

## Iterated Geostrophic Intermediate Models

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### ABSTRACT

Intermediate models contain physics between that in the primitive equations and that in the quasigeostrophic equations and are capable of representing subinertial frequency motion over  $O(1)$  topographic variations typical of the continental slope while filtering out high-frequency gravity-inertial waves. We present here a formulation for stratified flow of a set of new intermediate models, termed iterated geostrophic (IG) models, derived under the assumption that the Rossby number  $\epsilon$  is small. We consider the motion of a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, adiabatic primitive equations (PE) with a spatially variable Coriolis parameter and with weak biharmonic momentum diffusion. The IG models utilize the pressure field as the basic variable [as in the quasigeostrophic (QG) approximation], are capable of providing solutions of formally increasing accuracy in powers of  $\epsilon$  in a systematic manner, and are straightforward to solve numerically. The IG models are obtained by iteration, at a fixed time  $t = t_0$ , of the momentum and thermodynamic equations using the known pressure field  $\phi(\mathbf{x}, t)$ . The iteration procedure produces a sequence of estimates of increasing accuracy for the velocity components and for the time derivative of the pressure field  $\phi_t(\mathbf{x}, t_0)$ . The formulation is asymptotic in the sense that, given the pressure field, at each iteration the velocity components and  $\phi_t(\mathbf{x}, t_0)$  are formally determined to a higher order of accuracy in powers of  $\epsilon$ . The order of the IG model is specified by the predetermined fixed number of iterations  $N$ . Thus, a set of models is produced depending on the choice for  $N$ , and the different models are denoted by IGN. The value of  $\phi_t(\mathbf{x}, t_0)$  obtained from iteration  $N$  is used with a time difference scheme to advance the pressure field in time, and the process may be repeated. Energy and potential enstrophy conservation in the IG models are asymptotic. In the following companion paper (Allen and Newberger), the accuracies of several intermediate models, including IG2 and IG3, are investigated by a comparison of numerical finite-difference solutions to those of the primitive equations. For moderate Rossby number flows, it is found that IG2 gives approximate solutions of reasonable accuracy, with errors substantially smaller than those obtained from QG and several other intermediate models. The IG3 model is found to give extremely accurate approximate solutions for flows with Rossby numbers that range from moderately small to  $O(1)$ .

### 1. Introduction

Intermediate models (McWilliams and Gent 1980) contain physics between that in the primitive equations and that in the quasigeostrophic approximation. They are derived under the assumption that the Rossby number  $\epsilon$  is small and they filter out high-frequency gravity-inertial waves. They are capable of representing subinertial frequency flows over  $O(1)$  variations in bottom topography for which the quasigeostrophic approximation is not valid. The balance equations (Gent and McWilliams 1983) and the geostrophic momentum approximation (Hoskins 1975) are some of the better known and more widely used intermediate models. The general goal of this research is to investigate the possible use of intermediate models for studies of nonlinear mesoscale eddies and jets in coastal and near-coastal flow fields.

One issue regarding the properties of different intermediate models (McWilliams and Gent 1980) is the existence, as is the case for the primitive equations, of corresponding conservation equations for potential vorticity on fluid particles and for energy. The geostrophic momentum (GM) model has analogs of both these conservation relations. For the shallow-water equations (SWE), the balance equations (BE) conserve potential vorticity on fluid particles but do not have a conservation law for energy, whereas in the continuously stratified case in physical coordinates, the BE conserve energy (Lorenz 1960) but not potential vorticity. Recently Allen (1991) has shown that for a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, inviscid, adiabatic primitive equations (PE) with spatially variable Coriolis parameter, a model based on approximate momentum equations that is very close to the balance equations can be formulated such that an analog of potential vorticity is conserved on fluid particles and volume integrals of an appropriate energy density are conserved. The resulting approximate set of equations is referred to as a

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BEM model (balance equations based on momentum equations).

We note that many of the intermediate models being applied at present involve some ad hoc element in their derivation rather than following from an orderly asymptotic method that is capable, at least formally, of producing a sequence of approximate solutions of increasing accuracy. The balance equations (BE) are derived (Gent and McWilliams 1983) by a truncation of the equations for the vertical component of vorticity and for the horizontal divergence such that  $O(1)$  and  $O(\epsilon)$  terms are retained. Although this is a systematic truncation that appears to result in an accurate set of approximate equations, a method for obtaining higher-order approximations of increasing accuracy is not obvious. Moreover, the approximate momentum equations that correspond to the truncated vorticity and divergence equations do not follow directly from approximations to the PE momentum equations but involve implicitly defined force potential correction terms and are referred to as the equivalent momentum equations (Gent and McWilliams 1983). The BEM model derived by Allen (1991) involves truncation such that  $O(1)$ ,  $O(\epsilon)$ , and some  $O(\epsilon^2)$  terms are retained in the momentum equations. The GM model (Hoskins 1975) also retains  $O(1)$ ,  $O(\epsilon)$ , and some  $O(\epsilon^2)$  terms in the momentum equations (McWilliams and Gent 1980). Although all of these models are based on the assumption that  $\epsilon \ll 1$ , it is not clear that the derivations of any of them involve an asymptotic procedure that could be used to produce approximations of increasing accuracy.

We present here a formulation of a set of new intermediate models, termed "iterated geostrophic (IG) models," that are based on the assumption  $\epsilon \ll 1$ , utilize the pressure field as the basic variable as in the quasigeostrophic approximation, are capable of providing solutions of formally increasing accuracy in powers of  $\epsilon$  in a systematic manner, and are straightforward to solve numerically. The IG models are obtained by iteration, at a fixed time  $t = t_0$ , of the momentum and density equations using the known pressure field  $\phi(\mathbf{x}, t_0)$ . The iteration procedure produces a sequence of estimates of increasing accuracy for the velocity components and for the time derivative of the pressure field  $\phi_t(\mathbf{x}, t_0)$ . The formulation is asymptotic in the sense that, given the pressure field, at each iteration the velocity components and  $\phi_t(\mathbf{x}, t_0)$  are formally determined to a higher order of accuracy in powers of  $\epsilon$ . The order of the IG model is specified by the predetermined fixed number of iterations  $N$ . Thus, a set of models is produced depending on the choice for  $N$ , and the different models are denoted by IG $N$ . The value of  $\phi_t(\mathbf{x}, t_0)$  obtained from iteration  $N$  is used with a time difference scheme to advance the pressure field in time, and the process may be repeated.

The IG model formulation has some similarity in concept to higher-order quasigeostrophic initialization

procedures for the primitive equations (see e.g., Daley 1991). The IG model formulation is also related to the calculation of higher-order approximations in quasigeostrophic (QG) theory. The fact that in IG the pressure field  $\phi(\mathbf{x}, t_0)$  is not expanded in  $\epsilon$ , however, leads to important differences between the IG models and higher-order QG approximations. These differences are discussed further in section 4. We note that the IG models have an advantage for oceanographic data assimilation applications in that, similar to QG, they use the pressure field, which may be calculated from density and surface height measurements, as the basic variable. This compares to the PE where the horizontal velocity components and the density field are the basic variables, all of which must be specified for initial-value problems. This is also different than the BE and BEM models, for which the streamfunction, which is not as readily measured as the pressure field, is the basic variable. A possible shortcoming of the IG models is that both energy and potential enstrophy conservation are asymptotic rather than exact, as discussed in section 4. The effect of this feature on IG solution accuracy must be assessed from comparative numerical experiments.

In the following companion paper (Allen and Newberger 1993), we evaluate the accuracy of several different intermediate models for continuously stratified fluids, including IG2, IG3, BEM, BE, and GM, by a comparison of numerical solutions to those from PE for a flow problem involving the time-dependent development of unstable jets on an  $f$  plane in a doubly periodic region. This problem is an idealized representation of observed Coastal Transition Zone (CTZ) synoptic-scale flow fields studied previously with the assumption of QG dynamics by Allen et al. (1991), Pierce et al. (1991), and Walstad et al. (1991). For moderate Rossby number flows, it is found that IG2 gives approximate solutions of reasonable accuracy, with errors substantially smaller than those obtained from QG and several other intermediate models. The IG3 model is found to give extremely accurate approximate solutions for flows with Rossby numbers that range from moderately small to  $O(1)$ .

The outline of this paper is as follows. In section 2, we record the form of the primitive equations used both here and in Allen and Newberger (1993). In section 3, the IG models are derived. Properties of the IG models are discussed in section 4, and some analytical examples are presented in section 5.

## 2. Primitive equations

We consider the motion of a rotating, continuously stratified fluid governed by the hydrostatic, Boussinesq, adiabatic PE with a spatially variable Coriolis parameter and with weak biharmonic momentum diffusion. The notation is similar to that in Allen (1991). Dimensionless variables are used so that in Cartesian coordinates  $(x, y, z)$  the PE are

$$\nabla_{3D} \cdot \mathbf{u}_{3D} = 0, \tag{2.1a}$$

$$\epsilon \frac{D\mathbf{u}}{Dt} + f\mathbf{k} \times \mathbf{u} = -\nabla\phi - \epsilon\nu\nabla^4\mathbf{u}, \tag{2.1b}$$

$$0 = -\phi_z + \theta, \tag{2.1c}$$

$$\frac{D\theta}{Dt} + Sw = 0, \tag{2.1d}$$

where, for some applications, it is convenient to utilize, in place of (2.1c) and (2.1d), the equivalent equations,

$$0 = -p_z - \rho, \tag{2.1c'}$$

$$\frac{D\rho}{Dt} = 0, \tag{2.1d'}$$

and where

$$\mathbf{x} = (x, y, z), \tag{2.2a}$$

$$\mathbf{u}_{3D} = (u, v, \epsilon w), \quad \mathbf{u} = (u, v, 0), \tag{2.2b, c}$$

$$\nabla_{3D} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right), \quad \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right), \tag{2.2d, e}$$

$$\frac{D}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{u}_{3D} \cdot \nabla_{3D} \right), \tag{2.2f}$$

and  $\mathbf{k}$  is the unit vector in the vertical  $z$  direction. Note that the three-dimensional velocity vector  $\mathbf{u}_{3D}$  and gradient operator  $\nabla_{3D}$  are labeled with the subscript  $3D$ , whereas the horizontal velocity vector  $\mathbf{u}$  and gradient operator  $\nabla$  are not subscripted.

Dimensionless variables are formed using the characteristic values ( $L, H, U_0, f_0$ ) for, respectively, a horizontal length scale, vertical depth scale, horizontal velocity, and Coriolis parameter. The Rossby number

$$\epsilon = U_0/f_0L. \tag{2.3}$$

With dimensional variables denoted by primes, we have

$$(x, y) = (x', y')/L, \quad z = z'/H, \tag{2.4a, b}$$

$$(u, v) = (u', v')/U_0, \quad \epsilon w = w'L/(U_0H), \tag{2.4c, d}$$

$$t = t'U_0/L, \quad f(x, y) = f'(x', y')/f_0, \tag{2.4e, f}$$

so that  $(u, v, \epsilon w)$  are the dimensionless velocity components in the  $(x, y, z)$  directions,  $t$  is time, and  $f$  is the dimensionless Coriolis parameter. The dimensionless biharmonic eddy diffusion coefficient  $\nu = \nu'/U_0L^3$ . We include weak biharmonic momentum diffusion in the formulation because it is used in the numerical finite-difference solutions in Allen and Newberger (1993) (following, e.g., Norton et al. 1986) to provide dissipation at high wavenumbers in otherwise nearly inviscid flows.

