AN ABSTRACT OF THE THESIS OF

Michael L. Treuden for the degree of Doctor of Philosophy

in Mathematics presented on May 2, 1983

Title: Asymptotically Compact Operator Approximation Theory

Abstract approved: Redacted for Privacy

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Asymptotically compact sequences of operators arise from the approximate solution of differential and integral equations. Motivated by such problems, we study linear operator equations

$$(I-K)x = y \ , \quad (I-K_n)x_n = y \ , \quad n = 1,2,3,\ldots \ ,$$

in a Banach space $X$, where $K_n \to K$ pointwise and $\{K_n\}$ is asymptotically compact. If each $K_n$ is compact this reduces to the more fully studied case with $\{K_n\}$ collectively compact. We extend much of the theory for the collectively compact case to the asymptotically compact case. The analysis is based on the systematic use of measures of noncompactness as well as convergence and compactness concepts for sequences of sets.

The similarities of the two theories and the known examples motivate the conjecture:

$$\{K_n\} \text{ asymptotically compact } \Leftrightarrow$$

$$K_n = L_n + T_n, \ \{L_n\} \text{ collectively compact}, \|T_n\| \to 0 \ .$$

The principal result of this thesis proves the conjecture for a large class of spaces, which includes those of practical interest.

# Asymptotically Compact Operator Approximation Theory

by

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A THESIS

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Completed May 2, 1983

Commencement June 1983

APPROVED:

Redacted for Privacy

Professor of Mathematics in charge of major

Redacted for Privacy

Head of Department of Mathematics

Redacted for Privacy

Dean of Graduate School

Date thesis is presented May 2, 1983

Typed by Jane A. Tuor for Michael L. Treuden

ACKNOWLEDGMENT

First and foremost I wish to thank Philip M. Anselone for his generous support and guidance during both the research and writing phases of this thesis. I am also grateful to Joel Davis, William Firey, and Don Solmon for their interest and valued comments.

No acknowledgment would be complete without thanking my family. I dedicate this thesis to my mother and father.

The faculty, staff, and fellow graduate students have all contributed to make my stay at Oregon State University a thoroughly enjoyable one. To all those I have had the pleasure to meet, I say: Live long and prosper.

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# ASYMPTOTICALLY COMPACT OPERATOR APPROXIMATION THEORY

## I. INTRODUCTION AND SUMMARY

Asymptotically compact sequences of operators arise from the approximate solution of differential and integral equations. Motivated by such problems, we study linear operator equations

$$(1.1) \qquad (I-K)x = y \ , \quad (I-K_n)x_n = y \ , \ n = 1,2,3,\ldots \ ,$$

in a Banach space $X$, where $K_n \to K$ pointwise and $\{K_n\}$ is asymptotically compact, i.e., if the sequence of points $\{x_n\}$ is bounded, then each subsequence of $\{K_n x_n\}$ has a convergent subsequence. If each $K_n$ is compact this reduces to the more fully studied case with $\{K_n\}$ collectively compact. The collectively compact case is typified by numerical integration approximations of integral equations with continuous kernels. Weakly singular kernels lead to asymptotically compact operator approximations. We extend much of the theory for the collectively compact case to the asymptotically compact case. In particular, $I-K_n$ satisfies the Fredholm alternative for all $n$ large enough, and

$$(I-K_n)^{-1} \to (I-K)^{-1} \quad \text{on} \quad X \ ,$$

with practical error bounds, whenever $(I-K)^{-1}$ exists. There are implications for spectral properties of $K$

and $K_n$. The analysis is based on the use of convergence and compactness concepts for sequences of sets as well as a convenient measure of noncompactness defined for bounded sets. Definitions of these concepts along with their associated consequences and mutual relationships are presented in Chapter II.

To begin Chapter III we review some basic facts about contracting, compact, and semi-Fredholm operators. The majority of the chapter is devoted toward discussing bounded, linear operators on $X$ which diminish some measure of noncompactness, the so-called condensing operators. Special cases include contracting and compact operators. It is shown that if $T$ is condensing then $I-T$ satisfies the Fredholm alternative. The chapter concludes with some spectral properties of $T$ which depend on the measure of noncompactness of $T$. These results are generalizations of the classical situation when $T$ is compact, that is, the measure of noncompactness of $T$ is zero.

In Chapter IV we turn to the study of operator approximations. Coupled with a Fredholm alternative for condensing operators, the set convergence and compactness concepts of Chapter II provide efficient proofs of results concerning the mutual (unique) solvability of the equations (1.1) as well as the convergence

$(I-K_n)^{-1} \to (I-K)^{-1}$ (with practical error bounds) when the latter inverse exists. Finally, an example involving the approximate solution of a weakly singular integral equation is given.

It is very tempting, even on the basis of the terminology alone, to want to view asymptotically compact sequences as sequences of "almost" compact operators. The fact that the collectively and asymptotically compact approximation theories are virtually identical, along with the known examples (cf. Chapter IV), serve to strengthen this view. Thus, we have the conjecture:

$$\{K_n\} \text{ asymptotically compact} \iff$$
$$K_n = L_n + T_n \ , \ \{L_n\} \text{ collectively compact} \ , \ \|T_n\| \to 0 \ .$$

The principal result of this thesis proves this conjecture for a large class of spaces, which includes those of practical interest.

# II. PRELIMINARIES

## 1. Sequences of Sets

In this section we capuslize the set theoretic concepts and results needed in later chapters. The main emphasis revolves around the characterization of discretely compact sequences in two ways, in terms of set convergence and in terms of asymptotic total boundedness. Applications to linear subspaces will be particularly important.

Let $(X,\rho)$ be a metric space. Elements and subsets are $x_n$ , $x \in X$, $S_n$ , $S \subset X$ where $n \in N = \{1,2,3,...\}$ . Denote infinite subsets of $N$ by $N'$ , $N''$ , etc. Our first definition generalizes point convergence.

Definition 2.1.

For $S \subset X$ , $\varepsilon > 0$ , an ε-neighborhood of S is defined by

$$\Omega_\varepsilon(S) = \bigcup_{x' \in S} \{x \in X: \rho(x,x') < \varepsilon\} \quad \text{if} \quad S \neq \emptyset ,$$

and $\Omega_\varepsilon(\emptyset) = \emptyset$ . The sequence $\{S_n: n \in N\}$ is said to converge to S if $\forall \varepsilon > 0 \; \exists n(\varepsilon) \in N$ such that $S_n \subset \Omega_\varepsilon(S)$ whenever $n \geq n(\varepsilon)$ , in which case we write $S_n \to S$ .

These limits are not unique.

$$(2.1) \qquad S_n \to S \subset S' \Rightarrow S_n \quad S' .$$

Other immediate consequences of the definition are:

$$(2.2) \qquad S_n \to S , \quad S_n' \subset S_n \Rightarrow S_n' \to S ,$$

$$(2.3) \qquad S_n \to \emptyset \Longleftrightarrow S_n = \emptyset \quad \forall n \text{ large},$$

(2.4) Denote the closure of $S$ by $\bar{S}$. Then

$$\Omega_\varepsilon(S) = \Omega_\varepsilon(\bar{S}) \quad \forall \varepsilon > 0 ,$$

$$(2.5) \qquad S_n \to S \Longleftrightarrow \bar{S}_n \to \bar{S} \Longleftrightarrow \bar{S}_n \to S \Longleftrightarrow S_n \to \bar{S} .$$

The union of a fixed set and a convergent sequence is easily handled:

$$(2.6) \qquad S' \subset X, \quad S_n \to S \Rightarrow S_n \cup S' \to S \cup S' .$$

The above is a consequence of

$$\Omega_\varepsilon(S \cup S') \supset \Omega_\varepsilon(S) \cup S' \quad \forall \varepsilon > 0 .$$

Assertion (2.6) does not hold with union replaced by intersection. For if $S_n = [-\frac{1}{n}, 2 + \frac{1}{n}]$, $S = (0,2)$ are real intervals, and $S' = \{0,2\}$ then $S_n \to S$ but $S_n \cap S' = S' \not\to S \cap S' = \emptyset$ . However, note that $S_n \cap S' \to \bar{S} \cap S'$ . In fact, one can show $S_n \to S$ , $S'$ compact imply $S_n \cap S' \to \bar{S} \cap S'$ by using only the notion of set convergence. Though the proof is not difficult, we shall obtain the desired result as a consequence of the related idea of discretely compact sequences, which we define next.

Definition 2.2.

Define the cluster point sets:

$$\{x_n\}^* = \{x \in X: x_n \to x,\ n \in N'\}$$

$$\{S_n\}^* = \{x \in X: x_n \to x,\ x_n \in S_n,\ n \in N'\}\ .$$

The sequence $\{x_n\} = \{x_n: n \in N\}$ is d-compact (discretely compact) if each subsequence has a convergent subsequence, i.e., $\{x_n: n \in N'\}^* \neq \emptyset \quad \forall N' \subset N$ .
Analogously, $\{S_n\}$ is d-compact if
$\{S_n: n \in N'\}^* \neq \emptyset \quad \forall N' \subset N$ such that $S_n \neq \emptyset$ , $n \in N'$ .

It follows that the cluster point sets $\{x_n\}^*$ and $\{S_n\}^*$ are closed. Moreover,

(2.7) $\qquad \{S_n\}^* = \{\bar{S}_n\}^*$ ,

$\qquad \{S_n\}$ d-compact $\iff$ $\{S_n\}$ d-compact,

(2.8) $\qquad S_n \to S \Rightarrow \{S_n\}^* \subset \bar{S}$,

(2.9) $\qquad \{S_n \cap S\}^* \subset \{S_n\}^* \cap \bar{S}$.

The notions of d-compactness and the aforementioned set convergence are related:

Theorem 2.3.

$\{S_n\}$ d-compact $\iff S_n \to \{S_n\}^*$, $\{S_n\}^*$ compact.

proof: ($\Rightarrow$) Suppose $S_n \not\to \{S_n\}^*$. Then

$\exists \varepsilon_0 > 0$ , $N' \subset N$ , $x_n \in S_n$ such that $\mathrm{dist}(x_n, \{S_n\}^*) \geqq \varepsilon_0 > 0$ , $n \in N'$ . Now $\{S_n\}$ d-compact $\Rightarrow$ $\exists N'' \subset N'$ , $x \in X$ such that $x_n \to x$ , $n \in N''$ . Necessarily, $x \in \{S_n\}^*$ , which is a contradiction. To show $\{S_n\}^*$ is compact choose $\{x_m : m \in N\} \subset \{S_n\}^*$ . Then for each m there is an integer $n_m > n_{m-1}$ (take $n_0 = 0$) and an element $x_{n_m} \in S_{n_m}$ such that

$$(2.10) \qquad \rho(x_{n_m}, x_m) \to 0 \ , \quad m \in N \ .$$

The d-compactness of $\{S_n\}$ produces a convergent subsequence of $\{x_{n_m}\}$ , say $\{x_{n_m} : m \in N''' \subset N\}$ . It follows from (2.10) that $\{x_m : m \in N'''\}$ must have the same limit as $\{x_{n_m} : m \in N'''\}$ . Therefore, $\{S_n\}^*$ is compact.

($\Leftarrow$) Choose any subsequent $\{x_n \in S_n : n \in N'\}$ . Then $S_n \to \{S_n\}^*$ implies $\mathrm{dist}(x_n, \{S_n\}^*) \to 0$ , $n \in N'$ . Since $\{S_n\}^*$ is compact, there are elements $x^n \in \{S_n\}^*$ so that $\rho(x_n, x^n) = \mathrm{dist}(x_n, \{S_n\}^*) \to 0$ , $n \in N'$ and there exists a convergent subsequence of $\{x^n : n \in N'\}$ . Hence, $\{x_n : n \in N'\}$ has a convergent subsequence as well and the theorem is proved. ∎

<u>Corollary 2.4.</u>

Suppose $\{S_n\}$ d-compact. Then $\{S_n\}^*$ is the minimal closed set $S$ such that $S_n \to S$.

<u>proof</u>: $S_n \to \{S_n\}^*$ by Theorem 2.3. By (2.8), $\{S_n\}^* \subset S$ for any closed set $S$ such that $S_n \to S$. ■

<u>Corollary 2.5.</u>

Let $S_n \to S$ and $S'$ be compact. Then $S_n \cap S' \to \bar{S} \cap S'$.

<u>proof</u>: Since $S'$ is compact the sequence $\{S_n \cap S'\}$ is d-compact. By Theorem 2.3, (2.8), and (2.9) we obtain

$$S_n \cap S' \to \{S_n \cap S'\}^* \subset \{S_n\}^* \cap S' \subset \bar{S} \cap S' ,$$

as desired. ■

<u>Remark</u>: The closure of $S$ is needed. Refer to example on page 5.

The next theorem characterizes a sequence of sets which is, in some sense, a uniformly relatively compact sequence as a d-compact sequence of relatively compact sets.

<u>Theorem 2.6.</u>

$\overline{\bigcup_N S_n} = \bigcup_N \bar{S}_n \cup \{S_n\}^*$ and $\overline{\bigcup_N S_n}$ compact $\Leftrightarrow$ $\{S_n\}$ d-compact, $\bar{S}_n$ compact $\forall n$.

