AN ABSTRACT OF THE THESIS OF

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Title: Unique Determination of Acoustic Properties from Thermoacoustic Data.

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The extent to which thermoacoustic data determines the acoustic properties of an object was studied. In the case of one dimensional thermoacoustic imaging it was shown that constant acoustic profiles are uniquely specified from measurements. For a radial thermoacoustic problem we have shown that if the acoustic source is radial then constant acoustic speeds are uniquely determined from the data. The case of a radial thermoacoustic setup with a non-radial source was investigated. It was shown that constant acoustic profiles are uniquely determined within a special class of radial acoustic speeds. An investigation of the *interior transmission problem* was carried out. A variational form for this problem was found for a new class of refractive indexes, existence of an infinite discrete set of transmission eigenvalues was established. The study of the unique determination of the acoustic profiles used a relation between data generated from two distinct sound speeds and the spectrum of the transmission problem. The general transmission problem with a new class of refractive indexes was analyzed using Hardy's inequality. Results in the one dimensional and radial cases relate the transmission spectrum to the spectrum of an eigenvalue problem associated to a system of Sturm-Liouville equations. Asymptotic expansions of solutions to the Sturm-Liouville equations were used, together with a special boundary condition, to study this spectrum and derive the uniqueness results.

Unique Determination of Acoustic Properties from Thermoacoustic Data

by

Kyle Scott Hickmann

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Chair of the Department of Mathematics

Dean of the Graduate School

I understand that my thesis will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my thesis to any reader upon request.

Kyle Scott Hickmann, Author

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Academic

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UNIQUE DETERMINATION OF ACOUSTIC PROPERTIES FROM THERMOACOUSTIC DATA

1 INTRODUCTION

Recent developments in the detection of cancerous tissue have sought to identify unhealthy tissue by mapping the conductivity of a region of the body [23, 61, 52, 56, 84, 88]. It has been shown [53, 45, 22] that the electrical conductivity and other dielectric properties of cancerous tissue differ greatly from that of healthy tissue. Thermoacoustic tomography (TAT) is a developing medical imaging modality that seeks to exploit these properties for the detection of cancerous tissue. These imaging method combines the high resolution of ultrasound computed tomography (UCT) with the high contrast of microwave imaging modalities.

The process works by irradiating a body with a short duration ($\approx .5\,\mu$s) wide band microwave pulse, typically 3 GHz with wavelength $\approx 3\,$cm in tissue [84, 88, 56]. The cancerous tissue has a much higher electromagnetic absorption coefficient than the surrounding healthy tissue and therefore retains more of the microwave energy. This induces heating and thermoelastic expansion of the unhealthy tissue which causes the emission of ultrasound waves. Ultrasound transducers, typically in the frequency range of 2.25 MHz detecting ultrasound with a wavelength $\approx 1\,$mm in tissue [84, 88, 56], record the emitted waves on the boundary of the body being imaged. The goal is to recover the initial pressure disturbance generating the ultrasound field. This will be proportional to the electromagnetic absorption distribution throughout the imaged region.

In the mathematical study of TAT one assumes an imaging domain $D \subset \mathbb{R}^n$ and knowledge of the pressure field on the boundary,

$$g(x,t) = p(x,t)|_{\partial D\times[0,T]}.$$

Under certain physical assumptions, detailed in the next chapter, the field $p(x,t)$ satisfies the wave equation

$$\partial_t p(x,t) - c^2(x)\Delta p(x,t) = 0 \text{ on } \mathbb{R}^n \times \mathbb{R}_+$$
$$p(x,0) = f(x),\ p_t(x,0) = 0 \text{ on } \mathbb{R}^n.$$

One attempts to solve the inverse problem of reconstructing $f(x)$ in D from knowledge of $g(x,t)$ on $\partial D \times [0,T]$. It is often assumed that the acoustic profile of the region, $c(x)$, is constant [52, 84]. However, much work has gone in to dealing with the inverse problem when the region D contains significant acoustic variations [72, 2, 44, 43].

If the thermoacoustic imaging problem is considered with acoustic variations it is usually assumed that the variations are known ahead of time. This assumption, in practice, would rely on first determining the acoustic profile $c(x)$ through some other imaging method such as UCT. Some work has been done [90, 83] that suggests information about the acoustic properties of D can be reconstructed from the data collected in thermoacoustic tomography.

In this dissertation we study the extent to which the acoustic speed $c(x)$ is uniquely specified by the thermoacoustic data. It will be shown that the problem of unique determination of the acoustic profile in D from TAT measurements is related to a boundary value problem that arises in acoustic scattering theory called the interior transmission problem (ITP). If the acoustic speed of the domain is known to be *non-trapping* then we will be able to prove theorems having to do with the unique determination of the acoustic profile. These results rely on showing conditions under which the ITP spectrum is sufficiently sparse.

To study these types of uniqueness properties of TAT we compare the sets of data that could be generated by two distinct acoustic profiles. The question of uniqueness of the acoustic speed is posed as a question concerning which acoustic profiles generate distinct data. After showing a class of profiles that generate distinct TAT data we turn to a rigorous study of the ITP under previously unexamined assumptions. These assumptions are motivated by the uniqueness question in thermoacoustics. We prove that in this new setting the transmission spectrum is discrete, the ITP has a variational formulation, and an infinite discrete set of transmission eigenvalues exists.

The current theory of the ITP does not allow us to prove theorems when comparing two completely arbitrary profiles on the domain D. However, if we study the thermoacoustic problem in a single dimension more is known about the underlying differential equations in the transmission problem. Thus, we study the question of uniqueness of the sound speed for the one dimensional TAT problem. It will be shown that, for this problem, constant acoustic speeds can be uniquely determined from thermoacoustic data even when it is not known *a priori* that the acoustic speed is constant.

Motivated by the success in the one dimensional problem we apply the same methods to a study of the radial TAT problem in three dimensions on the unit ball. It is assumed the acoustic profile and the acoustic source are radially symmetric. With these assumptions we prove that constant acoustic profiles are uniquely determined by TAT data from radial sources even when it is not known *a priori* that the acoustic speed is constant.

The final chapter of this dissertation studies the unique determination of the acoustic profile with the assumption that the domain D and the acoustic speed are radially symmetric but the acoustic source is not. In this case, as in the previous two cases, we are able to show that to some extent constant acoustic profiles generate distinct data. However, in this case we must further restrict the class of radial acoustic speeds that constants are distinct in. It will be shown that among radial acoustic profiles satisfying a secondary integral condition constant speeds are uniquely determined.

We have organized this thesis as follows. In the next chapter, chapter 2, we introduce the background theory of thermoacoustic tomography, including the specific assumptions about the process that make the mathematical model a good approximation of the physical realization of thermoacoustic imaging. We also review uniqueness theorems and reconstruction algorithms used in TAT. In chapter 3 we show the relation of the unique determination of the acoustic speed to the ITP and prove a theorem stating sufficient conditions for two acoustic profiles to generate distinct data. Chapter 4 contains our treatment of the transmission spectrum under the assumption that the refractive index approaches the index of the surrounding homogeneous media with some degree of smoothness at ∂D. Chapter 5 details our study of the question of uniqueness in the one dimensional setting. We then move on, in chapter 6, to a study of the completely radial TAT problem where both the acoustic profile and initial impulse are assumed to be radially symmetric. Lastly we show the results involving a radial domain with a radial acoustic profile in chapter 7; here the initial impulse is not assumed to be radially symmetric. Our conclusion, in which we mention future directions of research and open problems in thermoacoustic tomography is in chapter 8.

2 BACKGROUND OF THERMOACOUSTIC TOMOGRAPHY

2.1 Introduction

In this chapter we present an overview of the physical basis of the thermoacoustic problem. The derivation of the problem presented here has been collected and synthesized from a few sources [70, 55, 57, 75] and is by no means new. However, we have had trouble finding a single source that shows the derivation of the thermoacoustic equation usually studied in the mathematics literature starting from the basic conservation laws and stress-strain relations of an elastic solid. This is presented below and a derivation from the basic equations like this should make it apparent that there are many unexplored problems left in this field. For a more in depth treatment of the physics of the thermoacoustic effect we refer the reader to [57, 75] and for background in linear continuum mechanics [70, 55].

For the convenience of the reader we also present some of the more well known theorems that have been recently developed for TAT. This should help to inform the reader of what is known in the theory and what open problems are left. It is not meant to be a complete literature review of the mathematics of thermoacoustic tomography for which we point the interested reader to the works [3, 52].

This chapter is laid out as follows. In section 2.2 we give a detailed derivation of the usual mathematical formulation of the TAT problem paying careful attention to the assumptions made. Section 2.3 presents the main theorems in the literature relating to the unique solvability of the thermoacoustic inverse problem. Results on well known inversion methods in TAT are presented in section 2.4.

2.2 Physical Setup of the Thermoacoustic Problem

2.2.1 Equations governing the thermoacoustic process.

For a continuous medium the *thermoacoustic* or *photoacoustic* effect is the phenomenon of electromagnetic waves inducing heating and subsequent thermoelastic expansion inside a body. This expansion causes acoustic waves which can then be measured on the boundary of an object [57, 75]. The equations that relate the heating of the tissue to the induced pressure field come from a system of conservation laws involving the stress and strain tensors.

Consider a body $D \subset \mathbb{R}^3$. The dimension is not important but fixing one is notationally much easier to deal with. At a point $\vec{x} = (x^1, x^2, x^3) \in D$ we define the time dependent displacement of the material at that point by $\vec{\delta}(\vec{x}, t) = (u^1(\vec{x}, t), u^2(\vec{x}, t), u^3(\vec{x}, t))$. The *strain tensor* is then defined as

$$\epsilon^{ij} = \frac{1}{2}\left(\frac{\partial u^j}{\partial x^i} + \frac{\partial u^i}{\partial x^j}\right). \tag{2.1}$$

The strain tensor represents infinitesimal deformations in the body D caused by the displacement $\vec{\delta}(\vec{x},t)$. The strain tensor is symmetric consisting of six distinct elements. The *extensional strains*, ϵ^{11}, ϵ^{22}, ϵ^{33}, and the *shear strains*, ϵ^{12}, ϵ^{13}, ϵ^{23}.

The displacement $\vec{\delta}(\vec{x},t)$ also induces stresses (forces) at each point $\vec{x} \in D$. The forces acting on each point in D are governed by the *stress tensor*, σ^{ij}, i, $j = 1, 2, 3$. This also is a symmetric tensor having six distinct elements given by the *normal stresses*, σ^{11}, σ^{22}, σ^{33}, and the *shear stresses*, σ^{12}, σ^{13}, σ^{23}. For a surface cutting through the body D at the point $\vec{x}$ with normal $\vec{n}$ the force per unit area acting on the surface is given by

$$\sigma\vec{n} = \begin{bmatrix} \sigma^{11} & \sigma^{12} & \sigma^{13} \\ \sigma^{12} & \sigma^{22} & \sigma^{23} \\ \sigma^{13} & \sigma^{23} & \sigma^{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}. \tag{2.2}$$

Define the velocity at a point in D by

$$\vec{v}(\vec{x},t) = v^i(\vec{x},t) = \partial_t u^i(\vec{x},t). \tag{2.3}$$

Then one may derive the system of conservation laws from only the definition of the strain tensor and Newton's laws of motion [55]:

$$\begin{aligned} \frac{\partial \epsilon^{ij}}{\partial t} - \frac{1}{2}\left(\frac{\partial v^i}{\partial x^j} + \frac{\partial v^j}{\partial x^i}\right) &= 0 \text{ for } i,\ j = 1,\ 2,\ 3 \\ \rho\frac{\partial \vec{v}}{\partial t} - \nabla\cdot\sigma &= 0. \end{aligned} \tag{2.4}$$

Here we have used the notation

$$\nabla\cdot\sigma = \begin{bmatrix} \frac{\partial\sigma^{11}}{\partial x^1} + \frac{\partial\sigma^{12}}{\partial x^2} + \frac{\partial\sigma^{13}}{\partial x^3} \\ \frac{\partial\sigma^{12}}{\partial x^1} + \frac{\partial\sigma^{22}}{\partial x^2} + \frac{\partial\sigma^{23}}{\partial x^3} \\ \frac{\partial\sigma^{13}}{\partial x^1} + \frac{\partial\sigma^{23}}{\partial x^2} + \frac{\partial\sigma^{33}}{\partial x^3} \end{bmatrix} \tag{2.5}$$

for the divergence of the tensor σ. The constant ρ is the density at a point $\vec{x} \in D$. This system of equations is not enough to completely specify the components of stress and strain and the components of velocity. Quickly we may count that there are nine total equations in the above system. However, there are six components for ϵ and σ and three components for the velocity resulting in 15 total unknowns. The remaining six equations necessary to completely specify all the components in the system come from the relations of ϵ to σ, the *stress-strain relations*. These come from the material properties of the body D and therefore will determine our assumptions in thermoacoustic tomography.

If we assume that the stresses and strains within the body result from only small deformations then the assumption of a linear relation of stress and strain will suffice. In this case Hooke's law yields the general relation

$$\sigma^{ij} = \sum_{k,l} C^{ijkl} \epsilon^{kl}. \tag{2.6}$$

We will assume that the material making up D is *isotropic*. This means that the reaction of the material to an applied displacement or force is independent of direction. This effectively means that the six distinct elements of ϵ and σ are related by a 6×6 matrix [55].

For an elastic medium this relation is found as follows. We expect that a small deformation in the x^i-direction will result in a proportional internal force in the x^i-direction, this is expressed as

$$\sigma^{ii} = E\epsilon^{ii} \text{ for } i = 1, 2, 3. \tag{2.7}$$

Here the constant E, *Young's modulus*, does not depend on direction since we have assumed the material is isotropic. For a small deformation in the x^i-direction we expect induced deformations in the x^j-directions, $i \neq j$, that are proportional to the deformation in the x^i-direction,

$$\epsilon^{jj} = -\nu\epsilon^{ii}, \tag{2.8}$$

where ν is *Poisson's ratio*. This yields the relation

$$\epsilon^{jj} = -\frac{\nu}{E}\sigma^{ii} \text{ for } i \neq j \tag{2.9}$$

$$\epsilon^{ii} = \frac{1}{E}\sigma^{ii}. \tag{2.10}$$

If each of the stresses σ^{ii} are applied simultaneously then we have the relation for normal stresses to extensional strains:

$$\begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \end{bmatrix}. \tag{2.11}$$

The shear stresses are related to the shear strains by

$$\begin{bmatrix} \epsilon^{12} \\ \epsilon^{13} \\ \epsilon^{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{2\mu} & 0 & 0 \\ 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & \frac{1}{2\mu} \end{bmatrix} \begin{bmatrix} \sigma^{12} \\ \sigma^{13} \\ \sigma^{23} \end{bmatrix} \tag{2.12}$$

with the *shear modulus*, μ. Putting the two previous equations together we find the relation between stress and strain for small deformations in isotropic materials to be

$$\begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \\ \epsilon^{12} \\ \epsilon^{13} \\ \epsilon^{23} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\mu} \end{bmatrix} \begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{13} \\ \sigma^{23} \end{bmatrix}. \tag{2.13}$$

It can be shown that for an elastic material that supports shear stresses

$$\mu = \frac{E}{2(1+\nu)}. \tag{2.14}$$

Using this the inverse of relation may be written as

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{13} \\ \sigma^{23} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \\ \epsilon^{12} \\ \epsilon^{13} \\ \epsilon^{23} \end{bmatrix}. \tag{2.15}$$

Here

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \tag{2.16}$$

Define invariants

$$\frac{1}{3}\mathrm{tr}(\sigma) = \frac{1}{3}\sum_i \sigma^{ii} \tag{2.17}$$

$$\mathrm{tr}(\epsilon) = \sum_i \epsilon^{ii}, \tag{2.18}$$

the *mean stress* and *volumetric strain* respectively. The mean stress represents the average force on a point in the domain D. The mean stress is related to the volumetric strain by the *bulk modulus*, K, through the equation

$$\frac{1}{3}\mathrm{tr}(\sigma) = K\,\mathrm{tr}(\epsilon). \tag{2.19}$$

One notices that through the stress-strain relation given in (2.15) that

$$K = \lambda + \frac{2}{3}\mu. \tag{2.20}$$

Thus, the strain to stress relation may be written as

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \\ \sigma^{12} \\ \sigma^{13} \\ \sigma^{23} \end{bmatrix} = \begin{bmatrix} K+\frac{4}{3}\mu & K-\frac{2}{3}\mu & K-\frac{2}{3}\mu & 0 & 0 & 0 \\ K-\frac{2}{3}\mu & K+\frac{4}{3}\mu & K-\frac{2}{3}\mu & 0 & 0 & 0 \\ K-\frac{2}{3}\mu & K-\frac{2}{3}\mu & K+\frac{4}{3}\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \\ \epsilon^{12} \\ \epsilon^{13} \\ \epsilon^{23} \end{bmatrix}. \tag{2.21}$$

Using this last relation the system (2.4) can be changed into a system of equations involving the stress σ and the velocity $\vec{v}$. This system has the form:

$$\begin{gathered} \frac{\partial \sigma^{ii}}{\partial t} - 2\mu\frac{\partial v^i}{\partial x^i} - \left(K - \frac{2}{3}\mu\right)\nabla\cdot\vec{v} = 0 \text{ for } i = 1,\, 2,\, 3 \\ \frac{\partial \sigma^{ij}}{\partial t} - \mu\left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j}\right) = 0 \text{ for } i \neq j \\ \rho\frac{\partial \vec{v}}{\partial t} - \nabla\cdot\sigma = 0. \end{gathered} \tag{2.22}$$

The material parameters may vary throughout the region D but not with respect to time, $K = K(\vec{x})$, $\mu = \mu(\vec{x})$, and $\rho = \rho(\vec{x})$. We see that this last system consist of nine equations and nine unknowns. This is a first order linear hyperbolic system for the distinct entries in the stress tensor $\sigma(\vec{x}, t)$ and the components of velocity $v^1(\vec{x}, t)$, $v^2(\vec{x}, t)$, and $v^3(\vec{x}, t)$. To derive the above system we have assumed the material making up the body D is an isotropic elastic solid and the deformations of D are assumed to be small.

In thermoacoustic tomography the stresses and velocities inside the medium are induced by thermoelastic expansion. We now analyze the effect of this thermoelastic expansion on the isotropic linear elasticity system (2.22). Let $T = T(\vec{x}, t)$ denote the time dependent temperature at a point in the domain D and let T_0 be the equilibrium temperature. Then the stress induced from heating of an isotropic medium is isotropic and proportional to the change in temperature from equilibrium [57]. In this case the stress strain relation contains the additional term $-K\beta(T - T_0)$ where K is the bulk modulus and β is the *thermal expansion coefficient* [57]. Therefore, the normal stresses are now related to the extensional strains by

$$\begin{bmatrix} \sigma^{11} \\ \sigma^{22} \\ \sigma^{33} \end{bmatrix} = \begin{bmatrix} K + \frac{4}{3}\mu & K - \frac{2}{3}\mu & K - \frac{2}{3}\mu \\ K - \frac{2}{3}\mu & K + \frac{4}{3}\mu & K - \frac{2}{3}\mu \\ K - \frac{2}{3}\mu & K - \frac{2}{3}\mu & K + \frac{4}{3}\mu \end{bmatrix} \begin{bmatrix} \epsilon^{11} \\ \epsilon^{22} \\ \epsilon^{33} \end{bmatrix} - K\beta(T - T_0)I \tag{2.23}$$

with I being the 3×3 identity matrix. It is important to note that our assumptions about the effect of the temperature on the internal stresses of the medium assures that there are no shear stresses induced through thermal expansion [57]. Using relation (2.23) in (2.4) now yields the system

$$\begin{gathered} \frac{\partial \sigma^{ii}}{\partial t} - 2\mu \frac{\partial v^i}{\partial x^i} - \left(K - \frac{2}{3}\mu\right) \nabla \cdot \vec{v} = -K\beta \frac{\partial T}{\partial t} \text{ for } i = 1,\ 2,\ 3 \\ \frac{\partial \sigma^{ij}}{\partial t} - \mu \left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j} \right) = 0 \text{ for } i \neq j \\ \rho \frac{\partial \vec{v}}{\partial t} - \nabla \cdot \sigma = 0. \end{gathered} \tag{2.24}$$

In TAT the heating of the material is caused by a short duration pulse of some type of electromagnetic radiation, usually either light (photoacoustic tomography) or microwave (thermoacoustic tomography). Let $I = I(\vec{x}, t)$ be the *intensity* of the EM radiation pulse as a function of time at $\vec{x} \in D$. The temperature induced by this pulse is governed by [57, 75]

$$\frac{\partial T}{\partial t} - \frac{\chi}{\rho c_p}\Delta T = \frac{\alpha}{\rho c_p} I \text{ on } D \times \mathbb{R}_+. \tag{2.25}$$

Here ρ is the density, χ is the *thermal conductivity* of the medium in D, c_p is the *specific heat* under constant pressure of the material, and α is the *electromagnetic absorption coefficient*. In general all of these constants can vary throughout a material but in TAT we will assume that at least the thermal conductivity and the specific heat are constant throughout D. The term $\kappa = \chi/\rho c_p$ is the *thermal diffusivity*. In TAT a very short duration pulse is used to illuminate the body D and κ is usually very small compared to the propagation speed of acoustic signals in the medium [57, 75, 84]. Therefore, the term $\kappa \Delta T$ in the diffusivity equation is neglected and it is assumed that

$$\frac{\partial T}{\partial t} = \frac{\alpha}{\rho c_p} I \text{ on } D \times \mathbb{R}_+. \tag{2.26}$$

Thus, in thermoacoustic tomography, internal stresses in D are governed by the system

$$\frac{\partial \sigma^{ii}}{\partial t} - 2\mu \frac{\partial v^i}{\partial x^i} - \left(K - \frac{2}{3}\mu\right) \nabla \cdot \vec{v} = -\frac{\alpha \beta K}{\rho c_p} I \text{ for } i = 1,\, 2,\, 3 \tag{2.27}$$

$$\frac{\partial \sigma^{ij}}{\partial t} - \mu \left(\frac{\partial v^j}{\partial x^i} + \frac{\partial v^i}{\partial x^j}\right) = 0 \text{ for } i \neq j$$

$$\rho \frac{\partial \vec{v}}{\partial t} - \nabla \cdot \sigma = 0.$$

The initial application TAT was developed for was detection of cancerous tissue in the breast. Since the human body is mostly water it is assumed that shear stresses are not supported. This means that we will assume the material making up the region D is *hydrostatic*. For such materials the shear stresses, and thus the shear modulus, are all zero. In this case

$$\sigma^{ii} = K \operatorname{tr}(\epsilon) \text{ for } i = 1,\, 2,\, 3 \tag{2.28}$$

we define the *hydrostatic pressure*

$$-p = \sigma^{11} = \sigma^{22} = \sigma^{33}. \tag{2.29}$$

Substituting these assumptions into the system (2.27) yields

$$\begin{aligned} \frac{\rho}{K}\frac{\partial p}{\partial t} + \rho\nabla\cdot\vec{v} &= \frac{\alpha\beta}{c_p} I \\ \frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho}\nabla p &= 0. \end{aligned} \tag{2.30}$$

Equations (2.30) make up the first model of propagation of pressure in D induced by an EM pulse. If one additionally assumes that the density is constant throughout the region of interest then the velocity $\vec{v}$ can be eliminated. Differentiate the first equation in (2.30) with respect to t and take the divergence of the second. This yields

$$\begin{aligned} \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} + \rho\nabla\cdot\frac{\partial \vec{v}}{\partial t} &= \frac{\alpha\beta}{c_p}\frac{\partial I}{\partial t} \\ \nabla\cdot\frac{\partial \vec{v}}{\partial t} + \frac{1}{\rho}\Delta p &= 0. \end{aligned}$$

Combining these we get a single equation for the hydrostatic pressure

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \Delta p = \frac{\alpha\beta}{c_p}\frac{\partial I}{\partial t}. \tag{2.31}$$

Therefore, $p(\vec{x}, t)$ satisfies a wave equation with acoustic speed

$$c(\vec{x}) = \sqrt{\frac{K(\vec{x})}{\rho}}. \tag{2.32}$$

In most of the TAT literature it has been assumed that both the bulk modulus, K, and the density, ρ, are constant. This implies that the pressure field induced from the TAT process satisfies a constant speed wave equation.

We have derived three distinct systems that can be used as a model for the thermoacoustic process depending on the specific assumptions about the material makeup of the body D being imaged. For clarity and convenience we collect these cases in the following theorem. The theorem is stated only for domains in $\mathbb{R}^3$ but generalizes easily to other dimensions.

Theorem 2.2.1.1 *Suppose the thermoacoustic process is carried out on a domain $D \subset \mathbb{R}^3$ with boundary $S = \partial D$. Suppose that the material making up D is* elastic and isotropic.

Also, assume the intensity, $I(\vec{x},t)$, of the EM source has very short duration compared to the thermal diffusivity of the material in D. Then the stress tensor $\sigma(\vec{x},t)$ and the velocity $\vec{v}(\vec{x},t)$ on D satisfy the system

$$\frac{\rho(\vec{x})}{K(\vec{x})}\left(\frac{\partial\sigma^{ii}}{\partial t}-2\mu(\vec{x})\frac{\partial v^i}{\partial x^i}-\left(K(\vec{x})-\frac{2}{3}\mu(\vec{x})\right)\nabla\cdot\vec{v}\right)=-\frac{\beta}{c_p}\alpha(\vec{x})I(\vec{x},t) \tag{2.33}$$
$$\frac{\partial\sigma^{ij}}{\partial t}-\mu(\vec{x})\left(\frac{\partial v^j}{\partial x^i}+\frac{\partial v^i}{\partial x^j}\right)=0 \text{ for } i\neq j$$
$$\rho(\vec{x})\frac{\partial\vec{v}}{\partial t}-\nabla\cdot\sigma=0.$$

Here $\rho(\vec{x})$ is density, $\mu(\vec{x})$ is the shear modulus, $K(\vec{x})$ is the bulk modulus, $\alpha(\vec{x})$ is the electromagnetic absorption coefficient, β is the thermal expansion coefficient, and c_p is the specific heat.

If we additionally assume that D is hydrostatic *then $\mu(\vec{x})=0$ and $\sigma^{ij}=0$ for $i\neq j$. The above system then reduces to*

$$\frac{\rho(\vec{x})}{K(\vec{x})}\frac{\partial p}{\partial t}+\rho(\vec{x})\nabla\cdot\vec{v}=\frac{\beta}{c_p}\alpha(\vec{x})I(\vec{x},t) \tag{2.34}$$
$$\frac{\partial\vec{v}}{\partial t}+\frac{1}{\rho(\vec{x})}\nabla p=0.$$

Where $p(\vec{x},t)=-\sigma^{11}(\vec{x},t)=-\sigma^{22}(\vec{x},t)=-\sigma^{33}(\vec{x},t)$ is the hydrostatic pressure.

Under the additional assumption of constant density *throughout D this reduces to the single equation for pressure*

$$\frac{1}{c^2(\vec{x})}\frac{\partial^2 p}{\partial t^2}-\Delta p=\frac{\beta}{c_p}\alpha(\vec{x})\frac{\partial I}{\partial t} \tag{2.35}$$

with acoustic speed

$$c(\vec{x})=\sqrt{\frac{K(\vec{x})}{\rho}}. \tag{2.36}$$

Each of the above equations must be satisfied in all of $\mathbb{R}^3\times\mathbb{R}_+$ with the assumption that $\alpha(\vec{x})\equiv 0$ and $K(\vec{x})\equiv K_0$, $\rho(\vec{x})\equiv\rho_0$ on $\mathbb{R}^3\setminus D$. All coefficients are assumed to be

$C^\infty(\mathbb{R}^3)$. *The medium is assumed to be at equilibrium before the EM pulse and therefore we add the condition*

$$\sigma(\vec{x},t) = 0,\ \vec{v}(\vec{x},t) = 0,\ p(\vec{x},t) = 0 \text{ for } t < 0 \text{ and } \vec{x} \in D. \tag{2.37}$$

In the wave equation case the extra condition $\frac{\partial p}{\partial t}(\vec{x},t) = 0$, *for* $t < 0$, *must also be satisfied.*

It is important to note that in the above theorem all of the equations must be satisfied in all of $\mathbb{R}^3$ and not only in the domain D. We will see, in the next section, that this is essential for a unique reconstruction to be possible in TAT.

2.2.2 The Cauchy problem for the thermoacoustic process.

The intensity $I(\vec{x},t)$ takes a special form in thermoacoustic tomography which we describe now [84]. One assumes that the electromagnetic pulse causing the thermoelastic expansion of the material in D can be constructed in such a way that the intensity is equally distributed throughout D. That is, one assumes that $I = I(t)$ is constant in space. The pulse duration is also assumed to be very short so it is usually a good approximation of a delta function. This means we take the intensity to be given by

$$I(\vec{x},t) = \delta(t). \tag{2.38}$$

This choice of pulse shape allows us to write the inhomogeneous hyperbolic equations, (2.34) and (2.35), as homogeneous Cauchy problems. The next theorem demonstrates this relation for the system (2.34). For equations (2.33) and (2.35) similar relations hold but we will not explicitly state them here.

Theorem 2.2.2.1 *For a domain* $D \subset \mathbb{R}^3$ *with material parameters* $K(\vec{x})$, $\rho(\vec{x})$, c_p, β, *and*

$\alpha(\vec{x})$ satisfying the conditions of theorem 2.2.1.1 the following two systems are equivalent:

$$\begin{aligned}\frac{\rho(\vec{x})}{K(\vec{x})}\frac{\partial p}{\partial t}+\rho(\vec{x})\nabla\cdot\vec{v}&=\frac{\beta}{c_p}\alpha(\vec{x})\delta(t)\ \ on\ D\times\mathbb{R}\\ \frac{\partial\vec{v}}{\partial t}+\frac{1}{\rho(\vec{x})}\nabla p&=0\ \ on\ D\times\mathbb{R}\\ p(\vec{x},t)\equiv 0,\ \ \vec{v}(\vec{x},t)\equiv 0\ \ ,\ \ t<0,\ \vec{x}\in D\end{aligned}\tag{2.39}$$

and

$$\begin{aligned}\frac{\rho(\vec{x})}{K(\vec{x})}\frac{\partial p}{\partial t}+\rho(\vec{x})\nabla\cdot\vec{v}&=0\ \ on\ D\times\mathbb{R}_+\\ \frac{\partial\vec{v}}{\partial t}+\frac{1}{\rho(\vec{x})}\nabla p&=0\ \ on\ D\times\mathbb{R}_+\\ p(\vec{x},0)=\frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})},\ \ \vec{v}(\vec{x},0)=0,\ \vec{x}\in D.\end{aligned}\tag{2.40}$$

That is, a solution $(p,\vec{v})$ satisfying (2.39) yields a solution, $(P,\vec{V})$, satisfying (2.40) and vice versa under a simple transformation.

Proof. First suppose that $(p(\vec{x},t),\vec{v}(\vec{x},t))$ satisfy (2.40). Let $H(t)$ be the Heaviside function on $\mathbb{R}$ and define

$$P(\vec{x},t)=p(\vec{x},t)H(t)\text{ and }\vec{V}(\vec{x},t)=\vec{v}(\vec{x},t)H(t).$$

Then $(P,\vec{V})$ satisfies (2.39). To see this first differentiate with respect to time to get

$$\begin{aligned}\partial_t P(\vec{x},t)&=p_t(\vec{x},t)H(t)+p(\vec{x},t)\delta(t)\\ &=p_t(\vec{x},t)H(t)+p(\vec{x},0)\delta(t)\\ &=p_t(\vec{x},t)+\frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}\delta(t)\text{ for }t\geq 0\end{aligned}$$

and

$$\begin{aligned}\partial_t\vec{V}(\vec{x},t)&=\vec{v}_t(\vec{x},t)H(t)+\vec{v}(\vec{x},t)\delta(t)\\ &=\vec{v}_t(\vec{x},t)H(t)+\vec{v}(\vec{x},0)\delta(t)\\ &=\vec{v}_t(\vec{x},t)\text{ for }t\geq 0.\end{aligned}$$

Also,

$$\nabla\cdot\vec{V}(\vec{x},t)=(\nabla\cdot\vec{v}(\vec{x},t))H(t)=\nabla\cdot\vec{v}(\vec{x},t)\text{ for }t\geq 0$$

and

$$\nabla P(\vec{x},t) = (\nabla p(\vec{x},t))H(t) = \nabla p(\vec{x},t) \text{ for } t \geq 0.$$

This is enough to see

$$\begin{aligned}\partial_t P(\vec{x},t) + K(\vec{x})\nabla \cdot \vec{V}(\vec{x},t) &= \partial_t p(\vec{x},t) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},t) + \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}\delta(t) \\ &= \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}\delta(t) \text{ for } t \geq 0\end{aligned}$$

and

$$\begin{aligned}\partial_t \vec{V}(\vec{x},t) + \frac{1}{\rho(\vec{x})}\nabla P(\vec{x},t) &= \partial_t \vec{v}(\vec{x},t) + \frac{1}{\rho(\vec{x})}\nabla p(\vec{x},t) \\ &= 0 \text{ for } t \geq 0.\end{aligned}$$

By definition of the Heaviside function $P(\vec{x},t) \equiv 0$, $\vec{V}(\vec{x},t) \equiv 0$ for $t < 0$ and therefore we may conclude that $(P, \vec{V})$ satisfy (2.39).

Now suppose $(p(\vec{x},t), \vec{v}(\vec{x},t))$ satisfy (2.39). Then automatically the differential system in (2.40) is satisfied for $t > 0$ since $\delta(t)$ is supported at $t = 0$. It remains to show that the initial conditions

$$p(\vec{x},0) = \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})},\ \vec{v}(\vec{x},0) = 0,\ \vec{x} \in D$$

are satisfied. To see this we first integrate both sides of the first equation in (2.39) with respect to time,

$$\begin{aligned}&\int_{\mathbb{R}} (\partial_t p(\vec{x},\tau) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},\tau))\, d\tau \\ &= \int_{\mathbb{R}} \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}\delta(\tau)\, d\tau = \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}.\end{aligned}$$

We now manipulate the left hand side to get

$$\begin{aligned}
&\int_{\mathbb{R}} (\partial_t p(\vec{x},\tau) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},\tau))\, d\tau \\
&= \lim_{t\to 0^+} \int_t^\infty (\partial_t p(\vec{x},\tau) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},\tau))\, d\tau \\
&\quad + \lim_{t\to 0^-} \int_{-\infty}^t (\partial_t p(\vec{x},\tau) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},\tau))\, d\tau \\
&= \lim_{t\to 0^-} \int_{-\infty}^t (\partial_t p(\vec{x},\tau) + K(\vec{x})\nabla \cdot \vec{v}(\vec{x},\tau))\, d\tau \\
&= \lim_{t\to 0^-} \int_{-\infty}^t \partial_t p(\vec{x},\tau)\, d\tau \\
&= \lim_{t\to 0^-} \lim_{s\to -\infty} \int_s^t \partial_t p(\vec{x},\tau)\, d\tau \\
&= \lim_{t\to 0^-} \lim_{s\to -\infty} (p(\vec{x},t) - p(\vec{x},s)) = p(\vec{x},0).
\end{aligned}$$

Here we have used that the differential equation equals zero for $t > 0$ since $\delta(t)$ is supported at $t = 0$. We have also used that $\vec{v}(\vec{x},t) \equiv 0$ and $p(\vec{x},t) \equiv 0$ for $t < 0$. Combining the above two computations we see

$$p(\vec{x},0) = \frac{\beta}{c_p} \frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}.$$

A similar computation with the second equation in (2.39) shows that $\vec{v}(\vec{x},0) = 0$. Thus, the statement of the theorem holds.

■

With the same method of proof as that of theorem 2.2.2.1 equations (2.33) and (2.35) can be transformed into Cauchy problems with initial conditions if it is assumed that $I(\vec{x},t) = \delta(t)$. These statements are collected in the following subsection.

2.2.3 The object of measurement and reconstruction in thermoacoustic tomography.

Here we formally state the *thermoacoustic imaging problem* for the Cauchy problems governing the thermoacoustic effect. For each of the systems, (2.33), (2.34), and (2.35) we state what is measured and reconstructed in TAT.

In thermoacoustic tomography the domain D is irradiated with an EM pulse of the form $I(\vec{x},t)=\delta(t)$. Ultrasound transducers on the boundary ∂D then record the pressure field over time [84, 52, 88]. It has been demonstrated that cancerous tissue has a much higher conductivity than healthy tissue [22, 45, 53]. This has the effect of increasing the absorption coefficient $\alpha(\vec{x})$ for cancerous tissue. In medical applications of TAT one then attempts to reconstruct the term $\frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}$, this should give a decent image for $\alpha(\vec{x})$ if the variations in $\frac{K(\vec{x})}{\rho(\vec{x})}$ are small. Depending on the material properties of the medium in D, TAT is modeled by one of the following three systems.

Case 1: If D is assumed to be *elastic and isotropic* then the stress $\sigma(\vec{x},t)$ and velocity $\vec{v}(\vec{x},t)$ are governed by

$$\begin{aligned}
\frac{\rho(\vec{x})}{K(\vec{x})}\left(\frac{\partial\sigma^{ii}}{\partial t}-2\mu(\vec{x})\frac{\partial v^i}{\partial x^i}-\left(K(\vec{x})-\frac{2}{3}\mu(\vec{x})\right)\nabla\cdot\vec{v}\right)&=0 \\
\text{for } i=1,2,3 \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\
\frac{\partial\sigma^{ij}}{\partial t}-\mu(\vec{x})\left(\frac{\partial v^j}{\partial x^i}+\frac{\partial v^i}{\partial x^j}\right)&=0 \text{ for } i\neq j \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\
\rho(\vec{x})\frac{\partial\vec{v}}{\partial t}-\nabla\cdot\sigma&=0 \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\
\sigma^{ii}(\vec{x},0)&=-\frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})} \text{ for } i=1,2,3 \\
\sigma^{ij}(\vec{x},0)=0 \text{ for } i\neq j \text{ and } \vec{v}(\vec{x},0)&=0 \text{ on } \mathbb{R}^3.
\end{aligned} \tag{2.41}$$

In this case the pressure along the boundary ∂D is measured

$$h(\vec{x},t)=-[\sigma(\vec{x},t)\,\vec{n}(\vec{x})]\cdot\vec{n}(\vec{x}) \tag{2.42}$$

for $(\vec{x},t)\in\partial D\times\mathbb{R}_+$. Here $\vec{n}(\vec{x})$ is the unit outward normal to ∂D at a point $\vec{x}\in\partial D$.

In this case the measured pressure is the negative of the magnitude of the internal force on ∂D at the point $\vec{x}\in D$ in the direction of the unit outward normal to ∂D. From these measurements one attempts to reconstruct

$$f(\vec{x})=-\frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}$$

for $\vec{x}\in D$.

Case 2: If D is assumed to be *elastic, isotropic, and hydrostatic* then the pressure $p(\vec{x},t)$ and velocity $\vec{v}(\vec{x},t)$ are governed by

$$\begin{aligned} \frac{\rho(\vec{x})}{K(\vec{x})}\frac{\partial p}{\partial t} + \rho(\vec{x})\nabla\cdot\vec{v} = 0 \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\ \frac{\partial\vec{v}}{\partial t} + \frac{1}{\rho(\vec{x})}\nabla p = 0 \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\ p(\vec{x},0) = \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})},\ \vec{v}(\vec{x},0) = 0,\ \vec{x}\in\mathbb{R}^3. \end{aligned} \tag{2.43}$$

The pressure is then measured on ∂D so the data collected is

$$h(\vec{x},t) = p(\vec{x},t)|_{\partial D\times\mathbb{R}_+}.$$

From these measurements one attempts to reconstruct

$$f(\vec{x}) = \frac{\beta}{c_p}\frac{K(\vec{x})\alpha(\vec{x})}{\rho(\vec{x})}$$

for $\vec{x}\in D$.

Case 3: If D is assumed to be *elastic, isotropic, hydrostatic*, and has *constant density* then the pressure $p(\vec{x},t)$ and velocity $\vec{v}(\vec{x},t)$ are governed by

$$\begin{aligned} \frac{1}{c^2(\vec{x})}\frac{\partial^2 p}{\partial t^2} - \Delta p = 0 \text{ on } \mathbb{R}^3\times\mathbb{R}_+ \\ p(\vec{x},0) = \frac{\beta}{c_p}c^2(\vec{x})\alpha(\vec{x}),\ p_t(\vec{x},0) = 0,\ \vec{x}\in\mathbb{R}^3. \end{aligned} \tag{2.44}$$

Where the acoustic speed is defined by

$$c(\vec{x}) = \sqrt{\frac{K(\vec{x})}{\rho}}.$$

The pressure is then measured on ∂D so the data collected is

$$h(\vec{x},t) = p(\vec{x},t)|_{\partial D\times\mathbb{R}_+}.$$

From these measurements one attempts to reconstruct

$$f(\vec{x}) = \frac{\beta}{c_p}c^2(\vec{x})\alpha(\vec{x})$$

for $\vec{x} \in D$.

The first two cases remain, to the author's knowledge, unstudied. The third case is very well studied when the acoustic speed $c(\vec{x}) = 1$ or is at least constant [3, 35, 36, 84, 50, 88]. Recently the problem of reconstructing $f(\vec{x})$ when $c(\vec{x})$ is variable has gained interest [72, 44, 43, 90, 2, 83, 89]. However, the dependence of $f(\vec{x})$ on $c(\vec{x})$ when the acoustic profile is unknown has not been investigated. Since, in medical imaging applications, the function desired is $\alpha(\vec{x})$ this dependence is very important for future investigation. In actuality the EM pulse, $I(\vec{x}, t)$, is not uniformly distributed throughout D due to electromagnetic scattering caused by variations in the conductivity of the region D, which we are trying to reconstruct. This problem has recently attracted interest [56, 7, 8, 9] and has led to important advancements. For the remainder of this dissertation we will leave these problems unsolved and assume that the domain D satisfies the conditions of **case 3**. It will also be assumed that the EM pulse is well approximated by $\delta(t)$ and the function to be reconstructed, $f(\vec{x})$, does not depend on the acoustic speed.

2.3 Results on the Unique Determination of the Source

The current results on the question of the unique determination of the source in thermoacoustic tomography for a known acoustic speed are discussed. We mention only a few of the more well known results in the field and point the interested reader to the review paper [52] for a more complete discussion. First we state this problem in the form that the results pertain to.

From the previous section we have that a decent model of the thermoacoustic process on a domain $D \subset \mathbb{R}^n$ is given by a pressure field $p(x, t)$ satisfying

$$\frac{\partial^2 p}{\partial t^2} - c^2(x)\Delta p = 0 \,\text{on}\, \mathbb{R}^n \times \mathbb{R}_+ \tag{2.45}$$
$$p(x, 0) = f(x),\ p_t(x, 0) = 0,\ x \in \mathbb{R}^n.$$

The measurements collected are

$$h(x,t) = p(x,t)|_{\Gamma\times\mathbb{R}_+} \tag{2.46}$$

where $\Gamma \subset \partial D$. One attempts to reconstruct the initial source $f(x)$ from knowledge of $h(x,t)$ and asks if $f(x)$ is uniquely determined from this data. In general the answer to this question depends on the support of $f(x)$, the acoustic profile $c(x)$, and the measurement subset $\Gamma \subset \partial D$.

Notice that the forward problem governing the propagation of the ultrasound field is defined in all of $\mathbb{R}^n$. Without this condition there would be no hope of uniquely determining $f(x)$ from the data. To see this, suppose we formulated the incorrect problem

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - c^2(x)\Delta p = 0 \,\text{on}\; D\times\mathbb{R}_+ \\ p(x,0) = f(x),\; p_t(x,0) = 0,\; x\in D \end{aligned} \tag{2.47}$$

and collected the data

$$h(x,t) = p(x,t)|_{\partial D\times\mathbb{R}_+}. \tag{2.48}$$

Then for any function $g(x)$ defined on D we can uniquely solve the forward problem, {§7.2.2, [34]},

$$\begin{aligned} \frac{\partial^2 p}{\partial t^2} - c^2(x)\Delta p = 0 \,\text{on}\; D\times\mathbb{R}_+ \\ p(x,t) = h(x,t) \,\text{on}\; \partial D\times\mathbb{R}_+ \\ p(x,0) = g(x),\; p_t(x,0) = 0,\; x\in D. \end{aligned} \tag{2.49}$$

Thus, the data generated from the two different sources $f(x)$ and $g(x)$ would be identical. The reason this does not happen in the correct formulation of the problem is, by specifying that $p(x,t)$ must satisfy the equation in all of $\mathbb{R}^n$, we have in fact imposed *transmission* boundary conditions on the measurement boundary, ∂D. The imposition of these boundary conditions along with the measurement data allows the unique determination of $f(x)$.

The first uniqueness theorem we state shows that if $c(x) = 1$ then the source function is uniquely determined from pressure data collected on a subset of the boundary of the domain.

Theorem 2.3.0.1 *[36] Suppose D is a bounded, open, subset of $\mathbb{R}^n$, $n \geq 2$, with a smooth boundary ∂D, and $\overline{D}$ is strictly convex. Let $\Gamma \subset \partial D$ be a relatively open subset. Suppose $f \in C_0^\infty(\mathbb{R}^n)$ with $\text{supp}(f) \subset \overline{D}$ and $p(x,t)$ satisfies (2.45) with $c(x) = 1$. If $p(x,t) = 0$ for all $x \in \Gamma$ and $t > 0$ then $f \equiv 0$.*

Though the statement of the theorem is for $c(x) = 1$ it is not difficult to adapt the proof to any constant acoustic profile $c(x) = c$. We should also mention that the statement of the theorem in [36] uses the equation (2.45) with the alternate initial conditions

$$p(x,0) = 0, \; p_t(\vec{x},0) = f(x), \; x \in \mathbb{R}^n.$$

These initial conditions can be changed to our initial conditions by a standard transformation [35].

The two important requirements for this theorem are the assumptions, $supp(f) \subset \overline{D}$, and $c(x) = constant$. These conditions can be modified and the source can still be uniquely determined by the acoustic data. The next two uniqueness theorems we state deal with these cases. Since the next two theorems will involve variable acoustic profiles $c(x)$ we must first discuss some important requirements for allowable variable sound speeds. These requirements are necessary for physical reasons, to show the unique determination of $f(x)$, and for the reconstruction formulas in the next section to hold.

In the thermoacoustic literature the variable acoustic profile is usually assumed to satisfy the conditions:

1. $c(x) \in C^\infty(\mathbb{R}^n)$ with $0 < c < c(x)$ for $x \in \mathbb{R}^n$ and $supp(1 - c(x))$ is compact

2. $c(x)$ must be *non-trapping*

Throughout this manuscript the acoustic profiles will satisfy the above conditions. The first condition ensures that the acoustic profile is realistic and that outside of some compact set the medium has homogeneous acoustic properties. We will give a more rigorous mathematical statement of the non-trapping condition in the following chapter. For now the second condition will mean that the acoustic speed is such that any pressure disturbance will propagate outside of any compact domain as $t \to \infty$. With these conditions on a variable acoustic speed the following uniqueness theorem has been shown.

Theorem 2.3.0.2 *[2] For a source function $f \in C_0^\infty(\mathbb{R}^n)$ and a domain $D \subset \mathbb{R}^n$ with compact closure let $p(x,t)$ satisfy (2.45) with $c(x)$ satisfying the above two conditions. Then the data $p(x,t)|_{\partial D \times \mathbb{R}_+}$ uniquely determines $f(x)$ in D.*

Notice here that the data must be collected on a closed measurement surface ∂D enclosing a compact region in $\mathbb{R}^n$. The uniqueness however holds for a variable acoustic speed and the *supp*(f) is assumed to be compact but not necessarily contained in D.

For the statement of the next theorem we must define a notion of distance relative to the travel time of acoustic signals under propagation governed by (2.45). To do this we define the Riemannian metric $g_{ij}(x) = c^{-2}(x)\delta_{ij}$ on $\mathbb{R}^n$. The *distance* between two points x, $z \in \mathbb{R}^n$ is then defined to be the infimum of the lengths of all geodesics in the metric g_{ij} joining x and z, we write *dist*(x,z). The following uniqueness theorem then holds.

