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Title REDUCTION OF FREDHOLM INTEGRAL EQUATIONS WITH GREEN'S FUNCTION KERNELS TO VOLTERRA EQUATIONS

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G. F. Drukarev has given a method for solving the Fredholm equations which arise in the study of collisions between electrons and atoms. He transforms the Fredholm equations into Volterra equations plus finite algebraic systems. H. Brysk observes that Drukarev's method applies generally to a Fredholm integral equation $(I-\lambda G)u = h$ with a Green's function kernel.

In this thesis connections between the Drukarev transformation and boundary value problems for ordinary differential equations are investigated. In particular, it is shown that the induced Volterra operator is independent of the boundary conditions. The resolvent operator can be expressed in terms of the Volterra operator for regular λ. The characteristic values of G satisfy a certain transcendental equation. The Neumann expansion provides a means for approximating this resolvent and the characteristic values. To illustrate the theory several classical boundary value problems are

solved by this method. Also included is an appendix which relates the resolvent operator mentioned above and the Fredholm resolvent operator.

REDUCTION OF FREDHOLM INTEGRAL EQUATIONS WITH GREEN'S FUNCTION KERNELS TO VOLTERRA EQUATIONS

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CHAPTER I

INTRODUCTION

We shall be concerned with a certain method of solving particular Fredholm integral equations of the second kind,

$$u(x) - \lambda \int_0^1 G(x, s)u(s)ds = h(x). \tag{1.1}$$

We assume that G, h and u are continuous complex valued functions on their closed domains of definition.

Physicists are interested in obtaining solutions to integral equations of form (1. 1) which often arise in the study of collisions between electrons and atoms [2, 5]. One method physicists use to obtain approximate solutions to (1. 1) is to calculate one of the Born approximations which are truncations of the Neumann series solutions [2, p. 1536; 12, p. 1073]. The Born approximations are useful only when the Neumann series converges and, generally, this occurs only for sufficiently small values of the parameter λ in equation (1. 1). It is of physical interest to obtain solutions to (1. 1) for larger values of λ. Thus there is motivation to study other methods of solving Fredholm equations.

Integral equations (1.1) often come from ordinary differential equations with two point boundary conditions, e.g. a one dimensional scattering problem. In this case G is the Green's function associated with the given boundary value problem. Thus we are led to consider (1.1) with a Green's function type kernel,

$$G(x,s) = \begin{cases} V(s)f(s)g(x), & 0 \leq s \leq x \leq 1, \\ V(s)f(x)g(s), & 0 \leq x \leq s \leq 1. \end{cases} \tag{1.2}$$

In (1.2) let V, f and g be continuous complex valued functions defined on the closed unit interval and further suppose that $V \not\equiv 0$, $g \not\equiv 0$ and $f \not\equiv 0$.

G. F. Drukarev gave a novel method of solving (1.1) with the kernel of form (1.2) [2, p. 1536; 5, pp. 309-320]. He was able to transform the Fredholm equation into a Volterra equation and a finite algebraic system for certain constants.

H. Brysk observes that Drukarev's transformation of the Fredholm equation into a Volterra equation is possible because the kernel is of Green's function type [2, p. 1536]. Brysk further attempts to show that the solution of the Volterra equation leads to the solution of the Fredholm equation obtained by using the Fredholm resolvent [2, pp. 1537-1538].

Briefly, the transformation of a Fredholm equation with kernel (1. 2) depends on

$$(1.3) \qquad \int_0^1 G(x, s)u(s)ds = \int_0^x V(s)f(s)g(x)u(s)ds + \int_x^1 V(s)f(x)g(s)u(s)ds$$

$$= \int_0^x V(s)[f(s)g(x)-f(x)g(s)]\, u(s)ds + f(x)\int_0^1 V(s)g(s)u(s)ds.$$

Let

$$(1.4) \qquad K(x, s) = V(s)[f(s)g(x)-f(x)g(s)]\,, \quad 0 \leqq s \leqq x \leqq 1\ .$$

Then equation (1. 1) can be rewritten

$$(1.5) \quad u(x)-\lambda \int_0^x K(x,s)u(s)ds = h(x)+\lambda f(x)\int_0^1 V(s)g(s)u(s)ds\ .$$

The right member of (1. 5) is of the form

$$h(x) + cf(x)$$

where h and f are known and c is a constant depending on u and λ. Note that (1.5) is a Volterra equation for the unknown function u. If it is solved for u with c arbitrary, then substitution of the solution into (1.5) yields an equation for c. The technique we shall develop to solve (1.5) is somewhat analogous to the "shooting method" discussed by Henrici [7, pp. 345-346], (cf. Chapter VI). Brysk deals with the special case $h = f$ [2, p. 1537] and thus his Volterra equation has the form

$$u(x) - \lambda \int_0^x K(x,s)u(s)ds = f(x)[1 + \lambda \int_0^1 V(s)g(s)u(s)ds].$$

This is a brief summary of the results of Drukarev and Brysk dealing with the mathematical aspects of the problem. The author intends to set the problem in a more abstract setting and to extend the results obtained by Drukarev and Brysk.

As we will be dealing with integral equations with continuous kernels, it will be convenient to work in the complex Banach space C of continuous complex valued functions defined on the closed unit interval with the norm $||f|| = \max \{|f(x)| : x \in [0, 1]\}$. Capital letters will denote continuous linear mappings of C into itself. For example, define G and K by the equations

$$(1.6) \qquad (Gu)(x) = \int_0^1 G(x,s)u(s)ds,$$

and

(1.7) $$(Ku)(x) = \int_0^x K(x,s)u(s)ds$$

where G and K are given by (1.2) and (1.4). Now (1.1) may be expressed by

(1.8) $$(I - \lambda G)u = h$$

where I is the identity operator on C.

Capital Greek letters will denote continuous linear functionals; i.e. continuous linear mappings of C into the scalar field $\mathfrak{F}$. The set of all linear functionals on C forms a Banach space C^*. In particular, define $\Phi \epsilon C^*$ by the equation

(1.9) $$\Phi(u) = \int_0^1 V(s)g(s)u(s)ds$$

where V(s) and g(s) are as above. Note that $\Phi \neq 0$.

As a final notational convention, the symbols for elements of C will also be used to indicate mappings from the scalars into C. This convention is adopted because of the obvious isomorphism between C and these mappings: for each $f \epsilon C$ define the mapping $f: \mathfrak{F} \to C$ by

(1.10) $$(f\gamma)(x) = \gamma f(x)$$

where $\gamma \in \mathfrak{F}$. With this convention $f\Phi$ is a linear operator on C into C with the one dimensional range $\{\gamma f : \gamma \in \mathfrak{F}\}$. Thus the operator $f\Phi$ has rank one where the rank of an operator is the dimension of its range.

Now (1.3) and (1.4) may be expressed by

$$G = K + f\Phi, \tag{1.11}$$

and

$$(I-\lambda K)u = h + \lambda f\Phi(u) . \tag{1.12}$$

Thus, the Fredholm operator G has a decomposition into the sum of the Volterra operator K and the operator $f\Phi$ of finite rank.

In Chapters II and III of this thesis the above decomposition of a Fredholm operator with a Green's function kernel arising from an ordinary differential equation is investigated. In Chapters IV and VI, the solution of (1.12) is developed. Also included in Chapters IV and VI are examples worked out using the techniques inspired by Drukarev. Approximate solutions of (1.12) are discussed and error estimates given in Chapter V. Finally the solution of (1.12) is related to the Fredholm resolvent operator in the Appendix.

