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Title VIBRATION OF ELASTIC BARS

Abstract approved (Major Professor)

In this thesis we consider boundary and initial value problems associated with the equation

\((\partial^4_x + \partial^2_t)u = 0\), which describes the vibrations of elastic bars. The problems are reduced to integral equations with, in some cases, highly singular kernels. These equations are solved by iteration and lead to the solutions of the original problems when the given data are sufficiently smooth and exhibit (in the cases of infinite and semi-infinite bars) the proper growth at infinity.

This equation, though of fourth order, can be considered in some sense to have a "reduced order" of 2. This second order character is pointed out and explored, in particular, the connection of our equation with the
two Schrödinger equations \((\partial_x^2 + i\partial_t)u = 0\) permeates the whole problem.

Uniqueness questions are also discussed. In the infinite and semi-infinite cases we give examples to show that uniqueness of the solution fails to hold if the class of functions in which one seeks a solution is permitted to be too large.
VIBRATION OF ELASTIC BARS

by

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INTRODUCTION

Under appropriate restrictions of material and motion the equation governing the response of an elastic bar in vibration can be taken to be \((\partial_x^4 + \partial_t^2)u = 0\). Because of its mechanical importance there is a large amount of literature available on this equation, but it deals almost exclusively with the eigenvalue problem which is of considerable interest to mechanical engineers. The Schrödinger equations \((\partial_x^2 \pm i\partial_t)u = 0\), which are closely related to the equation above, have considerable literature devoted to them, but again the eigenvalue problems have been emphasized because of their practical importance to physicists. In this dissertation we consider the mathematical problems of existence, representation and uniqueness of solutions to \((\partial_x^4 + \partial_t^2)u = 0\) for three distinct boundary and initial value problems.

In Chapter 1 we consider the Cauchy problem (initial value problem or infinite bar problem) and obtain a solution and representation for a certain class of initial data. The results of this chapter are not new, being known to Pini [7] and apparently to Boussinesq (see [11]), but are included for completeness and to introduce the fundamental solutions which will be used in what follows.

Chapter 2 is devoted to the problem of the semi-infinite bar, or sometimes called the mixed problem, with
initial data \( u = u_t = 0 \) for \( x > 0 \) and boundary data 
\( u(0, t) = a(t) \) and \( u_x(0, t) = b(t) \). We point out that 
the more general problem of non-zero initial data can be 
reduced to a problem of the type above. We solve this 
problem by finding density functions which, when convo-
luted with the fundamental solutions, satisfy the dif-
ferential equation and which are related to the given data 
by systems of integral equations obtained by investigating 
the singularities of the fundamental solutions. It is 
apparent that the methods we use can be extended to other 
problems of this type and we catalogue these results with-
out details in the appendices. Special problems of this 
type have been considered by other methods by Sneddon 

In Chapter 3 we consider a boundary value problem 
for a finite bar. Although there are at least 36 "inter-
esting" possible boundary value problems for the finite 
bar (see [5, pp. 94-95]), we are content to solve only 
the problem, \( u = u_t = 0 \) initially and \( u = a_1, u_x = b_1 \) 
at \( x = 0 \) and \( u = a_2, u_x = b_2 \) at \( x = 2 \), confident 
that no essentially new difficulties will arise in the 
other cases. This problem was also considered by Pini 
[7] whose work overlaps with the results given here. Pini 
obtains a solution, as we do, by reducing the boundary 
value problem to the problem of solving a set of integral
equations. Pini proves the existence of a solution to these equations using transform theory while here the solutions are found by solving directly the integral equations. Further, our method is considerably simpler than that of Pini. We reduce the solution of the system of integral equations to solving the complex valued integral equation \( \varphi = f + \lambda k \ast \varphi \).

As a side problem we also investigate the related equation \( \varphi = f + \lambda k \ast \varphi \) and obtain the interesting result that the solution defined by the Neumann series exists in general only if \( |\lambda| < 1 \). This is apparently the first example of a Volterra integral equation exhibiting this behavior.

The fourth and final chapter is devoted to problems of uniqueness and representation. We discuss some of the more recent work, [2] and [6], on the uniqueness and existence of solutions of the Cauchy problem in rather general classes and we give examples to show that uniqueness fails in certain classes. A uniqueness theorem for the finite bar is given by Pini [8], but it is not easy to follow. We present a straightforward proof based upon the energy integral which is sufficient for our purposes. From this we obtain a uniqueness proof for the semi-infinite bar problem, but in view of the discussion for the infinite bar this result seems to be very inadequate.

It is clear that functions \( u \) having four
continuous derivatives which satisfy either 1)
\[(\alpha_x^2 + ia_t)u = 0 \quad \text{or} \quad 2) \quad (\alpha_x^2 - ia_t)u = 0\]
will also satisfy our equation, 3) \[(\alpha_x^4 + \alpha_t^2)u = 0, \quad \text{so that the sum of such a solution of 1) and of 2) will be a solution of 3).}]
Conversely we show here that if \( u \) is a solution of 3) in a region of the \((x,t)\)-plane, then it can be represented, at least locally, as the sum of solutions of 1) and 2).

The appendices are devoted to justifying an interchange of order of integration needed in Chapter 3, evaluation of certain integrals and cataloging operator matrices and their inverses for different boundary value problems for the semi-infinite bar.

Throughout this work numbers such as 1) refer to formulas within the chapter, 1.a) means formula a of Chapter 1, and I.a) means formula a of Appendix I.
1. The infinite bar problem

The problem of the infinite bar, or Cauchy problem, is to determine a function \( u(x,t) \) which satisfies the equation

\[
\delta_x^4 u + \delta_t^2 u = 0 ,
\]

for \( \{ (x,t) | t>0 \} \), with

\[
\lim_{t \to 0} u(x,t) = f_1(x) , \quad \lim_{t \to 0} \delta_t u(x,t) = f_2''(x) .
\]

Formally taking Fourier transforms with respect to \( x \), one obtains a formal solution

\[
u(x,t) = \frac{1}{\sqrt{4 \pi t}} \int_{-\infty}^{\infty} [ f_2(y) \cos \left( \frac{(x-y)^2}{4t} + \frac{\pi}{4} \right) \\
- f_1(y) \sin \left( \frac{(x-y)^2}{4t} + \frac{\pi}{4} \right) ] dy
\]

which is essentially that of Boussinesq (see [11]).

Set

\[
k(x,t) = - \frac{1}{\sqrt{\pi}} t^{-1/2} \exp \left[ \frac{ix^2}{4t} + \frac{i\pi}{4} \right] .
\]

Then the fundamental solutions

\[
C(x,t) = -\frac{1}{\sqrt{\pi}} t^{-1/2} \cos \left( \frac{x^2}{4t} + \frac{\pi}{4} \right)
\]

\[
S(y,t) = -\frac{1}{\sqrt{\pi}} t^{-1/2} \sin \left( \frac{x^2}{4t} + \frac{\pi}{4} \right)
\]

are respectively the real and imaginary parts of \( k(x,t) \).
2. **Properties of \( k(x,t) \) and verification of solution**

Immediately from the definition we see that \( k(x,t) \) satisfies for \( t > 0 \)

5) \[ \partial_x k(x,t) = \frac{ix}{2t} k(x,t) \]

6) \[ (\partial_x^2 + i\partial_t) k(x,t) = 0 \]

7) \[ (\partial_x^2 - i\partial_t) \bar{k}(x,t) = 0 \]

and hence

8) \[ \partial_x^4 + \partial_t^2 \begin{cases} C(x,t) \\ S(x,t) \end{cases} = 0 \]

**Lemma 1)** \[ \int_{-\infty}^{\infty} k(x-y, t) \, dy = \int_{-\infty}^{\infty} k(y, t) \, dy = -2i \]

**Proof:** \[ \int_{-\infty}^{\infty} k(y, t) \, dy = -e^{i\pi/4} (\pi t)^{-1/2} \int_{-\infty}^{\infty} \exp \left( \frac{iy^2}{4t} \right) \, dy \]

\[ = -\frac{2e^{i\pi/4}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{is^2} \, ds = -\frac{4e^{i\pi/4}}{\sqrt{\pi}} \int_{0}^{\infty} e^{is^2} \, ds \]

\[ = -\frac{4e^{i\pi/4}}{\sqrt{\pi/2}} e^{i\pi/4} = -2i \]

The question of existence and representation of solutions of the Cauchy problem will be considered in detail again in Chapter 4 with different emphasis. The following theorems are not new, being known to Boussinesq (see [11]) and Pini (see [7]), but are included for completeness. They are also not the most general, but adequate for the
present purposes.

Theorem 1) Let \( \varphi(x) \) be \( BV(-\infty, \infty) \) and \( \varphi(x) = O(x^{-1-a}) \), \( a > 0 \) as \( |x| \to \infty \), then

\[
\int_{-\infty}^{\infty} k(x-y,t) \varphi(y) dy \to i[\varphi](x) \quad \text{as} \quad t \to 0
\]

where \( i[\varphi](x) = \varphi(x + 0) + \varphi(x - 0) \).

Proof: Observe first that the integral exists and

\[
\int_{-\infty}^{\infty} k(x-y,t) \varphi(y) dy = \int_{-\infty}^{\infty} k(y,t) [\varphi(x-y) + \varphi(x+y)] dy.
\]

From Lemma 1)

\[
\int_{-\infty}^{\infty} k(x-y,t) \varphi(y) dy + i[\varphi](x) = \int_{-\infty}^{\infty} k(x-y,t) [\varphi(y) - \frac{1}{2}[\varphi](x)] dy.
\]

Then

\[
\left| \int_{-\infty}^{\infty} k(x-y,t) [\varphi(y) - \frac{1}{2}[\varphi](x)] dy \right|
\]

\[
= \left| \int_{-\infty}^{\infty} k(y,t) [\varphi(x-y) + \varphi(x+y) - [\varphi](x)] ds \right|
\]

\[
\leq \frac{2}{\sqrt{\pi}} \left| \int_{-\infty}^{\infty} \exp(is^2)[\varphi(x-s\sqrt{t}) + \varphi(x+s\sqrt{t}) - [\varphi](x)] ds \right|
\]

\[
= \frac{2}{\sqrt{\pi}} \left| \left( \int_0^S + \int_{-\infty}^0 \right) \cdots ds \right| = I_1 + I_2.
\]

Since \( \varphi \) is \( BV(-\infty, \infty) \) and \( \to 0 \) as \( |x| \to \infty \), there exists a constant \( B \) such that

\[
|\varphi(x-s\sqrt{t}) + \varphi(x+s\sqrt{t}) - [\varphi](x)| \leq B.
\]

Now from Hobson [3] page 623

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\( BV(a,b) \) means that the function is of bounded variation on the interval \((a,b)\).
\[ I_2 \leq \frac{2}{\sqrt{\pi}}(B + V_S^\infty [\varphi(x-s\sqrt{4t}) + \varphi(x+s\sqrt{4t}) - [\varphi](x)]) \sigma \]

where \( \sigma = \sup_{S \leq S' \leq S'} |\int_0^{S'} \exp(is^2) \, ds| \).

Since \( \int_0^\infty \exp(is^2) \, ds \) exists, \( \sigma \) and hence \( I_2 \) can be made arbitrarily small by choosing \( S \) sufficiently large.

Since \( \varphi \) is BV, \( t \) can be chosen so small that

\[ |\varphi(x-s\sqrt{4t}) + \varphi(x+s\sqrt{4t}) - [\varphi](x)| < \frac{\varepsilon \sqrt{\pi}}{4S} \]

uniformly for \( 0 \leq s \leq S \). Thus

\[ I_1 < \frac{\varepsilon \sqrt{\pi}}{4S} \frac{2}{\sqrt{\pi}} \int_0^S |\exp(is^2)| \, ds \leq \frac{\varepsilon}{2} \]

which completes the proof.

That 2) is indeed a solution of 1) will now be established.

**Theorem 2:** Suppose \( f_i'' \) is BV(\(-\infty, \infty\)) with

\[ f_i''(x) = O(x^{-5-\alpha}), \quad \alpha > 0 \quad (i = 1, 2) \]

Then 2) is a solution of 1) with \( \lim_{t \to 0} u(x, t) = f_1(x) \) and \( \lim_{t \to 0} \partial_t u(x, t) = \frac{1}{2}[f_2''](x) \),

where \( [f_2''](x) \) is the sum of the right and left hand limits at \( x \).

**Proof:** Let \( \varphi \) be a function with the properties of \( f_1 \) given above. Then for \( t > 0 \)
9) \( \partial_t \int_{-\infty}^{\infty} k(x-y,t)\varphi(y)dy = \int_{-\infty}^{\infty} \partial_t k(x-y,t)\varphi(y)dy \)

\( = i \int_{-\infty}^{\infty} k_{yy}(x-y,t)\varphi(y)dy \)

by 6) and the absolute and uniform convergence of the second integral for \( t \geq s > 0 \). Now from 9)

10) \( \partial_t \int_{-\infty}^{\infty} k(x-y,t)\varphi(y)dy = i \int_{-\infty}^{\infty} k(x-y,t)\varphi''(y)dy \)

by successive integrations by parts. Therefore, since \( \varphi(y) \) and \( \varphi''(y) \) satisfy the conditions of Theorem 1)

\( \int_{-\infty}^{\infty} k(x-y,t)\varphi(y)dy \rightarrow -2i\varphi(x) \)

and

\( \partial_t \int_{-\infty}^{\infty} k(x-y,t)\varphi(y)dy \rightarrow [\varphi''](x) \)

as \( t \rightarrow 0 \).

