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Title: BOUNDARY LAYER TRANSITION

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Transition phenomena commonly occur in nature. These arise either due to structural or behavioral changes in the medium. Examples for these abound in all applied sciences and to mention a few of these, we have, boundary layer, elastic-plastic deformation, and shocks. The present work is devoted to the study of laminar boundary layer transition. In this case, transition from the region near the surface of the body to the main stream takes place within a thin layer called the boundary layer. Although the basic properties of the fluid remain the same its behavior changes appreciably from the surface of the body to the main stream. Owing to the presence of spin, rotation or vorticity effects, the transition phenomenon is non-linear, irreversible and non-conservative and hence it cannot be treated satisfactorily by superposition or perturbation techniques. In this thesis an attempt is made to study the transition as an asymptotic phenomenon from the boundary layer.

The flow in the presence of any body is divided into two regions, (a) boundary layer, (b) all the region excepting the boundary layer, called the transition region. The classical boundary layer theory due to Prandtl, is based on his main assumptions that (a) in the boundary layer, the viscous and inertial forces are of the same order, (b) the transverse velocity in the case of a flat plate is taken of the same order as that of the transverse coordinate, (c) the variation of pressure in the boundary layer is negligible. On careful examination, it becomes clear that the above assumptions are not quite reasonable. In the present investigation the boundary layer thickness is estimated without making any of these assumptions since the ratio of the viscous to the inertial forces varies continuously from infinity near the boundary to zero at the outer edge of the boundary layer. Also the order of the transverse velocity need not be the same as that of the transverse coordinate and the continuity of pressure across the boundary layer comes out from the transition analysis and therefore it is not necessary to assume it.

By making an order of magnitude analysis, the boundary layer thickness for two dimensional flow is estimated in terms of two parameters. One of these parameters depends upon the relative order of magnitude of the viscous and inertia forces at the outer edge of the boundary layer and the other depends upon the order of vorticity allowable at the outer edge of the boundary layer.

The transition phenomenon in boundary flow is treated as an asymptotic phenomenon from the boundary layer. In order to study the transition region, a limiting form of the Navier-Stokes equations in three dimensions is obtained, which is called the transition equation. Owing to the importance of vorticity in the transition region, the transition equation is solved for the vorticity. The form of vorticity shows that in general the functions which govern the transition region are either subharmonic or superharmonic functions.

In classical two dimensional flow the study of cylindrical vortex is made by employing matching techniques. There does not exist any mathematical treatment of the spiral formation which exists in case of flow past a body at large Reynolds number. In the present thesis a study of two dimensional flow past a body at large Reynolds number is undertaken on the basis of transition analysis, thus obtaining a satisfactory mathematical treatment of various phenomena that occur in the boundary layer flow. The transition equations for axisymmetric and two dimensional flow are also obtained. Besides other known results, transition equation in two dimensions gives the stagnation points and the formation of spirals which is noticed in the flow of a real fluid past any body at large Reynolds number. Transition equation also gives the formation of cylindrical vortices. These vortices are given by the limiting form of the stream function and come out from the transition equation itself without the use of any matching process as is done in current literature. Hence it can be concluded that the transition equation is a global representation of different phenomena which exist in fluid flow past a body at large Reynolds number.

The transition concept is also extended to magnetohydrodynamics. A formula for the magnetohydrodynamic boundary layer thickness is obtained in terms of two parameters on the basis of a magnitude analysis. The transition equation for two dimensional magnetohydrodynamic case is also obtained, and its solution gives the spiral formations.

Boundary Layer Transition

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DEFINITIONS

In order to avoid the repetition of definitions of some of the important terms in the thesis, we find it convenient to give the definitions here for ready reference.

Reynolds Number

The term Reynolds number is used for the quantity $\frac{V\ell}{\nu}$, where

V = main stream velocity

ℓ = characteristic length

ν = kinematic viscosity.

On the basis of dimensional analysis Reynolds number is the ratio of inertia force to viscous forces. It is the characteristic of the particular flow under consideration and it is a non-dimensional number independent of space coordinates.

Vorticity

If $\vec{V}$ is the velocity vector at any point (x, y, z) then $\vec{\omega} = \text{curl } \vec{V}$ is called the vorticity vector or simply vorticity. The angular velocity of an infinitesimal element is equal to half the velocity.

Boundary Layer Thickness

It is the thickness of the layer in the immediate neighborhood of the body, which is characterized by the following two properties:

1. Relative order of magnitude of viscous and inertia forces at the outer edge of the boundary layer.
2. Order of vorticity allowable at the outer edge of the boundary layers.

BOUNDARY LAYER TRANSITION

I. INTRODUCTION

1.1 Preliminary Remarks

In the presence of a solid body at large Reynolds number the flow field exhibits a variety of distinct phenomena, which apparently seem to be unrelated. These phenomena include, boundary layer, shock wave, vortices, spiral formation in wake and stagnation points. Some of these phenomena have been studied mathematically as entities in themselves. When a particular phenomenon is singled out for study, it is done for mathematical simplicity. The splitting of the field into a number of linear problems is hypothetical so that each phenomenon can be studied by some mathematical technique. It seems possible that all these phenomena which exist in different parts of the flow field are part and parcel of one and the same field. As such it should be possible to give a global representation to these phenomena. Seth's (1962-1966) theory of transition phenomena is an attempt to bridge the gulf between seemingly unrelated phenomena. Later on it will be shown that most of the above mentioned phenomena for flow past a body at large Reynolds number are described by the transition equation and can be represented by subharmonic (superharmonic) functions.

1.2 Transition Phenomena

In general transition phenomena can be explained as follows: When a medium is subjected to internal and external stresses and body forces, then a stage comes when the medium yields and two states are obtained which are dovetailing into each other. If these successive states be denoted by A and B and the transition or the region of dovetailing by T, then A passes into B through T. The region T is usually referred to as transition region.

An attempt has been made by Seth (1962-1966) to study the transition which arises from the changes in the properties of a medium, for example, elastic-plastic deformation, creep, relaxation etc. If the initial state is defined by a set of field equations, it should be possible to identify the transition to the next neighboring state with the transition (critical) points of the differential system involved. An asymptotic solution at these points should give the results without assuming semi-empirical laws or ad hoc conditions which are otherwise found necessary to treat the transition. Seth has solved a number of problems in elastic-plastic deformations by using transition theory, and obtained a satisfactory scientific basis for explaining a number of irreversible phenomena in continuum mechanics. His results can be summarized as follows:

1. The transition points correspond either to infinite

contraction or infinite extension of line elements in the deformed state.

2. The asymptotic solutions at the transition points show that there exists a transition state, and the solutions for fully plastic state can be obtained by a limiting process. Thus the yield condition need not be assumed, it comes out of the field equations.

3. The transition stresses are different in tension and compression regions of the material. This idea has been carried further by Purushotama (1965) and Hulsurhar (1967). Purushotham has shown that plastic yielding can be identified with the degeneracy of the strain ellipsoids.

This treatment makes the assumption of yield conditions like that of von-Mises and Tresca, creep strain laws like that of Norton, Odqvist and Andrade and jump conditions for shock both unnecessary and redundant. If they exist they should come out of the field equations themselves.

In particular for fluid dynamics Lamb (1932, p. 684) has mentioned the following about transition region.

> ... the slightest observation is enough to show that the transition from the velocity of the surface to that of the fluid abreast of it is often affected within a very short space. In fact when a solid of fair easy shape, such as a sphere, or a cylinder or an aerofoil, moves through a mobile fluid... vorticity appears to be confined almost to a narrow band along the anterior portion of the surface

and to the wake. It is to the study, both dynamical and experimental, of this transition region, that the efforts of many investigators have for some time been directed.

Apparently the situation in the case of transition of flow from boundary layer to main stream seems to be different from that of elastic-plastic transition. But it can be visualized that basically a similar type of phenomenon occurs even in the boundary layer transition. Although the fluid remains the same, its intrinsic behavior changes appreciably from boundary layer to the main stream, where flow can be approximated by a non-viscous flow. In the case of fluid flow, the transition region is the entire region, excepting the boundary layer region. It will be shown later on that the transition equation, which governs the transition region, contains a host of phenomena, which are observed in different parts of the flow field.

In order to illustrate the treatment of transition phenomena, shock wave transition (Seth, 1964c) is discussed here. In the case of one dimensional steady viscous compressible flow, the equations of motion are

$$\rho u \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x} + \frac{4}{3} \frac{\partial}{\partial x} (\mu \frac{\partial u}{\partial x}), \qquad (1.2.1)$$

$$\rho u = m = \text{constant}. \qquad (1.2.2)$$

Assuming $p = f(\rho)$ and putting $P = \frac{\partial u}{\partial x}$, we get

$$m\frac{du}{dx} = -\frac{dp}{d\rho}\frac{d\rho}{dx} + \frac{4}{3}\frac{d}{dx}(\mu\frac{du}{dx})$$

$$= \frac{c^2 m}{u^2}\frac{du}{dx} + \frac{4}{3}\frac{d}{du}(\mu\frac{du}{dx})\frac{du}{dx},$$

that is,

$$mP = \frac{c^2 m}{u^2}P + \frac{4}{3}P\frac{d}{du}(\mu P),$$

or

$$mP(1-\frac{c^2}{u^2}) = \frac{4}{3}P\frac{d}{du}(\mu P),$$

where $c^2 = \frac{dp}{d\rho}$ = the local sound velocity.

Since $P \neq 0$, as otherwise $u = 0$. We have, if μ is taken as constant,

$$\frac{dP}{du} = \frac{3m}{4\mu}(1-\frac{c^2}{u^2}) = \frac{3m}{4\mu}(1-\frac{1}{M^2}), \tag{1. 2. 3}$$

or

$$\frac{dP_1}{du_1} = \frac{3R}{4}(1-\frac{1}{M^2}), \tag{1. 2. 4}$$

P_1, u_1 being the non-dimensional values of P and u respectively, R is the Reynolds number and M the Mach number. The three critical points, which can be interpreted as the transition points, are

1. $R \to \infty$, which corresponds to the boundary layer,

2. $R \to 0$, gives Stokes slow viscous motion of highly viscous fluids,

3. $M \to \pm 1$, which indicates the transition from subsonic to supersonic. It is significant that μ has not been taken to be zero and no particular specifying condition in the form $p = f(\rho)$ has been taken. Over the shock

$$M \to 1 ,$$

$$\frac{u^2}{c^2} \to 1 \Rightarrow \frac{m^2}{\rho^2} \to \frac{dp}{d\rho} ,$$

or

$$p \sim k - \frac{m^2}{\rho} ,$$

a relation which can be expected to be true for weak shock where the entropy change can be neglected.

