Let $X$ be a compact topological space and $\varphi:X \to X$ a continuous map. Denote by $G(X)$ (or simply by $G$ when $X$ is understood) the class of all open covers of $X$. For $\alpha \in G$ we define $N(\alpha)$ as the number of sets in a subcover of $\alpha$ of minimal cardinality. It is clear that $\varphi^{-1}\alpha = \{\varphi^{-1}(U): U \in \alpha\} \in G$. Let $\alpha, \beta \in G$. The join of $\alpha$ and $\beta$, written $\alpha \vee \beta$, is defined by $\alpha \vee \beta = \{U \cap V: U \in \alpha, V \in \beta\}$. More generally, if $\alpha_i \in G$, $i = 1, 2, \ldots, n$, then the join of the $\{\alpha_i\}_{i=1}^n$ is denoted and defined by $\bigvee_{i=1}^n \alpha_i = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$

$$= \{U_1 \cap U_2 \cap \ldots \cap U_n: U_i \in \alpha_i, \; i = 1, 2, \ldots n\}.$$  

Set $h(\alpha, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right)$ (it is well known that
this limit exists and is finite (see [2])). The topological entropy of \( \varphi \), written as \( h(\varphi) \), is then defined by

\[
h(\varphi) = \sup_{\alpha \in \mathcal{G}_f(X)} h(\alpha, \varphi).
\]

Topological entropy was first introduced in [2] as a notion analogous to the notion of measure theoretic entropy, and was defined for compact spaces. It is the main purpose of this investigation to extend this notion to noncompact spaces.

Suppose then, that \( X \) is a noncompact topological space. One way to calculate topological entropy on noncompact spaces is to first compactify the space \( X \) (say \( X^* \) is a compactification of \( X \)), extend the map \( \varphi: X \to X^* \) to \( \varphi^*: X^* \to X^* \), and define the topological entropy of \( \varphi \) to be \( h(\varphi^*) \); another way is to define

\[
h(\varphi) = \sup_{\alpha \in \mathcal{G}_f(X)} h(\alpha, \varphi),
\]

where \( \mathcal{G}_f(X) \) is the class of all finite open covers of \( X \); while still another method utilizes the notion of uniform spaces. Several definitions of topological entropy are given for noncompact spaces. We then prove some properties as well as some relationships between the various definitions.

Let \((X, \mathcal{J})\) be a topological space and let \( A \subseteq X \). Then we say \( A \) is nearly open if there exists a first category set \( B \) such that \( x \in A - B \implies x \in A^o \). The notation \( A^o \) means the interior of \( A \). 

A
nearly open cover is a cover whose members are nearly open sets. A one-to-one function \( \varphi \) from \( X \) onto \( X \) is called a near homeomorphism if there exists a first category set \( N_\varphi \) such that both \( \varphi \) and \( \varphi^{-1} \) are continuous on \( (X - N_\varphi) \). We say that \( \varphi: X \to X \) is a first category preserving transformation if for each first category set \( A \), both \( \varphi(A) \) and \( \varphi^{-1}(A) \) are first category sets.

Most of the definitions and basic properties of \([2]\) are valid for finite covers and first category preserving near homeomorphisms. Several properties involving these concepts are proven.

Now suppose \((X, \mathcal{J})\) is a compact metric space and \( A \subseteq X \) where \( A \) is a closed set having nonempty interior and let \( T:X \to X \) be a homeomorphism. The induced transformation \( T_A \) (first defined in \([16]\)) is defined, for \( T \) recurrent, by

\[
T_A(x) = T^K x \quad \text{where} \quad x, T^K(x) \in A \quad \text{and} \quad T^i x \not\in A \quad \text{for} \quad 0 < i < K.
\]

Some properties of \( T_A \) are shown and it is conjectured that

\[
h(T) = \eta(A, T) h(T_A)
\]

where

\[
\eta(A, T) = \sup_{x \in X} \lim_{n \to \infty} \inf \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(T^i x).
\]

Here \( \chi_A \) is the characteristic function of \( A \).
In [23] the concept of sequence entropy, as defined by Kushnirenko [21], is discussed. This concept has an obvious topological analog, and "extends" the results of [2] in a way quite different from that of the previous paragraphs.

Let \( X \) be a compact topological space and \( T:X \to X \) a continuous map and let \( \sigma = \langle t_n \rangle = \langle t_1, t_2, \ldots \rangle \) be a strictly increasing sequence of nonnegative integers.

Define \( h_\sigma(\alpha, T) = \limsup_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right) \).

The sequence entropy of \( T \) is then defined by \( h_\sigma(T) = \sup_{\alpha \in \mathcal{G}} h_\sigma(\alpha, T) \). A sequence \( \sigma = \langle t_n \rangle \) is said to have bounded gaps if there exists a positive constant \( K \) such that \( t_n - t_{n-1} < K \) for \( n \geq 1 \). Let \( d(\sigma) = \limsup_{n \to \infty} \frac{t_n}{n} \). We prove several properties using these concepts and in particular we compare \( h_\sigma(T) \) and \( h(T) \).

Even more generality may be obtained by letting \( \sigma = \langle t_n \rangle \) be any arbitrary sequence, and \( T \) a first category preserving near homeomorphism.
TOPOLOGICAL ENTROPY FOR NONCOMPACT SPACES

AND OTHER EXTENSIONS

by

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A THESIS

submitted to
Oregon State University

in partial fulfillment of
the requirements for the
degree of

DOCTOR OF PHILOSOPHY

June 1972
ACKNOWLEDGEMENTS

I wish to express my gratitude to Dr. James R. Brown for providing many helpful criticisms and suggestions during the preparation of this thesis; the numerous conferences we had were very beneficial.

I also wish to thank Dr. L. Wayne Goodwyn for his helpful suggestions and for the interest he showed in this work.
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We will use the following notation: $G(X)$, or simply $G$ when the meaning is clear, will denote the class of all open covers of $X$ while $G_f(X)$, or simply $G_f$, will denote the class of all finite open covers of $X$. Suppose that $X$ is a compact topological space and $\phi: X \to X$ is continuous. Let $\alpha_i \in G$ for $i = 1, 2, \ldots, n$. The join of the $\alpha_i$'s, is denoted and defined by

$$\bigvee_{i=1}^{n} \alpha_i = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$$

$$= \{U_1 \cap \ldots \cap U_n : U_i \in \alpha_i, \ i = 1, \ldots, n\}.$$

Clearly if $\alpha \in G$ then $\phi^{-1}\alpha = (\phi^{-1}(U) : U \in \alpha) \in G$. Let $\alpha \in G(X)$. We define $N_X(\alpha)$ (or simply $N(\alpha)$ when the space $X$ is understood) as the number of sets in a subcover of $\alpha$ of minimal cardinality. Set

$$h(\alpha, \phi) = \lim_{n \to \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} \phi^{-i} \alpha\right)$$

and

$$h(\phi) = \sup_{\alpha \in G} h(\alpha, \phi).$$
The quantity $h(\varphi)$ is called the topological entropy of $\varphi$ (see [2]).

Topological entropy was introduced as a notion analogous to measure theoretic entropy. For a good introduction to topological entropy and related reading see [2], [10], and [18]; while [4], [14], and [24] are good references for measure theoretic entropy.

Let $X$ be a noncompact Hausdorff space and $T: X \to X$ a continuous map. The purpose of this work is to extend the notion of topological entropy, originally defined for continuous mappings on compact spaces, to the less restrictive case of noncompact spaces. To calculate topological entropy for noncompact spaces, at least two approaches appear natural: one is to first compactify the space and consider the extension $T^*$ of $T$ to the compactification $X^*$ of $X$; and the second approach is consider only finite open covers of $X$. Our main result is as follows: let $X$ be a noncompact normal $T_1$-space, $T: X \to X$ a continuous map, $X^*$ the Stone-Cech compactification of $X$, and $T^*: X^* \to X^*$ the continuous extension of $T$ to $X^*$. Define $h^2(T) = h(T^*)$ and $h^3(T) = \sup_{\alpha \in \mathcal{D}_f(X)} h(\alpha, T)$. We prove $h^2(T) = h^3(T)$.

Since there are many ways to compactify a space one would rightly suspect many possible alternate definitions.
for topological entropy. We consider the one-point compactification, the Stone-Chech compactification and the Wallman compactification. Another approach, not involving a compactification, utilizes the notion of uniform spaces. This is done in Chapter IV. Both [13] and [5] also deal with topological entropy on noncompact spaces.

Most of the basic properties of [2] are valid for finite nearly open covers and first category preserving near homeomorphisms. In Chapter V we define and use these notions, interesting in their own right, and prove some properties along the way. Then we define the induced transformation and prove some of its properties. We give a conjecture relating the topological entropy of $T$ with that of the induced transformation.

Finally, we "extend" the concept of topological entropy in a different way. Let $\sigma$ be a sequence $\sigma = \langle t_1, t_2, \ldots \rangle$ such that $t_i > 0$ and $t_n < t_{n+1}$ for $n = 1, 2, \ldots$. Suppose also that $X$ is a compact space. We define

1) $h_{\sigma}(\alpha, T) = \limsup_{n \to \infty} \frac{1}{n} \log N\left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right)$, and

2) $h_{\sigma}(T) = \sup_{\alpha \in \mathcal{G}} h_{\sigma}(\alpha, T)$,

where $T: X \to X$ is a continuous map and call $h_{\sigma}(T)$ the $\sigma$-sequence entropy of $T$. Several propositions are proven and we compare $h_{\sigma}(T)$ and $h(T)$. 
But first, for the sake of completeness, we reproduce some of the material of [2] and [10].

In the remainder of this chapter, X will be a topological space.

1.1 Definition. Let $\alpha = \{G_i\}$ be a class of subsets of $X$ such that $A \subseteq \bigcup_i G_i$ for some $A \subseteq X$. Then $\alpha$ is called a cover of $A$, and an open cover if each $G_i$ is open.

1.2 Definition. For any two covers $\alpha$ and $\beta$ of $X$, we define their join, written $\alpha \vee \beta$, by

$$\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}.$$

Clearly $\alpha \vee \beta = \beta \vee \alpha$. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are covers of $X$, we write and define the join of $\{\alpha_i\}_{i=1}^n$ as

$$\bigvee_{i=1}^n \alpha_i = \alpha_1 \vee \ldots \vee \alpha_n = \{U_1 \cap \ldots \cap U_n : U_i \in \alpha_i\}$$

for $i = 1, \ldots, n$.

1.3 Definition. Let $\alpha$ and $\beta$ be covers of $X$. We say that $\alpha$ refines $\beta$, written as $\alpha \succ \beta$ or $\beta \prec \alpha$, if each member of $\alpha$ is a subset of some member of $\beta$.

1.4 Definition. Let $\alpha$ be a cover of $X$. Define $N(\alpha)$ as the number of sets in a subcover of $\alpha$ of minimal cardinality.
1.5 Definition. Let \( \varphi : X \to X \) be a function from \( X \) into itself and let \( \alpha \) be a cover of \( X \). Then \( \varphi^{-1} \alpha \) is defined by
\[
\varphi^{-1} \alpha = \{ \varphi^{-1}(U) : U \in \alpha \}.
\]

It is clear that \( \varphi^{-1} \alpha \) is also a cover of \( X \).

1.6 Proposition. Let \( \alpha \) and \( \beta \) be covers of \( X \). Then
\[
\alpha < \beta \Rightarrow N(\alpha) \leq N(\beta).
\]

Proof: Clearly if \( N(\alpha) = \infty \), then \( N(\beta) = \infty \). If \( N(\beta) = \infty \) then of course \( N(\alpha) \leq N(\beta) \). Suppose \( N(\beta) < \infty \).

Let \( \gamma \) be a subcover of \( \beta \) with \( N(\beta) \) members. Then for each \( U \) in \( \gamma \), there exists a \( U' \) in \( \alpha \) such that \( U \subseteq U' \). Then \( \{ U' : U \in \gamma \} \) is a subcover of \( \alpha \) with no more members than \( \gamma \) and so \( N(\alpha) \leq N(\beta) \).

The next proposition, though simple in appearance and proof, turns out to be remarkably useful.

1.7 Proposition. Let \( \alpha \) and \( \beta \) be covers of \( X \). Then
\[
N(\alpha \vee \beta) \leq N(\alpha)N(\beta).
\]

Proof: If at least one of \( N(\alpha) \) and \( N(\beta) \) is infinite, the proposition is obviously true. So suppose \( N(\alpha) < \infty \) and \( N(\beta) < \infty \). Let \( \alpha' \) be a subcover of \( \alpha \) with \( N(\alpha) \) members and \( \beta' \) be a subcover of \( \beta \) with \( N(\beta) \) members. Then \( \alpha' \vee \beta' \) is a subcover of \( \alpha \vee \beta \) with no more than
N(α)N(β) members and hence N(α ∨ β) ≤ N(α)N(β).

1.8 Proposition. Let φ:X → X be a map from X into itself. If α is a cover of X, then

\[ N(φ^{-1}α) \leq N(α). \]

Proof: The proof is clear if N(α) = ∞, so assume N(α) < ∞. Let γ be a subcover of α with N(α) members. Then \( φ^{-1}γ = \{φ^{-1}(U): U ∈ γ\} \) is a subcover of \( φ^{-1}α \) with no more members than γ. Therefore

\[ N(φ^{-1}α) \leq N(α). \]

1.9 Proposition. Let φ:X → Y be a map from a space X onto another space Y. If α is a cover of Y, then

\[ N(φ^{-1}α) = N(α). \]

Proof: If N(α) = ∞, then N(φ^{-1}α) = ∞ for if N(φ^{-1}α) < ∞ it is easy to show that N(α) < ∞. So assume N(α) < ∞. Let α' be a subcover of α with N(α) members. Then \( φ^{-1}α' \) is a subcover of \( φ^{-1}α \) with N(α) members. Hence N(φ^{-1}α) ≤ N(α). Next, let γ be a subcover of \( φ^{-1}α \) with N(φ^{-1}α) members. Then γ has the form γ = \( \{φ^{-1}(U): U ∈ γ'\} \) where γ' is a subset of α. But φ is onto Y implies γ' must be a cover. So N(α) ≤ N(φ^{-1}α).

1.10 Proposition. Let φ:X → X and let α be a finite cover of X. Then
\[
\lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right)
\]
exists, is finite, and in fact equals
\[
\inf \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right).
\]

Proof: For each \( n \in \mathbb{Z}^+ \) (\( \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \)), define
\[
H_n = \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right).
\]
Then, using the remark on page 9,
\[
H_{m+n} = \log N \left( \bigvee_{i=0}^{m+n-1} \varphi^{-i} \alpha \right) = \log N \left( \bigvee_{i=0}^{m-1} \varphi^{-i} \alpha \vee \varphi^{-m} \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right) \right)
\]
\[
\leq \log N \left( \bigvee_{i=0}^{m-1} \varphi^{-i} \alpha \right) + \log N \left( \varphi^{-m} \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right) \right)
\]
\[
\leq H_m + H_n, \text{ for } m, n \in \mathbb{Z}^+.
\]
Now using the facts that \( H_n > 0 \) and \( H_{m+n} \leq H_m + H_n \) for all \( m, n \in \mathbb{Z}^+ \), it is easy to show that \( \lim_{n \to \infty} \frac{1}{n} H_n \) exists and equals \( \inf \frac{1}{n} H_n \). See [28] for the proof.

1.11 Definition. Let \( \alpha \) be a finite cover of \( X \) and \( \varphi: X \to X \). Define
\[
h(\alpha, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right).
\]

1.12 Corollary. Let \( \alpha \) be a finite cover of \( X \) and \( \varphi: X \to X \). Then for each \( m \in \mathbb{Z}^+ \),
\[ h(\alpha, \varphi) \leq \frac{1}{m} \log N \left( \bigvee_{i=0}^{m-1} \varphi^{-i} \alpha \right). \]

Proof: The proof is clear.