<u>proof</u>: The containment $\overline{\cup S_n} \supset \cup \bar{S}_n \cup \{S_n\}^*$ is obvious. Since closed subsets of a compact set are compact, we have $\bar{S}_n$ compact $\forall n$ when $\overline{\cup S_n}$ is compact. Moreover,

$\{S_n\}$ is d-compact whenever $\overline{\cup S_n}$ is compact since each subsequence $\{x_n \in S_n : n \in N'\}$ is contained in $\overline{\cup S_n}$ . To show both the other containment and implication, let $\{x_m : m \in N\}$ be any sequence out of $\cup S_n$ and suppose $x_m \to x$ . If $\{x_m\}$ is contained in a finite union of the sets $\bar{S}_n$ then $x \in \cup \bar{S}_n$ . If not, then $x \in \{S_n\}^*$ and the desired inclusion follows. Now suppose that $\bar{S}_n$ is compact $\forall n$ and $\{S_n\}$ is d-compact. If $\{x_m\}$ is contained in a finite union of the sets $\bar{S}_n$ then the compactness of such a union produces a convergent subsequence. If not, then $\exists N'$ such that $x_m \in S_m$ , $m \in N'$ . Then the d-compactness of $\{S_n\}$ produces a convergent subsequence. ■

For our purposes it will be more useful to characterize d-compactness by an asymptotic version of total boundedness when $X$ is complete.

<u>Definition 2.7.</u>

$\{S_n\}$ is <u>asymptotically totally bounded</u> if $\forall \varepsilon > 0$ $n_\varepsilon \in N$ such that $\bigcup_{n \geq n_\varepsilon} S_n$ has a finite $\varepsilon$-net (in $X$) .

<u>Theorem 2.8.</u>

$\{S_n\}$ d-compact $\Rightarrow$ $\{S_n\}$ asymptotically totally bounded. The converse holds if $X$ is complete.

<u>proof</u>: ($\Rightarrow$) We have $S_n \to \{S_n\}^*$ and $\{S_n\}^*$ compact by Theorem 2.3. Then $\{S_n\}^*$ is totally bounded, i.e., $\forall \varepsilon > 0$ $\exists$ finite set $S_\varepsilon \subset X$ such that $\{S_n\}^* \subset \Omega_\varepsilon(S_\varepsilon)$ .

The set convergence $S_n \to \{S_n\}^*$ implies that $\exists n_\varepsilon \in N$ such that $n \geqq n_\varepsilon \Rightarrow S_n \subset \Omega_\varepsilon(\{S_n\}^*)$. Hence, $\bigcup_{n \geqq n_\varepsilon} S_n \subset \Omega_{2\varepsilon}(S_\varepsilon)$. Therefore, $\{S_n\}$ is asymptotically totally bounded.

($\Leftarrow$) Assume $X$ is complete and $\{S_n\}$ is asymptotically totally bounded. Then for each $m \in N$ $\exists$ a finite set $s^m \subset X$ and an integer $n_m$ so that $\bigcup_{n \geqq n_m} S_n \subset \Omega_{2^{-m}}(S^m)$. Choose any subsequence $\{x_n \in S_n\colon\ n \in N'\}$. Then there is an infinite set $N_1 \subset N'$ and an element $y_1 \in S^1$ such that $x_n \in \Omega_{2^{-1}}(y_1)\ \forall n \in N_1$. Furthermore, $\exists N_2 \subset N_1$, $y_2 \in S^2$ such that $x_n \in \Omega_{2^{-2}}(y_2)\ \forall n \in N_2$. In this way, we recursively obtain for each $m \in N$ infinite sets $N_m \subset N_{m-1}$ (take $N_0 = N$) and elements $y_m \in S^m$ such that $x_n \in \Omega_{2^{-m}}(y_m)\ \forall n \in N_m$. For $m \in N$ choose $n_m \in N_m$ so that $n_m > n_{m-1}$ (take $n_0 = 0$). By construction, $k < m \Rightarrow n_k, n_m \in N_k$, and consequently,

$$\begin{aligned}\rho(x_{n_k}, x_{n_m}) &\leqq \rho(x_{n_k}, y_k) + \rho(y_k, x_{n_m}) \\ &\leqq 2^{-k} + 2^{-k} = 2^{1-k} .\end{aligned}$$

Therefore, $\{x_{n_m}\}$ is Cauchy and must converge since $X$ is complete. ■

We now impose a linear structure on $X$ and indicate how d-compactness and the associated set convergence combine via Theorem 2.3 to establish results comparing the dimensions of various subspaces of $X$. To this end, let $E_n$ , $E$ , $F \subset X$ be linear subspaces of the normed linear space $(X,\|\cdot\|)$ . Denote the unit sphere in $X$ by $U = \{x \in X: \|x\| = 1\}$ . Note that $\{E_n\}^*$ is a closed subspace of $X$.

<u>Theorem 2.9.</u>

(a) $\{E_n \cap U\}^* = \{E_n\}^* \cap U$ .

(b) $\{E_n \cap U\}$ d-compact $\Rightarrow$ $\dim \{E_n\}^* < \infty$ .

<u>proof</u>: (a) ($\subset$) This follows from $\{E_n \cap U\}^* \subset \{E_n\}^*$ and the fact that the norm $\|\cdot\|$ is continuous.

($\supset$) Let $x \in \{E_n\}^* \cap U$. Then $\exists N'$ , $x_n \in E_n$ such that $x_n \to x$ , $n \in N'$ . We may choose each $x_n \neq 0$ . Then $\frac{x_n}{\|x_n\|} \in E_n \cap U$ and $\frac{x_n}{\|x_n\|} \to x$ , $n \in N'$ . Thus, $x \in \{E_n \cap U\}^*$ .

(b) Recall that if $E$ is a linear subspace of $X$ then $\dim E < \infty \iff E \cap U$ is compact (see e.g. Schechter [34] pp. 84-860). By Theorem 2.3 and (a) we have $\{E_n\}^* \cap U$ compact when $\{E_n\}$ is d-compact. This proves the assertion. ∎

<u>Lemma 2.10.</u>

Let $\{E_n\}^* \subset E$ and suppose that either $\dim F < \infty$ or $\{E_n \cap U\}$ d-compact , $F$ closed . Then
$E \cap F = \{0\} \Rightarrow E_n \cap F = \{0\}\ \forall n$ large .

<u>proof</u>: Either assumption gives $\{E_n \cap F \cap U\}$ d-compact .

By Theorems 2.3 and 2.9, and (2.9), we have

$$E_n \cap F \cap U \to \{E_n \cap F \cap U\}^* \subset \{E_n\}^* \cap F \cap U .$$

Thus, $E_n \cap F \cap U \to E \cap F \cap U$ . But $E \cap F \cap U = \emptyset$ and so (2.3) implies that $E_n \cap F \cap U = \emptyset$ $\forall n$ large . Therefore, $E_n \cap F = \{0\}$ $\forall n$ large . ∎

The last few results culminate in a theorem which compares the dimensions of subspaces.

<u>Theorem 2.11.</u>

Let $\{E_n \cap U\}$ be d-compact, $\{E_n\}^* \subset E$ . Then $\dim E_n \leqq \dim E$ $\forall n$ large . In particular, $\dim E_n \leqq \dim \{E_n\}^* < \infty$ $\forall n$ large .

<u>proof</u>: There is nothing to prove if $\dim E = \infty$ . Assume $\dim E < \infty$ . It can be shown that $X$ can be written as the direct sum $E \oplus F$, $F$ a closed subspace (see e.g. Taylor and Lay [39] p. 247). By Lemma 2.10, $E_n \cap F = \emptyset$ $\forall n$ large . Consequently, $\dim E_n \leqq \dim E$ $\forall n$ large . The last assertion follows from Theorem 2.9(b). ∎

The above theorem allows us to compare, in the context of operator approximations (cf. Chapter IV), numbers of linearly independent solutions between the linear operator equations $Ax = 0$ and $A_n x_n = 0$ by setting $E$ = nullspace of $A$ , and $E_n$ = nullspace of $A_n$.

Discretely compact sequences and the associated set convergence have been used in conjunction with nonlinear operator theory by Anselone and Ansorge [3].

## 2. Measures of Noncompactness

Let $(X,\rho)$ be a complete metric space.

**Definition 2.12.**

For $S \subset X$ bounded, define

$$\alpha(S) = \inf\{\varepsilon > 0 : S \subset \bigcup_{\text{finite}} U_k \text{ with diam } U_k < \varepsilon\} .$$

$\alpha$ is the Kuratowski measure of noncompactness.

Then

(2.11) $\quad 0 \leqq \alpha(S) = \alpha(\bar{S})$ ,

(2.12) $\quad \alpha(S) = 0 \Leftrightarrow \bar{S}$ compact ,

(2.13) $\quad S \subset U \Rightarrow \alpha(S) \leqq \alpha(U)$ ,

(2.14) $\quad \alpha(S \cup U) = \max\{\alpha(S), \alpha(U)\}$ ,

(2.15) $\quad \alpha(S \cap U) \leqq \min\{\alpha(S), \alpha(U)\}$ .

Kuratowski [25] uses $\alpha$ to show that if $\{S_n\}$ is a decreasing sequence of nonempty, closed and bounded subsets with $\alpha(S_n) \to 0$ then $\emptyset \neq \bigcap_{n=1}^{\infty} S_n$ is compact.

Now suppose $(X,\|\cdot\|)$ is a Banach space. Then

(2.16) $\quad \alpha(S + U) \leqq \alpha(S) + \alpha(U)$ , any $S , U \subset X$ ,

(2.17) $\quad \alpha(\lambda S) = |\lambda|\alpha(S)$ , $\lambda$ complex .

If $\overline{\text{conv}S}$ is the closed convex hull of $S$, Darbo [12] has shown that $\alpha(\overline{\text{conv}}\ S) = \alpha(S)$. We shall not need this result.

Definition 2.13.

For $S \subset X$ bounded define the Hausdorff measure of noncompactness of S by

$$\chi(S) = \inf\{\varepsilon > 0\text{: } S \text{ has a finite } \varepsilon\text{-net in } X\}\ .$$

It follows that $\chi$ enjoys properties (2.11)-(2.17) along with $\alpha$.

The above definition was given by Goldenstein, Gohberg, and Markus [16]. The terminology is motivated by the following observations. For $x \in X$, $r > 0$, set $B(x,r) = \{x' \in X\text{: } \|x-x'\| < r\}$. Define $d(S,U) = \inf \{r > 0\text{: } S \subset U + B(0,r)\}$, and set $D(S,U) = \max\{d(S,U), d(U,S)\}$, the Hausdorff metric. If $F = \{U \quad X\text{: } U \neq \emptyset, \bar{U} \text{ compact}\}$ then for $S \subset X$ bounded,

$$\chi(S) = \inf_{U \in F} D(S,U)\,.$$

See Banaś and Goebel [9] for details and further properties.

The measures $\alpha$ & $\chi$ are related. It is an easy exercise to show

$$\text{(2.18)} \qquad \chi(S) \leqq \alpha(S) \leqq 2\,\chi(S)\ .$$

In a linear setting $\chi$ is often easier to use. For example, if $B = B(0,1)$ and $\dim X = \infty$, then it requires substantial work to show $\alpha(B) = 2$ (see Furi and Vignoli [15]) while the proof of $\chi(B) = 1$, given below, is easy.

Theorem 2.14.

$\chi(B) = 1$ whenever $\dim X = \infty$.

proof: Clearly $\chi(B) \leqq 1$. Assume $\chi(B) < 1$. Then $\exists$ $\varepsilon \in (0,1)$ and $U = \{x_1, \ldots, x_n\} \subset X$ such that

$$B \subset \bigcup_{k=1}^{n} B(x_k, \varepsilon) = U + \varepsilon B .$$

Hence, $\chi(B) \leqq \chi(U) + \varepsilon\chi(B) = \varepsilon\chi(B)$ and therefore $\chi(B) = 0$, a contradiction to $\dim X = \infty$. ■

Remark: Goldenstein, Gohberg, and Markus [16] have obtained formulas for $\chi$ in various Banach spaces. For example, if $X$ has a basis $\{e_k\}_{k=1}^{\infty}$, i.e. for each $x \in X$ there is a unique sequence of scalars $\{\xi_k\}_{k=1}^{\infty}$ such that $x = \lim_{n\to\infty} \sum_{k=1}^{n} \xi_k e_k$, then for bounded $S \subset X$,

$$\frac{1}{a} \inf_{n} \sup_{x\in S} \| Q_n x \| \leqq \chi(S) \leqq \inf_{n} \sup_{x\in S} \| Q_n x \|$$

where $Q_n$ are the bounded linear projections given by $Q_n x = x - \sum_{k=1}^{n-1} \alpha_k e_k$, and $a = \lim_{n\to\infty} \| Q_n \|$. Though in general the above bounds are sharp, if $a = 1$ then

$$\chi(S) = \lim_{n\to\infty} \sup_{x\in S} \| Q_n x \| .$$

In particular, if $e_k = (\delta_{kj})$ and $X = c_o, \ell_p (1 \leqq p \leqq \infty)$ then, respectively,

$$\chi(S) = \limsup_{n\to\infty} \sup_{x\in S} \max_{k\geqq n} |\xi_k|$$

and

$$\chi(S) = \limsup_{n\to\infty} \sup_{x\in S} \left( \sum_{k=n}^{\infty} |\xi_k|^p \right)^{1/p} .$$

Also shown is that for $X = C[0,1]$ then

$$\chi(S) = \frac{1}{2} \lim_{\delta\to 0} \sup_{f\in S} \max_{\substack{|t_1-t_2| \leqq \delta \\ t_1,t_2 \in [0,1]}} |f(t_1)-f(t_2)| .$$

Contained in Banás and Goebel is a generalization to $C(U,\rho)$ where $(U,\rho)$ is a compact metric space.

The next result is a direct application of the definitions.

<u>Theorem 2.15.</u>

$\{S_n\}$ asymptotically totally bounded $\Leftrightarrow$

$$\chi\left(\bigcup_{j\geqq n} S_j\right) \longrightarrow 0 \text{ as } n \to \infty .$$

<u>Corollary 2.16.</u>

$\{S_n\}$ d-compact $\Rightarrow$ $\chi\left(\bigcup_{j\geqq n} S_j\right) \to 0$ ,

conversely if $X$ is complete.