From Equation (1. 2. 3) it is clear that if $\mu \to 0$ and $M \to \pm 1$ simultaneously $\frac{dP}{du}$ becomes indeterminate. This shown that viscosity is significant on the shock transition and should not be taken equal to zero. Again over the transition, $M \to \pm 1$, therefore

$$\frac{dP}{du} \to 0.$$

From Equation (1. 2. 1), we get

$$m\frac{du}{dx} = -\frac{dp}{dx},$$

or

$$m(u_2 - u_1) = p_1 - p_2,$$

which is a momentum equation. This has been obtained without taking $\mu = 0$. Thus we see that shock transition becomes obvious in the state plane of (P, U). This plane may not be the same in all cases. Since all continuous transformations are topological, the transition should come out explicitly in some state plane.

1.3 Prandtl's Boundary Layer Equations

At the turn of the nineteenth century three major areas of interest in fluid dynamics attracted the attention of many prominent research workers. These areas are gas dynamics, boundary layer flows and turbulence. Basically the problems involved in these three areas are similar, that is, the transition. In the first case, the transition is shock wave, in the other case from the boundary layer to the main flow and in the third case from laminar to turbulent flow. One of the major questions to be answered in the flow of real fluids in the presence of a solid boundary was to account for the drag experienced by the body. The classical non-viscous (Euler's) equations of motion could not account for the drag suffered by the body (Landau and

Lifshitz, 1966, p. 34). This is well known as D'Alembert's paradox. Prandtl in an effort to answer this problem suggested that flow near the boundary, where frictional forces dominate, is different from the flow away from the boundary where the flow can be approximated to non-viscous flow. Owing to viscous forces the velocity of the fluid at the boundary is zero and attains the value of the main stream velocity over a thin region. Thus the transition of flow from zero velocity at the wall to its full magnitude at some distance from it takes place in a very thin layer, the so called boundary layer. In this manner it was thought of that there are two regions to consider, even though the division between them is not very sharp.

1. A very thin layer in the immediate neighborhood of the body in which the velocity gradient normal to the wall $\frac{\partial u}{\partial y}$, is very large (boundary layer). In this region, however small the viscosity of the fluid may be, it exerts an essential influence in so far as the shearing stress $\tau = \mu \frac{\partial u}{\partial y}$ may assume very significant value.

2. In the remaining region no such large velocity gradients occur and the influence of viscosity is unimportant. In this region the flow may be regarded as frictionless and potential.

Boundary layer flow has been studied extensively by Prandtl (Schlichting, 1968). He illustrated his point of view by considering in detail the motion of an incompressible viscous fluid along a semi-infinite plate. Taking the origin to coincide with the leading edge of the plate and the x-axis along its length the Navier-Stokes equations in non-dimensional form are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1.3.1}$$

$$1 \qquad 1.\,1 \qquad \delta\,\frac{1}{\delta} \qquad\qquad\qquad 1 \qquad \frac{1}{\delta^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{1.3.2}$$

$$\delta \qquad 1\;\delta \qquad \delta\;1 \qquad\qquad\qquad \delta \qquad \frac{1}{\delta}$$

where $R = \frac{U\ell}{\nu}$ is the Reynolds number.

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.3.3}$$

In order to derive the boundary layer equation, Prandtl introduced certain approximations in the Navier-Stokes equations by carrying out an analysis of the order of magnitude of different terms. If δ is the boundary layer thickness then the orders of magnitude of different quantities are taken as

$$x \sim 1, \quad u \sim 1, \quad \frac{\partial}{\partial x} \sim 1$$

$$y \sim \delta, \quad v \sim \delta, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}$$

The orders of magnitude of different terms in Equations (1. 3. 1) and (1. 3. 2) are indicated below the respective terms. From the order consideration it is found that Equation (1. 3. 1) is more important than Equation (1. 3. 2) and the term

$$\frac{\partial^2 u}{\partial y^2} \gg \frac{\partial^2 u}{\partial x^2} .$$

Therefore Equation (1. 3. 2) may be neglected in comparison with Equation (1. 3. 1) and $\frac{\partial^2 u}{\partial x^2}$ is neglected in comparison with $\frac{\partial^2 u}{\partial y^2}$.
Thus Prandtl's boundary layer equations in non-dimensional form are

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{1}{R} \frac{\partial^2 u}{\partial y^2} , \tag{1. 3. 4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0. \tag{1. 3. 5}$$

Prandtl's consideration that viscous and inertia terms are of the same order of magnitude in the boundary layer, it follows that $\delta \sim R^{-1/2}$. From Equation (1. 3. 2) it is then clear that $\frac{\partial p}{\partial y} \sim \delta$. That is, the variation of pressure along the normal to the wall within the boundary layer may be neglected in considering boundary layer phenomena. Equations (1. 3. 4) and (1. 3. 5) are then the equations to be solved for the three unknown u, v and p. Thus mathematically speaking an indeterminacy enters into the problem. This is, however, overcome by taking the pressure distribution inside the boundary layer the same as at its outer edge, where the flow is supposed to be non-viscous.

Since very near the boundary the viscous effects are predominant and away from the boundary inertia terms are important, Prandtl made the further assumption that inside the boundary layer the viscous and inertia terms are of the same order of magnitude. This assumption has led to the semi-empirical formula for the thickness of the boundary layer as proportional to $R^{-1/2}$. This assumption, as we shall see later on, is not quite reasonable.

The mathematical indeterminacy caused by the Prandtl's order considerations did not do any appreciable harm to the theory because of the vast practical applicability of the theory and its close agreement with observed phenomena. However, the characterization of the boundary layer as an asymptotic phenomenon went a long way in establishing confidence in this theory. As Friedrichs (1955) points out, the approach to the boundary layer by characterizing it as an asymptotic phenomenon leads to a definite clarification of the issue but it does not yield a rigorous justifaication of this theory. Some of the main defects of the classical boundary layer theory are:

1. The truncated boundary layer equations provide a solution holding good only in the thin region surrounding the boundary. The smooth transition into the invisid flow away from the boundary is not accomplished.
2. The number of equations fall short of number of unknowns and this indeed is a mathematical indeterminacy. The borrowing of the pressure distribution in the boundary layer from the inviscid flow theory or from experiment, is another hypothesis which has yielded some results.
3. The assumption that inside the boundary layer the viscous and inertia forces are of the same order is not quite reasonable [Prandtl and Tietjen, p. 61]. In fact the viscous terms as compared to inertia terms are predominant near the boundary but are very small not very far from the boundary. Thus, however thin the boundary layer region may be, the ratio of the viscous to inertia terms is not uniformly equal to unity inside the boundary layer, as assumed by Prandtl, but it is a rapidly though continuously decreasing function. In fact the ratio of viscous to inertia terms drops from an infinite value at the boundary to a very small quantity much inside the boundary layer, and becomes vanishingly small at the outer edge of the layer.

The first application of Prandtl's boundary layer theory was made by Blasius (1908) in discussing the boundary layer along a flat plate. The difficulty inherited by taking the simplified equations of the boundary layer is evident in the solution obtained by Blasius. Firstly, the transverse component of velocity does not vanish at infinity and secondly the solutions are not true at the stagnation point.

1.4 Boundary Layer Thickness

As pointed out in the last section, the consideration that viscous and inertia forces are of the same order of magnitude in the boundary layer has yielded the boundary layer thickness of order $R^{-1/2}$. Proudman (1956) analyzed the flow between two rotating spheres and speculated that the order of the boundary layer thickness lies between $R^{-1/3}$ and $R^{-1/4}$. The difference between these two results is evident, because Prandtl's result does not take into account the vorticity which exists in the main stream where as Proudman's does. From these considerations it is clear that boundary layer thickness should depend upon two considerations:

1. The relative order of viscous and inertia forces at the outer edge of the boundary layer.
2. The order of vorticity allowable at the outer edge of the boundary layer.

Later on it will be shown in Chapter II, that in general, boundary layer thickness is given by

$$R^{-\frac{\alpha}{1+\beta}} \tag{1.4.1}$$

where

$$0 < \alpha < 1 \quad \text{and} \quad \beta > 1.$$

The parameter α depends upon the relative order of magnitude of the viscous and inertia forces in the neighborhood of boundary layer and the parameter β depends upon the vorticity allowable at the outer edge of the boundary layer. Prandtl's boundary layer thickness can be seen to be a limiting case of the above result when

$$\alpha \to 1,$$

and

$$\beta \to 1.$$

In the next article we shall discuss briefly the mathematical treatment of the occurrence of different types of vortices such as cylindrical, spherical and corner eddies as exist in literature.

1.5 Cylindrical Vortex, Spherical Vortex, Corner Eddies

The formation of various types of spirals or vortices during the flow in the presence of solid boundaries is a well known phenomenon. At large Reynolds number, due to boundary layer separation, these vortices move down stream forming the well known Kármán Vortex street. In the current literature these vortices are studied as isolated phenomena. It will be seen later on, in this thesis, that these spiral formations are actually included in the solution of the transition equation. Before discussing the transition concept it will be worthwhile to go through the existing treatment of these phenomena.

a. Cylindrical Vortex

If ψ, is the Lagrangian stream function, then the equation governing steady two dimensional flow of an inviscid fluid is

$$\nabla^2\psi = f(\psi).$$

In particular choosing $f(\psi) = k^2\psi$, we get

$$\nabla^2\psi = k^2\psi. \tag{1.5.1}$$

The solution of Equation (1.5.1) is

$$\psi = cJ_s(kr) \begin{matrix}\cos\\ \sin\end{matrix}\Big\} s\theta,$$

where J_s is the Bessel function of the first kind and of order s. Choosing $r = a$ as a fixed boundary, the possible values of k are given by

$$J_s(ka) = 0.$$

Suppose

$$\psi = cJ_1(kr) \sin \theta, \tag{1.5.2}$$

inside $r = a$ and outside (as usual for cylinder moving with velocity

$\vec{U}$)

$$\psi = U(r - \frac{a^2}{r}) \sin\theta. \qquad (1.5.3)$$

The values of ψ as given by Equations (1. 5. 2) and (1. 5. 3) will agree on $r = a$ if

$$J_1(ka) = 0.$$

Also the condition for continuity of tangential velocity gives

$$c = -\frac{2U}{kJ_0(ka)},$$

Hence if we impress on the system a velocity U opposite to the direction of the main stream, we get a host of cylindrical vortices travelling with velocity U, through a liquid which is at rest at infinity. The stream lines inside the vortices as shown by Lamb (1932, p. 288) are given in Figure 1.