Theorem 2.3.0.3 *[72] Suppose* $\operatorname{supp}(1-c) \subset D$ *and ∂D is strictly convex. Let $\Gamma \subset \partial D$ be a relatively open subset and define the measurement domain*

$$G = \{(x,t) : x \in \Gamma,\, 0 < t < s(x)\} \tag{2.50}$$

where $s(x)$ is a fixed continuous function on Γ. Fix a compact subset $K \subset D$ satisfying the condition

$$\forall\, x \in K\ \exists\, z \in \Gamma\ \ \textit{so that}\ \operatorname{dist}(x,z) < s(z). \tag{2.51}$$

Then if $\mathrm{supp}(f) \subset K$ *the function* f *is uniquely determined by the data* $p(x,t)|_G$ *where* $p(x,t)$ *satisfies (2.45).*

This means that the source function is uniquely determined in TAT for a variable acoustic speed with only partial data. This theorem now only holds for sources supported within some fixed compact set. Moreover, the compact set must satisfy the additional condition (2.51). This condition ensures that for every $x \in supp(f)$ at least one geodesic connects x to the measurement boundary Γ.

Thus, for the thermoacoustic process governed by (2.45) if one knows the acoustic speed then the acoustic source is uniquely determined by data collected on a subset of ∂D provided that we measure for a long enough time to receive signals from everywhere within the support of the source.

2.4 Results on Reconstruction of Acoustic Source

We review some of the methods of reconstructing the acoustic source in TAT using the model (2.45). The methods presented deal with both the constant acoustic speed case and the variable acoustic speed case. However, all of these methods require full boundary data and that the support of the acoustic source is contained inside the domain D. If the acoustic speed is variable it is assumed to satisfy the assumptions mentioned before theorem (2.3.0.2). Only a few methods in each of the better known classes of reconstruction algorithms for TAT are discussed. For a more complete review see [3, 52].

2.4.1 Reconstruction by Filtered Back-projection

It is well known that solutions of (2.45), with $c(x) = 1$, can be written in terms of spherical means of the source function $f(x)$. From [29], if $p(x,t)$ satisfies (2.45) in $\mathbb{R}^n \times \mathbb{R}_+$

then

$$p(x,t) = \frac{1}{(n-2)!}\partial_t^{n-1}\int_0^t r(t^2-r^2)^{(n-3)/2}(\mathcal{M}f)(x,r)\,dr. \tag{2.52}$$

The *spherical mean* operator is defined by

$$(\mathcal{M}f)(x,t) = \frac{1}{\omega_n}\int_{S_1(0)} f(x+t\theta)\,dS(\theta) \tag{2.53}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$, $S_1(0)$ is the unit sphere in $\mathbb{R}^n$, and $dS(\theta)$ is the surface element on $S_1(0)$.

For TAT conducted on a ball of radius r_0, $B_{r_0}(0) \subset \mathbb{R}^n$, reconstruction formulas based on inversion of the spherical mean have been shown to hold in all dimensions, $n \geq 2$, [35, 36, 50]. These inversion formulas are of filtered back-projection type. The first theorem is stated for the thermoacoustic operator

$$\mathcal{W}f(x,t) = p(x,t)|_{\partial B_{r_0}(0)\times\mathbb{R}_+} \tag{2.54}$$

with $p(x,t)$ defined by (2.52).

Theorem 2.4.1.1 *[35, 36] Let $f(x) \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, with* $\operatorname{supp}(f) \subset B_{r_0}(0)$. *Then for $x \in B_{r_0}(0)$*

$$f(x) = \frac{2}{r_0}(\mathcal{W}^* t\mathcal{W}f)(x). \tag{2.55}$$

Here the operator $\mathcal{W}^*$ is the formal L^2-adjoint of $\mathcal{W}$ defined on functions in $C^\infty(\partial B_{r_0}(0)\times\mathbb{R}_+)$ with sufficient decay in the second variable.

The papers [35, 36] show other possible inversion formulas for solutions of (2.45) with the source $f(x)$ either on the initial condition $p(x,0)$ or $\partial_t p(x,0)$. These have slightly different forms in even and odd dimensions but all are based around a study of the spherical mean operator. We have chosen to state the above form since it pertains to the initial conditions for TAT that we study in the rest of this thesis and because the above form holds in all dimensions. It should be noted here that these inversion formulas imply inversion formulas for the spherical mean operator itself. The next theorem we state is an alternate inversion formula for the spherical mean.

Due to the representation of the pressure field in terms of spherical means of the initial source (2.52) one can focus on just inverting the spherical mean transform when developing reconstruction formulas for constant speed thermoacoustic tomography. For completeness we state the following inversion formula proved in [50].

This inversion formula is stated in terms of Bessel functions of the first and second kind. To this end we define the functions

$$\begin{aligned} J(t) &= t^{1-n/2} J_{n/2-1}(t) \\ N(t) &= t^{1-n/2} N_{n/2-1}(t) \end{aligned} \tag{2.56}$$

where $J_{n/2-1}(t)$ and $N_{n/2-1}(t)$ are Bessel functions of order $n/2-1$ of the first and second kind respectively.

Theorem 2.4.1.2 *[50] Let $f(x) \in C_0^\infty(\mathbb{R}^n)$, $n \geq 2$, with* $\operatorname{supp}(f) \subset B_{r_0}(0)$. *Then for $x \in B_{r_0}(0)$,*

$$f(x) = \frac{1}{4(2\pi)^{n-1}} \nabla_y \cdot \int_{\partial B_{r_0}(0)} \vec{\nu}(y) h(y, |x-y|)\, dS(y) \tag{2.57}$$

where

$$\begin{aligned} h(y,t) = \int_0^\infty \Bigg\{ & N(rt) \left(\int_0^{2r_0} J(rt')(\mathcal{M}f)(y,t')\, dt' \right) \\ & - J(rt) \left(\int_0^{2r_0} N(rt')(\mathcal{M}f)(y,t')\, dt' \right) \Bigg\} r^{2n-3}\, dr \end{aligned} \tag{2.58}$$

and $\vec{\nu}(y)$ is the unit outward normal to $S_{r_0}(0)$ at $y \in S_{r_0}$.

This reconstruction procedure was discovered earlier in three dimensions and implemented in [88].

The above reconstruction methods, or inversion of the spherical mean operator, pertain to the TAT model only when the domain of measurement is assumed to be spherical and the acoustic speed is assumed constant. The implementation of the second reconstruction procedure was actually discussed, for arbitrary domains, in [50]. However, in this case the inversion procedures developed have the form of an infinite summation and thus do not have a closed form filtered back-projection type expression like the ones stated above. In this regard they are more like our next set of reconstruction methods.

2.4.2 Reconstruction by Eigenfunctions

Since the equation governing forward propagation of the ultrasound field, (2.45), is defined by the operator $A = -c^2(x)\Delta$ it makes sense that the eigenfunction expansion of this operator may be useful in determining reconstruction formulas. Indeed methods relying on this intuition were the first inversion formulas developed for the spherical mean operator [63] which relates to TAT if $c(x) = 1$. To state the reconstruction results we first develop some essential definitions.

For a domain $D \subset \mathbb{R}^n$ with boundary ∂D we define the Hilbert space $L^2(D, c^{-2}(x)\,dx)$. On this space the positive, self-adjoint operator $A = -c^2(x)\Delta$ on D with zero Dirichlet conditions on ∂D has an eigenbasis $\{\lambda_k^2,\ \psi_k(x)\}$. This basis is defined by

$$-c^2(x)\Delta\psi_k(x) = \lambda_k^2\psi_k(x) \text{ on } D$$
$$\psi_k(x) = 0 \text{ on } \partial D$$

for each k. The λ_k are all positive since A is positive and the eigenfunctions $\psi_k(x)$ form a basis for $L^2(D, c^{-2}(x)\,dx)$.

Suppose $p(x,t)$ now satisfies (2.45) on D for some initial impulse $f(x)$. Then the measured TAT data is $g(x,t) = p(x,t)|_{\partial D\times\mathbb{R}_+}$ and $p(x,t)$ satisfies

$$\frac{\partial^2 p}{\partial t^2} - c^2(x)\Delta p = 0 \,\text{on}\; D \times \mathbb{R}_+ \tag{2.59}$$
$$p(x,t) = g(x,t) \,\text{on}\; \partial D \times \mathbb{R}_+.$$
$$p(x,0) = f(x),\ p_t(x,0) = 0,\ x \in D.$$

We may define the *harmonic extension operator* $E : H^s(\partial D) \to H^{s+\frac{1}{2}}(D)$ defined by mapping a function $h(x) \in H^s(\partial D)$ to a solution $u(x) \in H^{s+\frac{1}{2}}(D)$ to

$$\Delta u = 0 \text{ on } D \tag{2.60}$$
$$u|_{\partial D} = h.$$

Applying this operator to the thermoacoustic data at each time $t \geq 0$ we see that the function $u(x,t) = p(x,t) - Eg(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial t^2} - c^2(x)\Delta u = -E\left(\frac{\partial^2 g}{\partial t^2}\right) \text{ on } D \times \mathbb{R}_+ \tag{2.61}$$
$$u(x,t) = 0 \,\text{on } \partial D \times \mathbb{R}_+.$$
$$u(x,0) = f(x) - Eg(x,0),\ p_t(x,0) = 0,\ x \in D.$$

As long as the acoustic speed $c(x)$ is non-trapping and constant outside of some bounded region in $\mathbb{R}^n$ then it can be shown [2] that $u(x,t)$ can be represented as

$$u(x,t) = \int_t^\infty A^{-\frac{1}{2}} \sin((t-\tau)A^{\frac{1}{2}})E\left(\frac{\partial^2 g}{\partial t^2}\right)(x,\tau)\,d\tau. \tag{2.62}$$

Here we have used the operators $A^{-\frac{1}{2}}$ and $\sin(tA^{\frac{1}{2}})$ defined in terms of an eigenexpansion by

$$A^{-\frac{1}{2}} = \sum_{k\geq 0} \frac{1}{\lambda_k}\psi_k(x) \tag{2.63}$$
$$\sin(tA^{\frac{1}{2}}) = \sum_{k\geq 0} \sin(t\lambda_k)\psi_k(x).$$

The following reconstruction formulas for $f(x)$ then hold.

Theorem 2.4.2.1 *[2] For a bounded domain $D \subset \mathbb{R}^n$ and an acoustic speed $c(x)$ that is non-trapping and constant outside of a bounded region the acoustic source $f(x)$ can be reconstructed inside D from measured acoustic data $g(x,t)$ on ∂D by*

$$f(x) = Eg(x,0) - \int_0^\infty A^{-\frac{1}{2}} \sin((t-\tau)A^{\frac{1}{2}})E\left(\frac{\partial^2 g}{\partial t^2}\right)(x,\tau)\,d\tau. \tag{2.64}$$

In terms of expansion in eigenfunctions of the operator A we have

$$f(x) = \sum_{k \geq 0} f_k(x)\psi_k(x) \tag{2.65}$$

where the coefficients can be reconstructed from the the data $g(x,t)$ by

$$\begin{aligned} f_k(x) &= \frac{1}{\lambda_k^2} g_k(0) - \frac{1}{\lambda_k^3}\int_0^\infty \sin(\lambda_k t) g_k''(t)\,dt \\ f_k(x) &= \frac{1}{\lambda_k^2} g_k(0) - \frac{1}{\lambda_k^2}\int_0^\infty \cos(\lambda_k t) g_k'(t)\,dt \\ f_k(x) &= -\frac{1}{\lambda_k}\int_0^\infty \sin(\lambda_k t) g_k(t)\,dt. \end{aligned} \tag{2.66}$$

With

$$g_k(t) = \int_{\partial D} g(x,t)\overline{\frac{\partial \psi_k}{\partial \nu}(x)}\,dx, \tag{2.67}$$

ν being the exterior unit normal to ∂D.

It should be noted that the above formulas represent a general form for the special types of reconstruction methods developed in [63] earlier. The above reconstruction formulas hold in very general settings. The domain D can be any shape and the acoustic speed can be variable as long as it is non-trapping and eventually homogeneous. The source function can also be non-zero outside of the measurement domain D. The drawback of these reconstruction formulas is the necessity of finding the eigenbasis $\{\lambda_k^2,\, \psi_k\}$. These are known for only specific sets of acoustic profiles and domains D. For example constant acoustic profiles and rectangular or radial domains. However, it has been demonstrated [51] that if the domain D is a square and the acoustic speed is constant then these formulas allow for very fast and accurate reconstructions.

2.4.3 Reconstruction by Time Reversal

The most generally applicable method of reconstructing the acoustic source $f(x)$ in thermoacoustic tomography is the method of *time reversal* or *back propagation* of the data.

This method allows for relatively easily implemented approximate reconstructions in the case of both constant and variable acoustic speeds. Until recently [72] this method had only been used to find approximate, though very good, reconstructions [44, 43, 12]. We will describe the approximate reconstruction and state some of the known error estimates associated with it [43]. We will then state a theorem explaining how this reconstruction can be made exact [72].

Assume we measure the acoustic data

$$g(x,t) = p(x,t)|_{\partial D \times [0,T]}$$

for a domain $D \subset \mathbb{R}^n$ with $p(x,t)$ satisfying (2.45) with initial source $f(x)$. Notice that now we assume that the data is measured only for a finite interval of time $[0,T]$. The acoustic speed $c(x)$ is again assumed to be non-trapping. It can be shown, and we state the exact theorem in the next chapter, that under this assumption the pressure field $p(x,t)$ has at least polynomial decay in the domain D and thus $p(x,t) \to 0$ as $t \to \infty$ for $x \in D$. This motivates an approximate reconstruction of the source by time reversal of the pressure field.

In time reversal the backwards Cauchy problem

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} - c^2(x)\Delta v = 0 \text{ on } D \times [0,T] \\ v(x,t) = g(x,t) \text{ on } \partial D \times [0,T] \\ v(x,T) = 0,\ v_t(x,T) = 0,\ x \in D \end{aligned} \tag{2.68}$$

is solved. The approximate reconstruction of the source $f(x)$ is then given by

$$(Ag)(x) = v(x,0) \text{ for } x \in D. \tag{2.69}$$

Notice that this reconstruction would be exact if we instead let $v(x,t)$ solve the backward problem

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} - c^2(x)\Delta v = 0 \text{ on } D \times [0,T] \\ v(x,t) = g(x,t) \text{ on } \partial D \times [0,T] \\ v(x,T) = p(x,T),\ v_t(x,T) = p_t(x,T),\ x \in D. \end{aligned} \tag{2.70}$$

However, we do not know the pressure field and its derivative on the interior of D at any time. The choice of the terminal data $v(x,T)=0$, $v_t(x,T)=0$ in the time reversal process is motivated by the decay of the pressure field as $t\to\infty$. Because of this property we expect that $(Ag)(x)\to f(x)$ as the measurement time $T\to\infty$.

This process can be made exact for a finite measurement time T if one makes the following change. Define the operator of harmonic extension with respect to the operator $c^2(x)\Delta$ by $P: H(\partial D)\to \tilde{H}(D)$ taking $h(x)\in H(\partial D)$ to $u(x)=Ph(x)\in\tilde{H}(D)$ satisfying

$$\begin{aligned} c^2(x)\Delta u = 0 \text{ on } D \\ u|_{\partial D} = h \end{aligned} \tag{2.71}$$

where $H(D)$ and $\tilde{H}(D)$ are suitable Hilbert spaces. This method is actually applicable for much more general acoustic operators than $c^2(x)\Delta$ [72].

A second time reversal operator is now defined using P. Let $\tilde{v}(x,t)$ solve the back propagation problem

$$\begin{aligned} \frac{\partial^2\tilde{v}}{\partial t^2} - c^2(x)\Delta\tilde{v} = 0 \,\text{on } D\times[0,T] \\ \tilde{v}(x,t) = g(x,t) \,\text{on } \partial D\times[0,T] \\ \tilde{v}(x,T) = (Pg)(x,T),\ \tilde{v}_t(x,T) = 0,\ x\in D, \end{aligned} \tag{2.72}$$

and define the corrected time reversal operator

$$(\tilde{A}g)(x) = \tilde{v}(x,0) \text{ for } x\in D. \tag{2.73}$$

For the statement of the theorem we define the forward TAT operator, mapping the source to the thermoacoustic data,

$$(\mathcal{L}f)(x,t) = p(x,t)|_{\partial D\times[0,T]} \tag{2.74}$$

where $p(x,t)$ solves (2.45). Then the following theorem detailing an exact reconstruction of $f(x)$ holds.

Theorem 2.4.3.1 *[72] Let $T > \text{diam}(D)$. Then $\tilde{A}\mathcal{L} = I - K$, where K is compact in $H^1(D)$ and $\|K\|_{H^1(D)} < 1$. Thus, $I - K$ can be inverted by a Neumann series so the reconstruction*

$$f(x) = \sum_{m=0}^{\infty} K^m \tilde{A}\mathcal{L} f(x) \tag{2.75}$$

holds.

Here the *diameter*, $\text{diam}(D)$, is defined to be the supremum of the lengths of all geodesics under the metric $g_{ij} = c^{-2}(x)\delta_{ij}$.

This reconstruction method still requires infinitely many successive iterations to be exact. Moreover it is difficult to implement the operator $K = I - \tilde{A}\mathcal{L}$ since this involves solving a forward and backward propagation problem along with a Dirichlet problem. However, even using the first few terms in the sum (2.75) should let us find a better approximation than just using the reversal operator A.

We now state the error estimates for the time reversal process, using the operator A, developed in [43]. It was noted above that even though the reconstruction using A is not exact for finite measurement time $T > 0$ it should become more exact as $T \to \infty$. In order for the backward propagation associated with A to allow error estimates we must modify it slightly by mollifying the data $g(x,t)$. To this end let $\phi_\epsilon(t)$ be a C^∞ function on $\mathbb{R}$ with $\text{supp}\, \phi_\epsilon \subset [0,T)$ and $\phi_\epsilon(t) = 1$ for $0 \le t < T - \epsilon$. Then let $v(x,t)$ satisfy

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} - c^2(x)\Delta v &= 0 \text{ on } D \times [0,T] \\ v(x,t) &= g(x,t)\phi_\epsilon(t) \text{ on } \partial D \times [0,T] \\ v(x,T) = 0,\ v_t(x,T) &= 0,\ x \in D. \end{aligned} \tag{2.76}$$

Define the mollified time reversal operator by

$$(A'g)(x) = v(x,0) \text{ for } x \in D.$$

Then the following error estimate was shown in [43].

Theorem 2.4.3.2 *[43] There exists T_0 such that for $T > T_0$ and $\epsilon > 0$ satisfying $T - \epsilon > T_0$, the following error estimates for the acoustic source $f(x)$ hold:*

$$\|f - A'g\|_{H_0^1(D)} \leq C(\epsilon)(T-\epsilon)^{1-n}\|f\|_{L^2(D)} \text{ for even } n, \tag{2.77}$$

$$\|f - A'g\|_{H_0^1(D)} \leq C(\epsilon)e^{-\delta(T-\epsilon)}\|f\|_{L^2(D)} \text{ for odd } n. \tag{2.78}$$

Here $\delta > 0$ and $C(\epsilon) = C/\min\{\epsilon^2, 1\}$ for some constant C.

2.5 Conclusion

This concludes our overview of the thermoacoustic process, its physical basis, mathematical formulation, theory of unique determination of the source, and some of the more well established reconstruction methods available. We have left much theory out including the results that exist in the partial data problem [6, 85, 12], the results on the stability of the determination of the source [72], and minimization methods of reconstruction [83, 90]. Also left unmentioned is a discussion of the feasibility of a uniform distribution of the EM field in the initial pulse and corrections in the case that it is non-uniform [9, 8, 7, 56].

3 UNIQUE DETERMINATION OF ACOUSTIC PROFILE IN TAT

3.1 Introduction

It has become apparent that, to at least some extent, the measurements of an ultrasound field acquired in thermoacoustic tomography determine the acoustic properties of the body being imaged. However, very few theorems have been proved about this aspect of the TAT process. The results in this direction, especially those concerning reconstruction, are very weak or purely heuristic [83, 44, 90].

In this chapter we prove results concerning when two different acoustic profiles on a body D generate distinct thermoacoustic data. These results rely on estimates of the decay of TAT data in time and on properties of the spectrum of a particular, non-classical, boundary value problem having its origin in scattering theory.

The determination of the acoustic properties of a body from thermoacoustic measurements would improve this imaging modality significantly. Classically, in thermoacoustic reconstruction, it has been assumed that the acoustic speed in the body is constant. It has been demonstrated [89] that this assumption can lead to many artifacts in an image if the medium being imaged has variations in acoustic speed. Also, for medical diagnosis, the acoustics themselves are of interest [63, 38], so having a way to reconstruct both the electromagnetic and acoustic properties of a body with a single imaging method would be desirable.

Recently, motivated by the above concerns, methods of TAT reconstruction have been developed [44, 43, 72] which take into account acoustic variations throughout the imaged domain. These results rely on the variations in acoustics being known *a priori*. So far these properties must first be reconstructed using ultrasound tomography before the thermoacoustic imaging methods may be implemented.

Previously, the problem of determining acoustic profiles using TAT has attracted little attention. There are a few reasons for this. First, the thermoacoustic method was developed to partially make up for the low contrast of ultrasound tomography (UT) methods. For many biological materials the variation in acoustic properties between healthy and cancerous tissue is too small to reliably diagnose healthy and unhealthy tissue. The variations in electromagnetic properties of healthy and unhealthy tissue are large enough for a more reliable diagnosis. So the assumption of constant acoustic properties seemed reasonable.

Moreover, the assumption of a constant acoustic profile justified the use of the spherical mean operator and therefore justified the development of filtered backprojection reconstruction methods. Considering the great deal of research already available in this area the constant acoustic assumption was very convenient.

The study of reconstructing acoustic properties from thermoacoustic measurements differs greatly from other imaging methods, namely UT and seismic imaging, in which sound speed is recovered. In both UT and seismic imaging one assumes that the sources generating the measured acoustic field are known, both location and pulse shape [63, 66, 74]. This is not the case for TAT, in which the main goal is to reconstruct the original acoustic source. More, in UT one usually attempts to recover only a Born approximation to the actual acoustic speed of the body [63, 71]. In seismic imaging it is usually assumed that the background velocity is constant and only discontinuities or sharp variations in the acoustic properties are recovered [66].

Given that the acoustic sources, and therefore the travel times, are unknown in TAT it seems, at first, that there is no hope of the measurement process determining the acoustics of the domain. However, heuristic minimization methods of reconstructing acoustic properties of a body from thermoacoustic data have been implemented with some success both numerically and experimentally [83, 90]. It has also been shown that [44] the sound speed is uniquely determined within a one parameter family of acoustic profiles from TAT data.

In this chapter we do not look for uniqueness results in the classical sense. Instead we will call a *condition for uniqueness* any condition concerning two acoustic profiles on a body D that imply the measurements generated by the respective profiles will be distinct. This idea can then be extended to a subset $\mathcal{D}$ of acoustic profiles. We may look for specific subsets of acoustic speeds in which each profile must generate distinct thermoacoustic measurements. This is equivalent to what was shown for one parameter families in [44].

To prove our results we will exploit two properties of the thermoacoustic data. First, for certain types of acoustic profiles it can be shown that the TAT data has rapid decay. This will imply that the temporal Fourier transform has convenient analytic properties.

Second, it will be shown that the measurements are related to the *interior transmission problem* of scattering theory. The spectrum of this problem is well studied and its properties will be used to justify our results.

The outline of this chapter is as follows. In section 3.2 we elaborate on the type of uniqueness studied and its relation to questions about the range of the thermoacoustic operator. Next, section 3.3, the analytic properties of the TAT data are studied using the decay of solutions to the wave equation and the temporal Fourier transform. We then introduce, section 3.4, the *interior transmission problem* and show how it is related to thermoacoustic measurements. Here we also show what properties of the *transmission spectrum* imply uniqueness results. The chapter is concluded, section 3.5, with our main uniqueness results in which specific conditions on acoustic profiles are given that ensure distinct thermoacoustic data is generated.

3.2 Results on the Range of $\mathcal{L}_c$

We discuss the relation of the range of the thermoacoustic operator, $\mathcal{L}_{c(x)}$, and the unique determination of the profile $c(x)$. For completeness we will then elaborate on some previous results concerning the range of this operator.

Consider a body $D \subset \mathbb{R}^3$ with a thermoacoustically induced ultrasound field $u(x,t)$. If the acoustic profile on $D \subset \mathbb{R}^3$ is given by a function $c(x)$ then the field must satisfy

$$\partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ on } \mathbb{R}^3 \times \mathbb{R}_+ \tag{3.1}$$
$$u(x,0) = f(x),\ \partial_t u(x,0) = 0 \text{ for } x \in \mathbb{R}^3$$

for some initial pressure disturbance caused by electromagnetic absorption, $f(x)$.

Definition 3.2.0.1 *The* thermoacoustic map on D with profile $c(x)$ *is then defined to be* *the operator*

$$\mathcal{L}_{c(x)} f(x) = u(x,t)|_{\partial D \times \mathbb{R}_+} \tag{3.2}$$

for all initial pressure disturbances $f(x)$*.*

For a given domain D each acoustic profile defines a new thermoacoustic map. For this investigation to remain in the realm of applications we only wish to consider this map for physically realistic profiles.

Definition 3.2.0.2 *An* acoustic profile *on a domain* D *is a smooth function* $c(x) \in C^\infty(\mathbb{R}^n)$ *such that*

i*)* $0 < \sigma < c(x) < \infty$ *for* $x \in D$ *for some* $\sigma > 0$

ii*)* $supp(1 - c(x)) \subset D$*.*

In TAT, unlike ultrasound or scattering theory, one does not know the pressure disturbance generating the measured ultrasound field. For this reason it has usually been assumed that the acoustic properties of the body are known *a priori*. If one does not assume that the acoustic profile is known then one can formulate the question of the determination of this property into a question about the range of the operator generating the data.

Proposition 3.2.0.1 *Suppose* $\mathcal{D}$ *is a subset of acoustic profiles on some domain* D*. Then a profile* $c(x) \in \mathcal{D}$ *is uniquely determined in* $\mathcal{D}$ *by thermoacoustic data if and only if*

$$\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) = \{0\}$$

for every $b(x) \in \mathcal{D}$ *with* $b(x) \neq c(x)$*.*

Proof. Suppose a profile $c(x)$ is not determined uniquely in $\mathcal{D}$ by thermoacoustic data. Let us assume we measure thermoacoustic data $h(x,t)$ generated from D with profile $c(x)$. Then there exists a unique initial pressure $f(x)$ so that $\mathcal{L}_{c(x)}f(x) = h(x,t)$ [72]. Since $c(x)$ is not determined uniquely there must exist a second profile $b(x) \in \mathcal{D}$ that can also generate the data $h(x,t)$. This implies the existence of a unique disturbance $g(x)$ so that $\mathcal{L}_{b(x)}g(x) = h(x,t)$. It follows that $\mathcal{L}_{c(x)}f(x) = \mathcal{L}_{b(x)}g(x)$ so $\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) \neq \{0\}$.

Now suppose for an acoustic speed $c(x)$ there exists $b(x) \neq c(x)$ with $b(x) \in \mathcal{D}$ such that

$$\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) \neq \{0\}.$$

Then there must be two initial pressures $f(x)$ and $g(x)$ so that $\mathcal{L}_{c(x)}f(x) = h(x,t) = \mathcal{L}_{b(x)}g(x)$. Thus, the TAT data $h(x,t)$ may be generated by D with acoustic profile $c(x)$ or by D with profile $b(x)$. Therefore, the profile $c(x)$ is not determined uniquely.

■

The above proposition demonstrates that determining the acoustic profile in thermoacoustic tomography is related to the problem of classifying the range of the restriction of forward solutions of the wave equation to boundaries of domains. This problem has been studied extensively by many authors [5, 4, 37] in the case when the domain is a ball of radius $r > 0$ centered at the origin, $B_r(0) \subset \mathbb{R}^n$, and the sound speed $c(x)$ is constant.

For a constant acoustic speed solutions of the forward thermoacoustic problem are directly related to the spherical means of the source function $f(x)$. We define the spherical mean of a function $f(x) \in C_0^\infty(D)$, with $D \subset \mathbb{R}^n$ to be

$$\mathcal{M}f(x,t) = \frac{1}{\omega_n} \int_{|\theta|=1} f(x + t\theta)\, d\theta, \tag{3.3}$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. If the domain D is in $\mathbb{R}^n$ and the sound speed is constant, $c(x) = 1$, then the solution $u(x,t)$ of (3.1) can be written in terms of the spherical mean as [29]

$$u(x,t) = \frac{\sqrt{\pi}}{2\Gamma(n/2)}\frac{\partial}{\partial t}\left(\frac{1}{2t}\frac{\partial}{\partial t}\right)^{(n-3)/2}(t^{n-2}\mathcal{M}f(x,t)). \tag{3.4}$$

Thus, questions about the range of the thermoacoustic operator can be formulated as questions about the range of the spherical mean operator in the case of constant a acoustic profile. It is this approach that the authors of [5, 4] take.

In odd dimensions Huygen's principle for solutions to the constant speed wave equation holds so the range of the thermoacoustic operator for sources, $f(x) \in C_0^\infty(\overline{B_r(0)})$, is a subspace of $\tilde{C}^\infty(S_r(0) \times [0, \infty))$; the set of restrictions to $S_r(0) \times [0, \infty)$ of smooth functions on $S_r(0) \times (-\infty, \infty)$ with support in $S_r(0) \times [0, 2r]$. The following characterization of the range of the thermoacoustic operator was then demonstrated in [37].

Theorem 3.2.0.3 *[37] In odd dimensions $n \geq 3$ with domain $B_r(0)$, the function $g(x,t) \in \tilde{C}^\infty(S_r(0) \times [0,\infty))$ is in the range of $\mathcal{L}_1$ if and only if $V_t(x,0) = 0$ for every $x \in \overline{B_r(0)}$. Here $V(x,t)$ is a solution of the back propagation problem*

$$\begin{aligned} V_{tt} - \Delta V = 0 \ &in\ \overline{B_r(0)} \times [0, 2r] \\ V(x, 2r) = 0,\ V_t(x, 2r) = 0\ &on\ \overline{B_r(0)} \\ V(x,t) = g(x,t)\ &on\ S_r(0) \times [0, 2r]. \end{aligned} \tag{3.5}$$

Another set of characterizations of the range of the TAT operator are formulated in terms of the range of $\mathcal{M}$. The results in [5] hold in dimension two while the results of [4] hold in all dimensions. We state one formulation of the latter here for background on the subject. This result characterizes the range by using a moment condition and an orthogonality condition so it is different than the result of [37] mentioned above. Though the result holds in all dimensions it is still assumed that the acoustic speed is constant, equal to one, and that the domain is the unit ball, $B_1(0)$.

Theorem 3.2.0.4 *[4] The function $g(x,t) \in C_0^\infty(S_1(0) \times [0,2])$ is equal to $\mathcal{M}f(x,t)$ for some $f(x) \in C_0^\infty(B_1(0))$ if and only if the following two conditions are satisfied:*

1.) For any non-negative integer k

$$M_k(x) = \int_0^2 t^{2k+n-1} g(x,t)\, dt \text{ for } x \in S_1(0)$$

has an extension to $\mathbb{R}^n$ *as a polynomial* $q_k(x)$ *of degree at most* $2k$.

2.) Let $-\lambda^2$ *be an eigenvalue of the Dirichlet Laplacian in* $B_1(0)$ *and* $\psi_\lambda(x)$ *the corresponding eigenfunction. Then the following orthogonality condition is satisfied:*

$$\int_{S_1(0)\times[0,2]} g(x,t)\partial_\nu \psi_\lambda(x) j_{n/2-1}(\lambda t) t^{n-1}\, dx\, dt = 0 \tag{3.6}$$

where $j_m(t)$ *is the spherical Bessel function of order* m.

Orthogonality conditions like this, without the moment conditions, were also found in [37] for the case where the dimension $n \geq 3$ was odd. It is interesting to note that it was shown in [4] that the moment conditions are unnecessary in odd dimensions. A necessary, but not sufficient, analog of the above characterization for variable acoustic profiles was found in [44] which we state here.

Theorem 3.2.0.5 *[44] Let* $\{\lambda_k^2\}_{k=1}^\infty$ *be the Dirichlet spectrum of the operator* $-c^2(x)\Delta$ *on the domain* $H^2(B_1(0)) \cap H_0^1(B_1(0))$ *with corresponding basis of orthonormal eigenfunctions* $\{\psi_k\}_{k=1}^\infty$ *in* $L^2(c^{-2}, B_1(0))$. *Then* $g(x,t) = \mathcal{L}_{c(x)} f(x,t)$ *on* $S_1(0) \times [0,\infty)$ *for some* $f(x) \in C_0^\infty(B_1(0))$ *implies*

$$\int_0^\infty \left(\int_{S_1(0)} g(x,t)\partial_\nu \psi_k(x)\, dx \right) \cos(\lambda_k t)\, dt = 0 \tag{3.7}$$

for all k.

One may be tempted to use the first two characterizations of the range of the TAT operator as tests for constant acoustic speed. For measured data $g(x,t)$ on $S_r(0) \times \mathbb{R}_+$ one could test either the condition of theorem 3.2.0.3 or conditions 1 and 2 of theorem 3.2.0.4. If these conditions were satisfied then there would exist some $f(x) \in C_0^\infty(B_r(0))$

that would generate the data $g(x,t)$ with constant sound speed. However, this does not rule out the possibility of the existence of a second acoustic source $\tilde{f}(x) \in C_0^\infty(B_r(0))$ and an acoustic profile $c(x)$ that together also generate $g(x,t)$. Thus, range characterizations of the above type are insufficient to yield information on the acoustic speed from TAT measurements. Of course if the conditions of theorems 3.2.0.3 and 3.2.0.4 are not satisfied then the acoustic speed can not be equal to one.

Even with the above consideration there is still some hope of using range characterizations of the thermoacoustic operator to yield useful information about the acoustic profile of a body being imaged. Consider the following situation. Suppose we could prove that two classes of acoustic profiles, say $\mathcal{D}$ and $\mathcal{D}'$, generated TAT operators with ranges having zero intersection. Then assume one could show necessary and sufficient conditions on the data $g(x,t)$ to be in the range of an operator with acoustic speed from $\mathcal{D}$ and similarly for $\mathcal{D}'$. The testing of these conditions would determine which class the acoustic profile of the body was in.

We will show, in the final chapter of this document, that in three dimensions the ranges of TAT operators from radially symmetric non-trapping acoustic speeds meeting an integral condition have zero intersection with the ranges of constant speed TAT operators. Therefore, if conditions were found that characterized the range of radial sound speed TAT operators the above test would yield valuable information.

3.3 Temporal Fourier Transform of the Acoustic Field

Two lemmas dealing with the temporal Fourier transform of the pressure field are presented. Their proofs rely on decay estimates discovered for solutions of the wave equation [82, 81, 60, 13, 67]. It is assumed that the acoustic profile is non-trapping and that the overlying dimension is odd.

Definition 3.3.0.3 *An acoustic profile $c(x)$ is said to be* non-trapping *if solutions, the* bicharacteristics*, to*

$$\begin{cases} \dot{x} = c^2(x)\xi \\ \dot{\xi} = -\frac{1}{2}\nabla(c^2(x))|\xi|^2 \\ x(0) = x_0,\, \xi(0) = \xi_0, \end{cases}$$

in $\mathbb{R}^{2n}_{x,\xi}$ have projections, rays, *in $\mathbb{R}^n_x$ tending to infinity as $t \to \infty$ as long as $\xi_0 \neq 0$.*

The information in TAT data that allows one to prove some uniqueness results concerning the acoustics of the body is contained in the frequency side of the data.

Definition 3.3.0.4 *The* temporal Fourier transform *of a function $u(x,t)$ is defined by*

$$\hat{u}(x,\omega) = \frac{1}{2\pi}\int_0^\infty u(x,t)e^{-i\omega t}\, dt.$$

Theorem 3.3.0.6 *[82, 81] If the non-trapping condition is satisfied, then for any multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, the following estimate holds for solutions of the thermoacoustic forward problem:*

$$\left|\partial^\alpha_{(t,x)} u(x,t)\right| \leq C\eta(t)\|f\|_{L^2},\ x \in D.$$

Here the function $\eta(t)$ that characterizes the decay is $t^{-n-\alpha_0+1}$ when the dimension n is even and $e^{-\epsilon t}$ when n is odd.

The statement of this theorem can also be found in [32]. In the statement of the result it appears that the exponential decay holds for all $n \geq 1$ odd. However, it is not clear to the author that such decay is justified for the case when $n = 1$. For this reason we will state the results of this manuscript assuming that exponential decay of the ultrasound field is not guaranteed in one dimension.

Lemma 3.3.0.1 *Consider a solution $u(x,t)$ of the forward thermoacoustic problem in domain $D \subset \mathbb{R}^n$, with $n > 1$ odd, and non-trapping acoustic profile $c(x)$. For each fixed*

$x \in D$ the temporal Fourier transform, $\hat{u}(x,\omega)$, is the restriction to $\mathbb{R}_+$ of a function analytic in an open set in $\mathbb{C}$ containing $\mathbb{R}_+$.

Proof. Let $u(x,t)$ be a solution of the forward thermoacoustic problem with impulse function $f(x)$ and non-trapping sound speed $c(x)$. By theorem 3.3.0.6 there exists an $\epsilon > 0$ so that

$$|u(x,t)| \leq Ce^{-\epsilon t}\|f\|_{L^2} \text{ for } x \in D.$$

Using this inequality we see, for $x \in D$ and $\omega \in \mathbb{R}_+$, the temporal Fourier transform of $u(x,t)$ is defined and finite,

$$\begin{aligned} |\hat{u}(x,\omega)| &= \left|\frac{1}{2\pi}\int_0^\infty u(x,t)e^{-i\omega t}\,dt\right| \\ &\leq \frac{1}{2\pi}\int_0^\infty |u(x,t)|\,dt \\ &\leq \frac{C}{2\pi}\int_0^\infty \|f\|_{L^2}e^{-\epsilon t}\,dt < \infty. \end{aligned}$$

It remains to show that, for a fixed $x \in D$, $\hat{u}(x,\omega)$ is analytic in an open set containing $\mathbb{R}_+$. Let $\mu = \omega + i\eta \in \mathbb{C}$ with $\eta < \epsilon$. Again using the estimate on the decay of $u(x,t)$ for fixed x we see

$$\begin{aligned} |\hat{u}(x,\mu)| &= \left|\frac{1}{2\pi}\int_0^\infty u(x,t)e^{-i(\omega+i\eta)t}\,dt\right| \\ &\leq \frac{1}{2\pi}\int_0^\infty |u(x,t)|e^{\eta t}\,dt \\ &\leq \frac{C}{2\pi}\int_0^\infty \|f\|_{L^2}e^{(\eta-\epsilon)t}\,dt < \infty \end{aligned}$$

since $\eta < \epsilon$. This shows that for fixed $x \in D$, $\hat{u}(x,\cdot)$ is defined by an integral which is absolutely convergent on the open set

$$\{\mu \in \mathbb{C} : \mathcal{I}m(\mu) < \epsilon\}.$$

Since this set contains $\mathbb{R}_+$ the lemma holds.

∎

Our results will actually be more concerned with investigating $\mathcal{R}e(\hat{u})(x,\omega)$ since it is actually this function which satisfies a particular partial differential equation at each fixed frequency.

Lemma 3.3.0.2 *Consider a solution $u(x,t)$ of the forward thermoacoustic problem in domain $D \subset \mathbb{R}^n$, with $n > 1$ odd, and non-trapping acoustic profile $c(x)$. For each fixed $x \in D$ the real part of the temporal Fourier transform, $\mathcal{R}e(\hat{u})(x,\omega)$, is the restriction to $\mathbb{R}_+$ of a function analytic in an open strip in $\mathbb{C}$ containing $\mathbb{R}_+$.*

Proof. As before we let $u(x,t)$ be a solution of the forward thermoacoustic problem with impulse function $f(x)$ and non-trapping profile $c(x)$. By theorem 3.3.0.6 there exists an $\epsilon > 0$ so that

$$|u(x,t)| \leq Ce^{-\epsilon t}\|f\|_{L^2} \text{ for } x \in D.$$

Define an open strip of width ϵ about $\mathbb{R}_+$ to be the set

$$\mathbb{R}_{+,\epsilon} := \{\mu \in \mathbb{C} : \mu = \omega + i\eta,\ \omega \in \mathbb{R}_+,\ |\eta| < \epsilon\}.$$

Using the above inequality, for $x \in D$ and $\mu \in \mathbb{R}_{+,\epsilon}$, $\mathcal{R}e(\hat{u})(x,\omega)$ satisfies the following inequalities

$$\begin{aligned}
|\mathcal{R}e(\hat{u})(x,\mu)| &= \left|\mathcal{R}e\left\{\frac{1}{2\pi}\int_0^\infty u(x,t)e^{-i\mu t}\,dt\right\}\right| \\
&= \left|\frac{1}{2\pi}\int_0^\infty u(x,t)\frac{e^{i\mu t}+e^{-i\mu t}}{2}\,dt\right| \\
&= \left|\frac{1}{4\pi}\int_0^\infty u(x,t)(e^{-\eta t}e^{i\omega t}+e^{\eta t}e^{-i\omega t})\,dt\right| \\
&\leq \frac{1}{4\pi}\int_0^\infty |u(x,t)|(e^{\eta t}+e^{-\eta t})\,dt \\
&\leq \frac{C}{4\pi}\int_0^\infty e^{-\epsilon t}\|f\|_{L^2}(e^{\eta t}+e^{-\eta t})\,dt.
\end{aligned}$$

For this last integral to converge we must have $-(\epsilon+\eta) < 0$ and $\eta - \epsilon < 0$. However, $\mu \in \mathbb{R}_{+,\epsilon}$, so $-\epsilon < \eta < \epsilon$. This shows that for fixed $x \in D$, $\mathcal{R}e(\hat{u})(x,\cdot)$ is defined by an integral which is absolutely convergent on the open strip $\mathbb{R}_{+,\epsilon}$. Since this set contains $\mathbb{R}_+$ the lemma holds.

■

The uniqueness results require the analyticity described in the previous lemma. In the case of $D \subset \mathbb{R}^n$, with n even, theorem 3.3.0.6 does not ensure exponential decay of the ultrasound field. However, the polynomial decay does give us some control.

Lemma 3.3.0.3 *Consider a solution $u(x,t)$ of the forward thermoacoustic problem in domain $D \subset \mathbb{R}^n$, with n even, and non-trapping acoustic profile $c(x)$. For each fixed $x \in D$ the temporal Fourier transform, $\hat{u}(x,\omega)$, is defined by an absolutely convergent integral in the lower half-plane,*

$$\mathbb{C}_- = \{\mu \in \mathbb{C} : \mathcal{I}m(\mu) \leq 0\}.$$

Proof. Let $u(x,t)$ be a solution of the forward thermoacoustic problem with impulse function $f(x)$ and non-trapping sound speed $c(x)$. Since $D \subset \mathbb{R}^n$, with $n \geq 2$ even, theorem 3.3.0.6 guarantees an $\alpha_0 > 0$ so that

$$|u(x,t)| \leq Ct^{-n-\alpha_0+1}\|f\|_{L^2} \text{ for } x \in D$$

for some constant C. We now take $\mu = \omega + i\lambda \in \mathbb{C}_-$ and examine the temporal transform. Let $\epsilon > 0$ be given. It follows that, for $x \in D$,

$$\begin{aligned}
|\hat{u}(x,\mu)| &\leq \frac{1}{2\pi}\left|\int_0^\infty u(x,t)e^{-i\mu t}\,dt\right| = \frac{1}{2\pi}\left|\int_0^\infty u(x,t)e^{-i(\omega+i\lambda)t}\,dt\right| \\
&= \frac{1}{2\pi}\left|\int_0^\infty u(x,t)e^{-i\omega t}e^{\lambda t}\,dt\right| \leq \frac{1}{2\pi}\int_0^\infty \left|u(x,t)e^{-i\omega t}e^{\lambda t}\right|\,dt \\
&= \frac{1}{2\pi}\int_0^\infty |u(x,t)|\,e^{\lambda t}\,dt \\
&= \frac{1}{2\pi}\int_0^\epsilon |u(x,t)|\,e^{\lambda t}\,dt + \frac{1}{2\pi}\int_\epsilon^\infty |u(x,t)|\,e^{\lambda t}\,dt \\
&\leq \frac{1}{2\pi}\int_0^\epsilon |u(x,t)|\,e^{\lambda t}\,dt + \frac{C}{2\pi}\int_\epsilon^\infty \|f\|_{L^2}t^{-n-\alpha_0+1}e^{\lambda t}\,dt.
\end{aligned}$$

Since $n \geq 2$ we have $-n-\alpha_0+1 < 0$ and therefore the second integral converges for all $\lambda < 0$. If $\lambda = 0$ then we have

$$\int_\epsilon^\infty t^{-n-\alpha_0+1}\,dt = \frac{-\epsilon^{2-(n+\alpha_0)}}{2-(n+\alpha_0)} + \frac{1}{2-(n+\alpha_0)}\lim_{t\to\infty} t^{2-(n+\alpha_0)} < \infty \tag{3.8}$$

as long as $(n+\alpha_0) \geq 2$ which holds for all $n \geq 2$ even. It follows that $\hat{u}(x,\mu)$ is defined by an absolutely convergent integral for every $\mu \in \mathbb{C}_-$.

■

3.4 Relation of Interior Transmission Problem to Determination of Acoustics in TAT

We state the interior transmission problem and describe its relation to the question of uniqueness of the acoustic profile.

Definition 3.4.0.5 *The pair $(u,v) \in H^2(D) \times H^2(D)$ is a solution to the* interior transmission problem *(ITP) relative to the acoustic profiles $c(x)$ and $b(x)$ in D if*

$$\begin{aligned} \Delta u + k^2 n_c(x) u = 0 \quad & in \quad D \\ \Delta v + k^2 n_b(x) v = 0 \quad & in \quad D \\ u = v \quad & on \quad \partial D \\ \partial_\nu u = \partial_\nu v \quad & on \quad \partial D. \end{aligned} \tag{3.9}$$

Here, ∂_ν represents the outward normal derivative on ∂D. The coefficient, $n_c(x) = \frac{1}{c^2(x)}$, is known as the refractive index*.*

If, for $k \in \mathbb{R}_+$, there exists a nontrivial solution $\{u,v\}$ of the interior transmission problem we call k a transmission eigenvalue*.*

For differing conditions on the two profiles, $c(x)$ and $b(x)$, one usually must restrict to looking for solutions in a subspace $\mathcal{H} \subset H^2(D)$. This is done to ensure that the problem can be formulated in an equivalent variational form which is much easier to work with. For now it will suffice to consider our solutions as lying in $H^2(D)$.

Remark 3.4.0.1 *Throughout the rest of this manuscript we will strive to adhere to the following notation, denote the real part of the frequency side of a pressure field by* $U(x,k) = \mathcal{R}e(\hat{u})(x,k)$.