CHAPTER II

GREEN'S FUNCTION FOR A SECOND ORDER DIFFERENTIAL EQUATION

In this chapter a brief outline of the construction of a Green's function for a boundary value problem arising from a second order ordinary differential equation is given. Then the integral operator arising from this construction is decomposed as in Chapter I. A close examination of the Volterra operator shows that it is independent of the boundary values. Further discussion is given to show the relation of this decomposition to more classical results of ordinary differential equations.

The Green's function $G(x, s)$ will be constructed for the second order differential operator

$$Lu = u'' + p_1 u' + p_2 u$$

where u is defined on $[0, 1]$ and $p_1, p_2 \epsilon C$ with boundary conditions

$$\alpha_1 u(0) + \alpha_2 u'(0) = 0, \qquad |\alpha_1| + |\alpha_2| > 0, \tag{2.1}$$

$$\beta_1 u(1) + \beta_2 u'(1) = 0, \qquad |\beta_1| + |\beta_2| > 0 . \tag{2.2}$$

We wish to solve the equation

(2.3) $$Lu = h \qquad (h \in C)$$

subject to boundary conditions (2.1) and (2.2). We assume that this boundary value problem has a unique solution.

Since $p_1, p_2 \in C$, it follows from the theory of ordinary differential equations that there exist two linearly independent functions u_1 and u_2 satisfying the homogeneous equation

(2.4) $$Lu = 0 \qquad [3, \text{ p. } 106].$$

Thus the Wronskian of u_1 and u_2 is nonzero; that is

$$W[u_1(x), u_2(x)] = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix} \neq 0$$

for $x \in [0, 1]$. The assumption that the solution to the boundary value problem is unique assures us that u_1 and u_2 can be chosen such that u_1 satisfies (2.1) and u_2 satisfies (2.2) [9, p. 378]. The method of variation of parameters yields a solution of (2.1)-(2.3) in the form

$$u(x) = \int_0^x \frac{u_1(s)u_2(x)}{W[u_1(s), u_2(s)]} h(s)d(s) + \int_x^1 \frac{u_1(x)u_2(s)}{W[u_1(s), u_2(s)]} h(s)ds$$

[9, pp. 378-379] .

Let

$$G(x,s) = \begin{cases} \dfrac{u_1(s)u_2(x)}{W[u_1(s),u_2(s)]}, & 0 \leq s \leq x \leq 1 , \\ \\ \dfrac{u_1(x)u_2(s)}{W[u_1(s),u_2(s)]}, & 0 \leq x \leq s \leq 1 . \end{cases}$$

Then

$$u(x) = \int_0^1 G(x,s)h(s)ds . \tag{2.5}$$

The function $G(x, s)$ is the Green's function associated with the differential operator L with boundary conditions (2. 1) and (2. 2).

Now using the decomposition developed in Chapter I the right side of (2. 5) may be rewritten

$$\int_0^1 G(x,s)h(s)ds = \int_0^x \frac{[u_1(s)u_2(x) - u_1(x)u_2(s)]}{W[u_1(s), u_2(s)]} h(s)ds$$

$$+ u_1(x) \int_0^1 \frac{u_2(s)}{W[u_1(s),u_2(s)]} h(s)ds .$$

Let

$$K(x,s) = \frac{u_1(s)u_2(x) - u_1(x)u_2(s)}{W[u_1(s),\ u_2(s)]}, \quad 0 \le s \le x \le 1 .$$

Then

$$(2.6) \quad \int_0^1 G(x,s)h(s)ds = \int_0^x K(x,s)h(s)ds + u_1(x)\int_0^1 \frac{u_2(s)h(s)ds}{W[u_1(s),u_2(s)]} .$$

The kernel of the Volterra operator has the property that it is invariant under linear combinations of the functions u_1 and u_2; that is, if

$$(2.7) \quad \begin{aligned} v_1(x) &= \gamma_1 u_1(x) + \gamma_2 u_2(x) \\ v_2(x) &= \delta_1 u_1(x) + \delta_2 u_2(x) \end{aligned}$$

and $\gamma_1\delta_2 - \gamma_2\delta_1 \neq 0$, then

$$K(x,s) = \frac{v_1(s)v_2(x) - v_1(x)v_2(s)}{W[v_1(s), v_2(s)]} .$$

This follows since

$$v_1(s)v_2(x)-v_1(x)v_2(s) = (\gamma_1\delta_2-\gamma_2\delta_1)(u_1(s)u_2(x)-u_1(x)u_2(s))$$

and

$$W[v_1(s), v_2(s)] = (\gamma_1\delta_2-\gamma_2\delta_1)W[u_1(s),\ u_2(s)] .$$

If boundary conditions (2. 1) and (2. 2) are changed to

(2. 1)' $\alpha_1' u(0) + \alpha_2' u'(0) = 0, \quad |\alpha_1'| + |\alpha_2'| > 0,$

(2. 2)' $\beta_1' u(1) + \beta_2' u'(1) = 0, \quad |\beta_1'| + |\beta_2'| > 0,$

then the linearly independent functions v_1 and v_2 satisfying the homogeneous equation (2. 4) and the boundary conditions (2. 1)' and (2. 2)' respectively can be expressed as linear combinations of u_1 and u_2. In other words they can be expressed as in equations (2. 7) for some γ_1, γ_2, δ_1 and δ_2. The solution to the new boundary value problem written in terms of u_1 and u_2 is

$$u(x) = \int_0^x K(x, s)h(s)ds + \frac{\gamma_1 u_1(x)+\gamma_2 u_2(x)}{\gamma_1\delta_2 - \gamma_2\delta_1} \cdot \int_0^1 \frac{[\delta_1 u_1(s)+\delta_2 u_2(s)]}{W[u_1(s), u_2(s)]} h(s)ds .$$

Therefore the kernel $K(x, s)$ is independent of the boundary conditions and the second term on the right varies with the boundary conditions.

It is easy to verify that

$$u_p(x) = \int_0^x K(x, s)h(s)ds$$

satisfies equation (2. 3) and the initial conditions $u(0) = u'(0) = 0$. In fact $u_p(x)$ is the "particular" solution to (2. 3) which can be found by the method of variation of parameters if any two linearly independent solutions to equation (2. 4) are given. On the other hand it is well known from the theory of ordinary differential equations that any solution to equation (2. 3) can be written

$$u = u_p + \alpha u_1 + \beta u_2 \qquad [9, \text{ p. } 356] .$$

In our case we have

$$\alpha = \int_0^1 \frac{u_2(s)}{W[u_1(s), u_2(s)]} \, h(s)ds$$

and $\beta = 0$. Thus the solution obtained via the Green's function and the decomposition gives the parameters α and β as functionals operating on the function h.

As a final remark it should be noted that boundary value problems with inhomogeneous boundary conditions can be transformed into boundary value problems with homogeneous boundary conditions. The term on the right side of (2. 3) is modified by this transformation, but the discussion is simpler for homogeneous boundary value problems. Thus the techniques used here apply with greater generality than indicated above.