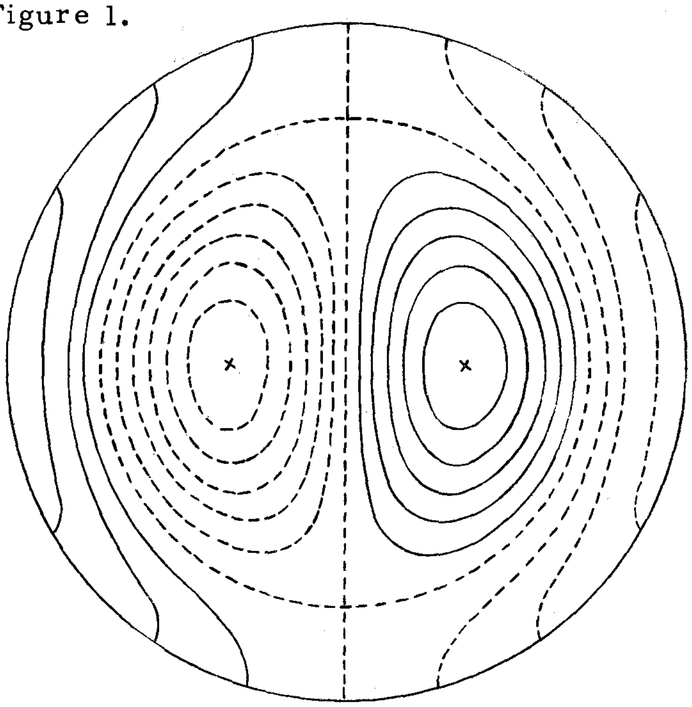

Figure 1. Stream lines inside a vortex.

b. Spherical Vortex

Hill's Spherical vortices are studies by considering the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial w^2} - \frac{1}{w}\frac{\partial\psi}{\partial w} = w^2 f(\psi), \tag{1.5.4}$$

where ψ is the Stokes stream function given by,

$$u = -\frac{1}{w}\frac{\partial\psi}{\partial w}, \qquad v = \frac{1}{w}\frac{\partial\psi}{\partial x},$$

and $w = (y^2+z^2)^{1/2}$ is the distance of any point from the axis of symmetry. In order to study formation of the spherical vortex two forms of stream function ψ are assumed as

$$\psi = \frac{1}{2} Aw^2(a^2 - r^2),$$

for points inside the sphere $r = a$, where $r^2 = x^2 + w^2$ and

$$\psi = \frac{1}{2} Uw^2(1 - \frac{a^3}{r^3}),$$

out side the sphere.

The two values of ψ agree when $r = a$. The condition for the continuity of tangential velocity gives

$$A = -\frac{3}{2}Ua^2.$$

So if we impress on the sytem a velocity U opposite the direction of main stream, we get a spherical vortex advancing with constant velocity U through the liquid, which is at rest at infinity.

c. Corner Eddies

It is well known that for a viscous fluid flow in a channel or pipe with an abrupt contraction, eddies occur at the corners immediately preceding that contraction. Formation of corner eddies has been discussed by Yih (1959) in the case of steady, rotational flow of an inviscid fluid in a two dimensional channel. The equation governing steady two dimensional flow of inviscid fluid is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = f(\psi). \tag{1.5.6}$$

The velocity distribution for upstream from the sink should be parabolic if it is the same as that for the laminar flow of a viscous fluid in a long channel. However, a parabolic distribution of the upstream velocity would make Equation (1. 5. 6) non-linear, but a cosine distribution, which is nearly parabolic makes Equation (1. 5. 6) linear. This choice of velocity distribution, boils down to the same thing as taking

$$f(\psi) = -\frac{1}{4}\pi^2\psi \tag{1.5.7}$$

With this value of $f(\psi)$, the solution of Equation (1.5.6) is

$$\psi = \left(\frac{2}{\pi}\right) \sin \frac{\pi}{2} y + \sum_{1}^{\infty} c_n \sin n\pi y \exp \{(n^2 - \frac{1}{4})^{1/2} \pi x\} ,$$

where

$$c_n = \frac{4}{n\pi^2} \left(1 + \frac{\cos n\pi}{4n^2 - 1}\right) .$$

The flow pattern for half of the channel is shown in Figure 2, in which the corner eddies appear.

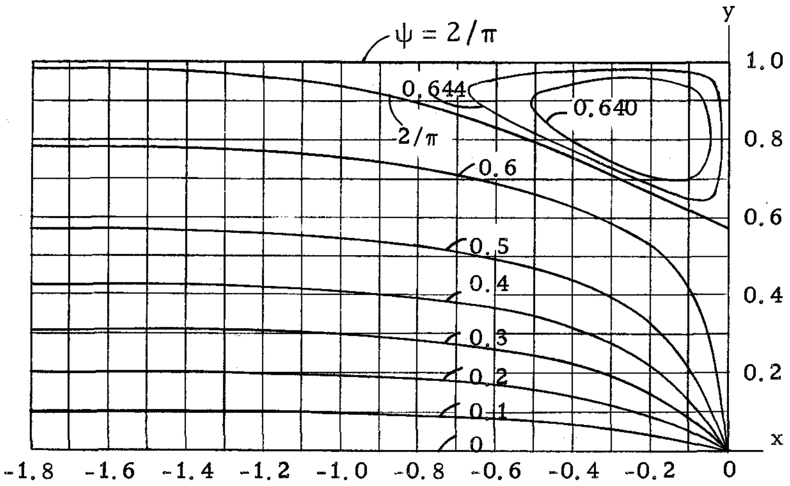

Figure 2. Corner eddies (Yih, 1959).

It has been pointed out by Yih (1959), that since the flow in the eddies does not originate at infinity there is no a priori reason why Equations (1.5.6) and (1.5.7) should govern the flow in the eddies. But from the analysis developed in Chapter IV for two dimensional flow it will be clear that Equation (1.5.6) is the transition equation. This equation is true at all points of the region excepting the boundary. In fact $\nabla^2\psi = f(\psi)$ is the global representation of different phenomena which occur in the flow past a body at large Reynolds number R. As such, Equation (1. 5. 6) should give not only eddies but other phenomena too.

Recently some attempts have also been made to study the numerical solution of Karman Vortex Street for flow past a rectangle (Francis and Jacob, 1965).

In the existing literature, each of the phenomena discussed above is treated by itself independent of others. These phenomena have not been studied as a continuous transition from the boundary layer. In Figures 5 and 6 it is clear that the boundary of the vortex is formed by the limiting form of the stream lines, which are obstructed by the presence of the body. It should be possible to study them as a limiting form of the flow field without assuming two stream functions and then matching them on the boundary.

It will be shown later on that the transition equation includes these phenomena. In particular it will be shown that the transition equation gives the formation of spirals, whose mathematical treatment does not exist in literature. Moreover, the transition equation will also include the main flow and the stagnation points. These transition fields will be shown to be subharmonic or superharmonic fields.

1.6 Objective of Present Study

A mathematical treatment of cylindrical vortex, spherical vortex and corner eddies exist in literature. Recently (Francis and Jacob, 1965) some attempts have also been made to study the numerical solution to the problem of the Kármán Vortex Street for flow past a rectangle. Formation of spirals in wake is physically a well understood phenomena (in two dimensions), but lacks a rigorous mathematical treatment. An attempt shall be made to give an analytical representation to these phenomena, which exist in flow past a body at large Reynolds number in terms of subharmonic (superharmonic) fields. In particular it will be shown that this representation includes formation of spirals and stagnation points, which has not been treated in literature.

It will also be shown that the result on cylindrical vortex, in the case of two dimensional flow of a non-viscous fluid is only a particular case of the transition equation, when the Reynolds number becomes large. This will also indicate that in general, spirals, vortices and the wake can exist in a flow of a real fluid.

A general solution of transition equation in three dimensions will be obtained in Chapter III, which will indicate that transition fields are subharmonic or superharmonic fields. The three dimensional transition equation will show why it is so difficult to deal with three dimensional boundary layer. The Beltrami flow in which vorticity lines are parallel to stream lines is found to be again a particular case of the transition equation.

The result of Proudman on boundary layer thickness for rotating spheres shows that a lacuna exists in Prandtl's boundary layer theory in which vortex motion in the main flow has not been taken into consideration. The general treatment of boundary layer thickness will include Prandtl's result as a limiting case, Proudman's result and those of turbulent flow as particular cases.

II. BOUNDARY LAYER THICKNESS

2.1 Preliminary Remarks

The starting point of boundary layer theory was to resolve the D'Alembert paradox of late 19th century. D'Alembert observed that when a solid body moved through a fluid flow pattern based on the inviscid theory agreed with the experimental results almost everywhere in the flow, but strangely enough the resistance experienced by the body was found to be zero. Prandtl (1904) in an attempt to resolve this dilemma suggested that the resistance to the body was caused by the viscosity of the fluid and the fluid flow near and away from the body were different in character. This could have been suspected since:

a) An inviscid fluid can slip along the wall, while the viscous fluid sticks to it.

b) Shearing stresses are ignored in a perfect fluid, while they vitally affect the motion of viscous flow.

Prandtl analyzed these fundamental differences and the behavior of inviscid and viscous fluids and suggested that the entire flow phenomena could be studied in two regions, one a very thin region near the surface of the body called bounday layer and the other away from the body where the inviscid fluid theory gives results with sufficient accuracy. In this manner there are two regions to consider, even though the division between them is not very sharp.

a) A thin layer in the immediate neighborhood of the body in which the velocity gradient normal to the wall, $\frac{\partial u}{\partial y}$, is very large. In this region the very small value of viscosity, μ, of the fluid exerts an essential influence in so far as the shearing stress $\tau = \mu \frac{\partial u}{\partial y}$, may assume large values.

b) In the remaining region, no such large velocity gradients occur and the influence of viscosity is unimportant. In this region, the flow is almost inviscid.

