a'c - c'a)/\Delta$$

Note that $\Delta = \Delta(1)$ of Section C. In particular, if the hypotheses of THEOREM 1 are satisfied, $\Delta \neq 0$.

G. NUMERICAL RESULTS

The two numerical methods outlined for solving differential equations with periodic boundary conditions, "shooting" and "quasi-linearization," were programmed in FORTRAN and a number of examples were run on the CYBER 70 machine at OSU. Plots 1, 2 and 3 were done with the GERBER plotter. The initial value method used is the classical RUNGE-KUTTA method of order four [2, 2.4].

Recall that with "shooting" we keep iterating until $|u_n(\beta) - u_n(\alpha)|$ and $|u_n'(\beta) - u_n'(\alpha)|$ are both "sufficiently small." Sufficiently small here means that

$$|u_n(\beta) - u_n(\alpha)| + |u_n'(\beta) - u_n'(\alpha)| < 10^{-7}$$

If we reach this point, the method has "converged" and we stop.

In the case of quasilinearization, convergence is taken to mean that

$$\max_{t_i}\{|u_{n+1}(t_i) - u_n(t_i)|, |u_{n+1}'(t_i) - u_n'(t_i)|\} < 10^{-7},$$

where t_i is the set of grid points.

In each example, unless noted, our initial guess is $u_1 \equiv 0$ for quasilinearization and $u_1(\alpha) = 0 \quad u_1'(\alpha) = 0$ for "shooting." The total central processor time used per problem depended on the complexity of the function f, the number of grid points used, and the number of iterations. (We used 50 evenly spaced grid points for all examples except for Plot 3, where we used 100). Typical central processor time per iteration with 50 grid points was around .5 seconds.

EXAMPLE 1. (See Plot 1).

$u'' = -6u - 120(t-1)^3 + 24$ with periodic boundary conditions on $[0, 2]$.

This linear problem has the explicit solution $u(t) = -20t^3 + 20t + 4$.

For this problem, as with all the linear problems run, both methods converged in one iteration, as theoretically they must.

EXAMPLE 2. $u'' = .1u - \sinh(u') - .95$ with periodic boundary conditions on $[-1, 1]$.

This problem has the unique (by THEOREM G) solution $u \equiv 9.5$.

Here, quasilinearization converged in a single step, and shooting required three steps.

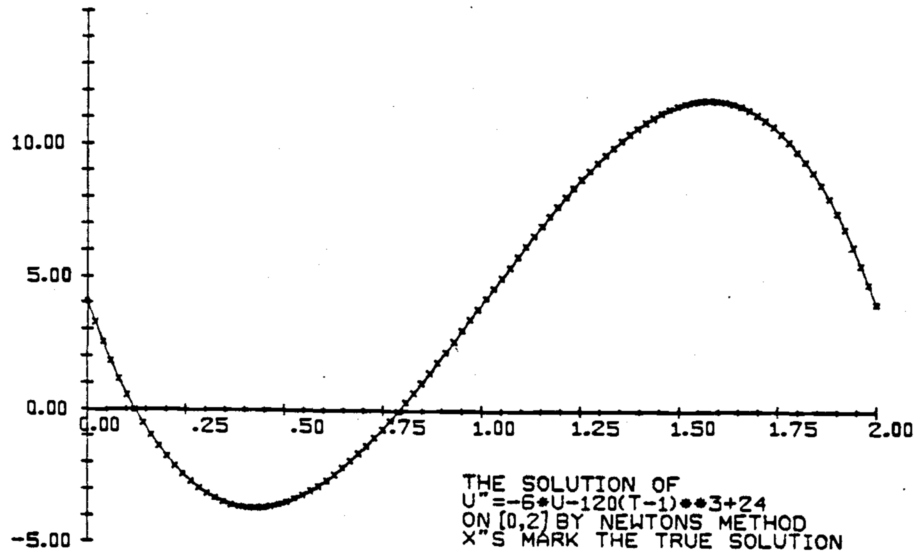
EXAMPLE 3: (See Plot 2).

$$u'' = 2(t^2 \tan(t) - u)(u' - 2t \tan(t) + 1) + 4t \sec^2 t$$

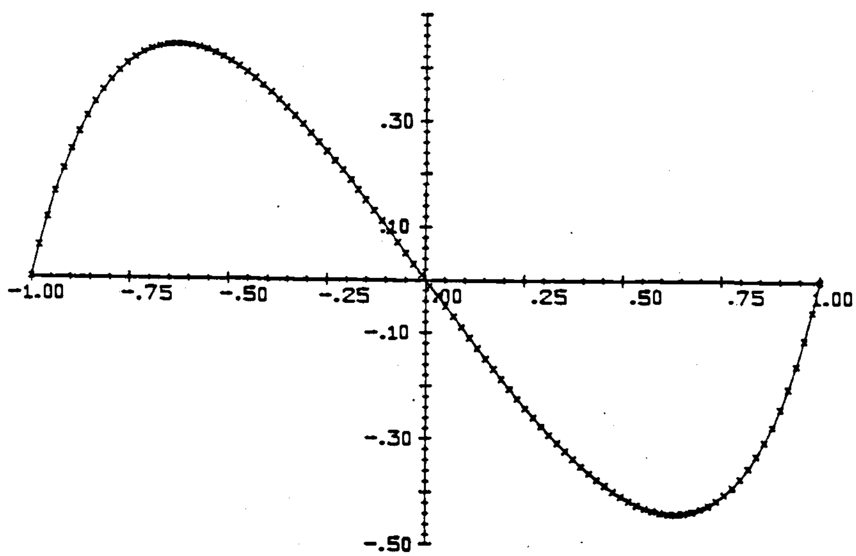
with periodic boundary conditions on $[-1, 1]$. This nonlinear problem has the explicit solution

$$u(t) = (t^2 - 1)\tan(t).$$

Quasilinearization converged in four iterations. Shooting failed to converge with zero initial data and even with the seemingly very good initial data $u_1(-1) = 0$ $\quad u_1'(-1) = 3.2$. It did converge after three iterations with the initial data $u_1(-1) = 0$ $u_1'(-1) = 3.12$. The true initial slope α satisfies $3.11 < \alpha < 3.12$.



PLOT 1.



PLOT 2.

THE SOLUTION OF
U"=2*(T*T*TAN(T)-U)*(U' -2*T*TAN(T)+1)+4*T*SEC(T)**2
ON [-1.,1.] BY QUASILINEARIZATION
X"S MARK THE TRUE SOLUTION

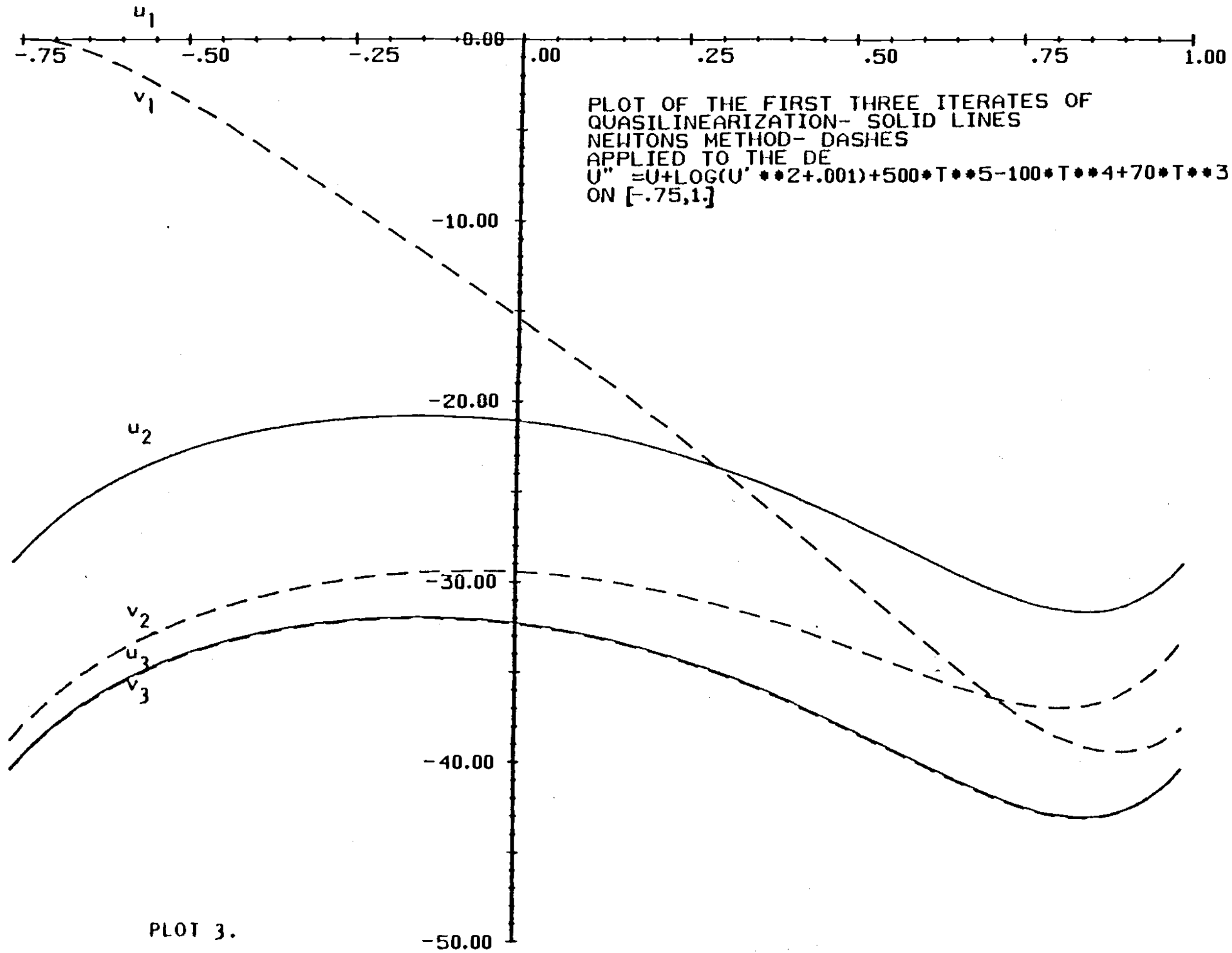
EXAMPLE 4: (See Plot 3).

$$u'' = u + \log(u'^2 + .001) + 500t^5 - 100t^4 + 70t^3$$

with periodic boundary conditions on $[-.75, 1]$. No explicit solution is known.

Plot 3 shows clearly how convergence of each method is taking place. The v_i's are successive iterates of shooting, and the u_i's are the iterates of quasilinearization. Note that each iterate of quasilinearization is periodic. After six iterations each (which the plot doesn't show) the two computed solutions were in five place agreement, and differed by less than 10^{-2} from u_3.

NOTE: Quasilinearization failed to converge for this problem when the interval $[-.75, 1]$ was replaced with $[-1, 1]$.


u1
v1
u2
v2
u3
v3
-.75
-.50
-.25
0.00
.00
.25
.50
.75
1.00
-10.00
-20.00
-30.00
-40.00
-50.00
PLOT OF THE FIRST THREE ITERATES OF
QUASILINEARIZATION- SOLID LINES
NEWTONS METHOD- DASHES
APPLIED TO THE DE
U" =U+LOG(U' **2+.001)+500*T**5-100*T**4+70*T**3
ON [-.75,1.]

PLOT 3.