The total dimensional density is given by

$$\rho_T = \rho_0 + \bar{\rho}'(z') - \theta'(x', t'), \tag{2.5}$$

where  $\rho_0$  is a constant reference density,  $\bar{\rho}'(z')$  the basic

undisturbed  $z'$ -dependent field, and  $\theta'$  the negative of the density fluctuation. We define

$$\bar{\rho}(z) = \bar{\rho}'(z')/\rho_C, \quad \theta = \theta'/\rho_C, \tag{2.6a, b}$$

and

$$\rho = \bar{\rho}(z) - \theta, \tag{2.7}$$

where  $\rho_C = p_C/(Hg)$ ,  $p_C = \rho_0 U_0 f L$ , and  $g$  is the acceleration of gravity. In addition, we write

$$\bar{\rho}_z = -S(z)/\epsilon, \quad S(z) = N^2(z)H^2/(f_0^2 L^2), \tag{2.8a, b}$$

where

$$N^2(z) = -g\bar{\rho}'_z/\rho_0 \tag{2.8c}$$

is the square of the basic Brunt-Väisälä frequency. Subscripts  $(x, y, z, t)$  denote partial differentiation. Pressure variables  $p, \bar{p}$ , and  $\phi$  are defined by nondimensionalizing with  $p_C$  such that

$$p = \bar{p}(z) + \phi, \tag{2.9}$$

where

$$\bar{p}_z = -\bar{\rho}, \quad \nabla p = \nabla\phi. \tag{2.10a, b}$$

We are primarily interested in the limit of small Rossby number,

$$\epsilon \ll 1, \tag{2.11}$$

with, in general,  $S = O(1)$ . It will also be assumed that the variations in  $f$  satisfy

$$|\nabla f| = O(\epsilon). \tag{2.12}$$

It is useful to express (2.1b) in the equivalent form,

$$\epsilon\mathbf{u}_t + \epsilon^2 w\mathbf{u}_z + (\epsilon\zeta + f)\mathbf{k} \times \mathbf{u} = -\nabla\mathcal{B} - \epsilon\nu\nabla^4\mathbf{u}, \tag{2.13}$$

where the vertical component of vorticity  $\zeta$ , the Bernoulli function  $\mathcal{B}$ , and the horizontal kinetic energy density  $\epsilon K$  are given by

$$\zeta = \zeta^{(z)} = v_x - u_y, \tag{2.14a}$$

$$\mathcal{B} = \phi + \epsilon K, \tag{2.14b}$$

$$K = \frac{1}{2}(u^2 + v^2) = \frac{1}{2}\mathbf{u} \cdot \mathbf{u} = \frac{1}{2}\mathbf{u}^2. \tag{2.14c}$$

The horizontal velocity vector may be represented as the sum of rotational and divergent components, such that

$$\mathbf{u} = \mathbf{k} \times \nabla\psi + \epsilon\nabla\chi, \tag{2.15a}$$

$$= \mathbf{u}_R + \epsilon\mathbf{u}_D, \tag{2.15b}$$

where

$$\mathbf{u}_R = (u_R, v_R, 0) = (-\psi_y, \psi_x, 0), \tag{2.15c}$$

$$\mathbf{u}_D = (u_D, v_D, 0) = (\chi_x, \chi_y, 0), \tag{2.15d}$$

and

$$\begin{aligned} \zeta &= \zeta^{(z)} = v_x - u_y = v_{Rx} - u_{Ry} \\ &= \nabla^2 \psi = \psi_{xx} + \psi_{yy}, \end{aligned} \quad (2.16)$$

$$D = u_x + v_y = \epsilon(u_{Dx} + v_{Dy}) = \epsilon \nabla^2 \chi, \quad (2.17)$$

where  $D$  is the horizontal divergence. The continuity equation (2.1a) then gives

$$w_z = -\nabla^2 \chi. \quad (2.18)$$

Boundary conditions for (2.1) at rigid boundaries, such as at the surface, bottom, or sidewalls, are normal flow,

$$\mathbf{u}_{3D} \cdot \mathbf{n}_{3D} = 0 \quad \text{at } \mathbf{x} = \mathbf{x}_S, \quad (2.19)$$

where  $\mathbf{n}_{3D} = \mathbf{n}_{3D}(\mathbf{x}_S)$  is the outward-pointing unit vector normal to the boundary surface at  $\mathbf{x} = \mathbf{x}_S$ . In addition, for nearly inviscid motion with weak biharmonic momentum diffusion, we utilize the additional free-slip boundary conditions

$$\mathbf{n} \cdot \nabla(\mathbf{u} \cdot \mathbf{t}) = 0, \quad (\mathbf{n} \cdot \nabla)^3(\mathbf{u} \cdot \mathbf{t}) = 0, \quad (2.20a, b)$$

$$(\mathbf{n} \cdot \nabla)^2(\mathbf{u} \cdot \mathbf{n}) = 0, \quad \text{at } \mathbf{x} = \mathbf{x}_S, \quad (2.20c)$$

where  $\mathbf{n}$  is a unit vector aligned along the horizontal component of  $\mathbf{n}_{3D}$  and  $\mathbf{t} = \mathbf{k} \times \mathbf{n}$  is a unit horizontal vector tangent to the boundary. For use later, we note that at vertical-wall horizontal boundaries (2.19) and (2.20a, b, c) imply

$$\mathbf{t} \cdot \nabla \psi - \epsilon \mathbf{n} \cdot \nabla \chi = 0, \quad (2.21)$$

$$\begin{aligned} \nabla^2 \psi &= \nabla^4 \psi = \mathbf{n} \cdot \nabla(\nabla^2 \chi) = 0, \quad \text{at } \mathbf{x} = \mathbf{x}_S. \\ & \quad (2.22a, b, c) \end{aligned}$$

The equation for the vertical component of vorticity (2.16) derived from (2.13) is

$$\begin{aligned} \frac{D}{Dt}(\epsilon \zeta + f) + \epsilon(\epsilon \zeta + f)\nabla^2 \chi + \epsilon^2(w_x v_z - w_y u_z) \\ = -\epsilon \nu \nabla^4 \zeta, \end{aligned} \quad (2.23a)$$

or

$$\begin{aligned} \zeta_t + J(\psi, \zeta) + f \nabla^2 \chi + \epsilon^{-1} J(\psi, f) \\ + \epsilon \nabla \cdot [w \nabla \psi_z + \zeta \nabla \chi] + \nabla \chi \cdot \nabla f \\ + \epsilon^2 J(w, \chi_z) = -\nu \nabla^6 \psi, \end{aligned} \quad (2.23b)$$

where the operator  $J(a, b) = a_x b_y - a_y b_x$  is the Jacobian. The equation for the horizontal divergence derived from (2.13) is

$$\begin{aligned} \nabla^2 \phi - \nabla \cdot (f \nabla \psi) - \epsilon 2J(\psi_x, \psi_y) + \epsilon J(\chi, f) \\ + \epsilon^2 [\nabla^2 \chi_t + \nabla^2 J(\psi, \chi) - J(\zeta, \chi) - J(w, \psi_z)] \\ + \epsilon^3 \nabla \cdot \left[ w \nabla \chi_z + \frac{1}{2} \nabla (|\nabla \chi|^2) \right] = -\epsilon^2 \nu \nabla^6 \chi. \end{aligned} \quad (2.24)$$

In terms of  $\psi$  and  $\chi$ , (2.1d) is

$$\theta_t + J(\psi, \theta) + Sw + \epsilon[\nabla \cdot (\theta \nabla \chi) + (w\theta)_z] = 0. \quad (2.25)$$

In the inviscid limit ( $\nu \equiv 0$ ), the primitive equations (2.1a, b, c', d') imply the conservation of potential vorticity  $Q$  on fluid particles,

$$\frac{D}{Dt} Q = 0, \quad (2.26)$$

where

$$Q = (\epsilon \zeta_{3D} + f \mathbf{k}) \cdot \nabla_{3D} \rho, \quad (2.27)$$

$$\begin{aligned} \zeta_{3D} &= \nabla_{3D} \times \mathbf{u} = (\zeta^{(x)}, \zeta^{(y)}, \zeta^{(z)}) \\ &= (-v_z, u_z, v_x - u_y), \end{aligned} \quad (2.28)$$

and where  $\rho$  is defined in (2.7). The relation (2.26) may be combined with (2.1a) to give the conservation equation,

$$Q_t^2 + \nabla_{3D} \cdot (\mathbf{u}_{3D} Q^2) = 0, \quad (2.29)$$

for potential enstrophy density  $Q^2$ . A similar equation may be derived for an arbitrary function  $Fn(Q)$  that replaces  $Q^2$  in (2.29). In bounded regions with boundary condition (2.19), (2.29) implies conservation of volume integrals of  $Q^2$ . The inviscid PE (2.1a, b, c', d') also imply the energy conservation equation,

$$E_t + \nabla_{3D} \cdot [\mathbf{u}_{3D}(p + E)] = 0, \quad (2.30)$$

$$E = \frac{\epsilon}{2} \mathbf{u}^2 + (z - \bar{z}_0) \rho, \quad (2.31)$$

where  $E$  is the energy density,  $\bar{z}_0$  is a constant reference level,  $\rho$  is defined in (2.7), and  $p$  is defined in (2.9). In bounded regions with boundary condition (2.19), (2.30) implies the conservation of volume integrals of  $E$ .

### 3. Iterated geostrophic intermediate models

We formulate here a set of iterated geostrophic intermediate models under the assumption (2.11) that  $\epsilon \ll 1$ . We concentrate on applications to synoptic-scale flow fields. Consequently, we consider a  $\beta$ -plane-type approximation in which

$$f = 1 + \epsilon \delta f(x, y), \quad (3.1)$$

and we make approximations consistent with the  $\rho$  scaling in (2.7) and (2.8a).

We start by writing (2.13) and (2.1d) in the form

$$\begin{aligned} u &= -\phi_y + \epsilon[-v_t - (\zeta + \delta f)u - K_y - \nu \nabla^4 v] \\ & \quad - \epsilon^2 w v_z, \end{aligned} \quad (3.2a)$$

$$v = \phi_x + \epsilon[u_t - (\zeta + \delta f)v + K_x + \nu \nabla^4 u] + \epsilon^2 w u_z, \quad (3.2b)$$

$$w = -S^{-1}[\phi_{zt} + (u\phi_z)_x + (v\phi_z)_y + \epsilon(w\phi_z)_z]. \quad (3.2c)$$

The iterated geostrophic models are derived by considering (3.2a,b,c) as the defining equations for  $u$ ,  $v$ , and  $w$ , respectively, with  $\epsilon \ll 1$ . The pressure field  $\phi(\mathbf{x}, t)$  is the basic variable and is considered known at a given time  $t = t_0$ . Thus, the  $O(1)$  geostrophic approximations  $(-\phi_y, \phi_x)$  are known for the horizontal velocity components  $(u, v)$ . These  $O(1)$  values may be substituted in the  $O(\epsilon)$  and  $O(\epsilon^2)$  terms on the right-hand sides of (3.2a, b, c) [recall  $w$  is already scaled with  $\epsilon$  (2.2b)] to produce more accurate approximations for  $u, v, w$ . The requirement that the approximations for  $(u, v, w)$  satisfy (2.1a) gives an equation that determines an approximation for  $\phi_t$ . This process may be continued in an iterative manner by substituting the new approximations for  $u, v, w$ , and  $\phi_t$  in the right-hand sides of (3.2a, b, c). Increasingly accurate approximations to  $u, v$ , and  $w$  at  $t = t_0$ , which satisfy (2.1a), are obtained at each iteration of (3.2a, b, c). This iteration procedure simultaneously determines increasingly accurate approximations for  $\phi_t(\mathbf{x}, t_0)$ . The order of the model is denoted by a predetermined fixed number of iterations  $N$ . After the final estimate for  $\phi_t(\mathbf{x}, t_0)$  is obtained at iteration  $N$ , that value is used with a time difference approximation to obtain the pressure field  $\phi(\mathbf{x}, t_1)$  at  $t_1 = t_0 + \Delta t$ , where  $\Delta t$  is the time step of the difference approximation. The process may then be repeated to find  $\phi(\mathbf{x}, t_2)$  at  $t_2 = t_1 + \Delta t$ , etc. Note, the pressure field  $\phi(\mathbf{x}, t)$  is always assumed to be known and is not expanded in  $\epsilon$ .