1.13 Proposition. Let \( \varphi: X \to X \) and let \( \alpha \) and \( \beta \) be finite covers of \( X \). Then

\[ \alpha < \beta \implies h(\alpha, \varphi) \leq h(\beta, \varphi). \]

Proof: For each \( n \in \mathbb{Z}^+ \), it is clear that

\[ \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha < \bigvee_{i=0}^{n-1} \varphi^{-i} \beta. \]

Hence by (1.6)

\[ \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right) \leq \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \beta \right). \]

Let \( n \to \infty \) to get the desired result.

1.14 Proposition. Let \( \alpha \) be any finite cover of \( X \) and \( \varphi: X \to X \). Then for each \( n \in \mathbb{Z}^+ \),

\[ h \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha, \varphi \right) = h(\alpha, \varphi). \]

Proof: Let \( \beta_n = \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \). Then for each \( m \in \mathbb{Z}^+ \),

\[ N(\beta_n \lor \varphi^{-1} \beta_n \lor \ldots \lor \varphi^{-m+1} \beta_n) \]

\[ = N(\alpha \lor \varphi^{-1} \alpha \lor \ldots \lor \varphi^{-m-n+2} \alpha). \]

Thus \( h(\beta_n, \varphi) = \lim_{m \to \infty} \frac{1}{m} \log N \left( \bigvee_{i=0}^{m-1} \varphi^{-i} \beta_n \right) \).
\[
\lim_{m \to \infty} \frac{m + n - 1}{m \log N} = \log \left( \sum_{i=0}^{(m+n-1)-1} \varphi^{-1}_i \right) = h(\alpha, \varphi).
\]

**1.15 Proposition.** If \( \varphi : X \to X \) is a homeomorphism and \( \alpha \) is any finite cover of \( X \), then for each \( n \in \mathbb{Z}^+ \),

\[
h \left( \bigvee_{i=-n}^{n} \varphi^{-1}_i \alpha, \varphi \right) = h(\alpha, \varphi).
\]

**Proof:** The proof is similar to that of (1.14).

**1.16 Definition.** By a flow we mean a pair \((X, T)\), where \( X \) is a compact Hausdorff space and \( T : X \to X \) is continuous.

**1.17 Definition.** Let \((X, T)\) and \((Y, S)\) be flows and let \( f : X \to Y \) be a continuous, onto map such that \( f \circ T = S \circ f \). Then we call \( f \) a flow homomorphism from the flow \((X, T)\) onto the flow \((Y, S)\). We also say that \((Y, S)\) is the homomorphic image of \((X, T)\).

**Remark.** Let \( \alpha \) and \( \beta \) be covers of \( X \) and \( \varphi : X \to X \). Then \( \varphi^{-1} (\alpha \vee \beta) = \varphi^{-1} \alpha \vee \varphi^{-1} \beta \).

**1.18 Proposition.** Let \( f \) be a flow homomorphism from \((X, T)\) onto \((Y, S)\), and let \( \alpha \) be an open cover of \( Y \). Then

\[
h(\alpha, S) = h(f^{-1} \alpha, T).
\]
Proof: Let \( n \in \mathbb{Z}^+ \). Then

\[
\begin{align*}
    f^{-1}\left(\bigvee_{i=0}^{n-1} S^{-i}a\right) &= \bigvee_{i=0}^{n-1} f^{-1}(S^{-i}a) \\
    &= \bigvee_{i=0}^{n-1} T^{-i}(f^{-1}a).
\end{align*}
\]

Hence, by (1.9), we have

\[
\frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} S^{-i}a\right) = \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}(f^{-1}a)\right).
\]

Now let \( n \to \infty \).

1.19 Definition. Let \( X \) be a compact topological space and \( \varphi: X \to X \) a continuous map. The topological entropy of \( \varphi \) is defined by

\[
h(\varphi) = \sup_{\alpha \in \mathcal{U}} h(\alpha, \varphi).
\]

The next proposition shows that topological entropy does not increase under homomorphic images.

1.20 Proposition. Let \( f \) be a flow homomorphism from the flow \((X, T)\) onto the flow \((Y, S)\). Then

\[
h(S) \leq h(T).
\]

Proof: This follows easily from (1.18).

1.21 Proposition. Let \( X \) be a compact topological space and \( \varphi: X \to X \) continuous. Then for \( k \) a positive integer

\[
h(\varphi^k) = kh(\varphi).
\]

Proof: \( h(\varphi^k) \geq h(\alpha \vee \varphi^{-1}a \vee \ldots \vee \varphi^{-k+1}a, \varphi^k) \)
1.22 Proposition. Let $X$ be a compact topological space and $\varphi : X \to X$ continuous. Also let $Y$ be a closed $\varphi$-invariant subset of $X$ and $\varphi \mid Y$ be the restriction of $\varphi$ to $Y$. Then

$$h(\varphi \mid Y) \leq h(\varphi).$$

Proof: Let $\alpha$ be any open cover of $X$. Then clearly $N_Y(\alpha) \leq N_X(\alpha)$ and hence

$$N_Y \left( \bigvee_{i=0}^{n-1} \varphi \mid Y^{-i} \alpha \right) = N_Y \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right) \leq N_X \left( \bigvee_{i=0}^{n-1} \varphi^{-i} \alpha \right).$$

This implies $h(\alpha, \varphi \mid Y) \leq h(\alpha, \varphi) \leq h(\varphi)$. Now for each open cover $\beta$ of $Y$, there exists an open cover $\alpha$
of \( X \) such that \( \beta = \alpha|Y \) where \( \alpha|Y = \{U \cap Y : U \in \alpha\} \).

Hence

\[
h(\varphi|Y) = \sup_{\beta \in \mathfrak{A}(Y)} h(\beta, \varphi|Y) \leq \sup_{\alpha \in \mathfrak{A}(X)} h(\alpha, \varphi) = h(\varphi).
\]

1.23 Theorem (The Product Theorem). Let \( X \) and \( Y \) be two compact topological Hausdorff spaces and let \( \varphi_1 : X \rightarrow X \), and \( \varphi_2 : Y \rightarrow Y \) be continuous. Then

\[
h(\varphi_1 \times \varphi_2) = h(\varphi_1) + h(\varphi_2)
\]

where \( \varphi_1 \times \varphi_2 : X \times Y \rightarrow X \times Y \) is defined by

\[
(\varphi_1 \times \varphi_2)(x, y) = (\varphi_1(x), \varphi_2(y)).
\]

Proof: The proof can be found in [11].

For completeness we will give the measure theoretic definitions. We let \( M(X, T) \) denote the set of all \( T \)-invariant regular probability measures on the Borel sets of \( X \). A measure \( \mu \) is \( T \)-invariant if

\[
\mu(A) = \mu(T^{-1}(A)) \quad \text{for every Borel set } A \subseteq X.
\]

Let \( \mu \in M(X, T) \) and \( \alpha \) be a finite measurable partition of \( X \), and set

1) \( H_\mu(\alpha) = -\sum_{U \in \alpha} \mu(U) \log \mu(U), \)

2) \( h_\mu(\alpha, T) = \lim_{n \to \infty} \frac{1}{n} H_\mu \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right), \) and
3) \[ h_\mu(T) = \sup \{ h_\mu(\alpha, T) : \alpha \text{ is a finite measurable partition of } X \}. \]

The quantity \( h_\mu(T) \) is called the measure theoretic entropy of \( T \) with respect to \( \mu \).

The next theorem shows that topological entropy bounds measure theoretic entropy.

1.24 Theorem (Goodwyn [12]). Let \( X \) be a compact Hausdorff space and \( T : X \to X \) a continuous map. Then if \( \mu \in \mathcal{M}(X, T) \),

\[ h_\mu(T) \leq h(T). \]

1.25 Remark. A recent result due to Goodman [A] shows that if \( X \) is compact Hausdorff and \( T : X \to X \) is continuous then

\[ h(T) = \sup_{\mu \in \mathcal{M}(X, T)} h_\mu(T). \]
CHAPTER II
BASIC DEFINITIONS AND PROPERTIES

Although many of our results are valid for $T$ merely continuous, we will assume, unless specified otherwise, that $T$ is, in fact, a homeomorphism throughout Chapters II and III.

2.1 Definition. A topological space $(X, \mathcal{J})$ is called zero-dimensional (see [25 p.98]) iff there is a base for $\mathcal{J}$ consisting of sets which are at the same time open and closed.

2.2 Definition. The one-point compactification of a topological space $(X, \mathcal{J})$ is the space $(X^*, \mathcal{J}^*)$ with $X^* = X \cup \{\infty\}$ where $\infty$ is distinct from every other point in $X$ and $\mathcal{J}^*$ consists of each member of the topology $\mathcal{J}$ and all subsets $U$ of $X^*$ such that $X^* - U$ is a closed and compact subset of $X$. See [17 p.150].

2.3 Definition. Let $X$ be a topological space and $F(X)$ the family of all continuous functions on $X$ to the closed unit interval $I$. Then $I^{F(X)}$ (the product of $I$ taken $F(X)$ times) is compact by the Tychonoff
Theorem. The evaluation map $e : X \to \mathcal{I}^F(X)$ defined by $x \mapsto e(x)$ whose $f$-th coordinate is $f(x)$ for each $f \in F(X)$ is a continuous map from $X$ into $\mathcal{I}^F(X)$.

If $X$ is a Tychonoff space, then $e$ is a homeomorphism of $X$ onto a subspace of $\mathcal{I}^F(X)$. The Stone-Cech compactification is the pair $(e, \beta(X))$ where $\beta(X)$ is the closure of $e(X)$ in $\mathcal{I}^F(X)$. See [17 p.152].

To simplify notation we will usually indicate a compactification of a space $X$ simply by $X^*$. Let $(X, \mathcal{J})$ be a noncompact Hausdorff space and $T : X \to X$ a homeomorphism. We now make several definitions.

2.4 Definition.

a) $h^0(T) = \sup_{Y} h(T|Y)$ where $Y$ ranges over all compact $T$-invariant subsets of $X$. If there are none, define $h^0(T) = 0$.

b) $h^1(T) = h(T^*)$ where $T^*$ is $T$ extended to the one point compactification by defining $T^*(x) = T(x)$ for $x \in X$ and $T^*(\infty) = \infty$. Note that $T$ must be a homeomorphism in order for $T^*$ to be continuous.

c) $h^2(T) = h(T^*)$ where $T^*$ is the unique continuous extension of $T$ to the Stone-Cech compactification $X^*$ of $X$. See [6 p.243]. Here $(X, \mathcal{J})$ is a completely regular $T_1$ topological space.

d) $h^3(T) = \sup_{\alpha \in G} h(\alpha, T)$. 

Remark. Suppose now that \(X\) is compact. It is clear that \(h^0(T) = h(T)\), \(h^1(T) = h(T)\), and \(h^3(T) = h(T)\). Consider now \(h^2(T) = h(T^*)\) defined in (2.4). The map (see Definition (2.3)) \(e:X \rightarrow \mathbb{F}(X)\) is a homeomorphism and since \(X\) is compact, \(e(X)\) is compact and \(e(X) = e(X)\). This implies that \(h^2(T) = h(T)\). Hence, if \(X\) is compact, our definitions agree with that of \(h(T)\) defined in [2].

We will now state and prove a few results. First we will need the following theorem of [2].

2.5 Theorem. Let \(X_1\) and \(X_2\) be two closed subsets of a compact space \(X\) such that \(X = X_1 \cup X_2\) and \(\varphi(X_1) \subseteq X_1, \varphi(X_2) \subseteq X_2\) for a continuous mapping \(\varphi\) of \(X\) into \(X\). Then

\[
h(\varphi) = \max\{h(\varphi_1), h(\varphi_2)\}
\]

where \(\varphi_1\) and \(\varphi_2\) are the restrictions of \(\varphi\) to \(X_1\) and \(X_2\) respectively.

2.6 Proposition. Let \(X\) be a locally compact Hausdorff space and \(T:X \rightarrow X\) a homeomorphism. Then

\[
h^0(T) \leq h^1(T).
\]

Proof: Let \(Y\) be any compact \(T\)-invariant subset of \(X\); \(X^*\) the one-point compactification of \(X\); \(T:X \rightarrow X\) be a homeomorphism; and \(T^*:X^* \rightarrow X^*\) be the continuous extension of \(T\) to \(X^*\). Then \(Y\) is closed in \(X^*\) and
invariant under $T^*$; $X^*$ is closed in $X^*$ and invariant under $T^*$; $X^* = Y \cup X^*$ and hence by (2.5) we have $h^0(T) \leq h(T^*) = h^1(T)$. Note that $Y$ is closed in $X^*$ for $Y$ is a compact subset of $X \subseteq X^*$; $X^*$ is Hausdorff since $X$ is locally compact Hausdorff; and a compact subset of a Hausdorff space is closed.

We will need the following result from dimension theory. The proof can be found in [22 p.48] and will not be repeated.

2.7 Theorem. Let $(X, \mathcal{J})$ be a zero-dimensional Hausdorff space. Then $\beta(X)$, the Stone-Chech compactification of $X$, is also zero-dimensional.

2.8 Proposition. Let $X$ be a zero-dimensional Hausdorff space. Then

$$h^2(T) \leq h^3(T).$$

Proof: Since $X$ is zero-dimensional Hausdorff, it is completely regular $T_1$ and so has a Stone-Chech compactification, $X^*$. Let $T^*$ be the continuous extension of $T$ to $X^*$, and let $\alpha$ be any open cover of $X^*$. Since $X^*$ is zero-dimensional (by (2.7)) we may assume that $\alpha$ is a finite, disjoint, open-closed cover of $X^*$. Then $N_{X^*}(\alpha) = N_X(\alpha)$ because $X$ is dense in $X^*$. Hence

$$N_{X^*}\left(\bigvee_{i=0}^{n-1} T^{*\alpha}_i\right) = N_X\left(\bigvee_{i=0}^{n-1} T^{\alpha}_i\right) = N_X\left(\bigvee_{i=0}^{n-1} T^{-\alpha}_i\right).$$
In the last equation we take logarithms, divide by \( n \), and let \( n \to \infty \) to get

\[
h(\alpha, T^*) = h(\alpha, T^*|X) = h(\alpha, T).
\]

Now take the supremum over all finite open covers of \( X \) to get \( h(\alpha, T^*) \leq h^3(T) \). But \( \alpha \) was arbitrary and hence \( h^2(T) \leq h^3(T) \).

The next result is of considerable importance since it shows that, for the calculation of topological entropy for normal \( T_1 \)-spaces, we obtain the same result using finite open covers of \( X \) that we obtain using the Stone-Chech compactification of \( X \).

**2.9 Proposition.** Let \( X \) be a normal \( T_1 \)-space. Then

\[
h^2(T) = h^3(T).
\]

**Proof:** Since the proof requires considerable preliminary material, it will be given in the next chapter.

**2.10 Proposition.** \( h^1(T) \leq h^3(T) \).

**Proof:** Let \( X^* \) be the one-point compactification of \( X \) and \( T^* \) the extension of \( T \) to \( X^* \) as defined previously. Let \( \beta \) be any open cover of \( X^* \) and let \( \alpha \) be the induced cover of \( X \). Then \( N_X(\alpha) = N_{X^*}(\beta) \) or \( N_X(\alpha) + 1 = N_{X^*}(\beta) \) and hence

\[
N_X \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) = N_{X^*} \left( \bigvee_{i=0}^{n-1} T^{*-i} \beta \right)
\] or
\[ N_X \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right) + 1 = N_{X^*} \left( \sum_{i=0}^{n-1} T^{*-i} \beta \right). \]

In either case, taking logarithms, dividing by \( n \), and letting \( n \to \infty \), we get

\[
\lim_{n \to \infty} \frac{1}{n} \log N_X \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right) = \lim_{n \to \infty} \frac{1}{n} \log \left[ N_X \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right) + 1 \right] \\
= \lim_{n \to \infty} \frac{1}{n} \log N_{X^*} \left( \sum_{i=0}^{n-1} T^{*-i} \beta \right).
\]

That is, \( h(\alpha, T) = h(\beta, T^*) \). Therefore

\[ h^3(T) = \sup_{\gamma \in \mathcal{G}_f(X)} h(\gamma, T) \geq h(\beta, T^*). \]

But \( \beta \) was arbitrary

and so \( h^3(T) \geq h(T^*) = h^1(T) \).