CHAPTER II. BIFURCATION THEORY

A. INTRODUCTION

To motivate the idea of bifurcation, consider the following boundary value problem:

$$u'' + \lambda u = 0 \qquad u(0) = 0 \qquad u(1) = 0$$

It is easily seen that the solutions of $u'' + \lambda u = 0 \quad u(0) = 0$ are precisely:

$$u(t) = a \sin \sqrt{\lambda}\, t \qquad \lambda>0$$
$$u(t) = At \qquad \lambda=0$$
$$u(t) = Ae^{\sqrt{-\lambda}t} - Ae^{-\sqrt{-\lambda}t} \qquad \lambda<0$$

where a, $A \in R$. In order to satisfy the second boundary condition, $u(1) = 0$; we must have that $A = 0$ or $a \sin \sqrt{\lambda} = 0$. To obtain non-trivial solutions to the problem, then, we must choose $\lambda = (n\pi)^2$ $n = 1, 2, 3. \ldots.$ If we plot all such solution pairs (λ, a) we obtain the following diagram:

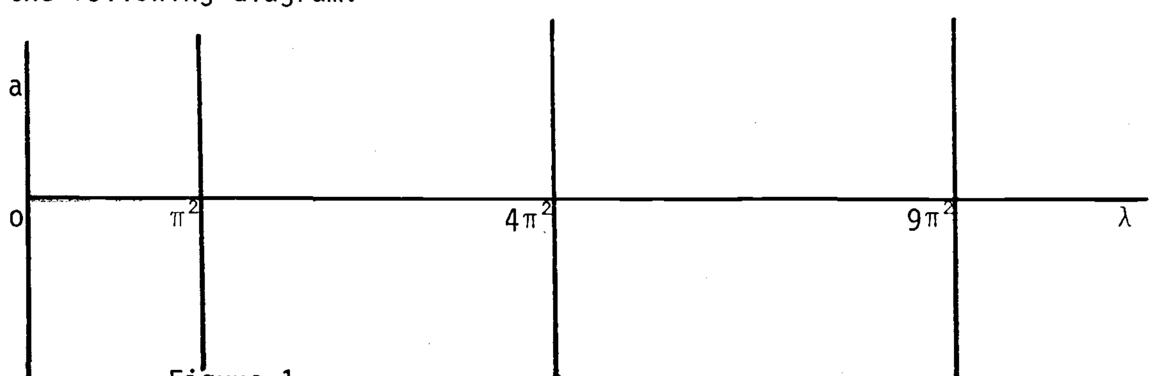


Figure 1.

We see there is a "splitting" of solutions at the critical values $\lambda_n = (n\pi)^2$. For this reason, we call these points "branch points" or "bifurcation points."

Bifurcation phenomena occur in many parts of physics. For example, consider the Euler buckling beam problem. Here we have a uniform rod of length 1 pinned at its ends, and subjected to a given compressive force. If we let v(s) be the height of the beam at arc length s, then the beam will satisfy the following equilibrium equation and boundary conditions:

$$v'' + \lambda v \sqrt{1 - v'^2} = 0 \qquad v(0) = 0 \qquad v(1) = 0$$

where λ is proportional to the applied force.

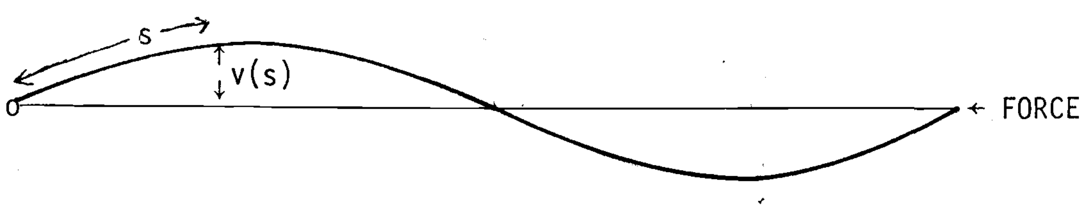


Figure 2.

Note that $v \equiv 0$ is a possible solution, the solution that corresponds to no buckling. But intuitively, as well as analytically, there are other solutions. Indeed, if we graph the possible solutions, $v'(0)$ vs. λ we obtain the following diagram (which we will justify later).

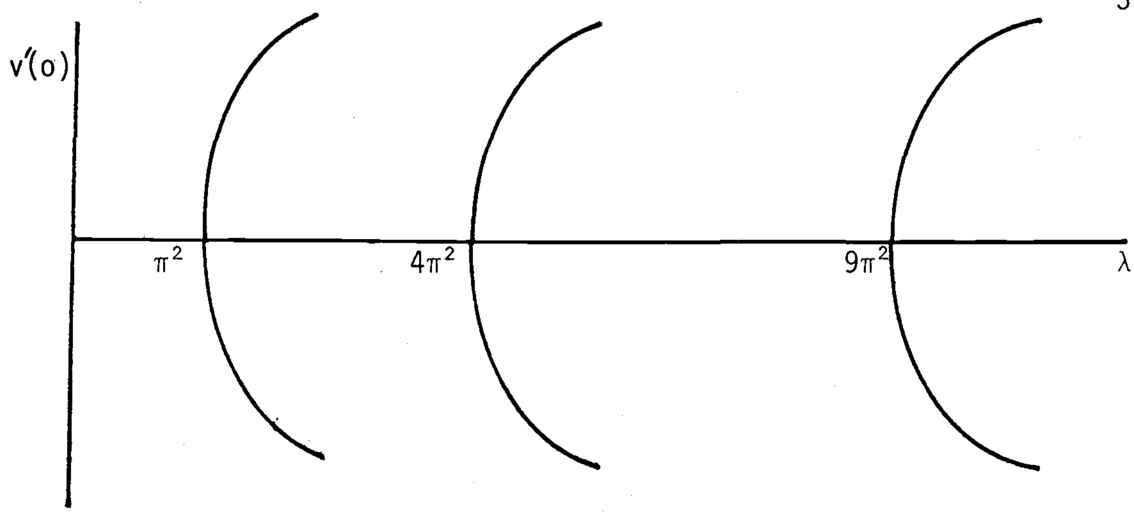


Figure 3.

The similarity of the two examples should be striking. Note that the branch points for the two problems are precisely the same. This should not be surprising. For $|v'|$ small, $|v|$ is also small. Therefore one would expect the linear problem $v'' + \lambda v = 0 \quad v(0) = v(1) = 0$ to be a good approximation to the nonlinear problem $v'' + \lambda v \sqrt{1 - v'^2} = 0$ $v(0) = v(1) = 0$.

Another similarity, not evident from the pictures, is that in each case, a point on the n-th branch (the branch emanating from $(n\pi)^2$) corresponds to a solution of its problem having $n - 1$ interior zeros on $[0, 1]$.

There are many approaches to bifurcation theory. Of note are: Degree Theory [19], Liaupunov-Schmidt Theory [16], [20] and Perterbation Theory [15], [20]. These theories are all quite general, and for us, needlessly complicated.

We set our sights a little lower. We wish to obtain the major results of bifurcation theory for a large class of physically important problems, in as elementary (and painless) a fashion as possible.

In the next section we describe a fairly general setting for bifurcation problems. We define what is meant by a branch point, and present a theorem that gives us necessary conditions for branching.

To prove sufficient conditions, we must specialize the problem.

B. SETTING

Let $F:B \times \mathbb{R} \to C$ be a continuous function, where $(B, \|\ \|_B)$ and $(C, \|\ \|_C)$ are Banach spaces. Suppose $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$.

DEFINITION: λ_0 is a <u>branch point</u> (or <u>bifurcation point</u>) for F if there is a sequence $(u_n, \lambda_n) \in B \times \mathbb{R}$ such that $F(u_n, \lambda_n) = 0$, $\lambda_n \to \lambda_0$, $\|u_n\| \to 0$ and $\|u_n\| \neq 0$ for all n.

DEFINITION: λ_0 is a <u>strong branch point</u> (or <u>strong bifurcation point</u>) for F if there is a continuous curve $\gamma:(-\varepsilon, \varepsilon) \to B \times \mathbb{R}$ such that $\gamma(t) = (u(t), \lambda(t))$, $F(\gamma(t)) = 0$, $\gamma(0) = (0, \lambda_0)$, and $\|u(t)\| \neq 0$ whenever $t \neq 0$.

Suppose that $F = F(u, \lambda)$ is continuously Frechét differentiable with respect to its first variable at the origin uniformly for λ in a bounded set. That is, we suppose F can be written as:

$$F(u, \lambda) = L(u, \lambda) + N(u, \lambda)$$

where $L(\cdot, \lambda):B \to C$ is a continuous linear operator, $\lambda \to L(\cdot, \lambda)$ is continuous, and $N(u, \lambda)$ satisfies

$$\lim_{\|u\| \to 0} \frac{\|N(u,\lambda)\|}{\|u\|} = 0$$

uniformly for λ in a bounded set. Then we have:

THEOREM 5: With F, L, and N as above, if $L^{-1}(\cdot, \lambda_0)$ exists and is continuous, then λ_0 is not a branch point for F.

PROOF: If λ_0 were a branch point for F, then by definition, there would be a sequence $(u_n, \lambda_n) \in B \times \mathbb{R}$ so that $F(u_n, \lambda_n) = 0$, i.e.,

$L(u_n, \lambda_n) + N(u_n, \lambda_n) = 0.$

If $L^{-1}(\cdot, \lambda_o)$ exists, then $L^{-1}(\cdot, \lambda)$ exists for all λ sufficiently close to λ_o. Consequently, there is a $n_1 > 0$ so that $L^{-1}(\cdot, \lambda_n)$ exists for all $n > n_1$. So for sufficiently large n we have:

$$N(u_n, \lambda_n) = - L(u_n, \lambda_n)$$

$$L^{-1}(N(u_n, \lambda_n), \lambda_n) = - u_n$$

$$\frac{||L^{-1}(N(u_n, \lambda_n), \lambda_n)||}{||u_n||} = \frac{||- u_n||}{||u_n||} = 1$$

But $$\lim_{n\to\infty} \frac{||L^{-1}(N(u_n, \lambda_n), \lambda_n)||}{||u_n||} = 0$$

because

$$\frac{||L^{-1}(N(u_n, \lambda_n), \lambda_n)||}{||u_n||} \leq ||L^{-1}(\cdot, \lambda_n)|| \frac{||N(u_n, \lambda_n)||}{||u_n||}$$

and $||L^{-1}(\cdot, \lambda_n)|| \to ||L^{-1}(\cdot, \lambda_o)||$ as $n \to \infty$. This contradiction proves the theorem.

C. NONLINEAR STURM-LIOUVILLE PROBLEMS

Henceforth, we will consider the nonlinear eigenvalue problem:

$$- u''(x) + q(x)u(x) = \lambda[a(x) - f(x, u(x), u'(x))]u(x) \qquad (1)$$

with the Sturm-Liouville boundary conditions:

$$\begin{aligned} a_1 u(0) + a_2 u'(0) &= 0 \\ b_1 u(1) + b_2 u'(1) &= 0 \end{aligned} \qquad (B.C.)$$

Unless stated to the contrary, we will assume that

a) $f: [0, 1] \times I \times J \to \mathbb{R}$ is continuous, locally Lipschitz, and $f(x, 0, 0) = 0$ (where $I, J \subseteq \mathbb{R}$ are open intervals containing 0);

b) a and q are continuous real valued functions on $[0, 1]$, and $a(x) > 0$;

c) the linear problem $- u'' + qu = \lambda au$ with the boundary conditions (B.C.) has an infinite number of simple eigenvalues $0 < \lambda_0 < \lambda_1 < \ldots$ with $\lim \lambda_n = \infty$ and such that the eigenfunction corresponding to λ_k has k simple zeros in $(0, 1)$.

A number of important remarks are in order: Hypotheses (a) and (b) guarantee that solutions of initial value problems for (1) are unique. Hypothesis (c), the assumptions on the linear eigenvalue problem, hold for the classical Sturm-Liouville systems, see [4, 10].

Note that the hypotheses are satisfied by the problem of Euler buckling, with $q \equiv 0$, $a \equiv 1$, $f(x, u, u') = 1 - \sqrt{1 - u'^2}$, and Dirichlet boundary conditions.

Finally, note that this problem fits into the framework of Section B. Let $F:C^2[0, 1] \times \mathbb{R} \to C[0, 1] \times \mathbb{R}^2$ be defined by:

$$F(u, \lambda) = (-u'' + qu - \lambda[a - f]u,\ a_1u(0) + a_2u'(0),\ b_1u(1) + b_2u'(1)).$$

Note that $F(0, \lambda) = 0$ for all λ and that F is continuously Frechét differentiable with respect to its first argument at the origin provided f is continuously differentiable.