We denote the variables corresponding to iteration number  $n = 1, 2, 3, \dots$ , with a subscript  $n$  (e.g.,  $u_n, v_n, w_n, \psi_n, \chi_n, \phi_n$ ), where

$$\mathbf{u}_{3Dn} = (u_n, v_n, \epsilon w_n), \quad \mathbf{u}_n = (u_n, v_n, 0), \quad (3.3a, b)$$

$$v_{nx} - u_{ny} = \zeta_n = \nabla^2 \psi_n, \quad (3.4a)$$

$$u_{nx} + v_{ny} = \epsilon \nabla^2 \chi_n. \quad (3.4b)$$

We require

$$\nabla_{3D} \cdot \mathbf{u}_{3Dn} = u_{nx} + v_{ny} + \epsilon w_{nz} = 0, \quad (3.5a)$$

so that

$$w_{nz} = -\nabla^2 \chi_n. \quad (3.5b)$$

In addition, boundary conditions (2.19), consistent with (3.5), are applied at each iteration (for  $n \geq 1$ ) so that

$$\mathbf{u}_{3Dn} \cdot \mathbf{n}_{3D} = 0 \quad \text{at} \quad \mathbf{x} = \mathbf{x}_S \quad (n \geq 1). \quad (3.6)$$

From (3.2a, b, c) we write

$$u_{n+1} = -\phi_y + \epsilon[-v_{nt} - (\zeta_n + \delta f)u_n - K_{ny} - \nu \nabla^4 v_n] - \epsilon^2 w_n v_{nz}, \quad (3.7a)$$

$$v_{n+1} = \phi_x + \epsilon[u_{nt} - (\zeta_n + \delta f)v_n + K_{nx} + \nu \nabla^4 u_n] + \epsilon^2 w_n u_{nz}, \quad (3.7b)$$

$$w_{n+1} = -S^{-1}[\phi_{ntz} + \nabla \cdot (\mathbf{u}_n \phi_z) + \epsilon(w_n \phi_z)_z], \quad (3.7c)$$

where  $n = 0, 1, 2, 3, \dots$ , and where

$$u_0 = -\phi_y, \quad v_0 = \phi_x, \quad w_0 = 0, \quad (3.8a, b, c)$$

$$\psi_0 = \phi, \quad \chi_0 = 0. \quad (3.9a, b)$$

The time derivatives  $v_{nt}$  and  $u_{nt}$  on the right-hand sides of (3.7a, b) are also evaluated iteratively by utilizing the time derivatives of (3.7a, b); that is,

$$u_{n+1t} = -\phi_{n+1ty} + \epsilon[-v_{ntt} - \zeta_n u_{nt} - (\zeta_n + \delta f)u_{nt}] - K_{ny} - \nu \nabla^4 v_{nt} - \epsilon^2[w_{nt} v_{nz} + w_n v_{ntz}], \quad (3.10a)$$

$$v_{n+1t} = \phi_{n+1tx} + \epsilon[u_{ntt} - \zeta_n v_{nt} - (\zeta_n + \delta f)v_{nt} + K_{tx}] + \nu \nabla^4 u_{nt} + \epsilon^2[w_{nt} u_{nz} + w_n u_{ntz}], \quad (3.10b)$$

for  $n = 0, 1, 2, 3, \dots$ , where

$$u_{0t} = -\phi_{0ty}, \quad v_{0t} = \phi_{0tx}, \quad w_{0t} = 0. \quad (3.11a, b, c)$$

From (3.7a, b) we obtain

$$\zeta_{n+1} = v_{n+1x} - u_{n+1y} = \nabla^2 \psi_{n+1} = \nabla^2 \phi + \epsilon \zeta'_{n+1}, \quad (3.12a)$$

where

$$\begin{aligned} \epsilon \zeta'_{n+1} = & \epsilon \{ (u_{nx} + v_{ny})_t + \nabla^2 K_n - [(\zeta_n + \delta f)v_n]_x \\ & + [(\zeta_n + \delta f)u_n]_y + \nu \nabla^4 (u_{nx} + v_{ny}) \} \\ & + \epsilon^2 \{ (w_n u_{nz})_x + (w_n v_{nz})_y \}. \end{aligned} \quad (3.12b)$$

Also, we obtain

$$\begin{aligned} \epsilon^{-1}(u_{n+1x} + v_{n+1y}) = & \nabla^2 \chi_{n+1} = -\nabla^2 \phi_{nt} \\ & - \nabla \cdot [(\zeta_n + \delta f)\mathbf{u}_n] - \nu \nabla^4 \zeta_n \\ & + \epsilon[-\zeta'_{nt} - (w_n v_{nz})_x + (w_n u_{nz})_y]. \end{aligned} \quad (3.13)$$

The condition (3.5), after the substitution of (3.13) and the  $z$  derivative of (3.7c), implies that  $\phi_{nt}$  satisfies the equation

$$\begin{aligned} \nabla^2 \phi_{nt} + (S^{-1} \phi_{ntz})_z = & -\nabla \cdot [(\zeta_n + \delta f)\mathbf{u}_n] \\ & - [S^{-1}[\nabla \cdot (\mathbf{u}_n \phi_z) + \epsilon(w_n \phi_z)_z]]_z - \nu \nabla^4 \zeta_n \\ & + \epsilon[-\zeta'_{nt} - (w_n v_{nz})_x + (w_n u_{nz})_y]. \end{aligned} \quad (3.14)$$

Values of  $\phi_{nt}$  are obtained from the solution of (3.14).

To facilitate comparison of the iterative geostrophic models with the balance equations and the BEM model (Allen 1991), it is useful to reformulate the approximations in terms of the variables  $\psi$  and  $\chi$  (2.15). These are also the variables used for all models in the computation of numerical solutions in Allen and Newberger (1993). Consequently, we record below (3.7a, b, c), (3.12), (3.13), and (3.14) written in terms of  $\psi$  and  $\chi$  [to simplify the notation, we omit the subscripts  $n$  from  $\psi, \chi$ , and  $\phi_t$  on the right-hand sides of (3.15)–(3.18) and will assume their presence is understood]:

$$u_{n+1} = -\phi_y + \epsilon\{-\psi_{xt} - J(\psi, \psi_x) + \delta f\psi_y - \nu\nabla^4\psi_x\} \\ + \epsilon^2\{-\chi_{yt} - w\psi_{xz} - \chi_x(\zeta + \delta f) - J_y(\psi, \chi) - \nu\nabla^4\chi_y\} + \epsilon^3\left\{-w\chi_{yz} - \frac{1}{2}(\chi_x^2 + \chi_y^2)_y\right\}, \quad (3.15a)$$

$$v_{n+1} = \phi_x + \epsilon\{-\psi_{yt} - J(\psi, \psi_y) - \delta f\psi_x - \nu\nabla^4\psi_y\} \\ + \epsilon^2\{\chi_{xt} - w\psi_{yz} - \chi_y(\zeta + \delta f) + J_x(\psi, \chi) + \nu\nabla^4\chi_x\} + \epsilon^3\left\{w\chi_{xz} + \frac{1}{2}(\chi_x^2 + \chi_y^2)_x\right\}, \quad (3.15b)$$

$$w_{n+1} = -S^{-1}[\phi_{zt} + J(\psi, \phi_z)] - \epsilon S^{-1}[\nabla \cdot (\phi_z \nabla \chi) + (w\phi_z)_z], \quad (3.15c)$$

and

$$\zeta_{n+1} = v_{n+1x} - u_{n+1y} = \nabla^2\psi_{n+1} = \nabla^2\phi + \epsilon\zeta'_{n+1}, \quad (3.16a)$$

$$\epsilon\zeta'_{n+1} = \epsilon\{-2J(\psi_x, \psi_y) - \nabla \cdot (\delta f \nabla \psi)\} + \epsilon^2\{\nabla^2\chi_t + J(\chi, \zeta + \delta f) + \nabla^2 J(\psi, \chi) - J(w, \psi_z) + \nu\nabla^6\chi\} \\ + \epsilon^3\left\{\nabla \cdot (w\nabla\chi_z) + \frac{1}{2}\nabla^2(\chi_x^2 + \chi_y^2)\right\}, \quad (3.16b)$$

$$\epsilon^{-1}(u_{n+1x} + v_{n+1y}) = \nabla^2\chi_{n+1} = -\nabla^2\phi_t - J(\psi, \zeta + \delta f) - \nu\nabla^6\psi + \epsilon\{-\zeta'_t - \nabla \cdot [w\nabla\psi_z + (\zeta + \delta f)\nabla\chi]\} \\ + \epsilon^2\{-J(w, \chi_z)\}, \quad (3.17)$$

where

$$\nabla^2\phi_{nt} + (S^{-1}\phi_{ntz})_z = -J(\psi, \zeta + \delta f) - [S^{-1}J(\psi, \phi_z)]_z - \nu\nabla^6\psi \\ + \epsilon\{-\zeta'_t - \nabla \cdot [w\nabla\psi_z + (\zeta + \delta f)\nabla\chi] - [S^{-1}[\nabla \cdot (\phi_z \nabla \chi) + (w\phi_z)_z]]_z\} + \epsilon^2\{-J(w, \chi_z)\}. \quad (3.18)$$

The order of a particular iterated geostrophic model is determined by the number of iterations  $N$ . We use the notation IG $N$  for the different models. In the following part of this section we derive models IG1, IG2, and IG3 utilizing the formulation involving  $\psi$  and  $\chi$ . For simplicity, we will assume that horizontal boundaries are vertical walls, so that boundary conditions (2.22) apply. The asymptotic nature of the IG model approximations is discussed further, with specific reference to IG2 and IG3, in section 4. Numerical solutions for IG2 and IG3 are obtained for doubly periodic  $f$ -plane problems in Allen and Newberger (1993).

#### a. Iterated geostrophic model IG1

In the formulation of these models we assume that  $t = t_0$  and that  $\phi(\mathbf{x}, t_0)$  is known. For consistency in notation, we first record the zeroth-order geostrophic approximation,

$$u_0 = -\phi_y, \quad v_0 = \phi_x, \quad w_0 = 0, \quad (3.19a, b, c)$$

$$\psi_0 = \phi, \quad \chi_0 = 0. \quad (3.20a, b)$$

From (3.15a, b, c) we obtain

$$u_1 = -\phi_y + \epsilon[-\phi_{0tx} - J(\phi, \phi_x) + \delta f\phi_y - \nu\nabla^4\phi_x], \quad (3.21a)$$

$$v_1 = \phi_x + \epsilon[-\phi_{0ty} - J(\phi, \phi_y) - \delta f\phi_x - \nu\nabla^4\phi_y], \quad (3.21b)$$

$$w_1 = -S^{-1}[\phi_{0tz} + J(\phi, \phi_z)]. \quad (3.21c)$$

The requirement that  $\mathbf{u}_{3D1}$  satisfy (3.5) gives

$$\nabla^2\phi_{0t} + (S^{-1}\phi_{0tz})_z = -J(\phi, \nabla^2\phi + \delta f) \\ - [S^{-1}J(\phi, \phi_z)]_z - \nu\nabla^6\phi. \quad (3.22)$$

Note IG1 has the same governing equation (3.22) as the QG approximation (e.g., Pedlosky 1987). The boundary condition (3.6) for  $\mathbf{u}_{3D1}$  in IG1, however, implies a condition on the normal derivative of  $\phi_{0t}$  and differs from the usual QG boundary condition. However, since (3.22) is derived from (3.5), the boundary condition (3.6) is quite naturally applied to (3.22), for example, in finite-difference form and results in a well-posed problem for  $\phi_{0t}$ . One implication of boundary condition (3.6) is that IG1 will support internal Kelvin waves (see section 5). For the frictional terms, (2.22a, b) give  $\nabla^2\phi = \nabla^4\phi = 0$  on vertical-wall horizontal boundaries. For the doubly periodic problems in Allen and Newberger (1993), IG1 solutions are the same as QG.