<u>proof</u>: Use above theorem and Theorem 2.8. ∎

It immediately follows that $\chi(S_n) \to 0$ when $\{S_n\}$ is d-compact. That $\chi(S_n) \to 0$ is not enough to guarantee $\{S_n\}$ is d-compact. Simply take $S_n = [2n, 2n+1] \subset R$, $n = 1,2,\ldots$ . Then $\chi(S_n) = 0$ but $\{S_n\}$ is not d-compact.

Theorem 2.15 forges the vital link relating asymptotically compact (or collectively compact) sequences of operators of Chapter IV to the theory of condensing operators which are defined in the next chapter.

The literature regarding measures of noncompactness is extensive. An axiomatic treatment, along with detailed bibliography, is given by Banás and Goebel [9].

# III. OPERATOR THEORY

## 1. Operator Fundamentals

The definitions and results given in this section set the stage for the remainder of this thesis. Ideas which will be used later are presented. In most cases proofs are not given. The reader is encouraged to consult any of a number of standard texts (e.g., Schechter [34], Taylor and Lay [39], and Yosida [48]).

Let $(X, \|\cdot\|)$ be a real or complex normed linear space with $B = \{x \in X: \|x\| \leqq 1\}$, the closed unit ball. The linear space of bounded (linear) operators $T: X \to X$ is denoted by $[X]$. It is equipped with the usual operator norm, $\|T\| = \sup\{\|Tx\| : x \in B\}$. When $X$ is complete than so is $[X]$. Let

$N(T)$ = nullspace of $T = \{x \in X: Tx = 0\}$,

$R(T)$ = range of $T = TX$. If $N(T) = \{0\}$ then $T$ is one-to-one and onto its range. In this case there is an operator $T^{-1}: R(T) \to X$ so that

$T^{-1}Tx = x \ \forall x \in X$, $TT^{-1}y = y \ \forall y \in R(T)$. We shall call $T^{-1}$ the <u>inverse of T</u>. Note that $T^{-1}$ may not be defined on all of $X$.

Recall that $U = \{x \in X: \|x\| = 1\}$. Then

(3.1) $\qquad \dim R(T) < \infty \iff TU$ compact ,

(3.2) $\qquad \exists T^{-1} \iff N(T) = \{0\} \iff 0 \notin TU$ ,

(3.3) $\qquad \exists T^{-1}$ bounded $\iff 0 \notin \overline{TU}$ , (3.4) $TU$ closed, $\exists T^{-1} \Rightarrow T^{-1}$ bounded.

(3.5) Let $\exists T^{-1}$ . Then $T^{-1}$ bounded $\Leftrightarrow$ $R(T)$ closed.

If $E$ is a linear subspace of $X$ , then a linear operator $P: X \to X$ is a <u>projection of $X$ onto $E$</u> if $P^2 = P$ and $R(P) = E$ . Then $X = E \oplus N(P)$ , $Px = x \ \forall x \in E$ , $Q = I - P$ is a projection of $X$ onto $N(P)$ , and $E = N(Q)$ . If $P$ is bounded, then $E$ and $N(P)$ are closed.

<u>Lemma 3.1.</u>

$\dim E < \infty \Rightarrow \exists$ bounded projection of $X$ onto $E$ .

<u>proof</u>: See Taylor and Lay [39] p. 247. ■

Thus, if $\dim E < \infty$ there exists a closed subspace $F$ so that $X = E \oplus F$ . The next theorem incorporates most of the preceding results.

<u>Theorem 3.2.</u>

Let $\dim N(T) < \infty$ . Then there exists a closed subspace $F$ of $X$ such that $X = N(T) \oplus F$ . Let $T_F: F \to R(T)$ denote the restriction of $T$ to $F$ . Then $\exists T_F^{-1}$ , $R(T) = R(T_F)$ , and

$$R(T) \text{ closed} \Leftrightarrow T_F^{-1} \text{ bounded} .$$

Let $P$ be the projection of $X$ onto $N(T)$ . If $Q = I - P$ , then $R(P) = N(T)$ , $R(Q) = F$ , $TP = 0$ , and $T = TQ = T_F Q$ .

Now let $X'$ be the <u>conjugate space of $X$</u>, i.e., the (Banach) space of bounded linear functionals on $X$. Then $T \in [X]$ gives rise to $T' \in [X']$, called the <u>conjugate of $T$</u>, defined by $(T'f)x = f(Tx)$, $f \in X'$, $x \in X$. Let $X'' = (X')'$ and $T'' = (T')'$. Then $X$ can be isometrically embedded into $X''$ by $J: X \to X''$ defined by $(Jx)f = f(x)$, $x \in X$, $f \in X'$, and

$$JT = T''J .$$

Define $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim (X/R(T)) = \operatorname{codim} R(T)$.

<u>Remark:</u> When $R(T)$ is closed and $\beta(T) < \infty$, it can be shown that $\beta(T) = \dim N(T')$.

Then

$$\alpha(T) = 0 \iff \exists T^{-1} , \tag{3.7}$$

$$\beta(T) = 0 \iff R(T) = X . \tag{3.8}$$

Clearly, $T$ is bijective if and only if $\alpha(T) = 0$, $\beta(T) = 0$. The next definition essentially gives a measure of how far a given operator is from being bijective.

<u>Definition 3.3.</u>

The <u>index of T</u> is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T) ,$$

when the right side is well-defined.

We state an important property of the index.

<u>Theorem 3.4.</u>

Let $T, L \in [X]$ have finite index. Then so does $TL$, and

$$\operatorname{ind}(TL) = \operatorname{ind}(T) + \operatorname{ind}(L) .$$

<u>proof</u>: See Schechter [34] pp. 111-113 or Taylor and Lay [39] pp. 253-254. ∎

## 2. Contractive and Compact Operators

In this section we primarily review the highlights of the Riesz-Schauder theory for compact operators. Contractive operators are mentioned for two reasons. First, the fact that $(I-T)^{-1} \in [X]$ if $T$ is contracting is needed to prove the stability of the index under small perturbations. Second, contracting operators provide, in some sense, one edge of a spectrum of operators, with the compact operators at the opposite edge and the condensing operators somewhere in between. No proofs are given. In almost every case generalizations will be presented in the more general situation when the operators are condensing.

<u>Definition 3.5.</u>

$T \in [X]$ is <u>contractive</u> if $\|T\| < 1$ .

If $T$ is contractive, it is well known that $\exists (I-T)^{-1} \in [X]$ given by the Neumann expansion

$$(I-T)^{-1} = \sum_{k=0}^{\infty} T^k \ .$$

Definition 3.6.

$K \in [X]$ is compact if $S \subset X$ bounded implies $\overline{KS}$ compact. Equivalently,

$$K \text{ compact} \iff \{Kx_n\} \text{d-compact } \forall \{x_n\} \text{ bounded}$$

$$\iff \chi(KS) = 0 \ \forall S \text{ bounded} .$$

It suffices if $S = B$ .

Examples.

1. $\dim R(K) < \infty \Rightarrow K$ compact .

2. Let $X = C[0,1]$ = continuous scalar-valued functions on $[0,1]$ with norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$ . Let $k(s,t) \in C([0,1] \times [0,1])$ . Then the integral operator on $C[0,1]$ defined by

$$(Kx)(s) = \int_0^1 k(s,t)\ x(t)\ dt\ ,\ 0 \leqq s \leqq 1\ ,$$

is compact. Using numerical intergration implies that the approximate operators

$$(K_n x)(s) = \sum_{j=1}^{n} w_{nj}\ k(s,t_{nj})\ x(t_{nj})\ ,\ n \in N\ ,$$

are each compact (being finite dimensional). Here,

$0 \leqq t_{nj} \leqq 1$ are the subdivision points and $w_{nj}$ are the (real or complex) weights. For example, using the rectangular integration rule gives $t_{nj} = j/n$ $w_{nj} = 1/n$ .

In Section II.1 it was stated that if $F$ is a subspace of $X$ then $\dim F < \infty \iff F \cap U$ compact. The next lemma, due to Riesz, provides a contrapositive argument that $F \cap U$ compact $\Rightarrow \dim F < \infty$ ($\dim F < \infty \Rightarrow F \cap U$ compact follows from the Heine-Borel Theorem).

<u>Lemma 3.7.</u> (F. Riesz)

Let $F \subsetneqq X$ be a closed linear subspace. Then $\forall\, \varepsilon \in (0,1) \exists x_\varepsilon \in U$ such that $\| x_\varepsilon - y \| \geqq 1 - \varepsilon \,\forall y \in F$ . If $\dim F < \infty$ we may take $\varepsilon = 0$ .

<u>proof</u>: See Schechter [34] p. 86 or Taylor and Lay [39] p. 64. ■

It immediately follows that

(3.9) $\qquad I$ compact $\iff \dim X < \infty$ .

Since $N(I-K) = \{x \in X \colon Kx = x\}$ , we obtain

(3.10) $\qquad K$ compact $\Rightarrow \alpha(I-K) < \infty$ .

We shall obtain another proof of (3.10), as well as the fact that $R(I-K)$ is closed when regular operators are considered in Section 4.

To conclude this section we list some standard results, all of which will be extended to the case when the operators are condensing instead of compact (cf. Section 5).

Theorem 3.8. (Schauder)

$K$ compact $\Leftrightarrow$ $K'$ compact.

Corollary 3.9. (Fredholm alternative)

$K$ compact $\Rightarrow$ $\operatorname{ind}(I-K) = 0$ .

Thus,

$$\exists (I-K)^{-1} \Leftrightarrow R(I-K) = X \Rightarrow (I-K)^{-1} \in [X] ,$$

$$\dim N(I-K) = \operatorname{codim} R(I-K) = \dim N[(I-K)'] < \infty .$$

Moreover, all of the above assertions hold with $I$ replaced by $\lambda I$ , $\lambda \neq 0$ a scalar.

Remark: The eigenvalue $\lambda = 0$ is exceptional since $\exists K^{-1} \in [X] \Leftrightarrow \dim X < \infty$ by (3.9).

Theorem 3.10.

If $K \in [X]$ is compact, then the set of eigenvalues form either a finite set or an infinite sequence which converges to zero.

## 3. Fredholm and Semi-Fredholm Operators

Fredholm operators are generalizations of the prototype $I-K$ , $K \in [X]$ compact. Fredholm operators have a finite index. Thus, they are important from the standpoint of solving operator equations. In this section Fredholm operators are characterized and a stability theorem on the index is proven.

Definition 3.11.

$T \in [X]$ is called a Fredholm operator (Fredholm for short) if $\alpha(T) < \infty$, $\beta(T) < \infty$. If $T$ has closed range and either $\alpha(T)$ or $\beta(T)$ is finite, then $T$ is a semi-Fredholm operator.

Motivated by the solution of operator equations, we shall be particularly interested in Fredholm operators of index zero. For if $T$ is Fredholm and $\operatorname{ind}(T) = 0$, then $T$ satisfies the Fredholm alternative (cf. Corollary 3.9).

Remark: Any bijective operator is Fredholm of index zero.

Next, we prove a few facts about Fredholm operators. References include Caradus, Pfaffenberger, and Yood [10]; Kato [24], Pietsch [32], Schechter [34]; Taylor and Lay [39].

Theorem 3.12.

$T \in [X]$ is Fredholm iff

$\exists T_0 \in [X]$ such that $\dim R(I-TT_0)$, $\dim R(I-T_0T) < \infty$.

<u>proof</u>: ($\Rightarrow$) There exist linear subspaces $E$, $F$ with $E$ closed and $\dim F < \infty$ such that $X = N(T) \oplus E = F \oplus R(T)$. Let $P_1$, $Q_1 = I-P_1$ be the complimentary projections of $X$ onto $N(T)$, $E$. Let $P_2$, $Q_2 = I-P_2$ be projections of $X$ onto $F$, $R(T)$. Note that $\exists\, T_E^{-1}$ bounded (cf. Theorem 3.2). Let $T_0 = T_E^{-1}Q_2$. Then

$$TT_0 = TT_E^{-1}Q_2 = I_{R(T)}Q_2 = Q_2 \;,\; I-TT_0 = P_2 \;,$$

$$T_0T = T_E^{-1}Q_2T = T_E^{-1}T = Q_1 \;,\; I-T_0T = P_1 \;.$$

($\Leftarrow$) Let $P_1 = I-T_0T$, $P_2 = I-TT_0$,

where $\dim R(P_i) < \infty$, $i = 1,2$. Then

$$N(T) \subset N(I-P_1) \;,\quad R(I-P_2) \subset R(T) \;.$$

By Corollary 3.9, $\alpha(I-P_1) < \infty$ and $\beta(I-P_2) < \infty$. Hence, we have $\alpha(T) < \infty$, $\beta(T) < \infty$ as desired. ■

<u>Remarks</u>:

1. The operator $T_0$ is sometimes referred to as a <u>pseudoinverse</u> or (left and right) <u>compact regulizer</u> of $T$.

2. The symmetry of Theorem 3.12 shows that $T$ is Fredholm. Furthermore, since $\mathrm{ind}(I-P_1) = \mathrm{ind}(T) + \mathrm{ind}(T_0) = 0$ we obtain

$$\mathrm{ind}(T_0) = -\mathrm{ind}(T) \;. \tag{3.11}$$

3. If $T_1$ , $T_2 \in [X]$ are such that $T_1T - I$ and $TT_2 - I$ are finite dimensional then $T$ is Fredholm.

The next theorem will be used to prove that the index is stable under small perturbations. It also shows that the set of Fredholm operators is an open set in $[X]$ .

Theorem 3.13.