In this case:

$$L(u,\lambda) = (-u'' + qu - \lambda au,\ a_1u(0) + a_2u'(0),\ b_1 u(1) + b_2u'(1))$$

and $N(u,\lambda) = F(u,\lambda) - L(u,\lambda)$. We claim that in this case, branching for F can only occur at the eigenvalues $\{\lambda_i\}$ of the linear problem: If $\lambda \neq \lambda_j$ $j = 1, 2, 3 \ldots$, the linear problem $L(u, \lambda) = 0$ has $u = 0$ as its only solution. Hence $L(\cdot, \lambda)$ is one-to-one and $L^{-1}(\cdot, \lambda)$ exists, is continuous, and is given by a Green's function [4, 10.15]. Hence, by THEOREM 5, λ is not a branch point for F. (This result is true even if f is not continuously differentiable, but merely locally Lipschitz. The proof, in this more general case, relies on properties of the function $\theta(1, \cdot, \cdot)$ which will be defined in the next section. The result follows from LEMMA 3 of Section E).

D. THE PRÜFER SUBSTITUTION

We wish to study the nontrivial solutions to equation (1) by the polar coordinate functions r and θ defined (up to an additive multiple of 2π in the case of θ) by the equations:

$$u(x) = r(x)\cos\theta(x) \tag{2}$$

$$u'(x) = r(x)\sin\theta(x) \tag{3}$$

We will derive a system of differential equations, satisfied by r and θ, equivalent to equation (1) in the case of nontrivial solutions. Note that if $r(x_0) = 0$ then $u(x_0) = u'(x_0) = 0$, and since solutions for initial value problems for (1) are unique, this implies $u \equiv 0$. Thus, if u(x) is a nontrivial solution to (1), $r(x)^2 = u(x)^2 + u'(x)^2$ never vanishes.

Assume u(x) is a nontrivial solution to (1). Define $r(x) = \sqrt{u(x)^2 + u'(x)^2}$ or $r(x) = -\sqrt{u(x)^2 + u'(x)^2}$ and then $\theta(x)$ is defined by (2), mod 2π. If we differentiate (2),

$$u' = r'\cos\theta - r'\sin\theta\theta' \tag{4}$$

and so, by (3)

$$r'\cos\theta - r\sin\theta\theta' = r\sin\theta, \text{ or}$$

$$r'\cos\theta = r\sin\theta\theta' + r\sin\theta \tag{5}$$

$$\theta' r\sin\theta = r'\cos\theta - r\sin\theta \tag{6}$$

so substituting into equation (1) we get,

$$-(r'\sin\theta + r\cos\theta\theta') + qr\cos\theta = \lambda[a - f]r\cos\theta \tag{7}$$

Multiplying (7) by $\frac{\cos\theta}{r}$ and using (5), we can get a problem for θ:

$$-(r'\cos\theta\sin\theta r^{-1} + \cos^2\theta\theta') + q\cos^2\theta = \lambda[a - f]\cos^2\theta$$

$$-((r\sin\theta\theta' + r\sin\theta)\sin\theta r^{-1} + \cos^2\theta\theta') = \lambda[a - f]\cos^2\theta - q\cos^2\theta$$

$$-(\sin^2\theta\theta' + \cos^2\theta\theta' + \sin^2\theta) = \lambda[a - f]\cos^2\theta - q\cos^2\theta$$

$$\theta' \doteq - \sin^2\theta - \lambda[a - f(x, r\cos\theta, r\sin\theta)]\cos^2\theta + q\cos^2\theta \qquad (8)$$

Multiplying (7) by $\sin\theta$ and using (6), we can get a problem for r:

$$-(r'\sin^2\theta + \theta' r\sin\theta\cos\theta) + qr\sin\theta\cos\theta = \lambda[a - f]r\sin\theta\cos\theta$$

$$-(r'\sin^2\theta + (r'\cos\theta - r\sin\theta)\cos\theta) = \lambda[a - f]r\sin\theta\cos\theta - qr\sin\theta\cos\theta$$

$$-(r'\sin^2\theta + r'\cos^2\theta - r\sin\theta\cos\theta) = \lambda[a - f]r\sin\theta\cos\theta - qr\sin\theta\cos\theta$$

$$r' \doteq r\sin\theta\cos\theta[1 + q - \lambda[a - f(x, r\cos\theta, r\sin\theta)]] \qquad (9)$$

Since all the steps in the derivation of the system (8), (9) are reversible when $r\neq 0$, a solution (r, θ) of the system (8), (9) with $r\neq 0$ gives a nontrivial solution u of equation (1) given by equation (2). This solution also satisifes (3).

Now define:

$$\alpha = \begin{cases} \arctan\left(-\frac{a_1}{a_2}\right) & a_2\neq 0 \\ \\ \frac{\pi}{2} & a_2 = 0 \end{cases}$$

$$\beta_k = \begin{cases} \arctan(-\frac{b_1}{b_2}) - k\pi & b_2 \neq 0 \\ \\ -\frac{\pi}{2} - k\pi & b_2 = 0 \end{cases}$$

where arctan takes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$.

If the functions r and θ satisfy the system (8), (9) (with a fixed λ), are defined over $[0, 1]$, and satisfy the boundary conditions $\theta(0) = \alpha$ $\quad \theta(1) = \beta_k$ for some k; then $u(x) = r(x)\cos\theta(x)$ will satisfy (1) as well as the Sturm-Liouville boundary conditions (B.C.).

From now on, fix the initial condition $\theta(0) = \alpha$. Our plan is to find a nontrivial solution of (1) which satisfies the boundary conditions (B.C.) by the following "shooting" method: we seek values $r(0) = \mu_0$ and $\overline{\lambda}$ so that the initial value problem (8), (9) with $\lambda = \overline{\lambda}$, and initial conditions $\theta(0) = \alpha$, $r(0) = \mu_0$ is such that its solution extends across $[0, 1]$ and also satisfies $\theta(1) = \beta_k$ for some integer k.

Note that such solutions can only exist for nonnegative k. Indeed $\theta(0) = \alpha \leq \pi/2$ and if $\theta(x_0) = \pi/2$, then $\theta'(x_0) = -1$ by (8) so $\theta(x) \leq \pi/2$ for all x. By the same argument, $\theta' = -1$ whenever $\theta = \pi/2 - k\pi$, hence θ is decreasing at the points $\theta = \pi/2 - k\pi$. This observation shows that if $\theta(1) = \beta_k$ then the function u defined by (2) has precisely k interior zeros in $[0, 1]$.

To analyze the shooting method outlined above, we let $\theta(x, \mu, \lambda)$, $r(x, \mu, \lambda)$ be the unique maximal solution to the system (8), (9) which satisfies the initial conditions $\theta(0) = \alpha$ and $r(0) = \mu$. $\theta(x, \mu, \lambda)$ and $r(x, \mu, \lambda)$ are well defined because solutions to initial value problems for the system (8), (9) are unique.

Now set f = 0 in equations (8) and (9) to obtain the following system:

$$\phi' = -\sin^2\phi - [\lambda a - q]\cos^2\phi \tag{10}$$

$$\rho' = \rho\cos\phi\sin\phi[1 + q - \lambda a] \tag{11}$$

This system is equivalent to the second order equation $-v'' + qv = \lambda av$. More precisely, if $v(x)$ is a nontrivial solution to $-v''(x) + q(x)v(x) = \lambda a(x)v(x)$, then $\rho(x)$ and $\phi(x)$ defined by:

$$v(x) = \rho(x)\cos\phi(x) \tag{12}$$

$$v'(x) = \rho(x)\sin\phi(x) \tag{13}$$

satisfy (10), (11) and $\rho(x) \neq 0$. Conversely, if (10), (11) hold with $\rho(x) \neq 0$, then (12) defines a nontrivial solution to $-v'' + qv = \lambda av$.

Furthermore, if we let $\phi(x, \lambda)$ be the solution of the initial value problem defined by equation (10) with the initial condition $\phi(0, \lambda) = \alpha$; then $\phi(x, \lambda) = \theta(x, 0, \lambda)$. This follows from the uniqueness of solutions to the initial value problem for the system (8), (9). Clearly $r(x, 0, \lambda) \equiv 0$ satisfies (9) when $\mu = 0$ and then the differential equation (8) for $\theta(x, 0, \lambda)$ is identical to (10), the equation for $\phi(x, \lambda)$, because $f(x, 0, 0) = 0$. Since $\phi(0, \lambda) = \theta(0, 0, \lambda) = \alpha$, we conclude that $\theta(x, 0, \lambda) = \phi(x, \lambda)$.

Since the system (10), (11) is linear, it has solutions which are defined on $[0, 1]$ regardless of the initial data. The domain of the functions $\theta(1, \cdot, \cdot)$ and $r(1, \cdot, \cdot)$ is an open subset of $\mathbb{R}^2$ by THEOREM E. Since solutions of the system (10), (11) extend across $[0, 1]$, $\phi(1, \lambda) = \theta(1, 0, \lambda)$ exists for all λ; hence the domain of

$\theta(1, \cdot, \cdot)$ (and of $r(1, \cdot, \cdot)$) contains the line $\mu = 0$, in the $\lambda\mu$-plane.

Recall that by hypotheses (c), the linear problem $-v'' + qv = \lambda av$ with the boundary conditions (B.C.) has a sequence of eigenvalues λ_j and eigenfunctions $v_j(x)$, $j = 0, 1, 2, \cdots$. It is a consequence of the linear Sturm-Liouville theory that the j-th eigenfunction $v_j(x) = \rho(x, \lambda_j)\cos\phi(x, \lambda_j)$ and that $\theta(1, 0, \lambda_j) = \phi(1, \lambda_j) = \beta_j$.

E. LOCAL BIFURCATION

In this section we prove our main result: solutions to the nonlinear Sturm-Liouville problem (1), (B.C.) do bifurcate at the critical values λ_j. We will require the following three lemmas:

LEMMA 1: Suppose $r(x, \mu, \lambda)$ and $\theta(x, \mu, \lambda)$ satisfy the system (8), (9) with the initial conditions $\theta(0, \mu, \lambda) = \alpha$ $r(0, \mu, \lambda) = \mu$. Restrict λ to lie in $[0, \lambda^*]$. If $\varepsilon>0$ there is a $\delta>0$ so that if $|\mu|<\delta$ then $|r(x, \mu, \lambda)|<\varepsilon$ for all $x\in[0, 1]$.

PROOF: Since the domains of $\theta(1, \cdot, \cdot)$ and $r(1, \cdot, \cdot)$ are open and contain the line $\mu = 0$, by compactness there will be a $\delta_1>0$ so that $r(1, \mu, \lambda)$ exists for all $|\mu|\le\delta_1$, $\lambda\in[0, \lambda^*]$. Since $r(1, 0, \lambda) = 0$ for all λ, the lemma follows from the uniform continuity of the function $r(1, \cdot, \cdot)$ on $[-\delta_1, \delta_1]\times[0, \lambda^*]$.

LEMMA 2: The function $\theta(1, 0, \lambda) = \phi(1, \lambda)$ is strictly decreasing in λ. Furthermore, if f is continuously differentiable and λ is restricted to $[0, \lambda^*]$ then there is a $\delta>0$ so that if $|\mu|<\delta$ then $\theta(1, \mu, \lambda)$ will be strictly decreasing in λ as well.

PROOF: The first assertion follows directly from the comparison theorem (THEOREM F). Hence we know that $\phi_\lambda(1, \lambda)\le0$. To deduce the second part, we want to show that the strict inequality $\phi_\lambda(1, \lambda)<0$ is true. From this we deduce that if f is continuously differentiable, $\theta_\lambda(1, \mu, \lambda)$ will exist and be continuous, and $\theta_\lambda(1, 0, \lambda) = \phi_\lambda(1, \lambda)$. Hence by uniform continuity of $\theta_\lambda(1, \cdot, \cdot)$ there will be a $\delta>0$ so that $\theta_\lambda(1, \mu, \lambda)<0$ for all $(\mu, \lambda)\in[-\delta, \delta]\times[0, \lambda^*]$.