#### b. Iterated geostrophic model IG2

We obtain  $\psi_1$ ,  $\chi_1$ , and  $w_1$  from (3.16), (3.17), and (3.5b):

$$\zeta_1 = \nabla^2\psi_1 = \nabla^2\phi - \epsilon[2J(\phi_x, \phi_y) + \nabla \cdot (\delta f \nabla \phi)], \quad (3.23a)$$

$$\nabla^2\chi_1 = -\nabla^2\phi_{0t} - J(\phi, \nabla^2\phi + \delta f) - \nu\nabla^6\phi, \quad (3.23b)$$

$$w_{1z} = -\nabla^2\chi_1. \quad (3.23c)$$

The equations for  $u_2$ ,  $v_2$ , and  $w_2$  follow from (3.15a, b, c) and are

$$u_2 = -\phi_y + \epsilon\{-\psi_{1tx} - J(\psi_1, \psi_{1x}) + \delta f\psi_{1y} - \nu\nabla^4\psi_{1x}\} + \epsilon^2\{-\chi_{1yt} - w_1\psi_{1xz} - \chi_{1x}(\zeta_1 + \delta f) - J_y(\psi_1, \chi_1) - \nu\nabla^4\chi_{1y}\} + \epsilon^3\left\{-w_1\chi_{1yz} - \frac{1}{2}(\chi_{1x}^2 + \chi_{1y}^2)_y\right\}, \quad (3.24a)$$

$$v_2 = \phi_x + \epsilon\{-\psi_{1ty} - J(\psi_1, \psi_{1y}) - \delta f\psi_{1x} - \nu\nabla^4\psi_{1y}\} + \epsilon^2\{\chi_{1xt} - w_1\psi_{1yz} - \chi_{1y}(\zeta_1 + \delta f) + J_x(\psi_1, \chi_1) + \nu\nabla^4\chi_{1x}\} + \epsilon^3\left\{w_1\chi_{1xz} + \frac{1}{2}(\chi_{1x}^2 + \chi_{1y}^2)_x\right\}, \quad (3.24b)$$

$$w_2 = -S^{-1}[\phi_{1tz} + J(\psi_1, \phi_z)] - \epsilon S^{-1}[\nabla \cdot (\phi_z \nabla \chi_1) + (w_1 \phi_z)_z]. \quad (3.24c)$$

Note that, alternatively on the right-hand sides of (3.24a, b),  $v_{1t} = \psi_{1tx} + \epsilon\chi_{1ty}$  and  $u_{1t} = -\psi_{1ty} + \epsilon\chi_{1tx}$  may be found directly by taking the time derivatives of (3.21a, b). We obtain  $\psi_{1t}$ ,  $\chi_{1t}$ , and  $w_{1t}$  from

$$\nabla^2\psi_{1t} = \nabla^2\phi_{1t} + \epsilon\zeta'_{1t}, \quad (3.25a)$$

$$\zeta'_{1t} = -2[J(\phi_{0tx}, \phi_y) + J(\phi_x, \phi_{0ty})] - \nabla \cdot (\delta f \nabla \phi_{0t}), \quad (3.25b)$$

$$\nabla^2\chi_{1t} = -\nabla^2\phi_{0t} - J(\phi_{0t}, \nabla^2\phi + \delta f) - J(\phi, \nabla^2\phi_{0t}) - \nu\nabla^6\phi_{0t}, \quad (3.26a)$$

$$w_{1tz} = -\nabla^2\chi_{1t}, \quad (3.26b)$$

which follow from the time derivatives of (3.23a, b, c). In addition, we obtain  $\phi_{0t}$  from

$$\nabla^2\phi_{0t} + (S^{-1}\phi_{0tz})_z = -J(\phi_{0t}, \nabla^2\phi + \delta f) - J(\phi, \nabla^2\phi_{0t}) - [S^{-1}[J(\phi_{0t}, \phi_z) + J(\phi, \phi_{0tz})]]_z - \nu\nabla^6\phi_{0t}, \quad (3.27)$$

which follows from the time derivative of (3.22).

The requirement that  $\mathbf{u}_{3D2}$  satisfies (3.5) gives the equation for  $\phi_{1t}$ :

$$\nabla^2\phi_{1t} + (S^{-1}\phi_{1tz})_z = -J(\psi_1, \zeta_1 + \delta f) - [S^{-1}J(\psi_1, \phi_z)]_z - \nu\nabla^6\psi_1 + \epsilon\{-\zeta'_{1t} - \nabla \cdot [w_1\nabla\psi_{1z} + (\zeta_1 + \delta f)\nabla\chi_1] - [S^{-1}[\nabla \cdot (\phi_z \nabla \chi_1) + (w_1\phi_z)_z]]_z + \epsilon^2\{-J(w_1, \chi_{1z})\}. \quad (3.28)$$

Boundary conditions on the normal derivative of  $\phi_{1t}$  in (3.28) are obtained from (3.6) with  $\mathbf{u}_{3D2}$  from (3.24). Boundary conditions on the normal derivative of  $\phi_{0t}$  in (3.27) may be obtained from (3.6) with  $\mathbf{u}_{3D1t}$ , where on the boundary  $\mathbf{u}_{3D1t}$  is approximated by the

time derivative of (3.21) using  $\phi_{0t}$  in the leading-order term. This differs from (3.10) but should be a consistent approximation because of the ordering of terms on the boundary implied by (3.6). In particular, it allows the solution of (3.27) for  $\phi_{0t}$  to be obtained independently of the solution of (3.28) for  $\phi_{1t}$ . For the friction terms, (2.22a, b) imply  $\nabla^2\psi_1 = \nabla^4\psi_1 = 0$  on the vertical-wall horizontal boundaries.

### c. Iterated geostrophic model IG3

We find  $\psi_2$ ,  $\chi_2$ , and  $w_2$  from (3.16), (3.17), and (3.5b):

$$\zeta_2 = \nabla^2\psi_2 = \nabla^2\phi + \epsilon\{-2J(\psi_{1x}, \psi_{1y}) - \nabla \cdot (\delta f \nabla \psi_1)\} + \epsilon^2\{\nabla^2\chi_{1t} + J(\chi_1, \zeta_1 + \delta f) + \nabla^2J(\psi_1, \chi_1) - J(w_1, \psi_{1z}) + \nu\nabla^6\chi_1\} + \epsilon^3\left\{\nabla \cdot (w_1\nabla\chi_{1z}) + \frac{1}{2}\nabla^2(\chi_{1x}^2 + \chi_{1y}^2)\right\}, \quad (3.29)$$

$$\nabla^2\chi_2 = -\nabla^2\phi_{1t} - J(\psi_1, \zeta_1 + \delta f) - \nu\nabla^6\psi_1 + \epsilon\{-\zeta'_{1t} - \nabla \cdot [w_1\nabla\psi_{1z} + (\zeta_1 + \delta f)\nabla\chi_1]\} + \epsilon^2\{-J(w_1, \chi_{1z})\}, \quad (3.30a)$$

$$w_{2z} = -\nabla^2\chi_2. \quad (3.30b)$$

The equations for  $u_3$ ,  $v_3$ , and  $w_3$  follow from (3.15a, b, c) and are

$$u_3 = -\phi_y + \epsilon\{-\psi_{2tx} - J(\psi_2, \psi_{2x}) + \delta f\psi_{2y} - \nu\nabla^4\psi_{2x}\} + \epsilon^2\{-\chi_{2yt} - w_2\psi_{2xz} - \chi_{2x}(\zeta_2 + \delta f) - J_y(\psi_2, \chi_2) - \nu\nabla^4\chi_{2y}\} + \epsilon^3\left\{-w_2\chi_{2yz} - \frac{1}{2}(\chi_{2x}^2 + \chi_{2y}^2)_y\right\}, \quad (3.31a)$$

$$v_3 = \phi_x + \epsilon\{-\psi_{2ty} - J(\psi_2, \psi_{2y}) - \delta f\psi_{2x} - \nu\nabla^4\psi_{2y}\} + \epsilon^2\{\chi_{2xt} - w_2\psi_{2yz} - \chi_{2y}(\zeta_2 + \delta f) + J_x(\psi_2, \chi_2) + \nu\nabla^4\chi_{2x}\} + \epsilon^3\left\{w_2\chi_{2xz} + \frac{1}{2}(\chi_{2x}^2 + \chi_{2y}^2)_x\right\}, \quad (3.31b)$$

$$w_3 = -S^{-1}[\phi_{2tz} + J(\psi_2, \phi_z)] - \epsilon S^{-1}[\nabla \cdot (\phi_z \nabla \chi_2) + (w_2\phi_z)_z]. \quad (3.31c)$$

We obtain  $\psi_{2t}$  from the time derivative of (3.29):

$$\nabla^2\psi_{2t} = \nabla^2\phi_{2t} + \epsilon\zeta'_{2t}, \quad (3.32a)$$

$$\zeta'_{2t} = -2[J(\psi_{1tx}, \psi_{1y}) + J(\psi_{1x}, \psi_{1ty})] - \nabla \cdot (\delta f \nabla \psi_{1t}) + \epsilon\{\nabla^2\chi_{1t} + J(\chi_{1t}, \zeta_1 + \delta f) + J(\chi_1, \zeta_{1t}) + \nabla^2J(\psi_{1t}, \chi_1) + \nabla^2J(\psi_1, \chi_{1t}) - J(w_{1t}, \psi_{1z}) - J(w_1, \psi_{1tz}) + \nu\nabla^6\chi_{1t}\}$$

$$+ \epsilon^2 \{ \nabla \cdot (w_{1t} \nabla \chi_{1z}) + \nabla \cdot (w_1 \nabla \chi_{1zt}) \\ + \nabla^2 (\chi_{1x} \chi_{1xt} + \chi_{1y} \chi_{1yt}) \}. \quad (3.32b)$$

The determination of  $\zeta'_{2t}$  requires  $\nabla^2 \chi_{1tt}$ , which may be obtained from the time derivative of (3.26a):

$$\nabla^2 \chi_{1tt} = -\nabla^2 \phi_{0tt} - J(\phi_{0tt}, \nabla^2 \phi + \delta f) \\ - 2J(\phi_{0t}, \nabla^2 \phi_{0t}) - J(\phi, \nabla^2 \phi_{0tt}) - \nu \nabla^6 \phi_{0tt}, \quad (3.33a)$$

where  $\phi_{0tt}$  is obtained from

$$\nabla^2 \phi_{0tt} + (S^{-1} \phi_{ttz})_z = -J(\phi_{0tt}, \nabla^2 \phi + \delta f) \\ - 2J(\phi_{0t}, \nabla^2 \phi_{0t}) - J(\phi_0, \nabla^2 \phi_{0tt}) \\ - \nu \nabla^6 \phi_{0tt} - [S^{-1} [J(\phi_{0tt}, \phi_z) \\ + 2J(\phi_{0t}, \phi_{0zt}) + J(\phi, \phi_{0ttz})]]_z, \quad (3.33b)$$

which follows from the time derivative of (3.27).

The requirement that  $\mathbf{u}_{3D3}$  satisfies (3.5) gives

$$\nabla^2 \phi_{2t} + (S^{-1} \phi_{2tz})_z = -J(\psi_2, \zeta_2 + \delta f) \\ - [S^{-1} J(\psi_2, \phi_z)]_z - \nu \nabla^6 \psi_2 \\ + \epsilon \{ -\zeta'_{2t} - \nabla \cdot [w_2 \nabla \psi_{2z} + (\zeta_2 + \delta f) \nabla \chi_2] \\ - [S^{-1} [\nabla \cdot (\phi_z \nabla \chi_2) + (w_2 \phi_z)_z]]_z \} \\ + \epsilon^2 \{ -J(w_2, \chi_{2z}) \}. \quad (3.34)$$

Boundary conditions on the normal derivative of  $\phi_{2t}$  in (3.34) are obtained from (3.6) with  $\mathbf{u}_{3D3}$  from (3.31). Boundary conditions on the normal derivative of  $\phi_{0tt}$  in (3.33) may be obtained from (3.6) with  $\mathbf{u}_{3D1tt}$ , where  $\mathbf{u}_{3D1tt}$  on the boundary is approximated by the second time derivative of (3.21) using  $\phi_{0tt}$  in the leading-order term. For the friction terms, (2.22a, b, c) imply  $\nabla^2 \psi_2 = \nabla^4 \psi_2 = \mathbf{n} \cdot \nabla (\nabla^2 \chi_1) = 0$  on vertical-wall horizontal boundaries.