Let $T \in [X]$ be Fredholm. Then $\exists\, \varepsilon > 0$ such that $L \in [X]$ with $\|L\| < \varepsilon$ implies that $T + L$ is Fredholm and $\mathrm{ind}(T+L) = \mathrm{ind}(T)$ .

proof: Refer to Theorem 3.12. Since $T_0 + P$ , $P \in [X]$ , $\dim R(P) < \infty$ , is also a compact regulizer we may assume $\|T_0\| \neq 0$ . Set $\varepsilon = \|T_0\|^{-1}$ . Then $\|T_0L\| < 1$ and $\|LT_0\| < 1$ whenever $\|L\| < \varepsilon$ . Hence the operators $I + T_0L$ and $I + LT_0$ are bijective with bounded inverses. Therefore,

$$\begin{aligned}(I + T_0L)^{-1}T_0(T+L) &= (I + T_0L)^{-1}(I-P_1+T_0L) \\ &= I - (I+T_0L)^{-1}P_1 ,\end{aligned}$$

and

$$\begin{aligned}(T+L)T_0(I+LT_0)^{-1} &= (I-P_2+LT_0)(I+LT_0)^{-1} \\ &= I - P_2(I+LT_0)^{-1} .\end{aligned}$$

This representation, along with the previous remark and Theorem 3.4, implies that $T + L$ is Fredholm and

$$\begin{aligned} \operatorname{ind}(T+L) &= -\operatorname{ind}(T_0) - \operatorname{ind}[(I+LT_0)^{-1}] \\ &= \operatorname{ind}(T) . \end{aligned}$$

This proves the theorem. ∎

Generalizations of Theorem 3.13 are well known. They extend the results to semi-Fredholm operators and/or weaken the bound on the perturbing operator. See Gohberg and Krein [17], and Kato [24]. While we shall not need such extentions, we shall indicate some particular generalizations based on measures of noncompactness in Section 5.

Next, we present a useful stability result for the index.

Corollary 3.14.

Suppose $T + \lambda L$ is Fredholm for each $0 \leqq \lambda \leqq 1$. Then $\operatorname{ind}(T) = \operatorname{ind}(L)$.

proof: Define $f:[0,1] \to R$ by $f(\lambda) = \operatorname{ind}(T+\lambda L)$ . Then $f$ is continuous by Theorem 3.13. But $f$ is integer-valued and therefore must be constant. ∎

## 4. Regular Operators

In this section we characterize those semi-Fredholm operators $A$ with $R(A)$ closed and $\alpha(A) < \infty$ .

Definition 3.15.

$A \in [X]$ is regular if $\bar{S}$ is compact whenever $S \subset X$ is bounded and $\overline{AS}$ is compact.

Examples:

1. $K$ compact $\Rightarrow$ $I - K$ regular .
2. $\exists A^{-1}$ bounded $\Rightarrow$ $A$ regular .

It follows from the definition that restrictions and products of regular operators are regular. Equivalent definitions are: $A$ regular $\Longleftrightarrow$

(3.12) $\{x_n\}$ bounded , $\{Ax_n\}$ d-compact $\Rightarrow$

$\{x_n\}$ d-compact ,

(3.13) $\{x_n\}$ bounded , $Ax_n \to y$ $\Rightarrow$ $\{x_n\}^* \neq \phi$ .

If $A$ is regular and $Ax_n \to y$ , where $\{x_n\}$ is bounded, then (3.13) gives $Ax = y$ $x \in \{x_n\}^*$ . Also,

(3.14) $A$ regular, $S$ closed and bdd $\Rightarrow$ $AS$ closed.

Consequently, $A$ regular $\Rightarrow$ $AU$ closed. Hence, by (3.4), (3.5),

(3.15) Whenever $\exists A^{-1}$,

$A$ regular $\Longleftrightarrow$ $A^{-1}$bdd $\Longleftrightarrow$ $R(A)$ closed .

For more on regular operators see Anselone and Ansorge [3], Gigorieff [19,20], Wolf [44]. Alternatively, in terms of the Hausdorff measure of noncompactness,

$$(3.16) \qquad A \text{ regular } \Longleftrightarrow$$
$$S \text{ bdd}\ , \quad \chi(AS) = 0 \quad \text{implies} \quad \chi(S) = 0\ .$$

A continuous mapping $f$ between topological spaces is called <u>proper</u> if $f^{-1}(S)$ is compact whenever $S$ is compact. Thus, $A$ is regular iff $A$ restricted to closed, bounded sets is proper. Yood [47] first showed that such proper operators have closed ranges and finite dimensional nullspaces. This result comprises the next theorem.

<u>Theorem 3.16.</u> (Wolf [44])

$A$ regular $\Longleftrightarrow$ $R(A)$ closed, $\dim N(A) < \infty$ .

<u>proof</u>: Since $A$ is regular,

$A(U \cap N(A)) = 0 \Rightarrow U \cap N(A)$ compact $\Rightarrow \dim N(A) < \infty$ .

Now assume $\dim N(A) < \infty$ . Form the decomposition

$X = N(A) \oplus F$ with corresponding bounded projections

$I = P + Q$ , $Q = I - P$ . Refer to Theorem 3.2. Since $P$ is compact, $Q$ is regular. Also, $A = A_F Q$ gives $A$ regular $\Longleftrightarrow$ $A_F$ regular because restrictions and products of regular operators are regular. But $A_F{}^{-1}$ exists and so by (3.5)

$$A_F \text{ regular} \Longleftrightarrow A_F{}^{-1}\text{bdd} \Longleftrightarrow R(A_F) = R(A) \text{ closed}\ .$$

The theorem follows. ∎

<u>Remark</u>: Lebow and Schechter [26] have extended (3.16) to:

$$A \text{ regular} \iff$$
$$\exists \text{ constant } C \ni \chi(S) \leq C\chi(AS) \ \forall \text{bdd } S \subset X .$$

## 5. Condensing Operators

In this section we extend previous results for contracting and compact operators to the more general case when the operators are condensing with respect to some measure of noncompactness. The Hausdorff measure of noncompactness plays a key role. In particular, if $K \in [X]$ is condensing, i.e., $\chi(KB) < 1$, then $\operatorname{ind}(I-K) = 0$.

First, some definitions and observations. Let $\mathcal{K}$ be the set of all compact $K \in [X]$.

<u>Definition 3.17</u>.

For $T \in [X]$ define $\|T\|_{\mathcal{K}} = \inf\{\|T+K\|: K \in \mathcal{K}\}$.

Then $\|T\|_{\mathcal{K}} = 0 \iff T$ compact. Thus, one can think of $\|T\|_{\mathcal{K}}$ as measuring the noncompactness of $T$. Gohberg and Krein [17] used this measure to (among other things) extend Theorem 3.13 to include more general perturbing operators. That is, if $T$ is Fredholm,

$\operatorname{ind}(T + L) = \operatorname{ind}(T)$ when $\|L\|_K$ is sufficiently small. They also considered closed, unbounded operators. Next we define other measures of noncompactness based on the set measures of noncompactness $\alpha$ and $\chi$.

Definition 3.18.

For $T \in [X]$ define the Hausdorff measure of noncompactness of $T$ by

$$\|T\|_\chi = \inf\{b : \chi(TS) \leqq b\,\chi(S) \ \ \forall \text{bdd } S \subset X\} .$$

Define the Kuratowski measure of noncompactness of $T$ by

$$\|T\|_\alpha = \inf\{b : \alpha(TS) \leqq b\alpha(S) \ \ \forall \text{bdd } S \subset X\} .$$

Straightforward consequences of the definitions are:

(3.17) $\|\cdot\|_K$, $\|\cdot\|_\chi$, $\|\cdot\|_\alpha$ are seminorms on $[X]$ and $\|T\|_K = 0 \iff \|T\|_\chi = 0 \iff \|T\|_\alpha = 0 \iff T$ compact,

(3.18) $\|T + K\|_K = \|T\|_K$, $\|T + K\|_\alpha = \|T\|_\alpha$,

$\|T + K\|_\chi = \|T\|_\chi \quad \forall K \in K$,

(3.19) $\|TL\|_K \leqq \|T\|_K \|L\|_K$, $\|TL\|_\alpha \leqq \|T\|_\alpha \|L\|_\alpha$,

$\|TL\|_\chi \leqq \|T\|_\chi \|L\|_\chi$,

(3.20) $\|T\|_\chi \leqq \|T\|_K \leqq \|T\|$.

Remark: In general the inequalities in (3.20) are sharp. Goldenstein and Markus [18] give an example for which

$\|T\|_\chi < 1 = \|T\|_K$ .

It it well known that when X is complete, $K$ is a closed (two-sided) ideal. Consequently, each of the above seminorms defines a norm on the quotient space $[X]/K$ , e.g., $\|T + K\| = \|T\|_K$ is the standard quotient norm. By (3.19), $[X]/K$ becomes a normed algebra with respect to any of the three norms. In particular, $([X]/K\ ,\ \|\cdot\|_K) \equiv C(X)$ (called the <u>Calkin algebra</u>; see [10]) is a Banach algebra. Apparently, $[X]/K$ is not generally complete with respect to these other norms. More on this in Chapter V.

One of the first to investigate $\|\cdot\|_\alpha$ was Darbo [12], who used the terminology <u>k-set contraction</u> to mean any operator T such that $\alpha(TS) \leq k\,\alpha(S)$ $\forall$ bounded $S \subset X$ . Independently, Goldenstein, Gohberg, and Markus [16] studied the seminorm $\|\cdot\|_\chi$ in conjunction with bounded linear operators. Regarding the constancy of the index, Goldenstein and Markus [18] have extended Theorem 3.13 to:

$$\begin{aligned} &T \text{ Semi-Fredholm}\ ,\ \|L\|_\chi \text{ sufficiently small} \Rightarrow \\ &T + L \text{ semi-Fredholm}\ ,\ \operatorname{ind}(T+L) = \operatorname{ind}(T)\ . \end{aligned}$$

Another result concerning the constancy of the index is given by:

$$T \text{ Fredholm },$$
$$\chi(LS) \leq k\chi(TS) \ \forall \text{bounded } S \subset X, \quad k < 1 \Rightarrow$$
$$T + L \text{ Fredholm }, \quad \operatorname{ind}(T+L) = \operatorname{ind}(T) .$$

See Sadovskii [33], Petryshyn and Fitzpatrick [31] for more details. The special case when $T = I$ is worth mentioning.

Other relevant contributions were made by Nussbaum [28], and Sedaev [35]. For Hilbert space results, see Stuart [36] and Webb [43].

Remark: Much of the work in recent years has been devoted to nonlinear operator theory. Vainikko and Sadovskii [41], and Nussbaum [29] formulated degree theories based on the above measures of noncompactness. For related results, including fixed point theorems, see Appell and Pera [8], Nussbaum [30], Petryshyn and Fitzpatrick [31], Sadovskii [33], Webb [42], and Wolf [46].

Two of the above seminorms are equivalent.

Theorem 3.19.

$$\frac{1}{2}\|T\|_\alpha \leq \|T\|_\chi \leq 2\|T\|_\alpha .$$

proof: Let $S \subset X$ be bounded. Then by (2.18) we have

$$\begin{aligned}\alpha(TS) &\leq 2\chi(TS) \leq 2\|T\|_\chi \ \chi(S) \\ &\leq 2\|T\|_\chi \ \alpha(S)\end{aligned}$$

and

$$\chi(TS) \leqq \alpha(TS) \leqq \|T\|_\alpha \ \alpha(S) \leqq 2\|T\|_\alpha \ \chi(S) \ . \quad \blacksquare$$

When working with linear operators, $\|\cdot\|_\chi$ is very convenient to deal with, as the next result demonstrates.

<u>Lemma 3.20.</u>

$$\|T\|_\chi = \chi(TB) \ .$$

<u>proof</u>: Since $\chi(B) \leqq 1$ we have

$$\chi(TB) \leqq \|T\|_\chi \ \chi(B) \leqq \|T\|_\chi \ .$$

To show the reverse inequality, let $S \subset X$ be bounded and let $\varepsilon > \chi(S)$ . Then

$\exists$ finite $\varepsilon$-net $\{x_1,\ldots,x_n\} \subset X$ of $S$ , i.e.,

$TS \subset \bigcup_{k=1}^{n} TB(x_k,\varepsilon)$ . For any $1 \leqq k \leqq n$ ,

$TB(x_k,\varepsilon) = \varepsilon \ TB(\frac{x_k}{\varepsilon},1) = \varepsilon \ TB + \frac{x_k}{\varepsilon}$ gives

$\chi(TS) \leqq \max_{1\leqq k\leqq n} \{\chi[TB(x_k,\varepsilon)]\} = \varepsilon \ \chi(TB)$ .

Since $\varepsilon > \chi(S)$ was arbitrary we are done. $\blacksquare$

The next theorem extends the result of Schauder on the compactness of the conjugate (cf. Theorem 3.8). We shall use the same symbol, $\chi$ , to denote the measure of noncompactness of sets in the conjugate spaces $X'$ and $X''$ , as well as in $X$ .