CHAPTER III

GREEN'S FUNCTION FOR AN N^{TH} ORDER DIFFERENTIAL EQUATION

In this chapter the results of the previous chapter are generalized to a boundary value problem arising from an n^{th} order ordinary differential equation. The more general results do not appear in as simple a form as the second order case considered in the previous chapter. In the second order case the Fredholm operator G admitted the decomposition

$$G = K + f\Phi .$$

In the n^{th} order case we obtain a decomposition

(3.1) $$G = K + \sum_{i=1}^{n} f_i \Phi_i .$$

However the Volterra operator is still independent of the boundary conditions.

The Green's function will be constructed for the n^{th} order differential operator

(3.2) $$Lu = \sum_{j=0}^{n} p_j \frac{d^{n-j}u}{dx^{n-j}}$$

where u is defined on $[0,1]$, $p_j \in C$, $j = 0, 1, \cdots, n$ and $p_0(x) > 0$ for $x \in [0,1]$ with boundary conditions

$$(3.3) \qquad U_i(u) = \sum_{j=0}^{n-1} [\alpha_{ij} u^{(j)}(0) + \beta_{ij} u^{(j)}(1)] = 0$$

$i = 1, 2, \cdots, n$. We wish to solve the boundary value problem given by

$$(3.4) \qquad Lu = h \qquad (h \in C)$$

and boundary conditions (3.3). As before we assume that this boundary value problem has a unique solution.

The usual definition of the Green's function for the operator L given in (3.2) with boundary conditions (3.3) is

(a) $G(x,s)$ and its derivatives up to and including the $(n-2)$ derivative are continuous for $0 \leqq x, s \leqq 1$,

(b) $\lim_{\epsilon \to 0^+} \{\frac{\partial^{n-1}}{\partial x^{n-1}} G(s+\epsilon, s) - \frac{\partial^{n-1}}{\partial x^{n-1}} G(s-\epsilon, s)\} = \frac{1}{p_0(s)}$,

(c) for each fixed $s \in [0,1]$ and all $x \neq s$ $L(G(x,s)) = 0$, $U_i(G) = 0$, $i = 1, 2, \cdots, n$ [8, p. 254].

In the region $0 \leqq s \leqq x \leqq 1$, we assume the Green's function $G(x,s)$ has the representation

$$G(x, s) = \sum_{i=1}^{n} a_i(s)u_i(x)$$

and in the region $0 \leqq x \leqq s \leqq 1$

$$G(x, s) = \sum_{i=1}^{n} b_i(s)u_i(x)$$

where $\{u_i(x): i = 1, 2, \cdots, n\}$ is a linearly independent set of solutions to the equation

$$Lu = 0.$$

Using conditions (a) and (b) unique solutions for the quantities

$$c_i(s) = a_i(s) - b_i(s)$$

$i = 1, 2, \cdots, n$ are obtained [8, pp. 254-255]. Let

$$K(x, s) = \sum_{i=1}^{n} c_i(s)u_i(x), \qquad 0 \leqq s \leqq x \leqq 1 .$$

Using condition (c) unique solutions for the $b_i(s)$ are obtained in terms of the $c_i(s)$ and the boundary terms [8, p. 255]. But $a_i(s) = c_i(s) + b_i(s)$, $i = 1, 2, \cdots, n$ and thus $a_i(s)$ and $b_i(s)$ can be found such that the assumed representation of $G(x, s)$ in the appropriate regions are satisfied. Therefore

$$G(x,s) = \begin{cases} \sum_{i=1}^{n} c_i(s)u_i(x) + \sum_{i=1}^{n} b_i(s)u_i(x), & 0 \leqq s \leqq x \leqq 1, \\ \\ \sum_{i=1}^{n} b_i(s)u_i(x), & 0 \leqq x \leqq s \leqq 1 \end{cases}$$

$$= \begin{cases} K(x,s) + \sum_{i=1}^{n} b_i(s)u_i(x), & 0 \leqq s \leqq x \leqq 1, \\ \\ \sum_{i=1}^{n} b_i(s)u_i(x), & 0 \leqq x \leqq s \leqq 1 . \end{cases}$$

Thus the solution to (3.4) with boundary values (3.3) can be represented by

$$(3.5) \quad u(x) = \int_0^1 G(x,s)h(s)ds = \int_0^x K(x,s)h(s)ds + \int_0^x \Big(\sum_{i=1}^{n} b_i(s)u_i(x)\Big)h(s)ds$$

$$+ \int_x^1 \Big(\sum_{i=1}^{n} b_i(s)u_i(x)\Big)h(s)ds$$

$$= \int_0^x K(x,s)h(s)ds + \sum_{i=1}^{n} u_i(x) \int_0^1 b_i(s)h(s)ds.$$

Let $(Gh)(x) = \int_0^1 G(x, s)h(s)ds$

$$(Kh)(x) = \int_0^x K(x, s)h(s)ds$$

and

$$\Phi_i(h) = \int_0^1 b_i(s)h(s)ds \qquad i = 1, 2, \cdots, n .$$

Then (3. 5) can be rewritten as

$$u = Gh = Kh + \sum_{i=1}^{n} u_i \Phi_i(h)$$

and we see that the Fredholm operator G admits the decomposition (3. 1) where K is a Volterra operator and the Φ_i are linear functionals. Further, we note that $K(x, s)$ is determined by the $c_i(s)$ which were given by conditions (a) and (b). But (a) and (b) are independent of the boundary conditions (3. 3). Hence $K(x, s)$ is independent of the boundary conditions.

The same remarks made about the inhomogeneous boundary conditions in Chapter II can be repeated here, so that there is no need to consider inhomogeneous boundary conditions separately.

CHAPTER IV

SOLUTION OF THE INTEGRAL EQUATION $(I-\lambda K)u = h+\lambda f\Phi(u)$

In this chapter the equation

$$(I-\lambda G)u = h \qquad (h \in C) \tag{4.1}$$

is solved assuming that the Fredholm operator G has the decomposition

$$G = K + f\Phi \tag{4.2}$$

where K is a Volterra operator, $f \in C$, $\Phi \in C^*$, $f \not\equiv 0$ and $\Phi \not\equiv 0$. This is the decomposition considered in Chapter II.

The solution obtained for (4.1) is a quotient of an operator and an entire function in λ. The zeros of this entire function comprise all of the characteristic values of G (characteristic values are inverses of eigenvalues). The resolvent operator obtained by solving (4.1) exists for all noncharacteristic values λ. Thus this resolvent and the Fredholm resolvent are equal [11, p. 15] . Also to be discussed are solutions of characteristic value problems for integral equations. Finally several examples of characteristic value problems are solved using techniques developed in this chapter.

At this point it is convenient to note that the operator G is an operator such that the Fredholm alternative holds for (4. 1). This follows since $G(x, s)$ is continuous. The Fredholm alternative asserts that (4. 1) has a unique solution for arbitrary $h \in C$ iff the homogeneous equation

$$(4.3) \qquad (I-\lambda G)u = 0$$

has only the zero solution [11, p. 46]. Nonzero solutions of (4. 3) are called eigenfunctions of the operator G and the corresponding λ are called characteristic values of G.