From the above results it is immediate that

2) \( u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} [C(x-y,t)f_2(y) - S(x-y,t)f_1(y)]dy \)

satisfies

\( \lim_{t \rightarrow 0} u(x,t) = f_1(t) \)

\( \lim_{t \rightarrow 0} \partial_t u(x,t) = \frac{1}{2} [f''_2](x) \)

From 9) and 10) it follows that

\( \partial_t u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} [f_2(y)\partial_y^2 S(x-y,t) + f_1(y)\partial_y^2 C(x-y,t)]dy \)
and hence that
\[ \partial_x^2 u(x,t) = -\frac{1}{2} \int_{-\infty}^{\infty} [f''(y)\partial_x^2 C(x-y,t) - f_1''(y)\partial_x^2 S(x-y,t)] \, dy. \]

But
\[ \partial_x^4 u(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} [f''(y)\partial_x^2 C(x-y,t) - f_1''(y)\partial_x^2 S(x-y,t)] \, dy. \]

Questions of uniqueness of the solutions of the Cauchy problem and more general existence theorems will be discussed in Chapter 4.
1. **Semi-infinite bar problem**

The semi-infinite bar problem is to determine a function \( u(x,t) \) which satisfies

\[
\begin{align*}
1) \quad \alpha_4^x u + \alpha_2^t u &= 0, \quad x > 0, \ t > 0 \\
\text{with} \\
\lim_{x \to 0} \alpha_x^i u(x,t) &= a_i(t), \\
\lim_{x \to 0} \alpha_x^j u(x,t) &= a_j(t), \quad t > 0, \\
\text{for } i,j = 0,1,2,3 \quad \text{and } i \neq j, \quad \text{and} \\
\lim_{t \to 0} u(x,t) &= f_1(x), \\
\lim_{t \to 0} \alpha_t^i u(x,t) &= f_2(x), \quad x > 0, \\
\end{align*}
\]

where \( f_i(x > 0) \) satisfies the conditions of Chapter 1.

The only problem we consider explicitly is \( i=0, j=1 \) where we set \( a_0(t) = a(t) \) and \( a_1(t) = b(t) \). It is sufficient to solve the problem with \( f_1 = f_2 = 0 \). If \( f_1 \) and \( f_2 \) are not zero, extend them smoothly to functions \( g_1 \) and \( g_2 \) defined for all \( x \) with \( g_i = f_i \) for \( x > 0 \) and such that the Cauchy problem with initial data \( g_1 \) and \( g_2 \) can be solved. Let \( u_1(x,t) \) be the solution of the Cauchy problem for 1) with initial data \( g_1 \) and \( g_2 \). If we now find a function \( u_2(x,t) \) satisfying 1) with
\[
\lim_{x \to 0} u_2(x,t) = a(t) - u_1(0,t)
\]

\[
\lim_{x \to 0} \partial_x u_2(x,t) = b(t) - \partial_x u_1(0,t), \quad t > 0
\]

and

\[
\lim_{t \to 0} u_2(x,t) = \lim_{t \to 0} \partial_t u_2(x,t) = 0, \quad x > 0
\]

then \( u(x,t) = u_1(x,t) + u_2(x,t) \) will be a solution of the original problem. In this chapter then it will always be assumed that \( f_1 = f_2 = 0 \).

We seek a solution to this problem in the form

2) \[
\begin{align*}
\lim_{t \to 0} \partial_t u_2(x,t) &= 0, \quad x > 0
\end{align*}
\]

then \( u(x,t) = u_1(x,t) + u_2(x,t) \) will be a solution of the original problem. In this chapter then it will always be assumed that \( f_1 = f_2 = 0 \).

We seek a solution to this problem in the form

2) \[
\begin{align*}
u(x,t) &= \int_0^t [C(x,t-s)\phi(s) + S(x,t-s)\psi(s)]ds
\end{align*}
\]

where

\[
u_1(x,t;\phi) = \text{Re } U(x,t;\phi)
\]

3)

\[
u_2(x,t;\phi) = \text{Im } U(x,t;\phi)
\]

and

4) \[
U(x,t;\phi) = \int_0^t k(x,t-s)\phi(s)ds,
\]

\( k(x,t) \) being defined by 1.3).

2. **Properties of** \( U(x,t;\phi) \)

Define \( K(x,t) \) by

5) \[
K(x,t) = \int_0^t k(x,u)du.
\]
By integration by parts we obtain from 1.3)
\[
K(x,t) = -e^{i\pi/4/\sqrt{\pi}} \left\{ \frac{4it^{3/2}}{x^2} \exp \left( \frac{ix^2}{4t} \right) - \frac{6i}{x^2} \int_0^t \exp \left( \frac{ix^2}{4u} \right) \sqrt{u} \, du \right\}
\]
and
\[
\partial_x K(x,t) = -e^{i\pi/4/\sqrt{\pi}} \left\{ -\frac{8it^{3/2}}{x^3} \exp \left( \frac{ix^2}{4t} \right) \right. \\
+ \frac{12i}{x^3} \int_0^t \exp \left( \frac{ix^2}{4u} \right) \sqrt{u} \, du - \frac{2\sqrt{t}}{x} \exp \left( \frac{ix^2}{4t} \right) \\
+ \frac{3}{x} \int_0^t \exp \left( \frac{ix^2}{4u} \right) \sqrt{u} \, du \right\}
= -\frac{2}{x} K(x,t) - \frac{2t}{x} k(x,t) + \frac{3}{x} K(x,t) .
\]
Thus
\[
6) \quad \partial_x K(x,t) = \frac{1}{x} K(x,t) - \frac{2t}{x} k(x,t) .
\]
Correspondingly,
\[
\partial_x^2 K(x,t) = -\frac{1}{x^2} K(x,t) + \frac{1}{x^2} K(x,t) - \frac{2t}{x^2} k(x,t) \\
+ \frac{2t}{x^2} k(x,t) - \frac{2t}{x} \partial_x k(x,t)
\]
or by 1.5) \( \partial_x^2 K(x,t) = -ik(x,t) \). Since \( \partial_t k(x,t) = k(x,t) \), we have
\[
7) \quad \partial_x^2 K(x,t) = -i\partial_t K(x,t) .
\]
On the other hand
\[
\int_0^t \partial_x k(x,s) ds = -ie^{i\pi/4/2\sqrt{\pi}} \int_0^t \frac{x}{2\sqrt{\pi}} \exp \left( \frac{ix^2}{4s} \right) ds \\
= -ixe^{i\pi/4/2\sqrt{\pi}} \left\{ 4i\sqrt{t}/x^2 \exp \left( \frac{ix^2}{4t} \right) \\
- \frac{2i}{x^2} \int_0^t \exp \left( \frac{ix^2}{4s} \right) \sqrt{s} \, ds \right\} .
\]
Therefore, by 5), 6) and 1.3)

8) \[ \int_0^t \partial_x k(x,s)ds = -\frac{2t}{x}k(x,t) + \frac{1}{x} K(x,t) = \partial_x K(x,t). \]

We are now ready to determine some properties of \( U(x,t;\varphi) \).

Theorem 1) If \( \varphi \) is continuous and BV on some finite interval \([0,T]\), then

9) \[ \lim_{x \to 0} U(x,t;\varphi) = -\frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2} \varphi(s)ds \]

10) \[ \lim_{x \to 0} \partial_x U(x,t;\varphi) = \varphi(t) \]

for \( 0 < t \leq T \) and

11) \[ \lim_{t \to 0^+} U(x,t;\varphi) = \lim_{t \to 0^+} \partial_x U(x,t;\varphi) = 0. \]

Proof: Since \( \varphi \) is bounded,

\[
U(x,t) = \int_0^t k(x,t-s)\varphi(s)ds
= -\frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^t \exp\left(\frac{ix}{4(t-s)}\right)(t-s)^{-1/2}\varphi(s)ds
\]

converges uniformly in \( x \) and

\[
\lim_{x \to 0} U(x,t) = -\frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^t (t-s)^{-1/2}\varphi(s)ds
\]

and

\[
\lim_{t \to 0} U(x,t) = 0.
\]

Consider next
\[
\int_0^t \delta_x k(x,t-s) \varphi(s) ds, \quad x > 0
\]

Since \( \varphi \) is BV,

12) \[
\left| \int_0^t \delta_x k(x,t-s) \varphi(s) ds \right| 
\leq (|\varphi(0)| + \int_0^t \varphi(s)) \sup_{0 \leq t' < t \leq t} |\int_0^{t''} \delta_x k(x,s) ds|
\]

by a mean value theorem in Hobson [3, page 623].
Therefore this integral exists for \( x > 0 \) and defines a function

\[
f(x) = \int_0^t \delta_x k(x,t-s) \varphi(s) ds, \quad x > 0.
\]

By integration by parts

\[
f(x) = \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^t \left\{ t^{1/2} e^{ix^2/4t} \varphi(0) - \int_0^t \exp\left(\frac{ix^2}{4s}\right) \left(\frac{1}{2} s^{-1/2} \varphi(t-s) ds + s^{1/2} d\varphi(t-s)\right) \right\}
\]

For \( x > 0 \) the first term on the right is continuous and since the integral is uniformly integrable in \( x \geq x_0 > 0 \), the second term is continuous and hence \( f(x) \) is continuous. Also

\[
f(x) = \int_0^t \delta_x k(x,t-s)[\varphi(s) - \varphi(t)] ds + \varphi(t) \int_0^t \delta_x k(x,t-s) ds,
\]

but

\[
\int_0^t \delta_x k(x,t-s) ds = -\frac{i e^{i\pi/4}}{2\sqrt{\pi}} x \int_0^t s^{-3/2} \exp\left(\frac{ix^2}{4s}\right) ds
\]
Replacing \( s \) by \( t-s \)

\[
\lim_{x \to 0} \int_0^t \partial_x k(x, t-s)[\varphi(s)-\varphi(t)] ds = -\frac{ie^{i\pi/4}}{\sqrt{\pi}} \left( \sqrt{\pi} e^{i\pi/4} \right) = 1 .
\]

Reverting \( s \) by \( t-s \)

\[
\left| \int_0^t \partial_x k(x, t-s)[\varphi(s)-\varphi(t)] ds \right| \\
\leq \frac{x}{2\sqrt{\pi}} \int_0^t s^{-3/2} \text{exp} \left( \frac{ix^2}{4s} \right)[\varphi(t-s)-\varphi(t)] ds \\
\]

which gives, using the same theorem as in 12),

\[
\leq \frac{x}{2\sqrt{\pi}} \int_0^t s^{-3/2} |\varphi(t-s)-\varphi(t)| ds \\
+ \frac{x}{2\sqrt{\pi}} \sup_{s=0}^{s=\delta} [\varphi(t-s)-\varphi(t)] \sup_{0 \leq t' \leq t \leq \delta} \left| \int_{t'}^{t''} s^{-3/2} e^{ix^2/4s} ds \right|
\]

but

\[
\sup_{t' \leq t \leq \delta} \left| \int_{t'}^{t''} s^{-3/2} e^{ix^2/4s} ds \right| = \sup_{t' \leq t \leq \delta} \left| \int_{t'}^{t''} v^{-1/2} e^{iv} dv \right|
\]

\[
\leq \frac{2}{x} \sup_{0 \leq T' \leq T''} \left| \int_{T'}^{T''} v^{-1/2} e^{iv} dv \right| = \frac{2}{x} M
\]

hence

\[
\left| \int_0^t \partial_x k(x, t-s)[\varphi(s)-\varphi(t)] ds \right| \leq \frac{x}{2\sqrt{\pi}} \int_0^t s^{-3/2} |\varphi(t-s)-\varphi(t)| ds \\
+ \frac{M}{\sqrt{\pi}} \sup_{s=0}^{s=\delta} [\varphi(t-s)-\varphi(t)] .
\]
Since \( \varphi(s) \) is continuous,

\[
V_{s=0}^{s=u} \left[ \varphi(t-s) - \varphi(t) \right]
\]
is continuous, so that for \( \delta \) sufficiently small

\[
V_{s=0}^{s=\delta} \left[ \varphi(t-s) - \varphi(t) \right] < \frac{\varepsilon}{2M\sqrt{\pi}}
\]

for arbitrary \( \varepsilon \). Then

\[
\lim_{x \to 0} \int_0^t \int_0^x k(x,t-s) [\varphi(s) - \varphi(t)] ds \leq \varepsilon,
\]

but since \( \varepsilon \) is arbitrary, the limit exists and is zero.

Therefore \( \lim_{x \to 0} f(x) = \varphi(t) \) and if we define \( f(0) = \varphi(t) \),

\( f(x) \) is continuous for \( x > 0 \). On the other hand

\[
\int_\eta^\xi f(x)dx = \int_\eta^\xi \int_0^t k(x, t-s) \varphi(s) ds \, dx
\]

\[
= \int_\eta^\xi \int_0^t k(x, s) \varphi(t-s) ds \, dx
\]

\[
= \int_\eta^\xi \left\{ \int_0^s k(x, u) du \varphi(t-s) \right\}_0^t - \int_0^t \int_0^s k(x, u) du \varphi(t-s) \right\} dx
\]

\[
= \int_\eta^\xi \left\{ \partial_x K(x,t) \varphi(0) - \int_0^t \partial_x K(x,s) d\varphi(t-s) \right\} dx
\]

from equation 8). Therefore

\[
\int_\eta^\xi f(x)dx = [K(\xi,t) - K(\eta,t)] \varphi(0) - \int_\eta^\xi \int_0^t \partial_x K(x,s) d\varphi(t-s) dx.
\]

But equation 6) says
\[ \partial_x K(x,s) = \frac{1}{x} K(x,s) - \frac{2s}{x} k(x,s) \]

and since
\[ \frac{2s}{x} k(x,s) = -2e^{i\pi/4} \sqrt{\pi} s^{1/2} e^{ix^2/4s} \]

and
\[ K(x,s) = \int_0^s k(x,u) \, du \]

\[ \partial_x K(x,s) \] is continuous for \( 0 \leq s \leq t, \eta \leq x \leq \xi \) and Fubini's theorem justifies interchanging the order of integration so that
\[ \int_\eta^\xi f(x) \, dx = [K(\xi,t) - K(\eta,t)] \varphi(0) - \int_0^t [K(\xi,s) - K(\eta,s)] \varphi(t-s) \, ds. \]

Now integrating by parts again
\[ \int_\eta^\xi f(x) \, dx = [K(\xi,t) - K(\eta,t)] \varphi(0) - \left\{ [K(\xi,s) - K(\eta,s)] \varphi(t-s) \right\}_0^t \]
\[ - \int_0^t [k(\xi,s) - k(\eta,s)] \varphi(t-s) \, ds \]
\[ = \int_0^t k(\xi,s) \varphi(t-s) \, ds - \int_0^t k(\eta,s) \varphi(t-s) \, ds \]

which by 4)
\[ = U(\xi, t; \varphi) - U(\eta, t; \varphi). \]

From 9) and the continuity of \( f(x) \)
\[ \int_0^\xi f(x) \, dx = U(\xi, t; \varphi) + e^{i\pi/4} \sqrt{\pi} \int_0^t (t-s)^{-1/2} \varphi(s) \, ds \]

and hence
\[ \partial_x U(x,t; \varphi) = f(x) \]
that is
\[ \partial_x U(x,t;\varphi) = \int_0^t \partial_x k(x,t-s)\varphi(s)ds \]
and
\[ \lim_{x \to 0} \partial_x U(x,t;\varphi) = \lim_{x \to 0} f(x) = \varphi(t) . \]

From 12) it is seen that \( \lim_{t \to 0^+} \partial_x U(x,t;\varphi) = 0 \).