Theorem 3.21. (Goldenstein and Markus [18])

$$\frac{1}{2}\|T\|_\chi \leqq \|T'\|_\chi \leqq 2\|T\|_\chi .$$

proof: Set $B' = \{f \in X' : \|f\| \leqq 1\}$ . Let $\varepsilon > 0$ . Then $\exists$ finite $(\|T\| + \frac{\varepsilon}{3})$ - net of $TB$ , say $\{y_i : 1 \leqq i \leqq m\}$ . Set $\Lambda = \{\lambda \in \mathbb{C} : |\lambda| \leqq \max\{\|y_i\|, 1\}\}$ , and partition $\Lambda$ into disjoint sets $\Delta_j$ , $1 \leqq j \leqq n$ , with diam $\Delta_j < \frac{\varepsilon}{3}$ . Then $f \in B'$ , $1 \leqq i \leqq m \Rightarrow f(y_i) \in \Delta_{k(i)}$ , $1 \leqq k(i) \leqq n$ . Divide $B'$ into equivalence classes $C_k$ , $1 \leqq k \leqq p$ , as follows:

$$f,g \in C_k \iff$$

$$f(y_i) \, , \quad g(y_i) \in \Delta_{k(i)} \quad \forall 1 \leqq i \leqq m .$$

We claim that diam $T'C_k < 2\|T\|_\chi + \varepsilon$ . For if $f,g \in C_k$ and if $x \in B$ then $\exists 1 \leqq i \leqq m$ such that

$$\begin{aligned} |f(Tx)-g(Tx)| &\leqq |f(Tx)-f(y_i)| + |f(y_i)-g(y_i)| \\ &\quad + |g(y_i) - g(Tx)| \\ &\leqq 2\|Tx - y_i\| + \frac{\varepsilon}{3} \\ &< 2(\|T\|_\chi + \frac{\varepsilon}{3}) + \frac{\varepsilon}{3} = 2\|T\|_\chi + \varepsilon . \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\|T'\|_\chi \leqq 2\|T\|_\chi$ .

The second part of the theorem follows from the first. From the above arguments we have $\|T''\|_\chi \leqq 2\|T'\|_\chi$ . If $J$ is the isometric embedding of $X$ into $X''$ then $\chi(J(S)) = \chi(S)$ for all bounded

$S \subset X$ . Moreover, as $JT = T''J$ , we have

$$\begin{aligned} \chi(TS) &= \chi(J(TS)) = \chi(T''(J(S))) \\ &\leqq 2\|T'\|_\chi \, \chi(JS) \\ &= 2\|T'\|_\chi \, \chi(S) \ . \quad \blacksquare \end{aligned}$$

<u>Remark</u>: The above proof showed that $B'$ , which has a diameter of $2$ , can be covered by finitely many sets each with a diameter of at most $\|T\|_\chi \cdot 2 + \varepsilon$ , $\varepsilon > 0$ . Nussbaum [28] has suitably modified the proof to show that if $S' \subset X'$ has diameter of at most $d$ , then $T'S'$ can be covered by finitely many sets whose diameters are at most $\|T\|_\chi \cdot d + \varepsilon$ , $\varepsilon > 0$ arbitrary. This yields

<u>Corollary 3.22.</u>

$$\|T'\|_\alpha \leqq \|T\|_\chi \ , \quad \|T\|_\alpha \leqq \|T'\|_\chi \ .$$

We are now in a position to derive the main results of this section.

<u>Definition 3.23.</u>

$T \in [X]$ is called $K$-condensing if $\|T\|_K < 1$ .

It can be shown, by using purely classical techniques, that $T$ $K$-condensing $\Rightarrow$ $I - T$ Fredholm of index zero (see Pietsch [32], Taylor and Lay [39]). We shall obtain the above assertion as a consequence of a more general result.

<u>Definition 3.24.</u>

$T \in [X]$ is called <u>$\chi$-condensing</u> (<u>$\alpha$- condensing</u>), if $\|T\|_\chi < 1$ ($\|T\|_\alpha < 1$).

<u>Remark</u>: By condensing, we shall mean $\chi$- condensing.

<u>Theorem 3.25.</u> (Nussbaum [28], Goldenstein, Gohberg and Markus [16])

If $T \in [X]$ and $T^n$ is condensing for some $n \geqq 1$ then $I - T$ is Fredholm of index zero.

<u>proof</u>: Let $S \subset X$ be bounded, $\overline{(I-T)S}$ compact. The identity $I = T^n + \sum_{k=0}^{n-1} T^k(I-T)$ gives $S \subset T^n S + \sum_{k=0}^{n-1} T^k(I-T)S$. Consequently,

$$\begin{aligned}\chi(S) &\leqq \|T^n\|_\chi \chi(S) + \sum_{k=0}^{n-1} \|T^k\|_\chi \chi[(I-T)S] ,\\ &= \|T^n\|_\chi \chi(S) .\end{aligned}$$

Since $\|T^n\|_\chi < 1$, it follows that $\chi(S) = 0$. Hence, $I-T$ is regular, i.e., $\dim N(I-T) < \infty$ and $R(I-T)$ is closed (cf. Theorem 3.16.). By Corollary 3.22,

$$\|(T^n)'\|_\alpha = \|(T')^n\|_\alpha < 1 .$$

The previous arguments with $\chi$ replaced by $\alpha$ and $S$ replaced by $S' \subset X'$ show that $(I-T)'$ is also regular. Therefore, since $R(I-T)$ is closed, $\dim N[(I-T)'] = \operatorname{codim} R(I-T) < \infty$. Consequently, $I-T$ is Fredholm. The same arguments show that $I - tT$ is Fredholm $\forall t \in [0,1]$. By the stability of the index, $\operatorname{ind}(I-T) = \operatorname{ind}(I) = 0$ ■

In light of (3.20) we have the

Corollary 3.26.

$$T \ \ K\text{-condensing} \Rightarrow \operatorname{ind}(I-T) = 0 \ .$$

Various authors have obtained Theorem 3.25 using different, but equivalent, seminorms, e.g. $\|\cdot\|_\alpha$ . The next theorem shows the equivalence of the different approaches.

Theorem 3.27.

If $T \in [X]$ then $\lim\limits_{n\to\infty} \|T^n\|_\chi^{1/n}$ exists and equals $r(T) \equiv \inf\limits_n \|T^n\|_\chi^{1/n}$ . Moreover, if $|\lambda| > r(T)$ then $\lambda I - T$ is Fredholm of index zero.

proof: We first show that $r(T) = \lim\limits_{n\to\infty} \|T^n\|_\chi^{1/n}$ . It suffices to show that $\limsup\limits_n \|T^n\|_\chi^{1/n} \leqq r(T)$ . Given any $\varepsilon > 0$ choose $m$ so that $\|T^m\|_\chi^{1/m} \leqq r(T) + \varepsilon$ . For each positive integer $n$ we have $n = pm+q$ where $0 \leqq q \leqq m-1$ , $p,q$ integers . Then by (3.19),
$\|T^n\|_\chi^{1/n} \leqq \|T^m\|_\chi^{p/m} \|T\|_\chi^{q/n} \leqq (r(T) + \varepsilon)^{mq/n} \|T\|_\chi^{q/n}$ .
Since $\frac{q}{n} \to 0$ and $\frac{mq}{n} \to 1$ as $n \to \infty$ we obtain $\limsup\limits_n \|T^n\|_\chi^{1/n} \leqq r(T) + \varepsilon$ . Since $\varepsilon$ was arbitrary, the desired inequality holds.

Now choose $|\lambda| > r(T)$ and $n$ large enough so that $|\lambda| > \|T^n\|_\chi^{1/n}$ . Then $\|(\lambda^{-1}T)^n\|_\chi < 1$ . From the previous theorem $I - \lambda^{-1}T$ is Fredholm of index zero. As a result, $\lambda I - T = (\lambda I)(I - \lambda^{-1}T)$ is Fredholm with $\mathrm{ind}(\lambda I-T) = \mathrm{ind}(\lambda I) + \mathrm{ind}(I-\lambda^{-1}T) = 0$ . ∎

Corollary 3.28.

Let $|||\cdot|||$ be any seminorm equivalent to $\|\cdot\|_\chi$ on $[X]$ . If $T^n$, $T \in [X]$ , is condensing with respect to $|||\cdot|||$ , i.e. $|||T^n||| < 1$ , for some $n \geqq 1$ then $I - T$ is Fredholm of index zero.

proof: By the equivalence of seminorms, $\exists m$ , $M > 0$ such that

$$m\|T\|_\chi \leqq |||T||| \leqq M\|T\|_\chi \qquad \forall T \in [X] .$$

Consequently,

$$r(T) = \lim_{n\to\infty} (m\|T^n\|_\chi)^{1/n} \leqq \lim_{n\to\infty} |||T^n|||^{1/n}$$
$$\leqq \lim_{n\to\infty} (M\|T^n\|_\chi)^{1/n} = r(T) .$$
∎

One such seminorm which has been investigated is defined by

$$|||T||| = \inf_{\mathrm{codim}\, F<\infty} \|T|_F\| .$$

Sedaev [35] showed that

$$|||T||| = |||T'|||_\chi \text{ and } \|T''\|_\chi \leq \|T\|_\chi .$$

Since $JB \subset B''$ we have

$$\|T\|_\chi = \chi(TB) = \chi(J(TB))$$
$$= \chi(T''(JB))$$
$$\leq \chi(T''B'') = \|T''\|_\chi \ .$$

Thus, $\|T\|_\chi = \|T''\|_\chi$ . By Theorem 3.21 it follows that

$$\tfrac{1}{2}\|T'\|_\chi \leq \|T''\|_\chi = \|T\|_\chi \leq 2\|T'\|_\chi$$

and so we obtain the equivalence

$$\tfrac{1}{2}\,|||T||| \leq \|T\|_\chi \leq 2|||T||| \ .$$

A direct proof, along with other properties, is given by Lebow and Schechter [26].

That $r(T)$ resembles a kind of spectral radius is no accident.

Note that $r(T + K) = r(T)$ $\forall K \in \mathcal{K}$ . Since this radius is unchanged under compact perturbations it is reasonable to believe that $r(T)$ may be related to the spectral radius of $T + \mathcal{K}$ as an element in the Calkin algebra $\mathcal{C}(X)$ . This is indeed the case. Let $X$ be a complex Banach space and recall that the <u>resolvent set</u> and <u>spectrum</u> of $T + \mathcal{K} \in \mathcal{C}(X)$ are defined by

$$\rho(T+\mathcal{K}) = \{\lambda \in \mathbb{C}\colon (\lambda I - T) + \mathcal{K} \text{ is invertible in } \mathcal{C}(X)\} \ ,$$
$$\sigma(T+\mathcal{K}) = \mathbb{C} \setminus \rho(T+\mathcal{K}) \ .$$

A standard result of Banach algebras is that $\sigma$ is compact and the <u>spectral radius</u> $r_\sigma(T+\mathcal{K}) \equiv \max_{\lambda \in \sigma(T+\mathcal{K})} |\lambda|$ is given by

$$r_\sigma(T+K) = \lim_{n\to\infty} \|(T+K)^n\|^{1/n}$$

$$= \lim_{n\to\infty} \|T^n\|_K^{1/n} .$$

Since $\|T\|_\chi \leqq \|T\|_K$ we have

$$r(T) \leqq r_\sigma(T+K) .$$

We show the reverse inequality. First of all, note that by Theorem 3.12

$$\rho(T+K) = \{\lambda\in\mathbb{C}: \lambda I-T \text{ is Fredholm}\} \equiv \Phi_T ,$$

the <u>Fredholm resolvent set of T</u> .

Thus,

$$(3.21) \qquad r_\sigma(T+K) = \max_{\lambda\notin\Phi_T} |\lambda| .$$

If $|\lambda| > r(T)$ we have by the previous theorem that $\lambda \in \Phi_T$ . Then $|\lambda| > r_\sigma(T+K)$ and therefore,

$$r(T) = r_\sigma(T+K) .$$

Alternatively written,

$$(3.22) \qquad \lim_{n\to\infty} \|T^n\|_\chi^{1/n} = \lim_{n\to\infty} \|T^n\|_K^{1/n} .$$

The above result can be found in Nussbaum [28] and Lebow and Schechter [26]. They also go on to show that $r(T)$ is equal to the spectral radius of the essential spectrum no matter which of the known definitions of the essential spectrum is used.

In some sense, equation (3.22) can be thought of as a partial converse to: $T$ $K$-condensing $\Rightarrow$ $T$ $\chi$-condensing. For if $\|T\|_\chi < 1$ then $\|T^n\|_K < 1$ $\forall n$ large. In general, however, $n > 1$. In particular, the two notions of condensing are equivalent in any Hilbert space.

<u>Theorem 3.29</u>.

Let $H$ be a (real or complex) Hilbert space. Then $\|T\|_\chi = \|T\|_K$ $\forall T \in [H]$.

<u>proof</u>: By (3.20), $\|T\|_\chi \leqq \|T\|_K$. Let $\varepsilon > 0$. Then $\exists$ a finite $(\|T\|_\chi + \varepsilon)$-net $\{y_1,\ldots,y_n\}$ of $TB$ by Lemma 3.20. Set $F = \text{span}\,\{y_1,\ldots,y_n\}$ and let $P$ be the orthogonal projection of $H$ onto $F$. Then $PT$ is compact and $\|Tx - PTx\| = \inf_{y\in F} \|Tx - y\|$ $\forall x \in B$. Since for each $x \in B$ $\exists y_k \in \{y_1,\ldots,y_n\} \subset F$ such that $\|Tx - y_k\| \leqq \|T\|_\chi + \varepsilon$ we see that

$$\|T - PT\| = \sup_{x\in B} \|Tx - PTx\| \leqq \|T\|_\chi + \varepsilon .$$

Since $\varepsilon$ was arbitrary the theorem follows. ∎

<u>Remark</u>: The above result is found in Webb [43] though our proof is much more elementary. His goal was to determine conditions on $T$ (e.g., $T$ self-adjoint or $T$ normal) to guarantee equality between $\|T\|_\alpha$, $\|T\|_\chi$, and the essential spectral radius.

Remark: Thus, in a Hilbert space, $T$ $\chi$- condensing implies that $\exists$ compact $K \in [H]$ such that $T = K + (T-K)$ , with $\|T-K\| < 1$ (i.e. $T-K$ is contractive). If $X$ is a complex Banach space then contained in Istratescu [22], Sadovskii [33], and Sedaev [35], is: $\|T\|_\chi < 1 \Rightarrow T = K+T_1$, where $K$ is finite-dimensional and $r_\sigma(T_1) < 1$ . This is extended to $X$ real via a complexification argument (cf. Sadovskii [33], Sedaev [35]).