From (4. 1) and (4. 2) it follows that

$$(4.4) \qquad (I-\lambda K)u = h + \lambda f\Phi(u)$$

is equivalent to (4. 1). Since K is a Volterra operator, $(I-\lambda K)^{-1}$ exists for all λ. Thus equation (4. 4) is equivalent to

$$(4.5) \qquad u = (I-\lambda K)^{-1}h + \lambda (I-\lambda K)^{-1}f\Phi(u).$$

If there exists u such that (4. 5) holds, then operating on both sides of (4. 5) by Φ and transposing yields the following equation for $\Phi(u)$;

$$(4.6) \qquad \Phi(u)\cdot [1-\lambda \Phi(I-\lambda K)^{-1}f] = \Phi(I-\lambda K)^{-1}h.$$

Equation (4. 6) has a unique solution for $\Phi(u)$ iff

$$d(\lambda) \equiv 1-\lambda\,\Phi(I-\lambda K)^{-1}f \neq 0.$$

Assuming $d(\lambda) \neq 0$, we can solve for $\Phi(u)$ and we obtain equation

(4.7) $$u = (I-K)^{-1}h + \frac{\lambda}{d(\lambda)}(I-\lambda K)^{-1}f\Phi(I-\lambda K)^{-1}h.$$

That is to say, if $d(\lambda) \neq 0$ then (4.7) is the unique solution to (4.1) which implies that λ is not a characteristic value of G. The contrapositive of this statement is, if λ is a characteristic value then $d(\lambda) = 0$. Now suppose that λ is not a characteristic value of G. Then (4.1) holds for arbitrary $h \in C$ which implies that (4.6) holds for arbitrary $h \in C$. If $d(\lambda) = 0$, then $\Phi = 0$ since $(I-\lambda K)^{-1}$ is a one to one mapping of C onto itself. But we assumed that $\Phi \neq 0$, hence $d(\lambda) \neq 0$. To summarize, the following two theorems are recorded.

<u>Theorem 4.1</u> λ is a characteristic value of G iff $d(\lambda) = 0$

<u>Theorem 4.2</u> If λ is not a characteristic value, then (4.7) gives the unique solution to (4.1).

Note that (4.1) has a unique solution iff the one dimensional system (4.6) has a unique solution. Thus the Fredholm alternative for the operator $(I-\lambda G)$ reduces to the Fredholm alternative for the one dimensional system (4. 6).

From Theorems 4. 1 and 4. 2, $(I-\lambda G)^{-1}$ exists iff $d(\lambda) \neq 0$ in which case

$$(I-\lambda G)^{-1} = (I-\lambda K)^{-1} + \frac{\lambda}{d(\lambda)}(I-\lambda K)^{-1} f \Phi (I-\lambda K)^{-1} . \tag{4.8}$$

As K is a Volterra operator

$$(I-\lambda K)^{-1} = \sum_{n=0}^{\infty} \lambda^n K^n$$

where $K^0 = I$ and $K^{n+1} = KK^n$. The series converges in the operator norm for all λ. Letting

$$f_\lambda = (I-\lambda K)^{-1} f$$

(4. 8) may be rewritten

$$(I-\lambda G)^{-1} = I + \lambda \frac{\sum_{n=0}^{\infty} \lambda^n [d(\lambda) K^{n+1} + f_\lambda \Phi K^n]}{d(\lambda)} . \tag{4.9}$$

The Fredholm resolvent operator Γ_λ of G is defined by

$$(I-\lambda G)^{-1} = I + \lambda \Gamma_\lambda$$

whenever $(I-\lambda G)^{-1}$ exists. Thus by (4. 9)

$$\Gamma_\lambda = \frac{\sum_{n=0}^{\infty} \lambda^n [d(\lambda) K^{n+1} + f_\lambda \Phi K^n]}{d(\lambda)} .$$

It might be remarked at this point that Brysk attempts to prove a similar result by showing that the numerator and denominator of his solution are the same as the numerator and denominator of the solution obtained via the Fredholm resolvent [2, pp. 1537-1538]. His proof is faulty, but a proof can be established using techniques developed by Manning [10], (cf. appendix).

In general $d(\lambda)$ is an entire function in λ since

$$d(\lambda) = 1 - \lambda \Phi (I-\lambda K)^{-1} f = 1 - \sum_{n=0}^{\infty} \lambda^{n+1} \Phi K^n f.$$

Thus there is some difficulty in attempting to use the equation $d(\lambda) = 0$ to calculate the characteristic values of G. However it is easier to calculate $d(\lambda)$ than to calculate the Fredholm determinant [11, p. 56]. More specifically, in making approximate calculations of characteristic values it may be easier to use a truncation of $d(\lambda)$ than to use a truncation of the Fredholm determinant.

Consider the characteristic value problem,

(4.10) $$(I-\lambda G)u = 0 .$$

By (4.2) this may be rewritten

$$(I-\lambda K)u = \lambda f \Phi(u)$$

or

$$u = \Phi(u)\lambda(I-\lambda K)^{-1}f .$$

Thus the general form of the eigenfunctions of G will be

(4.11) $$u_\lambda = \alpha_\lambda (I-\lambda K)^{-1} f = \alpha_\lambda \sum_{n=0}^{\infty} \lambda^n K^n f,$$

and u_λ will satisfy (4.10) only if $d(\lambda) = 0$.

To conclude this chapter two examples of classical differential eigenvalue problems are solved using the integral equation generated by the Green's function for the given eigenvalue problem. The first is the eigenvalue problem for the vibrating string problem. The second example is the heat equation in cylindrical coordinates: Bessel's equation with two boundary conditions. This boundary value problem does not have an ordinary Green's function since the coefficient of the highest derivative vanishes. However in this special case an integral equation for the eigenfunctions can be derived. Furthermore this integral equation has a kernel of the type considered in Chapter I. Thus we can solve this problem by methods developed in this chapter.

Example 4.1 Consider the vibrating string problem;

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}$$

where $v(x,t)$ is defined for $0 \leq x \leq 1$ and $t \geq 0$ with boundary conditions

$$v(0,t) = v(1,t) = 0 \ ,$$

and the initial condition

$$v(x,0) = g(x).$$

Separating variables, the following boundary value problem is obtained;

(4.12) $$u''(x) = -\lambda u(x), \qquad 0 \leq x \leq 1,$$

(4.13) $$u(0) = u(1) = 0 \ .$$

The Green's function associated with the differential operator $L = \frac{d^2}{dx^2}$ with boundary conditions (4.13) is

$$G(x,s) = \begin{cases} s(x-1), & 0 \leq s \leq x \leq 1 \ , \\ x(s-1), & 0 \leq x \leq s \leq 1 \ . \end{cases}$$

Thus the following characteristic value problem arises;

$$u(x) = -\lambda \int_0^1 G(x,s)u(s)ds . \tag{4.14}$$

Using the decomposition outlined in Chapter I we obtain

$$u(x) = \lambda \int_0^x (s-x)u(s)ds + \lambda x \int_0^1 (1-s)u(s)ds$$

or in symbolic form

$$u = \lambda Ku + \lambda f\Phi(u)$$

where $(Ku)(x) = \int_0^x (s-x)u(s)ds$, $f(x) = x$ and $\Phi(u) = \int_0^1 (1-s)u(s)ds$.

In order that there exist nontrivial u satisfying (4. 12) it is sufficient that

$$d(\lambda) = 1-\lambda\Phi(I-\lambda K)^{-1}f = 0.$$

Now

$$[(I-\lambda K)^{-1}f](x) = \sum_{n=0}^{\infty} \lambda^n (K^n f)(x) = \frac{\sin\sqrt{\lambda}\, x}{\sqrt{\lambda}} ,$$

$$\Phi(I-\lambda K)^{-1}f = \frac{1}{\lambda} - \frac{\sin\sqrt{\lambda}}{\lambda^{3/2}}$$

and thus

$$d(\lambda) = \frac{\sin\sqrt{\lambda}}{\sqrt{\lambda}} .$$

Hence $d(\lambda) = 0$ iff $\sqrt{\lambda} = \pm n\pi$ or $\lambda = n^2\pi^2$. Thus the eigenvalues of (4.12) are $\lambda_n = n^2\pi^2$. Also we have

$$u_n(x) = a_n \sin(n\pi x)$$

as eigenfunctions. Thus known results are obtained.