2. 2 Prandtl's Boundary Layer Theory

Prandlt illustrated his point of view by considering in detail the motion of incompressible viscous fluid along a semi-infinite plate. Taking the origin to coincide with the leading edge of the plate and x-axis along its length, the Navier-Stokes equations are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \qquad (2.\ 2.\ 1)$$

and

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right). \qquad (2.\ 2.\ 2)$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \qquad (2.\ 2.\ 3)$$

Equations (2. 2. 1) to (2. 2. 3) are made dimensionless by referring all velocities to the free stream velocity, V, and all linear dimensions to a characteristic length, L, of the body, the pressure by ρV^2 and time by L/V. Under these assumptions and retaining the same symbols for the dimensionless quantities as for their dimensional counterpart, the Equations (2. 2. 1) to (2. 2. 3) take the form,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2. 2. 4}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{2. 2. 5}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2. 2. 6}$$

where

$$R = \frac{VL}{\nu}.$$

In order to study viscous effects attention must be focussed in the immediate vicinity of the plate. If the dimensionless boundary layer thickness is δ/L, for which the symbol δ is retained, then the different quantities involved in the Equations (2. 2. 4) to (2. 2. 6) have the following estimates for order of magnitudes.

$$x \sim 1, \quad u \sim 1, \quad \frac{\partial}{\partial x} \sim 1,$$

$$y \sim \delta, \quad v \sim \delta, \quad \frac{\partial}{\partial y} \sim \frac{1}{\delta}.$$

An estimate of order of magnitude of different terms in Equations (2. 2. 4) and (2. 2. 5) is made in order to neglect some terms in comparison with others and to achieve a simplification of the equations of motion. Orders of different terms are shown below each term in the Equations (2. 2. 4) and (2. 2. 5). Order of magnitude of different terms in Equation (2. 2. 5) is much smaller than Equation (2. 2. 4). Therefore Equation (2. 2. 5) is neglected in comparison with Equation (2. 2. 4). Again in Equation (2. 2. 4) the term $\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$. Therefore $\frac{\partial^2 u}{\partial x^2}$ is neglected in comparison with $\frac{\partial^2 u}{\partial y^2}$. Therefore the simplified boundary layer equations are

$$\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\,\frac{\partial p}{\partial x} + \frac{1}{R}\,\frac{\partial^2 u}{\partial y^2}\,, \tag{2. 2. 7}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{2. 2. 8}$$

Furthermore since very near the boundary the viscous effects are predominant and away from the boundary inertia terms are more important, Prandtl made the further assumption that inside the boundary layer the viscous and inertia terms are of the same order of magnitude. This assumption led to the semi-empirical formula for the thickness of the boundary layer as proportional to $R^{-1/2}$.

Equations (2. 2. 7) and (2. 2. 8) are called Prandtl's boundary layer equations. These equations are to be solved for three unknowns u, v and p. Thus mathematically speaking an indeterminacy enters into the problem. This is however overcome by taking the pressure distribution inside the boundary layer the same as at its outer edge, where the inviscid flow phenomenon is supposed to hold.

2.3 Limitations of Prandtl's Boundary Layer Theory

The following are the limitations of the theory:

a) The order of the boundary layer equations is less than the order of the complete Navier-Stokes equations and hence one boundary condition has to be relaxed.

b) The number of boundary layer equations becomes less than the number of unknown functions and hence the pressure distribution inside the boundary layer is taken from the non-viscous fluid theory or from experiments.

c) The assumption that viscous and inertia forces are of the same order of magnitude in the boundary layer fixes the order of the boundary layer thickness as $R^{-1/2}$. This assumption does not seem to be justified, because the inertia forces are zero on the surface of the body and become very large on the outer edge of the boundary layer. Therefore, even though the boundary layer is a very thin region the ratio of the viscous to inertia forces change from a very large value near the outer edge of the boundary layer and does not remain uniformly equal to unity across the thickness of the boundary layer.

d) The transverse velocity component v and the transverse coordinate y, have both been assumed in the boundary layer to be of order δ which is not quite reasonable.

2.4 Formula for Boundary Layer Thickness

In the present investigation, boundary layer thickness is determined by making use of the complete Navier-Stokes equations, without making use of the Prandtl's boundary layer equations, which are a truncated form of Navier-Stokes equations. According to Prandtl, the boundary layer thickness is proportional to $R^{-1/2}$ and Proudman (1956) while analyzing the flow between two rotating spheres speculated that the order of the boundary layer thickness lies between $R^{-1/3}$ and $R^{-1/4}$. This shows that there is a definite need to examine the concept of boundary layer thickness.

The edge of the boundary layer is an arbitrary line in the fluid such that viscous effects (V) which are predominant near the boundary die out rapidly as we proceed away from it. For flow past a fixed obstacle large vorticity is present very near the boundary but vanishes asymptotically away from it. The formula for the thickness of the boundary layer should depend on what order of vorticity is regarded as negligible outside the layer and not merely on the Reynolds number.

It should also depend on the relative order of magnitude of the viscous and inertia forces. As pointed out earlier that the ratio $\frac{V}{I} = \frac{\text{inertia forces}}{\text{viscous forces}}$ changes from a very large value near the surface, to a very small value outside the boundary layer. By taking this ratio as one, the boundary layer thickness so obtained is much less than what the actual boundary layer thickness should be.

As observed before in Prandtl's boundary layer theory, the transverse velocity v and the transverse coordinate y are taken to be of the same order of magnitude in the boundary layer. These are small quantities but they need not be of the same order of magnitude. Therefore we start by assuming the relative order of magnitude of different quantities involved at the outer edge of the boundary layer.

Let δ be the boundary layer thickness, then

$$y \sim \delta \quad \text{and} \quad 0 < \delta << 1$$

Since the y-component of velocity, v, is much smaller than y, let

$$v \sim \delta^{\frac{1}{k}}, \quad 0 < k < 1 .$$

Also, let

$$u \sim \delta^{m}, \quad 0 < m < 1 .$$

From the equation of continuity (2.2.3), it follows that

$$x \sim \delta^{1+m-\frac{1}{k}}$$

These are the orders of magnitude of different terms in the neighborhood of the outer edge of the boundary layer, because δ, very near the plate has no meaning. Orders of different terms in the Navier-Stokes equation are as follows:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.4.1}$$

$$\delta^{\frac{1}{k}-1+m} \qquad \delta^{\frac{1}{k}+m-1} \qquad\qquad \delta^{\frac{2}{k}-2-m} \qquad \delta^{m-2}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.4.2}$$

$$\delta^{\frac{2}{k}-1} \qquad \delta^{\frac{2}{k}-1} \qquad\qquad \delta^{\frac{3}{k}-2-2m} \qquad \delta^{\frac{1}{k}-2}$$

Each term on the left hand side of Equation (2.4.1) is of order $\delta^{m+\frac{1}{k}-1}$ and terms on the right hand side are of the orders $\frac{1}{R}\delta^{\frac{2}{k}-2-m}$ and $\frac{1}{R}\delta^{m-2}$ respectively. Obviously

$$\frac{1}{R}\delta^{m-2} \gg \frac{1}{R}\delta^{\frac{2}{k}-2-m} .$$

We have to calculate δ, in the neighborhood of the outer edge of the boundary layer, because very near the surface δ has no meaning. At the outer edge of the boundary layer, inertia forces (I) are very large compared to viscous forces (V). That is

$$I \gg V. \tag{2.4.3}$$

It may be mentioned here that Equation (2. 4. 3) is not an assumption, but it arises from physical considerations.

Using Equation (2. 4. 1), the inequality (2. 4. 3) becomes

$$\delta^m \ \delta^{\frac{1}{k}-1} \gg \frac{1}{R}\,\delta^{m-2} \tag{2.4.4}$$

Since $\delta^m > 0$, therefore dividing throughout by δ^m we have from Equation (2. 4. 4)

$$\delta^{\frac{1}{k}-1} \gg \frac{1}{R}\,\delta^{-2}.$$

That is

$$R\delta^{\frac{1}{k}+1} \gg 1 \tag{2.4.5}$$

In view of the inequality (2. 4. 5), the latter can also be expressed as

$$R\delta^{\frac{1}{k}+1} \sim R^p \tag{2.4.6}$$

where $0 < p < 1$. The choice of p is restricted as in $0 < p < 1$, for if, $p > 1$, then Equation (2. 4. 6) gives

$$\delta^{\frac{1}{k}+1} \sim R^{p-1}$$

This would make δ large, which is contrary to the actual physical situation. It is also clear that the choice of p depends upon the relative order of magnitude of viscous and inertia forces. Therefore Equation (2. 4. 6) gives,

$$\delta^{\frac{1}{k}+1} \sim R^{p-1}$$

or

$$\delta \sim R^{-\alpha/(1+\beta)}, \qquad (2.\,4.\,7)$$

where

$$\alpha = 1 - p < 1,$$

and

$$\beta = \frac{1}{k} > 1.$$

The result given by Equation (2. 4. 7) is obtained by considering the Equation (2. 4. 1). Now from Equation (2. 4. 2), the order of magnitude of inertia terms (I) is $\delta^{\frac{2}{k}-1}$ and order of viscous terms (V) is $\frac{1}{R}\delta^{\frac{1}{k}-2}$. As in inequality (2. 4. 3), we again have $R\delta^{1+\frac{1}{k}} \gg 1$, which is the same as given by the inequality (2. 4. 5). Therefore this inequality will also give the same result as given by the expression (2. 4. 7).

Thus Equation (2. 4. 7) gives the boundary layer thickness in terms of two parameters, α, and, β. From the above arguments it is clear that the parameter α depends upon the relative order of magnitude of the inertia and viscous forces in the neighborhood of the outer edge of the boundary layer. In order to understand the physical meaning of β consider the uniform flow in the direction of x-axis and velocity U. The stream function ψ in this case will have the form

$$\psi = Uy(1-e^{-ny}).$$

For this choice of ψ the vorticity ζ is given by

$$\zeta \sim ne^{-ny}.$$

At the outer edge of the boundary layer, we have

$$y \sim \delta$$

and

$$\delta \sim R^{-\alpha/1+\beta}$$

Therefore

$$\zeta \sim ne^{-nR^{-\alpha/1+\beta}} \tag{2.4.8}$$

As discussed above the parameter α is known once the relative order of magnitude of viscous and inertia forces is fixed at the outer edge of the boundary layer. Knowing the value of the parameter α the Equation (2. 4. 8) then determines the value of the parameter β by fixing the order of vorticity ζ allowable at the outer edge of the boundary layer.

So we conclude that Equation (2. 4. 7) gives a formula for the boundary layer thickness in the case of viscous motion past a solid body at large Reynolds number, R. This formula contains two parameters, α and β. The choice of the parameter α depends upon the relative order of magnitude of viscous and inertia forces allowable at the outer edge of the boundary layer and the parameter β depends upon the order of vorticity allowable at the outer edge of the boundary layer.

It may be noticed here that the formula (2. 4. 7) for the boundary layer thickness is obtained without making any reference to any particular body under consideration. The length, L, which is used to make Navier-Stokes equations non-dimensional and which consequently enters in R is just the characteristic length of the body. Also no reference is made to the flow pattern. Therefore Equation (2. 4. 7) gives boundary layer thickness for flow past bodies of all shapes. Since the only hypothesis involved is that R is large, formula (2. 4. 8) for boundary layer thickness is also applicable in the case of transition from laminar into turbulent flow.

The restrictions on the parameters α and β involved in the formula (2. 4. 7) of boundary layer thickness is that $\alpha < 1$ and $\beta > 1$. Therefore the expression $\frac{\alpha}{1+\beta}$ is always less than $\frac{1}{2}$.