ϕ_λ satisfies the equation:

$$\phi_\lambda' = -2\phi_\lambda \sin\phi\cos\phi[1 - \lambda a + q] - a\cos^2\phi \tag{14}$$

with the initial condition $\phi_\lambda(0) = 0$, as can be seen by differentiating (10). Set $r(x) = 2\sin\phi(x)\cos\phi(x)[1 - \lambda a(x) + q(x)]$. We then have

$$\phi_\lambda' + r\,\phi_\lambda = -a\cos^2\phi \tag{15}$$

Mulitplying (15) by the integrating factor $e^{\int_0^x r(t)dt}$, integrating, and using the fact that $\phi_\lambda(0) = 0$, we get:

$$(e^{\int_0^x r(t)dt}\phi_\lambda(x))' = -a(x)\cos^2\phi(x)e^{\int_0^x r(t)dt}$$

$$\phi_\lambda(x) = -e^{-\int_0^x r(t)dt}\int_0^x a(t)\cos^2\phi(t)e^{\int_0^t r(s)ds}\,dt.$$

Now, since $a(x)>0$ and $\cos^2\phi(x)>0$ almost everywhere, we get that $\phi_\lambda(1)<0$, proving the result. (Note that $\cos^2\phi(x) \equiv 0$ on some interval would imply by (10) that $\phi'(x) \equiv 1$ on this interval which is impossible).

LEMMA 3: If $\lambda\neq\lambda_j$, there is a $\eta(\lambda)>0$ so that if $|\mu|<\eta$ then

$$\text{if } \lambda>\lambda_j \qquad \theta(1, \mu, \lambda)<\phi(1, \lambda_j) = \beta_j$$
$$\text{and if } \lambda<\lambda_j \qquad \theta(1, \mu, \lambda)>\phi(1, \lambda_j) = \beta_j.$$

PROOF: The proof in both cases is similar, so suppose $\lambda<\lambda_j$. By previous remarks, $\theta(1, \cdot, \cdot)$ is a continuous function with an open domain $D \subseteq \mathbb{R}^2$ containing the line $\mu = 0$ (see Figure 4). By LEMMA 2, $\theta(1, 0, \lambda)$ is strictly decreasing in λ and so $\theta(1, 0, \lambda)>\theta(1, 0, \lambda_j) = \phi(1, \lambda_j) = \beta_j$. By continuity of $\theta(1, \cdot, \cdot)$, there is an open ball $B\subseteq D$

of radius $\eta(\lambda)>0$ about $(0, \lambda)$ so that $\theta(1, \mu, \lambda)>\beta_j$ for each $(\mu, \lambda)\in B$. Hence the result.

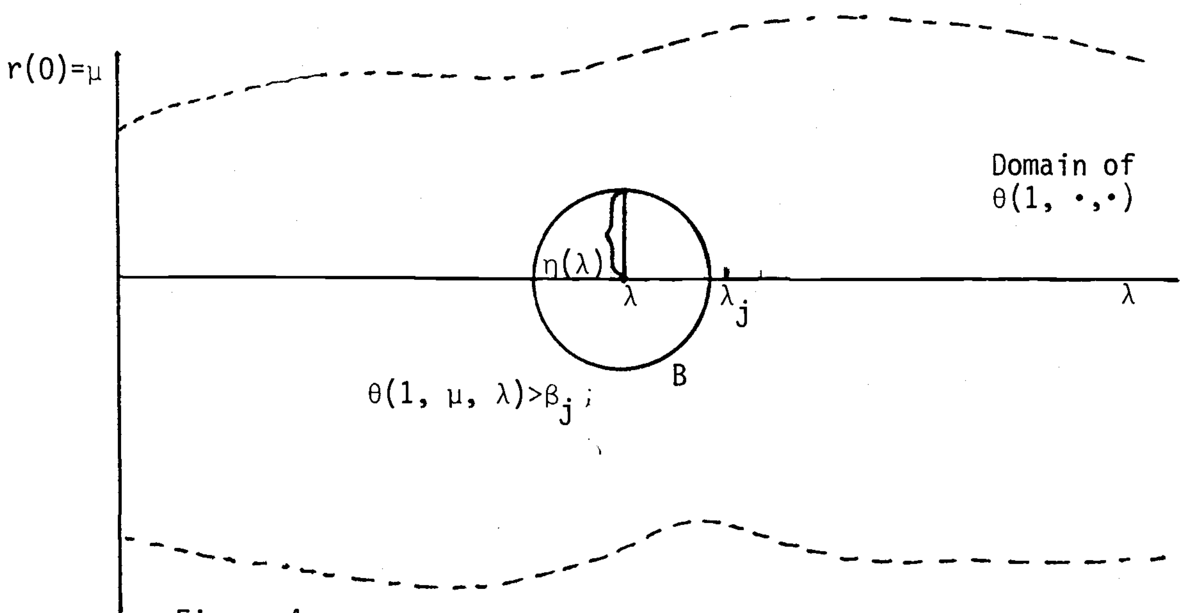


Figure 4.

We now prove our main result:

THEOREM 6: For each $j = 0, 1, 2, \cdots$, there is a $\delta(j)>0$ so that for $|\mu|<\delta(j)$ there is at least one solution $\lambda(\mu)$ of $\theta(1, \mu, \cdot) = \beta_j$. If, in addition, $\theta(1, \mu, \lambda)$ is strictly decreasing in λ for $|\mu|<\delta(j)$ there is a uniquely defined function $\mathcal{L}_j(\mu)$, which is continuous, and satisfies $\theta(1, \mu, \mathcal{L}_j(\mu)) = \beta_j$ and $\mathcal{L}_j(0) = \lambda_j$.

PROOF: (See Figure 5). Fix a $\underline{\lambda}<\lambda_j$ and $\overline{\lambda}>\lambda_j$. Take $0<\delta(j)<\min(\eta(\underline{\lambda}), \eta(\overline{\lambda}))$ where $\eta(\lambda)$ is chosen as in the previous lemma, and, in addition, so small that the function $\theta(1, \cdot, \cdot)$ is defined for all $(\mu, \lambda)\in[-\delta(j), \delta(j)]\times[\underline{\lambda}, \overline{\lambda}]$.

By LEMMA 3, $\theta(1, \mu, \underline{\lambda})>\beta_j$ and $\theta(1, \mu, \overline{\lambda})<\beta_j$ for all $|\mu|<\delta(j)$. By the continuity of $\theta(1, \cdot, \cdot)$ and the intermediate value property, there will be at least one value of λ, $\lambda(\mu)$, so that $\theta(1, \mu, \lambda(\mu)) = \beta_j$.

If $\theta(1, \mu, \cdot)$ is strictly decreasing (c.f. LEMMA 2), it is clear that there is precisely one value of λ, call it $\mathcal{L}_j(\mu)$ that satisfies $\theta(1, \mu, \mathcal{L}_j(\mu)) = \beta_j$. It is clear that the linear problem $\theta(1, 0, \cdot) = \beta_j$ has the unique solution $\lambda = \lambda_j$.

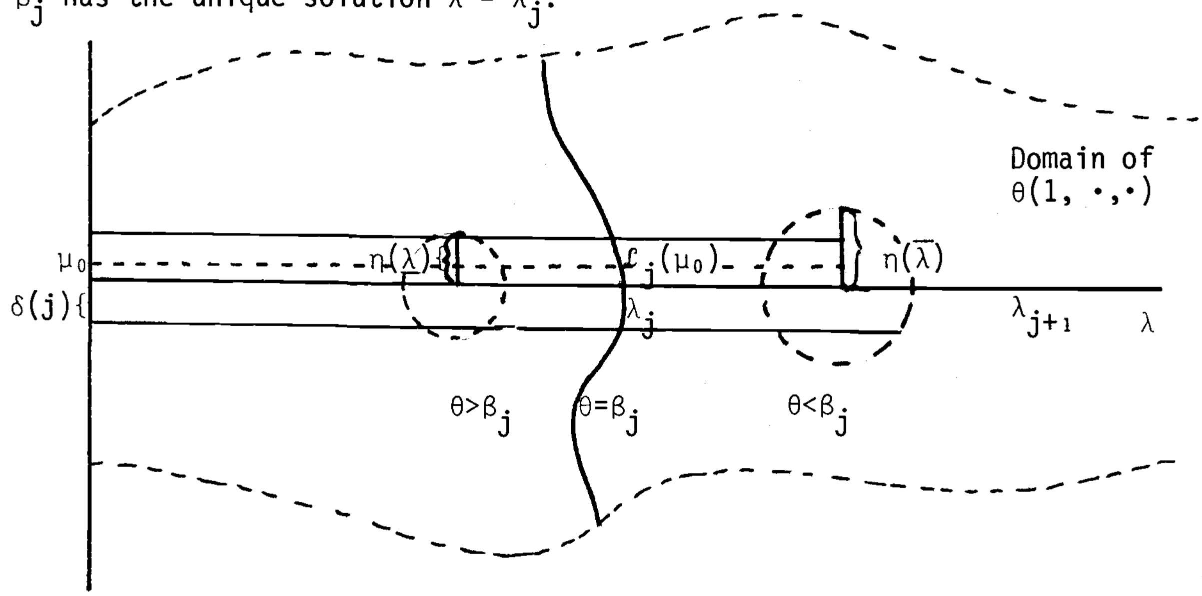


Figure 5.

It remains to check the continuity of $\mathcal{L}_j$ (see Figure 6). Fix a point $|\mu_0|<\delta(j)$ and let $\varepsilon>0$ be given, and small enough so that $(\mu_0, \mathcal{L}_j(\mu_0) - \varepsilon)$, $(\mu_0, \mathcal{L}_j(\mu_0) + \varepsilon)$ and the line segment joining these two points lies in the domain of $\theta(1, \cdot, \cdot)$.

By strict monotonicity of $\theta(1, \mu_0, \cdot)$, $\theta(1, \mu_0, \mathcal{L}_j(\mu_0) - \varepsilon)>\beta_j$ and $\theta(1, \mu_0, \mathcal{L}_j(\mu_0) + \varepsilon)<\beta_j$. Since $\theta(1, \cdot, \cdot)$ is continuous with an open domain, we can construct balls of some suitably small radius $\sigma>0$ so that $\theta(1, \mu, \lambda)$ is defined for all $|\mu - \mu_0| <\sigma$, $\mathcal{L}_j(\mu_0) - \varepsilon - \sigma\leq\lambda\leq \mathcal{L}_j(\mu_0) + \varepsilon + \sigma$; is strictly decreasing in λ for fixed μ in this region, and

$$\theta(1, \mu, \lambda)>\beta_j \text{ for all } (\mu, \lambda)\in B_\sigma(\mu_0, \mathcal{L}_j(\mu_0) - \varepsilon)$$
$$\theta(1, \mu, \lambda)<\beta_j \text{ for all } (\mu, \lambda)\in B_\sigma(\mu_0, \mathcal{L}_j(\mu_0) + \varepsilon).$$

By strict monotonicity of $\theta(1, \mu, \cdot)$, then, for each μ such that $|\mu - \mu_0|<\sigma$ there is a unique $\lambda = \mathcal{L}_j(\mu)$ so that $\theta(1, \mu, \mathcal{L}_j(\mu)) = \beta_j$, and further $|\lambda - \mathcal{L}_j(\mu_0)| = |\mathcal{L}_j(\mu) - \mathcal{L}_j(\mu_0)|<\varepsilon$, as desired.