Remark. Let \( (X, \mathfrak{T}) \) be a completely regular \( T_1 \)-space.

In this case, another proof of (2.10) results from noting that the one-point compactification is the homomorphic image of the Stone-Cech compactification and then applying (1.20) and (2.8). Thus, \( h^1(T) \leq h^2(T) \leq h^3(T) \).

2.11 Definition. A flow is a pair \( (X, T) \) where \( X \) is a compact Hausdorff space and \( T:X \to X \) is continuous.

If \( T \) is a homeomorphism then \( (X, T) \) is a cascade.

That is, a cascade is a transformation group where the acting group is the set of integers.

With the exception of (2.9) all of our results involving \( h^i(T), i = 0, 1, 2, 3 \) have, so far, not excluded the possibility of having strict inequality.
A natural question is whether \( h^i(T) \neq h^j(T) \) for some \( i \neq j \). We will now give a partial answer.

2.12 Example. Let \( X = \mathbb{Z} \), the set of integers and let \( T: X \to X \) be the shift defined by \( T(x) = x + 1 \). Robert Ellis [7] has shown that \( (X^*, T^*) \) (here \( X^* \) is the Stone-Čech compactification of \( X \)) is the universal point transitive cascade and hence we have

i) \( h^1(T) = 0 \)

ii) \( h^2(T) = h^3(T) = \infty \).

Proof: We will prove (i) and (ii).

i) We show later that \( h^1 \) is monotonic; that is, \( h^1(T|Y) \leq h^1(T) \) where \( Y \) is a closed \( T \)-invariant subset of \( X \). Let \( Q: \mathbb{R} \to \mathbb{R} \) (\( \mathbb{R} \), in this paper, will always be the set of real numbers) be defined by \( Q(x) = x + 1 \).

It is shown in [2] that \( h^1(Q) = h(Q^*) = 0 \). Hence \( h^1(T) \leq h^1(Q) = 0 \).

ii) Any point transitive cascade is the homomorphic image of the universal transitive cascade and so the universal transitive cascade has entropy greater than any homomorphic image. But there exist point transitive cascades of arbitrarily large entropy. Hence \( h^2(T) = h^3(T) = \infty \).

2.13 Definition. We say that \( h \) satisfies the power
formula if for any positive integer \( k \), \( h(T^k) = kh(T) \).

Note that \( h \) as defined in [2] satisfies the power formula.

2.14 Proposition. \( h^0(T^k) = kh^0(T) \).

Proof: The proof is trivial. Let \( T|_Y = T_Y \) where \( Y \) is any compact \( T \)-invariant subset of \( X \). By (1.21) we can write \( h(T_Y^k) = kh(T_Y) \). But if there are no compact \( T \)-invariant subsets of \( X \) (\( X \) is not compact), then there are no compact \( T^k \)-invariant subsets of \( X \); for if \( K \) is compact and \( T^k \)-invariant then \( \bigcup_{i=0}^{k-1} T^i K \) is \( T \)-invariant. Hence \( h^0(T^k) = 0 \), and \( h^0(T) = 0 \) by definition.

2.15 Proposition. \( h^1(T^k) = kh^1(T) \).

Proof: The proof is clear: since \( T^k = T^{k*} \),
\[ h^1(T^k) = h(T^{k*}) = h(T^{k*}) = kh(T^*) = kh^1(T). \]

We need the following lemma.

Lemma. Let \( X \) be a topological space and \( Y \) a Hausdorff space. Let \( f, g: X \to Y \) be continuous. If \( D \subseteq X \) is dense, and \( f|D = g|D \), then \( f = g \) on \( X \).

Proof: The proof can be found in [6 p.140].

2.16 Proposition. \( h^2(T^k) = kh^2(T) \).
Proof: It is clear that $T^k|X = T^k*X$. This implies, by the lemma, that $T^k = T^k*$ on $X*$. Hence $h^2(T^k) = h(T^k*) = h(T^k) = kh(T*) = kh^2(T)$.

2.17 Proposition. $h^3(T^k) = kh^3(T)$.

Proof: The proof is trivial.

2.18 Definition. Let $Y$ be a closed $T$-invariant subset of $X$. We say that $h$ has the monotonic property if $h(T|Y) \leq h(T)$.

Note that $h$ as defined in [2] has the monotonic property.

2.19 Proposition. $h^0$ has the monotonic property.

Proof: Let $Y$ be any closed, $T$-invariant subset of $X$ and set $T|Y = T_Y$. The problem is to show $h^0(T_Y) \leq h^0(T)$. Let $Y_1$ be any compact, $T_Y$-invariant subset of $Y$. Then $Y_1$ is also $T$-invariant, and so $h(T_Y|Y_1) = h(T|Y_1) \leq h^0(T)$. Varying $Y_1$ gives $h^0(T_y) \leq h^0(T)$. If there are no compact $T$-invariant subsets of $X$ then by definition $h^0(T) = 0$ and there is nothing to prove.

2.20 Proposition. $h^1$ has the monotonic property.

Proof: Let $Y$ be a closed $T$-invariant subset of $X$. We must show $h^1(T|Y) \leq h^1(T)$; that is, letting
Let $X$ be a completely regular, $T_1$-space; $Y$ a closed, $T$-invariant subset of $X$ where $T:X \to X$; and let $X^*$ be the Stone-Cech compactification of $X$. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $Y$ ($U_i$ open in $X$, $i = 1, 2, \ldots, n$) of minimal cardinality. Then there exists an open cover $\gamma$ of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.

Proof: Let $\alpha$ be as stated in the lemma:

$\alpha = \{U_1, U_2, \ldots, U_n\}$. Now $X$ is a subspace of $X^*$ and so there exist $G_1, G_2, \ldots, G_n$ open in $X^*$ such that $U_1 = X \cap G_1, \ldots, U_n = X \cap G_n$. Then the members of $\beta = \{G_1, \ldots, G_n\}$ are open in $X^*$ (but may not cover $X^*$). By minimality of $\alpha$ there exist $x_1, x_2, \ldots, x_n \in Y$ such that $x_i \in G_i$ but $x_i \not\in \bigcup_{i \neq j} G_j$, $i = 1, 2, \ldots, n$. Set $A = \{x_1, \ldots, x_n\}$. Then $A$ is closed in $X^*$ and hence $A^C$ is open in $X^*$. Then $\gamma = \{G_1, \ldots, G_n, A^C\}$ is an open cover of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$. 

Before proving that $h^2$ has the monotonic property, we prove the following lemma.

Lemma. Let $X$ be a completely regular, $T_1$-space; $Y$ a closed, $T$-invariant subset of $X$ where $T:X \to X$; and let $X^*$ be the Stone-Cech compactification of $X$. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $Y$ ($U_i$ open in $X$, $i = 1, 2, \ldots, n$) of minimal cardinality. Then there exists an open cover $\gamma$ of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.

Proof: Let $\alpha$ be as stated in the lemma:

$\alpha = \{U_1, U_2, \ldots, U_n\}$. Now $X$ is a subspace of $X^*$ and so there exist $G_1, G_2, \ldots, G_n$ open in $X^*$ such that $U_1 = X \cap G_1, \ldots, U_n = X \cap G_n$. Then the members of $\beta = \{G_1, \ldots, G_n\}$ are open in $X^*$ (but may not cover $X^*$). By minimality of $\alpha$ there exist $x_1, x_2, \ldots, x_n \in Y$ such that $x_i \in G_i$ but $x_i \not\in \bigcup_{i \neq j} G_j$, $i = 1, 2, \ldots, n$. Set $A = \{x_1, \ldots, x_n\}$. Then $A$ is closed in $X^*$ and hence $A^C$ is open in $X^*$. Then $\gamma = \{G_1, \ldots, G_n, A^C\}$ is an open cover of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$. 

T|Y = T_Y, our problem is to show $h(T^*_Y) \leq h(T^*_X)$. Note that $Y$ may not be closed in $X^*$. Either $Y$ is closed in $X^*$ or its closure is $Y^* = Y \cup \{\omega\}$. Hence $h^1(T^*_Y) = h(T^*_Y) \leq h(T^*_X) = h^1(T^*_X) = h^1(T)$. 

Before proving that $h^2$ has the monotonic property, we prove the following lemma.

Lemma. Let $X$ be a completely regular, $T_1$-space; $Y$ a closed, $T$-invariant subset of $X$ where $T:X \to X$; and let $X^*$ be the Stone-Cech compactification of $X$. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $Y$ ($U_i$ open in $X$, $i = 1, 2, \ldots, n$) of minimal cardinality. Then there exists an open cover $\gamma$ of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.

Proof: Let $\alpha$ be as stated in the lemma:

$\alpha = \{U_1, U_2, \ldots, U_n\}$. Now $X$ is a subspace of $X^*$ and so there exist $G_1, G_2, \ldots, G_n$ open in $X^*$ such that $U_1 = X \cap G_1, \ldots, U_n = X \cap G_n$. Then the members of $\beta = \{G_1, \ldots, G_n\}$ are open in $X^*$ (but may not cover $X^*$). By minimality of $\alpha$ there exist $x_1, x_2, \ldots, x_n \in Y$ such that $x_i \in G_i$ but $x_i \not\in \bigcup_{i \neq j} G_j$, $i = 1, 2, \ldots, n$. Set $A = \{x_1, \ldots, x_n\}$. Then $A$ is closed in $X^*$ and hence $A^C$ is open in $X^*$. Then $\gamma = \{G_1, \ldots, G_n, A^C\}$ is an open cover of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$. 

T|Y = T_Y, our problem is to show $h(T^*_Y) \leq h(T^*_X)$. Note that $Y$ may not be closed in $X^*$. Either $Y$ is closed in $X^*$ or its closure is $Y^* = Y \cup \{\omega\}$. Hence $h^1(T^*_Y) = h(T^*_Y) \leq h(T^*_X) = h^1(T^*_X) = h^1(T)$. 

Before proving that $h^2$ has the monotonic property, we prove the following lemma.

Lemma. Let $X$ be a completely regular, $T_1$-space; $Y$ a closed, $T$-invariant subset of $X$ where $T:X \to X$; and let $X^*$ be the Stone-Cech compactification of $X$. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $Y$ ($U_i$ open in $X$, $i = 1, 2, \ldots, n$) of minimal cardinality. Then there exists an open cover $\gamma$ of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.

Proof: Let $\alpha$ be as stated in the lemma:

$\alpha = \{U_1, U_2, \ldots, U_n\}$. Now $X$ is a subspace of $X^*$ and so there exist $G_1, G_2, \ldots, G_n$ open in $X^*$ such that $U_1 = X \cap G_1, \ldots, U_n = X \cap G_n$. Then the members of $\beta = \{G_1, \ldots, G_n\}$ are open in $X^*$ (but may not cover $X^*$). By minimality of $\alpha$ there exist $x_1, x_2, \ldots, x_n \in Y$ such that $x_i \in G_i$ but $x_i \not\in \bigcup_{i \neq j} G_j$, $i = 1, 2, \ldots, n$. Set $A = \{x_1, \ldots, x_n\}$. Then $A$ is closed in $X^*$ and hence $A^C$ is open in $X^*$. Then $\gamma = \{G_1, \ldots, G_n, A^C\}$ is an open cover of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$. 

T|Y = T_Y, our problem is to show $h(T^*_Y) \leq h(T^*_X)$. Note that $Y$ may not be closed in $X^*$. Either $Y$ is closed in $X^*$ or its closure is $Y^* = Y \cup \{\omega\}$. Hence $h^1(T^*_Y) = h(T^*_Y) \leq h(T^*_X) = h^1(T^*_X) = h^1(T)$. 

Before proving that $h^2$ has the monotonic property, we prove the following lemma.

Lemma. Let $X$ be a completely regular, $T_1$-space; $Y$ a closed, $T$-invariant subset of $X$ where $T:X \to X$; and let $X^*$ be the Stone-Cech compactification of $X$. Let $\alpha = \{U_1, U_2, \ldots, U_n\}$ be a finite open cover of $Y$ ($U_i$ open in $X$, $i = 1, 2, \ldots, n$) of minimal cardinality. Then there exists an open cover $\gamma$ of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.

Proof: Let $\alpha$ be as stated in the lemma:

$\alpha = \{U_1, U_2, \ldots, U_n\}$. Now $X$ is a subspace of $X^*$ and so there exist $G_1, G_2, \ldots, G_n$ open in $X^*$ such that $U_1 = X \cap G_1, \ldots, U_n = X \cap G_n$. Then the members of $\beta = \{G_1, \ldots, G_n\}$ are open in $X^*$ (but may not cover $X^*$). By minimality of $\alpha$ there exist $x_1, x_2, \ldots, x_n \in Y$ such that $x_i \in G_i$ but $x_i \not\in \bigcup_{i \neq j} G_j$, $i = 1, 2, \ldots, n$. Set $A = \{x_1, \ldots, x_n\}$. Then $A$ is closed in $X^*$ and hence $A^C$ is open in $X^*$. Then $\gamma = \{G_1, \ldots, G_n, A^C\}$ is an open cover of $X^*$ such that $N_Y(\alpha) \leq N_{X^*}(\gamma)$.
2.21 Proposition. \( h^2 \) has the monotonic property.

Proof: Let \( T_Y = T|_Y \) where \( Y \) is a closed \( T \)-invariant subset of \( X \). Note that \( Y \) may not be closed in \( X^* \). We show \( h^2(T_Y) \leq h^2(T) \); that is, \( h(T_Y) \leq h(T^*) \). To do this we first observe that \( h(\alpha, T^*|_Y) \leq h(\alpha, T^*) \) where \( \alpha \) is any open cover of \( X^* \). Now using this and the lemma together with (2.8) we get the following relation:

\[
\begin{align*}
  h^2(T_Y) &\leq h^3(T_Y) = \sup_{\beta \in \mathcal{G}_f(Y)} h(\beta, T_Y) = \sup_{\beta \in \mathcal{G}_f(Y)} h(\beta, T^*|_Y) \\
  &\leq \sup_{\gamma \in \mathcal{G}_f(X^*)} h(\gamma, T^*|_Y) \leq \sup_{\gamma \in \mathcal{G}_f(X^*)} h(\gamma, T^*) \\
  &= h(T^*) = h^2(T).
\end{align*}
\]

2.22 Proposition. \( h^3 \) has the monotonic property.

Proof: The proof is trivial.