Theorem 2) If \( \varphi \) is BV on a finite set \([0,T]\), then
\[ \partial_x^2 U(x,t;\varphi) = -i\partial_t U(x,t;\varphi) \]
on \([0,T]\) with \( x > 0 \).

Proof: By integration by parts
\[ U(x,t) = \varphi(0)K(x,t) + \int_0^t K(x,t-s)d\varphi(s) , \]
hence
\[ \partial_x U(x,t) = \varphi(0)\partial_x K(x,t) + \int_0^t \partial_x K(x,t-s)d\varphi(s) . \]
Therefore from 7)
\[ \partial_x^2 U(x,t) = -i\varphi(0)k(x,t) - i \int_0^t k(x,t-s)d\varphi(s) . \]

But
\[ \partial_t U(x,t) = \varphi(0)k(x,t) + \int_0^t k(x,t-s) d\varphi(s) . \]

Corollary: If \( \varphi \in C^1 \) and \( \varphi' \) is BV on \([0,T]\), then \( u_1 \) and \( u_2 \) of 3) satisfy 1). That is
\[ (\partial_x^4 + \partial_t^2)u_i = 0 \ (i=1,2) . \]

Proof: Under these conditions, 16) can be written
\[ \partial^2_x U(x,t;\varphi) = -i\varphi(0)k(x,t) - iU(x,t;\varphi') \]

and

\[ \partial^4_x U(x,t;\varphi) = -i\varphi(0)\partial^2_x k(x,t) - i\partial^2_x U(x,t;\varphi') \]

\[ = -\varphi(0)\partial_t k(x,t) - \varphi'(0)k(x,t) \]

\[ - \int_0^t k(x,t-s) \, d\varphi'(s) \]

from 1.6) and 16). But from 17)

\[ \partial^2_t U(x,t;\varphi) = \varphi(0)k_t(x,t) + \partial_t U(x,t;\varphi') \]

\[ = \varphi(0)k_t(x,t) + \varphi'(0)k(x,t) + \int_0^t k(x,t-s) \, d\varphi'(s) . \]

Therefore \( U \) satisfies 1) and hence the real and imaginary parts separately satisfy 1).

3. **Solution of the semi-infinite bar problem**

If 2) is a solution of 1) with

\[ u(x,0) = \partial_t u(x,0) = 0 \]

18) \[ u(0,t) = a(t) \]

\[ \partial_x u(0,t) = b(t) , \]

then with appropriate conditions on \( \varphi \) and \( \psi \) 9) and 10) imply

\[ -1/\sqrt{2\pi} \int_0^t \varphi(s)\sqrt{t-s} \, ds - 1/\sqrt{2\pi} \int_0^t \psi(s)\sqrt{t-s} \, ds = a(t) \]

19) \[ \varphi(t) = b(t) . \]
As in Riesz [9, page 10] we set

\[ I^\alpha \varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s) ds \]

by which system 19) becomes

\[ -\frac{1}{\sqrt{2}} I^{1/2} \varphi(t) - \frac{1}{\sqrt{2}} I^{1/2} \psi(t) = a(t) \]

\[ I^0 \varphi(t) = b(t) \]

or

20)

\[ M \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \]

where

\[ M = \begin{bmatrix} -\frac{1}{\sqrt{2}} I^{1/2} & -\frac{1}{\sqrt{2}} I^{1/2} \\ I^0 & 0 \end{bmatrix} \]

If \( b(t) \) and \( \frac{d}{dt} \int_0^t \frac{a(s)}{\sqrt{t-s}} ds \) have first derivatives which are continuous and BV, then the system

\[ \varphi(t) = b(t) \]

\[ \psi(t) = -\sqrt{2} \frac{d}{dt} \int_0^t \frac{a(s)}{\sqrt{t-s}} ds - b(t) \]

is a solution to system 18) where \( \varphi \) and \( \psi \) have first derivatives which are continuous and BV. Correspondingly, 20) becomes

21)

\[ \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} = M^{-1} \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \]
where

\[
M^{-1} = \begin{bmatrix}
0 & I^1 \\
-\sqrt{2}I^{-1/2} & -I^0
\end{bmatrix}
\]

We have just proved that if \( b(t) \) and \( \frac{d}{dt}a(s)\sqrt{t-s}ds \)
have first derivatives which are continuous and \( BV \) on
any finite interval \([0,T]\), then 2) is a solution to equa-
tion 1) satisfying conditions 18) where \( \varphi \) and \( \psi \)
are
determined by 19), and given by 21).

A catalogue of the operator matrices \( M \) and \( M^{-1} \)
for various other boundary value problems is given in
Appendix III.

4. Some special operators

For smooth \( \varphi \), define

\[
E^a_{\beta}(x)\varphi(t) = (C^a_{\beta}(x) + iS^a_{\beta}(x))\varphi(t)
\]

\[
= \frac{1}{T(a)} \int_0^t (t-s)^a-1 \exp(ix/(t-s))\varphi(s)ds .
\]

We will use only \( \alpha = \pm \frac{1}{2} \) and \( \beta = 0, \pm \frac{\pi}{4} \) in the para-
graphs that follow.

If \( \varphi \) has a continuous first derivative on \([0,T]\),
then

\[
E^{1/2}_{\pi/4} (x)I^{-1/2} \varphi(t) = \frac{e^{i\pi/4}}{\pi} \int_0^t \frac{\exp(ix/(t-u))}{\sqrt{(t-u)}} \frac{d}{du} \frac{\varphi(v)dv}{\sqrt{(u-v)}} du
\]

\[
= \frac{e^{i\pi/4}}{\pi} \int_0^t \frac{\exp(ix/(t-u))}{\sqrt{(t-u)}} (\varphi(0)/\sqrt{u})
\]
\[ + \int_{0}^{u} \varphi'(v) \sqrt{u-v} \, dv \, du \]

\[ = \frac{\varphi(0) e^{i\pi/4}}{\pi} \int_{0}^{t} \frac{\exp(ix/(t-u))}{\sqrt{u(t-u)}} \, du \]

\[ + \frac{e^{i\pi/4}}{\pi} \int_{0}^{t} \varphi'(v) \int_{v}^{t} \frac{\exp(ix/(t-u))}{\sqrt{1(t-u)(u-v)}} \, dv \, du . \]

If we integrate the second term by parts the expression becomes

\[-e^{i\pi/4} \int_{0}^{t} \varphi(v) \frac{d}{dv} \text{Erf}(e^{-i\pi/4} \sqrt{x/(t-v)}) \, dv \]

\[= \sqrt{(x/\pi)} \int_{0}^{t} \varphi(v) \exp(ix/(t-v)) (t-v)^{-3/2} \, dv \]

(see II.1). That is

23) \[ E_{\pi/4}(x) I^{-1/2} \varphi(t) = -2\sqrt{x} E_{0}^{-1/2} (x) \varphi(t) . \]

On the other hand, if \( \varphi \) is continuous and BV on \([0, T]\), then

\[ I^{-1/2} E_{\pi/4}^{1/2}(x) \varphi(t) = \frac{e^{i\pi/4}}{\sqrt{\pi}} \frac{d}{dt} \int_{0}^{t} \int_{0}^{s} \varphi(u) \exp(ix/(s-u)) \, duds . \]

Since the integrand is absolutely integrable, we can interchange the order of integration and the expression on the right becomes

\[ \frac{e^{i\pi/4}}{\sqrt{\pi}} \frac{d}{dt} \int_{0}^{t} \varphi(u) \int_{u}^{t} \exp(ix/(s-u)) \, ds \]

\[ \sqrt{[(t-s)(s-u)]} \]
\[
\frac{d}{dt} \text{Erfc} \left( e^{i\pi/4} \sqrt{\frac{\pi}{x(t-u)}} \right) = e^{i\pi/4} \sqrt{\frac{\pi}{x}} \exp \left( \frac{ix}{t-u} \right) (t-u)^{-3/2}.
\]

Let
\[
f_n(t) = \begin{cases} 
\frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^{t-1/n} \varphi(u)[\pi \text{Erfc} \left( e^{i\pi/4} \sqrt{\frac{\pi}{x(t-u)}} \right)] du, & t > \frac{1}{n} \\
0, & t < \frac{1}{n}
\end{cases}
\]

Then
\[
\frac{d}{dt} f_n(t) = \sqrt{\frac{\pi}{x}} \int_0^{t-1/n} \varphi(u)(t-u)^{-3/2} \exp \left( \frac{ix}{t-u} \right) du + e^{i\pi/4} \sqrt{\frac{\pi}{x}} \varphi(t-1/n) \text{Erfc}(e^{-i\pi/4} \sqrt{\frac{\pi}{x}})
\]

for \( t > \frac{1}{n} \) and \( \frac{d}{dt} f_n(t) = 0 \) for \( t < \frac{1}{n} \).

Therefore
\[
\left| \frac{d}{dt} f_n(t) - \sqrt{\frac{\pi}{x}} \int_0^t \varphi(u)(t-u)^{-3/2} \exp(ix/(t-u)) du \right|
\]
\[
= \begin{cases} 
\sqrt{\frac{\pi}{x}} \int_{t-1/n}^t \varphi(u)(t-u)^{-3/2} \exp(ix/(t-u)) du, & t > \frac{1}{n} \\
\sqrt{\frac{\pi}{x}} \int_0^{t} \varphi(u)(t-u)^{-3/2} \exp(ix/(t-u)) du, & t < \frac{1}{n}
\end{cases}
\]

Now by integration by parts.
\[
\sqrt{\frac{x}{\pi}} \left| \int_{t-1/n}^{t} \cdot \cdot \cdot \, du \right| = \sqrt{\frac{x}{\pi}} \left| \int_{0}^{1/n} \varphi(t-u)u^{-3/2}\exp\left(\frac{ix}{u}\right) \, du \right|
\]

\[
= \frac{1}{\sqrt{\pi x}} \left| \int_{0}^{1/n} \exp(inx)\varphi(t-1/n) \right|
\]

\[
- \int_{0}^{1/n} \exp\left(\frac{ix}{n}\right) \left[ \frac{1}{2\sqrt{u}} \varphi(t-u) \, du + \sqrt{u} \, d\varphi(t-u) \right]
\]

\[
\leq \frac{1}{\sqrt{\pi x}} \left[ B/\sqrt{n} + B/2 \int_{0}^{1/n} u^{-1/2} \, du + \sqrt{\n} \, T \varphi/\sqrt{n} \right]
\]

\[
\leq B' / \sqrt{n}
\]

where \( B \) is the maximum of \( \varphi \) on \([0,T]\) and \( B' \) is independent of \( t \). Also for \( t < \frac{1}{n} \),

\[
\sqrt{\frac{x}{\pi}} \left| \int_{0}^{t} \varphi(u)(t-u)^{-3/2}\exp(\frac{ix}{t-u}) \, du \right|
\]

\[
= \sqrt{\frac{x}{\pi}} \left| \int_{0}^{t} \varphi(t-u)u^{-3/2}\exp(\frac{ix}{u}) \, du \right|
\]

\[
= \frac{1}{\sqrt{\pi x}} \left| \varphi(0) \sqrt{t} \exp\left(\frac{ix}{t}\right) - \int_{0}^{t} \exp\left(\frac{ix}{u}\right) \left( \frac{1}{2\sqrt{u}} \right) \varphi(t-u) \, du \right.
\]

\[
+ \sqrt{u} \, d\varphi(t-u) \right|
\]

\[
\leq 2B\sqrt{t}/\sqrt{\pi x} + \sqrt{t} \, \n T \varphi/\sqrt{\pi x}
\]

\[
\leq B' / \sqrt{n}
\]

Therefore,

\[
\frac{d}{dt} f_n(t) \to \sqrt{\frac{x}{\pi}} \int_{0}^{t} \varphi(u)(t-u)^{-3/2}\exp(\frac{ix}{(t-u)}) \, du
\]

uniformly in \( t \) and hence
\[
\frac{d}{dt} \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_{0}^{t} \varphi(u) [\pi \operatorname{Erfc} (e^{-i\pi/4} \sqrt{x/(t-u)})] du
\]

\[= \sqrt{x/\pi} \int_{0}^{t} \varphi(u)(t-u)^{-3/2} \exp(ix/(t-u)) \, du\]

and thus by 22)

24) \[I^{-1/2} E_{\pi/4}^{1/2} (x) \varphi(t) = -2\sqrt{x} E_{0}^{-1/2} (x) \varphi(t)\]

for \(\varphi\) continuous and \(\text{BV}\).