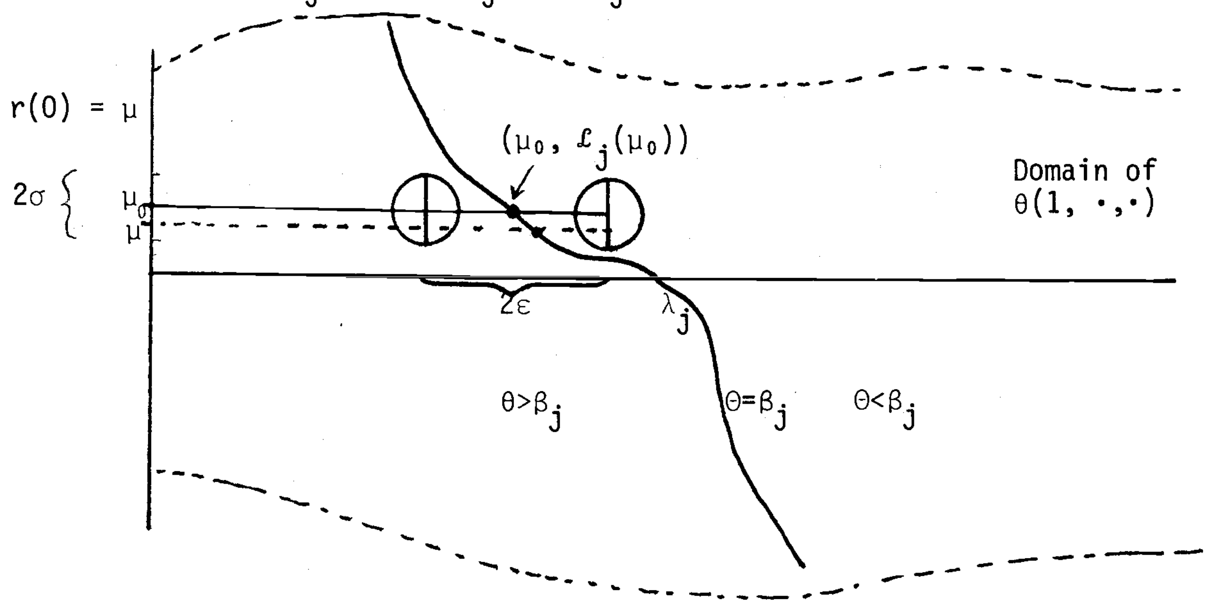


Figure 6.

We have thus shown that the numbers λ_j are branch points of problem (1), (B.C.). If f is continuously differentiable, we can get a stronger result, namely, that the numbers λ_j are strong branch points for problem (1), (B.C.).

THEOREM 7: Suppose in addition to hypotheses a, b, c of Section C, that f is continuously differentiable.
Then the functions $\mathcal{L}_j(\mu)$ as in the proof of THEOREM 6 are well defined for all $j = 0, 1, 2, \cdots$, and each $\mathcal{L}_j$ is differentiable.

PROOF: In the proof of LEMMA 2, we saw that $\theta_\lambda(1, 0, \lambda_j)<0$ for all $j = 0, 1, 2, \cdots$. Since $\theta(1, 0, \lambda_j) = \beta_j$, we can use the implicit function theorem [8, 3.11] to conclude the existence of numbers $\delta(j)>0$ and functions $\mathcal{L}_j$ that satisfy $\theta(1, \mu, \mathcal{L}_j(\mu)) = \beta_j$ for $|\mu|<\delta(j)$.

Furthermore, the functions $\mathcal{L}_j$ will be continuously differentiable.

E. BIFURCATION DIAGRAMS

The goal of a bifurcation diagram is to represent geometrically what is happening analytically. The idea is to map a (possibly very) large dimensional space and a parameter onto $\mathbb{R}^2$ retaining as much information as possible.

What is usually done for the more general problem $F(u, \lambda) = 0$ is that a functional ℓ is introduced and a "bifurcation diagram" consists of the pairs $(\lambda, \ell(u))$ such that $F(u, \lambda) = 0$. For example, in the Euler buckling beam problem we graphed λ vs. $v'(0)$. Different functionals, then, can lead to different "bifurcation diagrams."

In the case of the Sturm-Liouville problem we are considering, there is a natural choice: let $\ell(u) = r(0) = \mu$. This choice is a good one: for each point in the (λ, μ)-plane, we have a uniquely determined initial value problem, and hence, a unique function $u(x, \mu, \lambda)$. Further, if $\mu \neq 0$, $u(x, \mu, \lambda) \neq 0$.

In this section we would like to discuss some of the geometric aspects of our problem. We have shown that branching does occur at the eigenvalues of the linear problem. There remain a number of unanswered questions: What is the initial "shape" of the branches? How far do the branches extend? Is there "secondary" branching? Do the branches always represent "all" the solutions to the problem? (This last question is answered in the negative by EXAMPLE 2).

The global nature of the branches is a difficult question. For some interesting results, see [21]. Note that we can rule out secondary branching in regions where $\theta(1, \mu, \cdot)$ is strictly decreasing. (Recall that it was this assumption that enabled us to define $\mathcal{L}_j(\mu)$ uniquely).

We now present a result which partially answers the question of initial shape:

THEOREM 8: Suppose, in addition to the standing hypotheses on f, that $f(t, u, u')>0$ for $0<u^2 + u'^2<\delta$. Then for each λ_j there exists an $\varepsilon>0$ so that if $\lambda\in(\lambda_j, \lambda_j + \varepsilon)$ then problem (1), (B.C.) has at least two solutions having j interior zeros. Hence, if $\theta(1, \mu, \cdot)$ is strictly decreasing, the branching is initially to the right. (Likewise, if $f<0$ for $0<u^2 + u'^2<\delta$; and $\theta(1, \mu, \cdot)$ is monotone, the branching is initially to the left).

PROOF: The proof in both cases is similar, so we just consider $f>0$ (see Figure 7). Let $\theta = \theta(x, \mu, \lambda)$ and $r = r(x, \mu, \lambda)$ have their usual meanings. By LEMMA 1, and the hypotheses, there is a $\delta_1>0$ so that $f(x, r\cos\theta, r\sin\theta)>0$ provided $0<|\mu|<\delta$, and λ is restricted to lie in a closed bounded interval which contains λ_j in its interior. If $0<|\mu_0|<\delta$, we have:

$$\theta' = -\sin^2\theta - (\lambda_j a - q)\cos^2\theta + \lambda_j f\cos^2\theta$$
$$\theta' > - \sin^2\theta - (\lambda_j a - q)\cos^2\theta$$

and hence, by a comparison with (10), $\theta(1, \mu_0, \lambda_j)>\beta_j = \phi(1, \lambda_j)$.

Now, since $\theta(1, \cdot, \cdot)$ is continuous, there is an open ball $B_\varepsilon(\lambda_j, \mu_0)$ with center (λ_j, μ_0) and radius $\varepsilon>0$ so that if $(\lambda, \mu)\in B_\varepsilon(\lambda_j, \mu_0)$, $\theta(1, \mu, \lambda)>\beta_j$. So $\theta(1, \mu_0, \lambda)>\beta_j$ for λ in $(\lambda_j, \lambda_j + \varepsilon)$.

Fix λ in $(\lambda_j, \lambda_j + \varepsilon)$. By LEMMA 3, if $|\mu|$ is chosen small enough, $\theta(1, \mu, \lambda)<\beta_j$. By continuity of $\theta(1, \cdot, \cdot)$, there is a $\overline{\mu}$ between 0 and μ_0 such that $\theta(1, \overline{\mu}, \lambda) = \beta_j$. That there are at least two solutions comes from the fact that we have only specified $|\mu_0|$, i.e., the proof works for μ_0 and $-\mu_0$.

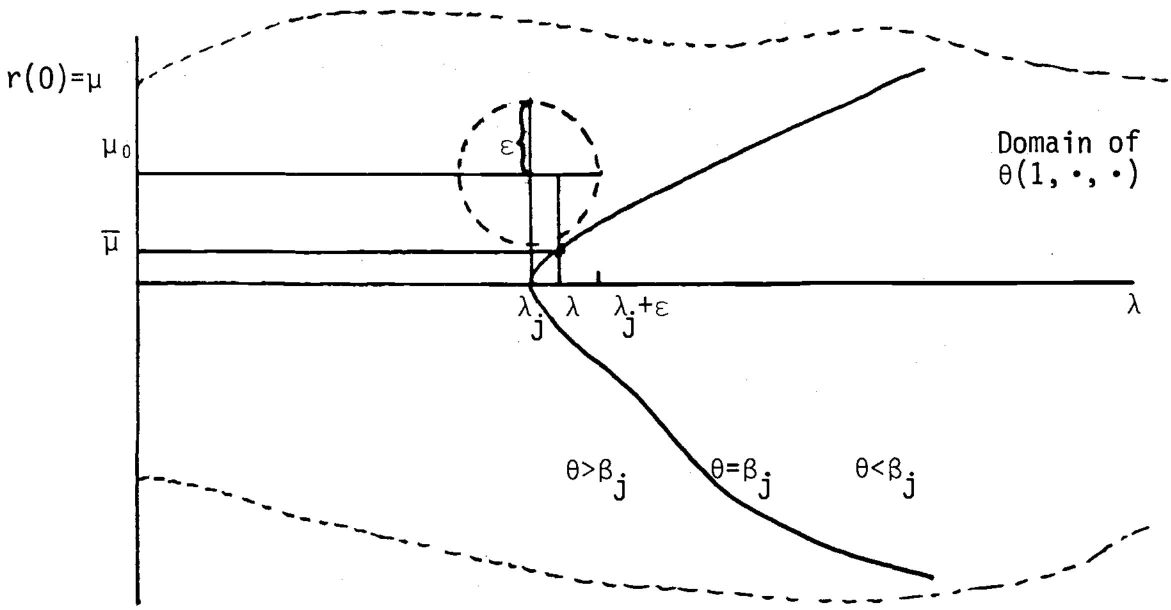


Figure 7.

COROLLARY 8.1: If in addition to the hypotheses of the previous theorem, f is continuously differentiable, $\mathcal{L}_j'(0) = 0$, and so the bifurcation curve intersects the λ-axis orthogonally.

PROOF: If $f(x, u, u')>0$ (or $f(x, u, u')<0$) for $0<u^2 + u'^2$ sufficiently small, we have that $f(x, \cdot, \cdot)$ has a local minimum or local maximum at (0, 0). If f is continuously differentiable, then, $f_u'(x, 0, 0) = f_{u'}(x, 0, 0) = 0$. By THEOREM 7, $\mathcal{L}_j(\mu)$ is continuously differentiable near $\mu = 0$. Differentiation of the equation

$$\theta(1, \mu, \mathcal{L}_j(\mu)) = \beta_j$$

gives

$$\theta_\mu(1, \mu, \mathcal{L}_j(\mu)) + \theta_\lambda(1, \mu, \mathcal{L}_j(\mu))\mathcal{L}_j'(\mu) = 0$$

$$\theta_\mu(1, 0, \lambda_j) + \theta_\lambda(1, 0, \lambda_j)\mathcal{L}_j'(0) = 0$$

Since $\theta_\lambda(1, 0, \lambda_j) = \phi_\lambda(1, \lambda_j)<0$, it follows that $\mathcal{L}_j'(0) = 0$ if $\theta_\mu(1, 0, \lambda_j) = 0$. θ_μ satisfies

$$\theta_\mu' \doteq -2\sin\theta\cos\theta[1 + q - \lambda(a - f)]\theta_\mu - \lambda\cos^2 \frac{d}{d\mu} f$$
$$\theta_\mu(0) = 0$$

which can be seen by differentiating (8). Evaluate this initial value problem at $(x, 0, \lambda_j)$ and note that $\frac{df}{d\mu}\big|_{\mu=0} = 0$ to see that $\theta_\mu(x, 0, \lambda_j)$ satisfies the problem

$$y' \doteq -2\sin\theta\cos\theta[1 + q - \lambda_j(a - f)]y$$
$$y(0) = 0.$$

Thus $\theta_\mu(x, 0, \lambda_j) \equiv 0$. In particular, $\theta_\mu(1, 0, \lambda_j) = 0$.

Note that the preceding corollary, together with THEOREM 8 justifies the bifurcation diagram we gave for the buckling beam problem (Figure 3). Recall we had:

$$-v'' = \lambda(1 - (1 - \sqrt{1-v'^2}))v \qquad v(0) = 0 \qquad v(1) = 0$$

Here $f(t, v, v') = 1 - \sqrt{1-v'^2}$, and so $f(t, v, v')>0$ if $v'\neq 0$. By THEOREM 8, the branching is to the right. Furthermore, f is continuously differentiable, and so by COROLLARY 8.1, the bifurcation curve is orthogonal to the λ-axis.

The hypothesis that $f_u(x, 0, 0)$ and $f_{u'}(x, 0, 0)$ exist in the preceding corollary is necessary, as the following example shows:

EXAMPLE 1: Consider the problem:

$$-u'' + \lambda(1 - (-\sqrt{u^2+u'^2}))u = 0 \qquad u(0) = 0 \qquad u(\pi) = 0$$

Note that $f(x, u, u') = -\sqrt{u^2+u'^2} \leq 0$ (so the branching is to the left), and that $f_u(x, 0, 0)$ and $f_{u'}(x, 0, 0)$ do not exist.