2.23 Definition. Let \( p \) be any positive integer. Let \( X_p = \{ x = (x(n)) : x(n) \in \{0, 1, \ldots, p - 1\} \text{ for } n \in \mathbb{Z} \} \). That is, \( X_p = \{0, 1, \ldots, p - 1\}^\mathbb{Z} \). Give \( X_p \) the product topology and let \( \sigma_p : X_p + X_p \) be the map defined by the rule

\[
\sigma_p(x)(n) = x(n + 1), \quad x \in X_p, \quad n \in \mathbb{Z}.
\]

The flow \( (X_p, \sigma_p) \) is called the symbolic flow on \( p \) symbols.
We now define a property which, quite interestingly, is not satisfied by all \( h^i, \ i = 0, 1, 2, 3. \)

\[ \text{2.24 Definition.} \text{ Let } X \text{ and } Y \text{ be topological spaces.} \]

We say that \( h \) has the continuous image property if, whenever the following diagram commutes (that is, \( \varphi \circ T = S \circ \varphi \)) then \( h(S) \leq h(T) \) where \( S, T \) are continuous and \( \varphi \) is continuous and onto.

\[ \begin{array}{ccc}
X & \xrightarrow{T} & X \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \xrightarrow{S} & Y
\end{array} \]

**Remark.** \( h \) as defined in [2] has the continuous image property.

\[ \text{2.25 Proposition.} \ h^0 \text{ does not have the continuous image property.} \]

**Proof:** The following counterexample is due to Goodwyn. Let \( (Y, S) \) be the symbolic flow on two symbols. Hence \( h^0(S) = h(S) = \log 2. \) Let \( Z \) be the integers and set \( X = Z \times Y. \) Define \( T:X \to X \) by \( T(n, y) = (n + 1, y). \) Then \( T \) is continuous and since there are no compact \( T \)-invariant subsets of \( X, \) \( h^0(T) = 0. \) Now let \( \varphi:X \to Y \) be defined by \( \varphi(n, y) = S^n(y). \) Clearly \( \varphi \) is continuous. Also \( \varphi \) is onto since \( \varphi(0, y) = y. \) Furthermore,
\( \varphi(T(n, y)) = \varphi(n + 1, y) = S^{n+1}(y) = S(\varphi(n, y)) \). Hence the following diagram commutes

\[
\begin{array}{ccc}
X & \xleftarrow{T} & X \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \xleftarrow{S} & Y
\end{array}
\]

but \( h^0(S) = \log 2 \) and \( h^0(T) = 0 \).

2.26 Proposition. \( h^1 \) does not have the continuous image property.

Proof: Again, consider the example of (2.25). Let \((Y, S)\) be the symbolic flow on two symbols. Hence \( h^1(S) = h(S) = \log 2 \). Then as before the following diagram commutes,

\[
\begin{array}{ccc}
X = Z \times Y & \xleftarrow{T} & X = Z \times Y \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \xleftarrow{S} & Y
\end{array}
\]

The map \( \varphi \) is continuous and onto, but we will show that \( h^1(T) = 0 \) in the following proposition.

2.27 Proposition. Let \((Y, S)\) be the symbolic flow on two symbols and \( Z \) the set of integers. Define \( T:Z \times Y \to Z \times Y \) by \( T(n, y) = (n + 1, y) \). Then \( h^1(T) = 0 \).

Proof: Let \( Q:R \to R \) be defined by \( Q(x) = x + 1 \). Now the one-point compactification \( R^* \) of the real line \( R \)
is the circle and $Q^*: \mathbb{R}^* \to \mathbb{R}^*$ defined by $Q^*(x) = Q(x)$ if $x \in \mathbb{R}$ and $Q^*(\infty) = \infty$, is a homeomorphism. It has been shown in [2] that $h^1(Q) = h(Q^*) = 0$. Define the shift $\bar{S}: \mathbb{Z} \to \mathbb{Z}$ on the integers by $\bar{S}(x) = x + 1$. Now $\mathbb{Z} \subseteq \mathbb{R}$ and $\mathbb{Z}$ is closed and $Q$-invariant. Since $h^1$ is monotonic we have $h(\bar{S}^*) = h^1(\bar{S}) \leq h^1(Q) = 0$.

Consider the following diagram.

$$
\begin{array}{ccc}
(Z \cup \{\infty\}) \times Y & \xrightarrow{\bar{S} \times I} & (Z \cup \{\infty\}) \times Y \\
\downarrow \varphi' & & \downarrow \varphi' \\
(Z \times Y) \cup \{\infty\} & \xrightarrow{(\bar{S} \times I)^*} & (Z \times Y) \cup \{\infty\}
\end{array}
$$

Then we have

i) $(Z \cup \{\infty\}) \times Y$ is a compact space; $\bar{S}^*: \mathbb{Z}^* \to \mathbb{Z}^*$ is defined by $\bar{S}^*(n) = \bar{S}(n) = n + 1$ if $n \in \mathbb{Z}$, and $\bar{S}^*(\infty) = \infty$. Here $\mathbb{Z}^* = \mathbb{Z} \cup \{\infty\}$, the one-point compactification of the integers. The map $I$ is the identity map. Then by the Product Theorem we have

$$h(\bar{S}^* \times I) = h(\bar{S}^*) + h(I) = 0 + 0 = 0.$$

ii) The map $\varphi'$ is defined by $\varphi'(n, y) = (n, y)$ and $\varphi'(\infty, y) = \infty$. Then $\varphi'$ is continuous and onto.

iii) The diagram commutes, for if

$n \neq \infty$, $\varphi'((\bar{S} \times I)(n, y)) = \varphi'(n + 1, y) = (n + 1, y)$; and

$(\bar{S} \times I)^*(\varphi'(n, y)) = (\bar{S} \times I)^*(n, y) = (\bar{S} \times I)(n, y) = (n + 1, y)$. If $n = \infty$, then $\varphi'((\bar{S} \times I)(\infty, y)) = \varphi'(\infty, y) = \infty$; and $(\bar{S} \times I)^*(\varphi'(\infty, y)) = (\bar{S} \times I)^*(\infty) = \infty$.  

iv) Hence \( 0 = h(S^* \times I) \geq h((S \times I)^*) = h^1(S \times I) = h^1(T). \)

We will need the following two well-known results from topology.

**Theorem A.** If \( X \) and \( Y \) are topological spaces then a function \( f \) from \( X \) onto \( Y \) is continuous iff \( \langle a_\nu \rangle \to a^* \) in \( X \) implies that \( \langle f(a_\nu) \rangle \to f(a^*) \) in \( Y \), where \( \langle a_\nu \rangle \) is a net in \( X \).

**Theorem B.** \( X \) is a Hausdorff space iff every convergent net in \( X \) has a unique limit. See [17 p.67].

**2.28 Proposition.** \( h^2 \) has the continuous image property.

**Proof:** Let \( X, Y \) be completely regular \( T_1 \)-spaces.

Consider the following diagram

\[
\begin{array}{ccc}
X & \xleftarrow{T} & X \\
\downarrow{\varphi} & & \downarrow{\varphi} \\
Y & \xleftarrow{S} & Y
\end{array}
\]

where \( T, S \) are continuous; \( \varphi \) is continuous and onto; and \( \varphi \circ T = S \circ \varphi \). The problem is to show \( h^2(S) \leq h^2(T) \) or equivalently, \( h(S^*) \leq h(T^*) \). We extend the above diagram by Stone-Chech compactifying each space and extending the corresponding maps. This gives the following diagram.
From topology (see [6] p.243) we know that $\varphi^*$ exists and is unique and continuous. It remains to show $\varphi^* \circ T^* = S^* \circ \varphi^*$. Let $x \in X^*$. If $x \in X$, then $\varphi^* \circ T^*(x) = \varphi \circ T(x) = S \circ \varphi(x) = S^* \circ \varphi^*(x)$. Let $x \in X^* - X$. Then there exists a net $\langle x_\nu \rangle \in X$ such that $\langle x_\nu \rangle \rightarrow x$. Then by Theorem A $\varphi \circ T(x_\nu) = \varphi^* \circ T^*(x_\nu) \rightarrow \varphi^* \circ T^*(x)$ and $S \circ \varphi(x_\nu) = S^* \circ \varphi^*(x_\nu) \rightarrow S^* \circ \varphi^*(x)$. Hence $T^* \circ T^* = S^* \circ \varphi^*$ by Theorem B. This implies $h(S^*) \leq h(T^*)$. The proof is complete.

2.29 Proposition. $h^3$ has the continuous image property.

Proof: The proof is the same as that of propositions (1.18) and (1.20).
In this chapter we consider yet another compactification in order to prove proposition (2.9). Two good references for the material of this section are [17 p.167] and [27]. We will use the following abbreviations:
i) f.i.p. means finite intersection property;
ii) w.r.t. means with respect to.

We will need the following result which we state without proof. No confusion need result by our use of the letter $G$ in this chapter with that of $G(X)$ sometimes also abbreviated $G$ in other chapters.

3.1 Problem 5R (Kelley [17 p.167]). Let $X$ be a $T_1$-space; $\mathcal{G}$ be the family of all closed subsets of $X$; and $w(X) = \{G \subseteq \mathcal{G} : G$ has the f.i.p. and is maximal in $\mathcal{G} \text{ w.r.t. the f.i.p.}\}$.

a) If $G \in w(X)$, then the intersection of two members of $G$ is a member of $G$; dually, if $A$ and $B \in \mathcal{G} - G$, then $A \cup B \in \mathcal{G} - G$.

b) Define $\varphi : X \to w(X)$ by $\varphi(x) = \{A : A \in \mathcal{G} \text{ and } x \in A\}$. Then $\varphi$ is one-to-one.

c) Let $U$ be any open (or closed) subset of $X$. Define
\( U^* = \{ G \in w(X) : A \subseteq U \text{ for some } A \in G \}. \) Then

i) \( w(X) - U^* = \{ G \in w(X) : X - U \in G \}; \) that is,

\( (U^*)^c = (U^c)^*. \)

ii) If \( U \) and \( V \) are open in \( X \), then

\( (U \cap V)^* = U^* \cap V^*, \ (U \cup V)^* = U^* \cup V^*. \)

d) Let \( w(X) \) have the topology with a base the family of all sets of the form \( U^* \) for \( U \) open in \( X \). Then

i) \( w(X) \) is compact,

ii) the map \( \varphi : X \rightarrow w(X) \) is continuous, and

iii) \( \varphi(X) \) is dense in \( w(X) \).

e) If \( X \) is normal, then \( w(X) \) is Hausdorff.

f) If \( f \) is a bounded continuous real-valued function on \( X \), then \( f \circ \varphi^{-1} \) may be extended continuously to all of \( w(X) \).

g) If \( w(X) \) is Hausdorff, then \((\varphi, w(X))\) is topologically equivalent to the Stone-Cech compactification of \( X \).

The pair \((\varphi, w(X))\) is called the Wallman compactification of \( X \).

3.2 Remark. As noted in [17 p.168] the correspondence \( U \rightarrow U^* \) preserves finite intersections and unions and the topology for \( X \) is carried into a base for the topology for \( w(X) \) by this correspondence.

We will now prove Proposition (2.9) but first we need a few lemmas.
3.3 Lemma. Let $X$ be a normal $T_1$-space and $\varphi: X \to w(X)$ be defined as in (3.1). Then if $U$ is open in $X$, 
$$U^* \cap \varphi(X) = \varphi(U).$$

Proof: Let $G \in U^* \cap \varphi(X)$. Now $G \in U^*$ implies $A \subseteq U$ for some $A$ in $G$. Also $G \in \varphi(X) = \{\varphi(x) : x \in X\} = \{\{A : A \in G \text{ and } x \in A\} : x \in X\}$. Now $\varphi(U) = \{\varphi(y) : y \in U\} = \{\{B : B \in \delta \text{ and } y \in B\} : y \in U\}$. Clearly $G \in \varphi(U)$, and so $U^* \cap \varphi(X) \subseteq \varphi(U)$. Now let $G \in \varphi(U)$. Then clearly $G \in \varphi(X)$. But $G \in U^*$ since $\{y\} \subseteq U$, and so $\varphi(U) \subseteq U^* \cap \varphi(X)$.

3.4 Lemma. Let $U$ and $V$ be open subsets of $X$. Then 
$$U \subseteq V \implies U^* \subseteq V^*.$$

Proof: Let $G \in U^* = \{G \in w(X) : A \subseteq U \text{ for some } A \in G\}$. Then $U \subseteq V \implies A \subseteq U \subseteq V \implies G \in V^*$.

3.5 Lemma. Let $X$ be a normal $T_1$-space. Define 
$$\alpha^* = \{U^* : U \in \alpha\}$$
where the members of $\alpha$ are open sets in $X$. Then $\alpha$ is a finite open cover of $X$ iff $\alpha^*$ is an open cover of $w(X)$.

Proof: Suppose $\alpha$ is a finite open cover of $X$. Then 
$$X \subseteq U_1 \cup U_2 \cup \ldots \cup U_n, U_i \in \alpha \text{ for } i = 1, \ldots, n.$$ 
Hence $X^* \subseteq (U_1 \cup U_2 \cup \ldots \cup U_n)^*$, and since $w(X) = X^*$, $w(X) \subseteq U_1^* \cup U_2^* \cup \ldots \cup U_n^*$. That is, $\alpha^*$ is a finite open cover of $X^*$. Next, suppose that $\alpha^*$ is an open
cover of $w(X)$. Now $w(X)$ is compact implies
\[ w(X) \subseteq U_1^* \cup U_2^* \cup \ldots \cup U_n^*, \text{ for } U_i^* \in \alpha^*, \ i = 1, 2, \ldots, n. \]
Thus $X^* \subseteq U_1^* \cup U_2^* \cup \ldots \cup U_n^* = (U_1 \cup U_2 \cup \ldots \cup U_n)^*$. This implies by (3.3) that $X \subseteq U_1 \cup U_2 \cup \ldots \cup U_n$, and hence $\alpha$ is a finite open cover of $X$.

3.6 Lemma. Consider the correspondence $U \leftrightarrow U^*$ where $U$ is open in $X$ and $U^*$ is open in $w(X)$. Let $\alpha$ be a finite open cover of $X$. Then
\[ N_X(\alpha) = N_{w(X)}(\alpha^*). \]
Proof: Let $\alpha'$ be a subcover of $\alpha$ having $N(\alpha)$ members. Then $\alpha'^*$ is a subcover of $\alpha^*$ with $N(\alpha)$ members. Hence $N(\alpha^*) \leq N(\alpha)$. Now $\alpha^*$ is a cover of $w(X)$. Let $\beta^*$ be a subcover of $\alpha^*$ with $N(\alpha^*)$ members. Then $\beta$ is a subcover of $\alpha$ with $N(\alpha^*)$ members and hence $N(\alpha) \leq N(\alpha^*)$.

3.7 Lemma. Let $X$ be a normal $T_1$-space and $T:X \to X$ a homeomorphism. Define $T^*:w(X) \to w(X)$ by
\[ T^*(G) = \{T(A):A \in G\}. \]
Then
\[ T^{-1}(A^*) = (T^{-1}(A))^*. \]
Proof: $T^{-1}(A^*) = \{G:T^*(G) \in A^*\}$
\[ = \{G:\{T(B):B \in G\} \in A^*\}. \]
We show
\[ \{G:\{T(B):B \in G\} \in A^*\} = (T^{-1}(A))^*. \]
Let $G$ belong to the left side of (1). Then
there exists a $C$ such that $C \in \{T(B) : B \in G \}$ and $C \subseteq A.$

There exists a $D$ in $G$ such that $C = T(D)$ and $C \subseteq A.$

Therefore $T^{-1}(C) \subseteq T^{-1}(A)$, $T^{-1}(C) = T^{-1}(T(D)) = D$, and $D \in G$. Now $D \subseteq T^{-1}(A)$, $D \in G$ implies $G \in (T^{-1}(A))^*.$

We have shown $T^{-1}(A^*) \subseteq (T^{-1}(A))^*$. Now let $G \in (T^{-1}(A))^*$. Then there exists an $E \in G$ such that $E \subseteq T^{-1}(A)$. Hence $T(E) \subseteq T(T^{-1}(A)) = A$, since $T$ is onto. Now $T(E) \in \{T(B) : B \in G \}$, and we have shown $(T^{-1}(A))^* \subseteq T^{-1}(A^*)$.