Example 4.2 Consider the heat equation in cylindrical coordinates;

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial s^2} + \frac{1}{s}\frac{\partial v}{\partial s}$$

where $v(s,t)$ is defined for $0 \le s \le 1$ and $t \ge 0$ with the boundary conditions

$$v(1,t) = 0, \qquad v(0,t) < \infty ,$$

and the initial condition

$$v(s,0) = g(s).$$

Separating variables the following boundary value problem is obtained;

(4.15) $$[su'(s)]' = -\lambda\, su(s), \quad 0 \le s \le 1,$$

$$u(1) = 0, \quad u(0) < \infty . \tag{4.16}$$

Although no ordinary Green's function exists for the operator $(Lu)(s) = [su'(s)]'$ (since the coefficient of u'' vanishes at $s = 0$), a function $G(s, r)$ can be found which is integrable with respect to the measure $\mu(dr) = rdr$. Since s appears in the right member of (4.15) it may be reasonable to try working in this measure space.

Proceeding formally, we notice that

$$u_1(s) = 1$$

satisfies the boundary condition at $s = 0$ and

$$u_2(s) = \log s$$

satisfies the boundary condition at $s = 1$. Furthermore u_1 and u_2 satisfy the homogeneous equation associated with (4.15). Carrying out the calculations in the same spirit as suggested in Chapter II we find a function

$$H(s, r) = \begin{cases} \log s, & 0 \leq r \leq s \leq 1, \\ \log r, & 0 \leq s \leq r \leq 1 . \end{cases}$$

Thus formally we expect that a solution to the equation

$$[su'(s)]' = sh(s) \tag{4.17}$$

satisfying boundary conditions (4.16) would be

$$u(s) = \int_0^1 H(s,r)h(r)rdr. \tag{4.18}$$

Let $G(s,r) = rH(s,r)$. Then (4.18) can be rewritten

$$u(s) = \int_0^1 G(s,r)h(r)dr . \tag{4.19}$$

The kernel $G(s,r)$ is continuous. Further a simple calculation shows that $u(s)$ as given by (4.19) satisfies (4.17) as well as the boundary conditions (4.16). Thus we expect that solutions to the characteristic value problem

$$u(s) = -\lambda \int_0^1 G(s,r)u(r)dr \tag{4.20}$$

will give eigenfunctions for (4.15). Clearly $G(s,r)$ is the same type of kernel as was encountered in Chapter I (cf. equation (1.2)). Thus we find that (4.20) can be written

$$u(s) = \lambda \int_0^s (r \log r - r \log s)u(r)dr + \lambda \int_0^1 (-r \log r)u(r)dr$$

or symbolically,

$$u = \lambda Ku + \lambda f \Phi(u)$$

where $(Ku)(s) = \int_0^s (r \log r - r \log s)u(r)dr$, $f(s) \equiv 1$ and

$$\Phi(u) = \int_0^1 (-r \log r)u(r)dr.$$

In order that there exist $u_\lambda \not\equiv 0$ satisfying (4. 20) it is sufficient that

$$d(\lambda) = 1 - \lambda \Phi(I-\lambda K)^{-1}f = 0 .$$

As before

$$[(I-\lambda K)^{-1}f](s) = 1 + \sum_{n=1}^{\infty} \lambda^n (K^n f)(s) .$$

By induction it can be verified that

$$(K^n f)(s) = \frac{(-1)^n s^{2n}}{2^2 \cdot 4^2 \cdot \cdots \cdot (2n)^2}, \quad n = 1, 2, \cdots$$

and therefore

$$[(I-\lambda K)^{-1}f](s) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{\lambda} s)^{2n}}{2^2 \cdot 4^2 \cdot \cdots \cdot (2n)^2} = J_0(\sqrt{\lambda}\, s) .$$

Furthermore

$$\lambda\,\Phi(I-\lambda K)^{-1}f = -\left[\ \sum_{n=1}^{\infty} \frac{(-1)^n(\sqrt{\lambda})^{2n}}{2^2\cdot 4^2\cdot \cdots \cdot (2n)^2}\ \right]$$

and thus

$$d(\lambda) = J_0(\sqrt{\lambda}).$$

As expected the squares of the eigenvalues of (4. 15) are the zeros of the Bessel function J_0 and the eigenfunctions are

$$u_k(s) = \alpha_k\, J_0(\sqrt{\lambda_k}\, s), \qquad k = 1, 2, \cdots .$$

CHAPTER V

APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION $(I-\lambda K)u = h + \lambda f\Phi(u)$

In this chapter $(I-\lambda G)^{-1}$ is approximated by truncating all of the series which appear in the expression on the right side of equation (4.9). Error estimates are calculated giving an error bound for the approximate solution. From this calculation an error bound for the approximate calculation of the characteristic values arises.

Equation (4.9) is

$$(I-\lambda G)^{-1} = I + \lambda \frac{\sum_{n=0}^{\infty} \lambda^n [d(\lambda)K^{n+1} + f_\lambda \Phi K^n]}{d(\lambda)}$$

where

$$d(\lambda) = 1 - \lambda \Phi \sum_{n=0}^{\infty} \lambda^n K^n f \tag{5.1}$$

and

$$f_\lambda = \sum_{n=0}^{\infty} \lambda^n K^n f \; . \tag{5.2}$$

Truncating the series which appear on the right side of (4.9) we obtain

$$(I-\lambda G)_m^{-1} = I + \lambda \frac{\sum_{n=0}^{m} \lambda^n [d_m(\lambda)K^{n+1} + f_{\lambda m} \Phi K^n]}{d_m(\lambda)}$$

where

(5.3) $$d_m(\lambda) = 1 - \lambda \Phi \sum_{n=0}^{m} \lambda^n K^n f$$

and

(5.4) $$f_{\lambda m} = \sum_{n=0}^{m} \lambda^n K^n f.$$

Let

$$\Delta(\lambda) = \lambda \sum_{n=0}^{\infty} \lambda^n [d(\lambda)K^{n+1} + f_\lambda \Phi K^n]$$

and $\Delta_m(\lambda)$ the analogous truncated expression. Then

$\lambda \Gamma_\lambda = \frac{\Delta(\lambda)}{d(\lambda)}$ and

$$||(I-\lambda G)^{-1} - (I-\lambda G)_m^{-1}|| = || \frac{\Delta}{d} - \frac{\Delta_m}{d_m} || = \frac{||d_m \Delta - d\Delta_m||}{|d| \cdot |d_m|}$$

$$\leqq \frac{|d-d_m| \cdot ||\Delta|| + |d| \cdot ||\Delta - \Delta_m||}{|d| \cdot |d_m|}$$