Lemma 3.4.0.4 *Let* $u(x,t)$ *satisfy*

$$\partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \tag{3.10}$$
$$u(x,0) = f(x),\ \partial_t u(x,t) = 0 \text{ on } \mathbb{R}^n.$$

Then for $k \in \mathbb{R}_+$ *the temporal Fourier transform* $\hat{u}(x,k)$ *satisfies*

$$\Delta\hat{u}(x,k) + k^2 n(x)\hat{u}(x,k) = -\frac{ik}{2\pi} n(x) f(x) \text{ for } x \in \mathbb{R}^n. \tag{3.11}$$

The real part, $U(x,k) = \mathcal{R}e(\hat{u})(x,k)$, *satisfies*

$$\Delta U(x,k) + k^2 n(x) U(x,k) = 0 \text{ for } x \in \mathbb{R}^n. \tag{3.12}$$

Proof. Suppose $u(x,t)$ satisfies 3.10, then for each $k \in \mathbb{R}_+$,

$$\widehat{\partial_t^2 u} - c^2(x)\Delta\hat{u} = 0.$$

Examining the first term we observe, through an integration by parts,

$$\begin{aligned}
\widehat{\partial_t^2 u}(x,k) &= \frac{1}{2\pi}\int_0^\infty \partial_t^2 u(x,t) e^{-ikt}\, dt \\
&= \lim_{s\to\infty} \frac{1}{2\pi}\partial_t u(x,t) e^{-ikt}|_0^s + \frac{ik}{2\pi}\int_0^\infty \partial_t u(x,t) e^{-ikt}\, dt \\
&= -\frac{1}{2\pi}\partial_t u(x,0) + \frac{ik}{2\pi}\int_0^\infty \partial_t u(x,t) e^{-ikt}\, dt.
\end{aligned}$$

Where we have again used the decay estimate of solutions to the wave equation in time for a fixed $x \in D$ to show that the limit at infinity vanishes. Integrating by parts again yields

$$\begin{aligned}
\widehat{\partial_t^2 u}(x,k) &= -\frac{1}{2\pi}\partial_t u(x,0) + \frac{ik}{2\pi}\int_0^\infty \partial_t u(x,t)e^{-ikt}\,dt \\
&= \lim_{s\to\infty}\frac{ik}{2\pi}u(x,t)e^{-ikt}|_0^s - \frac{1}{2\pi}\partial_t u(x,0) \\
&\qquad -\frac{k^2}{2\pi}\int_0^\infty u(x,t)e^{-ikt}\,dt \\
&= -\frac{ik}{2\pi}u(x,0) - \frac{1}{2\pi}\partial_t u(x,0) - k^2\hat{u}(x,k) \\
&= -\frac{ik}{2\pi}f(x) - k^2\hat{u}(x,k).
\end{aligned}$$

Thus,

$$-\frac{ik}{2\pi}f(x) - k^2\hat{u}(x,k) - c^2(x)\Delta\hat{u} = 0$$

so $\hat{u}$ satisfies

$$\Delta\hat{u} + k^2 n(x)\hat{u} = -\frac{ik}{2\pi}n(x)f(x) \text{ for } x \in \mathbb{R}^n.$$

Since our initial disturbance, $f(x)$, is real valued we see $U(x,k)$ satisfies the homogeneous Helmholtz equation

$$\Delta U(x,k) + k^2 n(x)U(x,k) = 0 \text{ for } x \in \mathbb{R}^n$$

for each $k \in \mathbb{R}_+$.

■

Note that for lemma 3.4.0.4 to hold we have assumed that the wave equation governing the ultrasound field has no attenuation term. This has the effect of ensuring that the refractive index is real. For attenuating media this is not the case since the wave equation that the ultrasound field satisfies will then include a damping term involving $\partial_t u(x,t)$. This adds a complex part to the refractive index after taking the Fourier transform.

Now, suppose we have performed two distinct thermoacoustic experiments on a body in domain D with acoustic profiles given by either $c(x)$ or $b(x)$. Then we have measured two ultrasound fields, $u(x,t)$ and $v(x,t)$, satisfying

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = f(x),\ \partial_t u(x,0) = 0 \text{ on } \mathbb{R}^n \\ \partial_t^2 v(x,t) - b^2(x)\Delta v(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ v(x,0) = g(x),\ \partial_t v(x,0) = 0 \text{ on } \mathbb{R}^n. \end{aligned} \tag{3.13}$$

If we assume that these profiles, with the respective initial disturbances $f(x)$ and $g(x)$, have generated identical thermoacoustic measurements then we must have

$$u(x,t)|_{\partial D \times \mathbb{R}_+} = v(x,t)|_{\partial D \times \mathbb{R}_+}.$$

It follows that for all $k \in \mathbb{R}_+$

$$U(x,k) = V(x,k) \text{ on } \partial D.$$

Therefore, when comparing to acoustic fields with identical TAT data, the Dirichlet data of the temporal frequency agree. To complete our connection to the ITP we now show that the Neumann data agrees as well.

Lemma 3.4.0.5 *Let $c(x)$ and $b(x)$ be two acoustic speeds relative to the domain D. For two initial impulses in D, $f(x)$ and $g(x)$, let $u(x,t)$ and $v(x,t)$ satisfy the following forward thermoacoustic problems:*

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \textit{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = f(x),\ \partial_t u(x,0) = 0 \textit{ on } \mathbb{R}^n, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \partial_t^2 v(x,t) - b^2(x)\Delta v(x,t) = 0 \textit{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ v(x,0) = g(x),\ \partial_t v(x,0) = 0 \textit{ on } \mathbb{R}^n. \end{aligned} \tag{3.15}$$

Suppose that the thermoacoustic data of each of these problems agree,

$$u(x,t) = h(x,t) = v(x,t) \textit{ for every } (x,t) \in \partial D \times \mathbb{R}_+. \tag{3.16}$$

Then, for $(x,t) \in \partial D \times \mathbb{R}_+$, we have

$$\partial_\nu u(x,t) = \partial_\nu v(x,t). \tag{3.17}$$

Proof. By our definition of acoustic speeds and impulse functions under consideration we must have

$$c(x) = b(x) = 1$$

and

$$f(x) = g(x) = 0$$

for every x in D^c. Then $u(x,t)$ must satisfy the exterior problem

$$\begin{aligned} \partial_t^2 u(x,t) - \Delta u(x,t) = 0 \quad &\text{for} \quad (x,t) \in D^c \times \mathbb{R}_+ \\ u(x,t) = h(x,t) \quad &\text{for all} \quad (x,t) \in \partial D \times \mathbb{R}_+ \\ u(x,0) = 0 = \partial_t u(x,0) \quad &\text{for} \quad x \in D^c. \end{aligned} \tag{3.18}$$

Likewise the function $v(x,t)$ must satisfy the same exterior problem. This problem is well posed and therefore has a unique solution in $D^c \times \mathbb{R}_+$. Thus, $v(x,t) = u(x,t)$ in the exterior of D which implies

$$\partial_\nu u(x,t) = \partial_\nu v(x,t).$$

∎

Theorem 3.4.0.7 *Let $c(x)$ and $b(x)$ be two acoustic profiles on the domain D. Let $u(x,t)$ and $v(x,t)$ be two thermoacoustic pressure fields, satisfying (3.14) and (3.15) respectively. If the thermoacoustic data agree, i.e. (3.16) holds, then the functions*

$$\begin{aligned} U(x,k) &= \mathcal{R}e(\hat{u})(x,k) \\ V(x,k) &= \mathcal{R}e(\hat{v})(x,k), \end{aligned}$$

satisfy the interior transmission problem

$$\begin{aligned} \Delta U(x,k) + k^2 n_c(x) U(x,k) = 0 \quad &\text{in} \quad D \\ \Delta V(x,k) + k^2 n_b(x) V(x,k) = 0 \quad &\text{in} \quad D \\ U(x,k) = V(x,k) \quad &\text{on} \quad \partial D \\ \partial_\nu U(x,k) = \partial_\nu V(x,k) \quad &\text{on} \quad \partial D, \end{aligned} \tag{3.19}$$

for every $k \in \mathbb{R}_+$.

Proof. By lemma 3.4.0.4 we observe that the partial differential equations in (3.19) must hold for each $k \in \mathbb{R}_+$. It remains to show that the boundary conditions hold for all $k \in \mathbb{R}_+$. However, the first condition is trivial since (3.16) is satisfied, which implies for $x \in \partial D$, $t \in \mathbb{R}_+$,

$$u(x,t) = h(x,t) = v(x,t).$$

Thus, for every k we must have

$$\hat{u}(x,k) = \hat{v}(x,k)$$

on ∂D which implies $U(x,k) = V(x,k)$ on the boundary of our domain. The same reasoning is used to show the second boundary condition holds. It suffices to observe that since lemma 3.4.0.5 implies $\partial_\nu u(x,t) = \partial_\nu v(x,t)$ on the boundary of D we have that

$$\widehat{\partial_\nu u}(x,k) = \widehat{\partial_\nu v}(x,k)$$

on ∂D. Because we are only taking the Fourier transform with respect to time

$$\widehat{\partial_\nu u}(x,k) = \partial_\nu \hat{u}(x,k).$$

Therefore,

$$\partial_\nu \hat{u}(x,k) = \partial_\nu \hat{v}(x,k),$$

and we conclude $\partial_\nu U(x,k) = \partial_\nu V(x,k)$ for $x \in \partial D$, $k \in \mathbb{R}_+$. Thus, for each $k \in \mathbb{R}_+$ $U(x,k)$ and $V(x,k)$ must satisfy (3.19).

■

We can now investigate the relation of the decay properties of the ultrasound fields to the transmission spectrum.

Theorem 3.4.0.8 *Consider a domain* D *with two acoustic profiles,* $c(x)$ *and* $b(x)$*. If the complement of the interior transmission spectrum has a finite cluster point in* $\mathbb{R}_+$ *then the intersection of the range of the thermoacoustic operators,* $\mathcal{L}_{c(x)}$ *and* $\mathcal{L}_{b(x)}$*, is zero.*

Proof. First, suppose that there exist two non-zero impulse functions, $f(x)$ and $g(x)$, in D such that $\mathcal{L}_{c(x)}f = \mathcal{L}_{b(x)}g$. This implies that the functions $u(x,t)$ and $v(x,t)$ satisfying

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ u(x,0) = f(x),\ \partial_t u(x,t) = 0 \text{ on } \mathbb{R}^n, \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \partial_t^2 v(x,t) - b^2(x)\Delta v(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ v(x,0) = g(x),\ \partial_t v(x,t) = 0 \text{ on } \mathbb{R}^n \end{aligned} \tag{3.21}$$

also satisfy

$$u(x,t)|_{\partial D\times\mathbb{R}_+} = v(x,t)|_{\partial D\times\mathbb{R}_+}.$$

By theorem 3.4.0.7 the functions $U(x,k)$ and $V(x,k)$ must satisfy the interior transmission problem

$$\begin{aligned} \Delta U(x,k) + k^2 n_c(x)U(x,k) = 0 \quad &\text{in} \quad D \\ \Delta V(x,k) + k^2 n_b(x)V(x,k) = 0 \quad &\text{in} \quad D \\ U(x,k) = V(x,k) \quad &\text{on} \quad \partial D \\ \partial_\nu U(x,k) = \partial_\nu V(x,k) \quad &\text{on} \quad \partial D \end{aligned} \tag{3.22}$$

for every $k \in \mathbb{R}_+$. By lemma 3.3.0.2 for each $x \in D$ the functions $U(x,k)$ and $V(x,k)$ are the restriction of a function analytic in an open strip in $\mathbb{C}$ containing $\mathbb{R}_+$.

At a value $k \in \mathbb{R}_+$ that is not a transmission eigenvalue the only solution to (3.9) is $U(x,k) \equiv 0 \equiv V(x,k)$. By hypothesis the complement of the interior transmission spectrum relative to $c(x)$ and $b(x)$ in D has a finite cluster point. Therefore, for each $x \in D$ the functions $U(x,k)$ and $V(x,k)$ are the restrictions of functions analytic in an open set about $\mathbb{R}_+$ that have a set of zeros with a finite accumulation point. From classical results on analytic functions this implies, for each $x \in D$, $U(x,k)$ and $V(x,k)$ are identically zero on $\mathbb{R}_+$. This is enough to show that $u(x,t)$ and $v(x,t)$ must be identically zero, which contradicts the choice of $f(x)$ and $g(x)$ non-zero. We can then conclude that the intersection of the range of $\mathcal{L}_{c(x)}$ and $\mathcal{L}_{b(x)}$ is zero.

■

3.5 Conditions for Unique Determination of Acoustics from Thermoacoustic Measurements

In light of theorem 3.4.0.8 we prove a weak condition on two acoustic profiles which ensures the transmission spectrum is bounded away from $k = 0$.

We consider the interior transmission problem on a bounded domain $D \subset \mathbb{R}^n$. The refractive indexes $n_c(x)$ and $n_b(x)$, related to two acoustic profiles $c(x)$ and $b(x)$, on D are then smooth functions such that $supp(1 - n_c)$, $supp(1 - n_b) \subset D$ and $0 < \sigma < n_c(x), n_b(x) < \infty$ in D for some constant $\sigma \in \mathbb{R}$. The *contrast* is defined to be the difference of the refractive indices being compared,

$$m(x) = n_c(x) - n_b(x).$$

Notice that the contrast satisfies $supp(m) \subset D$.

Suppose that $k \in \mathbb{R}_+$ is a transmission eigenvalue and that $w, v \in H^2_{loc}(D) \cap L^2(D)$ are the corresponding eigenfunctions satisfying

$$\begin{aligned} \Delta w + k^2 n_c(x) w = 0 \quad &\text{on} \quad D \\ \Delta v + k^2 n_b(x) v = 0 \quad &\text{on} \quad D \\ w = v \quad &\text{on} \quad \partial D \\ \partial_\nu w = \partial_\nu v \quad &\text{on} \quad \partial D. \end{aligned} \tag{3.23}$$

Then the function $u = w - v \in H^2_0(D)$ satisfies two important equations. We observe

$$\begin{aligned} \Delta u + k^2 n_c(x) u &= \Delta w + k^2 n_c(x) w - \Delta v - k^2 n_c(x) v \\ &= -\Delta v - k^2 n_b(x) v - k^2 n_c(x) v + k^2 n_b(x) v \\ &= -k^2 m(x) v, \end{aligned}$$

and similarly for $\Delta u + k^2 n_b(x)u$. It follows that $u = w - v$ satisfies

$$\Delta u + k^2 n_c(x)u = -k^2 m v \text{ on } D \tag{3.24}$$
$$\Delta u + k^2 n_b(x)u = -k^2 m w \text{ on } D.$$

Lemma 3.5.0.6 *For a transmission pair $w, v \in H^2_{loc}(D) \cap L^2(D)$ we have*

$$\int_D m(x) w v \, dx = 0. \tag{3.25}$$

Proof. We let $u = w - v \in H^2_0(D)$ and notice that

$$\begin{aligned} 0 &= \int_D u(\Delta w + k^2 n_c(x) w)\, dx = \int_D u \Delta w \, dx + k^2 \int_D n_c(x) u w \, dx \\ &= \int_D w \Delta u \, dx + k^2 \int_D n_c(x) u w \, dx + \int_{\partial D} (u \partial_\nu w - w \partial_\nu u)\, dx \\ &= \int_D w(\Delta u + k^2 n_c(x) u)\, dx = -k^2 \int_D m(x) w v \, dx. \end{aligned}$$

Here the first equality follows from the fact that w is part of a transmission pair. Then we apply Green's formula and use the fact that $u \in H^2_0(D)$. The last equality follows from the first equation in (3.24).

■

The lower bound for the transmission spectrum is in terms of λ_0, the first Dirichlet eigenvalue for the Laplacian in D. We write $n_i^* = \sup_D n_i(x)$ for $i = c, b$. Results like this have been known for a while [28, 16] though slightly different methods of proof have been used. It is worth noting that the results in [28, 16] assume that the contrast, $m(x)$, satisfies $m(x) > \delta > 0$ in D for some $\delta > 0$.

Theorem 3.5.0.9 *If $k \in \mathbb{R}_+$ is a transmission eigenvalue and $m(x) = n_c(x) - n_b(x) \geq 0$ then*

$$k \geq \sqrt{\frac{\lambda_0}{n_c^*}}.$$

If $m(x) = n_c(x) - n_b(x) \leq 0$ *then*

$$k \geq \sqrt{\frac{\lambda_0}{n_b^*}}.$$

Proof. Suppose $k \in \mathbb{R}_+$ is a transmission eigenvalue with the pair w, v. Then $u = w - v \in H_0^2(D)$ and we see

$$\begin{aligned}\int_D u(\Delta u + k^2 n_c(x)u)\,dx &= -k^2 \int_D m(x)uv\,dx \\ &= -k^2 \int_D m(x)wv\,dx + k^2 \int_D m(x)v^2\,dx \\ &= k^2 \int_D m(x)v^2\,dx\end{aligned}$$

by lemma 3.5.0.6. Applying Green's formula we see, since $u \in H_0^2(D)$,

$$-\int_D |\nabla u|^2\,dx + k^2 \int_D n_c(x)|u|^2\,dx = k^2 \int_D m(x)v^2\,dx. \tag{3.26}$$

If we assume $m(x) \geq 0$ in D then this implies

$$k^2 \int_D n_c(x)|u|^2\,dx - \int_D |\nabla u|^2\,dx \geq 0. \tag{3.27}$$

Since

$$\inf_{u \in H_0^2(D)} \frac{\int_D |\nabla u|^2\,dx}{\int_D |u|^2\,dx} \geq \lambda_0 \tag{3.28}$$

we see

$$(k^2 n_c^* - \lambda_0)\|u\|_{L^2(D)} \geq 0.$$

Therefore, k can not be a transmission eigenvalue if

$$k < \sqrt{\frac{\lambda_0}{n_c^*}}.$$

If $m(x) \leq 0$ then a similar argument works. We notice

$$\begin{aligned}\int_D u(\Delta u + k^2 n_b(x)u)\,dx &= -k^2 \int_D m(x)uw\,dx \\ &= -k^2 \int_D m(x)w^2\,dx + k^2 \int_D m(x)vw\,dx \\ &= -k^2 \int_D m(x)w^2\,dx\end{aligned}$$

by lemma 3.5.0.6. Applying Green's formula we see, since $u \in H_0^2(D)$,

$$-\int_D |\nabla u|^2 \, dx + k^2 \int_D n_b(x)|u|^2 \, dx = -k^2 \int_D m(x) w^2 \, dx \tag{3.29}$$

which implies

$$k^2 \int_D n_b(x)|u|^2 \, dx - \int_D |\nabla u|^2 \, dx \geq 0. \tag{3.30}$$

An application of the Poincare inequality again shows

$$(k^2 n_b^* - \lambda_0)\|u\|_{L^2(D)} \geq 0.$$

So, in this case, k can not be a transmission eigenvalue if

$$k < \sqrt{\frac{\lambda_0}{n_b^*}}.$$

■

Theorem 3.5.0.10 *Consider two profiles, $c(x)$ and $b(x)$, relative to a domain D. If*

$$c(x) - b(x) \geq 0 \text{ or } c(x) - b(x) \leq 0$$

in D then the thermoacoustic data generated by the domain D from the acoustics $c(x)$ cannot be generated by the domain D with the profile $b(x)$. That is, the intersection of the ranges of the operators $\mathcal{L}_{c(x)}$ and $\mathcal{L}_{b(x)}$ is zero for any two acoustic profiles whose difference does not change signs in D.

Proof. By renaming $c(x)$ and $b(x)$ without loss of generality we may assume $b(x) - c(x) \geq 0$ in D. Defining $n_c(x) = c^{-2}(x)$, $n_b(x) = b^{-2}(x)$ we see

$$n_c(x) - n_b(x) = \frac{1}{c^2(x)} - \frac{1}{b^2(x)} = \frac{b^2(x) - c^2(x)}{b^2(x)c^2(x)}.$$

Then $m(x) = n_c(x) - n_b(x) \geq 0$ if and only if $b^2(x) - c^2(x) \geq 0$ since the acoustic speeds we are studying are strictly positive. This happens if and only if $(b(x)+c(x))(b(x)-c(x)) \geq 0$ which holds by hypothesis.

Theorem 3.5.0.9 then implies that the transmission spectrum on D with respect to $c(x)$ and $b(x)$ is bounded below by $\sqrt{\frac{\lambda_0}{n_c^*}}$. Therefore the complement of the spectrum has a finite cluster point. By theorem 3.4.0.8 we then may conclude that

$$\mathcal{R}g(\mathcal{L}_{c(x)}) \cap \mathcal{R}g(\mathcal{L}_{b(x)}) = \{0\}.$$

We have shown that the domain D with acoustic profile $c(x)$ can not generate the same TAT data as D with profile $b(x)$.

■

If we are now given a set $\mathcal{D}$ of potential acoustic profiles on some domain D then as long as one speed $c(x) \in \mathcal{D}$ meets the condition of theorem 3.5.0.10 when compared to all other profiles in $\mathcal{D}$, it is determined uniquely by thermoacoustic measurements within $\mathcal{D}$. This is stated as a corollary.

Corollary 3.5.0.1 *Suppose thermoacoustic data $h(x,t)$ on $\partial D \times \mathbb{R}_+$ is generated by an acoustic profile $c(x) \in \mathcal{D}$ and $\mathcal{D}$ is a set of acoustics on D such that for every $b(x) \neq c(x)$ in $\mathcal{D}$ we have either*

$$c(x) - b(x) \geq 0 \text{ or } c(x) - b(x) \leq 0$$

on D. Then $c(x)$ is determined uniquely within $\mathcal{D}$ by the data $h(x,t)$.

The proof follows immediately from theorem 3.5.0.10. The importance here is that even if all the acoustics in the set $\mathcal{D}$ do not satisfy the change of sign condition the one that does satisfy the change of sign condition is determined uniquely. Of course since we would not know *a priori* whether or not the data is generated by this specific profile it is not very satisfying. However, for purposes of numerical experimentation this may well be useful in developing minimization methods to reconstruct the unique acoustic profile within a set.

If we are instead studying a set of acoustics $\mathcal{D}$ such that each pair of acoustic speeds satisfies the hypotheses of theorem 3.5.0.10 then we do not require *a priori* knowledge.

Corollary 3.5.0.2 *Suppose thermoacoustic data $h(x,t)$ on $\partial D \times \mathbb{R}_+$ is generated by an acoustic profile in some set $\mathcal{D}$. Assume also that for every pair $c(x)$, $b(x) \in \mathcal{D}$*

$$c(x) - b(x) \geq 0 \text{ or } c(x) - b(x) \leq 0$$

on D. Then the acoustic profile generating data $h(x,t)$ is determined uniquely in $\mathcal{D}$.

3.6 Conclusion

Results have been presented that relate the question of determination of the acoustic profile using thermoacoustic tomography to the decay properties of TAT data and the spectrum of the interior transmission problem. It was shown that if specific conditions were met by two acoustic profiles in a domain D then the TAT data they generate will be distinct. This condition arose from requirements on the sparsity of the transmission spectrum.

The rest of this manuscript will be devoted to a study of the ITP and its spectrum. The spectrum of the ITP depends on both the refractive indices $n_c(x)$ and $n_b(x)$ as well as the domain D. We will study properties of transmission eigenvalues for different subsets of acoustic profiles as well as different assumptions on the domain D. Though we are able to prove stronger results about the spectrum of the ITP in the following section the above theorems remain the strongest uniqueness results yet obtained for the case of arbitrary domains. For the case in which the imaging domain is an open ball we will prove stronger results.

4 RESULTS ON THE GENERAL INTERIOR TRANSMISSION PROBLEM

4.1 Introduction

This chapter investigates the interior transmission problem in a setting different from those considered by previous authors. It is shown that the transmission spectrum is discrete and the ITP has a variational formulation on a domain D with contrast $m(x)$ that vanishes to second order on ∂D. Also, it will be demonstrated that there exists an infinite discrete set of transmission eigenvalues in this case. In the work of other authors it has always been assumed that the contrast is zero outside of the domain of interest D but that this transition is discontinuous.

In previous studies of the interior transmission problem [25, 27, 69, 47, 17] it has been shown, through methods similar to those used here, that the transmission spectrum is discrete and bounded below. This was done by first demonstrating the transmission problem had a variational formulation on some hilbert space. The existence of transmission eigenvalues has been shown only relatively recently and relies heavily on the variational formulation [18, 19, 64]. However, previous authors have been mostly motivated by scattering theory in which it makes sense to assume that the acoustics of the body D being studied are different than that of the surrounding medium. For this reason the above work makes the assumption that the contrast is non-zero on the interior of the boundary of D. In other words, it has been assumed the contrast is essentially bounded away from zero on the domain.

The study of the ITP which follows is motivated by the work in the previous chapter having to do with the relation of the transmission problem to thermoacoustic tomography. In the thermoacoustic setting either the ultrasound transducers are in direct contact with the body being imaged or the body is immersed in some fluid having similar acoustic properties as the boundary of the body. In both cases it is assumed that there is no difference between the acoustics at the measurement boundary and the acoustics directly on the interior of the medium being imaged. Indeed, if there was a sharp change in acoustics at ∂D then there would be reflections which current models of TAT reconstruction do not account for. For this reason it has been important to develop a formulation of the ITP which does not assume that the contrast vanishes discontinuously, but instead transitions smoothly to the acoustic speed of the medium surrounding the domain being imaged. Due to the operators involved in a variational formulation of the transmission problem this is a nontrivial task.

We have already shown the transmission spectrum is bounded below for a contrast which does not change signs but is allowed to be zero on the interior of D. These transitions to zero were allowed to be smooth. The work of this chapter does not improve the uniqueness result of that section. However, we did not show that the transmission spectrum was discrete for those types of acoustics. At this point we have not been able to show that, for such a general contrast, the spectrum is discrete. For the thermoacoustic uniqueness results of the previous section it was only necessary to show that the complement of the transmission spectrum had a finite cluster point. This held since the dimension was odd and the acoustics were assumed to be non-trapping which implied exponential decay in time for the TAT data and made the temporal Fourier transform analytic in an open set containing $\mathbb{R}_+$.

If we wish to prove similar uniqueness results in even dimensions or for non-trapping acoustic profiles we must allow a slower rate of decay. Even though the decay rate is slower for these cases the ultrasound field does have some decay and therefore the temporal Fourier transform of the TAT data has some nice properties such as analyticity in a half-plane. It is hoped that the fact the transmission spectrum is discrete will be a strong enough condition to prove uniqueness results even if we do not have analyticity of the temporal transform in an open neighborhood of $\mathbb{R}_+$.

The methods used to prove our results rely on weighted Sobolev spaces. The weight coming from the inverse of the contrast function. It is this fact that makes the results below non-trivial since the contrast smoothly transitioning to zero at the boundary implies a singular weight on the Sobolev space. We make use of Hardy inequalities to analyze the singular nature of these spaces and show there exists a variational formulation of the ITP on such spaces.

Discreteness of the transmission spectrum is justified by decomposing our variational form into operators on weighted Sobolev spaces. The Hardy inequality is again used to prove compact imbedding theorems for these spaces. The imbedding theorems are then used to show that the operators are compact perturbations of a bounded operator. From that point the fact that the spectrum is discrete follows from analytic Fredholm theory.

To show the existence of an infinite discrete set of transmission eigenvalues we use methods first discovered in [18, 19, 64]. This includes rewriting the variational formulation as a generalized eigenvalue problem and studying the operators involved. An existence theorem for generalized eigenvalues proved in [18, 19] then yields our result once it is shown that the hypotheses are satisfied.

It is interesting in its own right that one can show the existence of an infinite discrete set of transmission eigenvalues for a singular contrast. Recently there has been work done [14, 16] in using transmission eigenvalues to reconstruct interior properties of a body so this result may be applicable in this area. However, the knowledge of the existence of transmission eigenvalues does not clearly pertain to questions in thermoacoustic tomography.

This chapter is organized as follows. In section 4.2 we present some background on the history of the interior transmission problem. The main body of this chapter, section 4.3, is divided up into three subsections. In the first we set up the notational conventions and definitions used throughout the chapter and give a precise statement of the Hardy inequality along with results arising from it. The second contains the proof of the variational formulation of the ITP. In the third part our main result of the fact that the transmission spectrum is discrete is presented. Lastly, we show there exists an infinite discrete set of transmission eigenvalues in section 4.4.

4.2 Background and Previous Work

In acoustic inverse scattering problems one seeks to reconstruct information about the interior of an object from knowledge of the acoustic fields in the exterior of an object. Usually the acoustic fields are assumed to be produced by acoustic plane waves of a narrow frequency band bombarding the object from known directions. A subset of the far field pattern is then measured. This process is done for many different directions and frequencies and the interior acoustic properties of the object are reconstructed from this data.

There are many variations of this problem. In this chapter we are concerned with the case of an acoustically penetrable object immersed in an otherwise homogeneous acoustic medium. The object is assumed to have acoustic variations and we wish to recover the acoustic profile of the interior or at least the support of these variations. This problem has a long history, a good overview is found in [26, 24] and references therein. Here we present the basic mathematical formulation of this problem and its reconstruction methods in the interest of showing how the interior transmission problem arises.

It is assumed that a domain $D \subset \mathbb{R}^3$ contains acoustic variations given by an acoustic profile $c(x)$ with $supp(1 - c(x)) \subset \overline{D}$. Acoustic waves in D satisfy

$$U_{tt}(x,t) - c^2(x)\Delta U(x,t) = 0 \text{ in } D \times \mathbb{R}_+.$$

If one considers only time harmonic acoustic waves $U(x,t) = \mathcal{R}e\{u(x)e^{-ikt}\}$ then $u(x)$ must satisfy

$$\Delta u(x) + k^2 n(x) u(x) = 0 \text{ in } D \tag{4.1}$$

with $n(x) = 1/c^2(x)$. The acoustic field is excited by an incident wave $u^i(x)$ originating from outside D. Acoustic variations in the domain cause part of this wave to be scattered, denoted $u^s(x)$. The total acoustic field $u(x)$ must then satisfy

$$\begin{aligned} \Delta u(x) + k^2 n(x) u(x) &= 0 \text{ on } \mathbb{R}^3 \\ u(x) &= u^i(x) + u^s(x) \text{ on } \mathbb{R}^3 \\ \lim_{r\to\infty} (\partial_r u^s - iku^s) &= 0. \end{aligned} \tag{4.2}$$

The last limit condition is the *Sommerfeld radiation condition* which ensures the time harmonic wave generated from $u^s(x)$ is outgoing. We assume the incident field is a plane wave from a known direction $d \in S_1(0)$, so $u^i(x) = e^{ikx\cdot d}$. The scattered field then satisfies [26]

$$u^s(x) = \frac{e^{ikr}}{r} u_\infty(\hat{x}, d) + \mathcal{O}(r^{-2}). \tag{4.3}$$

Here $r = |x|$ and $\hat{x} = x/|x| \in S_1(0)$. The term $u_\infty(\hat{x}, d)$ is the *far field* which depends on $(\hat{x}, d) \in S_1(0) \times S_1(0)$. In scattering theory it is the far field that is measured for some subset of *incident directions* d and *scattering directions* $\hat{x}$. This process is carried out for some subset of frequencies k. One then attempts to reconstruct $n(x)$ from $u_\infty(\hat{x}, d)$ known for all $(\hat{x}, d) \in S_1(0) \times S_1(0)$.

Methods of reconstruction, such as the *linear sampling method* and the *dual space method*, then rely on use of the *far field operator* $F : L^2(S_1(0)) \to L^2(S_1(0))$ defined by

$$Fg(\hat{x}) = \int_{S_1(0)} u_\infty(\hat{x}, d) g(d)\, dS(d). \tag{4.4}$$

The function $Fg(\hat{x})$ is then the far field of a solution to (4.2) with incident field given by the *Herglotz wave function*

$$v_g(x) = \int_{S_1(0)} e^{ikx\cdot d} g(d)\, dS(d). \tag{4.5}$$

It is known that the linear sampling and dual space methods of reconstruction are only guaranteed to work if F is injective with dense range. The following theorem [24, 28] connects this requirement with the interior transmission problem.

Theorem 4.2.0.11 *[24, 28] The far field operator is injective with dense range if and only if there does not exist a solution* v*,* $w \in C^2(D) \cap C^1(\overline{D})$ *of the ITP such that* v *is a Herglotz wave function for some* $g \neq 0 \in L^2(S_1(0))$.

For more detailed information on the relation of scattering theory to the ITP see [24, 28, 26, 25, 27, 47, 69]. Recall a solution of the *interior transmission problem* is a non-trivial pair (w, v) satisfying (3.9). In most of the literature, and in this chapter, the refractive index corresponding to w will be labelled $n(x)$ and the index corresponding to v will just be $n \equiv 1$. It was shown that the above BVP could be studied in an equivalent weak form as long as $n(x)$ is positive in D, either strictly greater or strictly less than one in D, and met certain decay conditions at ∂D in [25]. Later, assuming that $n(x) = 1$ only on a set of measure zero and was bounded away from one otherwise, it was shown that the interior transmission problem has a weak formulation, [69]. In both of these cases it was demonstrated that the set of transmission eigenvalues was discrete. More recently these results have been expanded to include the case where $n(x) = 1$ on open subsets of D [14, 15]. Also, the existence of an infinite discrete set of transmission eigenvalues, for certain classes of $n(x)$ has been demonstrated [19, 17, 18, 64]. However, it is always assumed that $n(x) \neq 1$ on ∂D and $n(x) - 1$ meets certain non-decay criteria close to ∂D or the boundary of the set where $n(x) = 1$. The exception is [47] in which a weak form of the BVP and the discreteness of the set of transmission eigenvalues was shown to hold for $n(x) - 1$ going to zero on ∂D like a power, $0 < \alpha < 1$, of the distance to the boundary.

4.3 Results for a More General Case

4.3.1 Notation and Supporting Results

Let $D \subset \mathbb{R}^n$ be bounded, open, with smooth boundary. We assume $n(x) \in C^2(\mathbb{R}^n)$ is such that $supp(n(x) - 1) \subset \overline{D}$ and $n(x) > 1$ in $int(D)$. Our results hold if we assume

$n(x) < 1$ in $int(D)$ if we replace the operators developed below with their negatives but we only demonstrate the former case. Let $m(x) = n(x) - 1$, then we assume that $m(x)$ decays, as x approaches ∂D, not faster than the square of the distance to ∂D. The results of this chapter hold if we let the contrast be defined be two variable refractive indexes, $m(x) = n_c(x) - n_b(x)$. However, the statements and proofs of the results in this section are made simpler by the choice $m(x) = n(x) - 1$. To formalize our decay assumptions on the contrast, for $\delta > 0$, set

$$U_\delta = \{x \in D : d(x, \partial D) \le \delta\}.$$

Definition 4.3.1.1 *We say $m(x)$ has* controlled decay at the boundary of degree α *if there exist a $\delta_0 > 0$ and $\alpha > 0$ such that for $\delta < \delta_0$*

$$Cd(x, \partial D)^\alpha \le m(x)$$

for all $x \in U_\delta$ for some C depending only on δ.

For such $m(x)$ we define the weighted Lebesgue space for $1 \le q < \infty$

$$L^2(D, m^{-q}\, dx) = \left\{ f \in L^2(D) : \int_D |f(x)|^2 \frac{dx}{m^q(x)} < \infty \right\},$$

and define the weighted Sobolev space

$$H^2_{m,q}(D) = \{f \in H^2_0(D) : \Delta f \in L^2(D, m^{-q}\, dx)\}.$$

When $q = 1$ we will just write $H^2_m(D)$. For $u, v \in L^2(D, m^{-q}\, dx)$ we have the inner product

$$(u, v)_{m,q} = \int_D uv \frac{dx}{m^q}$$

and for $u, v \in H^2_{m,q}(D)$ we take the inner product

$$(u, v)_{H^2_{m,q}} = (\Delta u, \Delta v)_{m,q}.$$

For $q = 1$ we will write $(\cdot,\cdot)_m = (\cdot,\cdot)_{m,1}$ and $(\cdot,\cdot)_{H^2_m} = (\cdot,\cdot)_{H^2_{m,1}}$. Also, we will use $(\cdot,\cdot)$ to denote the pairing in $L^2(D)$. With these inner products the spaces $L^2(D, m^{-q}\, dx)$ and $H^2_{m,q}(D)$ are Hilbert spaces [79, 80, 49, 78].

Contrary to the previous chapter, here we are more careful about the spaces in which our solutions to the ITP lie. For this reason we have the following definition.

Definition 4.3.1.2 *If for some $k \in \mathbb{R}_+$ there exists a nontrivial pair $w, v \in H^2_{loc}(D) \cap L^2(D)$ satisfying (3.9) with $n_c(x) = n(x)$ and $n_b(x) \equiv 1$ then we call k a* transmission *eigenvalue.*

The proofs below rely heavily on the Hardy inequality. We state this, without proof, in a version suited to our needs [62, 40, 31, 54, 48].

Theorem 4.3.1.1 *For $u \in C_0^\infty(D)$, $1 < p < \infty$, and $\beta > 1$ there exists a $C > 0$ depending only on p, β, and D so that*

$$\int_D \frac{|u(x)|^p}{d(x, \partial D)^\beta}\, dx \le C \int_D |\nabla u(x)|^p d(x, \partial D)^{p-\beta}\, dx. \tag{4.6}$$

We prove a few corollaries to theorem 4.3.1.1. The purpose of these is to develop imbedding results useful in studying the (ITP). Our first result essentially states that for $m(x)$ distance like of degree $\alpha = 2$ we have that $H_0^1(D)$ is boundedly imbedded in $L^2(D, m^{-1}\, dx)$.

Corollary 4.3.1.1 *For $u \in H_0^1(D)$ and $m(x)$ having controlled decay at the boundary of degree $\alpha = 2$ there exists some $C > 0$, independent of $u(x)$, so that*

$$\int_D \frac{|u(x)|^2}{m(x)}\, dx \le C \int_D |\nabla u(x)|^2\, dx. \tag{4.7}$$

Proof. Notice that for $m(x)$ having controlled decay at the boundary of degree $\alpha = 2$ and $u \in H_0^1(D)$ we have

$$\int_D \frac{|u(x)|^2}{m(x)}\, dx \le C \int_D \frac{|u(x)|^2}{d(x, \partial D)^2}\, dx.$$

In theorem 4.3.1.1 now take $p = 2$ and $\beta = 2$ and the result follows.

■

Corollary 4.3.1.2 *For $u \in H_0^1(D)$ and $m(x)$ having controlled decay at the boundary of degree $\alpha = 2$ there exist some $C > 0$, independent of $u(x)$, so that*

$$\int_D \frac{|u(x)|^2}{m^2(x)}\, dx \leq C \int_D \frac{|\nabla u(x)|^2}{d(x, \partial D)^2}\, dx. \tag{4.8}$$

Proof. By the assumptions on $m(x)$ and theorem 4.3.1.1 we see

$$\int_D \frac{|u(x)|^2}{m^2(x)}\, dx \leq C \int_D \frac{|u(x)|^2}{d(x, \partial D)^4}\, dx \leq C' \int_D \frac{|\nabla u(x)|^2}{d(x, \partial D)^2}\, dx. \tag{4.9}$$

■

The next lemma states that $H_0^2(D)$ is boundedly imbedded in $L^2(D, m^{-2}\, dx)$. We will use a multi-index notation for derivatives. For the coordinate $k = 1, 2, \ldots, n$ and a positive integer $i \in \mathbb{N}$ we write $\partial_k^i u(x) = \frac{\partial^i u}{\partial x_k^i}$. If $i = 1$ we will just write $\partial_k u(x) = \frac{\partial u}{\partial x_k}$. Higher order derivatives will be denoted by a multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ and the derivative will be written as

$$\partial^\alpha u(x) = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u(x).$$

The *order* of the multi-index will be defined as

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n.$$

Lemma 4.3.1.1 *For $u \in H_0^2(D)$ and $m(x)$ having controlled decay at the boundary of degree $\alpha = 2$ there exist some $C > 0$, independent of $u(x)$, so that*

$$\int_D \frac{|u(x)|^2}{m^2(x)}\, dx \leq C \int_D |\Delta u(x)|^2\, dx. \tag{4.10}$$

Proof. First we notice that for any index $s = 1, 2, \ldots, n$ and $u \in H_0^2(D)$ the Hardy inequality with $p, \beta = 2$ yields

$$\int_D \frac{|\partial_s u|^2}{d(x, \partial D)^2}\, dx \leq C \int_D |\nabla(\partial_s u)|^2\, dx. \tag{4.11}$$

As a consequence of Kadlec's formula {[77], pg. 340}, for $u \in H^2(D)$ with trace, $u|_{\partial D} = 0$, and $\partial D \in C^2$, the estimate

$$\sum_{|\sigma|=2} \int_D |\partial^\sigma u|^2\, dx \le \int_D |\Delta u|^2\, dx + C \int_{\partial D} |\partial_\nu u|^2\, dS \tag{4.12}$$

holds. For $u \in H_0^2(D)$ this estimate becomes

$$\sum_{|\sigma|=2} \int_D |\partial^\sigma u|^2\, dx \le \int_D |\Delta u|^2\, dx. \tag{4.13}$$

Observe that

$$|\nabla u|^2 = \sum_{s=1}^n |\partial_s u|^2.$$

Combining this with (4.11) gives, for $s = 1, 2, \ldots, n$,

$$\int_D \frac{|\partial_s u|^2}{d(x,\partial D)^2}\, dx \le C \int_D |\nabla(\partial_s u)|^2\, dx \le C \sum_{|\sigma|=1} \int_D |\partial^\sigma \partial_s u|^2\, dx. \tag{4.14}$$

Using (4.13) now yields

$$\begin{aligned} \int_D \frac{|\partial_s u|^2}{d(x,\partial D)^2} &\le C \sum_{|\sigma|=1} \int_D |\partial^\sigma \partial_s u|^2\, dx && (4.15) \\ &\le C \sum_{|k|=2} \int_D |\partial^k u|^2\, dx \le C \int_D |\Delta u|^2\, dx. && (4.16) \end{aligned}$$

After this it follows that

$$\int_D \frac{|\nabla u|^2}{d(x,\partial D)^2}\, dx \le \sum_{s=1}^n \int_D \frac{|\partial_s u|^2}{d(x,\partial D)^2}\, dx \le C' \int_D |\Delta u|^2\, dx. \tag{4.17}$$

Now the statement follows from corollary 4.3.1.2.

■

4.3.2 Variational Formulation

This section is devoted to the development of a weak formulation of the ITP. An equivalent weak formulation is reached on the space $H^2_{m,2}(D)$. An implied, but not equivalent, weak form is also shown to hold on the space $H^2_m(D)$. Similar results have been arrived at previously [25, 69, 47, 64, 19]. However, all previous results either require $m(x)$ to be bounded away from zero or require $m(x)$ to go to zero discontinuously or like a power $\alpha < 1$. Here we allow $m(x)$ to decay not faster than the square of the distance to ∂D. This allows us to assume that $m(x) \in C^2(D)$ but still equal to zero on ∂D. For the remainder of this manuscript we will use $\tau = k^2$.

Lemma 4.3.2.1 *Assume that $n(x) \in C^2(D)$ is such that $n(x) \geq \sigma > 0$ in D and $m(x) = n(x) - 1$ satisfies either $m(x) > 0$ or $m(x) < 0$ in $\text{int}(D)$. Suppose also that $m(x)$ has controlled decay at the boundary of degree 2. Then if $k \in \mathbb{R}_+$ is an interior transmission eigenvalue relative to the domain D there exists a nontrivial $z \in H^2_m(D)$ such that*

$$\mathcal{F}_\tau(z, \phi) = \int_D m^{-1}(\Delta + \tau)z(\Delta + \tau n)\phi \, dx = 0$$

for all $\phi \in H^2_m(D)$.

Proof. First, assume $k \in \mathbb{R}_+$ is a transmission eigenvalue for D so there exists nontrivial $w, v \in H^2_{loc}(D) \cap L^2(D)$ satisfying (3.9). Let $z = w - v$, so $z \in H^2_0(D)$ and $z \neq 0$. Then z satisfies

$$(\Delta + \tau)z = -\tau m w \tag{4.18}$$

and therefore $m^{-1}mw = w$, a.e. in D, implies $m^{-1}(\Delta + \tau)z \in H^2_{loc}(D) \cap L^2(D)$ and

$$-\tau w = m^{-1}(\Delta + \tau)z.$$

We wish to use Green's formula to perform an integration by parts using $w \in H^2_{loc}(D) \cap L^2(D)$. To do this we will appeal to a result by Lions and Magenes, see [11] for a review. This result uses the additional regularity guaranteed by the fact that w is a solution of the ITP.

Notice that since $w \in L^2(D)$ and $\Delta w + \tau n w = 0$ we also have $\Delta w \in L^2(D)$. The result by Lions and Magenes then assures that the traces, $w|_{\partial D}$ and $\partial_\nu w|_{\partial D}$, are defined as continuous extensions of the trace for C^2 functions and their normal derivatives on ∂D. It also asserts that $w|_{\partial D} \in H^{-1/2}(\partial D)$ and $\partial_\nu w|_{\partial D} \in H^{-3/2}(\partial D)$. For any $v \in H^2(D)$ the standard trace theorem for $H^2(D)$ functions states that the trace of v is again defined with $v|_{\partial D} \in H^{3/2}(\partial D)$ and $\partial_\nu v|_{\partial D} \in H^{1/2}(\partial D)$. Green's formula can then be interpreted in terms of pairing between the spaces $H^{3/2}(\partial D)$ and $H^{-3/2}(\partial D)$, and the spaces $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$.

Green's formula then holds in a weak form for $u \in L^2(D)$ with $\Delta u \in L^2(D)$ and $v \in H^2(D)$ as

$$\int_D v\Delta u\, dx = \int_D u\Delta v\, dx + \langle v, \partial_\nu u\rangle_{H^{3/2}(\partial D)} - \langle u, \partial_\nu v\rangle_{H^{1/2}(\partial D)}.$$

Here $\langle \cdot,\cdot\rangle_{H^{3/2}(\partial D)}$ is the pairing between $H^{3/2}(\partial D)$ and its dual space $H^{-3/2}(\partial D)$ and $\langle \cdot,\cdot\rangle_{H^{1/2}(\partial D)}$is a pairing between $H^{1/2}(\partial D)$ and its dual space $H^{-1/2}(\partial D)$.

Taking the above mentioned results into account we see that for every $\phi \in H^2_m(D)$ we have

$$\begin{aligned}\int_D m^{-1}(\Delta+\tau)z(\Delta+\tau n)\phi\, dx &= -\tau\int_D w(\Delta+\tau n)\phi\, dx\\ &= -\tau\int_D \phi(\Delta+\tau n)w\, dx\\ &\quad -\tau\int_{\partial D}(w\partial_\nu\phi - \phi\partial_\nu w)\, dS = 0\end{aligned}$$

with the last integrals interpreted as pairings between dual spaces.

The last two terms vanish since $\phi \in H^2_m(D) \subset H^2_0(D)$ and $\Delta w + \tau n w = 0$ on D. Thus, $z \neq 0 \in H^2_0(D)$ satisfies $\mathcal{F}_\tau(z,\phi) = 0$ for all $\phi \in H^2_m(D)$. It remains to show that $z \in H^2_m(D)$. We use an appeal to the Hardy inequality, or rather its corollary, to justify this.

Notice, from (4.18), $m^{-1}(\Delta+\tau)z \in L^2(D)$. This clearly implies $(\Delta+\tau)z \in L^2(D, m^{-1}\, dx)$. From the Hardy inequality, corollary 4.3.1.1,

$$\int_D |z|^2\frac{dx}{m} \le C\int_D |\nabla z|^2\, dx. \tag{4.19}$$

Thus,

$$\|z\|_m \leq C\|\nabla z\| < \infty$$

since $z \in H_0^2(D)$. But this implies $z \in L^2(D, m^{-1}\,dx)$, therefore we must have $\Delta z \in L^2(D, m^{-1}\,dx)$ and we conclude that $z \in H_m^2(D)$.

■

In the previous lemma, the existence of a nonzero $z \in H_m^2(D)$ satisfying $\mathcal{F}_\tau(z, \phi) = 0$ for all $\phi \in H_m^2(D)$ does not imply that $k \in \mathbb{R}_+$ is a transmission eigenvalue. However, if we require additional regularity on such a z we can show that $k \in \mathbb{R}_+$ is indeed a transmission eigenvalue.

Lemma 4.3.2.2 *Assume that $n(x) \in C^2(D)$ is such that $n(x) \geq \sigma > 0$ in D and $m(x) = n(x) - 1$ satisfies either $m(x) > 0$ or $m(x) < 0$ in $\operatorname{int}(D)$. Suppose also that $m(x)$ has controlled decay at the boundary of degree 2. If, for $k \in \mathbb{R}_+$, there exists a nontrivial $z \in H_m^2(D)$ such that*

$$\mathcal{F}_\tau(z, \phi) = \int_D m^{-1}(\Delta + \tau)z(\Delta + \tau n)\phi\,dx = 0$$

for all $\phi \in H_m^2(D)$ and $m^{-1}z$, $m^{-1}\Delta z \in L^2(D)$ then $k \in \mathbb{R}_+$ is a transmission eigenvalue.