3.8 Lemma. $T^*:w(X) \to w(X)$ defined by $T^*(G) = \{T(A) : A \in G \}$ is continuous.

Proof: Let $U^*$ be a basic open set in $w(X)$. Then $T^{-1}(U^*) = (T^{-1}(U))^*$ is open since $T^{-1}(U)$ is a basic open set in $X$.

3.9 Proposition. Let $X$ be a normal $T_1$-space, and $T:X \to X$ a homeomorphism. Then

$$h^2(T) = h^3(T).$$

Proof: Since $X$ is normal, we know by (3.1e) that $w(X)$ is Hausdorff and by (3.1g) that $w(X)$ is topologically equivalent to the Stone-Chech compactification of $X$. Let $\alpha$ be any finite open cover of $X$. Then

$$N_X \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) = N_w(X) \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right)^*$$
\[
= N_w(X) \left( \sum_{i=0}^{n-1} (T^{-i} \alpha)^* \right)
\]
\[
= N_w(X) \left( \sum_{i=0}^{n-1} T^{-i} \alpha* \right). \quad \text{Hence}
\]
\[
\frac{1}{n} \log N_X \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right) = \frac{1}{n} \log N_w(X) \left( \sum_{i=0}^{n-1} T^{-i} \alpha* \right).
\]
Letting \( n \to \infty \), we get
\[
h(\alpha, T) = h(\alpha^*, T^*).\]
Hence \( h(\alpha, T) \leq h(T^*) = h^2(T) \) and so \( h^3(T) \leq h^2(T) \).

Now let \( \beta \) be any open cover of \( w(X) \). Since \( \{U^* : U \text{ is open in } X\} \) forms a base for the topology of \( w(X) \), we can refine \( \beta \) by a finite (since \( w(X) \) is compact) open cover of the form \( \alpha^* = \{U^* : U \text{ is open in } X\} \).

Hence \( \beta < \alpha^* \) and so
\[
h(\beta, T^*) \leq h(\alpha^*, T^*) = h(\alpha, T) \leq h^3(T).
\]
But \( \beta \) was arbitrary and so \( h^2(T) = h(T^*) \leq h^3(T) \).

The proof is complete.

**Remark.** We conjecture that \( h^2(T) = h^3(T) \) for all completely regular spaces. This would validate Proposition (3.9) to an even larger class.
In this chapter we consider the calculation of topological entropy on noncompact spaces using a method that does not depend upon first compactifying the space as was done in Chapters II and III. This is accomplished by using the notion of uniform spaces. Our definition of $h^*(T)$ is motivated by the work done in [5]. A good reference for the material of this chapter is [17 p.174-199].

The use of the letters $H, K, K_1, K_2, \ldots$ will be reserved for compact subsets of $X$. For completeness we will state the basic definitions and properties (from [17 p.174-199]) of uniform spaces that we need.

A relation $U$ is a set of ordered pairs. The inverse relation $U^{-1} = \{(x, y) : (y, x) \in U\}$. If $U = U^{-1}$, then $U$ is called symmetric. The composition $U \circ V$ is defined by $U \circ V = \{(x, z) : \text{there exists } y \text{ such that } (x, y) \in V \text{ and } (y, z) \in U\}$. Also, $\Delta(X) = \Delta = \{(x, x) : x \in X\}$ is called the identity relation or the diagonal. Let $A \subseteq X$. Then $U[A] = \{y : (x, y) \in U \text{ for some } x \in A\}$.

4.1 Definition. A uniformity for a set $X$ is a nonempty
family $\mathcal{U}$ of subsets of $X \times X$ such that

a) $U \in \mathcal{U} \Rightarrow \Delta \subseteq U$;

b) $U \in \mathcal{U} \Rightarrow U^{-1} \in \mathcal{U}$;

c) if $U \in \mathcal{U}$, then there exists $V \in \mathcal{U}$ such that $V \circ V \subseteq U$;

d) if $U, V \in \mathcal{U}$, then $U \cap V \in \mathcal{U}$; and

e) if $U \in \mathcal{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called a uniform space.

A set $X$ may have many different uniformities; the largest of these is the family of those subsets of $X \times X$ which contain the diagonal, and the smallest is the family whose only member is $X \times X$. We also need the following definition.

4.2 Definition. Let $X$ be the set of real numbers. Then the usual uniformity for $X$ is the family $\mathcal{U}$ defined by

$$\mathcal{U} = \{U \subseteq X \times X : \{(x, y) : |x - y| < r\} \subseteq U \text{ for some } r > 0\}.$$ 

4.3 Definition. Let $(X, \mathcal{U})$ be a uniform space. The uniform topology $\mathcal{J}$ of the uniformity $\mathcal{U}$, is defined by

$$\mathcal{J} = \{T \subseteq X : \text{for each } x \in T, \text{ there exists } U \in \mathcal{U} \text{ such that } U[x] \subseteq T\}.$$ 

It is easy to show that $\mathcal{J}$ is a topology.

4.4 Definition. A cover $\alpha$ of a subset $A$ of a uniform space $(X, \mathcal{U})$ is a uniform cover iff for each $x \in A$, there exists a $U$ in $\mathcal{U}$ such that $U[x] \subseteq G$ for some $G$ in $\alpha$. That is, 

$$\{U[x] : x \in A\} \text{ refines } \alpha.$$
4.5 Definition. Let \((X, \mathcal{U})\) be a uniform space and \(\mathcal{U}\) a uniform cover of \(X\). Let \(T: X \to X\) be uniformly continuous. We define

\[
h_K(\alpha, T) = \lim_{n \to \infty} \frac{1}{n} \log N_K \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) \text{ and}
\]

\[
h^*(T) = \sup_{\alpha \text{ uniform cover}} \sup_{K \subseteq X} h_K(\alpha, T),
\]

where \(K\) is any compact subset of \(X\). We will call \(h^*(T)\) the uniform topological entropy of \(T\).

A given uniformity \(\mathcal{U}\) always generates a topology \(\mathcal{J}\); but given a topological space \((X, \mathcal{J})\), does there exist a uniformity \(\mathcal{U}\) such that \(\mathcal{U}\) generates \(\mathcal{J}\)?

4.6 Definition. A given topological space \((X, \mathcal{J})\) is uniformizable if there exists a uniformity \(\mathcal{U}\) of \(X\) such that \(\mathcal{U}\) generates \(\mathcal{J}\).

4.7 Theorem. Let \((X, \mathcal{J})\) be a topological space.

Then \(\mathcal{J}\) is the uniform topology for some uniformity \(\mathcal{U}\) of \(X\) iff \((X, \mathcal{J})\) is completely regular (see [17 p.88]).

Hence uniformizability is equivalent to complete regularity.

4.8 Remark. A compact Hausdorff space is completely regular. We show that in the case of a compact Hausdorff space \(X\) our definition of \(h^*(T)\) agrees with that of \(h(T)\) defined in [2]. From topology we know a compact
Hausdorff space possesses a unique uniformity. Let $\alpha$ be a uniform cover of $X$, and $K_1, K_2$ be compact subsets of $X$. Then clearly

$$K_1 \subseteq K_2 \implies N_{K_1} \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right) \subseteq N_{K_2} \left( \sum_{i=0}^{n-1} T^{-i} \alpha \right)$$

$$\implies h_{K_1}(\alpha, T) \leq h_{K_2}(\alpha, T).$$

Therefore

$$h^*(T) = \sup_{\alpha \text{ uniform cover}} \sup_{K \subseteq X} h_K(\alpha, T)$$

$$= \sup_{\alpha \text{ uniform cover}} h_X(\alpha, T)$$

$$= \sup_{\alpha \in \mathcal{U}(X)} h(\alpha, T)$$

$$= h(T).$$

The third equality follows since every open cover of a compact Hausdorff space is a uniform cover. See [17 p.199].

4.9 Example. Let $X = \mathbb{R}$ (the real numbers) and $T: X \to X$ be defined by $T(x) = 2x$ and suppose $X$ has the usual uniformity $\mathcal{U}$. Then

$$h^*(T) = \log 2.$$

Proof: $\mathcal{U}$ contains all subsets $U \subseteq X \times X$ such that $U_\varepsilon = \{(x, y) : |x - y| < \varepsilon\} \subseteq U$ for some $\varepsilon > 0$. So each $U_\varepsilon \in \mathcal{U}$. The topology generated by $U_\varepsilon$ is

$$\alpha_\varepsilon = \{B(x, \varepsilon) : x \in X\}$$

where $B(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$. 
Now, it clearly suffices to consider the uniform covers \( \alpha_\varepsilon \). Also, since \( H \subseteq K \Rightarrow N_H \left( \bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon \right) \leq N_K \left( \bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon \right) \) (hence \( h_H(\alpha_\varepsilon, T) \leq h_K(\alpha_\varepsilon, T) \)) and since any compact subset of \( X \) is contained in a compact set of the form \( K_m = [-m\varepsilon, m\varepsilon] \) for \( m \in \mathbb{Z}^+ \), it suffices to consider compact subsets of the form \( K_m \) where \( m \in \mathbb{Z}^+ \). Then for any positive integer \( n \),

\[
N_{K_m} \left( \bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon \right) = 2^{n-1} N_{K_m}(\alpha_\varepsilon). \quad \text{Hence}
\]

\[
\frac{1}{n} \log N_{K_m} \left( \bigvee_{i=0}^{n-1} T^{-i}\alpha_\varepsilon \right) = \frac{n-1}{n} \log 2 + \frac{1}{n} \log N_{K_m}(\alpha_\varepsilon).
\]

Let \( n \to \infty \). Then \( h_{K_m}(\alpha_\varepsilon, T) = \log 2 \) for any positive integer \( m \) and any \( \varepsilon > 0 \). Hence \( h^*(T) = \log 2 \).

Remark. If \( \nu \) is the uniformity of all neighborhoods of the diagonal in \( \mathbb{R} \times \mathbb{R} \), then \( h^*(T) = \infty \).

4.10 Example. Under the conditions of Example (4.9), let \( T(x) = x + 1 \). Then

\[ h^*(T) = 0. \]

If \( T \) is the identity then

\[ h^*(T) = 0. \]

Proof: The proof is identical to that of (4.9) except
that now for each positive integer \( n \), we have

\[
N_{K_m} \left( \sum_{i=0}^{n-1} T^{-i} \alpha \xi \right) = N_{K_m} (\alpha \xi). \quad \text{Hence}
\]

\[
\frac{1}{n} \log N_{K_m} \left( \sum_{i=0}^{n-1} T^{-i} \alpha \xi \right) = \frac{1}{n} \log N_{K_m} (\alpha \xi). \quad \text{Let} \ n \to \infty \text{ to get}
\]

\[
h_K (\alpha \xi, T) = 0, \quad \text{for all} \ m \in \mathbb{Z}^+ \text{ and any} \ \xi > 0.
\]

Therefore \( h^* (T) = 0. \)

**Remark.** Let \( X = \mathbb{Z} \), the set of integers and let \( T : X \to X \) be defined by \( T(n) = n + 1. \) Then

\[
h^* (T) = 0.
\]

**4.11 Proposition.** \( h^* (T^m) = mh^* (T), \ m \in \mathbb{Z}^+. \)

**Proof:** The proof is similar to that of (1.21).

**4.12 Definition.** Let \((X, U)\) and \((Y, V)\) be uniform spaces and let \( f : (X, U) \to (Y, V). \) Then \( f \) is uniformly continuous relative to \( U \) and \( V \) iff for each \( V \) in \( V \), the set

\[
\{ (x, y) : (f(x), f(y)) \in V \} \in U.
\]

Or equivalently, let \( f_2 = f \times f : X \times X \to Y \times Y \) be defined by

\[
f_2(x, y) = (f(x), f(y)). \quad \text{Then} \ f \text{ is uniformly continuous iff for each} \ V \text{ in} \ V, \text{ there exists} \ U \text{ in} \ U \text{ such that} \ f_2(U) \subseteq V.
\]

**4.13 Definition.** Let \( f : X \to Y \) be a one-to-one map such that \( f \) and \( f^{-1} \) are uniformly continuous. Then \( f \) is
called a uniform isomorphism and \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) are said to be uniformly equivalent.

The collection of uniform spaces may be divided into equivalence classes consisting of uniformly equivalent spaces.

4.14 Definition. A property which when possessed by one uniform space is also possessed by every uniformly isomorphic space is called a uniform invariant.

4.15 Theorem. Each uniformly continuous function is continuous relative to the uniform topology, and hence each uniform isomorphism is a homeomorphism (see [17 p.181]).

4.16 Proposition. Let \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) be uniform spaces. Consider the following diagram

\[
\begin{array}{c}
(X, \mathcal{U}) \xrightarrow{T} (X, \mathcal{U}) \\
\downarrow \varphi \quad \quad \quad \quad \downarrow \varphi \\
(Y, \mathcal{V}) \xleftarrow{S} (Y, \mathcal{V})
\end{array}
\]

where \(\varphi, S\) and \(T\) are uniform isomorphisms such that \(\varphi \circ T = S \circ \varphi\). Let \(\alpha\) be a uniform cover of \(Y\) and \(K \subseteq Y\). Then

\[
h_K(\alpha, S) = h_{\varphi^{-1}(K)}(\varphi^{-1} \alpha, T).
\]

Proof: \(N_K \left( \bigvee_{i=0}^{n-1} S^{-i} \alpha \right) = N_{\varphi^{-1}(K)} \left( \bigvee_{i=0}^{n-1} \varphi^{-1} S^{-i} \alpha \right)\)
\[
\begin{align*}
\log N & = \varphi^{-1}(K) \left( \bigvee_{i=0}^{n-1} \varphi^{-1}(S^{-i}x) \right) \\
& = N \varphi^{-1}(K) \left( \bigvee_{i=0}^{n-1} T^{-i}(\varphi^{-1}x) \right).
\end{align*}
\]

Hence \( \frac{1}{n} \log N_K \left( \bigvee_{i=0}^{n-1} S^{-i}x \right) = \frac{1}{n} \log N \varphi^{-1}(K) \left( \bigvee_{i=0}^{n-1} T^{-i}(\varphi^{-1}x) \right) \).

Let \( n \to \infty \) to get the result.

4.17 Proposition. Uniform topological entropy is a uniform invariant. That is, under the conditions of (4.16), we have

\[ h^*(S) = h^*(T). \]

**Proof:** The proof is clear from (4.16).

4.18 Proposition. Consider the following diagram

\[
\begin{array}{ccc}
(X, \mathcal{U}) & \xleftarrow{T} & (X, \mathcal{U}) \\
\downarrow \varphi & & \downarrow \varphi \\
(Y, \mathcal{V}) & \xleftarrow{S} & (Y, \mathcal{V})
\end{array}
\]

where \((X, \mathcal{U})\) and \((Y, \mathcal{V})\) are uniform spaces; \(T\) and \(S\) are uniformly continuous; \(\varphi\) is a uniformly continuous onto map such that \(\varphi \circ T = S \circ \varphi\), and \(\varphi^{-1}(K)\) is compact for any \(K \subseteq Y\). Then

\[ h^*(S) \leq h^*(T). \]
Proof: Let $\alpha$ be any uniform cover of $Y$ and $K \subseteq Y$. (The letters $H$ and $K$ are reserved for compact sets.) Then as in (4.16) we again get

$$h_K(\alpha, S) = h_{\varphi^{-1}K}(\varphi^{-1}\alpha, T).$$

Hence

$$\sup_{K \subseteq Y} h_K(\alpha, S) = \sup_{K \subseteq Y} h_{\varphi^{-1}K}(\varphi^{-1}\alpha, T),$$

$$\leq \sup_{H \subseteq X} h_H(\varphi^{-1}\alpha, T),$$

$$\leq \sup_{\beta \text{ uniform cover}} \sup_{H \subseteq X} h_H(\beta, T),$$

$$= h^*(T).$$

But $\alpha$ was arbitrary and so $h^*(S) \leq h^*(T)$.