In addition

25) \[E_{\pi/4}^{1/2} (x) I^{1/2} \varphi(t) = I^{1/2} E_{\pi/4}^{1/2} (x) \varphi(t)\]

\[= e^{i\pi/4/\pi} \int_{0}^{t} \varphi(u) du \int_{u}^{t} \exp(ix/(t-v)) \sqrt{(t-v)(v-u)} \, dv\]

for which II.1) gives

26) \[E_{\pi/4}^{1/2} (x) I^{1/2} \varphi(t) = e^{i\pi/4} \int_{0}^{t} \varphi(u) \operatorname{Erfc}(e^{-i\pi/4} \sqrt{x/(t-u)}) du\]

or where \(\Phi'(t) = \varphi(t)\)

26') \[E_{\pi/4}^{1/2} (x) I^{1/2} \varphi(t) = -e^{i\pi/4} \Phi(0) \operatorname{Erfc}[e^{-i\pi/4} \sqrt{x/t}]\]

\[+ 2\sqrt{x} E_{0}^{-1/2} (x) \varphi(t) .\]

5. **Two special cases**

From 22) and 4), we see that

27) \[U(x,t;\varphi) = -E_{\pi/4}^{1/2} (x^2/4) \varphi(t)\]
so that 2) can be written
\[ u(x,t) = -C_{\pi/4}(x^2/4)\psi(t) - S_{\pi/4}(x^2/4)\varphi(t) \]
or
\[ u(x,t) = -[C_{\pi/4}(x^2/4), S_{\pi/4}(x^2/4)] \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}. \]

For the boundary conditions given by 18)
\[ u(x,t) = -[C_{\pi/4}(x^2/4), S_{\pi/4}(x^2/4)] \begin{bmatrix} 0 & I^0 \\ -\sqrt{2} & I^{-1/2} \end{bmatrix} \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} 
= -[\sqrt{2} S_{\pi/4}^{-1/2}, C_{\pi/4}^{-1/2} - S_{\pi/4}^{-1/2}] \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}, \]
\[ u(x,t) = \sqrt{2} S_{\pi/4}^{1/2}(x^2/4)I^{-1/2}a(t) + S_{\pi/4}^{1/2}(x^2/4)b(t). \]

For the boundary value problem
\[ u(0,t) = a(t) \]
\[ \partial_x^2 u(0,t) = c'(t), \quad c(0) = 0, \]
we have from Appendix III
\[ M^{-1} = -\frac{1}{\sqrt{2}} \begin{bmatrix} I^{-1/2} & I^{1/2} \\ I^{-1/2} & -I^{1/2} \end{bmatrix} \]
from which one obtains
\[ u(x,t) = xC^{-1/2}(x^2/4)c(t) - xS^{-1/2}(x^2/4)a(t). \]

When \( c(t) = 0 \) we have exactly the formulas obtained by Sneddon [10, page 115] by Fourier transform methods which he ascribes to Boussinesq (see [11]).
CHAPTER 3

The Finite Bar

1. The finite bar problem

The problem of the finite bar is to determine a function \( u(x,t) \) which satisfies

1) \[ (\partial_x^4 + \partial_t^2)u = 0 \quad 0<x<2, \quad 0<t \]

with the additional conditions that \( u(x,0) = \partial_t u(x,0) = 0 \) and

\[ u(0,t) = a_1(t), \quad u(2,t) = a_2(t) \]

2) \[ \partial_x u(0,t) = b_1(t), \quad \partial_x u(2,t) = b_2(t). \]

The more general problem with \( u(x,0) = f_1(x) \) and \( \partial_t u(x,0) = f_2(x), \quad |f_1| + |f_2| \neq 0 \), can be reduced to the above problem just as in the semi-infinite bar problem if \( f_i (i=1,2) \) is sufficiently smooth. Extend \( f_i (i=1,2) \) to the infinite line in a smooth way with sufficiently rapid decrease at infinity that the conditions of Chapter 1) are satisfied. Let \( u'(x,t) \) be the solution of the Cauchy problem with the extended data. Then \( u(x,t) = u'(x,t) + u''(x,t) \) will be a solution of 1) with non-zero initial data if \( u'' \) satisfies 1) with zero initial data and

\[ u''(0,t) = a_1(t) - u'(0,t), \quad u''(2,t) = a_2(t) - u'(2,t), \]
\[ \partial_x u''(0,t) = b_1(t) - \partial_x u'(0,t), \quad \partial_x u''(2,t) = b_2(t) - \partial_x u'(2,t). \]

But this is just the problem described in the previous paragraph.

We seek a solution in the form

\[ u(x,t) = u_1(x,t; \psi_1) + u_2(x,t; \psi_2) \]

\[ + u_1(2-x,t; \varphi_2) + u_2(2-x,t; \psi_2), \]

where \( u_i \) (\( i=1,2 \)) is defined by 2.3). If \( \varphi_i, \psi_i \) (\( i=1,2 \)) have first derivatives that are continuous and BV, then Theorem 2.1 and the corollary to Theorem 2.2 imply that 3) satisfies 1) and the initial conditions. By Theorem 2.1) \( \varphi_i, \psi_i \) and \( a_i, b_i \) (\( i=1,2 \)) must be related by

\[ - \frac{1}{\sqrt{2}} \int^{1/2} \varphi_1 - \frac{1}{\sqrt{2}} \int^{1/2} \psi_1 - C^{1/2}_{\pi/4} \varphi_2 - S^{1/2}_{\pi/4} \psi_2 = a_1 \]

\[ \varphi_1 + 2S^{-1/2}_{\pi/4} \varphi_2 - 2C^{-1/2}_{\pi/4} \psi_2 = b_1 \]

4)

\[ - \frac{1}{\sqrt{2}} \int^{1/2} \varphi_1 - \frac{1}{\sqrt{2}} \int^{1/2} \psi_1 - \frac{1}{\sqrt{2}} \int^{1/2} \varphi_2 - \frac{1}{\sqrt{2}} \int^{1/2} \psi_2 = a_2 \]

\[ - 2S^{-1/2}_{\pi/4} \varphi_1 + 2C^{-1/2}_{\pi/4} \psi_1 - \varphi_2 = b_2 \]

where \( C^a_{\rho} + iS^a_{\rho} = E^a_{\rho} = E^a_{\rho}(1) \) in 2.22). The system 4) is equivalent to the system obtained by Pini [7, page 101].
2. Reduction of the integral equations

By adding and subtracting the first and third and the second and fourth equations of 4) we can write the system as

\[-\frac{1}{\sqrt{2}} I^{1/2}(\varphi_1 \pm \varphi_2) - \frac{1}{\sqrt{2}} I^{1/2}(\psi_1 \pm \psi_2) \mp C^{1/2}(\varphi_1 \pm \varphi_2)\]

\[\mp S^{1/2}(\psi_1 \pm \psi_2) = a_1 \pm a_2\]

\[(\varphi_1 \pm \varphi_2) \pm 2S^{-1/2}(\varphi_1 \pm \varphi_2) \mp 2C^{-1/2}(\psi_1 \pm \psi_2)\]

\[= b_1 \mp b_2 .\]

If we now set

\[\varphi_1 + \varphi_2 = f_1 \quad \text{\(a_1 + a_2 = A_1\)}\]

\[\varphi_1 - \varphi_2 = f_2 \quad \text{\(a_1 - a_2 = A_2\)}\]

\[\psi_1 + \psi_2 = g_1 \quad \text{\(b_1 - b_2 = B_1\)}\]

\[\psi_1 - \psi_2 = g_2 \quad \text{\(b_1 + b_2 = B_2\)}\]

then 4) gives

\[-\frac{1}{\sqrt{2}} I^{1/2}f_1 - \frac{1}{\sqrt{2}} I^{1/2}g_1 - C^{1/2}f_1 - S^{1/2}g_1 = A_1\]

6)

\[f_1 + 2S^{-1/2}f_1 - 2C^{-1/2}g_1 = B_1\]

and

\[-\frac{1}{\sqrt{2}} I^{1/2}f_2 - \frac{1}{\sqrt{2}} I^{1/2}g_2 + C^{1/2}f_2 + S^{1/2}g_2 = A_2\]
\[
f_2 - 2S^{-1/2}f_2 + 2C^{-1/2}g_2 = B_2.
\]

System 6) can be rewritten as

\[
M \begin{bmatrix} f_1 \\ g_1 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} + N_1 \begin{bmatrix} f_1 \\ g_1 \end{bmatrix}
\]

where

\[
N_1 = \begin{bmatrix} C^{1/2} & S^{1/2} \\ \pi/4 & \pi/4 \\ -2S^{-1/2} & 2C^{-1/2} \\ \pi/4 & \pi/4 \end{bmatrix}.
\]

From 2.21) \( M^{-1} \) is known and

\[
M^{-1}N_1 = \begin{bmatrix} 0 & i^0 \\ -\sqrt{2}I^{-1/2} & i^0 \end{bmatrix} \begin{bmatrix} C^{1/2} & S^{1/2} \\ \pi/4 & \pi/4 \\ -2S^{-1/2} & 2C^{-1/2} \\ \pi/4 & \pi/4 \end{bmatrix}
\]

\[
= 2 \begin{bmatrix} -S^{-1/2} & C^{-1/2} \\ \pi/4 & \pi/4 \\ C^{-1/2} & S^{-1/2} \\ \pi/4 & \pi/4 \end{bmatrix}.
\]

Now if we set

\[
\begin{bmatrix} C_1 \\ D_1 \end{bmatrix} = M^{-1} \begin{bmatrix} A_1 \\ B_1 \end{bmatrix},
\]

then 6) becomes
7) \[
\begin{bmatrix}
    f_1 \\
    g_1
\end{bmatrix} = \begin{bmatrix}
    C_1 \\
    D_1
\end{bmatrix} + 2 \begin{bmatrix}
    -S_{-1/2}^{\pi/4} \\
    C_{-1/2}^{\pi/4}
\end{bmatrix} \begin{bmatrix}
    f_1 \\
    g_1
\end{bmatrix}.
\]

Further if we multiply the second equation by \( i \) and add the two equations, and if we set

\[ f(t) = f_1 + ig_1 \]

8) \[ a = C_1 + iD_1 \]

then 7) becomes

\[ f = a + 2iE_{-1/2}^{\pi/4} \bar{f} \]

or

9) \[ f(t) = a(t) - \int_0^t k(t-u) \bar{f}(u) \, du \]

where

10) \[ k(t) = \frac{ie^{i\pi/4}}{\sqrt{\pi}} t^{-3/2} \exp(i/t) \].

Correspondingly if

\[ g = f_2 + ig_2 \]

8') \[ b = C_2 + iD_2 \]

then 6') becomes

\[ g = b - 2iE_{-1/2}^{\pi/4} \bar{g} \]

or
3. Solution of the integral equations

We now solve the integral equation

\[ y(t) = a(t) - \int_0^t k(t-u) f(u) \, du. \]

Combining 9) and 11) we can write

\[
\begin{bmatrix}
  f(t) \\
g(t)
\end{bmatrix} = \begin{bmatrix}
a(t) \\
b(t)
\end{bmatrix} + \int_0^t k(t-u) \begin{bmatrix}
f(u) \\
g(u)
\end{bmatrix} \, du.
\]

By successive substitutions we obtain the Neumann series

\[ \varphi(t) = a(t) - \lambda k \ast \varphi(t) \]

where \( k(t) \) is defined by 10) and \( \ast \) means the convolution product \( f \ast h = \int_0^t f(t-u)h(u) \, du \). Equation 13) is just 9), 11) when \( \lambda = -1, +1 \) respectively.

Interchanging the order of integration we have

\[ k \ast (\bar{k} \ast a) = (k \ast \bar{k}) \ast a. \]

That this interchange is valid for \( a \) in some class of functions is not obvious, however, we show in Appendix I that for functions \( a \) which are continuous and BV the interchange is valid. The validity of 15) having been
proved, we evaluate $k \ast \bar{k}$ as

$$16) \quad k \ast \bar{k}(t) = \sqrt{2/\pi} t^{-3/2} \exp(-2/t)$$

(see II.2).