$u(x) = c\sin x$ is a solution for this problem as long as

$\lambda(1 + |c|) = 1.$

So for $c>0$ we have $c = \frac{1}{\lambda} - 1$

and $c<0$ we have $c = -\frac{1}{\lambda} + 1$

$c = u'(0) = r(0) = \mu$, so our bifurcation diagram looks like:

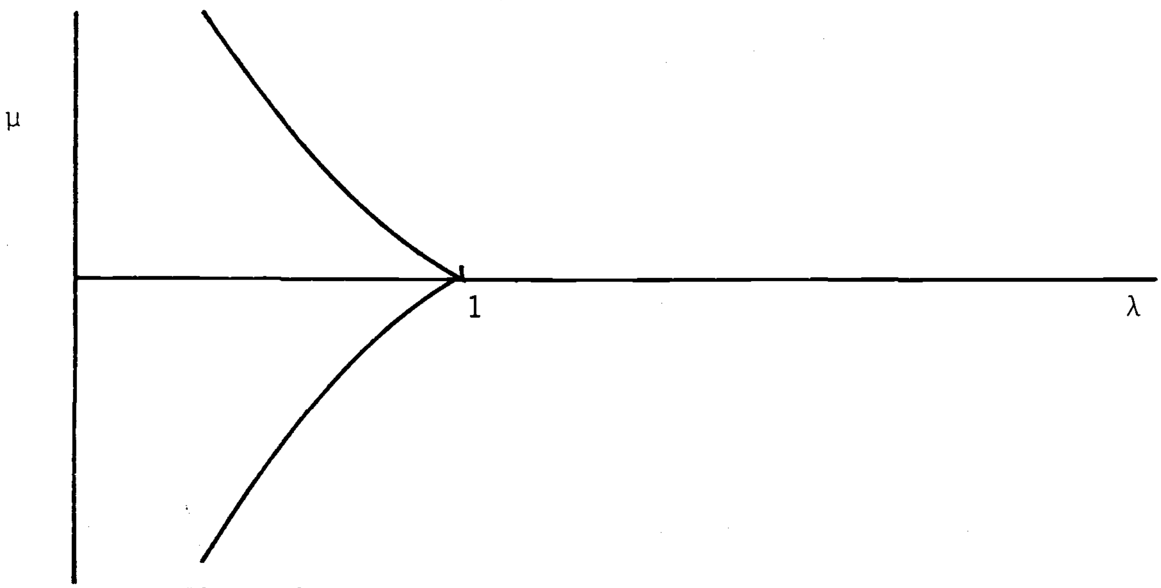


Figure 8.

EXAMPLE 2: The following example (due to Paul Rabinowicz [18] shows bifurcation at the value $\lambda = 1$, and a continuum of solutions emanating from it. Note that there are an infinite number of other "branches" not reachable from the main one.

The problem is:

$$-u'' = \lambda(1 + \sin\sqrt{u^2+u'^2})u \qquad u(0) = 0 \qquad u(\pi) = 0$$

Note that $u(x) = c\sin x$ is a solution, as long as $\lambda(1 + \sin c) = 1$, i.e., $\lambda = \frac{1}{1+\sin c}$. So we get the following diagram:

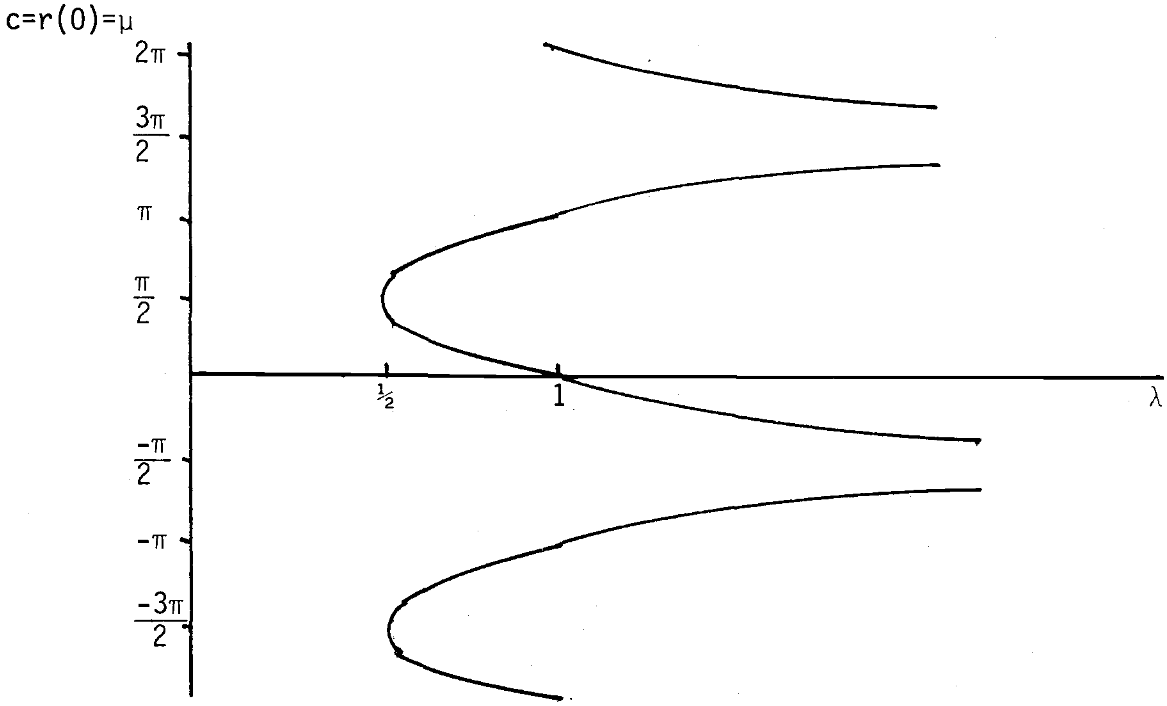


Figure 9.

Note that for this problem $f(x, u, u') = -\sin\sqrt{u^2+u'^2} = -\sin c$. Note that for small positive values of c, $f<0$; and so by THEOREM 8, we expect branching to the left. For small negative values of c, $f>0$ and so we expect branching to the right.

G. NUMERICAL METHODS AND EXAMPLES

Here we present a simple, yet effective, numerical method to compute bifurcation diagrams for a special case. Let $g:\mathbb{R}^4 \to \mathbb{R}$ be continuously differentiable. We wish to solve

$$u'' = g(x, u, u', \lambda) \qquad u(0) = 0 \qquad u(1) = 0, \tag{1}$$

a nonlinear Dirichlet problem with a parameter. Let $y(x, A, \lambda)$ be the solution of the initial value problem

$$u'' = g(x, u, u', \lambda) \qquad u(0) = 0 \qquad u'(0) = A \tag{2}$$

Our problem is to determine the locus of points $(A, \lambda)\in\mathbb{R}^2$ such that $y(1, A, \lambda) = 0$. As before, $y(1, \cdot, \cdot)$ is a continuously differentiable function with an open domain. We could use a two-dimensional Newton's method to find roots of $y(1, \cdot, \cdot)$, but in our case we desire control of the norm of the solution. (If we carried out the two-dimensional program for the problem $u'' = -[\lambda[a - f] - q]u$; we are likely to get the solution $u \equiv 0$).

To get a nontrivial solution to (1) then, the idea is to fix the initial condition $y'(0) = A$ and use a one-dimensional Newton's method in λ. That is, for each A, we form the function

$$h(\lambda) = y(1, A, \lambda)$$

and try to find its roots, by Newton's method.

The Newton's sequence becomes:

$$\lambda_{n+1} = \lambda_n - \frac{h(\lambda_n)}{h'(\lambda_n)} = \lambda_n - \frac{y(1,A,\lambda_n)}{y_\lambda(1,A,\lambda_n)}$$

So to carry out the iterative step, we must solve two initial value problems; (2), to compute $y(1, A, \lambda_n)$ and the following linear problem: to compute $y_\lambda(1, A, \lambda_n)$:

$$u_\lambda'' = g_u(t, u, u', \lambda_n)u_\lambda + g_{u'}(t, u, u', \lambda_n)u_\lambda + g_\lambda(t, u, u', \lambda_n)$$

$$u_\lambda(0) = 0 \qquad u_\lambda'(0) = 0$$

As before, we iterate until $|y(1, A, \lambda_n)|$ is sufficiently small.

Note that if $g(x, u, u', \lambda) = -(\lambda[a - f] - q)u$ as in the last section, we are guaranteed branching at the values $\overline{\lambda}_0, \overline{\lambda}_1, \cdots$, where $\overline{\lambda}_j$ is the j-th eigenvalue of the linear problem $-u'' + qu = \lambda au$ with Dirichlet boundary conditions. Hence in this case, there will be roots of $y(1, A, \cdot)$, $A \neq 0$ for λ close to $\overline{\lambda}_j$, if $0<|A|$ is sufficiently small.

To compute a bifurcation diagram, we choose $|A|>0$ small. If we want the j-th branch, we "shoot" with $\overline{\lambda}_j$, i.e., we set $\lambda = \overline{\lambda}_j$.

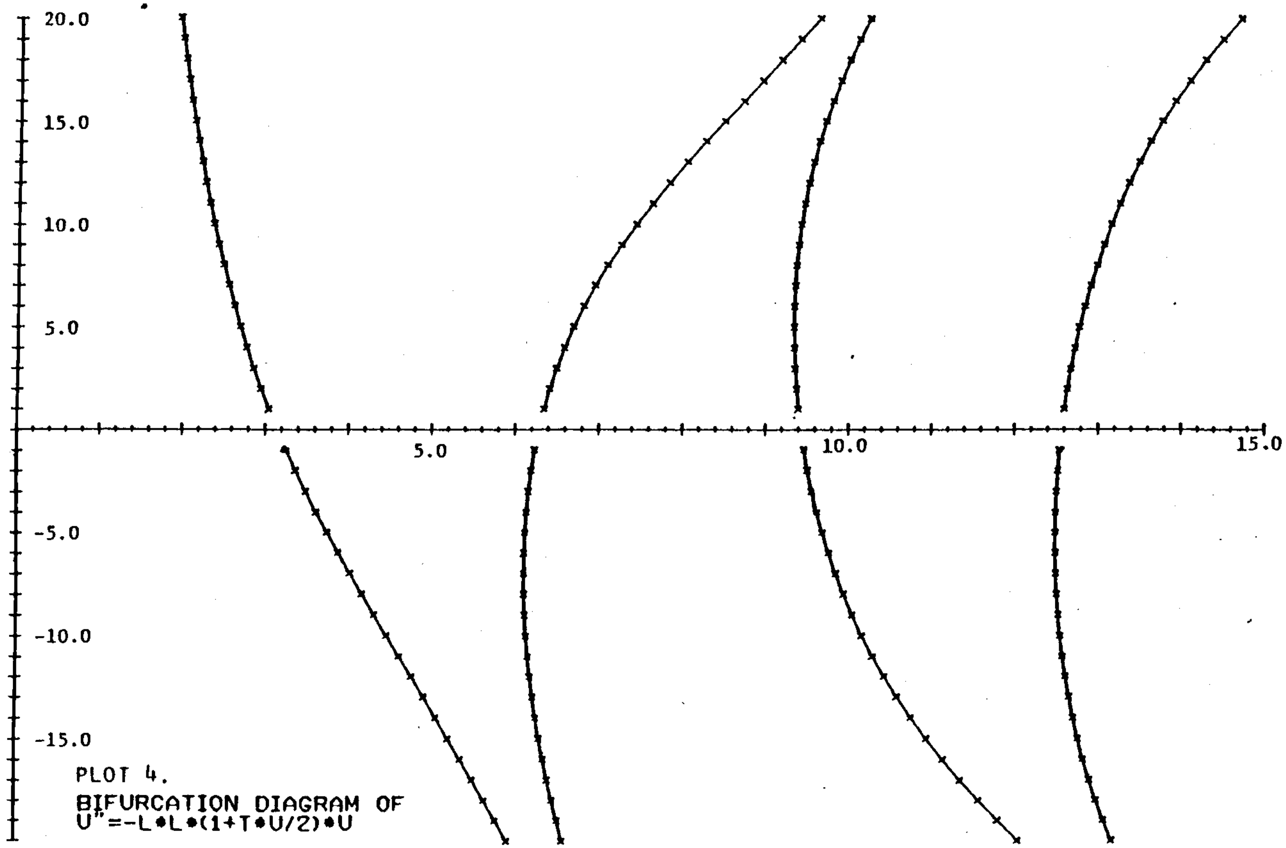
We then compute a root of $y(1, A, \cdot) = 0$. We repeat this process for different values of A, and generate a sequence of pairs (A_i, λ_{ji}) where $y(1, A_i, \lambda_{ji}) = 0$. We then connect these points to form an approximate "bifurcation diagram."