This shows that in general the thickness of the boundary layer is of order greater than $R^{-1/2}$, that is the boundary layer thickness is greater than that given by Prandtl. This is otherwise also clear, because the consideration that viscous and inertia terms are of the same order of magnitude, gives the value of δ at some distance from the surface of the body and not near the outer edge of the boundary layer.

The strongly limiting case, $\alpha = 1$ and $\beta = 1$, gives

$$\delta \sim R^{-1/2},$$

which is the boundary layer thickness as given by Prandtl. If the values of parameters α and β are chosen as,

$$\alpha = \frac{1}{2},$$

and

$$\beta = \frac{3}{2},$$

then Equation (2. 4. 7) gives

$$\delta \sim R^{-1/5}$$

which is the well known boundary layer thickness in the case of turbulent boundary layer. If the values of the parameters α and β are chosen as

$$\alpha = \frac{1}{2}$$

and

$$\beta = \frac{1}{2}$$

then Equation (2. 4. 7) gives

$$\delta \sim R^{-1/3}.$$

which is the boundary layer thickness as speculated by Proudman (1956).

In the above particular cases, it is clear that the formula for boundary layer thickness as given by Prandtl does not hold good. The reason is obvious, because Prandtl considered the flow along a flat plate and did not take into account the general type of flow in the main stream, which may have vorticity transported from the surface of the body.

Therefore the formula (2. 4. 7) for the boundary layer thickness is applicable for two dimensional flow past any body. The determination of thickness depends upon two considerations, firstly the relative order of the viscous and inertia terms near the outer edge of the boundary layer and secondly the order of vorticity allowable at the outer edge of the boundary layer.

Seth (1960a) has also analyzed the boundary layer thickness for flow along a flat plate. In this case the boundary layer thickness is given by the formula

$$\delta \sim R^{-k/2+\lambda},$$

where

$$0 < k < 1 \quad \text{and} \quad \lambda > 1,$$

k and λ, depending upon the vorticity, and the relative order of viscous and inertia forces allowable at the outer edge of the boundary layer.

The result obtained in Equation (2. 4. 7) is general and holds good for the boundary layer over bodies of any shape whatsoever.

III. TRANSITION CONCEPT AND GENERAL TRANSITION EQUATION

3.1 Preliminary Remarks

A large number of problems in fluid mechanics, elasticity, plasticity involve quick transitions and non-uniformity. They have been subjected to perturbation techniques which are not always satisfactory. A number of examples can be mentioned in which the field equations have been truncated or boundary conditions relaxed. This is particularly true of non-linear problems, for example, boundary layer, shock wave, stability problems, etc. In all of them an asymptotic phenomenon through transition can be notices. This asymptotic aspect has an extensive literature on the subject. Exhaustive references are given by Friedrichs (1955).

Physical problems also exhibit another type of transition, which arises from the changes in the properties of the medium, for example, elastic-plastic deformation, creep, relaxation, turbulence, boundary layer, etc. In these cases, if the initial state is defined by a set of field equations, it should be possible to identify the transition to the next neighboring state with the critical points of the differential system involved. An asymptotic solution at these points should give the transition state without assuming semi-empirical laws, or ad-hoc conditions which are otherwise found necessary to treat the change.

This treatment as done by Seth (1962 - 1966) makes the assumption of yield conditions like that of von-Mises and Tresca, creep strain laws like that of Norton and jump conditions for shock both unnecessary and redundant.

At transition the fundamental structure of the medium undergoes a change and gives rise to spin, rotation or vorticity effects. Non-conservative nature of spin forces makes transition phenomena both non-linear and irreversible. This also explains the existence of different types of spirals or vortex formation and wake at the boundary layer transition. It will be shown in Chapter IV that the transition equation gives the spiral and vortex formation in the case of two dimensional flow.

3. 2 Transition Concept

Every medium under the action of internal and external stresses and body forces begins to yield and two states are obtained which dovetail into each other. To be more explicit, if these successive states be denoted by A and B, and the transition or mid-state, or dovetailing state by T, then A passes into B through T. This dovetailing region T, is usually referred to as the transition region. But on a broader basis, every state in the region apart from the initial state should be called transition state. It is in this broader sense that the term transition will be used. In fluid mechanics, the whole of the region excepting the boundary layer region will be called the transition region. In a transition state the whole of the medium participates and the effect of a change is not confined to a particular line or region, as is usually assumed.

In order to study the transition region and to obtain the transition condition it may first be observed that it is an asymptotic phenomenon. Transition conditions should therefore be identified with some type of limiting concept in the field. This will involve some invariance relations and these should be obtained in terms of the invariance associated with the field. In other words some functional relation should exist between the invariants corresponding to the transition. In elastic-plastic transition this invariant relationship is known as the yield condition (Erigen, 1962, p. 294).

In fluid dynamics, the equation of motion for study flow are:

$$(\tau_{ij}+\rho u_i u_j),_j = 0$$

In the study of the boundary layer transition we consider the stress invariants in the main flow which is one-dimensional and which can be approximated to a non-viscous flow. In this case from the equations of motion it follows that there exists only one invariant viz

$I_1 = -p + \rho q^2$ since $\tau_{11} = -p$ and q is the main stream velocity. Thus the invariance relation corresponding to the transition may be taken as

$$-p + \rho q^2 = \text{const}$$

Since q remains very nearly the same on either side of the immediate vicinity of the outer edge of the boundary layer it follows that p may be taken as continuous across this outer edge of the boundary layer, a result which was assumed by Prandtl.

3.3 Subharmonicity of Transition Fields

The flow of a viscous fluid in the immediate vicinity of a body has a creeping motion known as Stokes flow and its stream function satisfies a biharmonic equation. Flow away from the boundary can be approximated to an inviscid flow and the stream function in this case satisfies a harmonic equation. Heuristically, therefore it may be expected that dovetailing fields are subharmonic or superharmonic fields, which reduce to harmonic and biharmonic fields as limiting cases. Even slight deviation from the harmonic and biharmonic fields should be in terms of allied subharmonic fields. These subharmonic or superharmonic fields are non-linear and hence it can be expected to explain the non-linear, non-conservative, and irreversible phenomena like spiral formations, vortex flow, etc. It will be shown in the next chapter that these fields do give the spiral formations, which are observed experimentally and have received very little global analytic treatment.

Owing to the existence of vast literature on harmonic functions, they are easy to handle. Harmonic functions also have nearly all nice properties like existence, uniqueness, superposability and a large number of known solutions. Superharmonic functions are non-linear and hence have not been so extensively explored as harmonic functions. References for these may be found in (Greenspan and Yoke, 1963) and recently some attempts have been made to solve these equations with computers.

It will be useful for further reference, to include here a rigorous definition and some known properties of subharmonic (superharmonic) functions.

Subharmonic Functions

Definition (Rado, 1949): Let $U(x, y)$ be a function defined in a domain G (connected open set), such that $-\infty \leqq U < \infty$ in G. That is, $-\infty$ is an admissible value of U, while $+\infty$ is not. Such a function is subharmonic in G if it satisfies the following conditions:

a) U is not identically equal to $-\infty$ in G.

b) U is upper semi-continuous in G. That is, for every point (x_0, y_0) in G and for every number $\lambda > u(x_0, y_0)$ there exists a $\delta = \delta(x_0, y_0) > 0$ such that $u(x, y) < \lambda$ for $[(x-x_0)^2 + (y-y_0)^2]^{1/2} < \delta$. Observe that for $u(x_0, y_0) = -\infty$

this condition implies that $u(x, y) \to -\infty$ for $(x,y) \to (x_0, y_0)$.

c) Let G' be any domain contained in G together with its boundary B'. Let $H(x, y)$ be harmonic in G', continuous in $G' + B'$ and $H \geqq U$ on B'. Whenever these assumptions are satisfied, we also have $H \geqq U$ in G'.

Superharmonic Function

A function V is superharmonic in a domain G if the function $U = -V$ is subharmonic there.

It has been proved (Rado, 1949, p. 13) that if $U(x, y)$ is of class C^2 (if its second derivative is continuous) and is a solution of $\nabla^2 U = P$ when P is a function of $x, y, u, u_x, \ldots$ then U is subharmonic in every domain G in which $P \geq 0$. Similarly it is superharmonic in every domain G in which $P \leq 0$.

Under certain conditions the uniqueness of subharmonic solutions can be proved. They can be expected to possess strong properties of smoothness provided the coefficients in the differential equation are sufficiently smooth. Subharmonicity is only a local property. The subharmonic functions obeys the maximum principle, that is, if they are subharmonic in a domain G, and have a maximum point in the interior of G, then they are only constant. Also if ψ_n's are subharmonic and α_n's are non-negative constants then $\Sigma \psi_n k_n$ is also subharmonic in G.

Note. Since subharmonic functions automatically have superharmonic functions as their counterparts, it will be sufficient to use just one of the terms only and the term subharmonic will be used in the subsequent chapters.

In the next section, an equation governing the transition fields in three dimensions is derived. The subharmonicity (superharmonicity) of the transition equation is established by using a rectangular cartesian coordinate system.

3.4 Three Dimensional Transition Equation

The Navier-Stokes equations for viscous, incompressible fluid without any heat transfer are given by

$$\rho \frac{D\vec{u}}{Dt} = \rho\vec{F} - \nabla p + \mu \nabla^2 \vec{u} \qquad (3.4.1)$$

where ρ is the density, μ the viscosity of the fluid and $\vec{F}$ the external body forces per unit volume, $\frac{D}{Dt}$ being the operator,

$$\frac{\partial}{\partial t} + \vec{u} \cdot \nabla$$

The vorticity vector $\vec{\omega}$ is defined as

$$\vec{\omega} = \nabla \times \vec{u} \qquad (3.4.2)$$

Making use of the identity,

$$\vec{u} \times (\nabla \times \vec{u}) = \frac{1}{2}\nabla(\vec{u} \cdot \vec{u}) - \vec{u} \cdot \nabla \vec{u} \qquad (3.4.3)$$

In Equation (3.4.1) we get

$$\frac{\partial u}{\partial t} - \vec{u} \times \vec{\omega} = \vec{F} - \nabla p_0 + \nu \nabla^2 u \qquad (3.4.4)$$

where

$$q^2 = \vec{u} \cdot \vec{u},$$

$$p_0 = p + \frac{1}{2}\rho q^2 = \text{stagnation pressure},$$

and

$$\nu = \frac{\mu}{\rho} \qquad = \text{kinematic viscosity}.$$

In the case of steady flow and in the absence of any external body force, Equation (3.4.4) becomes,

$$- \vec{u} \times \vec{\omega} = -\frac{1}{\rho}\nabla p_0 + \nu\nabla^2 \vec{u} \qquad (3.4.5)$$

Taking curl of both sides of Equation (3.4.5), we get

$$-\nabla \times (\vec{u} \times \vec{\omega}) = \nu \nabla^2(\nabla \times \vec{u}) = \nu\nabla^2 \omega. \qquad (3.3.6)$$

Writing Equation (3.3.6) in non-dimensional form and retaining the same symbols for the non-dimensional quantities as for their dimensional counterparts, we get

$$-\nabla \times (\vec{u} \times \vec{\omega}) = \frac{1}{R} \nabla^2 \vec{\omega} \qquad (3.3.7)$$

Where R is the Reynolds number.