Proof. Suppose for $k \in \mathbb{R}_+$ there is a nontrivial $z \in H_m^2(D)$ such that $\mathcal{F}_\tau(z, \phi) = 0$ for all $\phi \in H_m^2(D)$. Assume as well that $m^{-1}z$, $m^{-1}\Delta z \in L^2(D)$. We set

$$w = \frac{1}{\tau m}(\Delta + \tau)z.$$

Then $w \in L^2(D)$ and $v = w - z \in L^2(D)$. We observe

$$(w, (\Delta + \tau n)\phi) = \tau^{-1}(m^{-1}(\Delta + \tau)z, (\Delta + \tau n)\phi) = \tau^{-1}\mathcal{F}_\tau(z, \phi) = 0$$

for every $\phi \in H_m^2(D)$. But $C_0^\infty(D) \subset H_m^2(D)$ so

$$(w, (\Delta + \tau n)\phi) = 0$$

for every $\phi \in C_0^\infty(D)$. Thus, standard regularity results for elliptic operators [1] show that $w \in H^2_{loc}(D)$ and $(\Delta + \tau n)w = 0$ in $L^2(D)$. Since $z \in H^2_m(D) \subset H^2_{loc}(D)$ it follows that $v \in H^2_{loc}(D)$. One now sees

$$(\Delta + \tau)v = (\Delta + \tau)w - (\Delta + \tau)z = \tau m w - \tau m w = 0$$

in $L^2(D)$. Also, $z = w - v \in H^2_0(D)$ so the boundary conditions are satisfied and we have shown that $w, v \in H^2_{loc}(D) \cap L^2(D)$ are solutions of the interior transmission problem for $k \in \mathbb{R}_+$ in D. This implies k is a transmission eigenvalue.

■

Theorem 4.3.2.1 *Assume that $n(x) \in C^2(D)$ is such that $n(x) \geq \sigma > 0$ in D and $m(x) = n(x) - 1$ satisfies either $m(x) > 0$ or $m(x) < 0$ in $\mathrm{int}(D)$. Suppose also that $m(x)$ has controlled decay at the boundary of degree 2. Then $k \in \mathbb{R}_+$ is an interior transmission eigenvalue relative to the domain D if and only if there exists a nontrivial $z \in H^2_{m,2}(D)$ such that*

$$\mathcal{F}_\tau(z, \phi) = \int_D m^{-1}(\Delta + \tau)z(\Delta + \tau n)\phi \, dx = 0$$

for all $\phi \in H^2_{m,2}(D)$.

Proof. If $k \in \mathbb{R}_+$ is a transmission eigenvalue then, since $H^2_{m,2}(D) \subset H^2_m(D)$, the proof of lemma 4.3.2.1 shows that there exists a nontrivial $z \in H^2_0(D)$ such that $\mathcal{F}_\tau(z, \phi) = 0$ for every $\phi \in H^2_{m,2}(D)$. In the proof of lemma 4.3.2.1 it was shown that this z satisfied

$$(\Delta + \tau)z = -\tau m w$$

where $w \in H^2_{loc}(D) \cap L^2(D)$ was part of the solution to the ITP with eigenvalue k along with some $v \in H^2_{loc}(D) \cap L^2(D)$. Thus, as shown in lemma 4.3.2.1, we have $m^{-1}(\Delta + \tau)z \in L^2(D)$. However, lemma 4.3.1.1 shows that

$$\int_D \frac{|z|^2}{m^2} \, dx \leq C \int_D |\Delta z|^2 \, dx < \infty$$

since $z \in H_0^2(D)$. We may then conclude that $m^{-1}z \in L^2(D)$ which implies $m^{-1}\Delta z \in L^2(D)$. It follows that $z \in H^2_{m,2}(D)$ is nontrivial and the forward direction is verified.

If, for some $k \in \mathbb{R}_+$, there exists a $z \neq 0 \in H^2_{m,2}(D)$ such that $\mathcal{F}_\tau(z,\phi) = 0$ for every $\phi \in H^2_{m,2}(D)$ then the proof of lemma 4.3.2.2 shows that k is a transmission eigenvalue. This holds since $z \in H^2_{m,2}(D)$ implies $m^{-1}z$, $m^{-1}\Delta z \in L^2(D)$ and we still have $C_0^\infty(D) \subset H^2_{m,2}(D)$ is dense. Therefore the variational form over the space $H^2_{m,2}(D)$ is equivalent to the original interior transmission problem.

■

4.3.3 Discreteness of the Transmission Spectrum

Our goal is to justify the use of analytic Fredholm theory to show that the transmission spectrum is discrete. It is well known that the form, $\mathcal{F}_\tau(\cdot,\cdot)$, on $H^2_m(D) \times H^2_m(D)$ can be written as a sum of bounded forms [69, 28, 17]. We show that the form F_τ is a compact perturbation of the identity depending analytically on τ. From now on we let $\Delta_\tau = \Delta + \tau$ and $\Delta_n = \Delta + \tau n$. With this notation

$$\mathcal{F}_\tau(z,\phi) = (\Delta_\tau z, \Delta_n \phi)_m \text{ for } z,\, \phi \in H^2_m(D). \tag{4.20}$$

Observe that, for z, $\phi \in H^2_m(D)$, we have

$$\begin{aligned}\mathcal{F}_\tau(z,\phi) &= \int_D m^{-1}\Delta_\tau z \Delta_n \phi \\ &= \int_D m^{-1}(\Delta z \Delta \phi + \tau z \Delta\phi + \tau n \phi \Delta z + \tau^2 n z \phi).\end{aligned}$$

So, define the bounded bilinear forms on $H^2_m(D)$

$$\mathcal{S}^0(z,\phi) = (\Delta z, \Delta\phi)_m = (z,\phi)_{H^2_m} \tag{4.21}$$

$$\mathcal{S}^1(z,\phi) = (z, \Delta\phi)_m \tag{4.22}$$

$$\mathcal{S}^2(z,\phi) = (n\Delta z, \phi)_m \tag{4.23}$$

$$\mathcal{S}^3(z,\phi) = (nz, \phi)_m \tag{4.24}$$

and notice that

$$\mathcal{F}_\tau(z,\phi) = \mathcal{S}^0(z,\phi) + \tau\mathcal{S}^1(z,\phi) + \tau\mathcal{S}^2(z,\phi) + \tau^2\mathcal{S}^3(z,\phi).$$

This means of investigation of the problem was first seen in [25, 69] where it was shown to be useful in proving that the transmission spectrum is discrete. Note that, by theorem 4.3.2.1, one could study the ITP using the same operators defined on the space $H^2_{m,2}(D)$ as well.

Lemma 4.3.3.1 *The bilinear forms $\mathcal{S}^0$, $\mathcal{S}^1$, $\mathcal{S}^2$, $\mathcal{S}^3$ are bounded on $H^2_m(D)$.*

Proof. Let z, $\phi \in H^2_m(D)$. Then $\mathcal{S}^0(z,z) = \|z\|^2_{H^2_m}$ so is clearly bounded. By an application of Cauchy-Bunyakowsky-Schwarz and the Hardy inequality used above we see

$$|\mathcal{S}^1(z,\phi)| \quad = \quad |(z,\Delta\phi)_m| \le \|z\|_m\|\phi\|_{H^2_m}.$$

Again appealing to the Hardy inequality we observe that

$$\|z\|^2_m = \int_D \frac{|z|^2}{m}\,dx \le C\int_D |\nabla z|^2\,dx \le C\|z\|^2_{H^1}.$$

Since $z \in H^2_0(D)$ an application of Poincarés inequality yields

$$\|z\|^2_{H^1} \le C\int_D |\Delta z|^2\,dx \le C\int_D \frac{|\Delta z|^2}{m}\,dx \le C\|z\|^2_{H^2_m}.$$

Therefore we have that

$$\|z\|_m \le C\|z\|_{H^2_m}$$

and from this it follows that

$$|\mathcal{S}^1(z,\phi)| \le C\|z\|_{H^2_m}\|\phi\|_{H^2_m}.$$

Likewise one observes

$$|\mathcal{S}^2(z,\phi)| \le \sup_D n\,|\mathcal{S}^1(\phi,z)| \le C\sup_D n\|z\|_{H^2_m}\|\phi\|_{H^2_m}.$$

To show that the form $\mathcal{S}^3$ is bounded on $H^2_m(D)$ we again use the fact that $H^2_m(D)$ is boundedly imbedded in $L^2(D, m^{-1}\,dx)$. First observe that, by Cauchy-Bunyakowsky-Schwarz,

$$\begin{aligned} |\mathcal{S}^3(z,\phi)| &\leq |(nz,\phi)_m| \leq \sup_D n \left| \int_D z\phi \,\frac{dx}{m} \right| \\ &\leq \sup_D n \|z\|_m \|\phi\|_m \\ &\leq C \sup_D n \|z\|_{H^2_m} \|\phi\|_{H^2_m}. \end{aligned}$$

Thus, all these forms are bounded on $H^2_m(D)$.

■

Now use the Riesz representation theorem to define bounded operators on $H^2_m(D)$ by

$$\mathcal{S}^j(z,\phi) = (S^j z, \phi)_{H^2_m} \text{ for every } \phi \in H^2_m(D) \text{ for } j = 0, 1, 2, 3. \tag{4.25}$$

Note that $S^0 = I$, the identity operator on $H^2_m(D)$. Using these definitions, define the operator

$$F_\tau = S^0 + \tau S^1 + \tau S^2 + \tau^2 S^3 : H^2_m(D) \to H^2_m(D) \tag{4.26}$$

and note that $k \in \mathbb{R}_+$ is a transmission eigenvalue only if F_τ has nontrivial kernel by lemma 4.3.2.1.

Lemma 4.3.3.2 *$H^2_m(D)$ is compactly imbedded in $L^2(D, m^{-1}\,dx)$.*

Proof. Let $\{\phi_j\}$ be a sequence in $H^2_m(D)$ with $\|\phi_j\|_{H^2_m} \leq 1$. Then $\{\phi_j\}$ is certainly bounded in $H^2_0(D)$ as well. Since $H^2_0(D)$ is reflexive, by Alaoglu's theorem, there is a weakly convergent subsequence, which we also write as $\{\phi_j\}$, converging weakly to some $\phi \in H^2_0(D)$. However, $H^2_0(D)$ is also compactly imbedded in $H^1_0(D)$ so

$$\|\phi_j - \phi\|_{H^1} \to 0.$$

Then, appealing to the Hardy inequality,

$$\|\phi_j - \phi\|_m^2 = \|\frac{\phi_j - \phi}{\sqrt{m}}\|^2 \le C\|\phi_j - \phi\|_{H^1}^2 \to 0.$$

Thus, every bounded sequence in $H_m^2(D)$ has a subsequence which is convergent in $L^2(D, m^{-1}\,dx)$.

■

Lemma 4.3.3.3 *The operators S^1, S^2, S^3 are compact operators on $H_m^2(D)$.*

Proof. The proof is along the lines of [47]. Since $H_m^2(D)$ is reflexive it suffices to show that each operator is completely continuous at zero. Take a sequence $\{\phi_j\}$ converging weakly to zero in $H_m^2(D)$. Then

$$\begin{aligned}
\|S^1\phi_j\|_{H_m^2}^2 &= \mathcal{S}^1(\phi_j, S^1\phi_j) = \int_D \phi_j \Delta(S^1\phi_j)\frac{dx}{m} \\
&= \int_D \frac{\phi_j}{\sqrt{m}}\frac{\Delta(S^1\phi_j)}{\sqrt{m}}\,dx \le \|\phi_j\|_m \|\Delta(S^1\phi_j)\|_m \\
&\le \|\phi_j\|_m \|S^1\phi_j\|_{H_m^2}.
\end{aligned}$$

Since $\{\phi_j\}$ is bounded and S^1 is bounded on $H_m^2(D)$ we have that $\|S^1\phi_j\|_{H_m^2}$ is bounded. Also, since $H_m^2(D)$ is compactly imbedded in $L^2(D, m^{-1}\,dx)$ we have that $\|\phi_j\|_m \to 0$. Thus, $\|S^1\phi_j\|_{H_m^2} \to 0$. We conclude that S^1 is completely continuous and thus compact on $H_m^2(D)$. For $\{\phi_j\}$ as above we have

$$\begin{aligned}
\|S^2\phi_j\|_{H_m^2}^2 &= \mathcal{S}^2(\phi_j, S^2\phi_j) = \int_D n\Delta\phi_j(S^2\phi_j)\frac{dx}{m} \\
&\le \sup_D n\|\phi_j\|_{H_m^2}\|S^2\phi_j\|_m.
\end{aligned}$$

Since S^2 is a bounded operator $\{S^2\phi_j\}$ is bounded in $H_m^2(D)$ so the compactness of $H_m^2(D)$ in $L^2(D, m^{-1}\,dx)$ ensures that $\|S^2\phi_j\|_m \to 0$. The boundedness of $\|\phi_j\|_{H_m^2}$ lets us conclude $\|S^2\phi_j\|_{H_m^2} \to 0$ and thus that S^2 is compact. The proof for S^3 is similar. We observe

$$\begin{aligned}
\|S^3\phi_j\|_{H_m^2}^2 &= \mathcal{S}^3(\phi_j, S^3\phi_j) \\
&\le \sup_D n\|\phi_j\|_m\|S^3\phi_j\|_m.
\end{aligned}$$

Now $H^2_m(D)$ compact in $L^2(D, m^{-1}\,dx)$ lets us conclude $\|S^3\phi_j\|_{H^2_m} \to 0$. Therefore, S^3 is compact as well.

■

Theorem 4.3.3.1 *Assume that $n(x) \in C^2(D)$ is such that $n(x) \geq \sigma > 0$ in D and $m(x) = n(x) - 1$ satisfies either $m(x) > 0$ or $m(x) < 0$ in $\mathrm{int}(D)$. Suppose also that $m(x)$ has controlled decay at the boundary of degree 2. Then the interior transmission spectrum is discrete.*

Proof. By lemma 4.3.2.1 if $k \in \mathbb{R}_+$ is a transmission eigenvalue then F_τ has a nontrivial kernel in $H^2_m(D)$. We have shown in the above that this operator is the sum of an invertible operator and a compact operator. That is we have shown

$$F_\tau = S^0 + k^2(S^1 + S^2) + k^4 S^3$$

and S^0 is invertible on $H^2_m(D)$ while S^1, S^2, and S^3 are all compact on $H^2_m(D)$. Clearly the compact part of the operator F_τ depends analytically on $k \in \mathbb{R}_+$ since it actually depends polynomially on $k \in \mathbb{R}_+$.

The theorems of [[46], §7.1 - Thm. 1.9] then guarantee that F_τ has non-trivial kernel for every $k \in \mathbb{R}_+$ or on a discrete set. However, the lower bound on the set of transmission eigenvalues implied by theorem 3.5.0.9 with $n_c(x) = n(x)$ and $n_b(x) = 1$ ensures that the former can not happen and, therefore, the set of transmission eigenvalues is discrete.

■

4.4 Existence of an Infinite Set of Transmission Eigenvalues

The above results are useful for studying the unique determination of the acoustic profile in thermoacoustic tomography. However, since the ITP is of interest in a much larger field we collect some results on the existence of a discrete set of transmission eigenvalues in the setting outlined above. In the following we prove that there does indeed exist an infinite discrete set of transmission eigenvalues when the contrast, $m(x)$, is allowed to decay as x approaches ∂D. The main technique used here was first used in [19, 18, 17] and consists of decomposing the operator $\mathcal{F}_\tau(\cdot,\cdot)$ into a sum of positive definite self-adjoint operators and studying an associated generalized eigenvalue problem. First, define the bilinear forms on $H^2_{m,2}(D) \times H^2_{m,2}(D)$

$$\begin{aligned}\mathcal{A}_\tau(u,v) &= \int_D \frac{1}{m(x)}(\Delta+\tau)u(x)(\Delta+\tau)v(x)\,dx + \tau^2\int_D u(x)v(x)\,dx \qquad (4.27)\\ &= \left(\frac{1}{m}(\Delta+\tau)u,(\Delta+\tau)v\right) + \tau^2(u,v),\end{aligned}$$

and

$$\mathcal{B}(u,v) = \int_D \nabla u(x)\cdot\nabla v(x)\,dx = (\nabla u,\nabla v). \qquad (4.28)$$

Lemma 4.4.0.4 *For u, $v \in H^2_{m,2}(D)$ we have*

$$\mathcal{F}_\tau(u,v) = \mathcal{A}_\tau(u,v) - \tau\mathcal{B}(u,v). \qquad (4.29)$$

Proof. We simply compute

$$\begin{aligned}\mathcal{F}_\tau(u,v) &= \left(\frac{1}{m}(\Delta+\tau)u,(\Delta+\tau n(x))v\right)\\ &= \left(\frac{1}{m}(\Delta+\tau)u,(\Delta+\tau)v+\tau n(x)v-\tau v\right)\\ &= \left(\frac{1}{m}(\Delta+\tau)u,(\Delta+\tau)v\right) + ((\Delta+\tau)u,\tau v)\\ &= \mathcal{A}_\tau(u,v)+\tau(\Delta u,v) = \mathcal{A}_\tau(u,v)-\tau\mathcal{B}(u,v).\end{aligned}$$

■

Lemma 4.4.0.5 *On $H^2_{m,2}(D)$ we have that*

$$\mathcal{A}_\tau(u,v) = \mathcal{S}^0(u,v) + \tau[\mathcal{S}^1(u,v) + \mathcal{S}^1(v,u)] + \tau^2\mathcal{S}^3(u,v) \tag{4.30}$$

$$\mathcal{B}(u,v) = \mathcal{S}^1(v,u) - \mathcal{S}^2(u,v). \tag{4.31}$$

Both $\mathcal{A}_\tau(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ are bounded and symmetric bilinear forms on $H^2_{m,2}(D)$.

Proof. Simple computations show, for u, $v \in H^2_{m,2}(D)$,

$$\begin{aligned}\mathcal{A}_\tau(u,v) &= \left(\frac{1}{m}(\Delta+\tau)u, (\Delta+\tau)v\right) + \tau^2(u,v)\\ &= \left(\frac{1}{m}\Delta u, \Delta v\right) + \tau\left(\frac{1}{m}\Delta u, v\right)\\ &\quad + \tau\left(\frac{1}{m}u, \Delta v\right) + \tau^2\left(\frac{1}{m}u, v\right) + \tau^2(u,v)\\ &= \mathcal{S}^0(u,v) + \tau[\mathcal{S}^1(u,v) + \mathcal{S}^1(v,u)] + \tau^2\left(\left(1+\frac{1}{m}\right)u, v\right)\\ &= \mathcal{S}^0(u,v) + \tau[\mathcal{S}^1(u,v) + \mathcal{S}^1(v,u)] + \tau^2\mathcal{S}^3(u,v)\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}^1(v,u) - \mathcal{S}^2(u,v) &= \left(\frac{1}{m}v, \Delta u\right) - \left(\frac{n}{m}\Delta u, v\right)\\ &= \left(\left(\frac{1}{m} - \frac{n}{m}\right)\Delta u, v\right)\\ &= -(\Delta u, v) = \mathcal{B}(u,v).\end{aligned}$$

We have shown in lemma 4.3.3.1 that the forms $\mathcal{S}^i(\cdot,\cdot)$, $i = 0$, 1, 2, 3, are bounded on $H^2_m(D)$. Since $H^2_{m,2}(D) \subset H^2_m(D)$ is boundedly imbedded it follows that the $\mathcal{S}^i(\cdot,\cdot)$ are bounded on $H^2_{m,2}(D)$. Given the above expressions we can now conclude $\mathcal{A}_\tau(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ are bounded on $H^2_{m,2}(D)$.

■

Since both of the forms $\mathcal{A}_\tau(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ are bounded on $H^2_{m,2}(D)$ we may use the theorem of Riesz to define bounded operators $A_\tau : H^2_{m,2}(D) \to H^2_{m,2}(D)$ and $B : H^2_{m,2}(D) \to H^2_{m,2}(D)$ by

$$\mathcal{A}_\tau(u,v) = (A_\tau u, v)_{H^2_{m,2}(D)} \text{ for } u,\, v \in H^2_{m,2}(D) \tag{4.32}$$

$$\mathcal{B}(u,v) = (Bu, v)_{H^2_{m,2}(D)} \text{ for } u,\, v \in H^2_{m,2}(D). \tag{4.33}$$

Properties of these operators are summarized in the following lemma.

Lemma 4.4.0.6 *The operators $A_\tau : H^2_{m,2}(D) \to H^2_{m,2}(D)$ and $B : H^2_{m,2}(D) \to H^2_{m,2}(D)$ are bounded, self-adjoint operators. The operator A_τ is positive definite while the operator B is non-negative and compact.*

Proof. Both of the forms $\mathcal{A}_\tau(\cdot,\cdot)$ and $\mathcal{B}(\cdot,\cdot)$ are symmetric by definition therefore the operators they define are self-adjoint. Moreover, for all $u \in H^2_{m,2}(D)$ we see

$$\mathcal{B}(u,u) = (Bu,u)_{H^2_{m,2}(D)} = \int_D |\nabla u|^2\, dx \geq 0,$$

so B is non-negative. To see that B is compact we observe, from lemma 4.4.0.5, for $u,\, v \in H^2_{m,2}(D)$

$$\begin{aligned}
\mathcal{B}(u,v) &= (Bu,v)_{H^2_{m,2}(D)} \\
&= \mathcal{S}^1(v,u) - \mathcal{S}^2(u,v) \\
&= (S^1 v, u)_{H^2_m} - (S^2 u, v)_{H^2_m} \\
&= ((S^1)^* u, v)_{H^2_m} - (S^2 u, v)_{H^2_m} \\
&= ([(S^1)^* - S^2]u, v)_{H^2_m}.
\end{aligned}$$

If we restrict $(S^1)^*$ and S^2 to $H^2_{m,2}(D)$ then $B = (S^1)^* - S^2$. In lemma 4.3.3.3 it was shown that S^1 and S^2 are compact on $H^2_m(D)$. It follows that $(S^1)^*$ is then compact on $H^2_m(D)$ as well. One now observes that

$$H^2_{m,2}(D) \hookrightarrow H^2_m(D) \hookrightarrow L^2(D, m^{-1}\, dx)$$

with the first imbedding being bounded and the second imbedding being compact by lemma 4.3.3.2. This implies that the imbedding $H^2_{m,2}(D) \hookrightarrow L^2(D, m^{-1}\,dx)$ is compact and therefore both of the restricted operators

$$(S^1)^* : H^2_{m,2}(D) \to H^2_{m,2}(D) \tag{4.34}$$
$$S^2 : H^2_{m,2}(D) \to H^2_{m,2}(D)$$

are compact by the proof of lemma 4.3.3.3. It follows that the operator $B : H^2_{m,2}(D) \to H^2_{m,2}(D)$ is compact.

It only remains to show that $A_\tau : H^2_{m,2}(D) \to H^2_{m,2}(D)$ is positive definite. It suffices to show for $u \in H^2_{m,2}(D)$

$$\mathcal{A}_\tau(u,u) \geq 0$$

with equality only if $u = 0 \in H^2_{m,2}(D)$. However, it was shown in [18] for $u \in H^2_0(D)$

$$\mathcal{A}_\tau(u,u) \geq C_\tau \|u\|^2_{H^2_0}.$$

Since $H^2_{m,2}(D) \subset H^2_0(D)$ we only need to show that $\|u\|_{H^2_0} = 0$ implies $\|u\|_{H^2_{m,2}} = 0$. Suppose $u = 0$ in $H^2_0(D)$ and $u \in H^2_{m,2}(D)$. Then

$$\int_D |\Delta u|^2\,dx = 0 \text{ and } \int_D |\Delta u|^2 \frac{dx}{m^2} < \infty$$

and our goal is to demonstrate

$$\int_D |\Delta u|^2 \frac{dx}{m^2} = 0.$$

Let $\epsilon > 0$ and define the set

$$D_\epsilon = D \setminus \{x \in D : d(x, \partial D) < \epsilon\},$$

then there exists $C_\epsilon > 0$ so that $0 < \frac{1}{m^2} < C_\epsilon$ on D_ϵ. It then follows

$$0 = \int_D |\Delta u|^2\,dx \geq \int_{D_\epsilon} |\Delta u|^2\,dx \tag{4.35}$$
$$= \frac{C_\epsilon}{C_\epsilon} \int_{D_\epsilon} |\Delta u|^2\,dx \geq \frac{1}{C_\epsilon} \int_{D_\epsilon} |\Delta u|^2 \frac{dx}{m^2},$$

so

$$\int_{D_\epsilon} |\Delta u|^2 \frac{dx}{m^2} = 0$$

for every $\epsilon > 0$. It follows, dominated convergence, that $\|u\|_{H^2_{m,2}} = 0$. We may conclude that the operator A_τ is positive definite which finishes the proof.

■

We remark that in [18] it was shown that the equivalent form, $\mathcal{A}_\tau(\cdot,\cdot)$ on $H^2_0(D)$, was coercive. However, their assumptions on $m(x)$ were that it transitioned to zero at ∂D discontinuously. Here we have made different assumptions on $m(x)$ and thus have only been able to show the weaker conclusion of positive definiteness on $\mathcal{A}_\tau(\cdot,\cdot)$.

The next result uses methods outlined in [19] with only slight modifications due to the fact that we must work in the different space $H^2_{m,2}(D)$. The method relies on an appeal to the more general theory of generalized eigenvalue problems [68]. The following theorem was shown in [19] and we restate it here for reference.

Theorem 4.4.0.2 *[19, 68] Let $\tau \mapsto C_\tau$ be a continuous mapping from $(0, \infty)$ to the set of self-adjoint, positive definite, bounded linear operators on a separable Hilbert space U. Let K be a self-adjoint and non-negative compact bounded linear operator on U. If there exists two positive constants τ_0, $\tau_1 > 0$ satisfying the conditions*

(i) $C_{\tau_0} - \tau_0 K$ is positive on U

(ii) $C_{\tau_1} - \tau_1 K$ is non-positive on a k-dimensional subspace, W_k, of U,

then each of the equations $\lambda_j(\tau) = \tau$, $j = 1, 2, 3, \ldots, k$ has a solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue of C_τ with respect to K defined by

$$\ker(C_\tau - \lambda_j(\tau)K) \neq \{0\}.$$

Since we have shown that $F_\tau = A_\tau - \tau B$ on $H^2_{m,2}(D)$ we see that $\tau > 0$ defines a transmission eigenvalue if it is a solution to $\lambda_j(\tau) = \tau$ for any eigenvalue, λ_j, of A_τ with respect B. We have already shown that these operators satisfy the hypothesis of the theorem so it remains to show that conditions (i) and (ii) are met.

Theorem 4.4.0.3 *Assume that $n(x) \in C^2(D)$ is such that $n(x) \geq \sigma > 0$ in D and $m(x) = n(x) - 1$ satisfies either $m(x) > 0$ or $m(x) < 0$ in $\text{int}(D)$. Suppose also that $m(x)$ has controlled decay at the boundary of degree 2. Then there exists an infinite discrete set of transmission eigenvalues in D with respect to $n(x)$.*

Proof. The existence of τ_0 satisfying condition (i) of theorem 4.4.0.2 is equivalent to showing

$$\mathcal{F}_{\tau_0}(u,u) = (A_{\tau_0}u - \tau_0 Bu, u)_{H^2_{m,2}} > 0 \text{ for } u \in H^2_{m,2}(D) \tag{4.36}$$

for some τ_0. This was actually demonstrated in [28, 15], for completeness we give the argument here. For u, $\phi \in H^2_{m,2}(D)$

$$\begin{aligned}\mathcal{F}_\tau(u,\phi) &= \int_D m^{-1}(\Delta+\tau)u(\Delta+\tau n)\phi \\ &= \int_D m^{-1}(\Delta+\tau n)u(\Delta+\tau n)\phi - \tau\int_D (u\Delta\phi + \tau n u\phi) \\ &= \int_D m^{-1}(\Delta+\tau n)u(\Delta+\tau n)\phi - \tau\int_D (\tau n u\phi - \nabla u\cdot\nabla\phi)\end{aligned}$$

where we have used $(\Delta+\tau)u = (\Delta+\tau n)u - \tau m u$. Setting $\phi = u$ gives

$$\mathcal{F}_\tau(u,u) = \int_D m^{-1}|(\Delta+\tau n)u|^2 + \tau\int_D (|\nabla u|^2 - \tau n|u|^2).$$

The first term is always non-negative and

$$\int_D (|\nabla u|^2 - \tau n|u|^2) \geq \|u\|^2(\lambda_0(D) - \tau\sup_D n) \geq 0$$

as long as $\tau \leq \frac{\lambda_0(D)}{\sup_D n}$ where $\lambda_0(D)$ is the first Dirichlet eigenvalue on D of the laplacian. It follows that, for $\tau_0 < \frac{\lambda_0(D)}{\sup_D n}$,

$$\mathcal{F}_{\tau_0}(u,u) > 0 \text{ for } u \in H^2_{m,2}(D).$$

To show the existence of a τ_1 satisfying condition (ii) we first note that it suffices to show

$$\begin{aligned}\mathcal{F}_{\tau_1}(u,u) &= (A_{\tau_1}u - \tau_1 Bu, u)_{H^2_{m,2}} \\ &= \int_D \frac{1}{n-1}|(\Delta+\tau_1)u|^2 + \tau_1^2\int_D |u|^2 - \tau_1\int_D |\nabla u|^2 \le 0\end{aligned} \tag{4.37}$$

for some τ_1 on a finite dimensional subspace $V \subset H^2_{m,2}(D)$.

Now, if k, with $k^2 = \tau$, is a transmission eigenvalue relative to $n(x)$ on D then $\mathcal{F}_\tau(u,u) = 0$ for some $u \in H^2_{m,2}(D)$ or for $u \in H^2_0(D)$ if $n = n(x)$ is constant. We fix some $\sigma > 0$ and define the set

$$D_\sigma = D \setminus \{x \in D : d(x, \partial D) < \sigma\},$$

on which we set $n_\sigma = \inf_{D_\sigma} n(x) > 0$. Let k_{R,n_σ} denote the first transmission eigenvalue relative to the constant refractive index n_σ on a ball of radius $R > 0$. These eigenvalues satisfy the relation, for $\epsilon > 0$, $k_{\epsilon,n_\sigma} = \frac{1}{\epsilon}k_{1,n_\sigma}$. We now take $\epsilon > 0$ so that there exist $\theta = \theta(\epsilon)$ disjoint balls of radius ϵ inside D_σ, referred to by B^1_ϵ, $B^2_\epsilon, \ldots,$ B^θ_ϵ. In each of these balls k_{ϵ,n_σ} is the first transmission eigenvalue relative to (n_σ, B^j_ϵ), we call $u^j = v^j - w^j$ the corresponding eigenvector in each of these. We may extend each of the u^j by zero to all of D, write this extension as $\tilde{u}^j$ and notice that by the definition of a solution to the ITP we must have $\tilde{u}^j \in H^2_0(D)$. Moreover, since each $\tilde{u}^j = 0$ on $D \setminus D_\sigma$, $\tilde{u}^j \in H^2_{m,2}(D)$. It follows that the $\tilde{u}^j$ satisfy $\mathcal{F}_\tau(\tilde{u}^j, \tilde{u}^j) = 0$, for $\tau = k^2_{\epsilon,n_\sigma}$. Therefore, each element of the set $\left\{\tilde{u}^j\right\}_{j=1}^{\theta(\epsilon)} \subset H^2_{m,2}(D)$ satisfies

$$\int_{D_\sigma} \frac{1}{n_\sigma - 1}|(\Delta+\tau_1)\tilde{u}^j|^2 + \tau_1^2\int_{D_\sigma} |\tilde{u}^j|^2 - \tau_1\int_{D_\sigma} |\nabla\tilde{u}^j|^2 = 0 \tag{4.38}$$

for $\tau_1 = k^2_{\epsilon,n_\sigma}$.

Now define $V_{\theta(\epsilon)} = span\left\{\tilde{u}^j\right\}_{j=1}^{\theta(\epsilon)}$, a $\theta(\epsilon)$-dimensional subspace of $H^2_{m,2}(D)$ since the B^j_ϵ were disjoint. Taking an element $\tilde{u} \in V_{\theta(\epsilon)}$ we see

$$\begin{aligned}
&(A_{\tau_1}\tilde{u} - \tau_1 B\tilde{u}, \tilde{u})_{H^2_{m,2}} = \\
&\quad \int_D \frac{1}{n(x)-1}|(\Delta+\tau_1)\tilde{u}|^2 + \tau_1^2\int_D |\tilde{u}|^2 - \tau_1\int_D |\nabla\tilde{u}|^2 \\
&\quad = \int_{D_\sigma} \frac{1}{n(x)-1}|(\Delta+\tau_1)\tilde{u}|^2 + \tau_1^2\int_{D_\sigma} |\tilde{u}|^2 - \tau_1\int_{D_\sigma} |\nabla\tilde{u}|^2 \\
&\quad \leq \int_{D_\sigma} \frac{1}{n_\sigma-1}|(\Delta+\tau_1)\tilde{u}|^2 + \tau_1^2\int_{D_\sigma} |\tilde{u}|^2 - \tau_1\int_{D_\sigma} |\nabla\tilde{u}|^2 = 0.
\end{aligned} \tag{4.39}$$

This shows that $A_{\tau_1} - \tau_1 B$, with $\tau_1 = k^2_{\epsilon,n_\sigma}$, is non-positive on a $\theta(\epsilon)$-dimensional subspace of $H^2_{m,2}(D)$. So by theorem 4.4.0.2 there exists $\theta(\epsilon)$ transmission eigenvalues in the interval $[\tau_0, \tau_1]$. However, both $\theta(\epsilon)$ and $\tau_1 = k^2_{\epsilon,n_\sigma}$ go to ∞ as ϵ approaches zero. Therefore, we have shown the existence of an infinite discrete set of transmission eigenvalues.

■

4.5 Conclusion

We have demonstrated that the ITP has an equivalent weak formulation on the weighted Sobolev space $H^2_{m,2}(D)$ when the index of refraction is equal to the surrounding homogeneous medium on the boundary of an object. The weak formulation was shown to hold by use of the Hardy inequality which was shown to control imbedding properties of weighted Sobolev spaces. Discreteness of the transmission spectrum was then proven using these imbedding properties and an operator form of the weak formulation. It was demonstrated that one can use the variational formulation on $H^2_{m,2}(D)$ to show the existence of an infinite discrete set of transmission eigenvalues.

A case that is not dealt with in the above is the case when there are *cavities* within the interior of the body D. A place where $m(x) = 0$ with $m(x)$ transitioning C^2 to zero on the boundary of these cavities. This case was treated if the transition is discontinuous in [15] where a variational form was found and, in this case, it has been shown that there exists an infinite set of transmission eigenvalues [18]. However, if $m(x)$ is allowed to smoothly transition to zero at these cavities the variational treatment established in this paper and [18, 15] breaks down. This is due to the fact that the Hardy inequality no longer yields information at the interior boundaries of cavities since the function $u = w - v$ itself does not necessarily have to vanish there. In the future it would be favorable to extend to this case.

Also, since we have been able to extend the allowable smoothness of a function $n(x)$ which is equal to unity on ∂D using weighted Sobolev spaces one could hope to extend to the case of C^∞ refractive indices in this class. However, this extension is not immediate.

The specific case of interior transmission treated in this section is necessary to extend uniqueness results for TAT since it is usually assumed in thermoacoustics that the acoustic profile of the imaging domain transitions smoothly to the acoustic speed of the surrounding medium.

5 UNIQUENESS OF ACOUSTIC PROFILE IN TAT: THE ONE DIMENSIONAL CASE

5.1 Introduction

In chapter 3 we found sufficient conditions on acoustic profiles, $c(x)$ and $b(x)$, for the intersection of the ranges of thermoacoustic operators, $\mathcal{L}_c$ and $\mathcal{L}_b$, to be zero. These conditions were derived from studying the interior transmission problem of scattering theory and its spectrum of positive real transmission eigenvalues. The restrictions on the acoustic speeds were taken to ensure that this spectrum was discrete or at least bounded away from zero.

The condition on $c(x)$ and $b(x)$ was that the difference, $n_c(x)-n_b(x) = m(x)$, did not change sign in the domain D. In the study of the transmission spectrum this assumption plays an important role since the operators involved in the proofs are constructed using $m^{-1}(x)$ [25, 27, 28, 69] so the operators involved become unbounded when $m(x)$ changes signs.

There is a body of work that does not make the above assumption but instead restricts itself to the study of the interior transmission spectrum in the case of a spherically symmetric acoustic speed [58, 59]. In this case one may first reduce the ITP to a one dimensional equivalent problem using the symmetry of the acoustic profile. The transmission problem then takes the form of a Sturm-Liouville eigenvalue problem after a change of variables. Once this is done the question of whether or not a given $k \in \mathbb{R}_+$ is a transmission eigenvalue amounts to studying existence of solutions to a system of Sturm-Liouville equations.

Given the effectiveness of this approach at circumventing the need for the difference $m(x)$ to not change signs it seems interesting to apply it to the case of one dimensional thermoacoustic tomography. It will be shown that the application of the methods used in [58, 59] yields a new condition on acoustic profiles $c(x)$ and $b(x)$ for the intersection of the ranges of the operators $\mathcal{L}_{c(x)}$ and $\mathcal{L}_{b(x)}$ to have zero intersection. This condition also seems only sufficient and we have been unable, so far, to find a necessary condition for separating the ranges of two thermoacoustic operators even in the case of radial symmetry.

This chapter is organized as follows. In section 5.2, we show how the interior transmission problem in the Helmholtz equation form can be reduced to a Sturm-Liouville type problem. Next, section 5.3, we show how the study of transmission eigenvalues in this case is equivalent to the study of a special boundary condition on solutions to the Sturm-Liouville problem. We then examine a particular asymptotic expansion of the forward solution of the Sturm-Liouville problem following methods used in [65] in section 5.4. This is then used, section 5.5, along with the boundary condition to derive conditions on the acoustic speeds that will the associated transmission spectrum is discrete.

5.2 Reduction to a Sturm-Liouville Problem

Let $c(x)$ be an acoustic profile relative to the unit interval, $[0,1]$. Consider a solution $u(x,t)$ to

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x)\partial_x^2 u(x,t) = 0 \quad &\text{in} \quad \mathbb{R}\times\mathbb{R}_+ \\ u(x,0) = f(x),\ \partial_t u(x,0) = 0 \quad &\text{in} \quad \mathbb{R}. \end{aligned} \tag{5.1}$$

For each $x \in [0,1]$ the real part of the temporal Fourier transform, $U(x,k) = \mathcal{R}e(\hat{u})(x,k)$, is analytic in a strip about $\mathbb{R}_+$ as long as there is exponential decay in time of the ultrasound field. By taking the real part of the temporal Fourier transform of the solution to (5.1), we see $U(x,k)$ satisfies

$$\partial_x^2 U(x,k) + k^2 n(x) U(x,k) = 0 \text{ for } x \in [0,1] \tag{5.2}$$

for each $k \in \mathbb{R}_+$ with the refractive index $n(x) = \frac{1}{c^2(x)}$. Under a change of variables equation (5.2) is an eigenvalue problem for a Sturm-Liouville operator.

Lemma 5.2.0.7 *Let $U(x,k)$ be a smooth function satisfying (5.2). Set*

$$\eta = \int_0^x \sqrt{n(s)}\,ds \ \textit{and}\ z(\eta,k) = (n(x))^{1/4} U(x,k).$$

Then $z(\eta,k)$ satisfies

$$\ddot{z}(\eta,k) + (k^2 - p(\eta))z(\eta,k) = 0 \textit{ for } \eta \in [0,C] \tag{5.3}$$

where

$$p(\eta) = \frac{1}{4}\frac{\ddot{n}(x)}{n(x)^2} - \frac{5}{16}\frac{\dot{n}(x)^2}{n(x)^3} \tag{5.4}$$

and $C = \int_0^1 \sqrt{n(s)}\,ds$.

For ease of notation we will usually drop the dependence of $z(\eta,k)$ and $U(x,k)$ on k and write instead $z(\eta)$ and $U(x)$.

Proof. For the given change of variables we compute

$$\dot{z}(\eta) = \frac{1}{4}\frac{\dot{n}(x)}{n(x)^{5/4}}U(x) + n(x)^{-1/4}U'(x),$$

and

$$\begin{aligned}\ddot{z}(\eta) &= \left(\frac{1}{4}\frac{\ddot{n}(x)}{n(x)^{7/4}} - \frac{5}{16}\frac{\dot{n}(x)^2}{n(x)^{11/4}}\right)U(x) + n(x)^{-3/4}U''(x)\\ &= \left(\frac{1}{4}\frac{\ddot{n}(x)}{n(x)^2} - \frac{5}{16}\frac{\dot{n}(x)^2}{n(x)^3}\right)z(\eta) - k^2 n(x)^{1/4}U(x)\\ &= \left(\frac{1}{4}\frac{\ddot{n}(x)}{n(x)^2} - \frac{5}{16}\frac{\dot{n}(x)^2}{n(x)^3}\right)z(\eta) - k^2 z(\eta).\end{aligned}$$

Letting

$$p(\eta) = \frac{1}{4}\frac{\ddot{n}(x)}{n(x)^2} - \frac{5}{16}\frac{\dot{n}(x)^2}{n(x)^3},$$

the new function $z(\eta)$ satisfies

$$\ddot{z}(\eta) + (k^2 - p(\eta))z(\eta) = 0 \text{ for } \eta \in [0, C]$$

with $C = \eta(1) = \int_0^1 \sqrt{n(s)}\, ds$.

■

Theorem 5.2.0.4 *For two acoustic speeds* $c(x)$ *and* $b(x)$ *relative to the domain* $[0,1]$ *the transmission spectrum is a subset of the set of all* $k \in \mathbb{R}_+$ *such that there is a nontrivial pair* $(z_1, z_2) \in H^2[0,1] \times H^2[0,1]$ *satisfying*

$$\begin{aligned}&\ddot{z}_1(\eta) + (k^2 - p_1(\eta))z_1(\eta) = 0 \quad on \quad [0, C]\\ &\ddot{z}_2(\xi) + (k^2 - p_2(\xi))z_2(\xi) = 0 \quad on \quad [0, B]\\ &z_1(0) = z_2(0),\ z_1(C) = z_2(B)\\ &\dot{z}_1(0) = \dot{z}_2(0),\ \dot{z}_1(C) = \dot{z}_2(B).\end{aligned} \tag{5.5}$$

In the above theorem we have used refractive indexes $n_c(x) = 1/c^2(x)$, $n_b(x) = 1/b^2(x)$ and

$$p_1(\eta) = \frac{1}{4}\frac{\ddot{n}_c(x)}{n_c(x)^2} - \frac{5}{16}\frac{\dot{n}_c(x)^2}{n_c(x)^3} \tag{5.6}$$
$$p_2(\xi) = \frac{1}{4}\frac{\ddot{n}_b(x)}{n_b(x)^2} - \frac{5}{16}\frac{\dot{n}_b(x)^2}{n_b(x)^3}.$$

The end points C and B are given by

$$C = \int_0^1 \sqrt{n_c(s)}\, ds \tag{5.7}$$
$$B = \int_0^1 \sqrt{n_b(s)}\, ds.$$

Proof. Consider a value $k \in \mathbb{R}_+$ such that there exists a nontrivial pair $(u, v) \in H^2[0,1] \times H^2[0,1]$ satisfying the interior transmission problem

$$\begin{aligned} \partial_x^2 u + k^2 n_c(x) u = 0, &\qquad x \in [0,1] \\ \partial_x^2 v + k^2 n_b(x) v = 0, &\qquad x \in [0,1] \\ u(0) = v(0), &\qquad u(1) = v(1) \\ \partial_x u(0) = \partial_x v(0), &\qquad \partial_x u(1) = \partial_x v(1). \end{aligned} \tag{5.8}$$

By lemma 5.2.0.7 we may change variables in the differential equations using

$$\eta = \int_0^x \sqrt{n_c(s)}\, ds,\ \ z_1(\eta) = (n_c(x))^{1/4} u(x)$$

and

$$\xi = \int_0^x \sqrt{n_b(s)}\, ds,\ \ z_2(\xi) = (n_b(x))^{1/4} v(x).$$

Then the pair $(z_1, z_2) \in H^2[0,1] \times H^2[0,1]$ satisfies the differential equation (5.5). Since we have assumed

$$n_c(0) = n_c(1) = n_b(0) = n_b(1) = 1$$

we see

$$z_1(0) = u(0) = v(0) = z_2(0)$$

$$z_1(C) = u(1) = v(1) = z_2(B).$$

Recalling,

$$\dot{z}_1(\eta) = \frac{1}{4}\frac{\dot{n}_c(x)}{n_c(x)^{5/4}}u(x) + n_c(x)^{-1/4}\partial_x u(x)$$

and the fact that $\dot{n}_c(0) = \dot{n}_c(1) = 0$ one observes

$$\begin{aligned} \dot{z}_1(0) &= \frac{1}{4}\frac{\dot{n}_c(0)}{n_c(0)^{5/4}}u(0) + n_c(0)^{-1/4}\partial_x u(0) \\ &= \partial_x u(0) = \partial_x v(0) \\ &= \frac{1}{4}\frac{\dot{n}_b(0)}{n_b(0)^{5/4}}v(0) + n_b(0)^{-1/4}\partial_x v(0) \\ &= \dot{z}_2(0). \end{aligned} \tag{5.9}$$

Likewise, at the right endpoints,

$$\begin{aligned} \dot{z}_1(C) &= \frac{1}{4}\frac{\dot{n}_c(1)}{n_c(1)^{5/4}}u(1) + n_c(1)^{-1/4}\partial_x u(1) \\ &= \partial_x u(1) = \partial_x v(1) \\ &= \frac{1}{4}\frac{\dot{n}_b(1)}{n_b(1)^{5/4}}v(1) + n_b(1)^{-1/4}\partial_x v(1) \\ &= \dot{z}_2(B). \end{aligned} \tag{5.10}$$

We have demonstrated that if $k \in \mathbb{R}_+$ is a transmission eigenvalue then there exist z_1 and z_2 satisfying (5.5).

■

5.3 The Determinant Condition

We examine the effect of the two fundamental solutions to the Sturm-Liouville problem on the boundary conditions in the one dimensional ITP. This will let us derive a boundary condition that must be satisfied at a transmission eigenvalue. Let $(X, Y) \in H^2[0, C] \times H^2[0, B]$ satisfy (5.5) and define the fundamental solutions X_1, X_2 and Y_1, Y_2

to satisfy the initial value problems

$$\ddot{X}_i + (k^2 - p_X(\eta))X_i = 0, \quad i = 1, 2 \tag{5.11}$$
$$X_1(0) = \dot{X}_2(0) = 1$$
$$\dot{X}_1(0) = X_2(0) = 0$$

and

$$\ddot{Y}_i + (k^2 - p_Y(\xi))Y_i = 0, \quad i = 1, 2 \tag{5.12}$$
$$Y_1(0) = \dot{Y}_2(0) = 1$$
$$\dot{Y}_1(0) = Y_2(0) = 0.$$

Where the coefficients p_X and p_Y come from the two different acoustic profiles $c(x)$ and $b(x)$. The solutions X and Y can then be written as

$$X = a_1X_1 + a_2X_2,$$
$$Y = b_1Y_1 + b_2Y_2$$

for some constants a_1, a_2, b_1, b_2.