Note that for application of boundary condition (3.6) evaluation of the velocity components  $u_3$  and  $v_3$  is necessary, and we also need  $\chi_{2t}$ , which may be obtained from the time derivative of (3.30a). That, in turn, requires  $\phi_{1tt}$  and  $\zeta'_{1tt}$ , which may be found from the time derivatives of (3.28) and (3.25b), respectively. Boundary conditions for  $\phi_{1tt}$  are obtained from (3.6) with  $\mathbf{u}_{3D2t}$ , where  $\mathbf{u}_{3D2t}$  on the boundary is approximated by the time derivative of (3.24) using  $\phi_{1t}$  in the leading-order term. The determination of  $\chi_{2t}$  is not required in the numerical solutions for IG3 calculated by Allen and Newberger (1993) in a periodic domain since it does not appear in (3.34) and since the evaluation of  $u_3$ ,  $v_3$ , and  $w_3$  is not necessary. Consequently, for brevity we omit writing the equations for  $\chi_{2t}$ ,  $\phi_{1tt}$ , and  $\zeta'_{1tt}$ , which may, in any case, be easily derived.

In a periodic domain, once  $\phi_{2t}$  is found from the solution of (3.34),  $\chi_3$  (and also  $w_3$ ) may be readily obtained from

$$\nabla^2 \chi_3 = -\nabla^2 \phi_{2t} - J(\psi_2, \zeta_2 + \delta f) - \nu \nabla^6 \psi_2 \\ + \epsilon \{ -\zeta'_{2t} - \nabla \cdot [w_2 \nabla \psi_{2z} + (\zeta_2 + \delta f) \nabla \chi_2] \} \\ + \epsilon^2 \{ -J(w_2, \chi_{2z}) \}. \quad (3.35)$$

#### 4. Discussion

Some of the general properties of the iterated geostrophic models may be seen most easily by looking specifically at IG1, IG2, and IG3. In IG1,  $\phi_{0t}$  is determined by (3.22) to  $O(1)$ , and  $u_1$ ,  $v_1$  in (3.21a, b) are determined to  $O(\epsilon)$ . Accordingly,  $\psi_1$  (3.23a) is determined to  $O(\epsilon)$ , and  $\chi_1$  (3.23b) and  $w_1$  (3.23c) to  $O(1)$ . In IG2,  $\phi_{1t}$  is determined by (3.28) to  $O(\epsilon)$  since on the right-hand side of (3.28) all of the  $O(\epsilon)$  terms are included and all of the variables in the  $O(1)$  terms are known to  $O(\epsilon)$ . Also,  $u_2$  and  $v_2$  (3.24a, b) are determined to  $O(\epsilon^2)$ ,  $\psi_2$  (3.29) to  $O(\epsilon^2)$ , and  $\chi_2$  (3.30a) and  $w_2$  (3.30b) to  $O(\epsilon)$ . We note that the velocity components  $u_2$  and  $v_2$  give the horizontal divergence in the continuity equation (3.5) that, together with  $w_{2z}$ , implies the equation (3.28) for  $\phi_{1t}$ . The horizontal divergence of  $u_2$  and  $v_2$  also occurs in the vorticity equation (3.30a). On the other hand, in IG2 the effective advection velocities in the vorticity (3.30a) and heat equations [combined in (3.28)] are  $u_1$ ,  $v_1$ , and  $w_1$ .

Proceeding in a similar manner for IG3, it can be argued that  $\phi_{2t}$  is determined by (3.34) to  $O(\epsilon^2)$ ,  $u_3$  and  $v_3$  by (3.31a, b) to  $O(\epsilon^3)$ ,  $\psi_3$  to  $O(\epsilon^3)$ , and  $\chi_3$  and  $w_3$  (3.35) to  $O(\epsilon^2)$ . Again, the divergence of  $u_3$ ,  $v_3$ ,  $w_3$  satisfies the continuity equation (3.5) that results in (3.34) for  $\phi_{2t}$ , and the horizontal divergence of  $u_3$  and  $v_3$  occurs in the vorticity equation (3.35). The advection velocities in the vorticity (3.35) and density equations [combined in (3.34)] are  $u_2$ ,  $v_2$ , and  $w_2$ . It is clear that at each iteration  $u_n$ ,  $v_n$ , (and  $\psi_n$ ) are determined to  $O(\epsilon^n)$ , and  $\phi_{n-1t}$ ,  $w_n$ , (and  $\chi_n$ ) to  $O(\epsilon^{n-1})$ . It is also clear that this procedure can be continued in a systematic manner for any number of iterations  $N$  to find formally increasingly accurate approximations in powers of  $\epsilon$  to  $\phi_{nt}$ .

For IG1, we obtain  $\phi_{0t}$  from the solution to (3.22). For IG2, we find  $\phi_{1t}$  from (3.28) and  $\phi_{0tt}$  from (3.27). It is useful to outline the solution procedures for IG1, IG2, and IG3 in terms of the required inversions of the linear QG operator defined by

$$\mathcal{L}(\phi_t) = \nabla^2 \phi_t + (S^{-1} \phi_{tz})_z. \quad (4.1)$$

We summarize the equations involving the operator  $\mathcal{L}$  for IG1, IG2, and IG3 below, using the notation  $R_0$ ,  $R_1$ , and  $R_2$  for the right-hand sides of (3.22), (3.28), and (3.34), respectively,

IG1:

$$\mathcal{L}(\phi_{0t}) = R_0; \quad (4.2)$$



IG2:

$$\mathcal{L}(\phi_{1t}) = R_1, \tag{4.3a}$$

$$\mathcal{L}(\phi_{0t}) = R_{0t}; \tag{4.3b}$$

IG3:

$$\mathcal{L}(\phi_{2t}) = R_2, \tag{4.4a}$$

$$\mathcal{L}(\phi_{0t}) = R_{0t}, \tag{4.4b}$$

$$\mathcal{L}(\phi_{1t}) = R_{1t}. \tag{4.4c}$$

We see that in general IG2 requires three inversions of  $\mathcal{L}$  per time step (for  $\phi_{0t}$ ,  $\phi_{1t}$ ,  $\phi_{0t}$ ). IG3 requires these three inversions plus an additional three (for  $\phi_{2t}$ ,  $\phi_{0t}$ ,  $\phi_{1t}$ ), giving a total of six inversions of  $\mathcal{L}$  per time step. For numerical solutions in a horizontally doubly periodic domain, as in Allen and Newberger (1993), IG2 requires only two inversions of  $\mathcal{L}$  per time step (for  $\phi_{0t}$ ,  $\phi_{1t}$ ), while IG3 requires five (for  $\phi_{0t}$ ,  $\phi_{1t}$ ,  $\phi_{0t}$ ,  $\phi_{2t}$ ,  $\phi_{0t}$ ). The simplicity relative to other intermediate models in the application of IG2, which requires only three (or two) inversions of the linear QG operator  $\mathcal{L}$  per time step, may be noted (Allen and Newberger 1993).

The IG model formulation has some similarity to the derivation of higher-order corrections in the quasi-geostrophic approximation. The fact that the pressure field is not expanded in the IG models, however, provides an important difference. For comparison with IG2, we record in appendix A the second-order quasi-geostrophic approximation (QG2), which includes the lowest (first)-order QG solution and the next (second)-order correction. Note that in QG2 the lowest-order QG approximation  $\phi_{(0)}$  evolves independently of the second-order correction  $\phi_{(1)}$ . In contrast, in the IG models the effects of higher-order terms are included in the estimate of the pressure time derivative and are thus incorporated in the basic pressure field at each time step. Consequently, the IG models can account for the cumulative effect on the flow evolution of small terms acting over large times in a manner that the higher-order QG approximations cannot. Numerical solutions to QG2 are included in Allen and Newberger (1993) to directly demonstrate the differences in results compared to IG2 and the other models. It is found that in general IG2 is substantially more accurate than QG2. The differences between IG2 and QG2 are also illustrated in section 5 by application of both approximations to the rotating elliptical eddy (Rodon) problem for the shallow-water equations.

It is useful to consider the manner in which the IG equations represent a steady, time-independent solution. Consider IG2 in the inviscid limit ( $\nu = 0$ ). A steady solution for IG2 would require  $\phi_{1t} = 0$ . Note that, by a comparison of (3.28) for  $\phi_{1t}$  and (3.22) for  $\phi_{0t}$ , this would in general be accompanied by  $\phi_{0t} \neq 0$ .

However, since the difference between the right-hand sides of (3.28) and (3.22) is  $O(\epsilon)$ ,  $\phi_{1t} = 0$  will imply  $\phi_{0t} = O(\epsilon)$ . This in turn from (3.25) will imply  $\psi_{1t} = O(\epsilon^2)$ . Since  $\phi_{1t}$  is determined to  $O(\epsilon)$ ,  $\psi_{1t} = O(\epsilon^2)$  is consistent with  $\phi_{1t} = 0$ . For IG3, a steady state requires  $\phi_{2t} = 0$ . In a similar manner, this implies  $\phi_{1t} = O(\epsilon^2)$  and consequently  $\psi_{2t} = O(\epsilon^3)$ , consistent with  $\phi_{2t} = 0$ .

The iteration procedure that we have defined in (3.7) for  $\mathbf{u}_{3Dn+1}$  involves keeping all terms involving  $\mathbf{u}_{3Dn}$  on the right-hand sides of (3.7a, b, c) regardless of relative magnitudes in terms of powers of  $\epsilon$ . This seems desirable because it involves little extra work and it results in a clean formulation with the same form for the equations (3.7) at each iteration. This also allows a straightforward assessment of errors in an evaluation of model conservation of potential vorticity on fluid particles and conservation of volume integrals of energy. Using the preceding arguments with regard to order of determination of  $\mathbf{u}_{3Dn}$  in powers of  $\epsilon$ , we define

$$\mathbf{u}_{3Dn+1} - \mathbf{u}_{3Dn} = \epsilon^{n+1} \mathbf{u}'_{3Dn}, \tag{4.5}$$

where  $\mathbf{u}'_{3Dn} = O(1)$ . The iteration equations (3.7) may then be written (for  $\nu = 0$ ) as,

$$\epsilon \frac{D_n \mathbf{u}_n}{Dt} + f \mathbf{k} \times \mathbf{u}_n = -\nabla \phi - \epsilon^{n+1} f \mathbf{k} \times \mathbf{u}'_n, \tag{4.6a}$$

$$\frac{D_n \theta}{Dt} + S w_n = -\epsilon^{n+1} S w'_n, \tag{4.6b}$$

where

$$\frac{D_n}{Dt} = \left( \frac{\partial}{\partial t} + \mathbf{u}_{3Dn} \cdot \nabla \right). \tag{4.6c}$$

Thus, for IG $N$ , where  $N = n + 1$  in (4.6), the momentum (4.6a) and thermodynamic (4.6b) equations are the same in the  $\mathbf{u}_{3Dn-1}$  variables as the PE equations (2.1b, d) with error terms of  $O(\epsilon^N)$ . This implies that IG $N$  will have conservation equations for potential vorticity and energy involving the  $N - 1$  variables, that is, for  $Q_{N-1}$ ,  $E_{N-1}$ , with error terms of  $O(\epsilon^N)$ . As a consequence, potential vorticity and energy conservation equations for the iterated geostrophic models in general are asymptotic and not exact (McWilliams and Gent 1980). The consequences of that fact for the performance of the IG models have to be determined by an assessment of the IG models' solution accuracy. In Allen and Newberger (1993) it is shown that IG3 can produce extremely accurate approximate solution to PE, while IG2 can give reasonable accuracy that is a great deal better than that of QG and of several other intermediate models. These results indicate that possession of asymptotic, rather than exact, conservation laws may not be of critical importance for intermediate model performance, at least in the mesoscale-type problems considered in Allen and Newberger (1993).