Equation (3. 3. 7) is true at all points of the region. In the transition region, R is large. Since viscous effects are small compared to the inertial effects in that region. In order to obtain the transition equation we must take the limiting form of Equation (3. 3. 7) when R is large. Physically $\nabla^2 \omega$, can be interpreted as a measure of the difference between the value of ω at a point and the average value of ω in an infinitesimal neighborhood of this point (Hopf, 1948, p. 63). From the physical considerations it is obvious that in the transition region vorticity is not changing abruptly from point to point. Therefore $\nabla^2 \omega$ is small. Since the transition region does not include the boundary, the limiting form of Equation (3. 3. 7) when R is large yields

$$\nabla \times (\vec{u} \times \vec{\omega}) = 0. \tag{3. 3. 8}$$

The transition Equation (3. 3. 8) is a non-linear differential equation. A solution of this non-linear differential Equation (3. 3. 8) in terms of vorticity can be obtained as follows:

Let

$$\vec{u} = \text{curl } \vec{T}, \tag{3. 3. 9}$$

where

$$\vec{u} = \vec{i}\, u + \vec{j}\, v + \vec{k}\, w$$

and

$$T = \vec{i}\,F + \vec{j}\,G + \vec{k}\,H,$$

where $\vec{i}, \vec{j}, \vec{k},$ are unit vectors along x, y, z-axis respectively. From Equation (3. 3. 9), we get

$$u = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z},$$

$$v = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x},$$

$$w = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}$$

Making use of Equation (3. 3. 9) Equation (3. 3. 2) gives

$$\vec{\omega} = \nabla \times \vec{u} = \nabla \times (\nabla \times \vec{T}) = -\nabla^2 \vec{T} \qquad (3.\ 3.\ 10)$$

provided $\vec{T}$ is chosen such that

$$\nabla \cdot \vec{T} = 0.$$

Making use of Equations (3. 3. 9) and (3. 3. 10), Equation (3. 3. 8) reduces to

$$\nabla \times [(\nabla \times \vec{T}) \times \nabla^2 \vec{T}] = 0 \qquad (3.\ 3.\ 11)$$

Equation (3. 3. 11) shows that vorticity $(\omega = -\nabla^2 \vec{T})$ plays a significant role in the transition. A particular solution of Equation (3. 3. 11) is

$$\nabla^2 \vec{T} = (\nabla \times \vec{T})\phi_1 \qquad (3.\ 3.\ 12)$$

where ϕ_1 is a scalar function. It will be shown in Chapter IV, that the transition equation in two dimensions is $\nabla^2\psi = f(\psi)$. This suggests that the general solution should contain a particular solution of the type

$$\nabla^2\vec{T} = \phi_2'(\vec{T}\cdot\vec{T})\vec{T}.$$

These considerations show that the solution of the transition equation (3. 3. 11) may be put in the form

$$\nabla^2\vec{T} = (\phi_1)(\nabla \times \vec{T}) + 2\phi_2'(\vec{T}\cdot\vec{T})[\vec{T}+\vec{S}] \qquad (3.\ 3.\ 13)$$

where the vector $\vec{S}$ is to be suitably chosen, so as to satisfy the differential Equation (3. 3. 11). Now

$$\nabla^2 T \times (\nabla \times \vec{T}) = 2\phi_2'(\vec{T}\cdot\vec{T})[\vec{T}\times(\nabla\times\vec{T}) + \vec{S}\times(\nabla\times\vec{T})]$$

$$= 2\phi_2'(\vec{T}\cdot\vec{T})[\tfrac{1}{2}\nabla(\vec{T}\cdot\vec{T}) - (\vec{T}\cdot\nabla)T + \vec{S}\times(\nabla\times\vec{T})] \qquad (3.\ 3.\ 14)$$

If we choose $\vec{S}$, such that

$$-(\vec{T}\cdot\nabla)\vec{T} + \vec{S}\times(\nabla\times\vec{T}) = 0 \qquad (3.\ 3.\ 15)$$

then Equation (3. 3. 14) becomes,

$$\nabla^2\vec{T}\times(\nabla\times\vec{T}) = \phi_2'(\vec{T}\cdot\vec{T})[\nabla(\vec{T}\cdot\vec{T})],$$

$$= \nabla\{\phi_2(\vec{T}\cdot\vec{T})\},$$

Therefore Equation (3. 3. 11) gives,

$$\nabla \times [\nabla^2 T \times (\nabla \times \vec{T})] = \text{curl}\,(\text{grad}\,\phi_2(\vec{T} \cdot \vec{T})) = 0.$$

Therefore the solution of Equation (3. 3. 11) is given by Equation (3. 3. 13) provided $\vec{S}$ satisfies the Equation (3. 3. 15). In order to find the vector $\vec{S}$, let

$$\vec{S} = \vec{i}\,s_1 + \vec{j}\,s_2 + \vec{k}\,s_3,$$

Hence

$$(\vec{T} \cdot \nabla)\vec{T} = (\vec{i}\,s_1 + \vec{j}\,s_2 + \vec{k}\,s_3) \times (\nabla \times \vec{T}) \tag{3. 3. 15b}$$

From (3. 3. 15b), it follows that

$$\theta F = s_2 w - s_3 v, \tag{3. 3. 16}$$

$$\theta G = s_3 u - s_1 w, \tag{3. 3. 17}$$

$$\theta H = s_1 v - s_2 u, \tag{3. 3. 18}$$

where the operator θ is given by

$$\theta = F\frac{\partial}{\partial x} + G\frac{\partial}{\partial y} + H\frac{\partial}{\partial z}.$$

Multiplying Equations (3. 3. 16), (3. 3. 17) and (3. 3. 18) by u, v and w respectively and adding, we get

$$u\theta F + v\theta G + w\theta H = 0;$$

or

$$(\nabla \times \vec{T}) \cdot [(\vec{T} \cdot \nabla)\vec{T}] = 0 \tag{3. 3. 19}$$

Again, multiplying Equations (3. 3. 16), (3. 3. 17) and (3. 3. 18) by s_1, s_2 and s_3 respectively and adding, we get

$$s_1\theta F + s_2\theta G + s_3\theta H = 0;$$

or

$$\vec{S}\ [(\vec{T}\cdot\nabla)\vec{T}] = 0 \qquad (3.\ 3.\ 20)$$

From the two independent Equations (3. 3. 19) and (3. 3. 20), s_2 and s_3 can be expressed in terms of s_1. The values of s_2 and s_3 in terms of s_1 are as follows:

$$s_2 = \frac{1}{u}(s_1 v - \theta H)$$

$$s_3 = \frac{1}{u}(s_1 w + \theta G)$$

Making use of these values of s_2 and s_3, the general solution of Equation (3. 3. 11) may be written in the form

$$\nabla^2 T = (\phi_1)(\nabla \times \vec{T}) + 2\phi_2'(\vec{T}\cdot\vec{T})[\vec{T} + \frac{1}{u}\{s_1(\nabla \times \vec{T}) - \vec{j}\,\theta H + \vec{k}\,\theta G\}] \qquad (3.\ 3.\ 21)$$

Since ϕ_1 is an arbitrary function, the terms $(\phi_1)(\nabla \times \vec{T})$ and $\frac{s_1}{u}(\nabla \times \vec{T})$ can be combined together, and the Equation (3. 3. 21) then becomes

$$\nabla^2\vec{T} = (\phi_1')(\nabla \times \vec{T}) + 2\phi_2'(\vec{T}\cdot\vec{T})[\vec{T} + \frac{1}{u}(-\vec{j}\,\theta H + \vec{k}\,\theta G)]. \qquad (3.\ 3.\ 22)$$

The right-hand-side of Equation (3. 3. 22) contains $\vec{T}$ and its first order partial derivatives. Hence the transition field given by $\vec{T}$ are subharmonic or superharmonic, according as

$$\nabla^2 \vec{T} \geq 0,$$

or

$$\nabla^2 \vec{T} \leq 0.$$

This establishes that the transition fields are subharmonic (or superharmonic).

The transition equation as given by Equation (3. 3. 22) is obviously a very complicated equation and hence in general, not very many conclusions can be drawn. But it does give some interesting results. Firstly the Equation (3. 3. 22) contains the solution

$$\nabla^2 \vec{T} = (\phi_1{}')(\nabla \times \vec{T})$$

as a particular case, which shows that stream lines are parralel to the vortex lines. This type of flow is known as Beltrami flows.

Secondly, boundary layer separation in two dimensional flows is a well understood phenomenon. This has been explained as being caused by back flow of the fluid. But this explanation for the phenomenon of boundary layer separation in three dimensional flow is not valid. From the foregoing transition analysis, it is possible to explain in general all transition phenomena including three dimensional boundary layer separation as arising from the subharmonicity (superharmonicity) of the transition fields.

IV. TWO DIMENSIONAL BOUNDARY LAYER TRANSITION

4.1 Preliminary Remarks

In nature, as time passes almost all fields steady themselves out and thus remain solenoidal fields. When these fields come in contact with some obstacle, their solenoidal character is changed. Particularly in fluid dynamics when a fluid comes in contact with some obstruction at large velocity, the solenoidal character of the field is destroyed. Two types of flow patterns are observed. One very near the boundary, called the boundary layer and the other away from the boundary that is from boundary layer to infinity. In the first region vorticity is being rapidly diffused from a large value near the boundary to zero at large distances from the boundary. As has been pointed out in the last chapter, this second region will be referred to as the transition region. To be specific if the whole region is denoted by W and the boundary layer region by B, then $W - B$ is the transition region. A study of the transition region will be made by identifying it as an asymptotic phenomenon from the boundary.

The transition equation which will be obtained in Section 4.3 will have the same limitations as pointed out by Reid (1965). These can be summarized as follows:

a) In this approach one tries to express the solution of the given equation asymptotically in terms of the solutions of a similar but simpler comparison equation. The success of this method depends to a large extent on the particular form chosen for the comparison equation. If for example, too simple a form is chosen for the comparison equation, the resulting approximation to the solutions of the given equation may be inadequate. More precisely the domains of validity of the approximate solutions may not contain the boundary points.

b) Approximations obtained in this way to a particular solution of the given equation are usually valid only in limited domain.