Theorem 5.3.0.5 *For C, B, p_X, and p_Y defined above the value $k \in \mathbb{R}_+$ is a transmission eigenvalue only if*

$$W(X_1(C), Y_2(B)) + W(Y_1(B), X_2(C)) = 2. \tag{5.13}$$

Where

$$W(X_1(C), Y_2(B)) = X_1(C)\dot{Y}_2(B) - Y_2(B)\dot{X}_1(C)$$
$$W(Y_1(B), X_2(C)) = Y_1(B)\dot{X}_2(C) - X_2(C)\dot{Y}_1(B).$$

Proof. Suppose $(X, Y) \in H^2[0, 1] \times H^2[0, 1]$ satisfy (5.5). Then the boundary conditions imply

$$X(0) = Y(0) \text{ and } \dot{X}(0) = \dot{Y}(0), \tag{5.14}$$
$$X(C) = Y(B) \text{ and } \dot{X}(C) = \dot{Y}(B). \tag{5.15}$$

If a_1, a_2, b_1, b_2 satisfy

$$X = a_1X_1 + a_2X_2,$$
$$Y = b_1Y_1 + b_2Y_2$$

then at the left endpoint (5.14) yields

$$X(0) = a_1X_1(0) + a_2X_2(0) = a_1 = b_1 = b_1Y_1(0) + b_2Y_2(0) = Y(0),$$
$$\dot{X}(0) = a_1\dot{X}_1(0) + a_2\dot{X}_2(0) = a_2 = b_2 = b_1\dot{Y}_1(0) + b_2\dot{Y}_2(0) = \dot{Y}(0).$$

The boundary conditions at the right endpoint then require

$$X(C) = a_1X_1(C) + a_2X_2(C) = a_1Y_1(B) + a_2Y_2(B) = Y(B),$$
$$\dot{X}(C) = a_1\dot{X}_1(C) + a_2\dot{X}_2(C) = a_1\dot{Y}_1(B) + a_2\dot{Y}_2(B) = \dot{Y}(B).$$

So in order for the value $k \in \mathbb{R}_+$ to be a transmission eigenvalue there must exist a nontrivial pair $(a_1, a_2) \in \mathbb{R}^2$ satisfying the matrix equation

$$\begin{bmatrix} X_1(C) - Y_1(B) & X_2(C) - Y_2(B) \\ \dot{X}_1(C) - \dot{Y}_1(B) & \dot{X}_2(C) - \dot{Y}_2(B) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For this to be possible the determinant of this matrix must be equal to zero,

$$\det \begin{pmatrix} X_1(C) - Y_1(B) & X_2(C) - Y_2(B) \\ \dot{X}_1(C) - \dot{Y}_1(B) & \dot{X}_2(C) - \dot{Y}_2(B) \end{pmatrix} = 0.$$

We can expand this determinant out and use the Wronskian identity for the fundamental solutions of the Sturm-Liouville problems [65],

$$W(X_1, X_2)(x) = 1 \text{ in } [0, C], W(Y_1, Y_2)(x) = 1 \text{ in } [0, B].$$

This yields, with some abuse of notation,

$$0 = 2 - (X_1(C)\dot{Y}_2(B) - Y_2(B)\dot{X}_1(C)) - (Y_1(B)\dot{X}_2(C) - X_2(C)\dot{Y}_1(B))$$
$$= 2 - W(X_1(C), Y_2(B)) - W(Y_1(B), X_2(C)).$$

So we have shown that if $k \in \mathbb{R}_+$ is a transmission eigenvalue then

$$W(X_1(C), Y_2(B)) + W(Y_1(B), X_2(C)) = 2.$$

■

Notice that the above condition is certainly satisfied for every $k \in \mathbb{R}_+$ if $c(x) = b(x)$ on $[0, 1]$ as we would expect. So this condition does give us a way to relate the transmission spectrum to the differences in the two acoustic speeds $c(x)$ and $b(x)$. Of course condition (5.13) is only helpful if we can find expressions for the fundamental solutions of the Sturm-Liouville problem which let us analyze their dependence on $k \in \mathbb{R}_+$.

5.4 Forward Solution

We have developed a determinant condition involving fundamental solutions to a Sturm-Liouville problem that is necessary for k to be a transmission eigenvalue. This was studied very extensively in [65] and for a more complete treatment of this development along with much more see this reference. The next theorem about the fundamental solutions to the perturbed eigenvalue problem was proven in [65].

Theorem 5.4.0.6 *The fundamental solutions z_1 and z_2 satisfying (5.3) along with the initial conditions*

$$\begin{aligned} z_1(0) &= \dot{z}_2(0) = 1 \\ \dot{z}_1(0) &= z_2(0) = 0 \end{aligned} \tag{5.16}$$

can be written in the form

$$\begin{aligned} z_1(x,k,p) &= C_0(x,k,p) + \sum_{n\geq 1} C_n(x,k,p) \\ z_2(x,k,p) &= S_0(x,k,p) + \sum_{n\geq 1} S_n(x,k,p), \end{aligned} \tag{5.17}$$

where the n^{th} term in the sum is a multilinear form in p. The terms in each sum satisfy *the recurrence relations*

$$\begin{aligned} C_0(x,k,p) &= \cos(kx) = c_k(x) \\ C_n(x,k,p) &= \int_0^x \frac{\sin(k(x-t))}{k} p(t) C_{n-1}(t,k,p)\,dt, \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} S_0(x,k,p) &= \frac{\sin(kx)}{k} = s_k(x) \\ S_n(x,k,p) &= \int_0^x \frac{\sin(k(x-t))}{k} p(t) S_{n-1}(t,k,p)\,dt. \end{aligned} \tag{5.19}$$

Moreover, the solutions z_1 *and* z_2 *satisfy the integral equations*

$$z_1(x,k,p) = \cos(kx) + \int_0^x \frac{\sin(k(x-t))}{k} p(t) z_1(t,k,p)\,dt \tag{5.20}$$

$$z_2(x,k,p) = \frac{\sin(kx)}{k} + \int_0^x \frac{\sin(k(x-t))}{k} p(t) z_2(t,k,p)\,dt. \tag{5.21}$$

Proof. We show, only formally, that the first expression in (5.17) holds with (5.18) satisfied. For a proof that the sum converges see [65]. The justification of the second equation in (5.17) is analogous. Assume

$$z_1(x,k,p) = C_0(x,k,p) + \sum_{n\geq 1} C_n(x,k,p)$$

satisfies

$$\ddot{z}_1(x,k,p) + (k^2 - p(x))z_1(x,k,p) = 0 \text{ on } [0,1]$$

$$z_1(0) = 1,\ \dot{z}_1(0) = 0.$$

Substituting the sum into the differential equation we see

$$\begin{aligned} &\ddot{z}_1(x,k,p) + (k^2 - p(x))z_1(x,k,p) = \\ &\quad \left[\ddot{C}_0(x,k,p) + k^2 C_0(x,k,p)\right] \\ &\quad + \sum_{n\geq 1}\left[\ddot{C}_n(x,k,p) + k^2 C_n(x,k,p) - p(x)C_{n-1}(x,k,p)\right] = 0. \end{aligned}$$

We require each $C_n(x,k,p)$ to be of order n in $p(x)$ therefore each term in the summation must vanish so we have

$$\ddot{C}_0(x,k,p) + k^2 C_0(x,k,p) = 0$$
$$\ddot{C}_n(x,k,p) + k^2 C_n(x,k,p) = p(x)C_{n-1}(x,k,p).$$

The initial conditions (5.16) are then satisfied if we take $C_0(x,k,p) = c_k(x)$. This implies that $C_n(0,k,p) = 0$ and $\dot{C}_n(0,k,p) = 0$ for all $n \geq 1$. The second equation can then be solved using variation of parameters. Since the two fundamental solutions to $(\partial_x^2 + k^2)z = 0$ are $c_k(x)$ and $s_k(x)$, for $n \geq 1$ we must have

$$\begin{aligned} C_n(x,k,p) &= \int_0^x \frac{c_k(x)s_k(t) - s_k(x)c_k(t)}{W(c_k,s_k)(t)} p(t)C_{n-1}(t,k,p)\,dt \\ &= \int_0^x \frac{\sin(k(x-t))}{k} p(t)C_{n-1}(t,k,p)\,dt. \end{aligned}$$

We have used $W(c_k,s_k)(t) = 1$. This justifies the expression for $z_1(x,k,p)$. To see that $z_1(x,k,p)$ satisfies the integral equation, assuming the sum converges uniformly, one computes

$$\begin{aligned} z_1(x,k,p) - c_k(x) &= \sum_{n\geq 1} C_n(x,k,p) \\ &= \sum_{n\geq 1} \int_0^x \frac{\sin(k(x-t))}{k} p(t)C_{n-1}(t,k,p)\,dt \\ &= \int_0^x \frac{\sin(k(x-t))}{k} p(t) \sum_{n\geq 1} C_{n-1}(t,k,p)\,dt \\ &= \int_0^x \frac{\sin(k(x-t))}{k} p(t) z_1(t,k,p)\,dt. \end{aligned}$$

■

Let us examine the above expression for $C_n(x,k,p)$ more closely. Notice,

$$C_1(x,k,p) = \int_0^x \frac{\sin(k(x-t))}{k} p(t)\cos(kt)\,dt. \tag{5.22}$$

So,

$$C_2(x,k,p) = \int_0^x \frac{\sin(k(x-t_2))}{k} p(t_2) C_1(t_2,k,p)\, dt_2$$
$$= \int_0^x \frac{\sin(k(x-t_2))}{k} p(t_2) \left(\int_0^{t_2} \frac{\sin(k(t_2-t_1))}{k} p(t_1)\cos(kt_1)\, dt_1 \right) dt_2$$
$$= \int_0^{x=t_3} \int_0^{t_2} \cos(kt_1) \left(\prod_{i=1}^{2} \frac{\sin(k(t_{i+1}-t_i))}{k} p(t_i) \right) dt_1\, dt_2.$$

This gives way to the general expressions

$$C_n(x,k,p) = \int_0^{x=t_{n+1}} \cdots \int_0^{t_2} \cos(kt_1) \left(\prod_{i=1}^{n} \frac{\sin(k(t_{i+1}-t_i))}{k} p(t_i) \right) dt_1 \ldots dt_n \tag{5.23}$$

and

$$S_n(x,k,p) = \int_0^{x=t_{n+1}} \cdots \int_0^{t_2} \frac{\sin(kt_1)}{k} \left(\prod_{i=1}^{n} \frac{\sin(k(t_{i+1}-t_i))}{k} p(t_i) \right) dt_1 \ldots dt_n. \tag{5.24}$$

If the coefficient function, $p(x)$, is smooth enough the integrals involved in the definitions of $C_n(x,k,p)$ and $S_n(x,k,p)$ can be expanded in terms of inverse powers of k using trigonometric identities and integration by parts. Here we show the beginning of this process for $C_1(x,k,p)$. Using $2\sin(a)\cos(b) = \sin(a+b) + \sin(a-b)$,

$$\begin{aligned} C_1(x,k,p) &= \frac{1}{k}\int_0^x \sin(k(x-t))\cos(kt)p(t)\,dt \\ &= \frac{\sin(kx)}{2k}\int_0^x p(t)\,dt + \frac{1}{2k}\int_0^x \sin(k(x-2t))p(t)\,dt. \end{aligned} \tag{5.25}$$

If we perform integration by parts on the second term we see

$$\begin{aligned} C_1(x,k,p) &= \frac{\sin(kx)}{2k}\int_0^x p(t)\,dt + \frac{\cos(kx)}{4k^2}(p(x)-p(0)) \\ &\quad -\frac{1}{4k^2}\int_0^x \cos(k(x-2t))p'(t)\,dt. \end{aligned} \tag{5.26}$$

If we assume that the coefficient function is C^∞ then we may integrate by parts indefinitely each time obtaining a term that depends on a higher inverse power of the eigenvalue k. We may do this for each term in the series expressions for z_1 and z_2 in (5.17). After a lengthy calculation this will yield asymptotic expansions for the two fundamental solutions. These asymptotic series will have the form

$$\begin{aligned} z_1(x,k,p) &\sim \sum_{n=0}^{\infty} a_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{\infty} b_n(x)\sin(kx)k^{-n} \\ z_2(x,k,p) &\sim \sum_{n=0}^{\infty} c_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{\infty} d_n(x)\sin(kx)k^{-n}. \end{aligned}$$

The fact that the solutions z_1 and z_2 may be expressed as an asymptotic series in the above form was discussed in [65] and [33]. We justify a simpler method to compute the coefficients. Our approach is similar to the method in [33] and is slightly more efficient than the method outlined in [65]. First let us review a little about asymptotic series, a good reference is found in [86, 33].

A sequence of functions, $\{\phi_n(k)\}_{n=0}^{\infty}$, is *asymptotic as* $k \to \infty$ if for all $n \geq 0$ we have

$$\lim_{k\to\infty} \frac{\phi_{n+1}(k)}{\phi_n(k)} = 0.$$

If the functions depend on a parameter in some interval, $x \in I \subset R$, then the sequence, $\{\phi_n(k;x)\}_{n=0}^{\infty}$, is *asymptotic as* $k \to \infty$ *uniformly in* $x \in I$ if for all $n \geq 0$ we have

$$\lim_{k\to\infty} \frac{\phi_{n+1}(k;x)}{\phi_n(k;x)} = 0$$

uniformly in $x \in I$. A function $f(x,k)$ will be said to have an *asymptotic expansion as* $k \to \infty$ in terms of the asymptotic sequence $\{\phi_n(k;x)\}_{n=0}^{\infty}$ for each $x \in I$ if for each $x \in I$ there exists coefficients $\{a_n(x)\}_{n=0}^{\infty}$ such that for each $N \geq 0$

$$\lim_{k\to\infty} \phi_N(k;x)^{-1} \left| f(x,k) - \sum_{n=0}^{N} a_n(x)\phi_n(k;x) \right| = 0.$$

This can otherwise be expressed as

$$f(x,k) = \sum_{n=0}^{N} a_n(x)\phi_n(k;x) + \mathcal{O}(\phi_{N+1}(k;x))$$

for $x \in I$ as $k \to \infty$. We will then use the notation

$$f(x,k) \sim \sum_{n=0}^{\infty} a_n(x)\phi_n(k;x).$$

It is shown in [33, 86] that if $f(x,k)$ has an asymptotic expansion then it is unique.

If $P(k,x,\partial_x)$ is a differential operator on the interval $0 \leq x \leq C$ then we will say that a solution, $f(x,k)$, of

$$P(k,x,\partial_x)f(x,k) = 0 \text{ on } [0,C]$$

has an asymptotic expansion

$$f(x,k) \sim \sum_{n=0}^{\infty} a_n(x)\phi_n(k;x)$$

if for all $N \geq 0$ and $x \in [0,C]$

$$\sum_{n=0}^{N} P(k,x,\partial_x)[a_n(x)\phi_n(k;x)] = \mathcal{O}(\phi_{N+1}(k;x))$$

as $k \to \infty$. Any initial or boundary conditions must also be satisfied asymptotically by the series. We are now able to justify the following theorem.

Theorem 5.4.0.7 *The two fundamental solutions of the equation*

$$\ddot{z} + (k^2 - p(x))z = 0 \text{ on } [0,1],$$

z_1 and z_2, may be expressed as an asymptotic series in k having the following form,

$$z_1(x,k,p) \sim \sum_{n=0}^{\infty} a_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{\infty} b_n(x)\sin(kx)k^{-n} \quad (5.27)$$

$$z_2(x,k,p) \sim \sum_{n=0}^{\infty} c_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{\infty} d_n(x)\sin(kx)k^{-n}. \quad (5.28)$$

The coefficients a_n, b_n, c_n, and d_n satisfy the recursive system of ordinary differential equations

$$\dot{a}_0(x) = 0,\ \dot{a}_{n+1} = \frac{1}{2}(\ddot{b}_n - p(x)b_n) \text{ for } n \geq 0 \quad (5.29)$$

$$\dot{b}_0(x) = 0,\ \dot{b}_{n+1} = \frac{1}{2}(p(x)a_n - \ddot{a}_n) \text{ for } n \geq 0$$

$$a_0(0) = 1,\ a_n(0) = 0 \text{ for } n \geq 1$$

$$b_0(0) = 0,\ b_{n+1}(0) = -\dot{a}_n(0) \text{ for } n \geq 0$$

and

$$\begin{aligned}\dot{c}_0(x) = 0,\ \dot{c}_{n+1} &= \frac{1}{2}(\ddot{d}_n - p(x)d_n)\ \textit{for}\ n \geq 0 \\ \dot{d}_0(x) = 0,\ \dot{d}_{n+1} &= \frac{1}{2}(p(x)c_n - \ddot{c}_n)\ \textit{for}\ n \geq 0 \\ c_0(0) = 0,\ c_n(0) &= 0\ \textit{for}\ n \geq 1 \\ d_0(0) = 0,\ d_1(0) = 1,\ d_{n+1}(0) &= -\dot{c}_n(0)\ \textit{for}\ n \geq 1.\end{aligned} \tag{5.30}$$

Proof. We show the recursive system of differential equations for the coefficients in the expansion for z_1 hold. The proof that the system for the coefficients of the expansion for z_2 is similar. By [33, 65] there exist unique coefficients $a_n(x)$ and $b_n(x)$ such that for all $N \geq 0$ and $x \in [0,1]$

$$z_1(x,k,p) = \sum_{n=0}^{N} a_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{N} b_n(x)\sin(kx)k^{-n} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right).$$

The initial condition $z_1(0,k,p) = 1$ yields

$$z_1(0,k,p) = 1 \sim \sum_{n=0}^{\infty} a_n(0)k^{-n}.$$

So we must require

$$a_0(0) = 1,\ \text{and}\ a_n(0) = 0,\ \text{for}\ n \geq 1.$$

Differentiating $z_1(x,k,p)$ and collecting like powers of k yields, for all $N \geq 0$ and $x \in [0,1]$,

$$\begin{aligned}\partial_x z_1(x,k,p) &= \sum_{n=0}^{N} \dot{a}_n(x)\cos(kx)k^{-n} + \sum_{n=0}^{N} \dot{b}_n(x)\sin(kx)k^{-n} \\ &\quad - \sum_{n=0}^{N} a_n(x)\sin(kx)k^{-n+1} + \sum_{n=0}^{N} b_n(x)\cos(kx)k^{-n+1} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right) \\ &= kb_0(x)\cos(kx) - ka_0(x)\sin(kx) \\ &\quad + \sum_{n=0}^{N} [\dot{a}_n(x) + b_{n+1}(x)]\cos(kx)k^{-n} \\ &\quad + \sum_{n=0}^{N} \left[\dot{b}_n(x) - a_{n+1}(x)\right]\sin(kx)k^{-n} + \mathcal{O}\left(\frac{1}{k^{N+1}}\right).\end{aligned}$$

The initial condition $\dot{z}_1(0,k,p)=0$ now yields

$$\dot{z}_1(0,k,p)=0\sim kb_0(0)+\sum_{n=0}^{\infty}\left[\dot{a}_n(0)+b_{n+1}(0)\right]k^{-n}.$$

So we require

$$b_0(0)=0,\ \text{and}\ b_{n+1}(0)=-\dot{a}_n(0)\ \text{for}\ n\geq 0.$$

We may differentiate $z_1(x,k,p)$ a second time,

$$\begin{aligned}
\partial_x^2 z_1(x,k,p) &= \\
& k\dot{b}_0(x)\cos(kx)-k\dot{a}_0(x)\sin(kx)-k^2b_0(x)\sin(kx)-k^2a_0(x)\cos(kx)\\
&+\sum_{n=0}^{N}\left[\ddot{a}_n(x)+\dot{b}_{n+1}(x)\right]\cos(kx)k^{-n}+\sum_{n=0}^{N}\left[\ddot{b}_n(x)-\dot{a}_{n+1}(x)\right]\sin(kx)k^{-n}\\
&-\sum_{n=0}^{N}\left[\dot{a}_n(x)+b_{n+1}(x)\right]\sin(kx)k^{-n+1}+\sum_{n=0}^{N}\left[\dot{b}_n(x)-a_{n+1}(x)\right]\cos(kx)k^{-n+1}\\
&+\mathcal{O}\left(\frac{1}{k^{N+1}}\right)\\
&= -k^2b_0(x)\sin(kx)-k^2a_0(x)\cos(kx)-k\left(2\dot{a}_0(x)+b_1(x)\right)\sin(kx)\\
&+k\left(2\dot{b}_0(x)-a_1(x)\right)\cos(kx)+\sum_{n=0}^{N}\left[\ddot{a}_n(x)+2\dot{b}_{n+1}(x)-a_{n+2}(x)\right]\cos(kx)k^{-n}\\
&+\sum_{n=0}^{N}\left[\ddot{b}_n(x)-2\dot{a}_{n+1}(x)-b_{n+2}(x)\right]\sin(kx)k^{-n}\\
&+\mathcal{O}\left(\frac{1}{k^{N+1}}\right).
\end{aligned}$$

Now if we plug the expressions for the asymptotic expansion of $z_1(x,k)$ and $\partial_x^2 z_1(x,k)$ into the differential equation for $z_1(x,k)$ we see

$$\begin{aligned}
0 &= \partial_x^2 z_1(x,k)+(k^2-p(x))z_1(x,k)\\
&\sim k2\dot{b}_0(x)\cos(kx)-k2\dot{a}_0(x)\sin(kx)\\
&\quad+\sum_{n=0}^{\infty}\left[\ddot{a}_n(x)+2\dot{b}_{n+1}(x)-p(x)a_n(x)\right]\cos(kx)k^{-n}\\
&\quad+\sum_{n=0}^{\infty}\left[\ddot{b}_n(x)-2\dot{a}_{n+1}(x)-p(x)b_n(x)\right]\sin(kx)k^{-n}.
\end{aligned}$$

This then forces

$$\dot{a}_0(x) = 0, \;\; \dot{b}_0(x) = 0,$$

and

$$\dot{a}_{n+1}(x) = \frac{1}{2}\left[\ddot{b}_n(x) - p(x)b_n(x)\right] \text{ for } n \geq 0$$
$$\dot{b}_{n+1}(x) = \frac{1}{2}\left[p(x)a_n(x) - \ddot{a}_n(x)\right] \text{ for } n \geq 0.$$

This concludes our result.

■

The above theorem gives an efficient way to compute the desired asymptotic expansions of the fundamental solutions of the Sturm-Liouville problem. We compute the first few terms of these asymptotic expansions here. To begin we will compute the expansion for the fundamental solution $z_1(x, k, p)$. Here and throughout the rest of this document we use the notation

$$Q(x) = \int_0^x p(t)\, dt$$

and

$$T(x) = \int_0^x p^2(t)\, dt.$$

The computations are carried out under the assumption that $p(x)$ vanishes to infinite order at the endpoints $x = 0$ and $x = 1$.

For $a_0(x)$ we have $\dot{a}_0(x) = 0$ and $a_0(0) = 1$ which implies $a_0(x) = 1$. Similarly $\dot{b}_0(x) = 0$ and $b_0(0) = 0$ implies $b_0(x) = 0$. For $a_1(x)$,

$$\dot{a}_1(x) = \frac{1}{2}(\ddot{b}_0 - p(x)b_0), \;\; a_1(0) = 0$$

so $a_1(x) = 0$. The next term satisfies

$$\dot{b}_1(x) = \frac{1}{2}(p(x)a_0 - \ddot{a}_0), \;\; b_1(0) = -\dot{a}_0(0) = 0.$$

Thus, $\dot{b}_1(x) = \frac{1}{2}p(x)$ and we see

$$b_1(x) = \frac{1}{2}Q(x).$$

To compute $a_2(x)$, solve

$$\dot{a}_2(x) = \frac{1}{2}(\ddot{b}_1 - p(x)b_1),\ \ a_2(0) = 0.$$

Since

$$\ddot{b}_1(x) = \frac{1}{2}\dot{p}(x)$$

we see

$$\dot{a}_2(x) = \frac{1}{4}\dot{p}(x) - \frac{1}{4}p(x)Q(x).$$

Upon integration one observes

$$a_2(x) = \frac{1}{4}p(x) - \frac{1}{8}Q^2(x)$$

where we have used the fact that

$$\frac{d}{dx}Q^2(x) = 2p(x)Q(x).$$

Straightforward computations now show that $b_2(x) = 0$ and $a_3(x) = 0$. To compute $b_3(x)$ we first calculate derivatives of $a_2(x)$. The result is

$$\begin{aligned}\dot{a}_2(x) &= \frac{1}{4}(\dot{p}(x) - p(x)Q(x))\\ \ddot{a}_2(x) &= \frac{1}{4}(\ddot{p}(x) - \dot{p}(x)Q(x) - p^2(x)).\end{aligned}$$

The coefficient is then calculated as follows

$$\begin{aligned}\dot{b}_3(x) &= \frac{1}{2}(\frac{1}{4}p^2(x) - \frac{1}{8}p(x)Q^2(x)) - \frac{1}{8}(\ddot{p}(x) - \dot{p}(x)Q(x) - p^2(x))\\ &= -\frac{1}{8}\ddot{p}(x) + \frac{1}{8}\dot{p}(x)Q(x) - \frac{1}{16}p(x)Q^2(x) + \frac{1}{4}p^2(x).\end{aligned}$$

We can integrate to get

$$\begin{aligned}
b_3(x) &= -\frac{1}{8}\dot{p}(x) - \frac{1}{48}Q^3(x) + \frac{1}{4}T(x) + \frac{1}{8}\int_0^x \dot{p}(t)\int_0^t p(s)\,ds\,dt \\
&= -\frac{1}{8}\dot{p}(x) - \frac{1}{48}Q^3(x) + \frac{1}{4}T(x) + \frac{1}{8}(p(x)Q(x) - T(x)) \\
&= -\frac{1}{8}\dot{p}(x) - \frac{1}{48}Q^3(x) + \frac{1}{8}T(x) + \frac{1}{8}p(x)Q(x).
\end{aligned}$$

Lastly, compute the derivatives of $b_3(x)$ to use in the derivation of $a_4(x)$,

$$\begin{aligned}
\dot{b}_3(x) &= -\frac{1}{8}\ddot{p}(x) + \frac{1}{4}p^2(x)\frac{1}{8}\dot{p}(x)Q(x) - \frac{1}{16}p(x)Q^2(x) \\
\ddot{b}_3(x) &= -\frac{1}{8}\dddot{p}(x) + \frac{5}{8}\dot{p}(x)p(x) + \frac{1}{8}\ddot{p}(x)Q(x) - \frac{1}{16}\dot{p}(x)Q^2(x) - \frac{1}{8}p^2(x)Q(x).
\end{aligned}$$

Finally,

$$\begin{aligned}
\dot{a}_4(x) &= \frac{1}{2}(\ddot{b}_3 - p(x)b_3) \\
&= \frac{1}{2}\left(-\frac{1}{8}\dddot{p}(x) + \frac{5}{8}\dot{p}(x)p(x)\right. \\
&\qquad \left. + \frac{1}{8}\ddot{p}(x)Q(x) - \frac{1}{16}\dot{p}(x)Q^2(x) - \frac{1}{8}p^2(x)Q(x)\right) \\
&\qquad - \frac{1}{2}\left(-\frac{1}{8}p(x)\dot{p}(x) - \frac{1}{48}p(x)Q^3(x) + \frac{1}{8}p(x)T(x) + \frac{1}{8}p^2(x)Q(x)\right) \\
&= -\frac{1}{16}p(x)T(x) + \frac{3}{8}\dot{p}(x)p(x) - \frac{1}{8}p^2(x)Q(x) + \frac{1}{4\cdot 24}p(x)Q^3(x) \\
&\qquad - \frac{1}{16}\dddot{p}(x) + \frac{1}{16}\ddot{p}(x)Q(x) - \frac{1}{32}\dot{p}(x)Q^2(x)
\end{aligned}$$

From which it follows,

$$
\begin{aligned}
a_4(x) = &-\frac{1}{16}\ddot{p}(x) + \frac{1}{16 \cdot 24}Q^4(x) + \frac{5}{16}\int_0^x \dot{p}(t)p(t)\,dt \\
&+\frac{1}{16}\int_0^x (\dot{p}(t)p(t) + \ddot{p}(t)Q(t))\,dt \\
&-\frac{1}{16}\int_0^x (p^2(t)Q(t) + \frac{1}{2}\dot{p}(t)Q^2(t))\,dt \\
&-\frac{1}{16}\int_0^x (p^2(t)Q(t) + p(t)T(t))\,dt \\
= &-\frac{1}{16}\ddot{p}(x) + \frac{1}{16 \cdot 24}Q^4(x) + \frac{5}{2 \cdot 16}p^2(x) \\
&+\frac{1}{16}\dot{p}(x)Q(x) - \frac{1}{2 \cdot 16}p(x)Q^2(x) - \frac{1}{16}Q(x)T(x) \\
= &\frac{1}{16}\left(\frac{1}{24}Q^4(x) - \ddot{p}(x) + \frac{5}{2}p^2(x)\right. \\
&\left.+\dot{p}(x)Q(x) - \frac{1}{2}p(x)Q^2(x) - Q(x)T(x)\right).
\end{aligned}
$$

Collecting all the expressions for the coefficients arrived at above we see

$$
\begin{aligned}
z_1(x,k,p) = \cos(kx) &+ \frac{\sin(kx)}{2k}Q(x) + \frac{\cos(kx)}{4k^2}\left(p(x) - \frac{1}{2}Q^2(x)\right) \\
&+\frac{\sin(kx)}{8k^3}\left(p(x)Q(x) - p'(x) + T(x) - \frac{1}{6}Q^3(x)\right) \\
&+\frac{\cos(kx)}{16k^4}\left(p'(x)Q(x) - p''(x) + \frac{5}{2}p^2(x) - \frac{1}{2}p(x)Q^2(x)\right. \\
&\qquad \left. - Q(x)T(x) + \frac{1}{24}Q^4(x)\right) \\
&+\mathcal{O}\left(\frac{1}{k^5}\right).
\end{aligned} \tag{5.31}
$$

A similar process is carried out for the coefficients in the expansion of $z_2(x,k,p)$. In the end one computes the expansion to be,

$$\begin{aligned} z_2(x,k,p) = {} & \frac{\sin(kx)}{k} - \frac{\cos(kx)}{2k^2}Q(x) + \frac{\sin(kx)}{4k^3}\left(p(x) - \frac{1}{2}Q^2(x)\right) \\ & + \frac{\cos(kx)}{8k^4}\left(p'(x) - p(x)Q(x) - T(x) + \frac{1}{6}Q^3(x)\right) \\ & + \frac{\sin(kx)}{16k^5}\left(p'(x)Q(x) - p''(x) + \frac{5}{2}p^2(x) - \frac{1}{2}p(x)Q^2(x)\right. \\ & \left. - Q(x)T(x) + \frac{1}{24}Q^4(x)\right) \\ & + \mathcal{O}\left(\frac{1}{k^6}\right). \end{aligned} \tag{5.32}$$

These specific forms are used, along with our determinant condition, to examine the one dimensional transmission spectrum.

5.5 Acoustic Profile Uniqueness Condition

Conditions on two acoustic speeds $c(x)$ and $b(x)$ that ensure the transmission spectrum relative to these acoustic speeds on the interval $[0,1]$ is discrete for large $k \in \mathbb{R}_+$ are shown. The conditions we derive are sufficient to ensure that the transmission spectrum on $[0,1]$ for two acoustic speeds is discrete for large k. We have not yet been able to find necessary and sufficient conditions.

Given two acoustic speeds $c(x)$ and $b(x)$ on $[0,1]$ we wish to analyze the set of $k \in \mathbb{R}_+$ such that there exists nontrivial solutions $u(x,k)$, $v(x,k)$ satisfying (5.8). Through a change of variables we have shown, theorem 5.2.0.4, that this set of $k \in \mathbb{R}_+$ is contained in the set of k such that there exist nontrivial solutions $X(\eta,k)$, $Y(\xi,k)$, satisfying (5.5). Again we call to attention the fact that since we are assuming the acoustic profiles $c(x)$ and $b(x)$ satisfy $supp(1-c(x))$, $supp(1-b(x)) \subset [0,1]$ the functions $p_X(\eta)$ and $p_Y(\xi)$, the coefficient functions in (5.5), must vanish to infinite order at the endpoints η, $\xi = 0$ and $\eta = C$, $\xi = B$ respectively.

We have shown that a necessary and sufficient condition for a given $k \in \mathbb{R}_+$ to allow nontrivial solutions to (5.5) is that the determinant condition (5.13) be satisfied for the fundamental solutions to the two differential equations involved in (5.5). Using the asymptotic expansions (5.31) and (5.32) computed in the previous section, we see the fundamental solutions involved are

$$\begin{aligned} Y_1(\xi,k) = \cos(k\xi) &+ \frac{\sin(k\xi)}{2k}Q_Y(\xi) + \frac{\cos(k\xi)}{4k^2}\left(p_Y(\xi) - \frac{1}{2}Q_Y^2(\xi)\right) \\ &+\frac{\sin(k\xi)}{8k^3}\left(p_Y(\xi)Q_Y(\xi) - p_Y'(\xi) + T_Y(\xi) - \frac{1}{6}Q_Y^3(\xi)\right) \\ &+\frac{\cos(k\xi)}{16k^4}\left(p_Y'(\xi)Q_Y(\xi) - p_Y''(\xi) + \frac{5}{2}p_Y^2(\xi)\right. \\ &\qquad\left. -\frac{1}{2}p_Y(\xi)Q_Y^2(\xi) - Q_Y(\xi)T_Y(\xi) + \frac{1}{24}Q_Y^4(\xi)\right) \\ &+\mathcal{O}\left(\frac{1}{k^5}\right), \end{aligned} \tag{5.33}$$

$$\begin{aligned} Y_2(\xi,k) = \frac{\sin(k\xi)}{k} &- \frac{\cos(k\xi)}{2k^2}Q_Y(\xi) + \frac{\sin(k\xi)}{4k^3}\left(p_Y(\xi) - \frac{1}{2}Q_Y^2(\xi)\right) \\ &+\frac{\cos(k\xi)}{8k^4}\left(p_Y'(\xi) - p_Y(\xi)Q_Y(\xi) - T_Y(\xi) + \frac{1}{6}Q_Y^3(\xi)\right) \\ &+\frac{\sin(k\xi)}{16k^5}\left(p_Y'(\xi)Q_Y(\xi) - p_Y''(\xi) + \frac{5}{2}p_Y^2(\xi)\right. \\ &\qquad\left. -\frac{1}{2}p_Y(\xi)Q_Y^2(\xi) - Q_Y(\xi)T_Y(\xi) + \frac{1}{24}Q_Y^4(\xi)\right) \\ &+\mathcal{O}\left(\frac{1}{k^6}\right) \end{aligned} \tag{5.34}$$

corresponding to the equation involving $p_Y(\xi)$ and

$$\begin{aligned} X_1(\eta,k) = \cos(k\eta) &+ \frac{\sin(k\eta)}{2k}Q_X(\eta) + \frac{\cos(k\eta)}{4k^2}\left(p_X(\eta) - \frac{1}{2}Q_X^2(\eta)\right) \\ &+\frac{\sin(k\eta)}{8k^3}\left(p_X(\eta)Q_X(\eta) - p_X'(\eta) + T_X(\eta) - \frac{1}{6}Q_X^3(\eta)\right) \\ &+\frac{\cos(k\eta)}{16k^4}\left(p_X'(\eta)Q_X(\eta) - p_X''(\eta) + \frac{5}{2}p_X^2(\eta)\right. \\ &\qquad\left. -\frac{1}{2}p_X(\eta)Q_X^2(\eta) - Q_X(\eta)T_X(\eta) + \frac{1}{24}Q_X^4(\eta)\right) \\ &+\mathcal{O}\left(\frac{1}{k^5}\right), \end{aligned} \tag{5.35}$$

and

$$X_2(\eta,k) = \frac{\sin(k\eta)}{k} - \frac{\cos(k\eta)}{2k^2}Q_X(\eta) + \frac{\sin(k\eta)}{4k^3}\left(p_X(\eta) - \frac{1}{2}Q_X^2(\eta)\right) \tag{5.36}$$
$$+\frac{\cos(k\eta)}{8k^4}\left(p_X'(\eta) - p_X(\eta)Q_X(\eta) - T_X(\eta) + \frac{1}{6}Q_X^3(\eta)\right)$$
$$+\frac{\sin(k\eta)}{16k^5}\left(p_X'(\eta)Q_X(\eta) - p_X''(\eta) + \frac{5}{2}p_X^2(\eta)\right.$$
$$\left. -\frac{1}{2}p_X(\eta)Q_X^2(\eta) - Q_X(\eta)T_X(\eta) + \frac{1}{24}Q_X^4(\eta)\right)$$
$$+\mathcal{O}\left(\frac{1}{k^6}\right)$$

corresponding to the equation involving $p_X(\eta)$. Here we have used the notation

$$Q_Y(\xi) = \int_0^\xi p_Y(t)\,dt,\ T_Y(\xi) = \int_0^\xi p_Y^2(t)\,dt \tag{5.37}$$
$$Q_X(\eta) = \int_0^\eta p_X(t)\,dt,\ T_X(\eta) = \int_0^\eta p_X^2(t)\,dt.$$

Since we have assumed that p_X and p_Y vanish to infinite order at the endpoints of the intervals $[0, C]$ and $[0, B]$, respectively, the fundamental solutions $(X_i,\, Y_i)$ take a much simpler form in the evaluation of the determinant condition.

We see

$$Y_1(B,k) = \cos(kB) + \frac{\sin(kB)}{2k}Q_Y(B) - \frac{\cos(kB)}{4k^2}\frac{1}{2}Q_Y^2(B) \tag{5.38}$$
$$+\frac{\sin(kB)}{8k^3}\left(T_Y(B) - \frac{1}{6}Q_Y^3(B)\right)$$
$$+\frac{\cos(kB)}{16k^4}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^5}\right),$$

$$Y_2(B,k) = \frac{\sin(kB)}{k} - \frac{\cos(kB)}{2k^2}Q_Y(B) - \frac{\sin(kB)}{4k^3}\frac{1}{2}Q_Y^2(B) \tag{5.39}$$
$$+\frac{\cos(kB)}{8k^4}\left(\frac{1}{6}Q_Y^3(B) - T_Y(B)\right)$$
$$+\frac{\sin(kB)}{16k^5}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^6}\right),$$

$$\begin{aligned}\dot{Y}_1(B,k) = &-k\sin(kB) + \frac{1}{2}\cos(kB)Q_Y(B) + \frac{\sin(kB)}{4k}\frac{1}{2}Q_Y^2(B) \\ &+\frac{\cos(kB)}{8k^2}\left(T_Y(B) - \frac{1}{6}Q_Y^3(B)\right) \\ &-\frac{\sin(kB)}{16k^3}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^4}\right),\end{aligned} \tag{5.40}$$

and

$$\begin{aligned}\dot{Y}_2(B,k) = &\cos(kB) + \frac{\sin(kB)}{2k}Q_Y(B) - \frac{\cos(kB)}{4k^2}\frac{1}{2}Q_Y^2(B) \\ &-\frac{\sin(kB)}{8k^3}\left(\frac{1}{6}Q_Y^3(B) - T_Y(B)\right) \\ &+\frac{\cos(kB)}{16k^4}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^5}\right).\end{aligned} \tag{5.41}$$

The analogous equations for $X_i(C,k)$ are

$$\begin{aligned}X_1(C,k) = &\cos(kC) + \frac{\sin(kC)}{2k}Q_X(C) - \frac{\cos(kC)}{4k^2}\frac{1}{2}Q_X^2(C) \\ &+\frac{\sin(kC)}{8k^3}\left(T_X(C) - \frac{1}{6}Q_X^3(C)\right) \\ &+\frac{\cos(kC)}{16k^4}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^5}\right),\end{aligned} \tag{5.42}$$

$$\begin{aligned}X_2(C,k) = &\frac{\sin(kC)}{k} - \frac{\cos(kC)}{2k^2}Q_X(C) - \frac{\sin(kC)}{4k^3}\frac{1}{2}Q_X^2(C) \\ &+\frac{\cos(kC)}{8k^4}\left(\frac{1}{6}Q_X^3(C) - T_X(C)\right) \\ &+\frac{\sin(kC)}{16k^5}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^6}\right),\end{aligned} \tag{5.43}$$

$$\begin{aligned}\dot{X}_1(C,k) = &-k\sin(kC) + \frac{1}{2}\cos(kC)Q_X(C) + \frac{\sin(kC)}{4k}\frac{1}{2}Q_X^2(C) \\ &+\frac{\cos(kC)}{8k^2}\left(T_X(C) - \frac{1}{6}Q_X^3(C)\right) \\ &-\frac{\sin(kC)}{16k^3}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^4}\right),\end{aligned} \tag{5.44}$$

and

$$\dot{X}_2(C,k) = \cos(kC) + \frac{\sin(kC)}{2k} Q_X(C) - \frac{\cos(kC)}{4k^2}\frac{1}{2}Q_X^2(C) \\ -\frac{\sin(kC)}{8k^3}\left(\frac{1}{6}Q_X^3(C) - T_X(C)\right) \\ +\frac{\cos(kC)}{16k^4}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^5}\right). \tag{5.45}$$

Theorem 5.5.0.8 *For two acoustic speeds $c(x)$ and $b(x)$ on $[0,1]$ the interior transmission spectrum in $\mathbb{R}_+$ has the property that for $k \in \mathbb{R}_+$ large enough there exist intervals in $\mathbb{R}_+$ that are free of transmission eigenvalues as long as one of the following two conditions is satisfied:*

(i) $\int_0^1 \sqrt{n_c(s)}\, ds \neq \int_0^1 \sqrt{n_b(s)}\, ds$

(ii) $\int_0^1 \frac{(c'(s))^2}{c(s)}\, ds \neq \int_0^1 \frac{(b'(s))^2}{b(s)}\, ds.$

Proof. The first step is to apply the determinant condition (5.13) using the above form for the fundamental solutions and their derivatives. Recall that $k \in \mathbb{R}_+$ is a transmission eigenvalue as long as

$$0 = 2 - W(Y_1(B), X_2(C)) + W(Y_2(B), X_1(C))$$

or

$$0 = 2 - (Y_1(B)\dot{X}_2(C) - \dot{Y}_1(B)X_2(C)) + (Y_2(B)\dot{X}_1(C) - \dot{Y}_2(B)X_1(C)). \tag{5.46}$$

Expand this condition using the asymptotic expansions (5.38)-(5.45). We compute, using the identities $\cos(a-b) = \cos a \cos b + \sin a \sin b$ and $\sin(a-b) = \sin a \cos b - \cos a \sin b$,

$$\begin{aligned}
Y_1(B)\dot{X}_2(C) - \dot{Y}_1(B)X_2(C) &= \cos(k(B-C)) \\
-\frac{\sin(k(B-C))}{2k}(Q_X(C) - Q_Y(B)) &- \frac{\cos(k(B-C))}{4k^2}(Q_X(C) - Q_Y(B))^2 \\
-\frac{\sin(k(B-C))}{4k^3}&\left((T_X(C) - T_Y(B)) + \frac{1}{6}(Q_Y(B) - Q_X(C))^3\right) \\
+\frac{\cos(k(B-C))}{8k^4}&((T_X(C) - T_Y(B))(Q_Y(B) - Q_X(C)) \\
&+\frac{1}{24}(Q_Y(B) - Q_X(C))^4) \\
&+\mathcal{O}\left(\frac{1}{k^5}\right).
\end{aligned} \tag{5.47}$$

For the second term we get

$$\begin{aligned}
Y_2(B)\dot{X}_1(C) - \dot{Y}_2(B)X_1(C) &= -\cos(k(B-C)) \\
+\frac{\sin(k(B-C))}{2k}(Q_X(C) - Q_Y(B)) &+ \frac{\cos(k(B-C))}{4k^2}(Q_X(C) - Q_Y(B))^2 \\
+\frac{\sin(k(B-C))}{4k^3}&\left((T_X(C) - T_Y(B)) + \frac{1}{6}(Q_Y(B) - Q_X(C))^3\right) \\
-\frac{\cos(k(B-C))}{8k^4}&((T_X(C) - T_Y(B))(Q_Y(B) - Q_X(C)) \\
&+\frac{1}{24}(Q_Y(B) - Q_X(C))^4) \\
&+\mathcal{O}\left(\frac{1}{k^5}\right).
\end{aligned} \tag{5.48}$$

After substituting the above expressions into (5.46) and collecting like powers of k one finds

$$\begin{aligned}
0 = 2 - 2\cos(k(B-C)) &+ \frac{\sin(k(B-C))}{k}(Q_X(C) - Q_Y(B)) \\
&+\frac{\cos(k(B-C))}{4k^2}(Q_X(C) - Q_Y(B))^2 \\
&+\frac{\sin(k(B-C))}{4k^3}\left((T_X(C) - T_Y(B)) + \frac{1}{6}(Q_Y(B) - Q_X(C))^3\right) \\
&-\frac{\cos(k(B-C))}{8k^4}((T_X(C) - T_Y(B))(Q_Y(B) - Q_X(C)) \\
&\quad+\frac{1}{24}(Q_Y(B) - Q_X(C))^4) \\
&+\mathcal{O}\left(\frac{1}{k^5}\right).
\end{aligned} \tag{5.49}$$

If condition (i) is satisfied then

$$\int_0^1 \sqrt{n_c(s)}\, ds \neq \int_0^1 \sqrt{n_b(s)}\, ds. \tag{5.50}$$

By definition this implies that $C \neq B$. However, one then concludes that

$$2 - 2\cos(k(B - C)) \neq 0 \tag{5.51}$$

for intervals of $k \in \mathbb{R}_+$. If one now focuses on the first two terms in the above expansion of the determinant condition,

$$0 = 2 - 2\cos(k(B - C)) + \mathcal{O}\left(\frac{1}{k}\right), \tag{5.52}$$

we see that for large values of $k \in \mathbb{R}_+$ there exist intervals where this quantity is nonzero. Therefore, for large k condition (i) implies the existence of intervals in $\mathbb{R}_+$ that are free of transmission eigenvalues.

If condition (i) is not satisfied then $B = C$ so the above asymptotic expansion (5.49) of condition (5.13) becomes

$$\begin{aligned} 0 &= \frac{1}{4k^2}\left(Q_X(C) - Q_Y(B)\right)^2 \\ &\quad -\frac{1}{8k^4}\left((T_X(C) - T_Y(B))(Q_Y(B) - Q_X(C)) + \frac{1}{24}(Q_Y(B) - Q_X(C))^4\right) \\ &\quad +\mathcal{O}\left(\frac{1}{k^5}\right). \end{aligned}$$

If we assume $Q_X(C) \neq Q_Y(B)$ then

$$\frac{1}{4k^2}\left(Q_X(C) - Q_Y(B)\right)^2 \neq 0.$$

For large enough k the asymptotic expansion becomes

$$0 = \frac{1}{4k^2}\left(Q_X(C) - Q_Y(B)\right)^2 + \mathcal{O}\left(\frac{1}{k^4}\right). \tag{5.53}$$

Thus, for large k the above can not vanish identically. So for large enough $k \in \mathbb{R}_+$ there exist intervals in $\mathbb{R}_+$ free of transmission eigenvalues.

We now show $Q_X(C) \neq Q_Y(B)$ is implied by condition (ii). Recall

$$Q_X(C) = \int_0^C p_X(\eta)\,d\eta.$$

Where $C = \int_0^1 \sqrt{n_c(s)}\,ds$ and

$$p_X(\eta) = \frac{1}{4}\frac{\ddot{n}_c(x)}{(n_c(x))^2} - \frac{5}{16}\frac{(\dot{n}_c(x))^2}{(n_c(x))^3} \tag{5.54}$$

with $\eta(x) = \int_0^x \sqrt{n_c(s)}\,ds$.