An additional notable feature of the IG model formulation is the requirement (3.6) that  $\mathbf{u}_{3Dn}$  ( $n \geq 1$ ) satisfy the no-normal flow boundary condition at rigid surfaces. Since the governing equation for  $\phi_n$  (3.14) is obtained from the continuity equation (3.5) for  $\mathbf{u}_{3Dn+1}$ , the correct boundary conditions for (3.14) may be readily applied during the derivation of (3.14) in finite-difference form by incorporating (3.6) in the formulation. Note that satisfaction of the boundary conditions (3.6) also automatically implies that for  $\nu \equiv 0$  all flux terms involving  $\mathbf{u}_{3Dn}$  ( $n \geq 1$ ) in (4.6a, b), or in the implied equations for  $E_n$  and  $Q_n^2$ , are correctly equal to zero on rigid boundaries where (3.6) applies. Note that this result does not hold for the  $n = 0$  variables in IG1 since  $(u_0, v_0)$  do not satisfy (3.6).

Although the formulation of the IG models in section 3 has been for application to synoptic-scale flow fields, it seems that a similar procedure could be followed for larger-scale flows on gyre scales. The basic assumption would remain  $\epsilon \ll 1$  (2.11), where the length scale in  $\epsilon$  is characteristic of the synoptic scale. In addition, the assumption  $|\nabla f| = O(\epsilon)$  (2.12) would be retained. The equations (3.2) used in the iteration procedure would be modified to allow for  $O(1)$  variations in  $f$  and in the vertical structure of the density field over the larger gyre scale by writing

$$fu = -\phi_y + \epsilon[-v_t - \zeta u - K_y - \nu \nabla^4 v] - \epsilon^2 wv_z, \tag{4.7a}$$

$$fv = \phi_x + \epsilon[u_t - \zeta v + K_x + \nu \nabla^4 u] + \epsilon^2 wu_z, \tag{4.7b}$$

$$w = \hat{S}^{-1}[\phi_{zt} + u\phi_{zx} + v\phi_{zy}], \tag{4.7c}$$

where

$$\hat{S}(\mathbf{x}, t) = S(z) + \epsilon\phi_z. \tag{4.7d}$$

A sequence of gyre-scale IG models could be produced by iteration of (4.7) following a procedure similar to that in section 3.

**5. Examples**

A few simple analytical solutions to the IG models may be found for idealized problems. Three of these that illustrate the nature of the IG approximations are given in the following.

*a. Circular vortex*

First, we consider inviscid ( $\nu \equiv 0$ ), steady flow on an  $f$  plane ( $\delta f = 0$ ) for a  $z$ -independent circular vortex with a pressure field given by

$$\phi = \frac{1}{4}(x^2 + y^2). \tag{5.1}$$

The only relevant PE equation in this idealized situation is (2.24), which simplifies to

$$\nabla^2 \phi - \nabla^2 \psi - \epsilon 2J(\psi_x, \psi_y) = 0. \tag{5.2}$$

With  $\phi$  given by (5.1), the streamfunction that satisfies (5.2) has the form,

$$\psi = \frac{\Psi_{PE}}{4}(x^2 + y^2), \tag{5.3}$$

where the constant  $\Psi_{PE}$  satisfies

$$\frac{1}{2} \epsilon \Psi_{PE}^2 + \Psi_{PE} - 1 = 0, \tag{5.4}$$

with solution

$$\Psi_{PE} = \epsilon^{-1}[-1 + (1 + 2\epsilon)^{1/2}]. \tag{5.5}$$

The IG models are obtained from (5.2) by iteration in the form

$$\nabla^2 \psi_{n+1} = \nabla^2 \phi - \epsilon 2J(\psi_{nx}, \psi_{ny}), \quad n = 0, 1, \dots, \tag{5.6}$$

where  $\psi_0 = \phi$ . The IG model solutions found from (5.6) are identical to those obtained by solving (5.4) for  $\Psi_{PE}$  by iteration for  $\epsilon \ll 1$ . With

$$\psi_n = \frac{\Psi_n}{4}(x^2 + y^2), \tag{5.7}$$

we find

$$\Psi_0 = 1, \quad \Psi_1 = 1 - \frac{1}{2} \epsilon, \tag{5.8a, b}$$

$$\Psi_2 = 1 - \frac{1}{2} \epsilon \Psi_1^2, \tag{5.8c}$$

$$\Psi_3 = 1 - \frac{1}{2} \epsilon \Psi_2^2. \tag{5.8d}$$

We plot  $\Psi_{PE}$  and  $\Psi_n$  ( $n = 0, \dots, 3$ ) as a function of  $\epsilon$  in Fig. 1. The improvement in the approximation of  $\Psi_n$  to  $\Psi_{PE}$  for larger  $|\epsilon|$  as  $n$  increases is clearly shown.

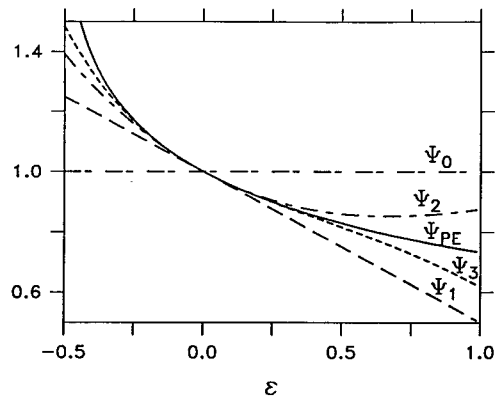


FIG. 1. Amplitude of the streamfunction as a function of  $\epsilon$  in the steady circular vortex problem with pressure field (5.1) for the PE [ $\Psi_{PE}$  in (5.3)] and for the IG models [ $\Psi_n$  in (5.7)] with  $n = 0, \dots, 3$ .

*b. Linear internal Kelvin wave*

The next example is a linear internal Kelvin wave on an  $f$  plane ( $\delta f = 0$ ). The linear PE are (2.1a, c) and (2.1a, c) and

$$\epsilon u_t - v = -\phi_x, \tag{5.9a}$$

$$\epsilon v_t + u = -\phi_y, \tag{5.9b}$$

$$\phi_{zt} + Sw = 0. \tag{5.9c}$$

The IG models are obtained from

$$u_{n+1} = -\phi_y + \epsilon[-v_{nt}], \tag{5.10a}$$

$$v_{n+1} = \phi_x + \epsilon[u_{nt}], \tag{5.10b}$$

$$w_{n+1} = -S^{-1}\phi_{ntz}. \tag{5.10c}$$

Iteration of (5.10) gives

IG1:

$$u_1 = -\phi_y + \epsilon[-\phi_{0tx}], \quad v_1 = \phi_x + \epsilon[-\phi_{0ty}], \tag{5.11a, b}$$

$$\mathcal{L}(\phi_{0t}) = \nabla^2\phi_{0t} + (S^{-1}\phi_{0tz})_z = 0; \tag{5.12}$$

IG2:

$$u_2 = -\phi_y + \epsilon[-\phi_{1tx} + \epsilon\phi_{0uty}], \tag{5.13a, b}$$

$$v_2 = \phi_x + \epsilon[-\phi_{1ty} - \epsilon\phi_{0ux}], \tag{5.13a, b}$$

$$\mathcal{L}(\phi_{1t}) = 0, \quad \mathcal{L}(\phi_{0ut}) = 0; \tag{5.14a, b}$$

IG3:

$$u_3 = -\phi_y + \epsilon[-\phi_{2tx} + \epsilon\phi_{1uy} + \epsilon^2\phi_{0oux}], \tag{5.15a}$$

$$v_3 = \phi_x + \epsilon[-\phi_{2ty} - \epsilon\phi_{1ux} + \epsilon^2\phi_{0uty}], \tag{5.15b}$$

$$\mathcal{L}(\phi_{2t}) = \epsilon^2\nabla^2\phi_{0ut}, \quad \mathcal{L}(\phi_{0ut}) = 0, \quad \mathcal{L}(\phi_{1ut}) = 0. \tag{5.16a, b, c}$$

We assume that the fluid occupies the semi-infinite region ( $-\infty < x < \infty, 0 \leq y < \infty, -1 \leq z \leq 0$ ). The variables are expanded in vertical normal modes; for example,

$$\phi = \sum_{j=0}^{\infty} \Phi_{(j)}(x, y, t)g_{(j)}(z), \tag{5.17}$$

where

$$(S^{-1}g_{(j)z})_z + \lambda_{(j)}^2g_{(j)} = 0, \tag{5.18a}$$

$$g_{(j)z} = 0 \quad \text{at} \quad z = 0, -1. \tag{5.18b}$$

Henceforth, we will consider only the first baroclinic mode ( $j = 1$ ) and drop the subscript ( $j$ ) on  $\Phi, g$ , and  $\lambda$  ( $\lambda \geq 0$ ).

The exact first baroclinic mode internal Kelvin wave solution to the linear PE (5.9) with sinusoidal varia-

tions in  $x$  that satisfies  $v(y = 0) = 0$  and  $\Phi(y \rightarrow \infty) = 0$  is, in terms of the pressure,

$$\Phi = \exp[-\lambda y + i(kx - \sigma_{PE}t)], \tag{5.19a}$$

where

$$\epsilon\sigma_{PE} = k/\lambda, \tag{5.19b}$$

and where we assume unit amplitude for  $\Phi(y = 0)$ .

The corresponding Kelvin wave solutions to the IG models are obtained from (5.11)–(5.16). The equations (5.12) for  $\phi_{0t}$ , (5.14a) for  $\phi_{1t}$ , and (5.16a) for  $\phi_{2t}$  are solved with boundary conditions at  $y = 0$  of  $v_1 = 0, v_2 = 0, v_3 = 0$ , respectively. The equations (5.14b) for  $\phi_{0ut}$ , (5.16b) for  $\phi_{0ut}$ , and (5.16c) for  $\phi_{1ut}$  are solved with boundary conditions at  $y = 0$  (see sections 3b, c) of

$$0 = \phi_{0tx} - \epsilon\phi_{0uty}, \quad 0 = \phi_{0ux} - \epsilon\phi_{0uty}, \tag{5.20a, b}$$

$$0 = \phi_{1tx} = \epsilon\phi_{1ty} - \epsilon^2\phi_{0ux}, \tag{5.20c}$$

respectively. We find

IG1:

$$\Phi_0 = \exp[-\alpha y + i(kx - \sigma_0 t)], \tag{5.21a}$$

$$\alpha = (\lambda^2 + k^2)^{1/2}, \tag{5.21b}$$

$$\epsilon\sigma_0 = k/\alpha; \tag{5.21c}$$

IG2:

$$\Phi_1 = \exp[-\alpha y + i(kx - \sigma_1 t)], \tag{5.22a}$$

$$\epsilon\sigma_1 = (k/\alpha)[1 + (\epsilon\sigma_0)^2]; \tag{5.22b}$$

IG3:

$$\Phi_2 = \exp[-\alpha y + i(kx - \sigma_2 t)](1 + \gamma_2 y), \tag{5.23a}$$

$$\gamma_2 = \frac{\lambda^2(\epsilon\sigma_0)^2}{2\alpha}, \tag{5.23b}$$

$$\epsilon\sigma_2 = (k/\alpha) \left[ 1 + \frac{1}{2}(\epsilon\sigma_0)^2 + \frac{3}{2}(\epsilon\sigma_0)^4 \right]. \tag{5.23c}$$

Note that for IG2 (or IG3) all variables oscillate at the frequency  $\sigma_1$  (or  $\sigma_2$ ) although the estimates for  $\phi_{0t}/\phi$  and  $\phi_{1t}/\phi$  determined by the iterative procedure at a fixed time involve  $i\sigma_0$  and  $i\sigma_1$ , respectively.