Since $r(K) = 0$ $\forall K \in \mathcal{K}$ we see that Theorem 3.27 directly extends Corollary 3.9, i.e., $|\lambda| > r(T) \Rightarrow$ either $(\lambda I-T)^{-1} \in [X]$ or $\lambda$ is an eigenvalue of finite multiplicity. Correspondingly, we extend Theorem 3.10.

Theorem 3.30.

Let $T \in [X]$ . Then for each $\varepsilon > 0$ $\{\lambda \in \mathbb{C}: \lambda \text{ is an eigenvalue of } T \text{ with } |\lambda| \geq r(T) + \varepsilon\}$ is finite.

proof: Suppose $T$ has infinitely many distinct eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ with $|\lambda_k| \geq r(T) + \varepsilon$ . Then we may choose corresponding eigenvectors $x_k$ so that $\{x_k\}_{k=1}^{\infty}$ is linearly independent. Let $X_k = \text{span}\,\{x_1,\ldots,x_k\}$ . For $k \geq 1$ there exist, by Riesz's lemma, $y_k \in X_k \cap U$ so that $\|y_k-x\| \geq 1$ $\forall x \in X_{k-1}$ . Since $T^n y_k \in X_k$ and $(\lambda_k^n I-T^n)y_k \in X_{k-1}$ for each $k \geq 1$ and any $n \geq 1$ we see that

$$T^n y_m - T^n y_k = \lambda_m^n (y_m - v) \ , \ 1 \leqq k < m \ , \ n \geqq 1 \ ,$$

where $v = \lambda_m^{-n}[(\lambda_m^n I - T^n)y_m + T^n y_k] \in X_{m-1}$ . It follows that

$$\| T^n y_m - T^n y_k \| \geqq | \lambda_m |^n \geqq (r(T) + \varepsilon)^n \ .$$

This contradicts the fact that for $n$ sufficiently large,

$$\| T^n \| = \chi(T^n B) < (r(T)+\varepsilon)^n \ . \ \blacksquare$$

Remark: This is a direct adaptation of the proof given in the compact case (cf. Anselone [1]).

# IV. OPERATOR APPROXIMATIONS

## 1. Stable Approximations

In this section we begin the comparison of equations $Ax = y$ and $A_n x_n = y$, where $A_n, A \in [X]$ and $\{A_n\}$ satisfies certain hypotheses. These hypotheses will gradually be strengthened in the following two sections in order to obtain for all $n$ large:

$$
(4.1)\qquad
\begin{array}{l}
R(A) = X \iff R(A_n) = X \, , \\
\exists A^{-1} \in [X] \iff \exists A_n^{-1} \in [X] \text{ uniformly bounded} \, , \\
A_n^{-1} \to A^{-1} \text{ on } X \text{ with practical error bounds} \, .
\end{array}
$$

Let $A_n, A \in [X]$. Denote pointwise convergence by $A_n \to A$. In terms of the cluster point sets given in Chapter II,

$$
(4.2)\qquad A_n \to A \Rightarrow \{N(A)\}^* \subset N(A) \, , \; \{R(A)\}^* \supset R(A) \, .
$$

The special cases $N(A) = \{0\}$, $R(A) = X$ are worth mentioning.

Definition 4.1.

$\{A_n\}$ is stable if $\exists A_n^{-1}$ bounded uniformly $\forall n$ large.
We say $\{A_n\}$ converges stably to $A$, and write $A_n \overset{s}{\to} A$, if $A_n \to A$, $\{A_n\}$ is stable, and $R(A_n) = X$ $\forall n$ large.

Remark: In general, $A_n^{-1}$ is only defined on $R(A_n)$.

However, if $A_n \xrightarrow{s} A$ then $A_n^{-1} \in [X]$ $\forall n$ large .

By analogy with (3.3)

(4.3) $\{A_n\}$ stable $\iff$ $0 \notin \{A_nU\}^*$ .

Similar reasoning as in (4.2) yields $\overline{AU} \subset \{A_nU\}^*$ . Consequently,

(4.4) $A_n \xrightarrow{s} A \Rightarrow \exists A^{-1}$ bounded .

The identity $A_n^{-1} - A^{-1} = A_n^{-1}(A - A_n)A^{-1}$ gives

(4.5) $A^{-1}, A_n^{-1} \in [X]$ , $A_n \xrightarrow{s} A$ , $R(A) = X$ $\Rightarrow$
$A_n^{-1} \to A^{-1}$ on $X$ .

In this case we obtain convergence of approximate solutions $x_n$ to the true solution $x$ , i.e.,

(4.6) $\|x_n - x\| \leqq \|A_n^{-1}\| \, \|Ax - A_nx\|$ .

<u>Remark</u>: This convergence depends on the pointwise convergence $A_n \to A$ . However, to be of practical value we would want a uniform estimate for the norms $\|A_n^{-1}\|$ . Another shortcoming is the need to assume $R(A_n) = X$ and $R(A) = X$ . Armed with a Fredholm alternative (i.e. $\mathrm{ind}(A) = \mathrm{ind}(A_n) = 0$) we shall overcome these difficulties in Section 3.

## 2. Regular Approximations

In this section we continue the program begun in Section 1, namely, to determine hypotheses on $\{A_n\}$ so that (4.1) is satisfied. Though regular approximations do not necessarily satisfy a Fredholm alternative, we do obtain stable approximations and so Section 1 applies.

Regular approximations occur in the numerical solution of (linear or nonlinear) differential and integral equations. References include Anselone and Ansorge [3,4], Chatelin [11], Grigorieff [19,20], Stummel [37], Vainikko [40], and Wolf [44].

Let $A_n$, $A \in [X]$ .

Definition 4.2.

$\{A_n\}$ is asymptotically regular if $\{S_n\}$ uniformly bounded, $\{A_nS_n\}$ d-compact $\Rightarrow$ $\{S_n\}$ d-compact .
Equivalent definitions are:

(4.7) $\{x_n\}$ bounded , $\{A_nx_n\}$ d-compact $\Rightarrow$ $\{x_n\}$ d-compact ,

(4.8) $\{x_n\}$ bounded , $A_nx_n \to y$ $\Rightarrow$ $\{x_n\}^* \neq \phi$ .

Define regular convergence by

$A_n \xrightarrow{r} A$: $A_n \to A$ , $\{A_n\}$ asymptotically regular .

Examples of regular convergence will be given in

Sections 3 and 4.

In view of (4.8) we obtain

$$(4.9) \qquad A_n \xrightarrow{r} A \ , \quad \{x_n\} \text{ bounded} \ , \quad A_n x_n \to y \quad \Rightarrow$$
$$y \in R(A) \ , \quad Ax = y \quad \forall x \in \{x_n\}^* \ .$$

Also, it is clear that any subsequence of an asymptotically regular sequence is asymptotically regular. Now suppose $A_n \xrightarrow{r} A$ and $A_n x_n \to y$ with $\{x_n\}$ bounded. Then we may choose positive integers $k(n) > k(n-1)$ such that $A_{k(n)} x_n \to y$ . Since $\{A_{k(n)}\}$ is asymptotically regular, we have $\{x_n\}^* \neq \emptyset$ . Therefore, by (3.13)

$$(4.10) \qquad A_n \xrightarrow{r} A \quad \Rightarrow \quad A \text{ regular} \ .$$

Next, we indicate connections between regular and stable approximations.

<u>Lemma 4.3.</u>

Let $A_n \xrightarrow{r} A$ . Then

a) $S \subset X$ closed , bounded $\Rightarrow$ $\{A_n S\}^* = AS$ ,

b) $\exists A^{-1} \Leftrightarrow \{A_n\}$ stable .

<u>proof</u>: (a) Similar reasoning as in (4.2) yields $\{A_n S\}^* \supset AS$ . Let $y \in \{A_n S\}^*$ . Then $\exists N' \subset N$ and $\{x_n : n \in N'\} \subset S$ so that $A_n x_n \to y$ . By analogy with (4.9), $y \in AS$ .

(b) In view of (a), $\{A_n U\}^* = AU$ . The result now follows from (4.3). ∎

Theorem 4.4. (Vainikko [40])

$$A_n \overset{r}{\to} A, \exists A^{-1}, R(A_n) = X \ \forall n \text{ large} \Rightarrow$$

$$A_n \overset{s}{\to} A, \quad R(A) = X, \text{ and } A_n^{-1} \to A^{-1}.$$

proof: According to Lemma 4.3, $A_n \overset{s}{\to} A$. To show $R(A) = X$, let $y \in X$. Then $\exists x_n$ such that $x_n = A_n^{-1}y \ \forall n$ large. Since $\{A_n\}$ is stable, $\{x_n\}$ is bounded. Now apply (4.9). The pointwise convergence $A_n^{-1} \to A^{-1}$ follows directly from (4.5). ■

Aided by a Fredholm alternative we shall alleviate the assumption $R(A_n) = X$ in Theorem 4.4 when collectively or asymptotically compact sequences of operators are involved in the next section. Furthermore, practical error bounds for the convergence $A_n^{-1} \to A^{-1}$ are given.

It should be pointed out that even if $A^{-1}$ does not exist, we have $\dim N(A) < \infty$. Are the approximate operators as well behaved? The next theorem gives an answer.

Theorem 4.5.

Let $A_n \overset{r}{\to} A$. Then

(a) $N(A_n) \cap U \to \{N(A_n) \cap U\}^* = N(A) \cap U$,

(b) $\dim N(A_n) \leqq \dim\{N(A_n)\}^* \leqq \dim N(A) < \infty \ \forall n$ large.

proof: (a) Since $\{N(A_n) \cap U\}$ is uniformly bounded and $\{A_n(N(A_n) \cap U)\} = \{0\}$ is d-compact, we obtain $\{N(A_n) \cap U\}$ d-compact. The rest follows by Theorems

2.3 and 2.9.

(b) This is an immediate consequence of (a) and Theorem 2.11. ∎

Remark: If $A_n \xrightarrow{r} A$ then $R(A)$ is closed as well. The same can be said for $R(A_n)$ $\forall n$ large (cf. Anselone and Treuden [7]). Therefore, by Theorem 3.16,

$$A_n \xrightarrow{r} A \Rightarrow A_n \text{ regular } \forall n \text{ large} .$$

## 3. Collectively and Asymptotically Compact Approximations

In this section we extend some of the collectively compact compact approximation theory to the asymptotically compact case. In particular, both a Fredholm alternative and practical error bounds are given.

Definition 4.6.

$\{K_n\}$ is collectively compact if

$S$ bounded $\Rightarrow$ $\overline{\bigcup_n K_n S}$ compact .

Definition 4.7.

$\{K_n\}$ is asymptotically compact if

$S$ bounded $\Rightarrow$ $\{K_n S\}$ d-compact .

It suffices if $S = B$ . Equivalent sequential forms are

(4.11) $\{K_n\}$ is collectively compact iff

$\{x_n\}$ bounded $\Rightarrow$ $\{K_{m_n} x_n\}$ d-compact $\{m_n\}$ ,

(4.12) $\{K_n\}$ is asymptotically compact iff

$\{x_n\}$ bounded $\Rightarrow$ $\{K_n x_n\}$ d-compact .

Define <u>collectively compact convergence</u> by

$K_n \overset{cc}{\to} K$: $K_n \to K$ , $\{K_n\}$ collectively compact , and

<u>asymptotically compact convergence</u> by

$K_n \overset{ac}{\to} K$: $K_n \to K$ , $\{K_n\}$ asymptotically compact .

Note that $\{K_n\}$ collectively compact implies each $K_n$ is compact. This need not be the case when asymptotically compact sequences are considered. It is an easy exercise to show that $\|K_n\| \to 0 \Rightarrow \{K_n\}$ asymptotically compact. Hence $\{\frac{1}{n} I\}$ is asymptotically compact but $\frac{1}{n} I$ is never compact when $\dim X = \infty$ . Anselone and Ansorge [3] have shown that the lack of compactness of the individual operators is the only difference between collectively and asymptotically compact sequences.

<u>Theorem 4.8.</u> (Anselone and Ansorge [3])

Let $K$ , $K_n \in [X]$ . Then

(a) $\{K_n\}$ collectively compact $\Longleftrightarrow$

$\{K_n\}$ asymptotically compact and $K_n$ compact $\forall n$ ,

(b) $K_n \overset{cc}{\to} K \Rightarrow K_n \overset{ac}{\to} K \Rightarrow K$ compact .

proof: (a) From the definitions and Theorem 2.6 we have

$$\{K_n\} \text{ collectively compact } \Leftrightarrow \overline{\cup_n K_n B} \text{ compact}$$

$$\Leftrightarrow \{K_n B\} \text{ d-compact and } \overline{K_n B} \text{ compact } \forall n$$

$$\Leftrightarrow \{K_n\} \text{ asymptotically compact, } K_n \text{ compact } \forall n .$$

(b) In view of (a) it suffices to shown that $K_n \overset{ac}{\to} K \Rightarrow K$ compact . By (4.2) we have $KB \subset \{K_n B\}^*$. By Theorem 2.3 , $\{K_n B\}^*$ is compact. The theorem follows. ■

We stated that $\mathcal{K}$ is an ideal in $[X]$ . In order to give an analogous result for asymptotically compact sequences it is convenient to define continuous convergence by

$$A_n \overset{c}{\to} A: \quad x_n \to x \Rightarrow A_n x_n \to Ax .$$

Continuous convergence replaces pointwise convergence in Definitions 4.6, 4.7 when nonlinear operators are considered (see Anselone and Ansorge [3]). Continuous convergence implies pointwise convergence (simply set $x_n = x$). The converse holds when the operators are linear by the uniform boundedness theorem.