These definitions can be unravelled back to our original acoustic speeds $c(x)$ and $b(x)$ to yield condition (ii). We start by noticing

$$\dot{\eta}(x) = \sqrt{n_c(x)}.$$

So $d\eta = \dot{\eta}(x)\,dx = \sqrt{n_c(x)}\,dx$ and, if we look at $x(\eta)$, we see $x(0) = 0$, $x(C) = 1$. Therefore, by a change of variables

$$\begin{aligned} Q_X(C) &= \int_0^C p_X(\eta)\,d\eta \\ &= \int_0^1 p_X(\eta(x))\dot{\eta}(x)\,dx \\ &= \int_0^1 p_X(x)\sqrt{n_c(x)}\,dx \\ &= \frac{1}{4}\int_0^1 \left(\frac{\ddot{n}_c(x)}{(n_c(x))^{3/2}} - \frac{5}{4}\frac{(\dot{n}_c(x))^2}{(n_c(x))^{5/2}}\right)dx. \end{aligned}$$

However,

$$\frac{\ddot{n}_c(x)}{(n_c(x))^{3/2}} = -2c''(x) + 6\frac{(c'(x))^2}{c(x)}$$
$$\frac{(\dot{n}_c(x))^2}{(n_c(x))^{5/2}} = 4\frac{(c'(x))^2}{c(x)}$$

which shows us that

$$\frac{\ddot{n}_c(x)}{(n_c(x))^{3/2}} - \frac{5}{4}\frac{(\dot{n}_c(x))^2}{(n_c(x))^{5/2}} = -2c''(x) + \frac{(c'(x))^2}{c(x)}.$$

It follows that

$$Q_X(C) = \frac{1}{4}\int_0^1 \left(\frac{(c'(x))^2}{c(x)} - 2c''(x)\right)dx$$

and by the symmetry of the argument

$$Q_Y(B) = \frac{1}{4}\int_0^1 \left(\frac{(b'(x))^2}{b(x)} - 2b''(x)\right)\,dx.$$

Examining these closer we notice that since $(1 - c(x))$ and $(1 - b(x))$ vanish along with all their derivatives at $x = 0$ and $x = 1$ the integrals

$$\int_0^1 c''(x)\,dx \quad \text{and} \quad \int_0^1 b''(x)\,dx$$

are equal to zero. Thus, it follows that the condition $Q_X(C) \neq Q_Y(B)$ is equivalent to condition (ii),

$$\int_0^1 \frac{(c'(x))^2}{c(x)}\,dx \neq \int_0^1 \frac{(b'(x))^2}{b(x)}\,dx.$$

■

These two conditions do not seem to answer the question of unique determination of the acoustic profiles completely. However, if either one, or both, of these conditions is satisfied, and the respective ultrasound fields have exponential decay, then the interval $[0, 1]$ with the sound speed $c(x)$ cannot generate the same thermoacoustic data as the interval $[0, 1]$ with the acoustic speed $b(x)$. We still cannot show that the thermoacoustic data, even in the one dimensional case on $[0, 1]$, uniquely determines the acoustic speed from which the data was generated. We can say that the ranges of thermoacoustic operators from differing constant speeds are disjoint and also a constant speed thermoacoustic operator has a range disjoint from a variable acoustic speed thermoacoustic operator.

First we must discuss the decay of ultrasound fields in the one dimensional case. It is clear [81, 82] that there is exponential decay in any bounded domain, $D \subset \mathbb{R}^n$, for solutions of the Cauchy problem for the wave equation

$$\partial_t^2 u(x,t) - c^2(x)\Delta u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+$$
$$u(x,0) = f(x),\ u_t(x,0) = 0 \text{ in } \mathbb{R}^n$$

if $\mathrm{supp}(f(x)) \subset D$, $c(x)$ is non-trapping, and the dimension $n \geq 3$ is odd. However, it is not clear to the author that this decay holds in dimension $n = 1$.

For the purposes of using the sparsity of the one dimensional transmission spectrum to prove a uniqueness result for the one dimensional thermoacoustic problem we have the following decay result.

Lemma 5.5.0.8 *Suppose $c(x)$ is an acoustic speed relative to $[0,1]$ on $\mathbb{R}$ and $f(x)$, $g(x) \in C_0^\infty([0,1])$. Suppose $u(x,t)$ satisfies*

$$\partial_t^2 u(x,t) - c^2(x)\partial_x^2 u(x,t) = 0 \text{ in } \mathbb{R} \times \mathbb{R}_+$$
$$u(x,0) = f(x),\ u_t(x,0) = 0 \text{ in } \mathbb{R}$$

and $v(x,t)$ satisfies

$$\partial_t^2 v(x,t) - \partial_x^2 v(x,t) = 0 \text{ in } \mathbb{R} \times \mathbb{R}_+$$
$$v(x,0) = g(x),\ v_t(x,0) = 0 \text{ in } \mathbb{R}.$$

If for all $t \in \mathbb{R}_+$ we have

$$u(0,t) = v(0,t) = g_0(t) \text{ and } u(1,t) = v(1,t) = g_1(t)$$

then there exists some $T > 0$ such that for $t > T$ we have $u(x,t) = 0$ for $x \in [0,1]$.

Proof. The proof is based on the fact that since $v(x,t)$ satisfies a constant speed wave equation the strong Huygen's principle holds. This implies that there exists some $T' > 0$ such that $v(x,t) = 0$ for $x \in [0,1]$ if $t \geq T'$. Thus, $v(0,t) = 0$ and $v(1,t) = 0$ for $t \geq T'$. Since the boundary data on the time cylinder for $v(x,t)$ and $u(x,t)$ are equal $u(0,t) = 0$ and $u(1,t) = 0$ for $t \geq T'$.

Outside of $[0,1]$ the fields $u(x,t)$ and $v(x,t)$ both satisfy the *exterior problem*

$$\partial_t^2 u(x,t) - \partial_x^2 u(x,t) = 0 \text{ in } [0,1]^c \times \mathbb{R}_+$$
$$u(x,0) = 0,\ u_t(x,0) = 0 \text{ in } [0,1]^c$$
$$u(0,t) = g_0(t),\ u(1,t) = g_1(t) \text{ in } \mathbb{R}_+.$$

Thus, $u(x,t) = v(x,t)$ for $x \in [0,1]^c$ and $t \in \mathbb{R}_+$. Since $g_0(t) = 0$ for $t \geq T'$ there must exist some $\epsilon_0 > 0$ and $\delta_0 > 0$ such that $u(x,t) = v(x,t) = 0$ for $t \geq T' + \epsilon_0$ and $x \in (-\delta_0, 0]$. This implies that $\partial_x u(0,t) = 0$ for $t \geq T' + \epsilon_0$.

In the one dimensional case the set of boundary data can also be used as Cauchy data. The above discussion implies the existence of some $t_0 > 0$ for which the solution $u(x,t)$ satisfies the Cauchy problem

$$\begin{aligned} \partial_x^2 u(x,t) - \frac{1}{c^2(x)} \partial_t^2 u(x,t) &= 0 \text{ in } [0,1] \times \{t > t_0\} \\ u(0,t) = 0, \ u_x(0,t) &= 0 \text{ in } t > t_0. \end{aligned}$$

Since this problem is strictly hyperbolic the solution at the boundary $x = 1$ has a finite domain of dependence on the initial data. Therefore, there exists some $T_1 > t_0$ such that $u(x,t) = 0$ in the cone

$$\{(x,t) : x \in [0,1], \ t \geq t_0 \text{ and } x \geq c_0|t - t_0|\}$$

with $c_0 = T_1 - t_0$. Now we have shown $u(x,T_1) = 0$ and $\partial_t u(x,T_1) = 0$ for $x \in [0,1]$ and $\partial_t u(x,T_1) = 0$. It follows that $u(x,t)$ satisfies

$$\begin{aligned} \partial_t^2 u(x,t) - c^2(x) \partial_x^2 u(x,t) &= 0 \text{ in } [0,1] \times \{t \geq T_1\} \\ u(x,T_1) = 0, \ u_t(x,T_1) &= 0 \text{ in } [0,1] \\ u(0,t) = 0 = u(1,t) &\text{ for } t \geq T_1. \end{aligned}$$

We conclude that $u(x,t) \equiv 0$ for $(x,t) \in [0,1] \times \{t \geq T_1\}$.

■

Theorem 5.5.0.9 *If the acoustic profile on $[0,1]$ is a constant then it is uniquely determined by the thermoacoustic data. That is, for a constant speed c and any other acoustic speed $b(x)$ on $[0,1]$, including constant speeds not equal to c, the intersection of the range of the thermoacoustic operators $\mathcal{L}_c$ and $\mathcal{L}_{b(x)}$ is zero.*

Proof. By the remarks made in previous chapters and the decay result of lemma 5.5.0.8 we only need to show that the complement of the transmission spectrum associated with

$$\begin{aligned} \partial_x^2 u + \frac{k^2}{c^2} u = 0, &\qquad x \in [0,1] \\ \partial_x^2 v + k^2 n_b(x) v = 0, &\qquad x \in [0,1] \\ u(0) = v(0), &\qquad u(1) = v(1) \\ \partial_x u(0) = \partial_x v(0), &\qquad \partial_x u(1) = \partial_x v(1). \end{aligned} \tag{5.55}$$

contains intervals if $b(x) \neq c$. However, by theorem 5.5.0.8 this happens as long as either

$$\int_0^1 \sqrt{n_c(s)}\, ds \neq \int_0^1 \sqrt{n_b(s)}\, ds \tag{5.56}$$

or

$$\int_0^1 \frac{(c'(s))^2}{c(s)}\, ds \neq \int_0^1 \frac{(b'(s))^2}{b(s)}\, ds \tag{5.57}$$

is satisfied.

For c and $b(x)$ to generate the same thermoacoustic data the second condition implies

$$\int_0^1 \frac{(b'(s))^2}{b(s)}\, ds = \int_0^1 \frac{(c'(s))^2}{c(s)}\, ds = 0. \tag{5.58}$$

Since the acoustic profiles in question are assumed to be positive the integrand $\frac{(b'(x))^2}{b(x)} > 0$ on $[0,1]$ so we must have $b'(x) = 0$ which means $b(x) = b$ a constant. Now the first condition implies the requirement

$$\int_0^1 \frac{ds}{c} = \frac{1}{c} = \int_0^1 \frac{ds}{b} = \frac{1}{b}. \tag{5.59}$$

This is enough to imply that $b(x) = c$. Therefore, if the acoustic profile is a constant then there are no other acoustic speeds that can generate the same thermoacoustic data.

■

We conclude this section by elaborating on the set of *possible non-uniqueness* determined by the two conditions in theorem 5.5.0.8. One notices that the condition requiring $c(x)$ to be equal to 1 in a neighborhood of $x = 0$ and $x = 1$ allows us to find other acoustic speeds on $[0, 1]$ such that neither condition (i) or (ii) of the above theorem are satisfied. To see this let $c(x)$ be a non-constant acoustic speed on $[0, 1]$ so there exists some $\epsilon > 0$ such that $c(x) = 1$ for $x \in [0, 1]$ with $x < \epsilon$ or $(1 - x) < \epsilon$. Now let $|\alpha| < \epsilon$ and define $b_\alpha(x) = c(x - \alpha)$. Then $b_\alpha(x)$ satisfies all necessary conditions to be an acoustic speed on $[0, 1]$, namely $b_\alpha(x)$ and all its derivatives vanish at the endpoints $x = 0$ and $x = 1$. Moreover, because of the translation invariance of the integrals in conditions (i) and (ii) the functions $c(x)$ and $b_\alpha(x)$ do not satisfy either of condition (i) or (ii). Clearly this does not contradict theorem 5.5.0.9 since the translation of a constant acoustic profile is still a constant acoustic profile so we do not generate a different acoustic speed.

6 ACOUSTIC UNIQUENESS IN TAT: RADIAL ACOUSTICS WITH RADIAL SOURCE

6.1 Introduction

Here we will show that in $\mathbb{R}^3$, if the thermoacoustic setup is completely radially symmetric; meaning that the domain D, the acoustic profile $c(x)$, and the source $f(x)$ are all radially symmetric, then we can derive conditions for uniqueness analogous to the preceding chapter. In particular we will demonstrate that, in this category of thermoacoustic measurements, constant acoustic profiles are uniquely determined from TAT measurements.

In previous chapters we have shown the relation of the ITP to the unique determination of acoustic speeds from TAT data. It was shown, in chapter 5, that this relation could be used in the one dimensional thermoacoustic problem to derive conditions for uniqueness on two acoustic profiles. This resulted in a proof that in a single dimension constant profiles generate distinct thermoacoustic measurements and therefore are uniquely determined. Our methods used a reduction of the transmission problem to a system of Sturm-Liouville problems for which we had explicit forms for fundamental solutions. In this chapter we will demonstrate that the assumption of radial symmetry, along with the assumption of the dimension $n = 3$, allows one to reduce the thermoacoustic problem to an equivalent set of Sturm-Liouville problems.

We assume that the ultrasound field, $u(x,t)$, generated by the thermoacoustic process satisfies

$$\partial_t^2 u(x,t) - c(|x|)\Delta u(x,t) = 0 \text{ on } B_1(0) \times \mathbb{R}_+$$
$$u(x,0) = f(|x|),\ u_t(x,0) = 0 \text{ on } B_1(0).$$

The TAT data is then collected on $S_1(0)$. It is important here that $B_1(0) \subset \mathbb{R}^3$ since this assumption not only ensures rapid decay of the ultrasound field but also allows us to make a change of variables that will reduce the *dimension* of the uniqueness problem.

It is obvious that we are restricting the thermoacoustic procedure in three ways. First, we are assuming that the body being imaged is spherical. This has been a common assumption in the mathematics of TAT in the past and therefore does not seem like much of an imposition. Second, we are assuming that the acoustic profile depends only on the radius $r = |x|$. Third, we have assumed that the acoustic source also depends only on the radial variable.

The assumption that the acoustic profile is radial is somewhat applicable since, for medical imaging, large variations in acoustic speed are caused by changes in tissue type which is somewhat layered. However, in assuming that the acoustic source is radial we have restricted the applicability of the results much more. As has been stated before the goal of thermoacoustics is to locate unhealthy tissue by mapping the absorption of EM radiation, this absorption is then proportional to the initial acoustic source. In assuming that the source is radial we have basically assumed that we know *a priori* that the unhealthy tissue is located in the center of the body. For this reason, though the results here will provide insight into the question of uniqueness in general, this assumption is not immediately relevant to applications.

Our method of proof here relies on reducing the dimension of the problem by exploiting the radial symmetry. We will make a change of variables in our ultrasound field

$$r = |x|,\ u(r,t) = a_0 \frac{z(r,t)}{r}.$$

In dimension three only, this has the effect of changing the laplacian into a second derivative with respect to the radius. This assumption on the overlying dimension is not completely necessary since the above change of variables is really just the first term in the expansion of the solution in spherical harmonics. In other dimensions the same expansion still works. The differential equation resulting from the equivalent change of variables is not as well behaved as in the $\mathbb{R}^3$ case but it is still able to be dealt with. Indeed this idea will motivate our results in the next chapter.

Even though the assumption of a radial source limits the application of these results in the three dimensional setting the methods of this chapter may have use in other areas of acoustic inverse problems. It will be shown that the radial ITP is equivalent to the one dimensional ITP with an extra boundary condition imposed on one endpoint. The introduction of this extra boundary condition has the effect of limiting the admissible solutions to the Sturm-Liouville problem involved. This actually lets us derive more information then in the general one dimensional case. There are many other one dimensional acoustic inverse problems that are of interest, each of these assume different boundary conditions on the endpoint of the interval under study. Thus, the relation of the assumed boundary condition to the ITP spectrum may have application in this area.

This chapter is organized as follows. In section 6.2 we will give an exact formulation of our assumptions and problem of study. We will also demonstrate the reduction of this radial case to dimension one. In section 6.3 we derive a determinant boundary condition for the transmission spectrum. This condition is equivalent to the determinant condition derived in the previous chapter. Our new set of conditions for uniqueness of two acoustic profiles and the results on determining constant acoustics are contained in section 6.4. Section 6.5 contains our ideas for further study and a discussion of the results.

6.2 Notation and Statement of the Problem

Let $x \in \mathbb{R}^3$, $r = |x|$, and $\theta = x/|x|$. We will denote the unit ball as $B_1(0) \subset \mathbb{R}^3$ and its boundary as $S_1(0) = \partial B_1(0)$. For a radially symmetric acoustic profile $c(x) = c(r) \in C^\infty(\mathbb{R}^3)$ with $supp(1 - c(r)) \subset B_1(0)$ the TAT operator, restricted to the domain of radial sources $f(r) \in C^\infty(\mathbb{R}^3)$ with $supp(f) \subset B_1(0)$, is defined by

$$\mathcal{L}_{c(r)} f(\theta, t) = u(\theta, t)|_{S_1(0)\times\mathbb{R}_+}$$

where $u(x, t)$ satisfies

$$\partial_t^2 u(x,t) - c^2(r)\Delta u(x,t) = 0 \text{ on } \mathbb{R}^3 \times \mathbb{R}_+$$
$$u(x,0) = f(r),\ u_t(x,0) = 0 \text{ on } \mathbb{R}^3.$$

The radial symmetry of this problem allows us to assume $u(x, t)$ is a function of the variable $r = |x|$.

As before we investigate the relation of the ranges of $\mathcal{L}_{c(r)}$ and a second thermoacoustic operator, $\mathcal{L}_{b(r)}$, arising from a radial profile $b(r) \in C^\infty(\mathbb{R}^3)$ with $supp(1-b(r)) \subset B_1(0)$. Given the results in chapter 3 this leads to the study of the radial interior transmission problem

$$\Delta U(r,k) + k^2 n_c(r) U(r,k) = 0 \text{ on } B_1(0) \tag{6.1}$$
$$\Delta V(r,k) + k^2 n_b(r) V(r,k) = 0 \text{ on } B_1(0)$$
$$U(1,k) = V(1,k),\ \partial_r U(1,k) = \partial_r V(1,k).$$

Recall the refractive indices are defined as $n_c(r) = c^{-2}(r)$, $n_b(r) = b^{-2}(r)$. It is important to note that the reduction to the radial ITP problem comes from assuming that the acoustic profile of $B_1(0)$ is a function of r as well as our source term $f(r)$. Thus, we assume that the TAT data is generated by a radially symmetric ultrasound field. The real part of the temporal Fourier transform of the acoustic field will then satisfy (6.1). That is, in the above $U(r,k) = \mathcal{R}e(\hat{u}(r,k))$, to use the notation of chapter 3, and similarly for $V(r,k)$ coming from $B_1(0)$ with acoustic profile $b(r)$.

Lemma 6.2.0.9 *For $k \in \mathbb{R}_+$ let $U(r = |x|, k) \in C^2(B_1(0)) \cap C^1(\overline{B_1(0)})$ satisfy*

$$\Delta U(r,k) + k^2 n(r) U(r,k) = 0 \text{ in } B_1(0) \tag{6.2}$$

with a radially symmetric refractive index $n(r)$. Then there exists some $a_0 \in \mathbb{R}$ such that

$$U(r,k) = a_0 \frac{z(r,k)}{r}, \tag{6.3}$$

where $z(r,k)$ satisfies

$$\ddot{z}(r,k) + k^2 n(r) z(r,k) = 0 \text{ for } r \in [0,1] \tag{6.4}$$
$$z(0,k) = 0,\ \dot{z}(0,k) = 1.$$

Proof. Since $U(r,k)$ is radial we may always write $U(r,k) = a_0 \left(\frac{r a_0^{-1} U(r,k)}{r}\right)$. So there is some $z(r,k)$ such that (6.3) holds. It remains to show that $z(r,k)$ satisfies the ODE and the initial conditions.

To see that $z(r,k)$ satisfies the equation in (6.4) we recall that in $\mathbb{R}^3$ the laplacian can be rewritten in spherical coordinates as

$$\Delta = \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\Delta_S \tag{6.5}$$

where Δ_S is the Laplace-Beltrami operator defined as the restriction of the Laplacian in $\mathbb{R}^3$ to $S_1(0)$. Now we set $U(r,k) = r^{-1} z(r,k)$ and identify the equation $z(r,k)$ must satisfy. One computes

$$\begin{aligned}\Delta U(r,k) + k^2 n(r) U(r,k) &= \Delta(a_0 r^{-1} z(r,k)) + k^2 n(r) a_0 r^{-1} z(r,k) \\ &= a_0 \partial_r^2 (r^{-1} z(r,k)) + a_0 \frac{2}{r}(r^{-1} z(r,k)) + k^2 n(r) a_0 r^{-1} z(r,k) \\ &= a_0 r^{-1}(\ddot{z}(r,k) + k^2 n(r) z(r,k)) = 0.\end{aligned}$$

Thus,

$$\ddot{z}(r,k) + k^2 n(r) z(r,k) = 0 \text{ for } r \in [0,1]$$

and it remains to show that $z(r,k)$ must satisfy the initial condition. This equation has continuous coefficients so has a two dimensional space of solutions spanned by the functions $z_1(r,k)$ and $z_2(r,k)$ satisfying the IVP

$$\ddot{z}_i(r,k) + k^2 n(r) z_i(r,k) = 0 \text{ for } r \in [0,1],\ i = 1,\ 2$$
$$z_1(0,k) = \dot{z}_2(0,k) = 0,\ \ \dot{z}_1(0,k) = z_2(0,k) = 1.$$

The function $U(r,k)$ is twice differentiable in $B_1(0)$ so $U(0,k)$ must be bounded. This implies

$$\lim_{r\to 0} U(r,k) = \lim_{r\to 0} a_0 \frac{z(r,k)}{r} < \infty \tag{6.6}$$

which forces $z(0,k) = 0$. We conclude that $z(r,k)$ must be a multiple of $z_1(r,k)$ so there exists $a_0 \in \mathbb{R}$ such that $U(r,k) = a_0 r^{-1} z(r,k)$ for $z(r,k)$ satisfying

$$\ddot{z}(r,k) + k^2 n(r) z(r,k) = 0 \text{ for } r \in [0,1]$$
$$z(0,k) = 0,\ \dot{z}(0,k) = 1.$$

■

In the above it is interesting to notice that for the solution $U(r,k)$ to be a radial function on $B_1(0)$ it must be possible to extend $U(r,k)$ to an even function in the variable $r \in \mathbb{R}$. This is implied by the above expression since we now have

$$\begin{aligned} \lim_{r\to 0^+} \partial_r U(r,k) &= \lim_{r\to 0^+} a_0 \left(\frac{\dot{z}(r,k)}{r} - \frac{z(r,k)}{r^2} \right) \\ &= a_0 \lim_{r\to 0^+} \frac{\dot{z}(r,k)}{r} - a_0 \lim_{r\to 0^+} \frac{1}{r}\frac{z(r,k)}{r} \\ &= a_0 \lim_{r\to 0^+} \frac{\dot{z}(r,k)}{r} - a_0 \lim_{r\to 0^+} \frac{\dot{z}(r,k)}{r} = 0. \end{aligned} \tag{6.7}$$

Lemma 6.2.0.10 *For $k \in \mathbb{R}_+$ let $z(r,k)$ satisfy*

$$\ddot{z}(r,k) + k^2 n(r) z(r,k) = 0 \text{ for } r \in [0,1]$$
$$z(0,k) = 0,\ \dot{z}(0,k) = 1.$$

Set

$$\eta = \int_0^r \sqrt{n(s)}\, ds, \tag{6.8}$$
$$w(\eta,k) = (n(r))^{1/4} z(r,k).$$

Then $w(\eta,k)$ satisfies

$$\ddot{w}(\eta,k) + (k^2 - p(\eta))w(\eta,k) = 0 \text{ for } \eta \in [0,C] \tag{6.9}$$
$$w(0,k) = 0,\ \dot{w}(0,k) = n(0)^{-1/4}$$

where

$$p(\eta) = \frac{1}{4}\frac{\ddot{n}(r)}{n(r)^2} - \frac{5}{16}\frac{\dot{n}(r)^2}{n(r)^3} \tag{6.10}$$

and $C = \int_0^1 \sqrt{n(s)}\, ds$.

Proof Since $z(r,k)$ satisfies (6.4) the fact that $w(\eta,k) = (n(r))^{1/4} z(r,k)$ satisfies the ODE in (6.9) has already been shown in the previous chapter. To see that the initial conditions are satisfied one easily checks $w(0,k) = (n(0))^{1/4} z(0,k) = 0$. Recalling that

$$\dot{w}(\eta,k) = \frac{1}{4}\frac{\dot{n}(r)}{n(r)^{5/4}} z(r,k) + n(r)^{-1/4}\dot{z}(r,k)$$

we see $\dot{w}(0,k) = n(0)^{-1/4}\dot{z}(0,k) = n(0)^{-1/4}$.

■

One should observe that a key difference between the ITP characterization in this chapter and the one dimensional problem of the last chapter is that the boundary conditions in the Sturm-Liouville form of the ITP only occur at the right endpoint. Since lemma 6.2.0.10 implies that the left boundary condition is determined the dimension of the problem is reduced. This lets us recover more information regarding the uniqueness question.

Theorem 6.2.0.10 *For two radial acoustic profiles $c(r)$ and $b(r)$ in $B_1(0)$ the transmission spectrum is a subset of the set of all $k \in \mathbb{R}_+$ such that there is a non-trivial pair $(w_1, w_2) \in C^2[0,C] \times C^2[0,B]$ satisfying*

$$\begin{aligned}\ddot{w}_1(\eta,k) + (k^2 - p_1(\eta))w_1(\eta,k) = 0 \ \textit{on}\ [0,C] \\ \ddot{w}_2(\xi,k) + (k^2 - p_2(\xi))w_2(\xi,k) = 0 \ \textit{on}\ [0,B] \\ w_1(0,k) = 0 = w_2(0,k), \\ w_1(C,k) = w_2(B,k),\ \dot{w}_1(C,k) = \dot{w}_2(B,k).\end{aligned} \tag{6.11}$$

Here we have defined

$$p_1(\eta) = \frac{1}{4}\frac{\ddot{n}_c(r)}{n_c(r)^2} - \frac{5}{16}\frac{\dot{n}_c(r)^2}{n_c(r)^3}$$
$$p_2(\xi) = \frac{1}{4}\frac{\ddot{n}_b(r)}{n_b(r)^2} - \frac{5}{16}\frac{\dot{n}_b(r)^2}{n_b(r)^3}$$

with indices of refraction $n_c(r)$ and $n_b(r)$. The endpoints are given by

$$C = \int_0^1 \sqrt{n_c(s)}\, ds$$
$$B = \int_0^1 \sqrt{n_b(s)}\, ds.$$

Proof. Suppose $k \in \mathbb{R}_+$ is a transmission eigenvalue so there are non-trivial $U(r,k)$ and $V(r,k)$ satisfying (6.1). By lemma 6.2.0.9 there exist a_1, $a_2 \in \mathbb{R}$ such that

$$U(r,k) = a_1 r^{-1} z_1(r,k) \text{ and } V(r,k) = a_2 r^{-1} z_2(r,k)$$

for $z_1(r,k)$ and $z_2(r,k)$ satisfying

$$\ddot{z}_1(r,k) + k^2 n_c(r) z_1(r,k) = 0 \text{ for } r \in [0,1]$$
$$\ddot{z}_2(r,k) + k^2 n_b(r) z_2(r,k) = 0 \text{ for } r \in [0,1]$$
$$z_1(0,k) = z_2(0,k) = 0,\ \dot{z}_1(0,k) = \dot{z}_2(0,k) = 1.$$

If we now define $w_1(\eta, k)$ and $w_2(\xi, k)$ from $z_1(r, k)$ and $z_2(r, k)$ by the transformations

$$\eta = \int_0^r \sqrt{n_c(s)}\, ds \text{ and } w_1(\eta, k) = (n_c(r))^{1/4} a_1 z_1(r, k),$$
$$\xi = \int_0^r \sqrt{n_b(s)}\, ds \text{ and } w_2(\xi, k) = (n_b(r))^{1/4} a_2 z_2(r, k),$$

lemma 6.2.0.10 ensures that the w_i satisfy the ODE in (6.11) and the boundary conditions at the left endpoint. To see that the right endpoint boundary conditions are satisfied we first note that $w_1(C, k) = (n_c(1))^{1/4} a_1 z_1(1, k)$ and $w_2(B, k) = (n_b(1))^{1/4} a_2 z_2(1, k)$. Since $n_c(1) = 1 = n_b(1)$ this shows

$$w_1(C, k) = a_1 z_1(1, k) = U(1, k) = V(1, k) = a_2 z_2(1, k) = w_2(B, k).$$

For the boundary condition on the derivative we first observe that

$$\dot{U}(r, k) = a_1 r^{-1} \dot{z}_1(r, k) - a_1 r^{-2} z_1(r, k)$$
$$\dot{V}(r, k) = a_2 r^{-1} \dot{z}_2(r, k) - a_2 r^{-2} z_2(r, k),$$

so

$$\dot{U}(1, k) = a_1 \dot{z}_1(1, k) - a_1 z_1(1, k) = a_2 \dot{z}_2(1, k) - a_2 z_2(1, k) = V(1, k).$$

But this implies $a_1 \dot{z}_1(1, k) = a_2 \dot{z}_1(1, k)$ since $a_1 z_1(1, k) = a_2 z_2(1, k)$. Differentiating w_1 and w_2 gives

$$\dot{w}_1(\eta, k) = a_1 \frac{1}{4} \frac{\dot{n_c}(r)}{n_c(r)^{5/4}} z_1(r, k) + n_c(r)^{-1/4} a_1 \dot{z}_1(r, k)$$
$$\dot{w}_2(\xi, k) = a_2 \frac{1}{4} \frac{\dot{n_b}(r)}{n_b(r)^{5/4}} z_2(r, k) + n_b(r)^{-1/4} a_2 \dot{z}_2(r, k),$$

from which it follows that

$$\dot{w}_1(C, k) = a_1 \dot{z}_1(1, k) = \dot{U}(1, k) = \dot{V}(1, k) = a_2 \dot{z}_2(1, k) = w_2(B, k)$$

since $n_c(1) = 1 = n_b(1)$ and $\dot{n}_c(1) = 0 = \dot{n}_b(1)$. Therefore, $w_1(\eta, k)$ and $w_2(\xi, k)$ satisfy (6.11).

■

6.3 Derivation of the Determinant Condition

We derive a necessary condition for $k \in \mathbb{R}_+$ to be a transmission eigenvalue relative to radially symmetric profiles $c(r)$ and $b(r)$ on $B_1(0)$. This condition arises from studying possible solutions of (6.11) and is analogous to the one dimensional determinant condition of the previous chapter. Like the condition in the previous chapter this result relies on the fundamental solutions of the equation

$$\ddot{w}(\eta, k) + (k^2 - p(\eta))w(\eta, k) = 0,\ \eta \in [0, \infty).$$

The set of solutions to this equation is spanned by two fundamental solutions, w_1 and w_2, satisfying the initial conditions

$$w_1(0, k) = \dot{w}_2(0, k) = 0$$
$$\dot{w}_1(0, k) = w_2(0, k) = 1.$$

From theorem 6.2.0.10 solutions corresponding to functions satisfying the radially symmetric transmission problem must satisfy $w(0, k) = 0$. This implies that if $(X, Y) \in C^2[0, C] \times C^2[0, B]$ is a non-trivial solution to the radially symmetric transmission problem,

$$\ddot{X}(\eta, k) + (k^2 - p_X(\eta))X(\eta, k) = 0 \text{ on } [0, C] \tag{6.12}$$
$$\ddot{Y}(\xi, k) + (k^2 - p_Y(\xi))Y(\xi, k) = 0 \text{ on } [0, B]$$
$$X(0, k) = 0 = Y(0, k),$$
$$X(C, k) = Y(B, k),\ \dot{X}(C, k) = \dot{Y}(B, k),$$

then there exists a_1, $a_2 \in \mathbb{R}$ such that

$$X(\eta, k) = a_1 X_1(\eta, k) \text{ and } Y(\xi, k) = a_2 Y_1(\xi, k). \tag{6.13}$$

Here $X_1(\eta, k)$ and $Y_1(\xi, k)$ are fundamental solutions satisfying

$$\ddot{X}_1(\eta, k) + (k^2 - p_X(\eta))X_1(\eta, k) = 0,\ \ \eta \in [0, C] \tag{6.14}$$
$$X_1(0, k) = 0,\ \dot{X}_1(0, k) = 1$$

and

$$\ddot{Y}_1(\xi,k) + (k^2 - p_Y(\xi))Y_1(\xi,k) = 0, \;\; \xi \in [0,B] \tag{6.15}$$
$$Y_1(0,k) = 0, \; \dot{Y}_1(0,k) = 1.$$

In the above $p_X = p_1$, $p_Y = p_2$, C and B correspond to two acoustic profiles $c(r)$ and $b(r)$ with the definitions used in theorem 6.2.0.10.

Theorem 6.3.0.11 *The number $k \in \mathbb{R}_+$ is a transmission eigenvalue on $B_1(0)$ relative to the radial acoustic profiles $c(r)$ and $b(r)$ only if*

$$W(Y_1(B,k), X_1(C,k)) = Y_1(B,k)\dot{X}_1(C,k) - \dot{Y}_1(B,k)X_1(C,k) = 0. \tag{6.16}$$

Proof. If $(X,Y) \in C^2[0,C] \times C^2[0,B]$ satisfy (6.12) then

$$X(C,k) = Y(B,k) \text{ and } \dot{X}(C,k) = \dot{Y}(B,k).$$

So there exists a_1, $a_2 \in \mathbb{R}$ such that

$$X = a_1 X_1 \text{ and } Y = a_2 Y_1$$

with X_1 and Y_1 satisfying (6.14) and (6.15) respectively. These facts imply

$$a_1 X_1(C,k) = a_2 Y_1(B,k)$$
$$a_1 \dot{X}_1(C,k) = a_2 \dot{Y}_1(B,k).$$

In matrix form this is

$$\begin{bmatrix} X_1(C,k) & -Y_1(B,k) \\ \dot{X}_1(C,k) & -\dot{Y}_1(B,k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For this to happen for non-zero a_1 and a_2 we require

$$\det \begin{pmatrix} X_1(C,k) & -Y_1(B,k) \\ \dot{X}_1(C,k) & -\dot{Y}_1(B,k) \end{pmatrix} = 0.$$

But this is exactly the condition

$$W(Y_1(B,k), X_1(C,k)) = \\ Y_1(B,k)\dot{X}_1(C,k) - \dot{Y}_1(B,k)X_1(C,k) = 0.$$

■

This condition is much simpler to investigate than the condition arrived at in the previous chapter. It is clear that this simplification comes from the required boundary condition at the left endpoint so in some ways the study of the transmission spectrum contained here is less general. We mention that one could investigate the effect of specifying other boundary conditions at the endpoint on the transmission spectrum.

6.4 Conditions for Uniqueness

Conditions of uniqueness on two radial acoustic profiles $c(r)$ and $b(r)$ on $B_1(0)$ that imply thermoacoustic data generated from $c(r)$ must be distinct from thermoacoustic data generated from $b(r)$ are demonstrated. We wish to study the set of $k \in \mathbb{R}_+$ for which there exists a non-trivial solution to (6.1). By theorem 6.2.0.10 this implies the existence of a non-trivial solution $(X, Y) \in C^2[0, C] \times C^2[0, B]$ of (6.12). Theorem 6.3.0.11 states that such a $k \in \mathbb{R}_+$ must satisfy the determinant condition

$$W(Y_1(B,k), X_1(C,k)) = \\ Y_1(B,k)\dot{X}_1(C,k) - \dot{Y}_1(B,k)X_1(C,k) = 0$$

where $X_1(\eta, k)$ and $Y_1(\xi, k)$ are the fundamental solutions of the Sturm-Liouville equations satisfying (6.14) and (6.15) respectively. In the previous chapter we derived an asymptotic form for the fundamental solution $z(x, k)$ of

$$\ddot{z}(x,k) + (k^2 - p(x))z(x,k) = 0,\ x \in [0, \infty)$$

$$z(0,k) = 0,\ \dot{z}(0,k) = 1$$

given by

$$
\begin{aligned}
z(x,k,p) = {} & \frac{\sin(kx)}{k} - \frac{\cos(kx)}{2k^2}Q(x) + \frac{\sin(kx)}{4k^3}\left(p(x) - \frac{1}{2}Q^2(x)\right) \\
& + \frac{\cos(kx)}{8k^4}\left(p'(x) - p(x)Q(x) - T(x) + \frac{1}{6}Q^3(x)\right) \\
& + \frac{\sin(kx)}{16k^5}\left(p'(x)Q(x) - p''(x) + \frac{5}{2}p^2(x) - \frac{1}{2}p(x)Q^2(x)\right. \\
& \quad \left. - Q(x)T(x) + \frac{1}{24}Q^4(x)\right) \\
& + \mathcal{O}\left(\frac{1}{k^6}\right)
\end{aligned}
\tag{6.17}
$$

where

$$
Q(x) = \int_0^x p(t)\,dt \tag{6.18}
$$

and

$$
T(x) = \int_0^x p^2(t)\,dt. \tag{6.19}
$$

This implies that the expansions of $X_1(\eta,k)$ and $Y_1(\xi,k)$ are given by

$$
\begin{aligned}
X_1(\eta,k) = {} & \frac{\sin(k\eta)}{k} - \frac{\cos(k\eta)}{2k^2}Q_X(\eta) + \frac{\sin(k\eta)}{4k^3}\left(p_X(\eta) - \frac{1}{2}Q_X^2(\eta)\right) \\
& + \frac{\cos(k\eta)}{8k^4}\left(p'_X(\eta) - p_X(\eta)Q_X(\eta) - T_X(\eta) + \frac{1}{6}Q_X^3(\eta)\right) \\
& + \frac{\sin(k\eta)}{16k^5}\left(p'_X(\eta)Q_X(\eta) - p''_X(\eta) + \frac{5}{2}p_X^2(\eta)\right. \\
& \quad \left. - \frac{1}{2}p_X(\eta)Q_X^2(\eta) - Q_X(\eta)T_X(\eta) + \frac{1}{24}Q_X^4(\eta)\right) \\
& + \mathcal{O}\left(\frac{1}{k^6}\right)
\end{aligned}
\tag{6.20}
$$

and

$$Y_1(\xi,k) = \frac{\sin(k\xi)}{k} - \frac{\cos(k\xi)}{2k^2}Q_Y(\xi) + \frac{\sin(k\xi)}{4k^3}\left(p_Y(\xi) - \frac{1}{2}Q_Y^2(\xi)\right) \qquad (6.21)$$
$$+\frac{\cos(k\xi)}{8k^4}\left(p_Y'(\xi) - p_Y(\xi)Q_Y(\xi) - T_Y(\xi) + \frac{1}{6}Q_Y^3(\xi)\right)$$
$$+\frac{\sin(k\xi)}{16k^5}\left(p_Y'(\xi)Q_Y(\xi) - p_Y''(\xi) + \frac{5}{2}p_Y^2(\xi)\right.$$
$$\left.-\frac{1}{2}p_Y(\xi)Q_Y^2(\xi) - Q_Y(\xi)T_Y(\xi) + \frac{1}{24}Q_Y^4(\xi)\right)$$
$$+\mathcal{O}\left(\frac{1}{k^6}\right).$$

It is assumed in this work that the acoustic profiles become constant at the boundary, $S_1(0)$. Therefore $supp(1 - c(|x|)) \subset B_1(0)$, $supp(1 - b(|x|)) \subset B_1(0)$. This has the effect of ensuring that $p_X(\eta)$ and $p_Y(\xi)$ vanish, along with all of their derivatives, at $\eta = C$ and $\xi = B$ respectively. The determinant condition is a condition on X_1 and Y_1 at these endpoints so the asymptotic expansions in k of X_1 and Y_1 used in the determinant condition are simplified,

$$X_1(C,k) = \frac{\sin(kC)}{k} - \frac{\cos(kC)}{2k^2}Q_X(C) - \frac{\sin(kC)}{4k^3}\frac{1}{2}Q_X^2(C) \qquad (6.22)$$
$$+\frac{\cos(kC)}{8k^4}\left(\frac{1}{6}Q_X^3(C) - T_X(C)\right)$$
$$+\frac{\sin(kC)}{16k^5}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^6}\right),$$

$$\dot{X}_1(C,k) = \cos(kC) + \frac{\sin(kC)}{2k}Q_X(C) - \frac{\cos(kC)}{4k^2}\frac{1}{2}Q_X^2(C) \qquad (6.23)$$
$$-\frac{\sin(kC)}{8k^3}\left(\frac{1}{6}Q_X^3(C) - T_X(C)\right)$$
$$+\frac{\cos(kC)}{16k^4}\left(\frac{1}{24}Q_X^4(C) - Q_X(C)T_X(C)\right) + \mathcal{O}\left(\frac{1}{k^5}\right)$$

and

$$Y_1(B,k) = \frac{\sin(kB)}{k} - \frac{\cos(kB)}{2k^2}Q_Y(B) - \frac{\sin(kB)}{4k^3}\frac{1}{2}Q_Y^2(B) \qquad (6.24)$$
$$+\frac{\cos(kB)}{8k^4}\left(\frac{1}{6}Q_Y^3(B) - T_Y(B)\right)$$
$$+\frac{\sin(kB)}{16k^5}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^6}\right),$$

$$\dot{Y}_1(B,k) = \cos(kB) + \frac{\sin(kB)}{2k}Q_Y(B) - \frac{\cos(kB)}{4k^2}\frac{1}{2}Q_Y^2(B) \tag{6.25}$$
$$-\frac{\sin(kB)}{8k^3}\left(\frac{1}{6}Q_Y^3(B) - T_Y(B)\right)$$
$$+\frac{\cos(kB)}{16k^4}\left(\frac{1}{24}Q_Y^4(B) - Q_Y(B)T_Y(B)\right) + \mathcal{O}\left(\frac{1}{k^5}\right).$$

Theorem 6.4.0.12 *For two radially symmetric acoustic speeds $c(r)$ and $b(r)$ on $B_1(0)$ the radial interior transmission spectrum in $\mathbb{R}_+$ has the property that for large enough $k \in \mathbb{R}_+$ there exists intervals in $\mathbb{R}_+$ that are free of transmission eigenvalues as long as one of the following three conditions are satisfied:*

(i) $\int_0^1 \sqrt{n_c(r)}\,dr \neq \int_0^1 \sqrt{n_b(r)}\,dr$

(ii) $\int_0^1 \frac{(c'(r))^2}{c(r)}\,ds \neq \int_0^1 \frac{(b'(r))^2}{b(r)}\,dr$

(iii) $\int_0^1 \left(4c(r)(c''(r))^2 + \frac{(c'(r))^4}{c(r)}\right)\,dr \neq \int_0^1 \left(4b(r)(b''(r))^2 + \frac{(b'(r))^4}{b(r)}\right)\,dr.$

Proof. To begin deriving these three conditions we apply the asymptotic expansions (6.22)-(6.25) to the determinant condition (6.16) for the radial case. We see that, after some simplification,

$$0 = W(Y_1(B,k), X_1(C,k)) \tag{6.26}$$
$$= Y_1(B,k)\dot{X}_1(C,k) - \dot{Y}_1(B,k)X_1(C,k)$$
$$= \frac{\sin(k(B-C))}{k} + \frac{\cos(k(B-C))}{2k^2}(Q_X(C) - Q_Y(B))$$
$$-\frac{\sin(k(B-C))}{8k^3}(Q_X(C) - Q_Y(B))^2$$
$$+\frac{\cos(k(B-C))}{8k^4}(T_X(C) - T_Y(B))$$
$$-\frac{\cos(k(B-C))}{48k^4}(Q_X(C) - Q_Y(B))^3$$
$$+\frac{\sin(k(B-C))}{16k^5}\left[\frac{1}{24}(Q_X(C) - Q_Y(B))^4\right.$$
$$\left. - \ (Q_X(C) - Q_Y(B))(T_X(C) - T_Y(B))\right] + \mathcal{O}\left(\frac{1}{k^6}\right).$$

As before we see that each term in this condition corresponds to an inverse power of k and a multiple of $\sin(k(B-C))$ or $\cos(k(B-C))$. Whether or not each term in this expansion is zero will determine conditions (i)-(iii).

We show the implications of conditions (i)-(iii) successively. To begin assume condition (i),

$$\int_0^1 \sqrt{n_c(r)}\, dr \neq \int_0^1 \sqrt{n_b(r)}\, dr.$$

By definition then $B \neq C$ so the first term in (6.26) is not identically zero, $\frac{\sin(k(B-C))}{k} \neq 0$ for intervals of $k \in \mathbb{R}_+$. The determinant condition can then be written as

$$0 = \frac{\sin(k(B-C))}{k} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

It follows that, for large enough $k \in \mathbb{R}_+$, there are intervals of $k \in \mathbb{R}_+$ such that

$$0 \neq \frac{\sin(k(B-C))}{k} + \mathcal{O}\left(\frac{1}{k^2}\right).$$

Thus, one may conclude that for large $k \in \mathbb{R}_+$ there exists intervals free of transmission eigenvalues.

If condition (i) is not satisfied then $B = C$ so the determinant condition becomes

$$\begin{aligned} 0 = &\frac{1}{2k^2}(Q_X(C) - Q_Y(B)) + \frac{1}{8k^4}(T_X(C) - T_Y(B)) \\ &-\frac{1}{48k^4}(Q_X(C) - Q_Y(B))^3 + \mathcal{O}\left(\frac{1}{k^6}\right). \end{aligned}$$

Therefore, by the same argument as in the previous case, there will exist intervals free of transmission eigenvalues in $\mathbb{R}_+$ as long as $Q_X(C) \neq Q_Y(B)$.

This is actually implied by condition (ii). To see this we unravel the definitions and recall,

$$Q_X(C) = \int_0^C p_X(\eta)\, d\eta,$$

where $C = \int_0^1 \sqrt{n_c(r)}\, dr$ and

$$p_X(\eta) = \frac{1}{4}\frac{\ddot{n}_c(r)}{n_c(r)^2} - \frac{5}{16}\frac{\dot{n}_c(r)^2}{n_c(r)^3}.$$

However, $\eta(r) = \int_0^r \sqrt{n_c(r)}\,dr$ so $\dot{\eta}(r) = \sqrt{n_c(r)}$ and

$$\begin{aligned} Q_X(C) &= \int_0^C p_X(\eta)\,d\eta \\ &= \int_0^1 p_X(\eta(r))\dot{\eta}(r)\,dr \\ &= \frac{1}{4}\int_0^1 \left(\frac{\ddot{n}_c(r)}{n_c(r)^{3/2}} - \frac{5}{4}\frac{\dot{n}_c(r)^2}{n_c(r)^{5/2}}\right) dr. \end{aligned}$$

Since $n_c(r) = c^{-2}(r)$ one easily computes

$$\frac{\ddot{n}_c(r)}{n_c(r)^{3/2}} = -2c''(r) + 6\frac{(c'(r))^2}{c(r)}$$

and

$$\frac{5}{4}\frac{\dot{n}_c(r)^2}{n_c(r)^{5/2}} = 5\frac{(c'(r))^2}{c(r)}.$$

The integrand then becomes

$$\frac{\ddot{n}_c(r)}{n_c(r)^{3/2}} - \frac{5}{4}\frac{\dot{n}_c(r)^2}{n_c(r)^{5/2}} = -2c''(r) + \frac{(c'(r))^2}{c(r)},$$

and we now have

$$Q_X(C) = \frac{1}{4}\int_0^1 \left(\frac{(c'(r))^2}{c(r)} - 2c''(r)\right) dr.$$

For $Q_Y(B)$ the exact same argument shows

$$Q_Y(B) = \frac{1}{4}\int_0^1 \left(\frac{(b'(r))^2}{b(r)} - 2b''(r)\right) dr.$$

Since $(1 - c(r))$, $(1 - b(r))$ have support contained in $[0, 1)$ we can eliminate the terms involving second derivatives in the above two expressions. To see this calculate the integral

$$\int_0^1 c''(r)\,dr = c'(1) - c'(0) = -c'(0),$$

$c(r)$, and $b(r)$ are both radial functions that are C^∞ on $B_1(0)$, therefore their derivatives at $r = 0$ must be zero and we conclude

$$\int_0^1 c''(r)\,dr = 0 \quad \text{and} \quad \int_0^1 b''(r)\,dr = 0.$$

It follows that condition (ii),

$$\int_0^1 \frac{(c'(r))^2}{c(r)}\,ds \neq \int_0^1 \frac{(b'(r))^2}{b(r)}\,dr,$$

implies $Q_X(C) \neq Q_Y(B)$ and that there are intervals in $\mathbb{R}_+$ free of transmission eigenvalues.