To find an error bound we need upper bounds for $|d-d_m|$, $||\Delta||$, $|d|$ and $||\Delta-\Delta_m||$ and a lower bound for $|d|$. Let $|K(x,s)| \leqq M$, $(0 \leqq s \leqq x \leqq 1)$. Then

$$|K_n(x,s)| \leqq \frac{M^n(x-s)^{n-1}}{(n-1)!} \qquad [11, p. 16].$$

It is easily verified that

(5.5)
$$||K^n f|| \leqq \frac{M^n ||f||}{n!}$$

for any $f \epsilon C$. Thus

(5.6)
$$||K^n|| \leqq \frac{M^n}{n!} \quad .$$

From (5.1) and (5.5) we have

$$|d(\lambda)| \leqq 1 + C_\lambda$$

where $C_\lambda = |\lambda| \cdot ||\Phi|| \cdot ||f|| e^{|\lambda|M}$ and from (5.1), (5.3) and (5.5)

(5.7)
$$[d(\lambda) - d_m(\lambda)| \leqq |\lambda| \cdot ||\Phi|| \cdot ||f|| \cdot \epsilon_m(\lambda)$$

where $\epsilon_m(\lambda) = \sum_{n=m+1}^{\infty} \frac{(|\lambda|M)^n}{n!}$. From (5.2) and (5.5) it follows that

$$||f_\lambda|| \leqq ||f|| \cdot e^{|\lambda|M} .$$

Again using (5. 5) it is easy to show that

$$||f_\lambda - f_{\lambda m}|| \leqq ||f|| \epsilon_m(\lambda) .$$

Now

$$||\Delta|| = ||\lambda \sum_{n=0}^{\infty} \lambda^n [dK^{n+1} + f_\lambda \Phi K^n]|| \leqq |d| \sum_{n=-1}^{\infty} |\lambda|^{n+1} ||K^{n+1}||$$

$$+ |\lambda| \cdot ||f_\lambda|| \cdot ||\Phi|| \cdot \sum_{n=0}^{\infty} |\lambda|^n ||\mathbf{K}^n|| \leqq [1+2C_\lambda] e^{|\lambda|M} .$$

A short calculation shows that

$$||\Delta - \Delta_m|| \leqq |d| \sum_{n=m+1}^{\infty} |\lambda|^{n+1} ||K^{n+1}|| + |\lambda| \cdot ||f_\lambda|| \cdot ||\Phi|| \sum_{n=m+1}^{\infty} |\lambda|^n ||K^n||$$

$$+ |d - d_m| \sum_{n=0}^{m} |\lambda|^{n+1} ||K^{n+1}||$$

$$+ |\lambda| \cdot ||f_\lambda - f_{\lambda m}|| \cdot ||\Phi|| \cdot \sum_{n=0}^{m} |\lambda|^n ||K^n|| .$$

Then using (5. 6) and the estimates

$$\sum_{n=m+1}^{\infty} |\lambda|^{n+1} ||K^{n+1}|| \leqq \epsilon_m(\lambda),$$

$$\sum_{n=0}^{m} \frac{(|\lambda|M)^{n+1}}{(n+1)!} \leqq e^{|\lambda|M},$$

$$\sum_{n=0}^{m} \frac{(|\lambda|M)^{n}}{n!} \leqq e^{|\lambda|M}$$

we obtain

$$||\Delta - \Delta_m|| \leqq (1 + 4C_\lambda)\epsilon_m(\lambda) .$$

Then

$$||(I-\lambda G)^{-1} - (I-\lambda G)_m^{-1}|| \leqq \frac{1+6C_\lambda + 6C_\lambda^2}{|d_m||d|} \epsilon_m(\lambda).$$

To complete the error analysis a lower bound must be found for $|d|$. From (5.7)

$$|d_m| - |d| \leqq |\lambda| \cdot ||\Phi|| \cdot ||f|| \epsilon_m(\lambda)$$

or

$$|d| \geqq |d_m| - |\lambda| \cdot ||\Phi|| \cdot ||f|| \epsilon_m(\lambda) .$$

For fixed λ, assuming λ is not a characteristic value, we have

$$\epsilon_m(\lambda) \to 0 \quad \text{as} \quad m \to \infty$$

and further $|d(\lambda)| > 0$. Since $d_m(\lambda) \to d(\lambda)$ as $m \to \infty$ we have for sufficiently large m that $|d_m| - |\lambda| \cdot ||\Phi|| \cdot ||f|| \epsilon_m(\lambda) > 0$. Thus for sufficiently large m, the final form of the error is

$$||(I-\lambda G)^{-1} - (I-\lambda G)_m^{-1}|| \leqq \frac{1 + 6C_\lambda + 6C_\lambda^2}{|d_m|(|d_m| - |\lambda| \cdot ||\Phi|| \cdot ||f|| \cdot \epsilon_m(\lambda))} \epsilon_m(\lambda).$$

If (5.3) is used to compute approximate characteristic values on any compact subset of the scalar field, then (5.7) gives a bound on the error. Thus we have that the characteristic values can be uniformly approximated on compact sets.

Due to results obtained in the appendix an error analysis developed by Glahn [6, pp. 7-16] also applies to this problem. The two methods of analyzing the error are not directly comparable since different parameters appear in the two methods. Thus more work could be done here.

CHAPTER VI

SOLUTION OF THE INTEGRAL EQUATIONS

$$(I-\lambda K)u = h + \lambda \sum f_i \Phi_i(u)$$

In this chapter we shall solve the equation

$$(6.1) \qquad (I-\lambda G)u = h \qquad (h \in C)$$

where G has the decomposition

$$(6.2) \qquad G = K + \sum_{i=1}^{n} f_i \Phi_i .$$

As before K is a Volterra operator, $f_i \in C$, $i = 1, 2, \cdots, n$, $\Phi_i \in C^*$, $i = 1, 2, \cdots, n$ and the f_i and Φ_i are linearly independent. Results similar to those in Chapter IV are obtained. However, instead of the Fredholm alternative reducing to the alternative for a one dimensional system, it reduces to the alternative for an n-dimensional algebraic system. Again the solution is expressible as a quotient of an operator and an entire function of λ. Also similar to the case dealt with in Chapter IV, the zeros of this entire function comprise all of the characteristic values of the operator G. Finally an example is worked using the techniques of this chapter.

In order to obtain the solution in a form comparable to the solution in Chapter IV, it is necessary to introduce certain notation. Let

$$G = K + f^n \Phi^n \tag{6.3}$$

where $f^n : \mathfrak{F}^n \to C$ is defined by

$$f^n \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_i f_i$$

and $\Phi^n : C \to \mathfrak{F}^n$ is defined by

$$\Phi^n(u) = \begin{bmatrix} \Phi_1(u) \\ \Phi_2(u) \\ \cdot \\ \cdot \\ \cdot \\ \Phi_n(u) \end{bmatrix}$$

where the f_i and Φ_i are as above. Thus $f^n \Phi^n$ maps C into C. For any norm in $\mathfrak{F}^n$, e.g.

$$\left\| \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} \right\| = \max\{|a_i| : i = 1, 2, \cdots, n\},$$

f^n is a bounded linear mapping of $\mathfrak{F}^n$ into C and Φ^n is a bounded linear mapping of C into $\mathfrak{F}^n$. Hence $f^n\Phi^n$ is bounded.