We assume that scaling is such that  $\lambda = 1$  and plot the scaled phase velocities

$$c_{PE} = \epsilon\sigma_{PE}/k = 1, \quad c_n = \epsilon\sigma_n/k \quad (n = 0, 1, 2), \tag{5.24a, b}$$

versus  $k$  in Fig. 2. The frequency  $\sigma_{PE}$  increases with  $k$  so that, with this scaling,  $k = 1$  corresponds to an oscillation at the inertial frequency. We can see from Fig. 2 that for increasing  $N, c_{N-1}$  from the IGN models give better approximations to  $c_{PE}$  over a larger range of  $k$  starting at  $k = 0$ .

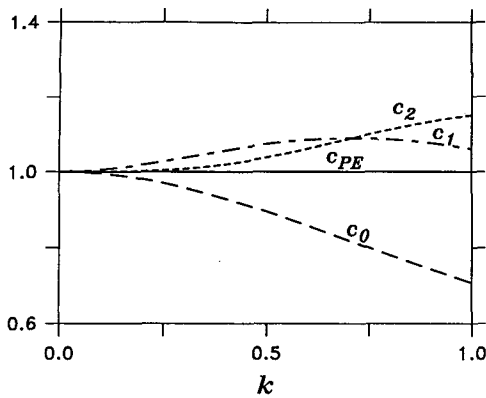


FIG. 2. The scaled phase velocities  $c_{PE}$  (5.24a) from the PE and  $c_n$  (5.24b) for  $n = 0, 1, 2$  from the IGN model solutions ( $N = n + 1$ ) to the linear internal Kelvin wave problem. The velocity  $c_2$  reaches a maximum value of 1.16 at  $k = 1.23$  and then decreases for increasing  $k$ .

### c. Rodon (shallow-water equations)

It is instructive to apply the IG2 model to the shallow-water equations for the case of a rotating elliptical eddy (Rodon) since an exact analytical solution exists for that problem (Cushman-Roisin et al. 1985; Cushman-Roisin 1987). The shallow-water equations (SWE), the IG2 model applied to the SWE, and the Rodon solution to the SWE are given in appendix B. In addition, it is useful to compare the results for the Rodon problem from the IG2 model with those obtained from the second-order quasigeostrophic approximation QG2. The QG2 equations for the SWE are also given in appendix B. The notation here is the same as that in Allen et al. (1990).

For the IG models applied to the SWE for the Rodon problem, we look for solutions for  $\eta$  (appendix B) in the form

$$\eta = N + \frac{1}{4}(R + Q)x^2 + \frac{1}{2}Sxy + \frac{1}{4}(R - Q)y^2. \quad (5.25)$$

From (B3a), we find

$$u_0 = -\frac{1}{2}Sx - \frac{1}{2}(R - Q)y, \quad (5.26a)$$

$$v_0 = \frac{1}{2}(R + Q)x + \frac{1}{2}Sy, \quad (5.26b)$$

and

$$\zeta_0 = v_{0x} - u_{0y} = R. \quad (5.26c)$$

From (B4a), we obtain

$$u_1 = -\frac{1}{2}Sx - \frac{1}{2}\left[(R - Q) + \frac{1}{2}\epsilon(S^2 + Q^2 - R^2)\right]y, \quad (5.27a)$$

$$v_1 = \frac{1}{2}\left[(R + Q) + \frac{1}{2}\epsilon(S^2 + Q^2 - R^2)\right]x + \frac{1}{2}Sy. \quad (5.27b)$$

The equation (B6) for  $\eta_{0t}$  implies

$$R_{0t} = S_{0t} = Q_{0t} = N_{0t} = 0. \quad (5.28)$$

From (5.27a, b) we obtain

$$\zeta_1 = v_{1x} - u_{1y} = R + \frac{1}{2}\epsilon(S^2 + Q^2 - R^2), \quad (5.29a)$$

$$u_{1x} + v_{1y} = 0. \quad (5.29b)$$

The equation (B10) for  $\eta_{1t}$  gives

$$R_{1t} = N_{1t} = 0, \quad (5.30a, b)$$

$$Q_{1t} + \frac{1}{2}\epsilon S(S^2 + Q^2 - R^2) = 0, \quad (5.30c)$$

$$S_{1t} - \frac{1}{2}\epsilon Q(S^2 + Q^2 - R^2) = 0. \quad (5.30d)$$

Since this is the final iteration, we can drop the subscript 1 in (5.30); that is, let  $Q_{1t} = Q_t$ , etc. Then (5.30c, d) imply

$$(Q^2 + S^2)_t = 0, \quad (5.31)$$

and the solutions to (5.30) are

$$R = R(0), \quad N = N(0), \quad (5.32a, b)$$

$$Q = \hat{Q}(0) \cos(\omega t + \Theta), \quad (5.32c)$$

$$S = \hat{Q}(0) \sin(\omega t + \Theta), \quad (5.32d)$$

where

$$\omega = -\frac{1}{2}\epsilon(R^2 - \hat{Q}^2(0)), \quad (5.33a)$$

$$\zeta_1 = R - \frac{1}{2}\epsilon(R^2 - \hat{Q}^2(0)), \quad (5.33b)$$

and  $\Theta$  is a constant initial value of the phase.

The values of  $\omega$  and  $\zeta_1$  in (5.33) agree with the lowest-order terms (B21a, c) from the small  $\epsilon$  expansion of the exact solution. Consequently, IG2 provides an approximate solution to the Rodon problem for  $\epsilon \ll 1$  with leading-order accuracy in both structure and rotation rate.

For the QG2 approximation (appendix B) we represent

$$\eta = \eta_{(0)} + \epsilon\eta_{(1)}, \quad (5.34)$$

and write  $\eta_{(0)}$  and  $\eta_{(1)}$  as

$$\eta_{(n)} = N_{(n)} + \frac{1}{4}(R_{(n)} + Q_{(n)})x^2 + \frac{1}{2}S_{(n)}xy + \frac{1}{4}(R_{(n)} - Q_{(n)})y^2, \quad n = 1, 2. \quad (5.35)$$

The equation (B13) for  $\eta_{(0)t}$  implies

$$R_{(0)t} = S_{(0)t} = Q_{(0)t} = N_{(0)t} = 0. \quad (5.36)$$

Thus, as found in Allen et al. (1990a), the QG solution is steady.

The equation (B14) for  $\eta_{(1)t}$  gives

$$R_{(1)t} = N_{(1)t} = 0, \quad (5.37a, b)$$

$$Q_{(1)t} = -\frac{1}{2} S_{(0)}(S_{(0)}^2 + Q_{(0)}^2 - R_{(0)}^2), \quad (5.37c)$$

$$S_{(1)t} = \frac{1}{2} Q_{(0)}(S_{(0)}^2 + Q_{(0)}^2 - R_{(0)}^2). \quad (5.37d)$$

Consequently, in general  $Q_{(1)}$  and  $S_{(1)}$  vary linearly with time, and the assumed expansion (5.34) for QG2 becomes invalid for  $t = O(\epsilon^{-1})$ .

The quasigeostrophic approximation procedure does not give useful approximate solutions in this case and would require generalization to include variations on the long  $\epsilon t$  time scale to be applicable for  $t \geq O(\epsilon^{-1})$ . The point of this example is that the IG2 model properly accounts for variations on the  $\epsilon t$  time scale and provides a valid approximate solution in a case where QG and QG2 fail.

*d. Comment*

Although these idealized examples are informative, the best test of the IG models involves evaluation of solution accuracy compared to PE for flow fields that are dynamically similar to those of the intended applications. In the following companion paper (Allen and Newberger 1993), numerical solutions from IG2 and IG3 are compared with those from PE, QG, QG2, and other intermediate models for nonlinear unstable jet flows similar to those observed in the coastal transition zone off northern California. As mentioned, both IG2 and IG3 perform well, with IG3 giving extremely accurate approximate solutions to PE for flows with Rossby numbers that range from moderately small to  $O(1)$ .

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APPENDIX A

**Second-Order Quasigeostrophic Approximation**

For comparison with the iterated geostrophic model IG2, we record the equations for the second-order

quasigeostrophic approximation (labeled QG2). The QG2 model includes the lowest (first)-order quasigeostrophic equations and the next (second)-order correction. Numerical solutions for QG2 are calculated in Allen and Newberger (1993) and compared with those from IG2 and the other models.

We derive the quasigeostrophic approximation (e.g., Pedlosky 1987) from the primitive equations in the form (2.18), (2.23), (2.24), (2.25), and (2.1c) with  $f$  given by (3.1). The variables are expanded as

$$\phi = \phi_{(0)} + \epsilon\phi_{(1)} + \dots, \quad \theta = \theta_{(0)} + \epsilon\theta_{(1)} + \dots, \quad (A1a, b)$$

$$\psi = \psi_{(0)} + \epsilon\psi_{(1)} + \dots, \quad (A1c)$$

$$\chi = \chi_{(1)} + \epsilon\chi_{(2)} + \dots, \quad w = w_{(1)} + \epsilon w_{(2)} + \dots. \quad (A2d, e)$$

At lowest (first)-order, the equations are

$$\nabla^2\chi_{(1)} + w_{(1)z} = 0, \quad (A2a)$$

$$\zeta_{(0)t} + J(\psi_{(0)}, \zeta_{(0)} + \delta f) + \nabla^2\chi_{(1)} = -\nu\nabla^4\zeta_{(0)}, \quad (A2b)$$

$$\zeta_{(0)} = \nabla^2\psi_{(0)} = \nabla^2\phi_{(0)}, \quad (A2c)$$

$$0 = -\phi_{(0)z} + \theta_{(0)}, \quad (A2d)$$

$$\theta_{(0)t} + J(\psi_{(0)}, \theta_{(0)}) + Sw_{(1)} = 0. \quad (A2e)$$

Combining (A2b) with the  $z$  derivative of (A2e) and using (A2a), we obtain the equation for  $\phi_{(0)t}$ :

$$[\nabla^2\phi_{(0)} + (S^{-1}\phi_{(0)z})_z]_t = -J(\phi_{(0)}, \zeta_{(0)} + \delta f) - [S^{-1}J(\psi_{(0)}, \theta_{(0)})]_z - \nu\nabla^6\phi_{(0)}. \quad (A3)$$

At second order, the equations are

$$\nabla^2\chi_{(2)} + w_{(2)z} = 0, \quad (A4a)$$

$$\begin{aligned} \zeta_{(1)t} + J(\psi_{(0)}, \zeta_{(1)}) + J(\psi_{(1)}, \zeta_{(0)} + \delta f) + \nabla^2\chi_{(2)} \\ + \nabla \cdot [w_{(1)}\nabla\psi_{(0)z} + (\zeta_{(0)} + \delta f)\nabla\chi_{(1)}] \\ = -\nu\nabla^4\zeta_{(1)}, \end{aligned} \quad (A4b)$$

$$\begin{aligned} \zeta_{(1)} = \nabla^2\psi_{(1)} = \nabla^2\phi_{(1)} - 2J(\psi_{(0)x}, \psi_{(0)y}) \\ - \nabla(\delta f\nabla\psi_{(0)}), \end{aligned} \quad (A4c)$$

$$0 = -\phi_{(1)z} + \theta_{(1)}, \quad (A4d)$$

$$\begin{aligned} \theta_{(1)t} + J(\psi_{(0)}, \theta_{(1)}) + J(\psi_{(1)}, \theta_{(0)}) + Sw_{(2)} \\ + [\nabla \cdot (\theta_{(0)}\nabla\chi_{(1)}) + (w_{(1)}\theta_{(0)})_z] = 0. \end{aligned} \quad (A4e)$$