Again by Appendix I the fourth term on the right in equation 14) gives

$$k \ast (\bar{k} \ast (k \ast a)) = k \ast ((\bar{k} \ast k) \ast a).$$

But 16) shows that $k \ast \bar{k}$ is also of the form considered in Appendix I so that

$$\bar{k} \ast ((k \ast k) \ast a) = (\bar{k} \ast (k \ast \bar{k})) \ast a.$$

Iterating this procedure we obtain

$$17) \quad \varphi = a + \lambda k \ast a + |\lambda|^2 k_2 \ast a + \lambda |\lambda|^2 k_3 \ast a + \cdots$$

where

$$k_{2n} = \sqrt{2/\pi} nt^{-3/2} \exp\left(-\frac{(2n)^2}{2t}\right)$$

$$18) \quad k_{2n+1} = \sqrt{2/\pi} \left(\frac{2n+1}{2} - \frac{i}{2}\right) t^{-3/2} \exp\left[-\frac{(2n+1)^2 - 1}{2t} + \frac{(2n+1)i}{t}\right].$$

On any finite interval $[0,T]$

$$|k_2| \leq \sqrt{2/\pi} t^{-3/2} \exp(-2/t) \leq M_2$$

and

$$|k_3| \leq 2\sqrt{2/\pi} t^{-3/2} \exp(-4/t) \leq M_3,$$

hence
\[ |k_{2n}(t)| \leq M_2^n \frac{t^{n-1}}{(n-1)!}, \quad n > 0 \]

and

\[ |k_{2n+1}(t)| \leq \int_0^t |k_3(t-u)||k_{2(n-1)}(u)| \, du \]

\[ \leq M_3 M_2^{n-1} \frac{(t)^{n-1}}{(n-1)!}, \quad n > 0. \]

Under the assumption that \( a(t) \) is continuous and \( BV \) let \( A = \max_{0 \leq t \leq T} a(t) \), then from 17)

\[ |\varphi| \leq |a| + |\lambda||k*\tilde{a}| + |\lambda|A(M_2+|\lambda|M_3) \sum_{n=1}^{\infty} (|\lambda| M_2)^{n-1} \frac{1}{(n-1)!}. \]

Therefore 17) is a uniformly convergent series for all \( \lambda \)

and \( n > 1 \) and

19) \( \varphi(t) = a(t) + \lambda k*\tilde{a}(t) + \sum_{n=1}^{\infty} |\lambda|^{2n}(k_{2n} * a(t) + \lambda k_{2n+1} * \tilde{a}(t)) \)

is continuous since 1.3 implies that \( k*\tilde{a}(t) \) and each term in the sum is continuous and the sum converges uniformly. Also, with \( \varphi \) defined by 19),

\[ \lambda k * \tilde{\varphi} = \lambda k*\tilde{a} + |\lambda|^2 k * (k*\tilde{a}) \]

\[ + \lambda k * [ \sum |\lambda|^{2u} (k_{2n} * a + \lambda k_{2n+1} * \tilde{a}) ] \]

\[ = \lambda k * \tilde{a} + |\lambda|^2 k_2 * a \]

\[ + \sum_{n=1}^{\infty} |\lambda|^{2n} [ \lambda k_{2n+1} * \tilde{a} + |\lambda|^2 k_{2n+2} * a ] \]
\[ y = a + \sum_{n=1}^{\infty} |\lambda|^{2n} (k_{2n} \ast a + \lambda k_{2n+1} \ast \tilde{a}) \]

\[ = -a + \varphi \]

by 15).

Thus we have proved that \( \varphi(t) \) defined by 19) is a continuous solution of equation 13) for all \( \lambda \) if \( a(t) \) is continuous and BV.

4. A related integral equation

The equation

20) \[ \varphi = a + \lambda k \ast \varphi \]

where \( k \) is again defined by 10) appears to be very similar to 13) but has remarkably different behavior. We obtain immediately the Neumann series

\[ \varphi = a + \lambda k \ast a + \lambda^2 k \ast (k \ast a) + \cdots \]

\[ = a + \lambda k \ast a + \lambda^2 (k \ast k) a + \cdots \]

or

21) \[ \varphi = a + \sum_{n=1}^{\infty} \lambda^n k_n \ast a \]

where from II.2)

22) \[ k_n'(t) = \frac{ne^{i\pi/4}}{\sqrt{n}} t^{-3/2} \exp\left(\frac{in^2}{t}\right). \]

The interchanges of order of integration are again justified by Appendix I.
The most striking feature of the "solution", 21), is that the series in general does not converge for $|\lambda| \geq 1$. Let $\lambda = 1$ and $a(t) \equiv 1$, then 21) gives

$$23) \quad \varphi(t) = 1 + \sum_{n=1}^{\infty} \int_{0}^{t} k_{n}^{'i}(s) \, ds.$$ 

By successive integration by parts

$$\int_{0}^{t} k_{n}^{'i}(s) \, ds = \frac{ne^{in/4}}{\sqrt{\pi}} \left\{ \frac{ie^{\pi/4} \sqrt{\pi}}{n^2} \exp \left( \frac{in^2}{t} \right) \right. \right.$$ 

$$+ \frac{t^{3/2} \exp \left( \frac{in^2}{t} \right)}{2n^4} + \frac{3}{4n^4} \left[ \int_{1/t}^{\infty} u^{-5/2} \exp \left( in^2 u \right) \, du \right] \right.$$ 

$$= \frac{ie^{i\pi/4} \sqrt{\pi}}{n^2} \exp \left( \frac{in^2}{t} \right) + O(n^{-3})$$

uniformly in $n$. Now let $t = \frac{1}{2 \pi m}$ where $m$ is any positive integer. Then

$$\int_{0}^{t} k_{n}^{'i}(s) \, ds = \frac{ie^{i\pi/4}}{n^{2} \sqrt{2 \pi m}} + O(n^{-3}).$$

Substituting into 23) we have

$$\varphi \left( \frac{1}{2\pi m} \right) = 1 + \frac{ie^{i\pi/4}}{\sqrt{2\pi m}} \sum_{n=1}^{\infty} \frac{1}{n} + \sum 0(n^{-3}).$$

The last sum converges like $n^{-3}$ while the first diverges so that the Neumann series fails to define $\varphi(t)$ for $t = \frac{1}{2 \pi m}$.

For $|\lambda| < 1$ the Neumann series 21) does give a
solution to 20) in a broad class of functions. In fact, if \( a(t) \) is continuous and \( BV \) on \([0,T]\) and \(|\lambda| < 1\), then 21) defines a continuous solution of 20).

To prove this result we use the mean value theorem for integrals introduced earlier (see 2.12) to obtain

\[
|\int_0^t k_n'(t-s)a(s)ds| \leq \frac{1}{n\sqrt{\pi}}(|a(0)| + V^T a)M
\]

where

\[
M = \sup \left( \sup_{0 \leq t' < t'' < t} (t-s)^{-3/2} \exp \left( \int_{t-s}^{t} ds \right) \right).
\]

That is,

\[
|\int_0^t k_n'(t-s)a(s)ds| \leq B
\]

where \( B \) is independent of \( t \). Thus 21) leads to

\[
|\sum_{n=1}^{\infty} \lambda^n k_n'*a| \leq B \sum_{n=1}^{\infty} \frac{\lambda^n}{n}
\]

which converges for \(|\lambda| < 1\) and hence 21) converges uniformly. Again by I.3) each term in 21) is continuous so that \( \varphi(t) \) is continuous. By an argument analogous to that used in the previous section we can show that 21) is indeed a solution of 20).

For \( \lambda = 1 \) it is possible to define a solution of 20) by 21) if \( a(0) = 0 \) and if \( a(t) \) has a first derivative which is continuous and \( BV \).
5. **Solution of the finite bar problem**

The solution $\varphi(t)$ of (13) defined by (19) is continuous but in general is not $BV$. For example, let $a(t) \equiv 1$. Then

$$k \ast \bar{a}(t) = \frac{ie^{-i\pi/4}}{\sqrt{\pi}} \int_0^t s^{-3/2} \exp(is) ds$$

$$= \frac{ie^{-i\pi/4}}{\sqrt{\pi}} \int_{1/t}^\infty u^{-1/2} \exp(iu) du$$

$$= - \frac{e^{-i\pi/4}}{\sqrt{\pi}} \{ t^{1/2} \exp(i/t) - \frac{1}{2} \int_{1/t}^\infty u^{-3/2} \exp(iu) du \}$$

The second term on the right has a bounded first derivative hence is $BV$ while the first term and hence $k \ast \bar{a}(t)$ is not $BV$. The conditions of the theorems of Chapter 2 are therefore not satisfied; the validity of the results, however, can be established directly.

**Theorem 1** If $a(t)$ is continuous and $BV$, then

$$24) \quad U(x,t;k \ast a(t)) = -U(x+2,t;a(t))$$

where $k(t)$ is defined by (10).

**Proof:** From 2.4) and 10)

$$U(x,t;k \ast a) = \frac{1}{\pi} \int_0^t (t-u)^{-1/2} \exp \left( \frac{i x^2}{4(t-u)} \right) \int_0^u (u-v)^{-3/2} \exp \left( \frac{i v}{u-v} \right) a(v) dv du$$

which after we interchange the order of integration
becomes
\[
\frac{1}{\pi} \int_{0}^{t} a(v) \int_{v}^{t} (t-u)^{-1/2} (u-v)^{-3/2} \exp \left( \frac{i x^2}{4(t-u)} + \frac{i}{u-v} \right) du dv
\]
\[
= \frac{e^{i \pi/4}}{\sqrt{\pi}} \int_{0}^{t} (t-v)^{-1/2} \exp \left( \frac{i(x+2)^2}{4(t-v)} \right) a(v) dv
\]
by II.3). Hence 24) follows from 2.4).

Now from 19) \( \varphi(t) - \lambda k * \bar{a}(t) \) is BV since \( a(t) \)
is BV by assumption and the remaining sum has by 18) a bounded first derivative. Therefore

\[
U(x,t;\varphi) = U(x,t;\varphi - \lambda k * \bar{a})
+ \lambda U(x,t;k * \bar{a})
\]

\[
= U(x,t;\varphi - \lambda k * \bar{a}) - \lambda U(x+2,t;\bar{a})
\]

From 2.9)
\[
\lim_{x \to 0} U(x,t;\varphi) = -e^{i \pi/4} I^{1/2}[\varphi(t) - \lambda k * \bar{a}(t)] + \lambda E^{1/2} \bar{a}(t)
\]
\[
= -e^{i \pi/4} I^{1/2}[\varphi - \lambda k * \bar{a}] - \lambda E^{1/2} E^{1/2} \bar{a}
\]
by 2.24). From 10) we can write this last term as
\[
e^{i \pi/4} I^{1/2}(\lambda k * \bar{a})
\]
so that \( \lim_{x \to 0} U(x,t;\varphi) = -e^{i \pi/4} I^{1/2} \varphi \).

Similarly by 2.13), 1.5), 1.3) and 10)

\[\text{II.3}\]

\[\text{That this interchange is permissible is not obvious but it is clear that the argument of Appendix I can be adapted to this case.}\]
\[
\lim_{x \to 0} \partial_x U(x,t;\varphi) = \lim_{x \to 0} \partial_x U(x,t;\varphi - \lambda k \ast \bar{a}) \\
- \lambda \lim_{x \to 0} \partial_x U(x+2,t;\bar{a})
\]
\[
= \varphi - \lambda k \ast \bar{a} - \lambda \partial_x U(2,t;\bar{a})
\]
\[
= \varphi - \lambda k \ast \bar{a} + \lambda k \ast \bar{a} = \varphi(t)
\]

From 2.11)
\[
\lim_{t \to 0^+} U(x,t;\varphi) = \lim_{t \to 0^+} \partial_x U(x,t;\varphi) = 0.
\]

If \(a(t)\) has a first derivative which is continuous and BV, then \(U(x,t;\varphi)\) satisfies 1) where \(\varphi\) is defined by 19). This follows since \(\varphi - \lambda k \ast \bar{a}\) has a first derivative which is continuous and BV so that each term of
\[
U(x,t;\varphi) = U(x,t;\varphi - \lambda k \ast \bar{a}) - \lambda U(x+2,t;\bar{a})
\]
satisfies the conditions of the corollary to Theorem 2.2 and hence \(U(x,t;\varphi)\) satisfies 1).

We have thus proved that if \(I^{-1/2}a_i\) and \(b_i\) (\(i=1,2\)) have first derivatives which are continuous and BV, then
3) is a solution of 1) satisfying conditions 2) where \(\varphi_i\) and \(\psi_i\) (\(i=1,2\)) are solutions of 4) defined by 19) with \(\varphi\), \(a\) and \(\lambda\) respectively \(f\), \(a\) and \(-1\) and \(g\), \(b\) and \(1\) as in equation 12).
Uniqueness and Related Topics

1. Decomposition of solutions

In this chapter we will observe that solutions to

\[ (a^4_x + a^2_t)u = 0 \]

behave in some sense like solutions to certain related second order equations. In particular, in this section we show that every solution of 1) can be represented at least locally as a sum of solutions to certain Schrödinger type second order equations.

Consider first the wave equation which factors as

\[ (a^2_x - a^2_t)u = (a_x + a_t)(a_x - a_t)u = 0. \]

It is well-known that any solution \( u(x,t) \) of the wave equation can be written as

\[ u(x,t) = u_1(x-t) + u_2(x+t) \]

and it is trivially clear that \( u_1(x-t) \) is a solution of \( (a_x + a_t)u = 0 \) and \( u_2(x+t) \) of \( (a_x - a_t)u = 0 \), for sufficiently smooth functions.

Analogously equation 1) factors,

\[ (a^4_x + a^2_t)u = (a^2_x + ia_t)(a^2_x - ia_t)u = 0. \]

By 1.3) and 1.2) the representation for the solution of
1) for the infinite bar problem can be written

\[ u(x, t) = -\frac{1}{2} \left[ \frac{1}{2} [k(x-y, t) + \tilde{k}(x-y, t)]f_2(y)dy \right. \]
\[ + \frac{1}{2} \left. \int_{-\infty}^{\infty} \frac{1}{2} [k(x-y, t) - \tilde{k}(x-y, t)]f_1(y)dy \right] \]

which with \( F(y) = f_2(y) + i f_1(y) \) becomes

2) \[ u(x, t) = -\frac{1}{4} \int_{-\infty}^{\infty} [k(x-y, t)F(y) + \tilde{k}(x-y, t)\bar{F}(y)]dy. \]

From equations 1.6), 1.7) and 1.9), along with the proof of Theorem 1.2, it follows that the two terms on the right of the above equation satisfy

3) \( (\partial_x^2 + i\partial_t)u = 0 \),

4) \( (\partial_x^2 - i\partial_t)u = 0 \)

respectively with appropriate initial data.

Pursuing this idea, we see that by 2.3) the representation 2.2) for the solution of the semi-infinite bar problem can be written

\[ u(x, t) = \frac{1}{2} [U(x, t; \varphi) + \overline{U(x, t; \varphi)}] + \]
\[ + \frac{1}{2i} [U(x, t; \psi) - \overline{U(x, t; \psi)}] \]

which with \( \Phi = \frac{1}{2}(\varphi - i\psi) \) becomes

\[ u(x, t) = U(x, t; \Phi) + \overline{U(x, t; \Phi)}. \]
However, from 2.4) $U$ and $\bar{U}$ are solutions of 3) and 4) respectively with appropriate boundary data.