Here we shall be considering exclusively steady two dimensional flow pattern. In case of two dimensional steady motion of fluid without heat transfer and in the absence of external force, the Navier-Stokes equations (3. 4. 1) take the form

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \frac{1}{R} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \tag{4. 1. 1}$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \frac{1}{R} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \tag{4. 1. 2}$$

where Equations (4. 1. 1) and (4. 1. 2) are in non-dimensional form and the velocity vector $\vec{q} = (u, v, 0)$. Eliminating p from Equations (4. 1. 1) and (4. 1. 2) by cross differentiation and using the stream function ψ (Lagrangian steam function), given by

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x},$$

one obtains a fourth order non-linear differential equation for ψ as

$$\nabla^4\psi + RJ(\psi, \nabla^2\psi) = 0, \tag{4.1.3}$$

where

$$J(\psi, \nabla^2\psi) = \begin{vmatrix} \dfrac{\partial\psi}{\partial x} & \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\nabla^2\psi}{\partial x} & \dfrac{\partial\nabla^2\psi}{\partial y} \end{vmatrix}.$$

The motion of the liquid depends upon the non-dimensional Reynolds number R, which is the flow characteristic of the particular fluid under consideration. When R changes, which can be affected either by changing the velocity or by any other factor, the flow pattern changes. When R exceeds approximately 10^5, the flow becomes turbulent.

Thus the two dimensional, incompressible viscous flow in the absence of external forces is governed by Equation (4.1.3). When a body is brought into the field, the flow pattern changes. Although the flow away from the body remains nearly uniform, near the body it changes and vortices, spiral formations and wake are observed in the flow field. The formation of vortex spirals in two dimensional cases is physically a well understood phenomenon and will be explained in the next section. In current literature this phenomenon lacks a rigorous mathematical treatment. In Section 4. 4 an attempt is made to give a mathematical treatment of this phenomenon.

4. 2 Physical Interpretation of Vortex Formations

In order to explain the formation of vortices, consider the flow about a circular cylinder as shown in Figure 3.

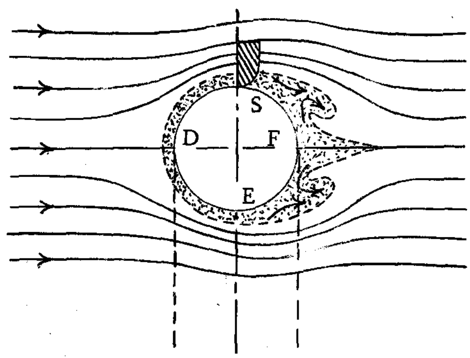

Figure 3a. Vortex formation about a circular cylinder (Streeter, 1961).

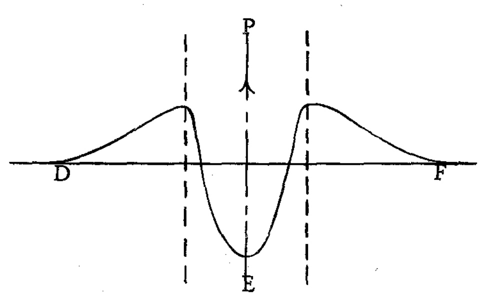

Figure 3b. Pressure diagram referred to in Figure (3a).

In frictionless flow, the fluid particles are accelerated on the upstream half from D to E and decelerated on the downstream half from E to F. Hence the pressure decreases from D to E and increases from E to F. When the flow is started up the motion in the first instant it is nearly frictionless, and remains so as long as the boundary layer remains thin. Outside the boundary layer there is a transformation of pressure into kinetic energy along DE, the reverse taking place along EF, so that a particle arrives at F with the same velocity as it had at D. A fluid particle which moves in the immediate vicinity of the wall in the boundary layer remains under the influence of the same pressure field as that existing outside, because the external pressure is impressed on the boundary layer. Owing to the large frictional forces in the thin boundary layer such a particle consumes so much of its kinetic energy on its path from D to E that the remainder is too small to surmount the "pressure hill" from E to F. Such a particle cannot move far into the region of increasing pressure between E and F and its motion is, eventually, arrested. The external pressure causes it then to move in the opposite direction. This reverse motion gives rise to vortex formation. The vortex becomes separated shortly afterwards and moves downstream in the fluid. This phenomenon is usually referred to as "boundary layer separation." At large distances from the body it is possible to distinguish a regular pattern of vortices which move alternately clockwise and counter clockwise, and which is known as a Kármán Vortex Street.

But in three dimensional flows which occur in real situations, separation of the flow can occur without the usual flow reversal and reduction of the wall shear stress to zero. Two-dimensional definitions of separation which commonly depend on these occurrences, are useless in three dimensional flow (Streeter, 1961). In order to eliminate confusions, a much more rigorous and general definition of separation is required. Such a definition has been developed by Eichelbrenner and Outdart (1954, 1955) and Maskell (1955).

4. 3 Transition Equation

As indicated in Equation (4. 1. 3), the two-dimensional, steady, incompressible, viscous flow in the absence of external forces, is governed by

$$\nabla^4\psi + RJ(\psi, \nabla^2\psi) = 0 \tag{4. 3. 1}$$

where R is the Reynolds number and ψ, the Lagrangian stream function.

Two critical points of the Equation (4. 3. 1) which can be interpreted as transition points are,

1. When R is small.
2. When R is large.

Case 1: When R is small which corresponds to the physical situation wherein viscous forces are large as compared to inertia forces, the limiting form of Equation (4. 3. 1) is

$$\nabla^4\psi = 0 \tag{4. 3. 2}$$

This corresponds to the slow viscous motion, which is usually referred to as creeping motion. In this case the fluid just creeps over the surface and no boundary layer separation takes place. Clearly Equation (4. 3. 2) contains both harmonic and biharmonic solutions. The harmonic part gives irrotational solution and the biharmonic gives the drag suffered by the body. As a smooth body can be made as small as we like, the biharmonic part should correspond to the solution for a concentrated force acting at a suitable point in the infinite liquid (Seth, 1958). Thus, a slow viscous motion can be expected to be a linear superposition of two solutions,

1. An irrotational solution.

and 2. Solution for a concentrated force acting at a point in an infinite liquid in the direction of motion of the body.

Case 2: When R is large. Physically this corresponds to the region in which inertia forces are large as compared to viscous forces. In this case Equation (4. 3. 1) takes the form

$$J(\psi, \nabla^2\psi) = 0 \tag{4. 3. 3}$$

Provided $\nabla^4\psi$ does not become large, when R is large. In order to understand the implication of $\nabla^4\psi$, it may be noted that $\nabla^4\psi = \nabla^2\omega$. Physically $\nabla^2\omega$, can be interpreted as a measure of the difference between the value of the ω at a point and the average value of ω in an infinitesimal neighborhood of this point (Hopf, 1948, p. 63). From physical considerations it is clear that the distribution of vorticity is uniform, that is, vorticity is not changing abruptly from point to point. Hence $\nabla^4\psi = \nabla^2\omega$ is small. Also as we shall see later on, the solution of Equation (4. 3.) is $\nabla^2\psi = f(\psi)$ and a particular form chosen for $f(\psi)$ will be $\frac{1}{n}e^{n\psi}$, where n is an integer. Therefore for this choice of $f(\psi)$,

$$\nabla^2\psi = \frac{1}{n}e^{n\psi}$$

$$\frac{\partial^2}{\partial x^2}(\nabla^2\psi) = ne^{n\psi}(\frac{\partial\psi}{\partial x})^2 + e^{n\psi}\frac{\partial^2\psi}{\partial x^2}$$

or

$$\nabla^4\psi = ne^{n\psi}\{(\frac{\partial\psi}{\partial x})^2+(\frac{\partial\psi}{\partial y})^2\} + e^{n\psi}\nabla^2\psi$$

$$= ne^{n\psi}(q)^2 + \frac{1}{n}e^{2n\psi}.$$

where $\vec{q}$ is the velocity vector. Therefore $\nabla^4\psi$ is finite at each point.

From the above two considerations it is clear that

$$\frac{\nabla^4\psi}{R} \rightarrow 0$$

when R is large.

$$R \rightarrow \infty.$$

Therefore when R is large, which is either due to V being large or μ

is small or both, the limiting form of Equation (4. 3. 1) is

$$J(\psi, \nabla^2\psi) = 0 \tag{4. 3. 5}$$

The following two points may be noticed.

1. When $\mu \rightarrow 0$, this makes R large and Equation (4. 3. 1) takes the form $J(\psi, \nabla^2\psi) = 0$ which holds throughout the region including the region near the surface of the immersed body. And this is the exact form of the differential system.

2. But in general when R is large, μ may or may not be small. The limiting form of Equation (4. 3. 1) is again $J(\psi, \nabla^2\psi) = 0$, which may be called the transition equation. This holds good at all points of the field, excepting the region very near the boundary, because near the boundary, viscous forces dominate the inertia forces.

   We shall be viewing Equation (4. 3. 5) from this point of view, that is, Equation (4. 3. 5) holds good everywhere except in a very thin layer near the body. But it is clear that in both cases away from the boundary, the fluid behavior may be approximated to that of a nonviscous fluid, which is also a justification for the Prandtl's assumption.

Therefore the transition equation is

$$J(\psi, \nabla^2\psi) = 0. \qquad (4.3.6)$$

Evidently Equation (4. 3. 6) is satisfied when

$$\nabla^2\psi = f(\psi), \qquad (4.3.7)$$

where $f(\psi)$ is an arbitrary function of ψ and also at all those regions where the velocity is zero. Since points of zero velocity are called stagnation points, the solution of transition equation contains the solution at stagnation points also.

Equation (4. 3. 7) is true in transition region, where $f(\psi)$ is any arbitrary function of ψ. The only restriction on $f(\psi)$ is that $|f(\psi)|$ should be finite. As has been pointed out in Section 3. 3, Equation (4. 3. 7) represents subharmonic or superharmonic functions according as

$$f(\psi) \geq 0$$

or

$$f(\psi) \leq 0$$

This again confirms the previous result that transition fields are either subharmonic or superharmonic fields.