Now suppose both conditions (i) and (ii) are not met. This implies both $B = C$ and $Q_X(C) = Q_Y(B)$. The asymptotic expansion of the determinant condition (6.26) becomes

$$0 = \frac{1}{8k^4}(T_X(C) - T_Y(B)) + \mathcal{O}\left(\frac{1}{k^6}\right).$$

If $T_X(C) \neq T_Y(B)$ then it follows that for large values of $k \in \mathbb{R}_+$ there are intervals free of transmission eigenvalues in $\mathbb{R}_+$. Like in the justification of condition (ii) we show that $T_X(C) \neq T_Y(B)$ is implied by condition (iii),

$$\int_0^1 \left(4c(r)(c''(r))^2 + \frac{(c'(r))^4}{c(r)}\right) dr \neq \int_0^1 \left(4b(r)(b''(r))^2 + \frac{(b'(r))^4}{b(r)}\right) dr.$$

To begin we unravel the definition of $T_X(C)$ as in the justification of condition (ii). Recall

$$T_X(C) = \int_0^C p_X^2(\eta)\,d\eta,$$

substituting the definition of $\eta(r)$ in and making a change of variables we see

$$\begin{aligned} T_X(C) &= \int_0^1 p_X^2(\eta(r))\dot{\eta}(r)\,dr \\ &= \int_0^1 p_X^2(r)\sqrt{n_c(r)}\,dr \\ &= \int_0^1 \left(\frac{1}{4}\frac{\ddot{n}_c(r)}{n_c(r)^2} - \frac{5}{16}\frac{\dot{n}_c(r)^2}{n_c(r)^3}\right)^2 \sqrt{n_c(r)}\,dr \\ &= \frac{1}{16}\int_0^1 \left(\frac{\ddot{n}_c(r)^2}{n_c(r)^{7/2}} - \frac{5}{2}\frac{\ddot{n}_c(r)\dot{n}_c(r)^2}{n_c(r)^{9/2}} + \frac{25}{16}\frac{\dot{n}_c(r)^4}{n_c(r)^{11/2}}\right) dr. \end{aligned}$$

Now $\dot{n}_c(r) = -2\dot{c}(r)c^{-3}(r)$ and $\ddot{n}_c(r) = 6\dot{c}^2(r)c^{-4}(r) - 2\ddot{c}(r)c^{-3}(r)$, from which we can compute

$$\frac{\ddot{n}_c(r)^2}{n_c(r)^{7/2}} = 36\frac{\dot{c}^4(r)}{c(r)} - 24\dot{c}^2(r)\ddot{c}(r) + 4c(r)\ddot{c}^2(r),$$

$$\frac{\ddot{n}_c(r)\dot{n}_c(r)^2}{n_c(r)^{9/2}} = 24\frac{\dot{c}^4(r)}{c(r)} - 8\dot{c}^2(r)\ddot{c}(r),$$

and

$$\frac{\dot{n}_c(r)^4}{n_c(r)^{11/2}} = 16\frac{\dot{c}^4(r)}{c(r)}.$$

Therefore the integrand in $T_X(C)$ becomes

$$\frac{\ddot{n}_c(r)^2}{n_c(r)^{7/2}} - \frac{5}{2}\frac{\ddot{n}_c(r)\dot{n}_c(r)^2}{n_c(r)^{9/2}} + \frac{25}{16}\frac{\dot{n}_c(r)^4}{n_c(r)^{11/2}} = \frac{\dot{c}^4(r)}{c(r)} - 4\dot{c}^2(r)\ddot{c}(r) + 4c(r)\ddot{c}^2(r).$$

We have shown

$$T_X(C) = \frac{1}{16}\int_0^1 \left(\frac{\dot{c}^4(r)}{c(r)} - 4\dot{c}^2(r)\ddot{c}(r) + 4c(r)\ddot{c}^2(r)\right)\, dr.$$

One immediately notices that the second term in this integral can be eliminated, we calculate

$$\int_0^1 \dot{c}^2(r)\ddot{c}(r)\, dr = \frac{1}{3}(\dot{c}^3(1) - \dot{c}^3(0)) = 0,$$

where the first term vanishes due to the conditions on the support of $c(r)$ and the second term vanishes since $c(r)$ must define a smooth acoustic speed in $B_1(0)$.

It follows that

$$T_X(C) = \frac{1}{16}\int_0^1 \left(\frac{\dot{c}^4(r)}{c(r)} + 4c(r)\ddot{c}^2(r)\right)\, dr.$$

An equivalent expression holds for $T_Y(B)$ so $T_X(C) \neq T_Y(B)$ implies condition (iii),

$$\int_0^1 \left(\frac{\dot{c}^4(r)}{c(r)} + 4c(r)\ddot{c}^2(r)\right)\, dr \neq \int_0^1 \left(\frac{\dot{b}^4(r)}{b(r)} + 4b(r)\ddot{b}^2(r)\right)\, dr.$$

■

Theorem 6.4.0.12 and its proof are completely analogous to the conditions derived in the one dimensional case of the previous chapter with the exception of condition (iii).

This means that we can justify the unique determination of constant acoustic profiles with the same argument used in that case. The proof is exactly the same as that of the previous chapter so we omit it here.

Theorem 6.4.0.13 *If the acoustic profile on $B_1(0)$ is a constant then it is uniquely determined among radial acoustic profiles by the thermoacoustic data generated from radial sources. That is, for a constant speed c and a second radially symmetric profile $b(r)$ on $B_1(0)$, including constant speeds not equal to c, the ranges of the operators $\mathcal{L}_c$ and $\mathcal{L}_{b(r)}$, restricted to the domain of radial acoustic sources, have zero intersection.*

Remark 6.4.0.1 *One should note that though this theorem is very similar to the result in the one dimensional case it is a bit weaker. That is, in the one dimensional case, to make the distinction between the ranges of the two thermoacoustic operators we required no restriction on the domain of acoustic sources. However, this does point to an interesting question about how the domain of the operator affects the distinctness of acoustic profiles.*

In this section we have proved sufficient conditions to have intervals free of transmission eigenvalues in $\mathbb{R}_+$. We have been able to show that, in the setting of this chapter, with an extra restriction on the domain of the TAT operator, constant acoustic speeds generate thermoacoustic data distinct from variable radial acoustic profiles and other constant profiles.

6.5 Conclusion

In summary, for the completely radial thermoacoustic problem data generated from a specific constant acoustic speed is distinct from data generated by any other radial acoustic profile. This was shown using conditions derived from coefficients in the expansion of a boundary condition for the radial transmission problem. It should be apparent at this point that each term in the expansion of our determinant condition, in both the one dimensional case and the radially symmetric case, will yield a condition akin to those in theorem 6.4.0.12. Each of these conditions are difficult to calculate explicitly and therefore we have only computed the first three here. The first two were enough to show that constant acoustic profiles are uniquely determined. This leads one to hypothesize that some subset of the conditions arising from these coefficients may be enough to show that the acoustic speeds are uniquely determined by the thermoacoustic data. To further investigate this matter it seems one would need to find a more efficient way of computing these coefficients.

We derived three conditions for uniqueness in theorem 6.4.0.12. However, we have not been able to use the third condition to show that a wider class than constant sound speeds are uniquely determined from thermoacoustic tomography data. This is disappointing since somehow this condition arises from the extra boundary condition the radial assumption imposes on the ITP. A deeper understanding of this condition should help to distinguish the differences in the acoustic determination problem for the radial three dimensional case and the one dimensional case.

Since the results of this section are really results about the one dimensional ITP with an imposed boundary condition at one endpoint it would be interesting to see what effect other boundary conditions have on the transmission spectrum. For instance reflection seismology [10] deals with acoustic vibrations in an interval with a reflecting boundary condition imposed at one endpoint. Therefore, it could be of interest to see if we can get more or less information about the transmission spectrum with this condition. One could then ask if there is a boundary condition that lets us uniquely determine the acoustic profile on the interval.

In the next section the ideas of this chapter are expanded on to determine conditions of uniqueness for radial acoustic speeds with general acoustic sources. The methods rely heavily on the expansion of solutions of the wave equation in terms of spherical harmonics, of which the results of this section are only the zeroth order case. This has the effect of removing the restriction of the domain of the TAT operator that was imposed in this chapter.

7 UNIQUENESS OF ACOUSTICS IN TAT: RADIAL ACOUSTICS, NON-RADIAL SOURCE

7.1 Introduction

In the previous two chapters we have demonstrated that asymptotic expansions of Sturm-Liouville equations can be used to study the set of transmission eigenvalues. It was shown in chapter 6 that if one looked for transmission eigenvalues among only radially symmetric solutions to the forward thermoacoustic problem it was possible to change coordinates to the one dimensional problem. This allowed us to derive conditions on two radially symmetric acoustic speeds that implied a sparse transmission spectrum with radial solutions. In this chapter our goal is to remove the assumption that our solutions to the ITP must be radial. We show conditions on two radial acoustic profiles that imply the transmission spectrum is sparse even when considering non-radial solutions. These conditions will allow us to show that constants are uniquely determined among a special class of radially symmetric acoustic profiles.

The question of uniqueness of the acoustic profile in thermoacoustic tomography for two radial sound speeds, $c(r)$ and $b(r)$, on the domain $B_1(0) \subset \mathbb{R}^d$ with $d \geq 3$ odd is studied. Given the results of chapter 3 this means that we seek conditions on $c(r)$ and $b(r)$ so that the ranges of the thermoacoustic operators $\mathcal{L}_{c(r)}$ and $\mathcal{L}_{b(r)}$ have zero intersection. Unlike our study of radial acoustic profiles in the previous chapter, we do not assume that the acoustic source is also radially symmetric. This is an important improvement in our assumptions since, as mentioned in that chapter, a radial source implies we already know which tissue is healthy and which is unhealthy in the medical imaging setting.

The above assumptions are physically realistic. The assumption of odd dimension allows the three dimensional case which is of significant physical interest. The assumption of a spherical body and radially symmetric acoustic profile is a mathematical idealization but serves as an important model case. Our method of proof relies on a reduction of the thermoacoustic problem to one dimension through an expansion of a solution to the forward Cauchy problem in spherical harmonics. The importance of the dimension $d \geq 3$ being odd is that in this case the coefficients in the expansion can be written in terms of spherical Bessel functions which in turn may be expressed as a finite trigonometric series. Thus, our determinant boundary condition can be written as an asymptotic expansion of sines and cosines.

The assumption of odd dimension may possibly be removed since we would still be able to write our determinant boundary condition for transmission eigenvalues in terms of regular Bessel functions. However, the zeros of products of the more general Bessel functions are more difficult to understand and therefore we have not been able to derive conditions on acoustic profiles from these expansions so far. Of course one would like to justify the uniqueness results of this chapter in dimension two since this is the dimension most suitable for numerical testing.

This chapter is organized as follows. In section 7.2 we rigorously define notation and our reduction to one dimension. Section 7.3 contains the reformulation of the determinant boundary condition for transmission eigenvalues relevant to the assumptions of this chapter. Fundamental solutions of the one dimensional problem and its asymptotic expansions are derived in section 7.4. The main results of this chapter concerning conditions for uniqueness on two radially symmetric acoustic profiles and the unique determination of constant acoustic profiles among a particular class of radial sound speeds are stated in section 7.5. Section 7.6 contains our summary and concluding remarks for this chapter.

7.2 Notation and Statement of the Problem

We use notation and definitions for radial acoustic profiles established in the previous chapter. The uniqueness of acoustic profiles for ultrasound fields $u(x,t)$ satisfying the forward problem,

$$\partial_t^2 u(x,t) - c^2(r)\Delta u(x,t) = 0 \text{ on } \mathbb{R}^3 \times \mathbb{R}_+ \qquad (7.1)$$
$$u(x,0) = f(x),\ u_t(x,0) = 0 \text{ on } \mathbb{R}^3$$

is studied. As before the TAT operator is defined by $\mathcal{L}_{c(r)} f(x,t) = u(x,t)|_{S_1(0)\times\mathbb{R}_+}$.

To begin we compare the TAT operator to a second thermoacoustic operator generated from an acoustic profile $b(x) = b(|x|)$ such that $1 - b(|x|) \in C_0^\infty(B_1(0))$. Refractive indexes are defined as $n_c(r) = \frac{1}{c^2(r)}$ and $n_b(r) = \frac{1}{b^2(r)}$. Given the results of chapter 3, relating the transmission spectrum to the question of uniqueness of the acoustic speed, we wish to know for which $k \in \mathbb{R}_+$ nontrivial solutions, $\{U, V\}$, of

$$\Delta U(x,k) + k^2 n_c(r) U(x,k) = 0 \text{ on } B_1(0) \qquad (7.2)$$
$$\Delta V(x,k) + k^2 n_b(r) V(x,k) = 0 \text{ on } B_1(0)$$
$$(U - V)|_{S_1(0)} = 0,\ \partial_\nu (U - V)|_{S_1(0)} = 0,$$

exist. Recall that $U(x,k)$ arises from the real part of the temporal Fourier transform of the ultrasound field generated with acoustic speed $c(r)$ and likewise for $V(x,k)$ with sound speed $b(r)$. It should be noted in this setup that we do not make the assumption, as we did in the previous chapter, that the acoustic source $f(x)$ is radial. Thus, we do not assume that the solutions $U(x,k)$ and $V(x,k)$, when they exist, will be radial.

Our first goal is to find a general form for solutions of

$$\Delta w(x,k) + k^2 n(r) w(x,k) = 0 \text{ on } B_1(0)$$

for some radially symmetric refractive index $n(r)$. We wish to study the radial part and angular part of the solution $w(x,k)$ separately so we suppress the dependence on k for now and write $w(x, k) = w(r, \theta)$. To derive a solution the first step is to expand $w(r, \theta)$ in terms of j^{th} degree spherical harmonics of order l, $Y_{jl}(\theta)$. This has the effect of allowing us to distinguish between the radial dependence and angular dependence.

Lemma 7.2.0.11 *For a radially symmetric, smooth, refractive index $n(r)$ the solution $w(r, \theta)$ of*

$$\Delta w + k^2 n(r) w = 0 \tag{7.3}$$

is given by

$$w(r, \theta) = \sum_{j=0}^{\infty} \sum_{l=0}^{M} f_{jl}(r) r^j Y_{jl}(\theta). \tag{7.4}$$

The coefficient functions, $f_{jl}(r)$, satisfy the Sturm-Liouville problem

$$\partial_r (r^\gamma \partial_r f_{jl}) + k^2 n(r) r^\gamma f_{jl} = 0 \text{ on } [0, 1], \tag{7.5}$$

where $\gamma = n + 2j - 1$.

Proof. It is well known that the set of spherical harmonics, $\{Y_{jl}(\theta)\}$ form an orthonormal basis on $L^2(S_1(0))$. Thus, for a function $w(r, \theta)$ satisfying $\Delta w + k^2 n(r) w = 0$ there must exist coefficient functions $f_{jl}(r)$ so that $w(r, \theta) = \sum_{j,l} f_{jl}(r) r^j Y_{jl}(\theta)$.

To find an equation that the $f_{jl}(r)$ satisfy we recall that the laplacian can be written in spherical coordinates on $\mathbb{R}^n$ as

$$\Delta = \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_S.$$

The term Δ_S is the Laplace-Beltrami operator satisfying, for $j \geq 0$,

$$\Delta_S Y_{jl}(\theta) = -j(j + n - 2) Y_{jl}(\theta).$$

This operator is defined as the restriction of the laplacian in $\mathbb{R}^n$ to the sphere $S_1(0) \subset \mathbb{R}^n$. Now fix a degree j, order l, and apply the operator $\Delta + k^2 n(r)$, in spherical coordinates, to $f_{jl}(r) r^j Y_{jl}(\theta)$. After some simplification we see that $f_{jl}(r)$ must satisfy

$$\partial_r^2 f_{jl}(r) + \frac{n + 2j - 1}{r} \partial_r f_{jl}(r) + k^2 n(r) f_{jl}(r) = 0 \text{ on } [0, 1].$$

Setting $\gamma = n + 2j - 1$ and multiplying through by r^γ this equation can be rewritten as

$$\partial_r(r^\gamma \partial_r f_{jl}) + k^2 n(r) r^\gamma f_{jl} = 0 \text{ on } [0,1].$$

■

Lemma 7.2.0.12 *Let $f_{jl}(r)$ be a smooth function satisfying (7.5). Make the change of variables*

$$\eta = \int_0^r \sqrt{n(\sigma)}\, d\sigma,\ X_{ml}(\eta) = [r^{2\gamma} n(r)]^{1/4} f_{jl}(r) \tag{7.6}$$

where

$$m = \frac{\gamma}{2} - 1 = j + \frac{n-3}{2}.$$

Then $X_{ml}(\eta)$ satisfies

$$-\ddot{X}_{ml}(\eta) + \left(\frac{m(m+1)}{\eta^2} + p_m(\eta)\right) X_{ml}(\eta) = k^2 X_{ml}(\eta) \text{ for } \eta \in [0, C] \tag{7.7}$$

with

$$p_m(\eta) = \frac{1}{4}\frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16}\frac{(\dot{n}(r))^2}{(n(r))^3} + m(m+1)\left(\frac{1}{r^2 n(r)} - \frac{1}{\eta^2}\right) \tag{7.8}$$

and $C = \int_0^1 \sqrt{n(s)}\, ds$.

Proof. We must compute the derivatives $\dot{X}_{ml}$ and $\ddot{X}_{ml}$. To begin,

$$\begin{aligned}\partial_r X_{ml}(\eta(r)) &= \eta'(r)\dot{X}_{ml}(\eta)\\ &= \frac{1}{4}(2\gamma r^{2\gamma-1} n(r) + r^{2\gamma}\dot{n}(r))(r^{2\gamma} n(r))^{-3/4} f_{jl}(r)\\ &\quad + (r^{2\gamma} n(r))^{1/4}(f_{jl})'(r)\end{aligned}$$

so $\eta'(r) = \sqrt{n(r)}$ implies

$$\dot{X}_{ml}(\eta) = \frac{2\gamma r^{2\gamma-1} n(r) + r^{2\gamma}\dot{n}(r)}{4 r^{3\gamma/2} n(r)^{5/4}} f_{jl}(r) + \frac{r^{\gamma/2}(f_{jl})'(r)}{n(r)^{1/4}}.$$

Repeating this process, and performing a much longer simplification, we also have

$$\begin{aligned}\ddot{X}_{ml}(\eta) &= -k^2 r^{\gamma/2}(n(r))^{1/4} f_{jl}(r) \\ &\qquad + \frac{1}{4} n(r)^{-1/2}\partial_r \left\{ 2\gamma \frac{r^{\gamma/2-1}}{n(r)^{1/4}} + r^{\gamma/2}\frac{\dot{n}(r)}{n(r)^{5/4}} \right\} f_{jl}(r) \\ &= -k^2 X_m(\eta) + \frac{1}{4} n(r)^{-1/2}\partial_r \left\{ 2\gamma \frac{r^{\gamma/2-1}}{n(r)^{1/4}} + r^{\gamma/2}\frac{\dot{n}(r)}{n(r)^{5/4}} \right\} f_{jl}(r).\end{aligned}$$

Here we have used that $f_{jl}(r)$ satisfies

$$\partial_r(r^\gamma \partial_r f_{jl}) + k^2 n(r) r^\gamma f_{jl} = 0$$

so

$$[r^{2\gamma} n(r)]^{-1/4}\partial_r\{r^\gamma (f_{jl})'(r)\} = -k^2 r^{\gamma/2}(n(r))^{1/4} f_{jl}(r).$$

This means that

$$\ddot{X}_{ml}(\eta) + k^2 X_{ml}(\eta) = \frac{1}{4} n(r)^{-1/2}\partial_r \left\{ 2\gamma \frac{r^{\gamma/2-1}}{n(r)^{1/4}} + r^{\gamma/2}\frac{\dot{n}(r)}{n(r)^{5/4}} \right\} f_{jl}(r).$$

The right hand side is

$$\begin{aligned}&\frac{1}{4} n(r)^{-1/2}\partial_r \left\{ 2\gamma \frac{r^{\gamma/2-1}}{n(r)^{1/4}} + r^{\gamma/2}\frac{\dot{n}(r)}{n(r)^{5/4}} \right\} f_{jl}(r) \\ &= \left\{ \frac{m(m+1)}{r^2 n(r)} + \frac{1}{4}\frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16}\frac{(\dot{n}(r))^2}{(n(r))^3} \right\} [r^{2\gamma} n(r)]^{1/4} f_{jl}(r) \\ &= \left\{ \frac{m(m+1)}{r^2 n(r)} + \frac{1}{4}\frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16}\frac{(\dot{n}(r))^2}{(n(r))^3} \right\} X_{ml}(\eta) \\ &= \left\{ p_m(\eta) + \frac{m(m+1)}{\eta^2} \right\} X_{ml}(\eta).\end{aligned}$$

It follows that $X_{ml}(\eta)$ satisfies the equation

$$\ddot{X}_{ml}(\eta) + k^2 X_{ml}(\eta) = \left\{ p_m(\eta) + \frac{m(m+1)}{\eta^2} \right\} X_{ml}(\eta)$$

or

$$-\ddot{X}_{ml}(\eta) + \left(p_m(\eta) + \frac{m(m+1)}{\eta^2} \right) X_{ml}(\eta) = k^2 X_{ml}(\eta).$$

We now see that $X_{ml}(\eta)$ satisfies

$$-\ddot{X}_{ml}(\eta) + \left(\frac{m(m+1)}{\eta^2} + p_m(\eta)\right) X_{ml}(\eta) = k^2 X_{ml}(\eta) \text{ for } \eta \in [0, C].$$

■

In order for the results of the rest of this chapter to hold the function $p_m(\eta)$ must be bounded for all $m \geq 0$.

Lemma 7.2.0.13 *For a radial acoustic profile $c(r)$ the coefficient function*

$$p_m(\eta) = \frac{1}{4}\frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16}\frac{(\dot{n}(r))^2}{(n(r))^3} + m(m+1)\left(\frac{1}{r^2 n(r)} - \frac{1}{\eta^2}\right) \tag{7.9}$$

with

$$\eta(r) = \int_0^r \sqrt{n(s)}\, ds \tag{7.10}$$

and $n(r) = c^{-2}(r)$ is bounded on the interval $[0, C]$, $C = \eta(1)$.

Proof. First, notice that since $c(r)$ is an acoustic profile it must be smooth, bounded, even in r, and bounded away from zero. Thus, $n(r)$ is smooth, bounded, even in r, and bounded away from zero. Moreover all derivatives of $c(r)$ and $n(r)$ must also be bounded. This implies that the first two terms in $p_m(\eta)$ are bounded on $[0, C]$. It remains to show that the term

$$\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)}$$

is bounded on $[0, C]$. For this we need only show

$$\lim_{r \to 0^+} \left| \frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)} \right| < \infty.$$

For any function, $f(r)$, with bounded $(n+1)^{st}$-derivative on $[0, r]$ we may use integration by parts to write

$$f(r) = \sum_{k=0}^{n} f^{(k)}(0)\frac{r^k}{k!} + \frac{1}{n!}\int_0^r (r-t)^n f^{(n+1)}(t)\, dt.$$

Applying this to $\eta^2(r)$ and $c^2(r)$ we see

$$\eta^2(r) = r^2 n(0) + \frac{1}{6}\int_0^r (r-t)^3 \left[\frac{d^4}{dr^4}\eta^2(r)\right]_{r=t} dt$$
$$c^2(r) = c^2(0) + r^2 c(0)c''(0) + \frac{1}{2}\int_0^r (r-t)^2 \left[\frac{d^3}{dr^3}c^2(r)\right]_{r=t} dt$$

where we have used $n'(0) = 0$, $c'(0) = 0$ since $c(r)$ is even in r. Using the mean value theorem for integrals this yields

$$\eta^2(r) = r^2 n(0) + \frac{r^4}{24}\left[\frac{d^4}{dr^4}\eta^2(r)\right]_{r=a} \quad \text{for some } 0 < a < r$$
$$c^2(r) = c^2(0) + r^2 c(0)c''(0) + \frac{r^3}{6}\left[\frac{d^3}{dr^3}c^2(r)\right]_{r=b} \quad \text{for some } 0 < b < r.$$

Upon calculation of these derivatives we see

$$\eta^2(r) = r^2 n(0) + r^4 D_4(a) \text{ for some } 0 < a < r$$
$$c^2(r) = c^2(0) + r^2 c(0)c''(0) + r^3 R_3(b) \text{ for some } 0 < b < r$$

with

$$D_4(a) = \left[\frac{\ddot{n}(a)}{6} - \frac{(\dot{n}(a))^2}{48n(a)} + \eta(a)\left(\frac{\dddot{n}(a)}{48n^3(a)} - \frac{\dot{n}(a)}{32(n(a))^{5/2}}\right)\right]$$
$$R_3(b) = c'(b)c''(b) + \frac{1}{3}c(b)c'''(b).$$

Both of the remainder terms $D_4(a)$ and $R_3(b)$ are bounded on $0 < a < 1$ and $0 < b < 1$ respectively since all derivatives of $c(r)$ are bounded on $[0,1]$ and $c(r)$ is bounded away from zero.

We use the above expansions to evaluate the limit. For each $0 < r < 1$

$$\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)} = \frac{\eta^2(r) - r^2 n(r)}{\eta^2(r) r^2 n(r)}$$

and

$$\frac{\eta^2(r) - r^2 n(r)}{r^2 n(r)} = n(0)\left(\frac{1}{n(r)} - \frac{1}{n(0)}\right) + r^2 \frac{D_4(a)}{n(r)}$$
$$= n(0)(c^2(r) - c^2(0)) + r^2 \frac{D_4(a)}{n(r)}$$
$$= r^2 \frac{c''(0)}{c(0)} + r^2 \frac{D_4(a)}{n(r)} + r^3 n(0) R_3(b)$$

for some $0 < a,\, b < 1$. Thus,

$$\begin{aligned}\lim_{r\to 0^+}\left|\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)}\right| &= \lim_{r\to 0^+}\left|\frac{r^2\frac{c''(0)}{c(0)} + r^2\frac{D_4(a)}{n(r)} + r^3 n(0) R_3(b)}{r^2 n(0) + r^4 D_4(a)}\right| \\ &= \lim_{r\to 0^+}\left|\frac{\frac{c''(0)}{c(0)} + \frac{D_4(a)}{n(r)} + r n(0) R_3(b)}{n(0) + r^2 D_4(a)}\right| \\ &= \left|c''(0)c(0) + \frac{D_4(0)}{n^2(0)}\right|.\end{aligned}$$

We have now shown that the term $\left(\frac{1}{r^2 n(r)} - \frac{1}{\eta^2(r)}\right)$ is bounded on $[0,1]$ and therefore $p_m(\eta)$ is bounded for $m \geq 0$ on $[0, C]$.

■

We will use $X_{ml}(\eta)$ to denote a solution of equation (7.7). As in the previous chapter the fact that the coefficient functions $f_{jl}(r)$ come from the spherical harmonic expansion of a solution to the wave equation imposes a boundary condition on the $X_{ml}(\eta)$ at $\eta = 0$.

Lemma 7.2.0.14 *If the solution, $w(r,\theta)$, of (7.3) is smooth at $r = 0$ then the coefficient functions, $f_{jl}(r)$, are bounded at $r = 0$ for all $j,\, l \geq 0$.*

Proof. Since

$$w(r,\theta) = \sum_{j=0}^{\infty}\sum_{l=0}^{M} f_{jl}(r) r^j Y_{jl}(\theta)$$

and the $\{Y_{jl}(\theta)\}$ form an orthonormal basis for $L^2(S_1(0))$ we have

$$\int_{S_1(0)} w(r,\theta) Y_{jl}(\theta)\, d\theta = f_{jl}(r) r^j.$$

Because $w(r,\theta) = w(x)$ is smooth at $x = 0$ we may form it's Taylor expansion up to order $j-1$ at $x = 0$ for $j \geq 1$. This has the form

$$w(r,\theta) = \sum_{|\alpha|\leq j-1} \frac{x^\alpha}{\alpha!}\left[\partial_x^\alpha w(x)\right]_{x=0} + R_j(x)$$

where we have used the multi-index notation for mixed partial derivatives of order less than or equal to $j-1$. The term $R_j(x)$ is the remainder term and satisfies

$$\lim_{|x|\to 0^+} \frac{R_j(x)}{|x|^j} < \infty.$$

The harmonic $Y_{jl}(\theta)$ is orthogonal to any polynomial of degree less than j so

$$\int_{S_1(0)} Y_{jl}(\theta) \sum_{|\alpha|\le j-1} \frac{x^\alpha}{\alpha!} \left[\partial_x^\alpha w(x)\right]_{x=0} \, d\theta = 0.$$

This implies

$$\int_{S_1(0)} w(r,\theta) Y_{jl}(\theta)\, d\theta = \int_{S_1(0)} Y_{jl}(\theta) R_j(x)\, d\theta$$

and we may conclude that

$$\lim_{r\to 0^+} r^{-j} \int_{S_1(0)} w(r,\theta) Y_{jl}(\theta)\, d\theta < \infty.$$

However, this shows $f_{jl}(r)$ must be bounded at $r=0$ for all $j,\, l \ge 0$.

■

Lemma 7.2.0.15 *In order for the solution, $w(r,\theta)$, of (7.3) to be smooth at $r=0$ the solutions, $X_{ml}(\eta)$, of (7.7) must satisfy the initial condition*

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_{ml}(\eta) < \infty \tag{7.11}$$

for all m and for each l.

Proof. By definition $\frac{\gamma}{2} = m+1$. From the change of variables defining $X_{ml}(\eta)$ it follows that

$$\begin{aligned} X_{ml}(\eta) = [r^{2\gamma} n(r)]^{1/4} f_{jl}(r) &= r^{\gamma/2} n(r)^{1/4} f_{jl}(r) \\ &= r^{m+1} n(r)^{1/4} f_{jl}(r). \end{aligned}$$

This implies that

$$\lim_{r\to 0} r^{-(m+1)} X_{ml}(\eta(r)) < \infty$$

by lemma 7.2.0.14 and the fact that $n(r)$ is bounded at $r = 0$. But $\eta(r) = \int_0^r \sqrt{n(s)}\, ds$ so

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_{ml}(\eta) = \lim_{r\to 0} r^{-(m+1)} X_{ml}(\eta(r)) < \infty.$$

■

Lemma 7.2.0.15 is important since it shows the necessary initial condition that solutions of interest to us must satisfy. This boundary condition also allows us to remove the dependence on the order of spherical harmonic from the coefficient functions $f_{jl}(r)$. The fundamental solutions to (7.7) satisfy specific boundary conditions at $\eta = 0$. This has been investigated previously in [20, 21, 39] and we state the results here.

The set of solutions to

$$\ddot{X}_m(\eta, k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta, k) = p_m(\eta) X_m(\eta, k) \text{ for } \eta \in [0, C]$$

is spanned by two fundamental solutions, say $X_{m1}(\eta, k)$ and $X_{m2}(\eta, k)$, satisfying the boundary conditions [20, 21, 39]

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_{m1}(\eta, k) = 1 \tag{7.12}$$

$$\lim_{\eta\to 0} \eta^{m} X_{m2}(\eta, k) = 1. \tag{7.13}$$

This means the solution $X_{m2}(\eta, k)$ has a singularity of order m at $\eta = 0$ while $X_{m1}(\eta, k)$ has a zero of order $(m + 1)$ at $\eta = 0$. From lemma 7.2.0.15 we see that $X_{m1}(\eta, k)$ will be related to the reformulation of the ITP since it satisfies the proper boundary condition. That is, if

$$X_{ml}(\eta, k) = [r^{2\gamma} n(r)]^{1/4} f_{jl}(r)$$

then we must have $\lim_{\eta\to 0^+} \eta^{-(m+1)} X_{ml}(\eta) < \infty$ and therefore there exists a constant a_{jl} such that

$$X_{ml}(\eta, k) = \tilde{a}_{jl} X_{m1}(\eta, k).$$

However, this implies that at each degree of spherical harmonic, $j \geq 0$, the coefficient functions may be written as

$$f_{jl}(r) = a_{jl} f_j(r)$$

for some constants a_{jl} and $f_j(r)$ satisfying

$$\partial_r(r^\gamma \partial_r f_j) + k^2 n(r) r^\gamma f_j = 0 \text{ on } [0, 1].$$

Now we may define

$$X_m(\eta, k) = a_m X_{m1}(\eta, k) = [r^{2\gamma} n(r)]^{1/4} f_j(r)$$

which satisfies

$$-\ddot{X}_m(\eta, k) + \left(\frac{m(m+1)}{\eta^2} + p_m(\eta)\right) X_m(\eta, k) = k^2 X_m(\eta, k) \text{ for } \eta \in [0, C] \quad (7.14)$$
$$\lim_{\eta \to 0^+} \eta^{-(m+1)} X_m(\eta, k) < \infty$$

with

$$p_m(\eta) = \frac{1}{4} \frac{\ddot{n}(r)}{(n(r))^2} - \frac{5}{16} \frac{(\dot{n}(r))^2}{(n(r))^3} + m(m+1) \left(\frac{1}{r^2 n(r)} - \frac{1}{\eta^2}\right)$$

and $C = \int_0^1 \sqrt{n(s)}\, ds$. It then follows that

$$X_{ml}(\eta, k) = a_{jl} X_m(\eta, k)$$

and the expansion for $w(r, \theta)$ is then written as

$$w(r, \theta) = \sum_{j=0}^{\infty} \sum_{l=0}^{M} a_{jl} f_j(r) r^j Y_{jl}(\theta).$$

7.3 Derivation of the Determinant Condition

We derive a one dimensional determinant form of the boundary conditions involved in the ITP (7.2). Following the organization of the previous two chapters, a one dimensional formulation of the ITP in terms of the Sturm-Liouville equation (7.14) will be justified.

Theorem 7.3.0.14 *For two radially symmetric acoustic profiles $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^n$ the transmission spectrum is a subset of the set of all $k \in \mathbb{R}_+$ such that, for some $m = j+\frac{1}{2}(n-3)$, there exists non-trivial solutions $(X_m(\eta,k), Z_m(\xi,k)) \in C^2((0,C]) \times C^2((0,B])$ satisfying*

$$\ddot{X}_m(\eta,k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta,k) = p_{1m}(\eta) X_m(\eta,k),\ \eta \in [0,C] \tag{7.15}$$
$$\ddot{Z}_m(\xi,k) + \left(k^2 - \frac{m(m+1)}{\xi^2}\right) Z_m(\xi,k) = p_{2m}(\xi) Z_m(\xi,k),\ \xi \in [0,B]$$
$$\lim_{\eta\to 0} \eta^{-(m+1)} X_m(\eta,k) < \infty, \quad \lim_{\xi\to 0} \xi^{-(m+1)} Z_m(\xi,k) < \infty$$
$$X_m(C,k) = Z_m(B,k), \quad \dot{X}_m(C,k) = \dot{Z}_m(B,k).$$

Here we restate the definitions for convenience. Recall that $n_c(r) = c^{-2}(r)$, $n_b(r) = b^{-2}(r)$ and

$$p_{1m}(\eta) = \frac{1}{4}\frac{\ddot{n}_c(r)}{(n_c(r))^2} - \frac{5}{16}\frac{(\dot{n}_c(r))^2}{(n_c(r))^3} + m(m+1)\left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right), \tag{7.16}$$

$$p_{2m}(\xi) = \frac{1}{4}\frac{\ddot{n}_b(r)}{(n_b(r))^2} - \frac{5}{16}\frac{(\dot{n}_b(r))^2}{(n_b(r))^3} + m(m+1)\left(\frac{1}{r^2 n_b(r)} - \frac{1}{\xi^2(r)}\right). \tag{7.17}$$

The right endpoints are defined by $C = \int_0^1 \sqrt{n_c(s)}\, ds$ and $B = \int_0^1 \sqrt{n_b(s)}\, ds$.

Proof. If $k \in \mathbb{R}_+$ is a transmission eigenvalue then there exists non-trivial solutions, $\{U(x,k), V(x,k)\}$, corresponding to (7.2). We write

$$w_1(r,\theta) = U(x,k)$$
$$w_2(r,\theta) = V(x,k)$$

from which the expansions

$$w_1(r,\theta) = \sum_{j=1}^{\infty}\sum_{l=-j}^{j} a_{j1}^l f_{j1}(r) r^j Y_{jl}(\theta)$$

and

$$w_2(r,\theta) = \sum_{j=1}^{\infty}\sum_{l=-j}^{j} a_{j2}^l f_{j2}(r) r^j Y_{jl}(\theta)$$

follow.

Since $U(1,k) = V(1,k)$ and $\partial_r U(1,k) = \partial_r V(1,k)$ and all of the spherical harmonics, $\{Y_{jl}(\theta)\}$, are linearly independent we must have

$$a^l_{j1} f_{j1}(1) = a^l_{j2} f_{j2}(1)$$
$$a^l_{j1} f'_{j1}(1) = a^l_{j2} f'_{j2}(1)$$

for all $j \in \mathbb{N}$ and $-j \leq l \leq j$. This clearly implies

$$f_{j1}(1) = f_{j2}(1)$$
$$f'_{j1}(1) = f'_{j2}(1)$$

for all $j \in \mathbb{N}$. Also, we have assumed that $U(x,k)$ and $V(x,k)$ are nonzero so there must exist at least one $j \in \mathbb{N}$ such that $f_{j1}(r)$ and $f_{j2}(r)$ are nonzero.

By lemma 7.2.0.12 the functions defined by

$$\eta = \int_0^r \sqrt{n_c(s)}\, ds,\ X_m(\eta) = [r^{2\gamma} n_c(r)]^{1/4} f_{j1}(r),$$

and

$$\xi = \int_0^r \sqrt{n_b(s)}\, ds,\ Z_m(\xi) = [r^{2\gamma} n_b(r)]^{1/4} f_{j2}(r)$$

satisfy the differential equations in (7.15). The growth conditions at the left endpoints are assured by lemma 7.2.0.15.

It remains to show that the conditions at the right endpoint are satisfied. First observe

$$\begin{aligned} X_m(C) &= X_m(\eta(1)) = [n_c(1)]^{1/4} f_{j1}(1) = f_{j1}(1) \\ &= f_{j2}(1) = [n_b(1)]^{1/4} f_{j2}(1) = Z_m(\xi(1)) = Z_m(B). \end{aligned}$$

Now recall, from the proof of lemma 7.2.0.12, that

$$\dot{X}_m(\eta) = \frac{2\gamma r^{2\gamma-1} n(r) + r^{2\gamma} \dot{n}(r)}{4 r^{3\gamma/2} n(r)^{5/4}} f_{j1}(r) + \frac{r^{\gamma/2} f'_{j1}(r)}{n(r)^{1/4}}.$$

A similar expression also holds for $\dot{Z}_m(\xi)$. Evaluating this at $r = 1$ we see

$$\begin{aligned}\dot{X}_m(C) = \dot{X}_m(\eta(1)) &= \frac{\gamma}{2} f_{j1}(1) + f'_{j1}(1) \\ &= \frac{\gamma}{2} f_{j2}(1) + f'_{j2}(1) = \dot{Z}_m(\xi(1)) = \dot{Z}_m(B).\end{aligned}$$

We have now demonstrated that (7.15) is satisfied for some m if k is a transmission eigenvalue.

■

Theorem 7.3.0.15 *For two radially symmetric acoustic speeds $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^n$, $k \in \mathbb{R}_+$ is a transmission eigenvalue only if*

$$\begin{aligned}&W(Z_{m1}(B,k), X_{m1}(C,k)) = \\ &\quad Z_{m1}(B,k)\dot{X}_{m1}(C,k) - \dot{Z}_{m1}(B,k)X_{m1}(C,k) = 0\end{aligned} \tag{7.18}$$

for some $m = j + \frac{1}{2}(n-3)$.

Here $X_{m1}(\eta, k)$ and $Z_{m1}(\xi, k)$ are fundamental solutions satisfying the differential equations in (7.15) corresponding to $c(r)$ and $b(r)$ respectively. They also satisfy the decay conditions

$$\begin{aligned}\lim_{\eta\to 0} \eta^{-(m+1)} X_{m1}(\eta, k) = 1 \\ \lim_{\xi\to 0} \xi^{-(m+1)} Z_{m1}(\xi, k) = 1.\end{aligned}$$

Proof. Suppose $k \in \mathbb{R}_+$ is a transmission eigenvalue relative to the domain $B_1(0)$ with sound speeds $c(r)$ and $b(r)$. Then, by theorem 7.3.0.14, there exists some $m = j + \frac{1}{2}(n - 3)$ so that there are non-trivial solutions $(X_m(\eta, k), Z_m(\xi, k)) \in C^2((0, C]) \times C^2((0, B])$ satisfying the ITP (7.15).

These solutions must satisfy the decay conditions

$$\begin{aligned}\lim_{\eta\to 0} \eta^{-(m+1)} X_m(\eta, k) < \infty, \\ \lim_{\xi\to 0} \xi^{-(m+1)} Z_m(\xi, k) < \infty.\end{aligned}$$

The set of solutions to

$$\ddot{X}_m(\eta, k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta, k) = p_{1m}(\eta) X_m(\eta, k) \text{ for } \eta \in [0, C]$$

is spanned by the fundamental solutions $X_{m1}(\eta, k)$ and $X_{m2}(\eta, k)$ satisfying

$$\lim_{\eta \to 0} \eta^{-(m+1)} X_{m1}(\eta, k) = 1$$

and

$$\lim_{\eta \to 0} \eta^m X_{m2}(\eta, k) = 1.$$

Similarly for the equation defining $Z_m(\xi, k)$. Therefore, there must exist constants a_1, $a_2 \in \mathbb{R}$ such that $X_m(\eta, k) = a_1 X_{m1}(\eta, k)$ and $Z_m(\xi, k) = a_2 Z_{m1}(\xi, k)$.

The boundary conditions

$$X_m(C, k) = Z_m(B, k) \text{ and } \dot{X}_m(C, k) = \dot{Z}_m(B, k)$$

then imply

$$a_1 X_{m1}(C, k) = a_2 Z_{m1}(B, k)$$
$$a_1 \dot{X}_{m1}(C, k) = a_2 \dot{Z}_{m1}(B, k).$$

In matrix form this becomes

$$\begin{bmatrix} X_{m1}(C, k) & -Z_{m1}(B, k) \\ \dot{X}_{m1}(C, k) & \dot{Z}_{m1}(B, k) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This can only be satisfied for nonzero a_1, a_2 if

$$\det \begin{bmatrix} X_{m1}(C, k) & -Z_{m1}(B, k) \\ \dot{X}_{m1}(C, k) & \dot{Z}_{m1}(B, k) \end{bmatrix} = 0.$$

The determinant condition follows when we rewrite the determinant as

$$W(Z_{m1}(B, k), X_{m1}(C, k)) =$$
$$Z_{m1}(B, k)\dot{X}_{m1}(C, k) - \dot{Z}_{m1}(B, k) X_{m1}(C, k) = 0.$$

■

Later on we will often use the shorter notation

$$W(Z_{m1}(B), X_{m1}(C)) = \\ Z_{m1}(B)\dot{X}_{m1}(C) - \dot{Z}_{m1}(B)X_{m1}(C) = 0$$

or, when it is obvious from context,

$$W(Z_m(B), X_m(C)) = \\ Z_m(B)\dot{X}_m(C) - \dot{Z}_m(B)X_m(C) = 0$$

where it will be understood that X_m and Z_m are the fundamental solutions with the proper decay condition at the left endpoint.

7.4 Derivation of Solutions

We develop a representation of solutions to the initial value problem

$$\ddot{X}_m(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta) = p_m(\eta)X_m(\eta) \quad \text{for} \quad \eta \in [0, C] \qquad (7.19)$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} X_m(\eta) = 1.$$

To begin studying (7.19) we first look at solutions to the unperturbed equation

$$\ddot{X}_m(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta) = 0 \text{ for } \eta \in [0, C]. \qquad (7.20)$$

Lemma 7.4.0.16 *Assume $n \geq 3$ is odd and let $m = j + \frac{1}{2}(n-3)$. Then there exists two linearly independent solutions to (7.20), $\phi_m(\eta)$ and $\psi_m(\eta)$, satisfying the initial conditions*

$$\lim_{\eta \to 0} \eta^{-(m+1)} \phi_m(\eta) = 1 \qquad (7.21)$$

and

$$\lim_{\eta \to 0} \eta^m \psi_m(\eta) = \frac{1}{2m+1}. \tag{7.22}$$

The wronskian of these two solutions is given by

$$W(\phi_m, \psi_m)(\eta) = 1. \tag{7.23}$$

Proof. The solutions of this equation are obtained from transformations of solutions to Bessel's equation, [20, 21, 39]

$$x^2 \ddot{w}(x) + x\dot{w}(x) + (x^2 - \nu)w(x) = 0. \tag{7.24}$$

Bessel's equation has two linearly independent solutions given by $J_\nu(x)$ and $N_\nu(x)$, Bessel functions of order ν of the first and second kind, respectively. These are both real valued smooth functions on $(0, \infty)$. The functions $J_\nu(x)$ are bounded at $x = 0$ while $N_\nu(x)$ becomes unbounded at $x = 0$.

Let $\nu = m + \frac{1}{2} = j + \frac{n}{2} - 1$ and define the functions

$$\phi_m(\eta) = \sqrt{k\eta}\, J_{m+\frac{1}{2}}(k\eta) \text{ and } \psi_m(\eta) = \sqrt{k\eta}\, N_{m+\frac{1}{2}}(k\eta).$$

Both ϕ_m and ψ_m satisfy (7.20), to see this we calculate the derivatives

$$\phi_m'(\eta) = \frac{\sqrt{k}}{2\sqrt{\eta}} J_{m+\frac{1}{2}}(k\eta) + k\sqrt{k\eta}\, J'_{m+\frac{1}{2}}(k\eta)$$

$$\phi_m''(\eta) = -\frac{\sqrt{k}}{4\eta^{3/2}} J_{m+\frac{1}{2}}(k\eta) + \frac{k^{3/2}}{\sqrt{\eta}} J'_{m+\frac{1}{2}}(k\eta) + k^{5/2}\sqrt{\eta}\, J''_{m+\frac{1}{2}}(k\eta).$$

Next notice $m(m+1) = (\nu - \frac{1}{2})(\nu + \frac{1}{2}) = \nu^2 - \frac{1}{4}$ so we can compute

$$\begin{aligned}
&\phi''_m(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\phi_m(\eta) \\
&= \phi''_m(\eta) + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)\phi_m(\eta) \\
&= -\frac{\sqrt{k}}{4\eta^{3/2}} J_{m+\frac{1}{2}}(k\eta) + \frac{k^{3/2}}{\sqrt{\eta}} J'_{m+\frac{1}{2}}(k\eta) \\
&\quad + k^{5/2}\sqrt{\eta}\, J''_{m+\frac{1}{2}}(k\eta) + \left(k^2 - \frac{\nu^2 - \frac{1}{4}}{\eta^2}\right)\sqrt{k\eta}\, J_{m+\frac{1}{2}}(k\eta) \\
&= \frac{\sqrt{k}}{\eta^{3/2}}\{(k\eta)^2 J''_{m+\frac{1}{2}}(k\eta) + k\eta\, J'_{m+\frac{1}{2}}(k\eta) + ((k\eta)^2 - \nu^2)\, J_{m+\frac{1}{2}}(k\eta)\} = 0
\end{aligned}$$

since $J_\nu(x)$ is a solution to (7.24). A similar process shows $\psi_m(\eta)$ also satisfies (7.20).

Our goal is to normalize the solutions ϕ_m and ψ_m to simplify later computations and to show that the correct decay and growth conditions are satisfied at $\eta = 0$. First, we call to attention that, we have assumed $n \geq 3$ is odd so m is an integer and $\nu = m + 1/2$ is not an integer. For integer m the spherical Bessel functions of first and second kind are defined by

$$j_m(x) = \sqrt{\frac{\pi}{2x}} J_{m+\frac{1}{2}}(x) \text{ and } y_m(x) = \sqrt{\frac{\pi}{2x}} N_{m+\frac{1}{2}}(x).$$

If we normalize and set $\phi_m(\eta) = \sqrt{\frac{\pi}{2}}\sqrt{k\eta}\, J_{m+\frac{1}{2}}(k\eta)$ and $\psi_m(\eta) = \sqrt{\frac{\pi}{2}}\sqrt{k\eta}\, N_{m+\frac{1}{2}}(k\eta)$, our solutions can be written as

$$\phi_m(\eta) = k\eta\, j_m(k\eta) \text{ and } \psi_m(\eta) = k\eta\, y_m(k\eta).$$

The behavior of spherical Bessel functions at zero is well known [87] and given by

$$\begin{aligned}
\lim_{x\to 0} x^{-m} j_m(x) &= \frac{1}{1 \cdot 3 \cdot 5 \ldots (2m+1)} \\
\lim_{x\to 0} x^{m+1} y_m(x) &= 1 \cdot 3 \cdot 5 \ldots (2m-1).
\end{aligned}$$

Define the normalization constant $\gamma_m = 1 \cdot 3 \cdot 5 \ldots (2m+1)$ and normalize our solutions so that

$$\phi_m(\eta) = k^{-m}\gamma_m \eta\, j_m(k\eta) \text{ and } \psi_m(\eta) = k^{m+1}\gamma_m^{-1}\eta\, y_m(k\eta).$$

Then

$$\lim_{\eta \to 0} \eta^{-(m+1)} \phi_m(\eta) = 1$$
$$\lim_{\eta \to 0} \eta^m \psi_m(\eta) = \frac{1}{2m+1}.$$

To demonstrate the condition on the wronskian note that $W(j_m, y_m)(x) = x^{-2}$, [87] from which it follows

$$\begin{aligned} W(\phi_m, \psi_m)(\eta) &= \phi_m(\eta)\psi_m'(\eta) - \phi_m'(\eta)\psi_m(\eta) \\ &= k^2\eta^2(j_m(k\eta)y_m'(k\eta) - j_m'(k\eta)y_m(k\eta)) = 1. \end{aligned}$$

■

From now on we will refer to the solutions derived in lemma 7.4.0.16 as $\phi_m(\eta)$ and $\psi_m(\eta)$. They will be defined by

$$\phi_m(\eta) = k^{-m}\gamma_m \eta\, j_m(k\eta) \tag{7.25}$$

and

$$\psi_m(\eta) = k^{m+1}\gamma_m^{-1} \eta\, y_m(k\eta) \tag{7.26}$$

where $\gamma_m = 1 \cdot 3 \cdot 5 \ldots (2m+1)$.