The extension of these results being obvious for the finite bar problem, we can show that any solution of 1) defined on some open set $G$ of the $(x,t)$-plane can be represented locally as the sum of solutions of 3) and 4).

Since $G$ is open, for $(x_0,t_0) \in G$, there exist numbers $\delta$, $t_1$, and $t_2$, with $t_1 < t_0 < t_2$, such that $G \supset \{(x,t) \mid |x-x_0| < \delta, \ t_1 < t < t_2\} = G'$. By the results of Chapter 1 and the remarks at the beginning of Chapter 3, we can obtain functions $v_1(x,t)$ and $v_2(x,t)$ which are solutions of 1) in the interior of $G'$ with $\delta^i_t v_1(x,t) = \delta^i_t u(x,t_1) \ (i=0,1)$ for $|x-x_0| < \delta$ and $v_1$ with a representation of the form 1.2), and $\delta^i_t v_2(x,t_1) = 0 \ (i=0,1)$, $\delta^i_x v_2(x_0 \pm \delta,t) = \delta^i_x u(x_0 \pm \delta,t) - \delta^i_x v_1(x_0 \pm \delta,t) \ (i=0,1)$ for $t_1 < t < t_2$ and $v_2$ with a representation of the form 3.3). If the solution to this problem is unique, which we prove below, then $u(x,t) = v_1(x,t) + v_2(x,t)$. The above discussion shows that $v_1$ and $v_2$ can be written as a sum of solutions of 3) and 4) so that $u(x,t)$ also admits this decomposition.
2. The Cauchy problem reconsidered

According to Chapter 1 we can find a solution to the Cauchy problem for 1) with representation 1.2) provided the initial data \(f_i(x) = 0(x^{-a})\), \(a > 0\) \((i=1,2)\), as \(|x| \to \infty\). It is clear that all solutions to the Cauchy problem are not thus obtained since \(u(x,t) = e^x \cos t\) is a solution of 1) with \(u(x,0) = e^x\), \(\partial_t u(x,0) = 0\).

In the nomenclature of Gel'fand and Šilov [2] equation 1) leads to a "regular" system of equations of reduced order \(p_0 = 2\) and hence by [6] has a solution for the class of functions which satisfy, for each \(\varepsilon > 0\), an inequality of the form

\[
|u(x,t)| \leq C(\varepsilon) e^{\varepsilon|x|^2}.
\]

And the solution has a representation in terms of a series of solutions, each of which has initial data of compact support, provided that the initial data is sufficiently smooth.\(^3\) Note that the class determined by 5) is the same as that for the heat equation so that the second order character of this equation is again reflected.

For the problem of the semi-infinite bar with boundary data like that discussed in Chapter 2 and initial

\(^3\)It is not clear on the basis of the text of [6] what order of smoothness is necessary, but it is clear that with sufficient smoothness a solution exists in the class given by 5).
data \( \partial_t^i u(x,0) = f_i(x) \) for \( x > 0 \) \((i=0,1)\) in the class given by 5) we can again obtain a solution in the class 5), if the \( f_i \) \((i=0,1)\) are sufficiently smooth. We extend the functions \( f_i \) \((i=0,1)\) to the infinite line in a smooth way without violating 5) and obtain a solution \( u_1(x,t) \) for this Cauchy problem from [6]. We then solve the boundary value problem considered in Chapter 2 with \( \partial_x^i u_2(0,t) = a_i(t) - \partial_x^i u_1(0,t) \) \((i=0,1)\). From the last section of Chapter 2 we can write the solution to this problem as

\[
 u_2(x,t) = \sqrt{2} \int_0^{\pi/4} \frac{1}{\sqrt{x}} \int_0^{1/2} \left[ a_0(t) - u_1(0,t) \right] + 2 \int_0^{\pi/2} \frac{x^2}{4} \left[ a_1(t) - \partial_x u_1(0,t) \right].
\]

It is clear that \( |u_2(x,t)| \leq B \) for some \( B \) so that the inequality 5) is satisfied.

3. **Uniqueness questions**

The uniqueness of solutions for the Cauchy problem in the class determined by 5) is established in [2]. That some condition such as 5) is necessary can be seen by considering the following example, patterned after the work of Tychonoff [12] and John [4]. Consider
6) \[ u(x,t) = \sum_{n=0}^{\infty} f_n(t) \frac{x^n}{n!}. \]

If 6) is to be a solution of 1), then

7) \[ f_{n+4} = -f_n. \]

Choose \( f_0(t) = e^{-1/t^2}, f_2(t) = \frac{d}{dt}(e^{-1/t^2}) = \frac{d}{dt}f_0(t), \)
and \( f_1 = f_3 = 0. \) Then 6) becomes

8) \[ u(x,t) = \sum_{p=0}^{\infty} e(p)f_0^{(p)}(t)x^{2p}(2p)! \]

where

\[ e(p) = \begin{cases} 
   1 & \text{if } p = 4k \text{ or } 4k+1 \\
   -1 & \text{if } p = 4k+2 \text{ or } 4k+3 
\end{cases}. \]

To estimate the order of growth of 8), we choose a circle

\[ |s-t| = \frac{t}{2} \]

about each \( t \) on the positive real axis of

the complex \( s \)-plane. Then

\[ f_0^{(p)}(t) = p!/(2\pi i) \int_\gamma (s-t)^{-1-p}\exp(-\frac{1}{s^2}) \, ds \]

where \( \gamma \) is the circle \( s = t + \frac{1}{2}te^{i0} \). Since the

mapping \( s \to s^{-1} \) carries circles into circles, \( \frac{1}{s} = \frac{4}{3t}(1 + \frac{1}{2}e^{i\phi}) \) and \( \text{Re}(s^{-2}) = \frac{16}{9}t^{-2}[(1+\frac{1}{2}\cos\phi)^2 - \frac{1}{4}\sin^2\phi] \),

that is \( \text{Re}(s^{-2}) \geq \frac{4}{9}t^{-2} \). Therefore
\[ |f_o^{(p)}(t)| \leq \frac{p!}{(2\pi)} \int \exp\left(-\frac{4}{9}t^{-2}\right) |s-t|^{-p-1} |ds| \]

and hence

9) \[ |f_o^{(p)}(t)| \leq 2^p p! t^{-p} \exp\left(-\frac{4}{9}t^{-2}\right). \]

From 8)
\[ |u(x,t)| \leq \sum_{p=0}^{\infty} \frac{2^p p!}{(2p)!} \left(\frac{x^2}{t}\right)^p \exp\left(-\frac{4}{9}t^{-2}\right), \]

but since \( 2^p p!/(2p)! \leq \frac{1}{p!} \),

10) \[ |u(x,t)| \leq \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{x^2}{t}\right)^p \exp\left(-\frac{4}{9}t^{-2}\right) = \exp\left[-\frac{1}{t}\left(\frac{4}{9t} - x^2\right)\right]. \]

Formally differentiating 8) term by term, we obtain
\[ a_t u(x,t) = \sum_{p=0}^{\infty} f^{(p+1)}(t) \frac{x^2}{t}^p / (2p)! \]

and
\[ |a_t u(x,t)| \leq \sum_{p=0}^{\infty} \frac{2^{p+1}(p+1)!}{(2p)! t} \left(\frac{x^2}{t}\right)^p \exp\left(-\frac{4}{9}t^{-2}\right), \]

but since \( 2^{p+1}(p+1)!/(2p)! \leq 8/p! \)

11) \[ |a_t u(x,t)| \leq \exp\left(-\frac{4}{9}t^{-2}\right) t^{-1} \sum_{p=0}^{\infty} \frac{8}{p!} \left(\frac{x^2}{t}\right)^p \]
\[ = \frac{8}{t} \exp\left[-\frac{1}{t}\left(\frac{4}{9t} - x^2\right)\right]. \]

These estimates show that on the domain \( D = \{ (x,t) | -R < x < R, t > 0 \} \) the series defining \( u(x,t) \) and \( a_t u(x,t) \)
converge uniformly and \( \lim_{t \to 0} u(x,t) = \lim_{t \to 0} \partial_t u(x,t) = 0 \).

Estimates corresponding to 11) for the formal second derivative \( \partial_t^2 u \) obtained by termwise differentiation show that the series of derivatives converges uniformly on \( D \) and hence that term by term differentiation is justified. By the construction of \( u(x,t) \), equation 1) is satisfied and since \( R \) is arbitrary, \( u(x,t) \) is a solution of 1) for \( t > 0 \) and all \( x \) with zero initial data and \( u(x,t) \neq 0 \). That is, we can not hope for uniqueness for functions that satisfy 10).

If we take only the terms of 8) that correspond to odd values of \( p \), we have the series

\[
12) \quad u(x,t) = \sum_{p=0}^{n} (-1)^p f(2p+1)(t) \frac{x^{4p+2}}{(4p+2)!}.
\]

which by 9) gives

\[
|u(x,t)| \leq \sum \frac{2^{2p+1}(2p+1)!}{(4p+2)!} t^{-2p-2} \exp\left(-\frac{4}{9}t^{-2}\right) \frac{x^{4p+2}}{(4p+2)!},
\]

but since \( \frac{2^{2p+1}(2p+1)!}{(4p+2)!} \leq \frac{2}{p!} \),

\[
|u(x,t)| \leq \sum \frac{2}{p!} \left(\frac{x}{t}\right)^2 \left(\frac{x^4}{t^2}\right)^p \exp\left(-\frac{4}{9}t^{-2}\right)
\]

\[
= 2\left(\frac{x}{t}\right)^2 \exp\left[ -\frac{1}{t^2} \left(\frac{4}{9} - x^4 \right) \right].
\]

Proceeding as in the previous paragraph we can obtain
estimates like 11) for $\partial_t u(x,t)$ and $\partial^2_t u(x,t)$ so that we have constructed a function which satisfies 1) with zero initial data and for each $t$ has exponential growth of order $\leq 4$.

The last example of the previous paragraph has the property that $u(0,t) = a_x u(0,t) = 0$. Thus we see that uniqueness fails for the semi-infinite bar problem for functions which for fixed $t$ have exponential growth of order $\leq 4$. It is not clear how to construct a function of order of growth $\leq 2$ which has $u(0,t) = u_x(0,t) = 0$, but if in 7) we take $f_0 = f_2 = 0$, $f_1(t) = e^{-1/t^2}$, and $f_3(t) = \frac{d}{dt} f_1(t)$, then

$$u(x,t) = \sum_{p=0}^{\infty} e(p) f_1^{(p)}(t) \frac{x^{2p+1}}{(2p+1)!}$$

will again be a solution of 1) with zero initial data and $u(0,t) = a_x^2 u(0,t) = 0$; and from 9)

$$|u(x,t)| \leq \exp[- \frac{1}{t} (\frac{4}{9t} - x^2)] .$$

It thus seems unlikely that the semi-infinite bar problem for any boundary data will be unique for functions of exponential growth of order $> 2$. That the solution for the semi-infinite bar problem is unique for some class of initial functions is not easy to prove directly and has
certainly not been established for the class implied by 5). We will obtain uniqueness in a much more limited class of functions from the uniqueness of the finite bar problem.

Pini [8, page 218] states a uniqueness theorem for the finite bar problem, but the argument is not easy to follow so we will give a straightforward argument based upon the energy integral. Our theorem includes all the cases covered by Pini, in particular the problem we consider in Chapter 3, but it is not adequate for all the "interesting" boundary value problems listed by Kirmser [5, pp. 94-95]. It will of course handle the boundary value problems obtained by variational methods from the energy integral. A uniqueness argument similar to that given in [5] can clearly be constructed which would cover all the problems listed in [5]; this approach would also reduce some of the continuity requirements of the theorem below, but we will be content with the following.

Theorem 1. If \( \partial_t \partial_x u, \partial_x^3 u \) are continuous in \( \{0 < x < 2, 0 < t\} \) and \( u \) is a solution of 1) in \( \{0 < x < 2, 0 < t\} \), then

\[ u(x,t) \equiv 0 \text{ is the only solution of 1) with } u(x,0) = u_t(x,0) = 0 \text{ and } \]

\[ 13) \quad u = \partial_x^i u = 0 \quad (i=1,2) \]

or

\[ 14) \quad \partial_x^3 u = \partial_x^i u = 0 \quad (i=1,2) \]
on $x = 0$ and $x = 2$.

Proof: Consider

$$E(t, \delta) = \int_{0+\delta}^{2-\delta} \left[(\partial_x^2 u)^2 + (\partial^2 u)^2\right] dx.$$ 

Then $E(0, \delta) = 0$ for $1 > \delta > 0$ and $t > 0$. For $\delta > 0$

$$\partial_t E(t, \delta) = 2\int_{0+\delta}^{2-\delta} (2\partial_t u \partial_x^2 u + 2\partial_x^2 u \partial_x^2 u) dx$$

$$= 2\int_{0+\delta}^{2-\delta} \partial_t u \partial_x^2 u + 2\int_{0+\delta}^{2-\delta} \partial_x^2 u \partial_x^2 u$$

$$-2[\partial_t u \partial_x^3 u - \partial_x^2 u \partial_x^2 u]_{0+\delta}^{2-\delta}$$

by integration by parts. Since $u$ is a solution of 1),

15) $\partial_t E(t, \delta) = -2[\partial_t u \partial_x^3 u - \partial_x^2 u \partial_x^2 u]_{0+\delta}^{2-\delta}$.

From the hypotheses of the theorem

$$\lim_{\delta \to 0} \partial_t E(t, \delta) = E'(t) = 0$$

hence $E(t) = 0$ and $\partial_t u = \partial_x^2 u = 0$ on $\{0 < x < 2, 0 < t\}$.