From (6.3) we see that (6.1) is equivalent to

$$(6.4) \qquad (I-\lambda K)u = h + \lambda f^n\Phi^n(u)$$

and hence to

$$(6.5) \qquad u = (I-\lambda K)^{-1}h + \lambda (I-\lambda K)^{-1}f^n\Phi^n(u).$$

Let $f_\lambda^n : \mathfrak{F}^n \to C$ be defined by

$$f_\lambda^n \begin{bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ \cdot \\ a_n \end{bmatrix} = \sum_{i=1}^{n} a_i (I-\lambda K)^{-1} f_i \quad .$$

Then (6.5) may be rewritten

$$u = (I-\lambda K)^{-1}h + \lambda f_\lambda^n \, \Phi^n(u) \; . \tag{6.6}$$

Assuming there exists a u such that (6.6) holds, we may operate on both sides of (6.6) with Φ^n to obtain

$$\Phi^n(u) = \Phi^n(I-\lambda K)^{-1}h + \lambda \Phi^n f_\lambda^n \, \Phi^n(u) \; . \tag{6.7}$$

The operator $\Phi^n f_\lambda^n$ is a linear mapping of $\mathfrak{F}^n$ into itself and may therefore be characterized by a matrix A_λ^n. Let I^n be the identity mapping of $\mathfrak{F}^n$ onto itself. Then (6.7) may be re-written

$$(I^n - A_\lambda^n)\Phi^n(u) = \Phi^n(I-\lambda K)^{-1}h \; . \tag{6.8}$$

The matrix equation (6.8) has a unique solution for $\Phi^n(u)$ iff $(I^n-A_\lambda^n)^{-1}$ exists iff $d^n(\lambda) = \det(I^n-A_\lambda^n) \neq 0$. Let $B_\lambda^n = \operatorname{adj}(I^n-A_\lambda^n)$ where $\operatorname{adj}(I^n-A_\lambda^n)$ is the transpose of the matrix of cofactors of $(I^n-A_\lambda^n)$. If $d^n(\lambda) \neq 0$, the solution for $\Phi^n(u)$ is given by

$$\Phi^n(u) = \frac{B_\lambda^n}{d^n(\lambda)} \, \Phi^n(I-\lambda K)^{-1}h \; .$$

If $d^n(\lambda) \neq 0$, then

(6. 9) $$u = (I-\lambda K)^{-1}h + \frac{\lambda}{d^n(\lambda)} f_\lambda^n B_\lambda^n \Phi^n (I-\lambda K)^{-1}h$$

is equivalent to (6. 1). By exactly the same reasoning as in Chapter IV we have the following two theorems.

<u>Theorem 6. 1</u> $d^n(\lambda) = 0$ iff λ is a characteristic value of G.

<u>Theorem 6. 2</u> If $d^n(\lambda) \neq 0$, then equation (6. 9) gives the solution to (6. 1).

As in Chapter IV we have if $d^n(\lambda) \neq 0$, then

(6. 10) $$(I-\lambda G)^{-1} = (I-\lambda K)^{-1} + \frac{\lambda}{d^n(\lambda)} f_\lambda^n B_\lambda^n \Phi^n (I-\lambda K)^{-1} .$$

It is of interest to note that (6. 10) is similar in form to (4. 8). In fact B_λ^n, the adjoint matrix, is equal to (1) if $n = 1$.

Finally as in Chapter IV we see that λ is a characteristic value of G iff $d^n(\lambda) = 0$. Now if $d^n(\lambda) = 0$, then

$$u_\lambda = (I-\lambda K)^{-1} f^n \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_n \end{bmatrix}$$

are the eigenfunctions of G where the α_i are appropriate constants.

As an example of a G of the form considered in this chapter we shall solve the characteristic value problem which arises from the transverse oscillations of a homogeneous bar clamped at one end and free at the other.

Example 6.1 The oscillations are determined by the equation

$$\frac{\partial^4 z}{\partial x^4} + \frac{\partial^2 z}{\partial t^2} = 0$$

where $z(x,t)$ is defined on the strip $\{(x,t):\ 0 \leq x \leq 1,\quad t \geq 0\}$ and the boundary conditions are

$$z(0,t) = z_x(0,t) = z_{xx}(1,t) = z_{xxx}(1,t) = 0$$

[13, pp. 26-29]. Assuming $z(x,t) = u(x)e^{i\omega t}$, we are led to the ordinary differential equation

(6.11) $$\frac{d^4u(x)}{dx^4} = \lambda^4 u(x), \qquad 0 \leq x \leq 1\ ,$$

with the boundary conditions

(6.12) $$u(0) = u'(0) = u''(1) = u'''(1) = 0.$$

The Green's function for the operator $L = \frac{d^4}{dx^4}$ with boundary conditions (6. 12) is

$$G(x,s) = \begin{cases} \dfrac{3s^2x - s^3}{3!}, & 0 \leq s \leq x \leq 1, \\ \dfrac{3sx^2 - x^3}{3!}, & 0 \leq x \leq s \leq 1. \end{cases}$$

We have the following integral equation for u;

$$u(x) = \lambda^4 \int_0^1 G(x,s)u(s)ds .$$

Decomposing the integral operator, we obtain

$$(6.13) \qquad u(x) - \lambda^4 \int_0^x \frac{(x-s)^3}{3!} u(s)ds = \lambda^4 \{ \int_0^1 su(s)ds \cdot \frac{x^2}{2!} + [-\int_0^1 u(s)ds] \frac{x^3}{3!} \} .$$

Here it may be of some interest to briefly mention Tricomi's treatment of this problem so that similarities in both methods can be compared. Initially, Tricomi develops a method by which the solution to an initial value problem can be written as a Volterra equation [13, pp. 18-19] . In the case of equation (6. 11), Tricomi assumes that $u''(0) = c_2$ and $u'''(0) = c_3$. Then by the above mentioned method for solving initial value problems, he arrives at the equation

(6. 14) $$u(x)-\lambda^4 \int_0^x \frac{(x-s)^3}{3!} u(s)ds = \lambda^4(c_2x^2/2! + c_3x^3/3!)$$

[13, p. 28]. These two methods are connected by the "shooting" method mentioned by Henrici [7, pp. 345-346]. Briefly the "shooting" method assumes that every boundary value problem is equivalent to an initial value problem. The solution to the boundary value problem is represented as a parameterized solution to the initial value problem (e. g. equation (6. 14) where the parameters are c_1 and c_2). Then the parameters are determined. The method used to obtain (6. 13) gives these parameters as linear functionals of u.

Now let us proceed to the problem of finding the characteristic values. Let (6. 13) be rewritten as

(6. 15) $$(I-\lambda^4K)u = \lambda^4 f^2 \Phi^2(u)$$

where

$$f^2 \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 x^2/2! + a_2 x^3/3!$$

and

$$\Phi^2(u) = \begin{bmatrix} \Phi_1(u) \\ \\ \Phi_2(u) \end{bmatrix} = \begin{bmatrix} \int_0^1 su(s)ds \\ -\int_0^1 u(s)ds \end{bmatrix} .$$

There exists a solution u if $d^2(\lambda) = 0$, where

$$d^2(\lambda) = \det \begin{bmatrix} 1 - \lambda^4\Phi_1(I-\lambda^4K)^{-1}\, x^2/2! & -\lambda^4\Phi_1(I-\lambda^4K)^{-1}\, x^3/3! \\ -\lambda^4\Phi_2(I-\lambda K)^{-1}\, x^2/2! & 1-\lambda^4\Phi_2(I-\lambda^4K)^{-1}\, x^3/3! \end{bmatrix}$$

$$= \det \begin{bmatrix} 1-\lambda^2/2 \int_0^1 s(\cosh\lambda s-\cos\lambda s)ds & (-\lambda)/2 \int_0^1 s(\sinh\lambda s-\sin\lambda s)ds \\ \lambda^2/2 \int_0^1 (\cosh\lambda s-\cos\lambda s)ds & 1+\lambda/2 \int_0^1 (\sinh\lambda s-\sin\lambda s)ds \end{bmatrix}$$

$$= \frac{1}{2}(1 + \cosh\lambda \cos\lambda).$$

Hence we obtain the anticipated result that the characteristic values are the zeros of the transcendental equation

$$1 + \cosh\lambda \cos\lambda = 0 \quad [13, \text{p. } 29].$$

BIBLIOGRAPHY

1. Allis, W. P. and Philip M. Morse. The effect of exchange on the scattering of slow neutrons from atoms. Physical Review 44: 269-276. 1933.