This scheme was programmed in FORTRAN, and a number of examples were run and the results plotted.

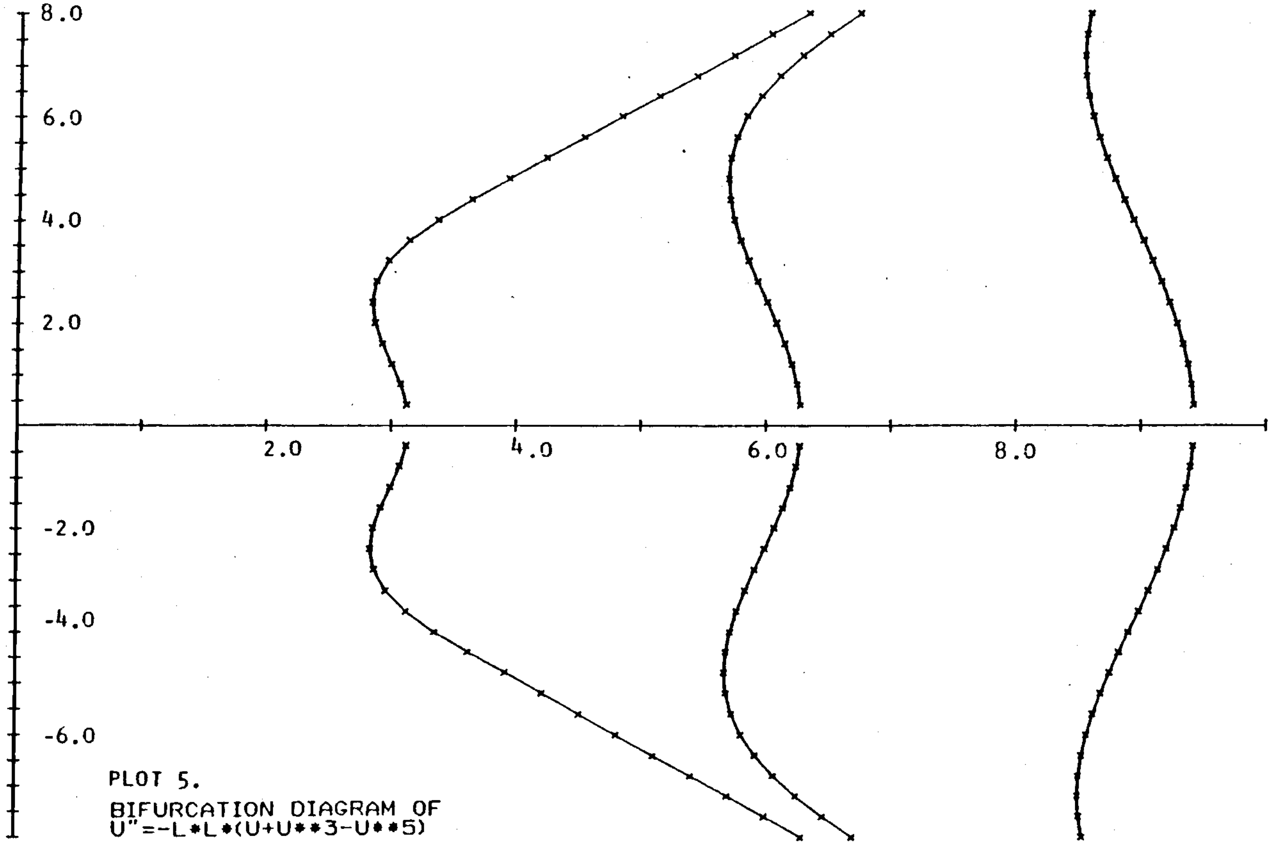
EXAMPLE 1: $u'' = -\lambda^2(1 + \frac{tu}{2})u \qquad u(0) = 0 \qquad u(1) = 0$. (See Plot 4). Here $f(t, u, u') = -\frac{tu}{2}$. The linear problem $u'' = -\lambda^2 u$ $u(0) = 0 \qquad u(1) = 0$ has eigenvalues $\lambda = \pi, 2\pi, 3\pi, \cdots$ so we expect bifurcation at these critical values. Moreover, if $u>0$ on $[0, 1]$,

$f(t, u, u')<0$, and if $u<0$ on $[0, 1]$, $f(t, u, u\ )>0$. In particular, any point on the branch emanating from π (the first branch) defines a problem with no interior zeros on $[0, 1]$. If $r(0) = u'(0)>0$ then, $u>0$ and so by THEOREM 8, we expect the "top part" of the first branch to branch left. Likewise, we expect the "bottom part" of the first branch to branch to the right. Notice that we cannot predict the behaviour of the other branches.

EXAMPLE 2: $u'' = -\lambda^2(1 + u^2 - u^4)u \qquad u(0) = 0 \qquad u(1) = 0$ (see Plot 5). Here $f(t, u, u') = -u^2 + u^4$. Once again, the linear problem has eigenvalues $\lambda = \pi, 2\pi, 3\pi, \cdots$, and so we expect branching at these critical values. Note that if (u, λ) is a solution to this problem, so is $(-u, \lambda)$ and so we expect the diagram to be symmetric about the line $\mu = 0$. Finally, notice that $f(t, u, u')<0$ for $0<|u^2 + u\ ^2|$ small, so by THEOREM 8, we expect branching to the left. (If $|u| >1$, however, $f(t, u, u')>0$ and so we would guess that the branches would eventually branch to the right).



PLOT 4.
BIFURCATION DIAGRAM OF
U"=-L*L*(1+T*U/2)*U



PLOT 5.
BIFURCATION DIAGRAM OF
U"=-L*L*(U+U**3-U**5)

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APPENDICES

APPENDIX I

FORTRAN PROGRAM FOR THE SHOOTING METHOD

```
      SUBROUTINE NEWTON
C
C   NEWTON IS A ROUTINE THAT WILL SOLVE SECOND ORDER D.E.≠S OF THE FORM
C       U≠=F(T,U,U≠) BY NEWTONS METHOD.
C    /INITIAL/  CONTAINS T1- THE LEFT HAND ENDPOINT OF THE INTERVAL,
C       STEP- THE NUMBER OF STEPS TO TAKE,
C       STEP1=STEP+1
C       AND H=THE STEPSIZE.
C    /GUESS/ CONTAINS THE INITIAL GUESS: X=U(T1) A=U(T1)
C       THE ROUTINE CALLS UPON RKUTTA TO INTEGRATE THE NONLINEAR PROBLEM AS
C       WELL AS THE TWO LINEAR PROBLEMS.  IT THEN UPDATES THE GUESS
C       U(T1),U≠(T1), BY NEWTONS METHOD.  THE ROUTINE STOPS WHEN
C       ABS(U(T2)-U(T1))+ABS(U≠(T2)-U≠(T1)) IS SUFFICIENTLY SMALL.
C
      INTEGER STEP,STEP1
      COMMON/SOL/U(100),DU(100)
      COMMON/INITIAL/T1,STEP,STEP1,H
      COMMON/GUESS/X,A
      DATA ERR/.0000001/
      DO 10 I=1,5
      CALL RKUTTA(XF,AF,GXF,GAF,HXF,HAF)
      PRINT 210
      PRINT 200,(U(K),K=1,STEP1)
      G=XF-X
      R=AF-A
      IF((ABS(G)+ABS(R)).LT.ERR) GO TO 20
      DET=(GXF-1.)*(HAF-1.)-GAF*HXF
      IF(ABS(DET).LT.ERR) GO TO 20
      X=X-(G*(HAF-1.)-R*GAF)/DET
   10 A=A-(R*(GXF-1.)-G*HXF)/DET
   20 PRINT 150,I,DET
      PRINT 220,X,A,XF,AF,GXF,GAF,HXF,HAF,G,R
  157 CONTINUE
      RETURN
  150 FORMAT(* AFTER ITERATION *,I3,* DET = *,F10.3)
  200 FORMAT(5(5X,F10.5))
  210 FORMAT(/)
  220 FORMAT(10(3X,F10.5))
      END
```

```
      SUBROUTINE RKUTTA(Y,A,UX,UA,DUX,DUA)
C
C      RKUTTA IS A RUNGE-KUTTA METHOD OF ORDER 4 USED BY SUBROUTINE NEWTON
C      TO INTEGRATE A NONLINEAR PROBLEM OF THE FORM  U≠≠=F(T,U,U≠).
C      IT CALLS SUBROUTINE LITTLE TO INTEGRATE A PAIR OF LINEAR PROBLEMS
C      EXTERNAL TO THE ROUTINE ARE THE THREE FUNCTIONS  F(T,U,U≠)
C      F2(T,U,U≠)              THE SECOND PARTIAL OF F AND
C      F3(T,U,U≠)             THE THIRD PARTIAL OF F.
C
      COMMON/SOL/U(100),DUL(100)
      COMMON/INITIAL/T1,STEP,STEP1,H
      COMMON/UVAL/YHT.AHT,YFT,AFT
      COMMON/GUESS/XS,AS
      REAL K1,K2,K3,K4,L1,L2,L3,L4
      INTEGER STEP,STEP1
      DATA INF/1000000./
      Y=XS
      A=AS
      UX=1.
      DUX=0.
      UA=0.
      DUA=1.
      T=T1
      DO 1 I=1,STEP
      U(I)=Y
      TH=T+H/2.
      TF=T+H
      K1=H*A
      L1=H*F(T,Y,A)
      K2=H*(A+L1/2.)
      L2=H*F(TH,Y+K1/2.,A+L1/2.)
      K3=H*(A+L2/2.)
      L3=H*F(TH,Y+K2/2.,A+L2/2.)
      K4=H*(A+L3)
      L4=H*F(TF,Y+K3,A+L3)

      YFT=Y+(K1+2.*K2+2.*K3+K4)/6.
      AFT=A+(L1+2.*L2+2.*L3+L4)/6.
      AHT=(AFT+A)/2.
      YHT=(YFT+Y)/2.
      CALL LITTLE(UX,DUX,T,TH,TF,Y,A)
      CALL LITTLE(UA,DUA,T,TH,TF,Y,A)
      Y=YFT
      A=AFT
      T=TF
      IF((ABS(A)+ABS(Y)).GT.INF) STOP
1     CONTINUE
      U(STEP1)=Y
      RETURN
      END

      SUBROUTINE LITTLE(V,DV,T,TH,TF,Y,A)
      COMMON/SOL/U(100),DU(100)
      COMMON/INITIAL/T1,STEP,STEP1,H
      COMMON/UVAL/YHT,AHT,YFT,AFT
      REAL K1,K2,K3,K4,L1,L2,L3,L4
      DATA INF/10000000./
      K1=H*DV
      L1=H*(F2(T,Y,A)*V+F3(T,Y,A)*DV)
      K2=H*(DV+L1/2.)
      L2=H*(F2(TH,YHT,AHT)*(V+K1/2.)+F3(TH,YHT,AHT)*(DV+L1/2.))
      K3=H*(DV+L2/2.)
      L3=H*(F2(TH,YHT,AHT)*(V+K2/2.)+F3(TH,YHT,AHT)*(DV+L2/2.))
      K4=H*(DV+L3)
      L4=H*(F2(TF,YFT,AFT)*(V+K3)+F3(TF,YFT,AFT)*(DV+L3))
      V=V+(K1+2.*K2+2.*K3+K4)/6.
      DV=DV+(L1+2.*L2+2.*L3+L4)/6.
      IF((ABS(V)+ABS(DV)).GT.INF) STOP
      RETURN
      END
```