Combining (A4b) with the  $z$  derivative of (A4e) and using (A4a) and (A4c), we obtain the equation for  $\phi_{(1)t}$ :

$$\begin{aligned} [\nabla^2\phi_{(1)} + (S^{-1}\phi_{(1)z})_z]_t \\ = -J(\psi_{(0)}, \zeta_{(1)}) - J(\psi_{(1)}, \zeta_{(0)} + \delta f) \\ - \nabla \cdot [w_{(1)}\nabla\psi_{(0)z} + (\zeta_{(0)} + \delta f)\nabla\chi_{(1)}] - \nu\nabla^4\zeta_{(1)} \end{aligned}$$

$$\begin{aligned}
& + 2[J(\psi_{(0)x}, \psi_{(0)y}) + J(\psi_{(0)x}, \psi_{(0)y})] \\
& + \nabla(\delta f \nabla \psi_{(0)t}) \\
& - \{S^{-1}[J(\psi_{(0)}, \theta_{(1)}) + J(\psi_{(1)}, \theta_{(0)}) \\
& \quad + \nabla \cdot (\theta_{(0)} \nabla \chi_{(1)}) + (w_{(1)} \theta_{(0)})_z]\}_z. \quad (A5)
\end{aligned}$$

In QG2,  $\phi = \phi_{(0)} + \epsilon \phi_{(1)}$ , where  $\phi_{(0)t}$  is found from the solution to (A3) and  $\phi_{(1)t}$  from the solution to (A5). In addition,  $\theta = \theta_{(0)} + \epsilon \theta_{(1)}$ , and  $\psi = \psi_{(0)} + \epsilon \psi_{(1)}$ . The variables  $\chi_{(1)}$  and  $w_{(1)}$  are involved in the advection terms in (A4b, e) along with  $\psi_{(1)}$ . Thus, we call  $\chi = \chi_{(1)}$  and  $w = w_{(1)}$ , although in a doubly periodic domain  $\chi_{(2)}$  may be calculated from (A4b) once  $\phi_{(1)t}$  is known, and then  $w_{(2)}$  may be found from (A4a).

#### APPENDIX B

##### Shallow-Water Equations

It is useful for understanding the IG models to apply them to the shallow-water equations (SWE) for the case of a rotating elliptical eddy (Rodon) where an exact analytical solution exists (Cushman-Roisin et al. 1985; Cushman-Roisin 1987). Accordingly, we derive IG2 for the SWE. We also derive QG2 and record the SWE Rodon solution. IG2 and QG2 are applied to the SWE Rodon problem in section 5. We restrict attention here to inviscid ( $\nu \equiv 0$ ) motion on an  $f$  plane with no bottom topography ( $h_B = 0$ ).

The shallow-water equations in dimensionless variables are

$$\epsilon F \eta_t + \nabla \cdot (h \mathbf{u}) = 0, \quad (B1a)$$

$$\epsilon(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) + \mathbf{k} \times \mathbf{u} = -\nabla \eta, \quad (B1b)$$

where

$$h = \epsilon F \eta + 1, \quad (B1c)$$

and  $\epsilon F \eta$  is the elevation of the free surface relative to the undisturbed depth of the fluid ( $= 1$ ). The variables and the nondimensionalization are the same as in Allen et al. (1990). The horizontal velocity vector  $\mathbf{u}$ , the horizontal gradient operator  $\nabla$ , the unit vertical vector  $\mathbf{k}$ , and the Rossby number  $\epsilon$  are as defined in section 2. The parameter  $F = f_0^2 L^2 / (gH)$ .

The IG models are obtained by iteration of (B1) in the form

$$\mathbf{u} = \mathbf{k} \times \nabla \eta + \epsilon \mathbf{k} \times [\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}], \quad (B2a)$$

$$\nabla \cdot \mathbf{u} = -\epsilon F [\eta_t + \nabla \cdot (\eta \mathbf{u})]. \quad (B2b)$$

##### a. Iterated geostrophic model IG1

From (B2) we start with

$$\mathbf{u}_0 = \mathbf{k} \times \nabla \eta, \quad (B3a)$$

which gives

$$\zeta_0 = \mathbf{k} \cdot \nabla \times \mathbf{u}_0 = \nabla^2 \eta, \quad \nabla \cdot \mathbf{u}_0 = 0. \quad (B3b, c)$$

Then we obtain

$$\mathbf{u}_1 = \mathbf{k} \times \nabla \eta + \epsilon \mathbf{k} \times [\mathbf{u}_{0t} + \mathbf{u}_0 \cdot \nabla \mathbf{u}_0], \quad (B4a)$$

$$\nabla \cdot \mathbf{u}_1 = \epsilon F [\eta_{0t} + \nabla \cdot (\eta \mathbf{u}_0)], \quad (B4b)$$

where

$$\mathbf{u}_{0t} = \mathbf{k} \times \nabla \eta_{0t}. \quad (B5)$$

Eliminating  $\mathbf{u}_1$  from (B4a, b) we find the governing equation for  $\eta_{0t}$ :

$$\nabla^2 \eta_{0t} - F \eta_{0t} = -J(\eta, \nabla^2 \eta - F \eta). \quad (B6)$$

##### b. Iterated geostrophic model IG2

We obtain  $\psi_1$  and  $\chi_1$  from (B4a); that is,

$$\mathbf{k} \cdot \nabla \times \mathbf{u}_1 = \zeta_1 = \nabla^2 \psi_1 = \nabla^2 \eta - 2\epsilon J(\eta_x, \eta_y), \quad (B7a)$$

$$\epsilon^{-1} \nabla \cdot \mathbf{u}_1 = \nabla^2 \chi_1 = -J(\eta, \nabla^2 \eta) - \nabla^2 \eta_{0t}. \quad (B7b)$$

From (B2), we obtain

$$\mathbf{u}_2 = \mathbf{k} \times \nabla \eta + \epsilon \mathbf{k} \times [\mathbf{u}_{1t} + \mathbf{u}_1 \cdot \nabla \mathbf{u}_1], \quad (B8a)$$

$$\nabla \cdot \mathbf{u}_2 = -\epsilon F [\eta_{1t} + \nabla \cdot (\eta \mathbf{u}_1)]. \quad (B8b)$$

Substituting  $\mathbf{u}_2$  from (B8a) into (B8b), and using

$$\zeta_{1t} = \nabla^2 \eta_{1t} - 2\epsilon [J(\eta_{0tx}, \eta_y) + J(\eta_x, \eta_{0ty})], \quad (B9)$$

we obtain the governing equation for  $\eta_{1t}$ :

$$\begin{aligned}
\nabla^2 \eta_{1t} - F \eta_{1t} = & -\nabla \cdot [\mathbf{u}_1 (\zeta_1 - F \eta)] \\
& + 2\epsilon [J(\eta_{0tx}, \eta_y) + J(\eta_x, \eta_{0ty})]. \quad (B10)
\end{aligned}$$

##### c. Quasigeostrophic and second-order quasigeostrophic approximations

In the quasigeostrophic approximation (e.g., Pedlosky 1987), the variables are expanded as

$$\eta = \eta_{(0)} + \epsilon \eta_{(1)} + \dots, \quad \mathbf{u} = \mathbf{u}_{(0)} + \epsilon \mathbf{u}_{(1)} + \dots. \quad (B11a, b)$$

It follows that

$$\mathbf{u}_{(0)} = -\mathbf{k} \times \nabla \eta_{(0)}. \quad (B12)$$

The governing equation for  $\eta_{(0)}$  is

$$\nabla^2 \eta_{(0)t} - F \eta_{(0)t} = -J(\eta_{(0)}, \nabla^2 \eta_{(0)}). \quad (B13)$$

At the next (second) order, we find

$$\begin{aligned}
\nabla^2 \eta_{(1)t} - F \eta_{(1)t} = & -\nabla \cdot [\mathbf{u}_{(0)} (\zeta_{(1)} - F \eta_{(1)})] \\
& - \nabla \cdot [\mathbf{u}_{(1)} (\zeta_{(0)} - F \eta_{(0)})] \\
& + 2[J(\eta_{(0)tx}, \eta_{(0)y}) + J(\eta_{(0)x}, \eta_{(0)ty})], \quad (B14)
\end{aligned}$$

where

$$\zeta_{(1)} = \nabla^2 \psi_{(1)} = \nabla^2 \eta_{(1)} - 2J(\eta_{(0)x}, \eta_{(0)y}), \quad (B15a)$$

$$\nabla^2 \chi_{(1)} = -J(\eta_{(0)}, \nabla^2 \eta_{(0)}) - \nabla^2 \eta_{(0)t}. \quad (B15b)$$

In the QG2 model,  $\eta = \eta_{(0)} + \epsilon\eta_{(1)}$ , where  $\eta_{(0)}$  is found from (B13) and  $\eta_{(1)}$  from (B14).

#### d. SWE Rodon solution

For the Rodon problem (Cushman-Roisin et al. 1985; Cushman-Roisin 1987) solutions to the SWE (B1) are sought in the form

$$u = U_1x + U_2y, \quad v = V_1x + V_2y, \quad (\text{B16a})$$

$$\eta = N + \frac{1}{2}Ax^2 + Bxy + \frac{1}{2}Cy^2, \quad (\text{B16b})$$

where the coefficients  $U_1$ , etc., are functions of time. It is convenient to use the variables

$$\zeta = v_x - u_y = V_1 - U_1, \quad D = u_x + v_y = U_1 + V_2, \quad (\text{B17a, b})$$

$$M = v_x + u_y = V_1 + U_2, \quad L = u_x - v_y = U_1 - V_2, \quad (\text{B17c, d})$$

$$R = \eta_{xx} + \eta_{yy} = A + C, \quad Q = \eta_{xx} - \eta_{yy} = A - C, \quad (\text{B17e, f})$$

$$S = 2\eta_{xy} = 2B. \quad (\text{B17g})$$

Substitution of (B16) in the SWE (B1) results in eight nonlinear ordinary differential equations for the eight coefficients. The resulting equations in terms of the variables defined in (B17) are given in Allen et al. (1990).

We consider motion for which

$$D = 0. \quad (\text{B18})$$

In that case, the following solution, corresponding to a steadily rotating elliptical eddy, (Rodon) may be found (Cushman-Roisin et al. 1985):

$$\zeta = \zeta(0), \quad R = R(0), \quad N = N(0), \quad (\text{B19a, b, c})$$

$$(L, S) = [\hat{L}(0), \hat{Q}(0)] \sin(\omega t + \Theta), \quad (\text{B19d})$$

$$(M, Q) = [-\hat{L}(0), \hat{Q}(0)] \cos(\omega t + \Theta), \quad (\text{B19e})$$

where

$$R = \zeta + \frac{\epsilon}{2}(\zeta^2 - \hat{L}(0)^2), \quad (\text{B20a})$$

$$\hat{Q}(0) = -\hat{L}(0)(1 + \epsilon\omega), \quad (\text{B20b})$$

$$\epsilon\omega = -\frac{1}{2}(1 - \epsilon\zeta) + \frac{1}{2}[(1 + \epsilon\zeta)^2 - 4\epsilon R]^{1/2}, \quad (\text{B20c})$$

and  $\Theta$  is a constant initial value of the phase.

For  $\epsilon \ll 1$ , (B20a, b, c) give

$$\omega = -\frac{1}{2}\epsilon(R^2 - \hat{Q}^2(0)) + O(\epsilon^2), \quad (\text{B21a})$$

$$\hat{L}^2(0) = \hat{Q}^2(0) + O(\epsilon^2), \quad (\text{B21b})$$

$$\zeta = R - \frac{1}{2}\epsilon(R^2 - \hat{Q}^2(0)) + O(\epsilon^2). \quad (\text{B21c})$$

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