4.19 Proposition. $h^*$ has the monotonic property.

Proof: Let $(X, U)$ be a uniform space, $T:X \to X$ uniformly continuous, and $Y$ a closed $T$-invariant subset of $X$. We show $h^*(T_Y) \leq h^*(T)$ where $T_Y = T|_Y$. Let $\alpha$ be any uniform cover of $Y$ open in the relative topology. Then for each $U$ in $\alpha$, there exists a $V_U$ open in $X$ such that $U = Y \cap V_U$. Then $\{V_U: U \in \alpha\}$ is also a uniform cover of $Y$. Set $\beta = \{V_U: U \in \alpha\} \cup \{Y^c\}$. Then $\beta$ is a uniform cover of $X$ such that $\beta|_Y = \alpha$. That is, for each uniform cover $\alpha$ of $Y$, there is a uniform cover $\beta$ of $X$ such that $\beta|_Y = \alpha$. Now let $H$ be any compact subset of $Y$. Then
\[ N_H \left( \bigvee_{i=0}^{n-1} T_Y^{-1} \alpha \right) = N_H \left( \bigvee_{i=0}^{n-1} T^{-1} \beta \right). \]

Hence \( h_H(\alpha, T_Y) = h_H(\beta, T) \). Therefore

\[
\sup_{H \subseteq Y} h_H(\alpha, T_Y) = \sup_{H \subseteq Y} h_H(\beta, T) \leq \sup_{K \subseteq X} h_K(\beta, T) \leq h^*(T).
\]

This implies \( h^*(T_Y) \leq h^*(T) \).

4.20 Definition. We say that \( h \) distinguishes between the transformations \( S \) and \( T \) if \( h(S) \neq h(T) \).

If \( \varphi, \psi : X \to X \) are minimal and continuous and \( X \) is not compact, then \( h^0 \) cannot distinguish between \( \varphi \) and \( \psi \). In this respect \( h^0 \) is quite inadequate. We want to test the ability of \( h^i, i = 0, 1, 2, 3, \) and \( * \) to distinguish between the transformations \( T, S : R \to R \) defined by \( T(x) = x + 1 \) and \( S(x) = 2x \). See (5.5).

4.21 Example. Let \( T, S : R \to R \) be defined by \( T(x) = x + 1 \), \( S(x) = 2x \) respectively. Then

\[
h^1(T) = 0,
\]

\[
h^1(S) = 0.
\]

Proof: The one-point compactification \( R^* \) of the real line \( R \) is the circle and both \( T^*, S^* : R^* \to R^* \) are
homeomorphisms. It is shown in [2] that $h(T^*) = h(S^*) = 0$.
This immediately implies that $h^0(T) = h^0(S) = 0$.

4.22 Example. Let $T, S: \mathbb{R} \to \mathbb{R}$ be as in (4.21). We have shown previously that

$$h^*(T) = 0,$$
$$h^*(S) = \log 2.$$  

Now $\mathbb{R}$ with the usual topology is a normal, $T_1$-space and hence $h^2(T) = h^3(T)$ and $h^2(S) = h^3(S)$. We will show that $h^3(S) = h^3(T) = h^2(S) = h^2(T) = \infty$. Hence $h^*$ distinguishes between the transformations $T$ and $S$, but $h^i$, $i = 0, 1, 2,$ and $3$ do not.

4.23 Example. Let $T: \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x + 1$. Then

$$h^2(T) = h^3(T) = \infty.$$  

Proof: From the preceding paragraph we know that $h^2(T) = h^3(T)$. Let $X^*$ be the Stone-Cech compactification of $X = \mathbb{R}$ and $T^*: X^* \to X^*$ be the unique continuous extension of $T$ to $X^*$. Note (see [7] and [8 p.167]) that $(X^*, T^*)$ is the universal point transitive cascade and hence $h^2(T) = h(T^*) = \infty$. Another proof can be obtained using the monotonicity of $h^2$: let $Z$ be the set of integers and note that $Z$ is a closed $T$-invariant subset of $\mathbb{R}$. Then by (2.12) $h^2(T|Z) = \infty$. Hence

$h^2(T) = \infty$. 

4.24 Definition. Let \( K \) be the circle group \( K = \{ z \in \mathbb{C} : |z| = 1 \} \), where \( \mathbb{C} \) is the set of complex numbers. Then \( K^n = K \times K \times \ldots \times K \) (n terms) is called the n-torus.

4.25 Lemma. Let \( G \) be any locally compact group, and \( E, F \) be subgroups of \( G \). Then
\[
E^\perp = F^\perp \implies \overline{E} = \overline{F}.
\]
(Note: \( \hat{G} \) is the dual group of \( G \), and \( E^\perp = \{ \gamma \in \hat{G} : \gamma(x) = 1 \text{ for each } x \text{ in } E \} \) is called the annihilator of \( E \).)

Proof: From continuity of the characters it follows that \( E^\perp = (\overline{E})^\perp \) and \( F^\perp = (\overline{F})^\perp \). Since \( \overline{E} \) and \( \overline{F} \) are closed subgroups of \( G \),
\[
((\overline{E})^\perp)^\perp = \overline{E}, \text{ and}
((\overline{F})^\perp)^\perp = \overline{F} \quad \text{(see [26 p.36]).}
\]
Hence
\[
\overline{E} = ((\overline{E})^\perp)^\perp = (E^\perp)^\perp = (F^\perp)^\perp = (\overline{F})^\perp.
\]

Choose a line \( L \) in \( \mathbb{R}^n \) passing through the origin such that \( L \) is not orthogonal to any lattice lines (lines joining points of \( \mathbb{Z}^n \)). Clearly, \( L \) is isomorphic to \( \mathbb{R} \). Project the line \( L \) onto \( K^n \) using the map \( \pi \) defined by:
\[ \pi(x) = \pi(x_1, \ldots, x_n) = (e^{2\pi ix_1}, \ldots, e^{2\pi ix_n}) \text{ for } x \in L. \]

**4.26 Lemma.** \( \pi L \) is a dense subgroup of \( K^n \).

**Proof:** We know that \( (\pi L) ^\perp = \{0\} \implies \overline{\pi L} = K^n \) from Lemma (4.25). Now \( Z^n = \bigwedge^n \) ([26 p.36]). We show the annihilator of \( \pi L \) is trivial. Let \( m \in Z^n = \bigwedge^n \), \( m \neq 0 \) and let \( x \in \pi L \). Then \( m(x) = \langle x, m \rangle = \exp 2\pi i(x \cdot m) \) (here \( (x \cdot m) \) is the dot product of \( x \) and \( m \)). By the choice of \( L \), \( (x \cdot m) \neq 0 \). We have two cases:

Case 1. If \( (x \cdot m) \in Z \), then \( \exp 2\pi i(x \cdot m) \neq 1 \) and we are done.

Case 2. If \( (x \cdot m) = k \in Z \), replace \( x \) by \( \epsilon x \) such that \( (\epsilon x \cdot m) \notin Z \). This is again case 1. Hence \( (\pi L) ^\perp = \{0\} \).

**4.27 Example.** Let \( S: \mathbb{R} \to \mathbb{R} \) be defined by \( S(x) = 2x \).

Then

\[ h^2(S) = h^3(S) = \infty. \]

**Proof:** Now the n-torus has a dense subgroup isomorphic to \( \mathbb{R} \). Consider the following diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xleftarrow{S} & \mathbb{R} \\
\downarrow{\pi} & & \downarrow{\pi} \\
K^n & \xleftarrow{S'} & K^n
\end{array}
\]

where \( S(x) = 2x \), and \( \pi \) is as defined previously, and \( S' \) (not yet determined) satisfies \( \pi \circ S = S' \circ \pi \). From
this it follows easily that $S'$ squares each component; that is, $S'(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2)$. Now $\pi$ induces a continuous map $\varphi$ from $R^*$ onto $K^n$ (here $R^*$ is the Stone-Cech compactification of $R$), and we can extend the above diagram to get

$$
\begin{array}{c}
R^* \leftarrow S^* \rightarrow R^* \\
\downarrow \varphi \quad \quad \downarrow \varphi \\
K^n \leftarrow S' \rightarrow K^n
\end{array}
$$

where, clearly, $\varphi \circ S^* = S' \circ \varphi$. Now the map from the 1-torus onto the 1-torus defined by $x \rightarrow e^{2\pi i(2x)}$ has the associated matrix $A = (2)$. This implies (see [3 p.67-79]) that $h(A) = \log 2$. Hence $h(S') = n \log 2$ by the Product Theorem. So for each positive integer $n$, $h(S^*) \geq n \log 2$. Therefore $h^2(S) = h^3(S) = h(S^*) = \infty$. 
CHAPTER V
THE INDUCED TRANSFORMATION

Throughout this chapter, X and Y will be compact metric spaces - although many of our results hold for the more general case of compact Hausdorff spaces as well - and we will not always require that \( \varphi: X \to Y \) (where Y may be X) be continuous and that \( \alpha \) be an open cover.

5.1 Definition. Let \( \varphi: X \to X \) be a function and \( \alpha \) any finite cover of \( X \). We define

\[
\begin{align*}
\text{i) } & h(\alpha, \varphi) = \lim_{n \to \infty} \frac{1}{n} \log N \left( n^{-1} \bigcup_{i=0}^{n-1} \varphi^{-i} \alpha \right), \\
\text{ii) } & \overline{\nu}(\varphi) = \sup_{\alpha} h(\alpha, \varphi),
\end{align*}
\]

where the supremum is taken over all finite covers of \( X \). The limit in (i) exists.

Let \( \alpha \) and \( \beta \) be finite covers of \( X \). Most of the basic definitions and properties of [2] are still valid; in particular, the following remain valid.

5.2 Properties.

1) \( \alpha < \alpha', \beta < \beta' \implies \alpha \vee \beta < \alpha' \vee \beta' \).

2) \( \alpha < \beta \implies N(\alpha) \leq N(\beta) \).
3) \( \alpha < \beta \implies N(\alpha \lor \beta) = N(\beta) \).

4) \( N(\alpha \lor \beta) \leq N(\alpha)N(\beta) \).

5) \( \alpha < \beta \implies \varphi^{-1}\alpha < \varphi^{-1}\beta \).

6) \( \varphi^{-1}(\alpha \lor \beta) = \varphi^{-1}\alpha \lor \varphi^{-1}\beta \).

7) \( \varphi: X \to X \implies N(\varphi^{-1}\alpha) \leq N(\alpha) \).

8) \( \varphi: X \to X \implies N(\varphi^{-1}\alpha) = N(\alpha) \).

9) \( \alpha < \beta \implies h(\alpha, \varphi) \leq h(\beta, \varphi) \).

10) Let \( \varphi_1: X \to X, \varphi_2: Y \to Y \), and \( f: X \to Y \) (\( f \) is onto) be functions such that \( f \circ \varphi_1 = \varphi_2 \circ f \) and let \( \alpha \) be any finite cover of \( Y \). Then

\[
h(\alpha, \varphi_2) = h(f^{-1}\alpha, \varphi_1).
\]

11) Under the conditions of (10),

\[
\overline{H}(\varphi_2) \leq \overline{H}(\varphi_1).
\]

12) \( h(\alpha, \varphi) \leq \frac{1}{m} \log N\left( \bigvee_{i=0}^{m-1} \varphi^{-i}\alpha \right), \ m \in \mathbb{Z}^+ \).

13) \( h(\alpha, \varphi) = h\left( \bigvee_{i=0}^{n-1} \varphi^{-i}\alpha, \varphi \right), \ n \in \mathbb{Z}^+ \).

14) \( \overline{H}(\varphi^k) = k\overline{H}(\varphi), \ k \in \mathbb{Z}^+ \).

We will use the following abbreviation to simplify the writing of statements and proofs.

**Notation.**

1) \( A^o \) is the interior of \( A \).

2) \( A^C \) is the complement of \( A \).

3) \( 1st. \text{ Cat.} \) means first category.
5.3 Definition. A closed (open) cover is a cover, each member of which is closed (open).

5.4 Definition. Let \( E \) be any closed subset of \( X \) having interior. We say that \( T:X \to X \) is recurrent if each \( x \) in \( E \) returns to each \( E \) infinitely often under both positive and negative iterations of \( T \).

See [15 p.10] and [9 p.6] for more results concerning recurrence.

5.5 Definition. We say that \( T:X \to X \) is minimal if \( X \) is the smallest closed \( T \)-invariant subset of \( X \).

5.6 Remark. If \( T \) is minimal, then \( T \) is recurrent, since every point visits every nonempty open set infinitely often.

We will assume in this chapter that \( T:X \to X \) in addition to being a 1st. Cat. preserving homeomorphism, is minimal and hence recurrent. See (5.18).

Now let \( E \subseteq X \) where \( E \) is a closed set having nonempty interior and let \( T:X \to X \). We make the following definition.

5.7 Definition.
1) \( r_E(x) = \inf\{k > 0: T^k(x) \in E\} \),
2) \( R_k(x) = \{x \in E: r_E(x) = k\} \), and
3) \( \rho_E = \{R_1(E), R_2(E), \ldots\} \).
5.8 Definition. The induced transformation (first defined in [16]) denoted by $T_E$ is defined by

$$T_E(x) = T_{E}(x), \quad x \in E.$$ 

Hence $T_E : E \to E$ is defined by $T_E(x) = T^k(x)$ where $x, T^k(x) \in E$ and $T^i(x) \notin E$ for $0 < i < k$. Note that $T_E$ may also be considered as a map from $X$ into $X$ which leaves the points of $X - E$ fixed. If $E = X$, then $T_E = T$. See [9 p.13] and [19] for more results concerning the induced transformation.

We now show some elementary properties of the induced transformation.

5.9 Proposition. $T_E : E \to E$ is one-to-one.

Proof: Suppose $T_E(x) = T_E(y)$. We show that $x = y$.

Now $T_E(x) = T^k(x)$ for some finite positive integer $k$ where $k$ is the smallest such integer. Similarly $T_E(y) = T^j(y)$ for some smallest finite positive integer $j$. Hence $T^k(x) = T^j(y)$ and if $j = k$, then $x = y$.

Suppose $j < k$, say $j + n = k$. Then $T^j(y) = T^{j+n}(x)$ and so $y = T^n(x)$. But $y \in E \implies T^n(x) \in E$ with $n < k$, which contradicts the definition of $T_E(x)$. Similarly we get a contradiction if $k < j$. So it must be the case that $j = k$ and hence $x = y$. That is, $T_E$ is one-to-one.
5.10 Proposition. \( T_E : E \to E \) is onto \( E \).

Proof: Let \( x \in E \). Since \( T \) is minimal, there exists a \( k > 0 \) such that \( (T^{-1})^k(x) \in E \) where \( k \) is the smallest such positive integer. Let \( y = T^{-k}(x) \). Then \( T_E(y) = x \) and so \( T_E \) is onto \( E \).

5.11 Proposition. \( T_E^{-1} = (T^{-1})_E \).

Proof: Let \( x \in E \). There exists a least positive integer \( k \) such that \( (T^{-1})^k(x) = y \in E \). Thus \( (T^{-1})_E(x) = T^{-k}(x) = y \). Hence \( T^k(y) = x \); that is, \( T_E(y) = T^k(y) = x \). This implies \( T^{-1}_E(x) = y = (T^{-1})_E(x) \).

Since \( x \) was arbitrary, the proof is complete.

Remark. \( T^n_E \neq (T^n)_E \) for all integers \( n \).

5.12 Proposition. Let \( F \subseteq E \subseteq X \) where \( F \) and \( E \) are closed subsets of \( X \) having interior. Then \( (T_E)_F = T_F \).