In particular, $u$ is independent of $t$ and since $u(x, 0) = 0$, $u(x, t) = 0$ for $\{0 < x < 2, 0 < t\}$.

The uniqueness on the more general set $\{0 < x < L, 0 < t\}$ is now easily obtained. Let $y = \frac{2}{L} x$ and $s = (\frac{2}{L})^2 t$.

Then $v(y, s) = u(x, t)$ satisfies
with \( \partial_y^4 v(2, s) = \partial_x^4 u(L, t) \) (i=0,1,2,3). The theorem above now applies to \( v(y, s) \). In particular 15) becomes

\[
E'(s) = -2[\partial_y^3 v_0 - \partial_y^2 v_0 \partial_y v_0]^2
\]

\[
= -2\left(\frac{L}{2}\right)^5[\partial_y^3 v_0 - \partial_x^2 v_0 \partial_x v_0]^L_0.
\]

Now replacing conditions 14) at \( L \) by \( \partial_x \partial_t u = o(x^{-3}) \) and \( \partial_x^3 u = o(x^{-3}) \) as \( x \to \infty \), \( E'(s) \to 0 \) as \( L \to \infty \).

Thus for the semi-infinite bar problem uniqueness is assured in the class of functions

\[
\partial_x \partial_t u = o(x^{-3}) , \quad \partial_x^3 u = o(x^{-3}) \quad \text{as} \quad x \to \infty.
\]

A uniqueness argument patterned after [5] may, in addition to the previous comments, require less restrictive conditions at \( \infty \) than the above, but it is clear that the improvement would still require that the function and certain of its derivatives vanish at \( \infty \). Thus the problem of uniqueness for the semi-infinite bar would seem to be in a very inadequate state in view of the results for the Cauchy problem.


APPENDIX I

In Chapter 3 it is necessary to invert the order of integration of certain improper integrals. The justification for this interchange is given here.

**Theorem** If \( f \) is continuous and BV on \([0,T]\) and \(|a| > 0, |b| > 0, \Re{(a)} \geq 0, \Re{(b)} \geq 0\), then for \(0 \leq t \leq T\)

\[
J_1 = \int_0^t (t-s)^{-3/2} \exp(-\frac{a}{t-s}) \int_0^s (s-u)^{-3/2} \exp(-\frac{b}{s-u}) f(u) du \, ds
\]

\[= \int_0^t f(u) \int_0^t [(t-s)(s-u)]^{-3/2} \exp(-\frac{a}{t-s} - \frac{b}{s-u}) ds \, du = J_2.
\]

**Proof:** The method of proof will be to show that \( J_2 \) exists, to restrict the domain to that on which interchange of order of integration is justified, and to show that the neglected terms give zero in the limit which will establish the existence of \( J_1 \) as well as the equality \( J_1 = J_2 \).

Of course, both \( a \) and \( b \) imaginary is the most interesting case.

The first integration of \( J_2 \) can be carried out explicitly. Let \( s = u + v \), then

\[
\int_u^t [(t-s)(s-u)]^{-3/2} \exp(-\frac{a}{t-s} - \frac{b}{s-u}) \, ds
\]

\[= \int_0^{t-u} [(t-u-v)v]^{-3/2} \exp(-\frac{a}{t-u-v} - \frac{b}{v}) \, dv
\]
by II.2). Therefore

\[ J_2 = C \int_0^t (t-u)^{-3/2} \exp\left(- \frac{c}{t-u}\right) f(u) \, du \]

which is equivalent to the first integration of \( J_1 \).

Consider now

2) \[ I(s) = \int_0^s (s-u)^{-3/2} \exp\left(- \frac{b}{s-u}\right) f(u) \, du \]

which by integration by parts is

\[ = \frac{1}{b} \exp\left(- \frac{b}{s}\right) s^{1/2} f(0) \]

\[ + \frac{1}{b} \int_0^s \exp\left(- \frac{b}{s-u}\right) \left[- \frac{1}{2}(s-u)^{-1/2} f(u) \, du + (s-u)^{1/2} df(u) \right] . \]

This last integral is absolutely convergent; hence \( J_2 \) and the first integral of \( J_1 \) exists.

Also

3) \[ |I| \leq \frac{f(0) \sqrt{s}}{|b| s} + \frac{M}{|b|} \int_0^s \frac{1}{2} u^{-1/2} \, du + \frac{1}{|b|} \int_0^s \sqrt{(s-u)} \, dV_s^b[f] \]

\[ \leq \frac{1}{|b|} (2M + V_{a}^b[f]) s^{1/2} \]

where \( M = \sup_{0 \leq t \leq T} |f(t)| \) and \( V_{a}^b[f] \) is the total variation of \( f \) on \([a,b]\). Thus the second integral of \( J_1 \) is not improper at the origin and its existence will be established.
if

$$\lim_{\delta \to 0} \int_{t-\delta}^{t} (t-s)^{-3/2} \exp(-\frac{a}{t-s}) I(s) ds$$

exists.

We define $J_1'$ by

$$4) \quad J_1' = \int_{t}^{t-\delta} (t-s)^{-3/2} \exp(-\frac{a}{t-s}) I(s) ds$$

$$= \int_{t-\delta}^{t} (t-s)^{-3/2} \exp(-\frac{a}{t-s}) I(s-\delta) ds$$

$$+ \int_{t-\delta}^{t} (t-s)^{-3/2} \exp(-\frac{a}{t-s}) I(s-\delta) \int_{s-\delta}^{s} (s-u)^{-3/2} \exp(-\frac{b}{t-s}) f(u) du \ ds.$$

The first term on the right of the last equality is a proper integral and the order of integration can be reversed yielding

$$\int_{t-\delta}^{t} (t-s)^{-3/2} \exp(-\frac{a}{t-s}) \int_{u+\delta}^{t} f(u) [(t-s)(s-u)]^{-3/2} \exp(-\frac{a}{t-s} - \frac{b}{s-u}) ds \ du$$

$$= \int_{t-\delta}^{t} ( \int_{u}^{t} - \int_{u}^{t+\delta} - \int_{t-\delta}^{t} ) \cdots ds \ du = J_2' - I_1 - I_2.$$

Putting this into 4) we obtain

$$5) \quad J_1' = J_2' - I_1 - I_2 + I_3$$

where

$$I_3 = \int_{t-\delta}^{t} \int_{s-\delta}^{s} [(t-s)(s-u)]^{-3/2} \exp(-\frac{a}{t-s} - \frac{b}{s-u}) f(u) du ds.$$
Since \( J_2 \) exists, \( \lim_{\delta \to 0} J_2' = J_2 \). We will show that
\[
\lim_{\delta \to 0} I_i = 0 \quad (i=1,2,3)
\]
from which it follows that \( J_1 \)
eexists and \( J_1 = J_2 \).

We integrate the first integral of \( I_3 \) by parts to obtain
\[
6) \int_{s-\delta}^{s} (s-u)^{-3/2} \exp(-\frac{b}{s-u}) f(u) du = \frac{1}{b} \exp(-\frac{b}{\delta}) \delta^{1/2} f(s-\delta)
\]
\[
- \frac{1}{2\delta} \int_{s-\delta}^{s} (s-u)^{-1/2} \exp(-\frac{b}{s-u}) f(u) du
\]
\[
+ \frac{1}{b} \int_{s-\delta}^{s} (s-u)^{1/2} \exp(-\frac{b}{s-u}) df(u)
\]
which substituted into \( I_3 \) gives
\[
I_3 = \frac{1}{b} (I_{31} - I_{32} + I_{33})
\]

In \( I_{31} \) we again integrate by parts to obtain
\[
7) I_{31} = \sqrt{\delta} \exp(-\frac{b}{\delta}) \{ \frac{\sqrt{(t-\delta)}}{a} \exp(-\frac{a}{t-\delta}) f(0)
\]
\[
- \frac{\sqrt{\delta}}{a} \exp(-\frac{a}{\delta}) f(t-2\delta) + \frac{1}{a} \int_{0}^{t-\delta} \exp(-\frac{a}{t-s})[\frac{1}{2}(t-s)^{-1/2} f(t-s) ds
\]
\[
+ (t-s)^{1/2} df(s-\delta)] \}
\]
If we substitute \( s = v + \delta \), then
\[ I_{31} = \sqrt{\delta} \exp(-\frac{b}{\delta}) \left\{ \frac{\sqrt{(t-\delta)}}{a} \exp(-\frac{a}{t-\delta})f(0) - \frac{\sqrt{\delta}}{a} \exp(-\frac{a}{\delta})f(t-2\delta) \right\} \\
+ \frac{1}{a} \int_0^{t-2\delta} \exp\left(-\frac{a}{t-\delta-v}\right) \left\{ -\frac{1}{2}(t-\delta-v)^{-1/2}f(v)dv + (t-\delta-v)^{1/2}df(v) \right\}. \]

Therefore

\[ |I_{31}| \leq \sqrt{\delta} \left( MA^2\left(\sqrt{\delta} + \sqrt{\gamma}\right) + \sqrt{\delta} \int_0^{t-2\delta} \frac{1}{2}(t-\delta-v)^{-1/2}dv \right. \]

\[ + \left. \sqrt{\delta} A^2 \int_0^{t-2\delta} (t-\delta-v)^{1/2}dv f(v) \right\}. \]

where \( A = \max \{ \exp(-\frac{a}{t}), \exp(-\frac{b}{\delta}) \} \) and \( M \) is defined in 3). But since \( t > t-2\delta - v \),

8) \[ |I_{31}| \leq \sqrt{\delta} \left( 2MA^2(\delta^{1/2} + (t-\delta)^{1/2}) + \sqrt{\delta} \sqrt{\gamma} A^2 \right) f(t|v) \].

Therefore \( I_{31} \to 0 \) as \( \delta \to 0 \).

In \( I_{32} \) we integrate the first integral by parts as in 6) and then set \( u = s - v \) so that

\[ I_{32} = \frac{1}{2} \int_0^{t-\delta} (t-s)^{-3/2} \exp\left(-\frac{a}{t-s}\right) \left\{ \frac{1}{b} \delta^{3/2} \exp(-\frac{b}{\delta})f(s-\delta) \right\} \\
- \frac{1}{b} \int_0^{\delta} \exp\left(-\frac{b}{v}\right) \left[ \frac{3v}{2}f(s-v)dv + v^{3/2}df(s-v) \right] \}. \]

Hence

\[ |I_{32}| \leq \frac{A}{2} \int_0^{t-\delta} (t-s)^{-3/2} \left\{ \frac{MA}{|b|} \delta^{3/2} + \frac{MA}{|b|} \frac{3}{2} \int_0^{\delta} v^{1/2}dv \right\}. \]
\[ + \frac{A}{|b|} \int_{0}^{\delta} v^{3/2} d\nu_{s}^{s-v} [f] \] 
\[ \leq \delta^{3/2} \frac{A^2 (2M + \nu_{0}^+[f])}{|b|} \frac{1}{2} \int_{0}^{t-\delta} (t-s)^{-3/2} ds , \]

that is

\[ I_{32} = K \delta^{3/2} (\delta^{-1/2} + (t-\delta)^{-1/2}) . \]

Therefore \( I_{32} \to 0 \) as \( \delta \to 0 \).

Since \( f \) is uniformly continuous on \([0, T]\),

\( \nu_{0}^{s}[f] \) is uniformly continuous on \([0, T]\) hence

\( \nu_{0}^{s}[f] - \nu_{0}^{s-\delta}[f] = o(1) \) as \( \delta \to 0 \) uniformly in \( s \). Now

\[ \left| \int_{s-\delta}^{s} \exp\left(\frac{-a}{s-u}\right)(s-u)^{1/2} df(u) \right| \leq A \delta^{1/2} (\nu_{0}^{s}[f] - \nu_{0}^{s-\delta}[f]) \]

so that

\[ |I_{33}| \leq \delta^{1/2} o(1) A^{2} \int_{0}^{t-\delta} (t-s)^{-3/2} ds \]

\[ = 2(1 + \delta^{1/2} (t-\delta)^{-1/2}) o(1) . \]

Therefore \( I_{33} \to 0 \) as \( \delta \to 0 \) and hence \( I_{3} \to 0 \) as \( \delta \to 0 \).

\( I_{1} \) and \( I_{2} \) are essentially the same as can be seen by substituting \( s = u + v \) in \( I_{1} \) and \( s = t - v \) in \( I_{2} \).
We then obtain

\[ I_1 = \int_0^{t-2\delta} f(u) \int_0^{\delta} \left[(t-u-v)v\right]^{-3/2} \exp\left(-\frac{a}{t-u-v} - \frac{b}{v}\right) dvdu \]

and

\[ I_2 = \int_0^{t-2\delta} f(u) \int_0^{\delta} [v(t-u-v)]^{-3/2} \exp\left(-\frac{a}{v} - \frac{b}{t-u-v}\right) dvdu. \]

Since these integrals have the same form, it will be sufficient to show that \( I_1 \rightarrow 0 \) as \( \delta \rightarrow 0 \).

Integrating the first integral in \( I_1 \) by parts we have

\[ I_1 = \int_0^{t-2\delta} \frac{1}{b} \exp\left(-\frac{b}{\delta}\right)(t-\delta-u)^{-3/2} \exp\left(-\frac{a}{t-\delta-u}\right) f(u) du \]

\[ - \frac{1}{b} \int_0^{t-2\delta} f(u) \int_0^{\delta} \exp\left(-\frac{b}{v}\right) \left[\frac{1}{2}v^{-1/2}(t-u-v)^{-3/2} \right. \]

\[ + \frac{3}{2}v^{1/2}(t-u-v)^{-5/2} - av^{1/2}(t-u-v)^{-7/2} \left. \right] \exp\left(-\frac{a}{t-u-v}\right) dvdu \]

\[ = \frac{1}{b}(I_{11} - I_{12} - I_{13} + I_{14}) \]

We integrate \( I_{11} \) by parts just as in 6) so that

\[ I_{11} = \sqrt{\delta} \exp\left(-\frac{b}{\delta}\right) \{ \frac{\sqrt{t-\delta}}{a} \exp\left(-\frac{a}{t-\delta}\right) f(0) \]

\[ - \frac{\sqrt{\delta}}{a} \exp\left(-\frac{a}{\delta}\right) f(t-2\delta) \]
which corresponds exactly to 7). Hence by 8) \( I_{11} \rightarrow 0 \) as \( \delta \rightarrow 0 \).