Lemma 7.4.0.17 *The solution of*

$$\ddot{X}_m(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta) = p_m(\eta) X_m(\eta) \text{ for } \eta \in [0, C] \tag{7.27}$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} X_m(\eta) = 1$$

satisfies the integral equation

$$X_m(\eta) = \phi_m(\eta) + \int_0^\eta G_m(\eta, t) p_m(t) X_m(t)\, dt \tag{7.28}$$

with

$$G_m(\eta, t) = \psi_m(\eta)\phi_m(t) - \psi_m(t)\phi_m(\eta). \tag{7.29}$$

Proof. Suppose $X_m(\eta)$ satisfies (7.27). Denote the right hand side of (7.28) as $R(\eta)$. Differentiating the right hand side of (7.28) we see

$$\begin{aligned}
R'(\eta) &= \frac{\partial}{\partial\eta}\left(\phi_m(\eta) + \int_0^\eta G_m(\eta,t)p_m(t)X_m(t)\,dt\right)\\
&= \phi_m'(\eta) + \int_0^\eta \partial_\eta G_m(\eta,t)p_m(t)X_m(t)\,dt\\
&= \phi_m'(\eta) + \int_0^\eta (\psi_m'(\eta)\phi_m(t) - \psi_m(t)\phi_m'(\eta))p_m(t)X_m(t)\,dt.
\end{aligned}$$

Differentiating a second time

$$\begin{aligned}
R''(\eta) &= \phi_m''(\eta) + \int_0^\eta (\psi_m''(\eta)\phi_m(t) - \psi_m(t)\phi_m''(\eta))p_m(t)X_m(t)\,dt\\
&\quad +(\psi_m'(\eta)\phi_m(\eta) - \psi_m(\eta)\phi_m'(\eta))p_m(\eta)X_m(\eta)\\
&= \phi_m''(\eta) + \psi_m''(\eta)\int_0^\eta \phi_m(t)p_m(t)X_m(t)\,dt\\
&\quad -\phi_m''(\eta)\int_0^\eta \psi_m(t)p_m(t)X_m(t)\,dt + W(\phi_m,\psi_m)(\eta)p_m(\eta)X_m(\eta)\\
&= \phi_m''(\eta) + p(\eta)X_m(\eta) + \psi_m''(\eta)\int_0^\eta \phi_m(t)p_m(t)X_m(t)\,dt\\
&\quad -\phi_m''(\eta)\int_0^\eta \psi_m(t)p_m(t)X_m(t)\,dt.
\end{aligned}$$

Now if we plug $R(\eta)$ into the differential equation we see

$$\begin{aligned}
&R''(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)R(\eta)\\
&= \phi_m''(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\phi_m(\eta) + p_m(\eta)X_m(\eta)\\
&\quad - \left(\phi_m''(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\phi_m(\eta)\right)\int_0^\eta \psi_m(t)p_m(t)X_m(t)\,dt\\
&\quad + \left(\psi_m''(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\psi_m(\eta)\right)\int_0^\eta \phi_m(t)p_m(t)X_m(t)\,dt\\
&= p_m(\eta)X_m(\eta).
\end{aligned}$$

If we evaluate the limit condition with $R(\eta)$ we see

$$\begin{aligned}
\lim_{\eta\to 0}\eta^{-(m+1)}R(\eta) &= \lim_{\eta\to 0}\eta^{-(m+1)}\phi_m(\eta)\\
&\quad + \lim_{\eta\to 0}\eta^{-(m+1)}\left(\int_0^\eta G_m(\eta,t)p_m(t)X_m(t)\,dt\right) = 1.
\end{aligned}$$

We have shown that $R(\eta)$ satisfies

$$R''(\eta) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) R(\eta) = p_m(\eta) X_m(\eta) \text{ for } \eta \in [0, C]$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} R(\eta) = 1.$$

Since this initial value problem has a unique solution we see $R(\eta) = X_m(\eta)$ and therefore $X_m(\eta)$ saitisfies

$$X_m(\eta) = \phi_m(\eta) + \int_0^\eta G_m(\eta, t) p_m(t) X_m(t)\, dt.$$

■

Note that in the above lemma we do not explicitly write the dependence of ϕ_m, ψ_m, and $G_m(\eta, t)$ on $k \in \mathbb{R}_+$. However, it should be noted that all of these functions do still depend on the eigenvalue k. In the following theorem the dependence on k is more relevant so we will change the notation accordingly.

Theorem 7.4.0.16 *For $m = j + \frac{1}{2}(d-3)$ with $d \geq 3$, odd, the solution, $X_m(\eta, k)$, of (7.27) can be expressed as*

$$X_m(\eta, k) = C_0(\eta, k, p_m) + \sum_{n \geq 1} C_n(\eta, k, p_m). \tag{7.30}$$

The terms $C_n(\eta, k, p_m)$ are n^{th} order multilinear forms in p_m satisfying

$$C_0(\eta, k, p_m) = \phi_m(\eta, k) \tag{7.31}$$
$$\ddot{C}_n + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) C_n = p_m(\eta) C_{n-1} \text{ for } n \geq 1 \tag{7.32}$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} C_n(\eta, k, p_m) = 0 \text{ for } n \geq 1. \tag{7.33}$$

These forms also satisfy the integral equations

$$C_n(\eta, k, p_m) = \int_0^\eta G_m(\eta, t; k) p_m(t) C_{n-1}(t, k, p_m)\, dt \tag{7.34}$$

with

$$G_m(\eta, t; k) = k\, \eta\, t \Gamma_m(k\eta, kt). \tag{7.35}$$

and

$$\Gamma_m(\eta, t) = y_m(\eta) j_m(t) - y_m(t) j_m(\eta). \tag{7.36}$$

Proof. This treatment follows the methods of [65]. Informally, we set

$$X_m(\eta, k) = C_0(\eta, k, p_m) + \sum_{n \geq 1} C_n(\eta, k, p_m)$$

where each $C_n(\eta, k, p_m)$ is a multi-linear form of degree n with respect to the function $p_m(\eta)$. To derive expressions for the terms in the above sum we first suppose $C_0(\eta, k, p_m)$ satisfies

$$\ddot{C}_0 + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) C_0 = 0$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} C_0(\eta, k, p_m) = 1.$$

This assumption implies we must have $C_0(\eta, k, p_m) = \phi_m(\eta, k)$. Thus, in order that $X_m(\eta, k)$ satisfy (7.27) we must have

$$\begin{aligned}
&\ddot{X}_m(\eta, k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta, k) \\
&= \left(\frac{\partial^2}{\partial \eta^2} + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\right) \left\{ C_0(\eta, k, p_m) + \sum_{n \geq 1} C_n(\eta, k, p_m) \right\} \\
&= \sum_{n \geq 1} \left\{ \left(\frac{\partial^2}{\partial \eta^2} + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)\right) C_n(\eta, k, p_m) \right\} \\
&= p_m(\eta) \sum_{n \geq 1} C_{n-1}(\eta, k, p_m) = p_m(\eta) X_m(\eta, k).
\end{aligned}$$

It follows that the $C_n(\eta, k, p_m)$ must satisfy

$$\ddot{C}_n + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) C_n = p_m(\eta) C_{n-1},$$
$$\lim_{\eta \to 0} \eta^{-(m+1)} C_n(\eta, k, p_m) = 0 \text{ for } n \geq 1.$$

Using variation of parameters this implies, for $n \geq 1$, the $C_n(\eta, k, p_m)$ must satisfy

$$C_n(\eta, k, p_m) = \int_0^\eta G_m(\eta, t; k) p_m(t) C_{n-1}(t, k, p_m) \, dt.$$

The kernel $G_m(\eta, t; k)$ simplifies,

$$\begin{aligned} G_m(\eta,t;k) &= \psi_m(\eta,k)\phi_m(t,k) - \psi_m(t,k)\phi_m(\eta,k) \\ &= k\,\eta\,t(y_m(k\eta)j_m(kt) - y_m(kt)j_m(k\eta)). \end{aligned}$$

Setting

$$\Gamma_m(\eta,t) = y_m(\eta)j_m(t) - y_m(t)j_m(\eta)$$

the theorem holds. To show that the sum in (7.30) actually converges we refer the reader to [65].

■

Theorem 7.4.0.17 *For $m = j + \frac{1}{2}(d-3)$ with $d \geq 3$, odd, the solutions of*

$$\ddot{X}_m(\eta,k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right) X_m(\eta,k) = p_m(\eta)X_m(\eta,k) \text{ for } \eta \in [0,C] \qquad (7.37)$$

$$\lim_{\eta\to 0} \eta^{-(m+1)} X_m(\eta,k,p_m) = 1$$

have asymptotic expansions

$$X_m(\eta,k) = \alpha_m \frac{\sin(k\eta)}{k^{(m+1)}} + \beta_m(\eta)\frac{\cos(k\eta)}{k^{(m+2)}} + \mathcal{O}\left(\frac{1}{k^{(m+3)}}\right) \qquad (7.38)$$

for $m \geq 0$ even and

$$X_m(\eta,k) = \alpha_m \frac{\cos(k\eta)}{k^{(m+1)}} + \beta_m(\eta)\frac{\sin(k\eta)}{k^{(m+2)}} + \mathcal{O}\left(\frac{1}{k^{(m+3)}}\right) \qquad (7.39)$$

for $m \geq 0$ odd. The coefficients in these expansions are given by

$$\alpha_m = \begin{cases} (-1)^{m/2}\gamma_m\Delta_m^1(0) & ,\ m \text{ even} \\ (-1)^{(m+1)/2}\gamma_m\Delta_m^1(0) & ,\ m \text{ odd} \end{cases} \qquad (7.40)$$

and

$$\beta_m(\eta) = \begin{cases} (-1)^{m/2}\gamma_m\Delta_m^1(0)\left[\eta^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)} - \frac{1}{2}(\Delta_m^1(0))^2\int_0^\eta p_m(t)\,dt\right] & ,\ m \text{ even} \\ (-1)^{(m+1)/2}\gamma_m\Delta_m^1(0)\left[\frac{1}{2}(\Delta_m^1(0))^2\int_0^\eta p_m(t)\,dt - \eta^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)}\right], & m \text{ odd.} \end{cases} \qquad (7.41)$$

Proof. The key to expanding each coefficient in the series (7.30) in terms of inverse powers of k is to work with trigonometric expansions of the spherical Bessel functions. To this end we define the coefficients

$$\begin{aligned} \Delta^1_m(l) &= \frac{(-1)^l (m+2l)!}{(2l)!\,(m-2l)!\,2^{2l}} \qquad (7.42)\\ \Delta^2_m(l) &= \frac{(-1)^l (m+2l+1)!}{(2l+1)!\,(m-2l-1)!\,2^{2l+1}},\ \Delta^2_0(0)=0, \end{aligned}$$

and the polynomials

$$\begin{aligned} A_m(z) &= \sum_{l=0}^{\lfloor m/2 \rfloor} z^{-(2l+1)} \Delta^1_m(l) \qquad (7.43)\\ B_m(z) &= \sum_{l=0}^{\lfloor (m-1)/2 \rfloor} z^{-(2l+2)} \Delta^2_m(l). \end{aligned}$$

Then, for $m \geq 0$, even, the spherical Bessel functions can be written as [87]

$$j_m(z) = (-1)^{m/2} \left\{ A_m(z) \sin z + B_m(z) \cos z \right\} \qquad (7.44)$$

$$y_m(z) = (-1)^{\frac{3m}{2}+1} \left\{ A_m(z) \cos z - B_m(z) \sin z \right\}. \qquad (7.45)$$

For $m \geq 0$, odd,

$$j_m(z) = (-1)^{\frac{m+1}{2}} \left\{ A_m(z) \cos z - B_m(z) \sin z \right\} \qquad (7.46)$$

$$y_m(z) = (-1)^{\frac{3}{2}(m+1)} \left\{ A_m(z) \sin z + B_m(z) \cos z \right\}. \qquad (7.47)$$

Using these forms it follows that

$$\begin{aligned} \Gamma_m(z,t) &= y_m(z) j_m(t) - y_m(t) j_m(z) \qquad (7.48)\\ &= \left\{ A_m(z) A_m(t) + B_m(z) B_m(t) \right\} \sin(z-t) \\ &\quad + \left\{ A_m(t) B_m(z) - B_m(t) A_m(z) \right\} \cos(z-t). \end{aligned}$$

Now consider the case $m \geq 0$, even. From equations (7.43)

$$\begin{aligned} A_m(z) &= z^{-1} \Delta^1_m(0) + \mathcal{O}(z^{-3}) \qquad (7.49)\\ B_m(z) &= z^{-2} \Delta^2_m(0) + \mathcal{O}(z^{-4}). \end{aligned}$$

Using these in (7.44) we see

$$j_m(k\eta)=(-1)^{m/2}\left\{\frac{1}{k\eta}\Delta_m^1(0)\sin(k\eta)+\frac{1}{k^2\eta^2}\Delta_m^2(0)\cos(k\eta)\right\}+\mathcal{O}(k^{-3}).$$

This is then used to develop the following asymptotic expansion

$$\begin{aligned}C_0(\eta,k,p_m) &= \frac{\gamma_m\eta}{k^m}j_m(k\eta) \\ &= \frac{(-1)^{m/2}\gamma_m}{k^m}\left\{\frac{1}{k}\Delta_m^1(0)\sin(k\eta)+\frac{1}{k^2\eta}\Delta_m^2(0)\cos(k\eta)\right\} \\ &\quad +\mathcal{O}\left(\frac{1}{k^{m+3}}\right).\end{aligned} \tag{7.50}$$

The term $C_1(\eta,k,p_m)$ is governed by the kernel $\Gamma_m(k\eta,kt)$. By (7.48) this has the asymptotic expression

$$\Gamma_m(k\eta,kt)=\frac{(\Delta_m^1(0))^2}{k^2\eta t}\sin(k(\eta-t))+\mathcal{O}\left(\frac{1}{k^3}\right). \tag{7.51}$$

Using the relation

$$C_1(\eta,k,p_m)=k\,\eta\int_0^\eta t\,\Gamma_m(k\eta,kt)p_m(t)C_0(t,k,p_m)\,dt \tag{7.52}$$

and the expansions (7.50) and (7.51) we see that the first order term has an expansion

$$\begin{aligned}C_1(\eta,k,p) &= \frac{(-1)^{m/2}\gamma_m(\Delta_m^1(0))^3}{k^{m+2}}\int_0^\eta \sin(k(\eta-t))\sin(kt)p_m(t)\,dt \\ &\quad +\mathcal{O}\left(\frac{1}{k^{m+3}}\right).\end{aligned} \tag{7.53}$$

The trigonometric identity $\sin a\sin b=\frac{1}{2}(\cos(a-b)-\cos(a+b))$ can be used to rewrite this as

$$C_1(\eta,k,p_m)=\frac{(-1)^{m/2+1}\gamma_m(\Delta_m^1(0))^3}{2k^{m+2}}\cos(k\eta)\int_0^\eta p_m(t)\,dt+\mathcal{O}\left(\frac{1}{k^{m+3}}\right). \tag{7.54}$$

Putting (7.50) and (7.54) together yields, for $m\geq 0$ even,

$$\begin{aligned}X_m(\eta,k,p_m) &= C_0(\eta,k,p_m)+\sum_{n\geq 1}C_n(\eta,k,p_m) \\ &= \frac{(-1)^{m/2}\gamma_m}{k^m}\left\{\frac{\Delta_m^1(0)\sin(k\eta)}{k}\right. \\ &\quad \left.+\frac{\cos(k\eta)}{k^2}\left[\eta^{-1}\Delta_m^2(0)-\frac{1}{2}(\Delta_m^1(0))^3\int_0^\eta p_m(t)\,dt\right]\right\} \\ &\quad +\mathcal{O}(k^{-(m+3)}).\end{aligned} \tag{7.55}$$

Of course a similar argument can be used to define an expansion for $X_m(\eta, k)$ in the case of $m \geq 0$, odd.

Define the following coefficients, for $m \geq 0$,

$$\alpha_m = \begin{cases} (-1)^{m/2}\gamma_m \Delta_m^1(0) & , m \text{ even} \\ (-1)^{(m+1)/2}\gamma_m \Delta_m^1(0) & , m \text{ odd} \end{cases}$$

and

$$\beta_m(\eta) = \begin{cases} (-1)^{m/2}\gamma_m \Delta_m^1(0) \left\{ \eta^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)} - \frac{1}{2}(\Delta_m^1(0))^2 \int_0^\eta p_m(t)\, dt \right\} & , m \text{ even} \\ (-1)^{(m+1)/2}\gamma_m \Delta_m^1(0) \left\{ \frac{1}{2}(\Delta_m^1(0))^2 \int_0^\eta p_m(t)\, dt - \eta^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)} \right\} & , m \text{ odd.} \end{cases}$$

Then the solution of the intitial value problem can then be expressed as

$$X_m(\eta, k, p_m) = \alpha_m \frac{\sin(k\eta)}{k^{m+1}} + \beta_m(\eta)\frac{\cos(k\eta)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right) \tag{7.56}$$

for $m \geq 0$ even and

$$X_m(\eta, k, p_m) = \alpha_m \frac{\cos(k\eta)}{k^{m+1}} + \beta_m(\eta)\frac{\sin(k\eta)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right) \tag{7.57}$$

for $m \geq 0$ odd.

■

Theorem 7.4.0.17 gives us an expression for the solution to the singular Sturm-Liouville problem that shows the asymptotic dependence on the eigenvalue k.

7.5 Conditions for Uniqueness

For two radially symmetric acoustic speeds $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^d$, $d \geq 3$ odd, we define the two sets of fundamental solutions

$$X_m(\eta, k) = \alpha_m \frac{\sin(k\eta)}{k^{m+1}} + \beta_m(\eta)\frac{\cos(k\eta)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ even}, \tag{7.58}$$

$$X_m(\eta, k) = \alpha_m \frac{\cos(k\eta)}{k^{m+1}} + \beta_m(\eta)\frac{\sin(k\eta)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ odd} \tag{7.59}$$

and

$$Z_m(\xi,k) = a_m\frac{\sin(k\xi)}{k^{m+1}} + b_m(\xi)\frac{\cos(k\xi)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right),\ m \text{ even}, \tag{7.60}$$

$$Z_m(\xi,k) = a_m\frac{\cos(k\xi)}{k^{m+1}} + b_m(\xi)\frac{\sin(k\xi)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right),\ m \text{ odd}. \tag{7.61}$$

The coefficients a_m and α_m are defined by (7.40) while the coefficients b_m and β_m are defined by (7.41). Here $X_m(\eta,k)$ and $Z_m(\xi,k)$ satisfy the initial value problems

$$\ddot{X}_m(\eta,k) + \left(k^2 - \frac{m(m+1)}{\eta^2}\right)X_m(\eta,k) = p_{1m}(\eta)X_m(\eta,k) \text{ for } \eta\in[0,C] \tag{7.62}$$
$$\lim_{\eta\to 0}\eta^{-(m+1)}X_m(\eta,k) = 1$$

and

$$\ddot{Z}_m(\xi,k) + \left(k^2 - \frac{m(m+1)}{\xi^2}\right)Z_m(\xi,k) = p_{2m}(\xi)Z_m(\xi,k) \text{ for } \xi\in[0,B] \tag{7.63}$$
$$\lim_{\xi\to 0}\xi^{-(m+1)}Z_m(\xi,k) = 1.$$

We restate here that the terms p_{1m}, p_{2m}, C, and B are defined, through the acoustic speeds $c(r)$ and $b(r)$, by

$$p_{1m}(\eta) = \frac{1}{4}\frac{\ddot{n}_c(r)}{(n_c(r))^2} - \frac{5}{16}\frac{(\dot{n}_c(r))^2}{(n_c(r))^3} + m(m+1)\left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right), \tag{7.64}$$

$$p_{2m}(\xi) = \frac{1}{4}\frac{\ddot{n}_b(r)}{(n_b(r))^2} - \frac{5}{16}\frac{(\dot{n}_b(r))^2}{(n_b(r))^3} + m(m+1)\left(\frac{1}{r^2 n_b(r)} - \frac{1}{\eta^2(r)}\right), \tag{7.65}$$

and $C = \int_0^1 \sqrt{n_c(s)}\,ds$ and $B = \int_0^1 \sqrt{n_b(s)}\,ds$. The index m is given again by $m = j + \frac{1}{2}(d-3)$ where j is the order of the spherical harmonic and $d \geq 3$, odd, is the dimension of the overlying space.

To study transmission eigenvalues recall that, by theorem 7.3.0.15, the number $k\in\mathbb{R}_+$ is a transmission eigenvalue relative to $c(r)$ and $b(r)$ on $B_1(0)\subset\mathbb{R}^d$ only if

$$W(Z_m(B,k), X_m(C,k)) = \tag{7.66}$$
$$Z_m(B,k)\dot{X}_m(C,k) - \dot{Z}_m(B,k)X_m(C,k) = 0$$

for some $m \geq 0$.

Therefore, we will need to evaluate X_m and Z_m, along with their derivatives, at the endpoints C and B respectively. At the right endpoints we have

$$X_m(C,k) = \alpha_m \frac{\sin(kC)}{k^{m+1}} + \beta_m(C)\frac{\cos(kC)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ even}, \tag{7.67}$$

$$X_m(C,k) = \alpha_m \frac{\cos(kC)}{k^{m+1}} + \beta_m(C)\frac{\sin(kC)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ odd}, \tag{7.68}$$

$$\dot{X}_m(C,k) = \alpha_m \frac{\cos(kC)}{k^{m}} - \beta_m(C)\frac{\sin(kC)}{k^{m+1}} + \mathcal{O}\left(\frac{1}{k^{m+2}}\right), \ m \text{ even}, \tag{7.69}$$

$$\dot{X}_m(C,k) = -\alpha_m \frac{\sin(kC)}{k^{m}} + \beta_m(C)\frac{\cos(kC)}{k^{m+1}} + \mathcal{O}\left(\frac{1}{k^{m+2}}\right), \ m \text{ odd}, \tag{7.70}$$

and

$$Z_m(B,k) = a_m \frac{\sin(kB)}{k^{m+1}} + b_m(B)\frac{\cos(kB)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ even}, \tag{7.71}$$

$$Z_m(B,k) = a_m \frac{\cos(kB)}{k^{m+1}} + b_m(B)\frac{\sin(kB)}{k^{m+2}} + \mathcal{O}\left(\frac{1}{k^{m+3}}\right), \ m \text{ odd}, \tag{7.72}$$

$$\dot{Z}_m(B,k) = a_m \frac{\cos(kB)}{k^{m}} - b_m(B)\frac{\sin(kB)}{k^{m+1}} + \mathcal{O}\left(\frac{1}{k^{m+2}}\right), \ m \text{ even}, \tag{7.73}$$

$$\dot{Z}_m(B,k) = -a_m \frac{\sin(kB)}{k^{m}} + b_m(B)\frac{\cos(kB)}{k^{m+1}} + \mathcal{O}\left(\frac{1}{k^{m+2}}\right), \ m \text{ odd}. \tag{7.74}$$

Using the above forms for the fundamental solutions and their derivatives at the right endpoints the determinant condition for transmission eigenvalues becomes

$$\begin{aligned} 0 = W(Z_m(B,k), X_m(C,k)) &= \alpha_m a_m \frac{\sin(k(C-B))}{k^{2m+1}} \\ \pm(\alpha_m b_m(B) - \beta_m(C) a_m)\frac{\cos(k(C-B))}{k^{2m+2}} &+ \mathcal{O}\left(\frac{1}{k^{2m+3}}\right) \end{aligned} \tag{7.75}$$

where the positive sign is taken for $m \geq 0$ even and the negative sign is taken for $m \geq 0$ odd.

Theorem 7.5.0.18 *For two radially symmetric acoustic profiles $c(r)$ and $b(r)$ on $B_1(0) \subset \mathbb{R}^d$, $d \geq 3$ odd, the interior transmission spectrum in $\mathbb{R}_+$ has the property that for large enough $k \in \mathbb{R}_+$ there exists intervals in $\mathbb{R}_+$ free of transmission eigenvalues as long as one of the following two conditions is satisfied:*

(i) $\int_0^1 \sqrt{n_c(r)}\,dr \neq \int_0^1 \sqrt{n_b(r)}\,dr$

(ii) $\int_0^1 \left(\frac{(c'(r))^2}{c(r)} + m(m+1)\frac{c'(r)}{r}\right) dr \neq \int_0^1 \left(\frac{(b'(r))^2}{b(r)} + m(m+1)\frac{b'(r)}{r}\right) dr \;\forall\; m \geq 0$

Proof. By theorem 7.3.0.15 we must show that if either condition (i) or (ii) hold then, for every $m \geq 0$, there exists intervals of $k \in \mathbb{R}_+$ such that $W(Z_m(B,k), X_m(C,k)) \neq 0$ as long as k is large enough. To simplify the proof we will assume m is even. The proof holds with only a slight change of sign if m is odd.

It was computed above that, for $m > 0$ even,

$$W(Z_m(B,k), X_m(C,k)) = \alpha_m a_m \frac{\sin(k(C-B))}{k^{2m+1}}$$
$$+(\alpha_m b_m(B) - \beta_m(C)a_m)\frac{\cos(k(C-B))}{k^{2m+2}} + \mathcal{O}\left(\frac{1}{k^{2m+3}}\right).$$

If condition (i) holds then, by definition, $C \neq B$ so we see

$$W(Z_m(C), X_m(B)) = \alpha_m a_m \frac{\sin(k(C-B))}{k^{2m+1}} + \mathcal{O}\left(\frac{1}{k^{2m+2}}\right).$$

This is enough to imply that for large enough $k \in \mathbb{R}_+$ there are intervals where $W(Z_m(C), X_m(B)) \neq 0$ since $C \neq B$ implies the existence of infinitely many intervals where $\sin(k(C-B)) \neq 0$ and for large enough k the term $\alpha_m a_m \frac{\sin(k(C-B))}{k^{2m+1}}$ is much larger in magnitude than every other term in the expansion of $W(Z_m(C), X_m(B))$. Therefore, if condition (i) is satisfied the statement holds.

Suppose (i) is not satisfied, thus implying $C = B$. Then the determinant becomes

$$W(Z_m(B,k), X_m(C,k)) = \frac{(\alpha_m b_m(B) - \beta_m(C)a_m)}{k^{2m+2}} + \mathcal{O}\left(\frac{1}{k^{2m+3}}\right).$$

The definitions of α_m and a_m do not depend on the coefficient functions p_{1m} and p_{2m} and therefore, $\alpha_m = a_m$. It follows, for large enough k, if

$$\beta_m(C) \neq b_m(B)$$

for all $m \geq 0$ then $W(Z_m(C), X_m(B)) \neq 0$ for all $m \geq 0$ and $k \in \mathbb{R}_+$ is not a transmission eigenvalue.

Our goal now is to show that $\beta_m(C) \neq b_m(B)$ for all $m \geq 0$ is implied by condition (ii). Recalling the definitions of these coefficients we see that $\beta_m(C) \neq b_m(B)$ means

$$\begin{aligned} C^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)} - \frac{1}{2}(\Delta_m^1(0))^2 \int_0^C p_{1m}(t)\,dt \neq \\ B^{-1}\frac{\Delta_m^2(0)}{\Delta_m^1(0)} - \frac{1}{2}(\Delta_m^1(0))^2 \int_0^B p_{2m}(t)\,dt. \end{aligned} \tag{7.76}$$

However, if $C = B$ then this is equivalent to

$$\int_0^C p_{1m}(t)\,dt \neq \int_0^B p_{2m}(t)\,dt \text{ for all } m \geq 0.$$

This condition is equivalent to condition (ii). We show this now by breaking up the integrands into the parts that depend on m and those that are independent of m. Remember

$$p_{1m}(\eta) = \frac{1}{4}\frac{\ddot{n}_c(r)}{(n_c(r))^2} - \frac{5}{16}\frac{(\dot{n}_c(r))^2}{(n_c(r))^3} + m(m+1)\left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right),$$

$$p_{2m}(\xi) = \frac{1}{4}\frac{\ddot{n}_b(r)}{(n_b(r))^2} - \frac{5}{16}\frac{(\dot{n}_b(r))^2}{(n_b(r))^3} + m(m+1)\left(\frac{1}{r^2 n_b(r)} - \frac{1}{\xi^2(r)}\right).$$

The equivalence with condition (ii) will be demonstrated by simplifying the integral for $p_{1m}(\eta)$. It has been shown in the previous two chapters, through a change of variables

$$\int_0^C \left(\frac{1}{4}\frac{\ddot{n}_c(r)}{(n_c(r))^2} - \frac{5}{16}\frac{(\dot{n}_c(r))^2}{(n_c(r))^3}\right) d\eta = \int_0^1 \frac{(c'(r))^2}{c(r)}\,dr. \tag{7.77}$$

Now we must apply the same change of variables to

$$\int_0^C \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta. \tag{7.78}$$

By lemma 7.2.0.13 this integral converges since the integrand is bounded near zero. This lets us break up the integral as follows, for $\epsilon > 0$ write

$$\begin{aligned}
\int_0^C \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta &= \int_\epsilon^C \frac{d\eta}{r^2 n_c(r)} - \int_\epsilon^C \frac{d\eta}{\eta^2} \\
&\quad + \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta \\
&= \frac{1}{C} - \frac{1}{\epsilon} + \int_{r(\epsilon)}^1 \frac{c(r)}{r^2}\, dr \\
&\quad + \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta \\
&= \frac{1}{C} - \frac{1}{\epsilon} - c(1) + \frac{c(r(\epsilon))}{r(\epsilon)} + \int_{r(\epsilon)}^1 \frac{c'(r)}{r}\, dr \\
&\quad + \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta \\
&= \frac{1-C}{C} + \frac{c(r(\epsilon))}{r(\epsilon)} - \frac{1}{\epsilon} + \int_{r(\epsilon)}^1 \frac{c'(r)}{r}\, dr \\
&\quad + \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta \\
&= \frac{1-C}{C} + \frac{c(r(\epsilon)) - c(0)}{r(\epsilon)} + \frac{c(0)\epsilon - r(\epsilon)}{\epsilon r(\epsilon)} + \int_{r(\epsilon)}^1 \frac{c'(r)}{r}\, dr \\
&\quad + \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta.
\end{aligned} \tag{7.79}$$

Now examine the behavior of each term as $\epsilon \to 0^+$. By lemma 7.2.0.13 it follows that

$$\lim_{\epsilon\to 0^+} \int_0^\epsilon \left(\frac{1}{r^2 n_c(r)} - \frac{1}{\eta^2(r)}\right) d\eta = 0. \tag{7.80}$$

We also see that since $c'(0) = 0$

$$\lim_{\epsilon\to 0^+} \frac{c(r(\epsilon)) - c(0)}{r(\epsilon)} = c'(0) = 0 \tag{7.81}$$

and

$$\int_0^1 \frac{c'(r)}{r}\, dr = \lim_{\epsilon\to 0^+} \int_{r(\epsilon)}^1 \frac{c'(r)}{r}\, dr < \infty. \tag{7.82}$$

Lastly, we change variables in the limit

$$\lim_{\epsilon\to 0^+} \frac{c(0)\epsilon - r(\epsilon)}{\epsilon r(\epsilon)} = \lim_{r\to 0^+} \frac{c(0)\eta(r) - r}{r\eta(r)}.$$

To evaluate this we will need the finite order Taylor expansion of $\eta(r)$,

$$\eta(r) = \frac{r}{c(0)} + \frac{r^3}{6}\left(\frac{\ddot{n}_c(a)}{2\sqrt{n_c(a)}} - \frac{(\dot{n}_c(a))^2}{4(n_c(a))^{3/2}}\right)$$

for some $0 < a < r$. Substituting this into the limit yields

$$\begin{aligned}\lim_{r\to 0^+}\frac{c(0)\eta(r)-r}{r\eta(r)} &= \lim_{r\to 0^+}\frac{\frac{r^3 c(0)}{6}\left(\frac{\ddot{n}_c(a)}{2\sqrt{n_c(a)}} - \frac{(\dot{n}_c(a))^2}{4(n_c(a))^{3/2}}\right)}{\frac{r^2}{c(0)} + \frac{r^4}{6}\left(\frac{\ddot{n}_c(a)}{2\sqrt{n_c(a)}} - \frac{(\dot{n}_c(a))^2}{4(n_c(a))^{3/2}}\right)} \\ &= \lim_{r\to 0^+}\frac{\frac{r c(0)}{6}\left(\frac{\ddot{n}_c(a)}{2\sqrt{n_c(a)}} - \frac{(\dot{n}_c(a))^2}{4(n_c(a))^{3/2}}\right)}{\frac{1}{c(0)} + \frac{r^2}{6}\left(\frac{\ddot{n}_c(a)}{2\sqrt{n_c(a)}} - \frac{(\dot{n}_c(a))^2}{4(n_c(a))^{3/2}}\right)} = 0.\end{aligned}$$

Putting all of the above together we see that

$$\int_0^C p_{1m}(\eta)\,d\eta = \int_0^1 \frac{(c'(r))^2}{c(r)}\,dr + m(m+1)\left[\frac{1-C}{C} + \int_0^1 \frac{c'(r)}{r}\,dr\right].$$

Thus, the condition

$$\int_0^C p_{1m}(\eta)\,d\eta \neq \int_0^B p_{2m}(\xi)\,d\xi$$

for all $m \geq 0$ is equivalent to

$$\begin{aligned}&\int_0^1 \frac{(c'(r))^2}{c(r)}\,dr + m(m+1)\left[\frac{1-C}{C} + \int_0^1 \frac{c'(r)}{r}\,dr\right] \\ &\qquad \neq \int_0^1 \frac{(b'(r))^2}{b(r)}\,dr + m(m+1)\left[\frac{1-B}{B} + \int_0^1 \frac{b'(r)}{r}\,dr\right].\end{aligned} \tag{7.83}$$

However, we have assumed that condition (i) does not hold so $C = B$ and therefore this can be shortened to

$$\begin{aligned}&\int_0^1 \frac{(c'(r))^2}{c(r)}\,dr + m(m+1)\int_0^1 \frac{c'(r)}{r}\,dr \\ &\qquad \neq \int_0^1 \frac{(b'(r))^2}{b(r)}\,dr + m(m+1)\int_0^1 \frac{b'(r)}{r}\,dr\end{aligned} \tag{7.84}$$

for all $m \geq 0$.

∎

This theorem provides conditions on two radial acoustic profiles that imply intervals free of transmission eigenvalues. Given our previous results on the relation of the transmission spectrum to the TAT acoustic uniqueness question this provides insight into when two such profiles generate distinct thermoacoustic measurements. For the case of distinguishing constants we are able to state this as a general result in the next theorem. The proof is almost the same as that given at the end of chapter 5 for the analogous theorem. However, since condition (ii) has changed and depends on $m > 0$ some modifications are in order. The major change comes from the fact that the integrand in condition (ii),

$$\frac{(b'(r))^2}{b(r)} + m(m+1)\frac{b'(r)}{r},$$

is not necessarily positive on $[0, 1]$. In order to use the methods of proof of chapter 5 we must assume extra hypotheses on the acoustic profiles being compared.

Theorem 7.5.0.19 *If the radial acoustic profile on $B_1(0) \subset \mathbb{R}^d$, $d \geq 3$ odd, is constant, $c(r) = c$, then it is uniquely determined by thermoacoustic data known to be generated by radially symmetric acoustic profiles, $b(r)$, satisfying*

$$\int_0^1 \frac{b'(r)}{r}\, dr \geq 0. \tag{7.85}$$

That is, for a constant acoustic profile c and a second radially symmetric profile $b(r)$ on $B_1(0) \subset \mathbb{R}^d$, $d \geq 3$ odd, satisfying (7.85) the ranges of the TAT operators $\mathcal{L}_c$ and $\mathcal{L}_{b(r)}$ have zero intersection.

Proof. Suppose the constant acoustic speed $c(r) = c > 0$ and a radial acoustic profile $b(r)$ satisfying (7.85) generate the same thermoacoustic data for some pair of acoustic source functions $f_1(x)$, $f_2(x)$. Then the transmission spectrum on $B_1(0)$ relative to the acoustic profiles c and $b(r)$ can not contain a finite cluster point. Thus, by theorem 7.5.0.18, conditions (i) and (ii) can not be satisfied. This implies two things. First,

$$\int_0^1 \sqrt{n_c(r)}\, dr = \int_0^1 \sqrt{n_b(r)}\, dr \tag{7.86}$$

which is equivalent to

$$\int_0^1 \frac{dr}{b(r)} = \frac{1}{c}. \tag{7.87}$$

Secondly, there exists some $m \geq 0$ such that

$$\int_0^1 \left(\frac{(c'(r))^2}{c(r)} + m(m+1)\frac{c'(r)}{r} \right) dr = \int_0^1 \left(\frac{(b'(r))^2}{b(r)} + m(m+1)\frac{b'(r)}{r} \right) dr. \tag{7.88}$$

Since $c(r)$ is constant this is the same as

$$\int_0^1 \left(\frac{(b'(r))^2}{b(r)} + m(m+1)\frac{b'(r)}{r} \right) dr = 0. \tag{7.89}$$

Certainly $\frac{(b'(r))^2}{b(r)} \geq 0$ on $[0,1]$ and (7.89) implies

$$\int_0^1 \frac{(b'(r))^2}{b(r)}\, dr = -m(m+1) \int_0^1 \frac{b'(r)}{r}\, dr \leq 0 \tag{7.90}$$

using condition (7.85). Thus, (7.89) implies

$$\int_0^1 \frac{(b'(r))^2}{b(r)}\, dr = 0$$

from which we can conclude $b'(r) = 0$. Therefore, $b(r)$ is a constant and (7.87) implies that $b(r) = c$.

■

Notice that the significant difference with this theorem and the analogous theorem in the previous chapter is the fact that we do not require the restriction of the domain of the TAT operators. Thus, in the class of radially symmetric acoustic speeds meeting condition (7.85) we may distinguish constant acoustic speeds from thermoacoustic data. It is no longer necessary to assume that the source that generated this data was radial. This is important since the assumption of a radial source in TAT is equivalent to physically assuming that the location of the unhealthy tissue is already known.

7.6 Conclusion

We have shown conditions of uniqueness for two radially symmetric acoustic profiles provided that the overlying dimension that the TAT measurements take place in is odd. It was demonstrated that these conditions were enough to imply that constant acoustic profiles are determined uniquely, in a special class of radial acoustic speeds, from thermoacoustic measurements.

The above results were arrived at by expanding the forward solution of the TAT problem in terms of increasing order spherical harmonics. This allowed us to reduce the problem to a one dimensional Sturm-Liouville problem for which we could derive explicit forms for the fundamental solution of interest.

It is worth restating that the primary significance of the results in this chapter when compared to the results of the preceding chapter is that here we do not assume a radially symmetric acoustic source in the thermoacoustic problem. This makes the results here more physically significant.

Without the extension of the above results to even dimensions the work will be somewhat difficult to test numerically. Therefore, it would be very desirable to try to fit the results to the two dimensional case. The difficulty is that then the solution X_{m1} would be given only in terms of Bessel functions of the first and second kind, J_ν and N_ν, which do not have *nice* expansions in terms of sines and cosines. However, given the abundance of known identities involving these functions it seems that it may be possible to develop usable asymptotic expansions of $W(Z_{m1}(B), X_{m1}(C))$ in terms of these functions. As mentioned at the end of the previous two chapters, each term in the asymptotic expansion of the determinant condition will yield a condition of uniqueness on the two acoustic profiles in question. In the future it would be desirable to understand the extent of new information contained in these successive terms.

8 CONCLUSION AND FUTURE DIRECTIONS

It has been demonstrated that thermoacoustic data arising from distinct acoustic profiles must be distinct regardless of the initial acoustic source provided that specific conditions are met between the two acoustic profiles. The connection with these results and the unique determination of the acoustic speed in TAT was detailed and explored.

For specific subsets of acoustic speeds it was shown that constant acoustic profiles were distinguished by their thermoacoustic data. This was demonstrated in the case of one dimensional TAT. It was shown that, in the case of a radial source on a radial domain with radial acoustic profiles, constant radial profiles generate distinct thermoacoustic measurements. If the TAT imaging process is carried out on a radial domain and the initial source is not radial then it was shown that constant radial profiles were uniquely determined among a special class of radial acoustic profiles. These results were justified by studying a special Sturm-Liouville eigenvalue problem. We looked at asymptotic expansions, with respect to the eigenvalue, of the fundamental solutions associated with this problem.

The determination of the uniqueness of the acoustic profile from TAT data was able to be analyzed by studying the spectrum of the interior transmission problem. Current theory does not deal with the spectrum of this problem if the smooth refractive index of the domain takes values both above and below the value of the surrounding homogeneous medium. Some special cases when the spectrum can be analyzed even when the refractive index takes values above and below the surrounding homogeneous refractive index were explored.

A variational formulation of the transmission problem was shown to exist for a contrast transitioning to zero on the boundary of the domain with some degree of smoothness. In this case we were able to show that the Hardy inequality can be exploited to let us prove that the transmission spectrum is discrete and that there exists an infinite discrete set of transmission eigenvalues.

The analysis carried out in this work does not arrive at necessary and sufficient conditions for the acoustic speed to be uniquely determined from TAT data in a class of acoustic profiles. The conditions that have been shown are only sufficient. A significant reason that necessary and sufficient conditions have not been shown is that we are unable to analyze the transmission spectrum in the case that the contrast function $m(x)$ changes signs continuously. The exception is the case of a single dimension or a radially symmetric profile in which case the spectrum can be analyzed through a special boundary condition.

This research motivates a few lines of inquiry into the improvement of thermoacoustic tomography and to a better understanding of inverse problems associated with hyperbolic operators in general. We end with some suggestions for future research into the problem of unique determination of an acoustic profile and to improvements in thermoacoustic tomography in general.

The case when the acoustic profile is non-trapping but the dimension is even has been left unexplored. In this case the measured data does not have the same analytic properties in the temporal frequency domain. However, we proved in chapter 3 that it is analytic in the half-plane so it may be possible to extend the above analysis to this case. Similarly we have not dealt with the question of uniqueness of the acoustic properties when the speed is trapping. In this case there is still decay of the energy in any bounded domain [13] but the decay does not admit closed form bounds for a general trapping acoustic profile. Thus, in this case a different method of analysis will probably be needed if some type of uniqueness exists at all.

The first result in the unique determination of acoustics from TAT data was in [44]. In this paper it was demonstrated that within a very particular one parameter family of acoustic profiles the acoustic speed was uniquely determined from thermoacoustic measurements. It would be interesting to apply the investigation using the transmission spectrum to try to characterize the one parameter families of acoustics which have this uniqueness property. At the very least one could hope to find other one parameter families of acoustics in which the acoustic profile is determined from TAT. Results of this kind would be very useful for developing minimization techniques to determine the acoustic properties of a body using TAT data since they would provide test cases for numerical methods.

In our treatment of the completely radial TAT problem, involving a radial sound speed and a radial acoustic source, we have proved that constant acoustic profiles are uniquely determined. The proof of this result was made possible by restricting the domain in which the TAT operators were allowed to act. That is, our assumptions restricted the acoustic source. Since the uniqueness question was formulated in terms of intersections of ranges of thermoacoustic operators it is not surprising that restricting the domain allows us to prove results. For this reason it seems worthwhile to further investigate the effect of restricting the domain of the TAT operator on the unique determination of the acoustic source.

First, given the success of the asymptotic study of the one dimensional case one may ask if these same lines of inquiry could be used to investigate a problem akin to the thermoacoustic problem on the interval but with the imposition of a different kind of boundary condition at one of the endpoints. For instance one could study the unique determination of the acoustic speed for the alternate problem,

$$\begin{aligned} \partial_t^2 p(x,t) - c^2(x)\Delta p(x,t) &= 0 \text{ on } \mathbb{R}\times\mathbb{R}_+ \\ p(0,t) &= 0 \text{ for } t\in\mathbb{R}_+ \\ p(x,0) = f(x),\ p_t(x,0) &= 0 \text{ on } \mathbb{R} \end{aligned} \tag{8.1}$$

with $\text{supp}(f)\subset(0,1)$ and measurements taken to be $g(t) = p(1,t)$ for $t\in\mathbb{R}_+$ where the goal is to reconstruct $c(x)$ and $f(x)$. This would be analogous to the TAT problem in a single dimension. An unknown interior source would still be the object of reconstruction. However, the condition $p(0,t) = 0$ changes the physical meaning of such a problem. More generally an inquiry into the same problem with the condition

$$b\,\partial_x p(0,t) + a\,p(0,t) = h(t) \tag{8.2}$$

for some constants a, $b \in \mathbb{R}$ and a specified function $h(t)$ would account for many types of physical situations and could yield some insight into how the boundary data depends on the acoustic profile $c(x)$. One could ask if there exists a boundary condition of the form (8.2) that implied the acoustic profile was uniquely determined by the data $g(t)$.

The minimization methods developed in [90, 83] yield good numerical results for reconstruction of the acoustic profile. It would be interesting to compare the effectiveness of these methods when restricted to a class of acoustic profiles in which constant speeds can be uniquely determined. Intuitively these methods should converge much faster if there is a uniquely determined acoustic profile. For instance one could compare convergence rates for the one dimensional problem with data generated by a constant acoustic speed to the convergence rates with data generated from a varying acoustic speed. Since constants are uniquely determined one should be able to demonstrate better convergence results for the constant speed generated data.

Our derivation of the equation governing the forward TAT process motivates lines of study unrelated to the unique determination of the acoustic profile. In general a region of biological tissue will propagate acoustic disturbances according to the equation (2.41) involving the stress tensor, velocity, bulk modulus, shear modulus, and density. Though we would expect the shear modulus to be small and the density to vary less than the bulk modulus these assumptions are not perfect. To begin studying the more general problem one could study the TAT imaging process governed by this equation for known constant density, shear modulus, and bulk modulus and determine if the initial pressure could be recovered uniquely. Since the available theory of TAT assumes the shear modulus is zero it would be interesting to prove stability results using the more general model with respect to small shear moduli.

Aside from improvements in thermoacoustic tomography some of the results in this manuscript improve the existing knowledge of the interior transmission problem and its spectrum. The existence of a variational form for the ITP in the case of a refractive index transitioning with some degree of smoothness to the surrounding homogeneous medium motivates an investigation into improved methods to deal with C^∞ transitions. In the analysis presented here we have not been able to improve the smoothness of the transition in the case of interior cavities at all. This would be an interesting case to deal with in the future since it focuses on the most difficult aspect of analyzing the transmission spectrum, the interior zeros of the contrast $m(x) = 1 - n(x)$.

Our results on conditions for existence of intervals free of transmission eigenvalues for the radial ITP problem when the solutions are not assumed to be radial are new. These conditions only held in odd dimensions since it was necessary to use spherical Bessel functions in the proofs. In the future it would be desirable to extend these results to even dimensions using regular Bessel functions.

One small aspect of this problem has been covered. I hope this leads to further investigation into methods to determine the acoustic properties of a body using thermoacoustic data.

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