Theorem 4.9.

Let $K$ , $L$ , $K_n$ , $L_n \in [X]$ with $K_n \overset{ac}{\to} K$ ,

$L_n \overset{ac}{\to} L$ . Suppose we have the scalar convergence $\alpha_n \to \alpha$ , $\beta_n \to \beta$. Then $\alpha_n K_n \overset{ac}{\to} \alpha K$ , $\beta_n L_n \overset{ac}{\to} \beta L$ , $\alpha_n K_n + \beta_n L_n \overset{ac}{\to} \alpha K + \beta L$ , and $K_n L_n \overset{ac}{\to} KL$ . Moreover, $L_n K \overset{ac}{\to} LK$ and $KL_n \overset{ac}{\to} KL$ .

<u>proof</u>: In all cases the continuous convergence is easy and is omitted. To show that each is an asymptotically compact sequence let $\{x_n\}$ be bounded and $N' \subset N$ . Then $\{Kx_n\}$ and $\{L_n x_n\}$ are also bounded. The asymptotic compactness of $\{K_n\}$ and $\{L_n\}$ produce $N'' \subset N'$ and elements $u$ , $v$ , $w \in X$ such that $K_n x_n \to u$ , $L_n x_n \to v$ , $L_n K x_n \to w$ for $n \in N''$ . By the continuous convergence $K_n \overset{c}{\to} K$ , $L_n \overset{c}{\to} L$ , and by the continuity of $K$ , it follows that $\alpha_n K_n x_n \to \alpha u$ , $\beta_n L_n x_n \to \beta v$ , $\alpha_n K_n x_n + \beta_n L_n x_n \to \alpha u + \beta v$ , $K_n L_n x_n \to Kv$ , $L_n K x_n \to w$ , and $KL_n x_n \to Kv$ . Hence the sequences $\{\alpha_n K_n\}$ , $\{\beta_n L_n\}$ , $\{\alpha_n K_n + \beta_n L_n\}$ , $\{K_n L_n\}$ , $\{L_n K\}$ , and $\{KL_n\}$ are asymptotically compact. ∎

As a special case of the previous theorem we note that

$$(4.13) \qquad K_n \overset{cc}{\to} K \ , \ \|L_n\| \to 0 \ \Rightarrow \ K_n + L_n \overset{ac}{\to} K \ .$$

We would expect $K_n + L_n \overset{cc}{\to} K$ if each $L_n$ is compact by Theorem 4.7. This is not the case in general. Simply let $L_n = c_n I$ , with $c_n \downarrow 0$ , and $\dim X = \infty$.

This situation occurs when the $K_n$ are numerical integration approximations to an integral operator with a weakly singular kernel. More on this in Section 4.

An extremely useful link between collectively or asymptotically compact sequences of operators and measures of noncompactness is given next.

<u>Theorem 4.10.</u>

$\{K_n\}$ collectively compact $\iff \chi(\bigcup_n K_n B) = 0$ ,

$\{K_n\}$ asymptotically compact $\iff \lim_{k\to\infty} \chi(\bigcup_{n\geq k} K_n B) = 0$ .

<u>proof</u>: The first is immediate by the definitions. See Theorem 2.16 for the other. ∎

<u>Corollary 4.11.</u>

If $\{K_n\}$ is asymptotically compact then $K_n$ is $\chi$-condensing $\forall n$ large , in which case $\operatorname{ind}(I-K_n) = 0$ .

<u>Remark</u>: We have applied the standard measure of noncompactness to sequences. There have been definitions given particularly to study sequences, the so-called discrete measures of noncompactness. A characterization of collectively or asymptotically compact sequences can be based on these measures, however, we shall not make use of these ideas. See Wolf [45,46] and Appell and Pera [8].

We now turn our attention to the study of the linear equations $(I-K)x = y$ , $(I - K_n)x_n = y$ where

$K_n \overset{ac}{\to} K$ . The organization for our presentation is essentially that for the collectively compact theory given by Anselone [1]. By Theorem 4.7, $K$ is compact and consequently $I - K$ is Fredholm of index zero, i.e., $I - K$ satisfies the Fredholm alternative. Thus, if $I - K$ is invertible the equation $(I-K)x = y$ is uniquely solvable for each $y$ and the solution depends continuously on $y$ . What can be said of the approximate equations $(I-K_n)x_n = y$ in this case? The next lemma helps answer this question.

<u>Lemma 4.12.</u>

$$K_n \overset{ac}{\to} K \Rightarrow (I-K_n) \overset{r}{\to} (I-K) .$$

<u>proof</u>: The pointwise convergence is clear. To show $\{I-K_n\}$ asymptotically regular it suffices to show that $\{x_n\}^* \neq \emptyset$ when $\{x_n\}$ is bounded and $(I-K_n)x_n \to y$ . Since $\{K_n\}$ is asymptotically compact $\exists N' \subset N$ , $z \in X$ such that $K_n x_n \to z$ for $n \in N'$ . Hence $x_n = K_n x_n + (I-K_n)x_n \to z + y$ for $n \in N'$ , which is what we wanted to prove. ■

<u>Theorem 4.13.</u>

Let $K_n \overset{ac}{\to} K$ . Then

$\exists (I-K)^{-1} \Leftrightarrow \exists (I-K_n)^{-1}$ uniformly bounded $\forall n$ large , in which case $(I-K_n)^{-1} \to (I-K)^{-1}$ on $X$ .

<u>proof</u>: From Lemmas 4.3 and 4.12 we have $\exists (I-K)^{-1} \Leftrightarrow \{I-K_n\}$ stable . By Corollary 4.11

$R(I-K_n) = X$ when $\exists (I-K_n)^{-1}$ $\forall n$ large . Thus, we may appeal to Theorem 4.4 to obtain the desired results. ∎

The package will be complete once we obtain practical error bounds, i.e., bounds on the convergence $(I-K_n)^{-1} \to (I-K)^{-1}$ when $(I-K)^{-1}$ exists. The next result is a significant step in obtaining such bounds. It generalizes the fact that pointwise convergence is uniform on each totally bounded set.

Lemma 4.14.

Let $T_n$ , $T \in [X]$ , with $T_n \to T$ and suppose $\{S_n\}$ is asymptotically totally bounded. Then $\| T_n x_n - Tx_n \| \to 0$ uniformly in the choice of sequences $\{x_n : x_n \in S_n\}$ .

proof: By the uniform boundedness theorem, $\exists b > 0$ so that $\| T_n \| \leqq b$ $\forall n$ and $\| T \| \leqq b$ . Let $\varepsilon > 0$ . Now $\exists n_1 \in N$ such that $\chi(\bigcup_{n \geqq n_1} S_n) < \frac{\varepsilon}{3b}$ (see Theorem 2.15). Let $\{y_1, \ldots, y_{n_2}\}$ be such an $\frac{\varepsilon}{3b}$ - net . Now $T_n \to T \Rightarrow \exists n_3 > n_1$ so that $n \geqq n_3$ implies $\| T_n y_k - Ty_k \| < \frac{\varepsilon}{3}$ $\forall 1 \leqq k \leqq n_2$ . Hence for each $x_n \in S$ and $n \geqq n_3$ we obtain

$$\begin{aligned} \| T_n x_n - Tx_n \| &\leqq \| T_n x_n - T_n y_k \| + \| T_n y_k - Ty_k \| + \| Ty_k - Tx_n \| \\ &\leqq (\| T_n \| + \| T \|) \| x_n - y_k \| + \| (T_n - T) y_k \| \\ &< 2\, b \frac{\varepsilon}{3b} + \frac{\varepsilon}{3} = \varepsilon \ . \end{aligned}$$

This proves the assertion. ∎

The next theorem summarizes the asymptotically compact approximation theory.

Theorem 4.15.

Let $K_n \overset{ac}{\to} K$. Then

(a) (Fredholm alternative) $\forall n$ large,

$R(I-K_n) = X \iff \exists (I-K_n)^{-1} \Rightarrow (I-K_n)^{-1} \in [X]$,

$\dim N(I-K_n) = \operatorname{codim} R(I-K_n) < \infty$.

(b) $\|(K_n-K)K\| \to 0$, $\|(K_n-K)K_n\| \to 0$ as $n \to \infty$.

(c) $\exists (I-K)^{-1} \iff \exists (I-K_n)^{-1}$ uniformly bounded $\forall n$ large, in which case there are practical error bounds:

Choose $n$ large enough so that $\|(K_n-K)K_n\| < 1$ and set $\Delta_n = \|(I-K)^{-1}\| \, \|(K_n-K)K_n\|$. Then $\Delta_n \to 0$,

$\Delta_n < 1 \Rightarrow \exists (I-K_n)^{-1} \in [X]$,

$$\|(I-K_n)^{-1}\| \leqq \frac{1+\|(I-K)^{-1}\| \, \|K_n\|}{1-\Delta_n}$$

and for each $y \in X$,

$$\|x_n-x\| = \|(I-K_n)^{-1}y - (I-K)^{-1}y\|$$

$$\leqq \frac{\|(I-K)^{-1}\| \, \|K_ny-Ky\| + \Delta_n \|x\|}{1-\Delta_n} \to 0 .$$

With the exception of choosing $n$ large enough so that $\|(K_n-K)K_n\| < 1$, all of the error analysis remains valid with $K_n$ and $K$ interchanged.

<u>proof</u>: (a) Immediate from Corollary 4.11.

(b) Use Lemma 4.14 with $S_n = KB$ and $S_n = K_n B$ .

(c) Choose $n$ so that $\| (K_n - K) K_n \| < 1$ . Since $K_n^2 = KK_n + (K_n - K) K_n$ we have by (3.17) and (3.20)

$$\begin{aligned} \| K_n^2 \|_\chi &\leqq \| KK_n \|_\chi + \| (K_n - K) K_n \|_\chi \\ &\leqq \| (K_n - K) K_n \| < 1 . \end{aligned}$$

Thus, $I-K_n$ is Fredholm of index zero by Theorem 3.27. The rest of the theorem is a consequence of (b) and is unchanged from the collectively compact theory presented by Anselone [1]. We omit the details. ∎

<u>Remark</u>: The convergence $\| (K_n - K) K_n \| \to 0$ plays a dual role. Not only does it provide criteria for determining when $(I-K_n)^{-1}$ exists but it establishes a bound on $n$ so that $I-K_n$ satisfies the Fredholm alternative, which was automatic in the collectively compact case. This may especially be desirable when $\| K_n \|_\chi$ may be difficult to compute.

<u>Remark</u>: We briefly mention that even if $(I-K)^{-1}$ doesn't exist then the "extent" that $(I-K_n)^{-1}$ doesn't exist, at least for $n$ large enough, is no worse by Theorem 4.5, i.e.

$$\dim N(I-K_n) \leqq \dim N(I-K) \quad \forall n \text{ large} .$$

Furthermore, if $n$ is also large enough so that
$\mathrm{ind}(I-K_n) = 0$ , we have
$\mathrm{codim}\, R(I-K_n) = \dim N[(I-K_n)'] < \infty$ , and
$R(I-K_n) = N[(I-K_n)']^{\perp}$ ., the set of <u>annihilators</u>,
i.e., $N(I'-K'_n)^{\perp} = \{x \in X: f(x) = 0 \quad f \in N[(I-K_n)']\}$ .
Thus, there is a concrete way of testing whether
$(I-K_n)x_n = y$ has a solution.

<u>Remark</u>: By virtue of Theorem 4.8 (or more generally Theorem 3.27) we have an immediate extension of the above theorem to the equations
$(\lambda I-K)x = y$ , $(\lambda_n I-K_n)x_n = y$ where $\lambda_n \to \lambda \neq 0$ .
Simply replace $K$ by $\lambda^{-1}K$ and $K_n$ by $\lambda_n^{-1}K_n$ .

## 4. An Example from Integral Equations

The example given involves the approximate solution of a weakly singular Fredholm integral equation on $C[0,1]$ ,

$$(4.14) \qquad x(s) - \int_0^1 k(s,t)\, x(t)\, dt = y(s) ,$$

where the kernel $k(s,t)$ is singular along the diagonal $s=t$ (e.g. $k(s,t) = |s - t|^{-1/2}, \ln|s-t|$) , based on the singularity subtraction technique of Kantorovich and Krylov [23] and the numerical integration of weakly singular functions developed by Anselone and Opfer [6]. Not only does this example illustrate the asymptotically compact theory, but it motivates a characterization which, at least in most spaces of practical interest, states that asymptotically compact sequences of linear operators can be thought of as perturbations of collectively compact sequences.