APPENDIX II

FORTRAN PROGRAM FOR QUASILINEARIZATION

```
      SUBROUTINE QUASI
C
C QUASI IS A ROUTINE THAT WILL SOLVE SECOND ORDER O.E.*S OF THE FORM
C  U**=F(T,U,U*)   VIA THE METHOD OF QUASILINEARIZATION.
C       THE SOLUTION AND ITS DERIVATIVE U AND DU ARE IN /SOL/ AND MUST BE SET
C       TO THE INITIAL GUESS BEFORE QUASI IS CALLED
C       IN  /INITIAL/ ARE THE VARIABLES
C         T1- THE LEFT HAND ENDPOINT OF THE INTERVAL,
C         STEP- THE NUMBER OF STEPS THE ROUTINE IS TO TAKE,
C         STEP1=STEP+1
C         AND H - THE STEPSIZE.
C THE METHOD INTEGRATES THREE LINEAR PROBLEMS BY SUBROUTINE LITTLE AND THE SOLUT
C  ION UN AT THE N-TH STEP IS A LINEAR COMBINATION OF THESE SOLUTIONS.  IF
C NORM(UN+1 -UN)= MAX( ABS(UN+1 -UN), ABS(UN+1* -UN*)) IS LESS THAN A SPECIFIED
C ERROR=ERR THE ROUTINE STOPS.
C N IS SET AT AN UPPER LIMIT OF 10
C
C       EXTERNAL TO THE ROUTINE ARE THE THREE FUNCTIONS   F(T,U,U*)
C       F2(T,U,U*)                 THE SECOND PARTIAL OF F AND
C       F3(T,U,U*)                THE THIRD PARTIAL OF F.
C
      REAL MAX,V1(100),DV1(100),V2(100),DV2(100),W(100),DW(100)
      REAL NORM
      INTEGER STEP,STEP1
      COMMON/SOL/U(100),DU(100)
      COMMON/INITIAL/T1,STEP,STEP1,H
      DATA ERR/.0000001/
      DO 10 I=1,3
      MAX=0.
      CALL LITTL (0.,1.,A,AP,V1,DV1,0.)
      CALL LITTL (1.,0.,B,BP,V2,DV2,0.)
      CALL LITTL (0.,0.,C,CP,W,DW,1.)
      DET=(1.-AP)*(1.-B)-A*BP
      PRINT 205,DET
      IF(ABS(DET).LT.ERR) GO TO 12
      C1=(CP-CP*B+C*BP)/DET
      C2=(C-C*AP+CP*A)/DET
      DO 5 J=1,STEP1
      Y=W(J)+C1*V1(J)+C2*V2(J)
      Z=DW(J)+C1*DV1(J)+C2*DV2(J)
      NORM=ABS(U(J)-Y)+ABS(DU(J)-Z)
      IF (NORM.GT.MAX) MAX=NORM
      U(J)=Y
      DU(J)=Z
    5 CONTINUE
      PRINT 220,I,MAX
      PRINT 200,(U(K),K=1,STEP1)
      IF(MAX.LT.ERR) GO TO 11
   10 CONTINUE
   11 CONTINUE
      RETURN
   12 PRINT 210
      GO TO 11
  200 FORMAT(5(5X,F10.5))
  205 FORMAT(* DET = *,F10.5)
  210 FORMAT(* DET IS ZERO, TOUGH LUCK*)
  220 FORMAT(* AFTER ITERATION *,I3,* MAX = *,F10.5)
      END
```

```
      SUBROUTINE LITTL (YS,AS,Y,A,V,DV,R)
C
C LITTL  IS A ROLTINE THAT WILL INTEGRATE OUR SECOND ORDER LINEAR PROBLEMS
C BY A RUNGE-KUTTA METHOD OF ORDER 4.  YS AND AS ARE THE INITIAL VALUE AND SLOPE
C Y AND A WILL BE THE FINAL VALUE AND SLOPE, V,DV ARE MATRICES CONTAINING THE SO
C LUTION AND R IS A FLAG- A NUMBER EITHER 0. OR 1. DEPENDING ON WHETHER THE
C HOMOGENEOUS OR THE INHOMOGENOUS PROBLEM IS TO BE INTEGRATED.
C
      INTEGER STEP,STEP1
      REAL V(100),DV(100),INF
      REAL K1,K2,K3,K4,L1,L2,L3,L4
      COMMON/INITIAL/TS,STEP,STEP1,H
      COMMON/SOL/U(100),DU(100)
      DATA INF/100000./
      T=TS
      Y=YS
      A=AS
      K1=0.
      K2=0.
      K3=0.
      K4=0.
      L1=0.
      L2=0.
      L3=0.
      L4=0.
      DO 1 I=1,STEP1
      Y=Y+(K1+2.*K2+2.*K3+K4)/6.
      A=A+(L1+2.*L2+2.*L3+L4)/6.
      IF((A*A+Y*Y).GT.INF) GO TO 2
      V(I)=Y
      DV(I)=A
      T1=T+H/2.
      T2=T+H
      YT=U(I)
      AT=DU(I)
      YFT=U(I+1)
      AFT=DU(I+1)
      YHT=(YFT+YT)/2.
      AHT=(AFT+AT)/2.
      K1=H*A
      L1=H*(F2(T,YT,AT)*Y+F3(T,YT,AT)*A
     X     +R*(F(T,YT,AT)-F2(T,YT,AT)*YT-F3(T,YT,AT)*AT))
      K2=H*(A+L1/2.)
      L2=H*(F2(T1,YHT,AHT)*(Y+K1/2.)+F3(T1,YHT,AHT)*(A+L1/2.)
     X     +R*(F(T1,YHT,AHT)-F2(T1,YHT,AHT)*YHT-F3(T1,YHT,AHT)*AHT))
      K3=H*(A+L2/2.)
      L3=H*(F2(T1,YHT,AHT)*(Y+K2/2.)+F3(T1,YHT,AHT)*(A+L2/2.)
     X     +R*(F(T1,YHT,AHT)-F2(T1,YHT,AHT)*YHT-F3(T1,YHT,AHT)*AHT))
      K4=H*(A+L3)
      L4=H*(F2(T2,YFT,AFT)*(Y+K3)+F3(T2,YFT,AFT)*(A+L3)
     X     +R*(F(T2,YFT,AFT)-F2(T2,YFT,AFT)*YFT-F3(T2,YFT,AFT)*AFT))
      T=T2
    1 CONTINUE
    2 RETURN
      END
```

APPENDIX III

FORTRAN PROGRAM TO GENERATE BIFURCATION DIAGRAMS

```
      PROGRAM RUFUS(OUTPUT,TAPE10=0)
C
C      THE FOLLOWING PROGRAM GENERATES A BIFURCATION DIAGRAM FOR THE PROBLEM
C   U≠=F(T,U,U≠,PARA) WITH DIRICHLET BOUNDARY CONDITIONS.
C     U≠≠=F(T,U,U≠) WITH DIRICHLET BOUNDARY CONDITIONS.
C     WE SHOOT WITH THE PARAMETER PARA
C     U CONTAINS THE COMPUTED EIGENFUNCTION.
C      A,B ARE MATRICES CONTAINING THE POINTS (PARA,U≠(0)).
C      STEP IS THE NUMBER OF STEPS THE INITIAL VALUE ROUTINE IS TO TAKE.
C      PARA1 IS THE INITIAL ≠SHOOTING≠ GUESS,
C
      COMMON /LAM/PARA
      INTEGER STEP,STEP1
      REAL U(200)
      REAL A(20),B(20)
      REAL M
      DATA TOL/.00001/
      DATA T,STEP/0.,50/
      STEP1=STEP+1
      H=1./FLOAT(STEP)
      X=0.
      PARA1=0.
C
C      THIS LOOP CONTROLS THE NUMBER OF BRANCHES.
C      INITIALIZE THE SHOOTING PARAMETER.
C
      DO 13 N3=1,3
      PARA1=PARA1+3.1
C
C      THIS LOOP PICKS UP THE POSITIVE AND NEGATIVE SIDES OF THE BRANCH.
C
      DO 13  N2=1,2
      PARA=PARA1
      M=(-1.)**N2
C
C      THIS LOOP CONTROLS THE NUMBER OF POINTS ON EACH SIDE OF THE BRANCH.
C
      DO 12 N1=1,20
      N4=N1-2
C      HERE WE USE A LINEAR INTERPOLATION TO GET A BETTER SHOOTING VALUE.
      IF(N1.GE.3) PARA=2.*PARA-B(N4)
C      HERE WE SET THE INITIAL SLOPE,
      A(N1)=M*FLOAT(N1)/2.5
C
C      IN THIS LOOP WE ITERATE USING NEWTONS METHOD IN THE VARIABLE PARA
C      IN ORDER TO SATISFY U≠(0)=0.
C
      DO 10 I=1,10
      CALL RKUTTA4(T,X,A(N1),H,STEP,U,FL)
      F=U(STEP1)
      IF((ABS(F).LT.TOL).OR.(ABS(FL).LT.TOL)) GO TO 11
C      NEWTONS METHOD TO UPDATE PARA.
      PARA=PARA-F/FL
   10 CONTINUE
C      SAVE THE COMPUTED VALUE OF PARA.
   11 B(N1)=PARA
   12 CONTINUE
      PRINT 160,(U(I),I=1,51)
  160 FORMAT(5(1X,F10.5))
   13 CONTINUE
      STOP
      END
```

```
      SUBROUTINE RKUTTA4(TS,XS,AS,H,ITER,U,UL)
C
C      RKUTTA4 IS A STANDARD ROUTINE TO INTEGRATE OUR INITIAL VALUE PROBLEMS.
C      IT SOLVES  U≠≠=F(T,U,U≠,PARA)  WITH  U(0)=0  U≠(L)=AS
C  AS WELL AS  THE LINEAR PROBLEM
C      V≠≠=F2(T,U,U≠,PARA)*V+F3(T,U,U≠,PARA)*V≠+F4(T,U,U≠,PARA)  V(0)=0 V≠(0)=0
C
      REAL K1,K2,K3,K4,L1,L2,L3,L4
      DIMENSION U(500)
      DATA INF/1000000./
      T=TS
      X=XS
      A=AS
      UL=0.
      DUL=0.
      DO 1 I=1,ITER
      U(I)=X
      T1=T+H/2.
      T2=T+H
      F2T=F2(T1,X,A)
      F3T=F3(T1,X,A)
      F4T=F4(T1,X,A)
      K1=H*DUL
      L1=H*(F2(T,X,A)*UL+F3(T,X,A)*DUL+F4(T,X,A))
      K2=H*(DUL+L1/2.)
      L2=H*(F2T*(UL+K1/2.)+F3T*(DUL+L1/2.)+F4T)
      K3=H*(DUL+L2/2.)
      L3=H*(F2T*(UL+K2/2.)+F3T*(DUL+L2/2.)+F4T)
      K4=H*(DUL+L3)
      L4=H*(F2(T2,X,A)*(UL+K3)+F3(T2,X,A)*(DUL+L3)+F4(T2,X,A))
      UL=UL+(K1+2.*K2+2.*K3+K4)/6.
      DUL=DUL+(L1+2.*L2+2.*L3+L4)/6.
      IF((ABS(UL).GT.INF).OR.(ABS(DUL).GT.INF)) GO TO 2
      K1=H*A
      L1=H*F(T,X,A)
      K2=H*(A+L1/2.)
      L2=H*F(T1,X+K1/2.,A+L1/2.)
      K3=H*(A+L2/2.)
      L3=H*F(T1,X+K2/2.,A+L2/2.)
      K4=H*(A+L3)
      L4=H*F(T2,X+K3,A+L3)
      X=X+(K1+2.*K2+2.*K3+K4)/6.
      A=A+(L1+2.*L2+2.*L3+L4)/6.
      IF((ABS(X).GT.INF).OR.(ABS(A).GT.INF)) GO TO 2
      T=T2
    1 CONTINUE
      U(ITER+1)=X
    2 RETURN
      END
```