Proof: Let \( x \in E^c = X - E \). Then clearly \( (T_E)_F(x) = x = T_F(x) \). Next suppose that \( x \in F \cap \left( \bigcup_{i=1}^{\infty} T^{-i}F \right) \): then \( T_F(x) \) is the first point of the orbit \{ \( x, T(x), T^2(x), \ldots \) \} that again lies in \( F \). But \( T_F(x) \) is also the first point of the orbit
\{x, T_E(x), T_E^2(x), \ldots\} that again lies in \(F\) since the set \(\{x, T_E(x), T_E^2(x), \ldots\}\) is a subset of the set \(\{x, T(x), T^2(x), \ldots\}\) where the points not in \(E\) are omitted. Hence for each \(x\) in \(X\), \(T_F(x) = (T_E)_F(x)\).

We now develop more preliminary material.

5.13 Definition. Let \(A \subseteq X\). Then \(A\) is called nearly open (n.o.) if there exists a first category set \(B\) such that

\[x \in A - B \implies x \in A^0.\]

5.14 Remark. An open set is nearly open; a closed set is nearly open; and hence, a open cover and a closed cover are nearly open covers - that is; covers each member of which is nearly open.

5.15 Definition. A one-to-one, onto map \(\varphi:X \to X\), is called a near homeomorphism if there exists a 1st. Cat. set \(N_\varphi\) such that both \(\varphi\) and \(\varphi^{-1}\) are continuous on \((X - N_\varphi)\). We may assume that \(\varphi^{-1}(N_\varphi) = N_\varphi\).

5.16 Remark. \(T_E:E \to E\) is continuous except on the boundary of \(E\) which is nowhere dense. Hence there exists a 1st. Cat. set on the complement of which each power \(T_E^n\) is continuous.

5.17 Remark. \(T_E:E \to E\) is a near homeomorphism (as
implied by (5.16)).

5.18 Definition. We say that $\varphi : X \to X$ is a 1st. Cat. preserving transformation if for each 1st. Cat. set $A$, both $\varphi(A)$ and $\varphi^{-1}(A)$ are 1st. Cat. sets.

5.19 Proposition. The countable union of nearly open sets is nearly open.

Proof: Let $A_n$ be a sequence of n.o. sets. We show that $A = \bigcup_{n=1}^{\infty} A_n$ is a n.o. set. Now for each $n$, $A_n$ is n.o. $\Rightarrow$ there exists a 1st. Cat. set $B_n$ such that $x \in A_n - B_n = x \in A_n^0$. Then $B = \bigcup_{n=1}^{\infty} B_n$ is a 1st. Cat. set and $x \in A - B = \exists \; n \; \text{such that} \; x \in A_n$ but $x \notin B_n$. So $x \in A_n - B_n$ which implies that $x \in A_n^0$ and hence $x \in A^0$.

5.20 Proposition. The finite intersection of nearly open sets is nearly open.

Proof: We prove the proposition for two sets. Let $A_1$ and $A_2$ be n.o. subsets of $X$. We show that $A = A_1 \cap A_2$ is n.o. $\Rightarrow$ there exists a 1st. Cat. set $B_1$ such that $x \in A_1 - B_1 = x \in A_1^0$ and $A_2$ is n.o. $\Rightarrow$ there exists a 1st. Cat. set $B_2$ such that $x \in A_2 - B_2 = x \in A_2^0$. Letting $B = B_1 \cup B_2$ we see that $B$ is a 1st. Cat. set and
\[ x \in A - B = (A_1 \cap A_2) - (B_1 \cup B_2) \]
\[ = (A_1 - B_1) \cap (A_2 - B_2) \]
\[ \implies x \in A_1^o \cap A_2^o = (A_1 \cap A_2)^o = A^o. \]

Let \( \varphi : X \rightarrow Y \) be a near homeomorphism and \( A \) a nearly open subset of \( Y \). It need not be true that \( \varphi^{-1}(A) \) is nearly open. We give an example of a near homeomorphism that is not semi-open (\( \varphi \) is semi-open if for each nonempty open set \( U \), \( \varphi(U) \) has interior - see [10p.15]) and at the same time shows that the inverse image of a nearly open set need not be nearly open.

The following example is due to J.R. Brown.

**Example.** An example of a map that is one-to-one, onto and nearly continuous (continuous except on a set of 1st. Cat.), but not semi-open. We proceed in three steps.

1) Consider the mapping of the unit interval onto the Cantor set given by

\[ \frac{x_1}{2} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \ldots \rightarrow \frac{y_1}{3} + \frac{y_2}{3^2} + \frac{y_3}{3^3} + \ldots \]

where

\[ y_k = \begin{cases} 0 & \text{if } x_k = 0 \\ 2 & \text{if } x_k = 1 \end{cases} \]

This map is one-to-one and continuous except at the dyadic rationals. We agree to take the terminating dyadic expansion at the rationals, so that the image includes right
hand end points of the excluded intervals but not the left hand end points. Note that the range is nowhere dense.

2) Next we scale down the domain of the above map to the interval \([0, \frac{1}{2})\). Divide \([\frac{1}{2}, 1)\) into a countable sequence of half-open intervals \([a_n, a_{n+1})\) with \(a_n < 1\).

We will map these linearly onto the excluded intervals; for example, \([a_1, a_2) = \left[\frac{1}{3}, \frac{2}{3}\right), [a_2, a_3) = \left[\frac{1}{5}, \frac{2}{5}\right), [a_3, a_4) = \left[\frac{7}{5}, \frac{8}{5}\right), \text{ etc.} \) Finally map \(1 \rightarrow 1\). The resulting map is clearly continuous except at a countable number of points (namely, the dyadic rationals in \([0, \frac{1}{2})\) and \(\{a_n\}, n = 1, 2, \ldots\) ). It is not semi-open since \(\psi((0, \frac{1}{2}))\) is nowhere dense, and hence has no interior.

3) The inverse map is continuous on the complement of the Cantor set, and hence is nearly continuous.

5.21 Remark. Let \(\psi : X \rightarrow X\) be a near homeomorphism.

Then the inverse image of a nearly open cover need not be a nearly open cover. To overcome this defect we will assume, for the remainder of this chapter, that \(\psi\) is a 1st Cat. preserving near homeomorphism.

5.22 Proposition. Let \(\psi : X \rightarrow X\). Then the inverse image of a nearly open set is nearly open. In particular the inverse image of an open set is nearly open. Hence if \(\alpha\) is a nearly open (or open) cover, then \(\psi^{-1}\alpha\) is a nearly open cover.
Proof: By definition, there exists a 1st. Cat. set \( N \varphi \) such that both \( \varphi \) and \( \varphi^{-1} \) are continuous on \( X - N \varphi \) and we assume \( \varphi^{-1}(N \varphi) = N \varphi \). Suppose \( A \) is a n.o. set. Then there exists a 1st. Cat. set \( B \) such that \( x \in A - B \Rightarrow x \in A^o \).

Let \( C = (A \cap N \varphi) \cup B \). Hence \( C \) is a 1st. Cat. set, and \( x \in A - C \Rightarrow x \in A^o \). Let \( y \in \varphi^{-1}(A - C) = \varphi^{-1}(A) - \varphi^{-1}(C) \). We show \( y \in (\varphi^{-1}A)^o \) and this will prove, since \( \varphi^{-1}(C) \) is a 1st. Cat. set, that \( \varphi^{-1}(A) \) is n.o. Now \( y \in \varphi^{-1}(A) \Rightarrow \varphi(y) \in A; \ y \not\in \varphi^{-1}(C) \Rightarrow \varphi(y) \not\in C \). That is, \( \varphi(y) \in A - C \) and hence \( \varphi(y) \in A^o \).

Let \( V \) be an open neighborhood of \( \varphi(y) \). Now \( \varphi \) is continuous at \( y \). Hence there exists an open neighborhood \( U \) of \( y \) such that \( \varphi(U) \subseteq V \), and so \( U \subseteq \varphi^{-1}(V) \subseteq \varphi^{-1}(A) \), which implies \( y \in (\varphi^{-1}A)^o \).

5.23 Remark. In [2], \( h(S) \) is defined for \( S \) a continuous map. We emphasize that we now permit \( S \) to be a 1st. Cat. preserving near homeomorphism. Hence we define \( h(\varphi) = \sup_{\alpha} h(\alpha, \varphi) \)

where the supremum is taken over all open covers of \( X \).

Abramov [1] has compared the measure theoretic entropy of an automorphism \( S \) with that of the induced transformation \( S_A \), with the following result. Let \( X \) be a Lebesgue space, \( A \subseteq X \), and \( S: X \to X \) an automorphism. Then, for \( \mu \) a normalized measure,
(5.24) \[ h_{\mu}(S) = \frac{h_{\mu}(S_A)}{\mu(A)}. \]

See [20] for another proof of (5.24).

It is conceivable that a formula analogous to (5.24) can be obtained for the topological case. We give an example showing that the formula \( h(T) = (T_A) \) is not always valid and offer a conjecture comparing \( h(T) \) with \( h(T_A) \) (the assumptions on \( T \) still stand, and hence \( T_A \) is a 1st. Cat. preserving near homeomorphism). Here \( A \) is a closed subset of \( X \) having interior.

The following example is due to L.W. Goodwyn.

5.25 Example. Assume that \((X, \overline{T})\) is a cascade that is minimal and totally minimal and has positive entropy, say \( \lambda \). Let \( Y \) be the disjoint union of two copies \( X_1 \) and \( X_2 \) of \( X \). Define \( S:Y \to Y \) by

\[
S(x) = \begin{cases} 
\text{the corresponding point in } X_2, & \text{for } x \text{ in } X_1. \\
\text{the point corresponding to } \overline{T}(x), & \text{for } x \text{ in } X_2.
\end{cases}
\]

Then \((Y, S)\) is minimal and \( S^2|X_1 = \overline{T} \), and \( S^2|X_2 = \overline{T} \). By (2.5),

\[
h(S^2) = \max\{h(S^2|X_1), h(S^2|X_2)\} = h(\overline{T}).
\]

Hence \( h(S) = \frac{1}{2} \lambda \). Now let \( A = X_1 \subseteq Y \). Then \( S_A = \overline{T} \) and so \( h(S_A) = h(\overline{T}) = \lambda \neq \frac{1}{2} \lambda = h(S) \).
5.26 Definition. A measure $\mu$ is $S$-invariant if 
$\mu(S^{-1}(A)) = \mu(A)$ for every Borel set $A \subseteq X$. We denote by $\mathcal{M}(X, S)$ the set of all $S$-invariant regular probability measures on the Borel sets of $X$.

5.27 Theorem (Goodwyn [12]). Let $S:X \to X$ be a continuous map. Then, for $\mu \in \mathcal{M}(X, S)$,
$$h_\mu(S) \leq h(S).$$

Theorem (5.27) is valid for $\varphi$.

5.28 Proposition. $h_\mu(\varphi) \leq h(\varphi)$.

Proof: The proof of (5.27) is valid.

5.29 Theorem (Goodman [A]). Let $S:X \to X$ be continuous. Then
$$h(S) = \sup_{\mu \in \mathcal{M}(X, S)} h_\mu(S).$$

5.30 Definition. Let $A \subseteq X$ and $S:X \to X$. Define
$$\eta(A, S) = \sup_{x \in X} \liminf_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(S^i(x)),$$
where $\chi_A$ is the characteristic function of $A$.

5.31 Proposition (Goodwyn[10]). Let $A$ be a closed subset of $X$. Then
$$\eta(A, S) = \sup_{\mu \in \mathcal{M}(X, S)} \mu(A).$$
Consider again equation (5.24). An interesting conjecture for the topological analog is

\[(5.32) \quad h(T) = \overline{\eta}(A, T) h(T_A).\]

**Remark.** In Example (5.25) \( \overline{\eta}(A, S) = \frac{1}{2} \) and hence (5.32) is valid for this example.

5.33 Proposition. \( h(T) \leq \overline{\eta}(A, T) h(T_A). \)

Proof: Using Abramov's formula (5.24), together with (5.31) and (5.28) we can write

\[
h_{\mu}(T) \leq h_{\mu}(T_A) \overline{\eta}(A, T)
\]

\[\leq h(T_A) \overline{\eta}(A, T).\]

This implies, using (5.29),

\[h(T) \leq h(T_A) \overline{\eta}(A, T).\]

It may be possible to prove the reverse inequality (if true) in a similar way. We have included the preliminary material since a topological argument is also feasible.

It turns out that the return partition

\[\rho_E = \{R_1(E), R_2(E), \ldots\}\]

is a nearly open cover. A topological proof would most certainly involve \( \rho_E \).

This leads to the consideration of countable nearly open covers. The following definitions may prove useful.

Let \( U \subseteq X \) and \( \epsilon > 0 \). Define (here \( d \) is a metric)

1) \( U_\epsilon = \{y : d(x, y) < \epsilon \ \text{for some} \ x \in U\}, \)
2) \( \alpha_\varepsilon = \{U_\varepsilon: U \in \alpha\} \) (here \( \alpha \) is a n.o. cover),

3) \( N_\varepsilon(\alpha) = N(\alpha_\varepsilon) \),

4) \( \tilde{h}(\alpha, T) = \lim_{n \to \infty} \frac{1}{n} \log \sup_{\varepsilon > 0} N_\varepsilon \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) \), and

5) \( \tilde{h}(T) = \sup_{\alpha} \tilde{h}(\alpha, T) \),

where the supremum is taken over all nearly open covers of \( X \).
In Chapter II we have extended the concept of topological entropy as defined in [2] to noncompact spaces. In this chapter we "extend" the results in a new direction.

In [23] the concept of sequence entropy, originally defined by Kushnirenko [21] is discussed. This has an obvious topological analog. Throughout this chapter \( X \) and \( Y \) will be a compact topological spaces, \( T: X \to X \) will be a continuous map, and \( \sigma \) will be any strictly increasing sequences of nonnegative integers;

\[ \sigma = \langle t_n \rangle = \langle t_1, t_2, \ldots \rangle. \]

6.1 Definition. We define \( h_\sigma (\alpha, T) \) by

\[ h_\sigma (\alpha, T) = \lim_{n \to \infty} \sup \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right). \]

6.2 Definition. The topological \( \sigma \)-sequence entropy (or simply sequence entropy) of \( T \) is defined by

\[ h_\sigma (T) = \sup_{\alpha \in G} h_\sigma (\alpha, T). \]
Remark. Let \( \sigma = \langle n \rangle = \langle 0, 1, 2, 3, \ldots, n - 1, n, \ldots \rangle \).

In this case the \( \lim \sup \) becomes the limit and we have

i) \( h_\sigma(a, T) = \lim_{n \to \infty} \frac{1}{n} \log N(a \vee T^{-1}a \vee \ldots \vee T^{-n+1}a) \)

\[ = h(a, T). \]

ii) \( h_\sigma(T) = h(T). \)

Hence the sequence \( \sigma = \langle n \rangle \) gives the original definition as defined in [2].

6.3 Proposition. Let \( \sigma = \langle n \rangle \) and \( k\sigma = \langle kn \rangle \). That is, \( \sigma = \langle 0, 1, 2, \ldots \rangle \); \( k\sigma = \langle 0, k, 2k, \ldots \rangle \), where \( k \in \mathbb{Z}^+ \). Then

i) \( h_{k\sigma}(a, T) = h_\sigma(a, T^k) = h(a, T^k) \);

ii) \( h_{k\sigma}(T) = kh_\sigma(T) = kh(T) \).