Since \( t - u - v \geq \delta, (t - u - v)^{-\alpha} \leq \delta^{-\alpha} (\alpha > 0) \), the integrals \( I_{1j} (j = 2, 3, 4) \) are absolutely integrable as double integrals. Fubini's theorem then permits us to interchange the order of integration. Interchanging the order of integration and integrating by parts, as in 6), we obtain

\[
I_{12} = \int_0^\delta \frac{1}{2} v^{-1/2} \exp(- \frac{b}{\sqrt{v}}) \{ \frac{1}{a} \exp(-\frac{a}{t-v}) \sqrt{t-v} f(0) \\
- \frac{1}{a} \exp(\frac{-a}{2\delta-v}) \sqrt{2\delta-v} f(t-2\delta) \\
+ \frac{1}{a} \int_0^{t-2\delta} \exp(\frac{-a}{t-v-u}) \{ - \frac{1}{2}(t-v-u)^{-1/2} f(u) du + (t-v-u)^{1/2} df(u) \} dv.
\]

Therefore

\[
|I_{12}| \leq \frac{A^2}{|a|} \int_0^\delta \frac{1}{2} v^{-1/2} [M(\sqrt{2\delta} + \sqrt{t}) + M \int_0^{t-2\delta} \frac{1}{2}(t-v-u)^{-1/2} du \\
+ \sqrt{t} \int_0^t [f] dv] dv.
\]

But

\[
\int_0^{t-2\delta} \frac{1}{2}(t-v-u)^{-1/2} du = \sqrt{(2\delta-v)} - \sqrt{(t-v)} \leq \sqrt{2\delta} + \sqrt{t}.
\]
hence

\[ |I_{12}| \leq \frac{A^2}{|a|} \left[ 2M(\sqrt{2} \delta + \sqrt{\delta}) + \sqrt{\delta} V^t_0 f \right] \sqrt{\delta} \]

and \( I_{12} \rightarrow 0 \) as \( \delta \rightarrow 0 \).

In \( I_{13} \) we interchange the order of integration and integrate by parts just as before. Therefore

\[ I_{13} = \int_0^\delta \frac{3}{2} v^{1/2} \exp\left(-\frac{b}{v}\right) \left\{ \frac{1}{a} (t-v)^{-1/2} \exp\left(\frac{-a}{t-v}\right) f(0) \right\} \]

\[ + \frac{1}{a} (2\delta-v)^{-1/2} \exp\left(\frac{-a}{2\delta-v}\right) f(t-2\delta) + \]

\[ + \int_0^{t-2\delta} \exp\left(\frac{-a}{t-u-v}\right) \left\{ \frac{1}{2} (t-v-u)^{-3/2} f(u) \right\} dv \]

and

\[ |I_{13}| \leq \frac{3}{2} \frac{A^2}{|a|} \int_0^\delta \sqrt{\delta} \left\{ M[(2\delta-v)^{-1/2} + (t-v)^{-1/2}] \right\} \]

\[ + M \int_0^{t-2\delta} \left( \frac{1}{2} (t-v-u)^{-3/2} du + \int_0^{t-2\delta} (t-v-u)^{-1/2} dV^u_0 f \right) dv. \]

But

\[ \frac{1}{2} \int_0^{t-2\delta} (t-v-u)^{-3/2} du = (2\delta-v)^{-1/2} - (t-v)^{-1/2} \]

\[ \leq (2\delta-v)^{-1/2} + (t-v)^{-1/2} \leq \delta^{-1/2} + (t-\delta)^{-1/2} \]

so that

\[ |I_{13}| \leq \frac{A^2}{|a|} \left[ 2M(\delta^{-1/2} + (t-\delta)^{-1/2}) + \delta^{-1/2} V^t_0 f \right] \delta^{3/2} \]

and hence \( I_{13} \rightarrow 0 \) as \( \delta \rightarrow 0 \).
Again in $I_{14}$ we interchange the order of integration and integrate by parts so that

$$I_{14} = \int_0^\delta \sqrt{v} \exp(- \frac{b}{v}) \left\{ \frac{1}{a} \exp(\frac{-a}{t-v})(t-v)^{-3/2} f(0) \right. $$

$$- \frac{1}{a} \exp(\frac{-a}{t-v})(2\delta-v)^{-3/2} f(t-2\delta) $$

$$+ \frac{1}{a} \int_0^{t-2\delta} \exp(\frac{-a}{t-u-v})[- \frac{3}{2}(t-u-v)^{-5/2} f(u) du $$

$$+ (t-u-v)^{-3/2} df(u) \right\} dv$$

which we write as

$$I_{14} = \frac{1}{a} [f(0)I_{141} - f(t-2\delta)I_{142} + I_{143} + I_{144}] .$$

Now

$$|I_{141}| \leq A^2 (t-\delta)^{-3/2} \int_0^\delta \sqrt{v} dv = \frac{2}{3} A^2 \delta^{3/2} (t-\delta)^{-3/2}$$

so that $I_{141} \to 0$ as $\delta \to 0$. And $I_{143}$ is just $I_{13}$ so that $I_{143} \to 0$ as $\delta \to 0$.

From the expression for $I_{144}$ we have immediately that

$$|I_{144}| \leq A^2 \delta \int_0^\delta \int_0^{t-2\delta} (t-u-v)^{-3/2} d\nu^u[f] dv .$$

If $\delta < 1$ and $p < 1$, $\eta = \delta^p > \delta$ and
\[
\int_{0}^{t-2\delta} (t-u-v)^{-3/2} d\nu_u^u[f] = \int_{0}^{t-2\eta} (t-u-v)^{-3/2} d\nu_u^u[f] + \\
+ \int_{t-2\eta}^{t-2\delta} (t-u-v)^{-3/2} d\nu_u^u[f]
\]

\[
\leq (2\eta-v)^{-3/2} \nu_o^t[f] + (2\eta-v)^{-3/2} (\nu_o^t - \nu_o^{t-2\eta})[f] \\
+ (2\delta - v)^{-3/2} (\nu_o^{t-2\delta} - \nu_o^t)[f]
\]

where \( t-2\eta \leq \xi \leq t-2\delta \), by the second mean value theorem for Riemann-Stieltjes integrals. Therefore

\[
|I_{144}| \leq \frac{2}{3} A^2 \delta^{3/2} \{(2\eta-v)^{-3/2} \nu_o^t[f] + \\
+ (2\eta-\delta)^{-3/2} (\nu_o^t - \nu_o^{t-2\eta})[f] + \delta^{-3/2} (\nu_o^{t-2\delta} - \nu_o^t)[f]\}
\]

\[
\leq \frac{2}{3} A^2 \delta^{3/2} \left[3(2\eta-\delta)^{-3/2} \nu_o^t[f] + \frac{2}{3} A^2 (\nu_o^{t-2\delta} - \nu_o^t)[f]\right]
\]

Since \( \nu_o^u[f] \) is continuous in \( u \), as observed earlier, we have

\[
|I_{144}| \leq 2A^2 \delta^{3(1-p)/2} (2-\delta^{1-p})^{-3/2} \nu_o^t[f] + o(1)
\]

as \( \delta \to 0 \). Therefore \( I_{144} \to 0 \) as \( \delta \to 0 \).

To complete the proof there remains only to show that

\( I_{142} \to 0 \) as \( \delta \to 0 \). Now
\[ I_{142} = \int_{0}^{\delta} \sqrt{v(2\delta-v)}^{-3/2} \exp(-\frac{b}{v} - \frac{a}{2\delta-v}) \, dv. \]

If \( \text{Re}(b) = \beta > 0 \), then

\[
|I_{142}| \leq A \delta^{-3/2} \exp(-\frac{\beta}{\delta}) \int_{0}^{\delta} \sqrt{v} \, dv
\]

\[ = \frac{2}{3} A \exp(-\frac{\beta}{\delta}). \]

Therefore, \( I_{142} \to 0 \) as \( \delta \to 0 \).

If \( b = i\beta \) and \( a \neq hi\beta \) (h>0), then setting \( v = \frac{2\delta}{w+1} \) we have

\[ I_{142} = \int_{1}^{\infty} (w+1)^{-1} w^{-3/2} \exp[-\frac{(w+1)(aw+i\beta)}{2\delta w}] \, dw \]

\[ = -2\delta \int_{1}^{\infty} \frac{d}{dw} \exp[-\frac{(w+1)(aw+i\beta)}{2\delta w}] \frac{\sqrt{w}}{(w+1)(aw^2-i\beta)} \, dw \]

which by integration by parts is

\[ = \frac{2\delta}{a-i\beta} \exp(-\frac{a+i\beta}{2\delta}) \]

\[ - 2\delta \int_{1}^{\infty} \exp[-\frac{(w+1)(aw+i\beta)}{2\delta w}] \frac{5aw^{5/2} - 5aw^{3/2} + 2i\beta w^{-1/2}}{2(w+1)^2(aw^2-i\beta)^2} \, dw \]

Since \( \exp[-\frac{(w+1)(aw+i\beta)}{2\delta w}] \leq \exp[-\frac{\text{Re}(\delta)}{\delta}] \) and the rational part of the integrand of the last integral is \( O(\delta^{-7/2}) \) as \( w \to \infty \), the last integral exists and hence
\[ |I_{142}| \leq \frac{2\delta}{|a - i\beta|} \exp(-\frac{\text{Re}(a)}{2\delta}) + B\delta \exp(-\frac{\text{Re}(a)}{\delta}) \]

where \( B \) is a constant. Therefore \( I_{142} \to 0 \) as \( \delta \to 0 \).

Finally if \( b = i\beta \) and \( a = ih\beta \) (\( h > 0 \)), then if we set \( v = \frac{2\delta}{w+1} \)

\[ I_{142} = \int_1^\infty (w+1)^{-1/2}w^{-3/2} \exp[-\frac{i\beta}{2\delta} \frac{(w+1)(hw+1)}{w}] \, dw. \]

By the methods of stationary phase [1, pp. 51-56] \( I_{142} = O(\delta) \) as \( \delta \to 0 \) so that \( I_{142} \to 0 \) as \( \delta \to 0 \).

From the above discussion we see that for all values of \( a \) and \( b \) of interest to us \( I_{142} \to 0 \) as \( \delta \to 0 \) and hence our proof is complete.
This appendix is devoted to the evaluation of certain integrals which we have used. The well-known formula from Laplace transformation theory for convolution integrals \( L[h \ast g] = L[h] L[g] \) will allow us to use Laplace transforms to evaluate our integrals since all three integrals are of convolution type. The page numbers and the numbers for the transform pairs refer to Erdélyi A., ed. Tables of integral transformation, vol. 1. McGraw-Hill, New York, 1954.

1) \[
\int_0^t w^{-1/2} (t-w)^{-1/2} \exp\left(\frac{ix}{t-w}\right) dw = \pi \text{Erfc}\left(\sqrt{x} e^{-i\pi/4}\right)
\]

From Erdélyi page 137 (1) we have \( L[w^{-1/2}] = \sqrt{\pi} s^{-1/2} \).

From page 146 (27)

\[
L[w^{-1/2} \exp\left(\frac{ix}{w}\right)] = \sqrt{\pi} s^{-1/2} \exp\left(-e^{-i\pi/4} \sqrt{x} \sqrt{s}\right).
\]

Therefore

\[
\int_0^t ... \, dw = L^{-1}\left\{\pi s^{-1} \exp\left(-e^{-i\pi/4} \sqrt{x} \sqrt{s}\right)\right\}
\]

which by page 245 (3) gives the stated result.

From page 146 (28)

\[
L\left[t^{-3/2} \exp\left(-\frac{a}{4t}\right)\right] = 2\sqrt{\pi} a^{-1/2} e^{-\sqrt{as}}
\]

for \( \text{Re}(a) > 0 \). This is not quite adequate for our next
two integrals. The transform pair remains valid for
Re (α) > 0 , |α| > 0 . If α = ε + iβ , then

\[ L\{t^{-3/2} \exp(- \frac{ε+iβ}{4t})\} = 2\sqrt{π} (ε+iβ)^{-1/2} \exp[(ε+iβ)^{1/2}/\sqrt{5}] \].

Integrating by parts we see clearly that the integral converges uniformly in ε so we can take the limit under the integral. This gives formula (28) page 146. With this we can obtain formulas 2) and 3) below with Re(α) ≥ 0, Re(β) ≥ 0, |α| > 0, |β| > 0.

2) \[ \int_0^t [(t-u)u]^{-3/2} \exp\left(- \frac{b}{t-u} - \frac{a}{u}\right) du \]

\[ = \sqrt{π} \left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}\right) t^{-3/2} \exp\left[- \frac{(\sqrt{a} + \sqrt{b})^2}{t}\right] \]

From page 146 (28)

\[ L\{u^{-3/2} \exp(- \frac{a}{u})\} = \sqrt{π} \ a^{-1/2} e^{-2\sqrt{a}s} \]

and

\[ L\{u^{-3/2} \exp(- \frac{b}{u})\} = \sqrt{π} \ b^{-1/2} e^{-2\sqrt{b}s} \].

Therefore

\[ \int_0^t \cdots du = L^{-1}\{π(ab)^{-1/2} \exp[-2\sqrt{s}(\sqrt{a} + \sqrt{b})]\} \]

and 2) follows again from page 146 (28).