2. Brysk, Henry. Determinantal solution of the Fredholm equation with Green's function kernel. Journal of Mathematical Physics 4: 1536-1538. 1963.

3. Coddington, Earl A. An introduction to ordinary differential equations. Englewood Cliffs, N. J., Prentice Hall, 1961. 292 p.

4. Davis, H. T. The theory of linear operators. Bloomington, Indiana, Principia Press, 1936. 628 p.

5. Drukarev, G. F. The theory of collisions of electrons and atoms. Soviet Physics. JETP 4: 309-320. 1957.

6. Glahn, Thomas Leroy. An error bound for an iterative method of solving Fredholm integral equations. Master's thesis. Corvallis, Oregon State University, 1954. 22 numb. leaves.

7. Henrici, Peter. Discrete variable methods in ordinary differential equations. New York, Wiley, 1962. 407 p.

8. Ince, E. L. Ordinary differential equations. New York, Longmans, Green and Company, 1927. 558 p.

9. Indritz, Jack. Methods in analysis. New York, Macmillan, 1963. 481 p.

10. Manning, Irwin. A theorem on the determinantal solution of the Fredholm equation. Journal of Mathematical Physics 5: 1223-1225. 1964.

11. Miklin, S. G. Integral equations and their applications to certain problems in mechanics, mathematical physics and technology. New York, Pergamon Press, 1957. 338 p.

12. Morse, Philip M. and Herman Feshbach. Methods of theoretical physics. 2 vol. New York, McGraw-Hill, 1953. 1978 p.

13. Tricomi, Francesco. Integral equations. New York, Interscience, 1957. 238 p.

APPENDICES

APPENDIX

COMPARISON OF THE FREDHOLM RESOLVENT OPERATOR AND EQUATION (4.9)

As remarked in Chapter IV, Brysk's assertion can be proved using tools developed by Manning. This result can be used to show that the numerator and denominator of

$$(1) \qquad \sum_{n=0}^{\infty} \lambda^n [d(\lambda) K^{n+1} + f_\lambda \Phi K^n] / d(\lambda)$$

are equal to the numerator and denominator respectively of the Fredholm resolvent operator. Theorems 4.1 and 4.2 then follow immediately from the equality of (1) and the Fredholm resolvent operator.

The Fredholm resolvent operator is given by

$$(2) \qquad \Gamma_\lambda = \left(\sum_{n=0}^{\infty} \lambda^n D^n \right) / \left(\sum_{n=0}^{\infty} \lambda^n d_n \right)$$

where $d_0 = 1$, $D_0 = G$,

$$d_n = (-1) n^{-1} \int_0^1 D_{n-1}(s, s) ds, \quad n = 1, 2, \cdots,$$

and the D_n are given by the recursion relation

(3) $$D_n = d_n G + GD_{n-1}, \quad n = 1, 2, \cdots,$$

[10, pp. 1223-1224; 11, p. 54]. Brysk's assertion was that

(4) $$\sum_{n=0}^{\infty} \lambda^n GK^n f/d(\lambda)$$

and $\Gamma_\lambda f$ were identical [2, pp. 1537-1538], (note that (4) is obtained from (1) by operating on f and simplifying). The tools developed by Manning are stated below in Lemmas 1 and 2 without proof.

Lemma 1 $\quad \{Gh\} = \Phi(h) \quad$ for $\quad h \in C$

where $\quad \{h\} = \lim_{x \to 0^+} h(x)/f(x)$.

Lemma 2 $\quad d_n = -\{D_{n-1} f\}, \quad n = 1, 2, \cdots.$

These tools yield

Theorem 1 $\quad d_{n+1} = -\Phi K^n f, \quad n = 0, 1, \cdots,$

and

$$GK^n f = D_n f, \quad n = 0, 1, \cdots .$$

Proof: (by induction)

For $n = 0$, $d_1 = -\{D_0 f\} = -\{Gf\} = -\Phi K^0 f$, and $GK^0 f = Gf = D_0 f$.

By Lemma 2

$$-d_{n+2} = \{D_{n+1} f\}$$

$$= \{d_{n+1} f + GD_n f\} \qquad \text{(by (3))}$$

$$= \{-\Phi(K^n f)Gf + G(GK^n f)\} \qquad \text{(by the inductive hypothesis)}$$

$$= -\Phi(K^n f)\{Gf\} + \{G(GK^n f)\}$$

$$= -\Phi(K^n f)\Phi(f) + \Phi(GK^n f) \qquad \text{(by Lemma 1)}$$

$$= -\Phi(K^n f)\Phi(f) + \Phi[(K+f\Phi)K^n f] \qquad \text{(by equation (4. 2))}$$

$$= \Phi K^{n+1} f .$$

Also by (3)

$$D_{n+1} f = d_{n+1} Gf + GD_n f$$

$$= -\Phi(K^n f)Gf + G(GK^n f) \qquad \text{(by the inductive hypothesis)}$$

$$= G(G-f\Phi)K^n f$$

$$= GKK^n f \qquad \text{(by equation (4. 2))}$$

$$= GK^{n+1} f.$$

Hence the conclusion follows by induction.

An interesting conclusion which may be drawn from Theorem 1 is

$$d(\lambda) = \sum_{n=0}^{\infty} d_n \lambda^n .$$

Using this we can write

$$\sum_{n=0}^{\infty} \lambda^n [d(\lambda)K^{n+1} + f_\lambda \Phi K^n] = \sum_{p=0}^{\infty} \lambda^p [\sum_{m=0}^{p} (d_m K^{p-m+1} + K^m f \cdot \Phi K^{p-m})] .$$

Then defining D'_p by

$$D'_p = \sum_{m=0}^{p} (d_m K^{p-m+1} + K^m f \cdot \Phi K^{p-m}),$$

it is easy to verify that $D'_0 = G$ and that D'_p satisfies the recursion relation (3). Hence (1) is identical to (2) and thus Theorems 4. 1 and 4. 2 follow immediately since these properties hold for the Fredholm resolvent operator.

This approach while being more tedious (the proof of Lemma 2 as proved by Manning is largely bookkeeping) yields the more satisfying result that $d_{n+1} = -\Phi K^n f$ and $D_p = D'_p$. This conclusion cannot be drawn from the remarks in Chapter IV leading up to the equality of the expression given in (1) and Γ_λ .

The method given in this thesis of solving equation (4. 1) does give greater insight into the nature of the decomposition and how it is used to solve the problem. Also it generalizes to operators with the type of decomposition considered in Chapter VI with greater ease (once the notation was developed the proof of Theorems 6. 1 and 6. 2 were identical to the proofs of Theorems 4. 1 and 4. 2).