Let $X = C[0,1]$, the space of continuous real-valued functions defined on $[0,1]$, with $\|x\| = \max_{0\leqq t\leqq 1} |x(t)|$. Define the integral operator

$$(4.15) \qquad (Kx)(s) = \int_0^1 k(s,t)\, x(t)\, dt\ , \quad 0 \leqq s \leqq 1\ .$$

If $k_s(\cdot) = k(s,\cdot)$ and

$$(4.16) \qquad k_s \in L^1[0,1],\ \|k_r - k_s\|_1 \to 0 \quad \text{as} \quad r \to s\ ,\ s \in [0,1]$$

then it is shown by Anselone [1] that $K \in [C[0,1]]$ and $K$ is compact. Such is the case for a kernel with a monotone symmetric singular factor:

$$k(s,t) = g(|s-t|)\, h(s,t)\ ,$$
$$h \in C([0,1] \times [0,1])\ ,$$
$$g \in L^1(0,1) \cap C(0,1]\ ,$$

and

$g \geqq 0$, $g$ nonincreasing on $(0,\delta]$, for some $\delta \in (0,1]$. Examples include $k(s,t) = |s-t|^{-1/2}$, $\ln|s-t|$. For $n=1,2,\ldots$ define continuous approximate kernels $k_n(s,t) = g_n(|s-t|)\, h(s,t)$ where

$$g_n \in C[0,1]\ , \quad g_n = g \quad \text{on } [\tfrac{1}{n},1]\ ,$$

$$0 \leqq g_n \leqq g\ , \quad g_n \text{ nonincreasing on } [0,\tfrac{1}{n}]\ ,$$

$$g_n(0) \geqq g_m(0) \quad \text{for} \quad n \geqq m\ .$$

For example, if $k(s,t) = |s-t|^{-1/2}$ then we may use truncation to define

$$k_n(s,t) = \begin{cases} \sqrt{n} & , 0 \leqq |s-t| \leqq \frac{1}{n} \\ |s-t|^{-\frac{1}{2}} , & \frac{1}{n} \leqq |s-t| \leqq 1\ . \end{cases}$$

More general kernels are considered by Anselone [2]. By means of a convergent quadrature rule define the approximate operators

$$(L_n x)(s) = \sum_{j=1}^{n} w_{nj}\ k_n(s,t_{nj})\ x(t_{nj})\ .$$

Since the $k_n$ are continuous we have $L_n \in [C[0,1]]$. Indeed, $L_n$ is compact since $\dim R(L_n) < \infty$. With further restrictions on the quadrature rule it can be shown that $L_n \overset{cc}{\to} K$ (see Anselone and Krabs [5], Anselone and Opfer [6]). Hence the collectively compact theory applies. However, the convergence $x_n \to x$ (when $(I-K)^{-1}$ exists) is usually slow due to the singular kernel. To obtain more rapid convergence, rewrite (4.14) as

(4.17) $x(s)-[\int_0^1 k(s,t)[x(t)-x(s)]dt + \int_0^1 k(s,t)x(s)dt]=y(s)$

and define operators

$$(K_n x)(s) = \sum_{j=1}^{n} w_{nj} k_n(s,t_{nj})[x(t_{nj})-x(s)] + x(s)\int_0^1 k(s,t)dt \ .$$

Then $K_n = L_n + (Ku - L_n u)I$ where $u \equiv 1$ . Hence, $K_n \overset{ac}{\to} K$ but $K_n \overset{cc}{\to} K$ only if $Ku-L_n u \equiv 0$ . Thus, the asymptotically compact theory applies. Furthermore, due to the singularity subtraction we would expect $(I-K_n)^{-1} \to (I-K)^{-1}$ to be faster than $(I-L_n)^{-1} \to (I-K)^{-1}$ when the latter inverses exist. Numerical examples substantiate this claim (see Anselone [2]).

It is worthwhile to note that $\| K_n - L_n \| = \| Ku - L_n u \| \to 0$ . That is, $K_n = L_n + T_n$ where $\{L_n\}$ is collectively compact and $\| T_n \| \to 0$ . The question of whether this decomposition is always valid for $\{K_n\}$ asymptotically compact is addressed in the next chapter.

## V. CHARACTERIZATION OF ASYMPTOTICALLY COMPACT SEQUENCES OF LINEAR OPERATORS

We have seen results involving the equation $(\lambda I-K)x = y$ , $\lambda \neq 0$ , when $K$ is compact extended to the case when $K$ has a sufficiently small measure of noncompactness (e.g. $\|K\|_\chi < |\lambda|$) . It is reasonable to ask how far operators with a small measure of noncompactness differ from compact operators. This question is easily answered when the measure $\|\cdot\|_K$ is considered. For if $\varepsilon > 0$ , $\|K\|_K < \varepsilon$ , there exists compact $L$ and bounded $T$ such that $K = L + T$ with $\|K-L\| = \|T\| < \varepsilon$ . Note that we need only consider $\varepsilon = 1$ . Now consider the measure of noncompactness $\|\cdot\|_\chi$ . Can we replace $\|\cdot\|_K$ with $\|\cdot\|_\chi$ and still obtain the same results? We can if the operators are defined on a Hilbert space, for then $\|\cdot\|_K = \|\cdot\|_\chi$ (cf. Theorem 3.29). Unfortunately, we can't in general. Goldenstein and Markus [18] give an example of an operator defined on a product of sequence spaces which is $\chi$-condensing but not $K$-condensing.

Recall that

(5.1) $\{K_n\}$ asymptotically compact $\Rightarrow$ $\|K_n\| \to 0$ as $n \to \infty$,

(5.2) $\{K_n\}$ collectively compact $\Longleftrightarrow$

$\{K_n\}$ asymptotically compact, $K_n$ compact $\forall n$ .

Thus, we are led to an analagous question relating asymptotically and collectively compact sequences, i.e., it is true that

(5.3) $\{K_n\}$ asymptotically compact $\Longleftrightarrow$

$\exists \{L_n\}$ collectively compact such that

$\|K_n - L_n\| \to 0$ as $n \to \infty$ ?

Sufficiency follows from Theorem 4.8 since $\{L_n\}$ , $\{K_n - L_n\}$ asymptotically compact imply $\{K_n - L_n + L_n\}$ asymptotically compact. To prove necessity it would suffice to find a constant $C > 0$ so that $\|\cdot\|_K \leqq C\|\cdot\|_X$ . For if such a constant exists,

$\{K_n\}$ asymptotically compact $\Rightarrow$

$\|K_n\|_X \to 0 \Rightarrow \|K_n\|_K \to 0 \Rightarrow$

$\exists \{L_n\} \subset K$ such that $\|K_n - L_n\| \to 0$ .

Since $\{L_n = K_n + (L_n - K_n)\}$ is asymptotically compact and each $L_n$ is compact we actually have $\{L_n\}$ collectively compact by Theorem 4.7. Note that by (3.20), $\|\cdot\|_X \leqq \|\cdot\|_K$ . Hence, such a constant exists of the seminorms $\|\cdot\|_X$ and $\|\cdot\|_K$ are equivalent if $[X]/K$ . The equivalence of these seminorms will be shown for a large class of spaces, which is the subject of the next definition.

<u>Definition 5.1.</u> (cf. Lindenstrauss and Tzafriri [27])

A Banach space X is said to have the

<u>compact approximation property</u> (abbr. C.A.P. ) if for each $\varepsilon > o$ and finite set of points $x_1,\ldots,x_n \in X$ there exists $K \in \mathcal{K}$ such that $\| x_k - Kx_k \| \leqq \varepsilon$, $1 \leqq k \leqq n$. If $1 \leqq \lambda \leqq \infty$, then $X$ has the <u>$\lambda$-compact approximation property</u> (abbr. $\lambda$-C.A.P. ) if $X$ has the C.A.P. with $\|K\| \leqq \lambda$.

<u>Lemma 5.2.</u> (Lebow and Schechter [26])

Let $X$ have the $\lambda$-C.A.P. Then

$$\|T\|_{\mathcal{K}} \leqq (\lambda+1) \|T\|_{\chi} \quad \forall T \in [X] .$$

<u>proof</u>: Let $\varepsilon > 0$. Then there is a finite $(\|T\|_{\chi} + \varepsilon)$ - net $\{y_1,\ldots,y_n\}$ of $TB$. Because $X$ has the $\lambda$-C.A.P. , there exists $K$ $\mathcal{K}$ with $\|I-K\| \leqq \lambda + 1$ and $\|y_k - Ky_k\| \leqq \varepsilon$, $1 \leqq k \leqq n$. If $x$ $B$ there is an element $y_k$ so that

$$\begin{aligned} \|(I-K)Tx\| &\leqq \|(I-K)(Tx-y_k)\| + \|(I-K)y_k\| \\ &\leqq (\lambda+1)(\|T\|_{\chi} + \varepsilon) + \varepsilon . \end{aligned}$$

Since $\varepsilon$ and $x$ were arbitrary, we obtain

$$\|T - KT\| \leqq (\lambda+1) \|T\|_{\chi} .$$

But $KT \in \mathcal{K}$, and so the proof is complete. ∎

Theorem 5.3.

Let $X$ have the $\lambda$-C.A.P. Then

$\{K_n\} \subset [X]$ asymptotically compact $\Longleftrightarrow$

$K_n = L_n + T_n$ , $\{L_n\} \subset [X]$ collectively compact,

$\|T_n\| \to 0$ .

proof: $(\Rightarrow)$ By Lemma 3.20 and Theorem 4.10, $\|K_n\|_X \to 0$ as $n \to \infty$ . By Corollary 5.2, $\|K_n\|_K \to 0$ as $n \to \infty$ . It follows that $\exists \{L_n\} \subset K$ such that $\|K_n - L_n\| \to 0$ as $n \to \infty$ . Set $T_n = K_n - L_n$ . Then $\|T_n\| \to 0$ and $\{T_n\}$ is asymptotically compact. In light of Theorem 4.9, $\{L_n = K_n - T_n\}$ is asymptotically compact. But since each $L_n$ is compact we have $\{L_n\}$ collectively compact by Theorem 4.8.

$(\Leftarrow)$ This assertion follows directly from Theorems 4.8 and 4.9. ∎

If $\lambda = 1$ and $K$ is finite dimensional in Definition 5.1 we obtain the metric approximation property (abbr. M.A.P. ) of Grothendieck [21]. The question of whether any Banach space possessed the M.A.P. was settled in the negative by Enflo [14] (see also Davie [13], Pietsch [32]) with an involved (!) counter example. Though the C.A.P. is apparently weaker, there are conditions for which $X$ fails to have the C.A.P. , and hence the $\lambda$-C.A.P. , that are based on Enflo's example (see Lindenstrauss and Tzafriri [27]). Also included therein is a result, due to Szankowski [38], that $\ell_p$ has a subspace without the C.A.P., $1 \leqq p \leqq 2$. Roughly speaking, a Banach space which is not isomorphically close to being a Hilbert space will always have a subspace which fails to have the C.A.P. It would be beyond the scope of this thesis to delve more deeply into the structure theory of Banach spaces. Suffice it to say that from the standpoint of applications, we have all the generality needed. For example, all Hilbert spaces, $L_p(\Omega,\mu)$ ($(\Omega,\mu)$ any measure space, $1 \leqq p \leqq \infty$), and $C(\Omega)$ ($\Omega$ any compact Hausdorff space) endowed with their standard norms have the M.A.P. (see e.g. Pietsch [32]).

We conclude this chapter with two special cases of Lemma 5.2 (and hence Theorem 5.3), both of which are general enough to include most classical Banach spaces.

<u>Theorem 5.4.</u> (Goldenstein and Markus [18])

Suppose $P_n \in [X]$ are finite dimensional projections with $P_n \to I$. Then there exists $0 < C < \infty$ such that

$$\|T\|_K \leqq C\|T\|_X \quad \forall T \in [X] .$$

<u>proof</u>: It suffices to show that $X$ has the $\lambda$-C.A.P. for some $\lambda$. Since $P_n \to I$, $\exists \lambda \in [1,\infty)$ such that $\sup_n \|P_n\| \leqq \lambda$. Let $\varepsilon > 0$ and $\{x_1,\ldots x_m\} \subset X$. Then

there exists $n_m \in N$ so that $n \geqq n_m \Rightarrow \|(I-P_n)x_k\| \leqq \varepsilon$, $1 \leqq k \leqq m$. Since $P_n$ is compact, $X$ has the $\lambda$-C.A.P. ■

Theorem 5.5

If $X$ has a basis then there exists $0 < C < \infty$ such that

$$\|T\|_K \leqq C\|T\|_X \quad \forall T \in [X] .$$

proof: Let $\{e_k\}_{k-1}^{\infty}$ be a basis for $X$. For $x \in X$ there exists a unique sequence of scalars $\{\xi_k\}$ such that $x = \sum_{k=1}^{\infty} \xi_k e_k$. Define $P_n \in [X]$ by $P_n x = \sum_{k=1}^{n} \xi_k e_k$, $n = 1,2,\ldots$. Now appeal to Theorem 5.4. ■

The space $C[0,1]$ of Section IV.4 has a (normalized) basis given by

$$e_0(t) = 1 , \quad e_1(t) = t ,$$

$$e_{2^n+k}(t) = \begin{cases} 0 & \text{for } t \notin \left(\frac{2k-2}{2^{n+1}}, \frac{2k}{2^{n+1}}\right) \\ 1 & \text{for } t = \frac{2k-1}{2^{n+1}} \\ \text{linear in} & \left[\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}\right] \text{ and } \left[\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}\right] \end{cases} ,$$

where $k = 1,2,\ldots,2^n$; $n = 0,1,2,\ldots$ .

We may define corresponding finite dimensional projections $P_n$, $n = 0,1,2,\ldots$, by linear interpolation at the points $\frac{2k-1}{2^{n+1}}$, $k = 1,2,\ldots,2^n$. Worth noting is that $\|P_n\| = 1$, $n = 0,1,2,\ldots$. Thus, by Lemma 5.2,

$$\text{(5.4)} \qquad \|T\|_\chi \leqq \|T\| \leqq 2\,\|T\|_\chi \quad \forall T \in [C[0,1]] \, .$$

Indeed, the bounded linear operators on any space with the M.A.P. satisfy (5.4).

Under the hypotheses of Theorems 5.4 and 5.5 we may assume that the approximating collectively compact sequence $\{L_n\}$ in (5.3) consists of finite dimensional operators. For if $\{P_n\}$ is a sequence of finite dimensional projections, $P_n \to I$, and $\{K_n\}$ is asymptotically compact, then by Theorem 4.14

$$\|(I-P_n)K_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

The above convergence implies that $\{P_nK_n-K_n\}$ is also asymptotically compact. But since each $P_nK_n$ is compact we have $\{P_nK_n\}$ collectively compact by Theorem 4.7.

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