Proof: Let \( a \) be any open cover of \( X \).

i) \( h_{k\sigma}(a, T) = \limsup_{n \to \infty} \frac{1}{n} \log N(a \vee T^{-k}a \vee T^{-2k}a \vee \ldots \vee T^{-k(n-1)}a) \)

\[ = h_\sigma(a, T^k) = h(a, T^k), \text{ for all } a \in G. \]

ii) \( h_{k\sigma}(T) = h_\sigma(T^k) = h(T^k) = kh(T) = kh_\sigma(T) \)

follows from (i) and the known property \( h(T^k) = kh(T) \).
6.4 Proposition. Let \( \alpha < \beta \) where \( \alpha, \beta \in G \). Then for any sequence \( \sigma \),

\[
h_{\sigma}(\alpha, T) \leq h_{\sigma}(\beta, T).
\]

Proof: For each \( n \in \mathbb{Z}^+ \) we have

\[
T^{-t_1} \alpha \vee \ldots \vee T^{-t_n} \alpha < T^{-t_1} \beta \vee \ldots \vee T^{-t_n} \beta.
\]

Then \( \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right) \leq \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \beta \right) \), which implies \( h_{\sigma}(\alpha, T) \leq h_{\sigma}(\beta, T) \).

6.5 Proposition. Let \( \alpha \in G \) and \( \sigma \) be any sequence. Then

\[
h_{\sigma}(\alpha, T) \leq \log N(\alpha).
\]

Proof: \( N(T^{-t_1} \alpha \vee \ldots \vee T^{-t_n} \alpha) \leq (N(\alpha))^n \). Therefore

\[
\frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha \right) \leq \log N(\alpha). \quad \text{Hence} \quad h_{\sigma}(\alpha, T) \leq \log N(\alpha).
\]

6.6 Proposition. Let \( \langle \alpha_n \rangle \) be a refining sequence of open covers; that is,

i) \( \alpha_n < \alpha_{n+1}, \; n = 1, 2, \ldots \).

ii) For each open cover \( \beta \), there exists an \( n \) such that \( \beta < \alpha_n \).

Then, for any sequence \( \sigma \),
\[ h_\sigma(T) = \lim_{n \to \infty} h_\sigma(\alpha_n, T). \]

**Proof:** By Proposition (6.4), \( h_\sigma(\alpha_n, T) \) is nondecreasing as \( n \) increases. Hence \( h_\sigma(T) = \lim_{n \to \infty} h_\sigma(\alpha_n, T). \)

6.7 **Proposition.** Let \( X, Y \) be compact topological spaces and \( T, \varphi, S \) be continuous maps (with \( \varphi \) onto) as shown in the diagram and suppose \( \varphi \circ T = S \circ \varphi. \)

\begin{align*}
\begin{array}{c}
X \\
\downarrow \varphi \\
Y
\end{array}
\begin{array}{c}
T \\
\downarrow \varphi \\
S \\
\downarrow \varphi \\
Y
\end{array}
\end{align*}

Let \( \alpha \) be any open cover of \( Y \). Then for any sequence \( \sigma \),

\[ h_\sigma(\alpha, S) = h_\sigma(\varphi^{-1} \alpha, T). \]

**Proof:** Let \( \sigma = \langle t_1, t_2, \ldots \rangle \). Then

\[ \varphi^{-1} \left( \bigvee_{i=1}^{n} S^{-t_i} \alpha \right) = \bigvee_{i=1}^{n} \varphi^{-1} (S^{-t_i} \alpha) = \bigvee_{i=1}^{n} T^{-t_i} (\varphi^{-1} \alpha). \]

Therefore \( \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} S^{-t_i} \alpha \right) = \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} (\varphi^{-1} \alpha) \right). \)

Hence \( h_\sigma(\alpha, S) = h_\sigma(\varphi^{-1} \alpha, T). \)

6.8 **Proposition.** Under the conditions of Proposition (6.7), we have for any sequence \( \sigma \),

\[ h_\sigma(S) \leq h_\sigma(T). \]
Proof: The proof is immediate: 
\[ h_\sigma(S) = \sup_{\alpha \in G(Y)} h_\sigma(\alpha, S) \]
\[ = \sup_{\alpha \in G(Y)} h_\sigma(\varphi^{-1}_a, T) \leq \sup_{\beta \in G(X)} h_\sigma(\beta, T) = h_\sigma(T). \]

**Lemma.** Let \( \alpha \) and \( \beta \) be open covers of \( X \) and \( Y \) respectively. Then 
\[ N(\alpha \times \beta) \leq N(\alpha)N(\beta). \]

Proof: Let \( \alpha' \) be a subcover of \( \alpha \) with \( N(\alpha) \) members and \( \beta' \) a subcover of \( \beta \) with \( N(\beta) \) members. Then \( \alpha' \times \beta' \) is a subcover of \( \alpha \times \beta \) with \( N(\alpha)N(\beta) \) members and hence \( N(\alpha \times \beta) \leq N(\alpha)N(\beta) \).

**6.9 Proposition.** Let \( \varphi_1: X \to X \) and \( \varphi_2: Y \to Y \) be continuous. Then \( \varphi_1 \times \varphi_2: X \times Y \to X \times Y \) defined by 
\[ \varphi_1 \times \varphi_2(x, y) = (\varphi_1 x, \varphi_2 y) \] satisfies, for any sequence \( \sigma \), 
\[ h_\sigma(\varphi_1 \times \varphi_2) \leq h_\sigma(\varphi_1) + h_\sigma(\varphi_2). \]

Proof: Let \( \gamma \) be any open cover of \( X \times Y \). In [2] it is shown that there exists an open cover of the form \( \alpha \times \beta \) (\( \alpha \) a cover of \( X \), \( \beta \) a cover of \( Y \)) such that \( \gamma < \alpha \times \beta \). Hence \( N(\gamma) \leq N(\alpha \times \beta) \leq N(\alpha)N(\beta) \). This implies 
\[ h_\sigma(\gamma, \varphi_1 \times \varphi_2) \leq h_\sigma(\alpha, \varphi_1) + h_\sigma(\beta, \varphi_2) \]
\[ \leq h_\sigma(\varphi_1) + h_\sigma(\varphi_2). \]
Hence 
\[ h_\sigma(\varphi_1 \times \varphi_2) \leq h_\sigma(\varphi_1) + h_\sigma(\varphi_2). \]
6.10 Proposition. Let \( X_1 \) and \( X_2 \) be two closed subsets of \( X \) such that \( X = X_1 \cup X_2 \) and \( \varphi(X_1) \subseteq X_1 \), \( \varphi(X_2) \subseteq X_2 \) for a continuous map \( \varphi: X \to X \). Then for any sequence \( \sigma \)

\[ h_\sigma(\varphi) = \max\{h_\sigma(\varphi_1), h_\sigma(\varphi_2)\} \]

Proof: The proof of [2] is valid if the sequence \( \langle 0, 1, 2, \ldots \rangle \) is replaced by \( \langle t_1, t_2, \ldots \rangle \).

6.11 Example. If \( T: X \to X \) (here \( X \) is a compact metric space) is an isometry from \( X \) onto \( X \), then for any sequence \( \sigma \)

\[ h_\sigma(T) = 0. \]

Proof: Let \( \alpha_p \) be the family of all open sets of diameter less than \( \frac{1}{p} \). Then \( \alpha_p \vee \alpha_p = \alpha_p \) and

\[ \varphi^{-1} \alpha_p = \alpha_p. \]

Hence \( \alpha_p = T^{-t_1} \alpha_p \vee \ldots \vee T^{-t_n} \alpha_p. \)

So

\[ \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-t_i} \alpha_p \right) = \frac{1}{n} \log N(\alpha_p). \]

Hence \( h_\sigma(\alpha_p, T) = 0, \)

which implies \( h_\sigma(T) = \lim_{p \to \infty} h_\sigma(\alpha_p, T) = 0. \)

6.12 Definition. A sequence \( \sigma = \langle t_1, t_2, \ldots, t_{n-1}, t_n, \ldots \rangle \) is said to have bounded gaps if there exists a constant \( K \) such that \( t_n - t_{n-1} < K \) for all \( n \geq 1 \).
Definition. For a sequence \( \sigma \), we define \( d(\sigma) \), called the density of the sequence, by

\[
d(\sigma) = \limsup_{n \to \infty} \left( \frac{t_n}{n} \right).
\]

Remark. Let \( \sigma = \langle t_1, t_2, \ldots \rangle \) be a sequence with bounded gaps. Then

\[
t_n - t_1 < nK \quad \text{for some constant} \quad K, \quad \text{and so}
\]

\[
\frac{t_n}{n} - \frac{t_1}{n} < K. \quad \text{Hence} \quad d(\sigma) \leq K.
\]

Let \( \sigma = \langle n \rangle = \langle 0, 1, 2, \ldots \rangle \). Then

\[
\frac{t_n}{n} = \frac{n - 1}{n} \quad \text{and} \quad d(\sigma) = 1.
\]

Let \( \sigma = \langle kn \rangle = \langle 0, k, 2k, \ldots \rangle \). Then

\[
\frac{t_n}{n} = \frac{k(n - 1)}{n} = \frac{kn - k}{n} \quad \text{and} \quad d(\sigma) = k.
\]

Definition. Let \( U \subseteq X \). The diameter of \( U \) is denoted and defined by

\[
diam(U) = \sup \{ \rho(x, y) : x, y \in U \}
\]

where \( \rho \) is a metric on \( X \).

Definition. Let \( \alpha \) be an open cover of \( X \). The diameter of \( \alpha \) is defined by

\[
diam(\alpha) = \sup_{U \in \alpha} \text{diam}(U).
\]

We will need the following simple lemma.

6.13 Lemma. Let \( \alpha \) be an open cover of the compact
metric space $X$ and suppose $\text{diam} \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) \to 0$ as $n \to \infty$. Then

$$h(T) = h(\alpha, T).$$

Proof: Recall from a previous result in Chapter I, that for each $n \in \mathbb{Z}^+$, and for any $\alpha \in G$,

$$h(\alpha, T) = h \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha, T \right).$$

Let $\alpha_n = \bigvee_{i=0}^{n-1} T^{-i} \alpha$, $n = 1, 2, 3, \ldots$. Then $\alpha_n < \alpha_{n+1}$ for $n = 1, 2, 3, \ldots$. This, together with the assumption that $\text{diam}(\alpha_n) \to 0$ as $n \to \infty$, implies that $\alpha_n$ is a refining sequence. Hence (see [2])

$$h(T) = \lim_{n \to \infty} h(\alpha_n, T) = \lim_{n \to \infty} h(\alpha, T) = h(\alpha, T).$$

We now prove a relationship between topological sequence entropy and topological entropy.

6.14 Proposition. Let $X$ be a compact metric space; $T: X \to X$ a homeomorphism; and let $\sigma$ be a sequence with bounded gaps. If there exists an $\alpha$ in $G$ such that

$$\text{diam} \left( \bigvee_{i=0}^{k-1} T^{-i} \alpha \right) \to 0$$

as $k \to \infty$, then

$$h_\sigma(T) = d(\sigma)h(T).$$
Proof: Let $\alpha_k = \sum_{i=0}^{k-1} T^{-i} \alpha$ where $\alpha \in G$ satisfies $\diam(\alpha_k) \to 0$ as $n \to \infty$. Then $\alpha_k < \alpha_{k+1}$, $n = 1, 2, \ldots$. By (6.13) $h(T) = h(\alpha, T)$. Now $\alpha_k$ is a refining sequence, and so $h_\sigma(T) = \lim_{k \to \infty} h_\sigma(\alpha_k, T)$. Since $\sigma$ has bounded gaps, there exists $K$ such that $t_n - t_{n-1} < K$, $n \geq 1$. Choose $k$ such that $k > K$. Then it is easy to show that

$$h_\sigma(\alpha_k, T) = \limsup_{n \to \infty} \frac{1}{n} \log N\left(\sum_{i=1}^{n} T^{-i} \alpha_k\right)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log N\left(\sum_{i=t_1}^{k+t-1} T^{-i} \alpha\right)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \log N\left(\sum_{i=0}^{t_n+k-t-1} T^{-i} \alpha\right)$$

$$= \limsup_{n \to \infty} \frac{1}{n} \left(\frac{t_n+k-t-1}{t_n+k-t_1}\right) \log N\left(\sum_{i=0}^{t_n+k-t-1} T^{-i} \alpha\right)$$

$$= \limsup_{n \to \infty} \left(\frac{t_n}{n}\right) \lim_{n \to \infty} \frac{1}{n} \log N\left(\sum_{i=0}^{t_n+k-t-1} T^{-i} \alpha\right)$$

$$= d(\sigma) h(\alpha, T)$$
Hence \( h_\sigma(T) = d(\sigma) h(T) \). Note in particular that if \( \sigma = \langle n \rangle \), then \( d(\sigma) = 1 \) and \( h_\sigma(T) = h(T) \).

Next we show that if \( d(\sigma) \) is finite then \( h_\sigma(T) \) is bounded above.

**6.15 Proposition.** Let \( \sigma \) be a sequence such that 
\( d(\sigma) < \infty \), and \( \alpha, T \) be as in (6.14). Then

\[ h_\sigma(T) \leq d(\sigma) h(T). \]

**Proof:** Let \( \alpha \) be an open cover of \( X \) as in proof of (6.14), and set \( \alpha_k = \bigvee_{i=0}^{k-1} T^{-i} \alpha \). Then it is easy to show that

\[ \bigvee_{i=1}^{n} T^{-i} \alpha_k < \bigvee_{i=t_1}^{k+t_n-1} T^{-i} \alpha. \]

Hence

\[ h_\sigma(\alpha_k, T) = \limsup_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=1}^{n} T^{-i} \alpha_k \right) \]
\[ \leq \limsup_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=t_1}^{k+t_n-1} T^{-i} \alpha \right) \]
\[ = \limsup_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{t_n+k-t_1-1} T^{-i} \alpha \right) \]
\[ = d(\sigma) h(\alpha, T) \quad \text{(as in the proof of (6.14))} \]
\[ = d(\sigma) h(T). \]

Now \( k \to \infty \) to get
6.16 Example. Let $X_i = \{0, 1\}$ be given the discrete topology. Then $X = \bigotimes_{i=-\infty}^{\infty} X_i$ is compact by virtue of the Tychonoff theorem. Let $x_i$ be the $i$th component of the sequence $x \in X$. Define $T: X \to X$, called the shift, by $T(x_i) = x_{i+1}$. Now $X$ is a compact metric space with metric $\rho$ where

$$
\rho(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^i}.
$$

Let $\alpha$ be the open cover of $X$ defined by

$$
\alpha = \{\{x:x_0 = 0\}, \{x:x_0 = 1\}\}.
$$

Set $\alpha_k = \bigvee_{i=-k}^{k} T^i \alpha$. Then $\alpha_1 < \alpha_2 < \ldots < \alpha_n < \ldots$ and $\text{diam}(\alpha_k) \to 0$ as $k \to \infty$. Hence the sequence $\{\alpha_k\}$ is refining. It is proven in (6.13) that $h(T) = h(\alpha, T)$ under these conditions. Let $\sigma$ be a sequence with bounded gaps. Then similar to Proposition (6.14) we can show that

$$
h_{\sigma}(T) = d(\sigma) h(T).
$$

But it is shown in [2] that $h(T) = \log 2$. Hence the sequence entropy of the shift defined above is

$$
h_{\sigma}(T) = d(\sigma) \log 2.
$$
6.17 Remark. Let $T: X \to X$ be a lst. Cat. preserving near homeomorphism where $X$ is a compact metric space. Let $\sigma = (\ldots, t_{-1}, t_0, t_1, \ldots)$ be any arbitrary sequence (including the sequence having $t_i = t_i(x)$, where $i$ is any integer) and define $h_\sigma(\sigma, T)$ and $h_\sigma(T)$ as usual. This generalizes our previous results and includes the topological entropy of the induced transformation. More research remains to be done using the above generalization.
BIBLIOGRAPHY


11. The product theorem for topological entropy. Lexington, Kentucky, University of Kentucky, Department of Mathematics, 1969. (Preprint)