4. 4 Solution of Two Dimensional Transition Equation--Spiral Formation

In cartesian coordinates transition Equation (4. 3. 7) becomes

$$\frac{\partial^2 x}{\partial x^2} + \frac{\partial^2 x}{\partial y^2} = f(\psi) \tag{4. 4. 1}$$

Putting $z = x + iy$ and $\bar{z} = x - iy$, Equation (4. 4. 1) reduces to

$$\frac{\partial^2 \psi}{\partial z \partial \bar{z}} = \phi'(\psi) \tag{4. 4. 2}$$

Where $4\phi'(\psi) = f(\psi)$ and prime denotes differentiation with respect to the variable ψ. Let

$$\psi = F(az + b\bar{z} + c) = F(X),$$

be a solution of Equation (4. 4. 2), then

$$ab \frac{d^2F}{dX^2} = \phi'(F),$$

multiplying both sides by $\frac{2dF}{dX}$ and integrating, we get

$$ab\left(\frac{dF}{dX}\right)^2 = 2\phi(F) + 2c_1,$$

where c_1 is a constant,

or

$$\frac{dF}{dX} = \pm \left(\frac{2}{ab}\right)^{1/2}[\phi(F) + c_1]^{1/2} \tag{4. 4. 3}$$

Again on integrating Equation (4. 4. 3), we get

$$az + b\bar{z} + c_3 = \pm\left(\frac{1}{2}ab\right)^{1/2}\int\frac{d\psi}{[\phi(\psi)+c_1]^{1/2}}, \qquad (4.4.4)$$

where $c_3 = c + c_2$. The solution of Equation (4. 4. 1) as given by Equation (4. 4. 4) depends on the particular form of function $f(\psi)$.

Solutions to Equation (4. 4. 1) when $f(\psi)$ is either a constant or a linear function of ψ, exists in literature. Numerical integration has also been carried out on digital computers (Greenspan and Yohe, 1963). This reference contains an extensive bibliography in this field. We shall consider here a non-linear form for $f(\psi)$. The particular case of

$$f(\psi) = \frac{4}{n} e^{n\psi} \qquad (4.4.5)$$

which corresponds to the Boltzmann distribution and in which n is a constant, is of great physical interest. Weymann (1961) has used the corresponding equation to discuss the electron density distribution from argon plasma in shock tubes.

In the present case $f(\psi)$ is the vorticity which plays an important role in the transition. Its distribution may be expected to approach the form given by Equation (4. 4. 5) after a sufficient lapse of time.

The transition equation in this particular case becomes,

$$\frac{\partial^2\psi}{\partial z\partial\bar{z}} = \frac{1}{n}e^{n\psi}. \tag{4. 4. 6}$$

Let

$$\psi = F[A(z)+B(\bar{z})] + C(z) + D(\bar{z}) \tag{4. 4. 7}$$

be a solution of Equation (4. 4. 6), where A, C and B, D are arbitrary functions of z and $\bar{z}$ respectively. Making use of the value of ψ, as given by Equation (4. 4. 7) in Equation (4. 4. 6), we get

$$F''(X)\ A'(z)\ B'(\bar{z}) = \frac{1}{n}e^{n[F(X)+C(z)+D(\bar{z})]}, \tag{4. 4. 8}$$

From Equation (4. 4. 8), we get

$$F''(X) = \frac{1}{n}e^{nF(X)}, \tag{4. 4. 9}$$

$$A'(z) = e^{nC(z)}, \tag{4. 4. 10}$$

$$B'(\bar{z}) = e^{nD(\bar{z})}. \tag{4. 4. 11}$$

Writing in full, from Equation (4. 4. 9) we get

$$\frac{d^2F}{dX^2} = \frac{1}{n}e^{nF(X)}.$$

Let

$$\frac{dF}{dX} = p.$$

Then Equation (4. 4. 9) takes the form

$$p\frac{dp}{dF} = \frac{1}{n}e^{nF(X)} . \tag{4.4.12}$$

On integrating Equation (4. 4. 12), we get

$$\frac{p^2}{2} = \frac{1}{n^2}e^{nF} + C^2 \tag{4.4.13}$$

where C is an arbitrary constant. Hence

$$\frac{dF}{dx} = \pm\frac{\sqrt{2}}{n}\sqrt{e^{nF}+C^2n^2} ,$$

or

$$\frac{n}{\sqrt{2}}\int\frac{dF}{\sqrt{e^{nF}+C^2n^2}} = \pm\int dX + C,$$

or

$$\pm X + C = \frac{n}{\sqrt{2}}\int\frac{dF}{\sqrt{e^{nF}+C^2n^2}}$$

$$= \frac{\sqrt{2}}{-nC}\sin h^{-1}(Cne^{-nF/2}).$$

Since X is the sum of two arbitrary functions, $\pm$ sign in front of X does not carry much sense. Thus we have

$$Cne^{-nF/2} = \sin h\ \{\frac{nC}{\sqrt{2}}(X+C_1)\}.$$

Making use of Equation (4. 4. 7), we get from the above equation

$$\sin h\,\{\frac{nC}{\sqrt{2}}(X+C_1)\} = Cne^{-\frac{n}{2}\{\psi - C(z) - D(\bar{z})\}}\,,$$

$$= Cne^{-\frac{n}{2}\psi}\, e^{\frac{n}{2}C(z)}\, C^{\frac{n}{2}D(\bar{z})}\,.$$

Making use of Equations (4. 4. 10) and (4. 4. 11), we get

$$Cne^{-\frac{n}{2}\psi} = \frac{\sin h\,\{\frac{nC}{\sqrt{2}}(X+C_1)\}}{\sqrt{A'(z)B'(\bar{z})}}\,. \tag{4. 4. 14}$$

Since $A(z)$ and $B(\bar{z})$ are arbitrary functions, the constant C can be absorbed in any one of them. Then we obtain

$$Cne^{-n\psi/2} = \frac{\sin h\,\{\frac{nC}{\sqrt{2}}(X)\}}{\sqrt{A'(z)B'(z)}}. \tag{4. 4. 15}$$

Taking the case where C is zero, we get

$$e^{n\psi} = \frac{2A'(z)B'(\bar{z})}{\{A(z)+B(\bar{z})\}^2}\,. \tag{4. 4. 16}$$

Thus Equations (4. 4. 14) and (4. 4. 15) are the solutions to the differential equation (4. 4. 6). The result given by Equation (4. 4. 16) is essentially the same due to Liouville (Forsyth, 1914, p. 555).

An infinite number of particular cases pertaining to different choices of the functions $A(z)$ and $B(\bar{z})$ can be built up. Some of them may be found to explain natural phenomena for which a number of ad hoc assumptions have to be made if only linear form was taken for $f(\psi)$. We shall now take some particular cases of these functions.

Case 1: Let

$$A(z) = \ln z \quad \text{and} \quad B(\bar{z}) = -\ln \bar{z} .$$

In this case Equation (4.4.16) becomes

$$e^{n\psi} = \frac{-2\frac{1}{z}\frac{1}{\bar{z}}}{\{\ln z - \ln \bar{z}\}^2} . \tag{4.4.17}$$

Now let

$$z = re^{i\theta},$$

$$\ln z = \ln r + i\theta,$$

$$\ln \bar{z} = \ln r - i\theta.$$

Hence Equation (4.4.17) becomes

$$e^{n\psi} = \frac{-2/r^2}{(2i\theta)^2} = \frac{1}{2r^2\theta^2} .$$

The stream lines are now given by

$$r^2\theta^2 = \text{constant}, \tag{4.4.18}$$

which are hyperbolic or reciprocal spirals. These spirals are shown in Figure 4.

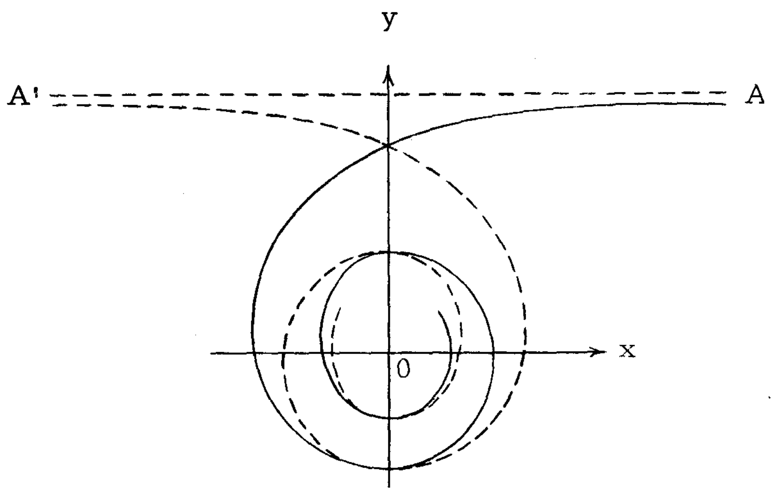

Figure 4. Hyperbolic spiral. $(r\theta = a)$

These types of spirals are noticed in the wake of two dimensional flow past any body at large Reynolds number. In particular these spirals are clearly visible in case of flow past a circular cylinder (Prandtl and Tietjen).

Case 2: Let

$$A(z) = z^{\pm m},$$

$$B(\bar{z}) = (\bar{z})^{\pm m},$$

where m is an integer. The stream lines in this case are found to be

$$r \cos m\theta = a$$

$$r \sin m\theta = a, \tag{4.4.19}$$

where a is a constant. These are Cote's spirals and are shown in Figure 5.

These types of spirals are also noticed in the wake of two dimensional flow past a body. In particular these can be noticed in the neighborhood of an oscillating circular cylinder (Schlichting, p. 414).

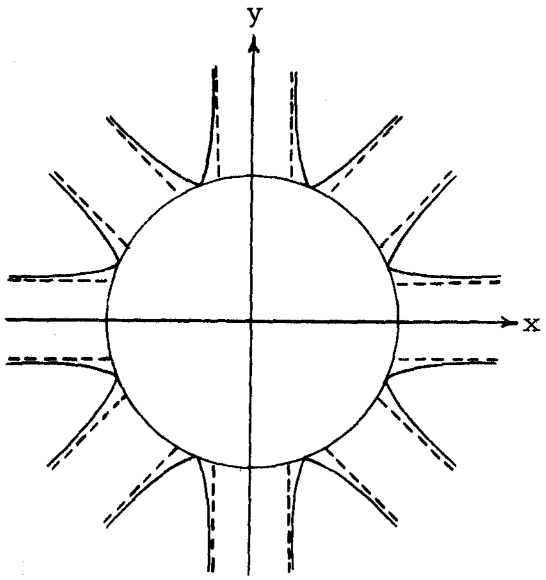

Figure 5. Cote's spiral. ($r \sin m\theta = a$)

Case 3: Let us choose

$$A(z) = \frac{U}{2i}\left(z + \frac{a^2}{z}\right),$$

$$B(\bar{z}) = -\frac{U}{2i}\left(\bar{z} + \frac{a^2}{\bar{z}}\right).$$

Then we have

$$A(z) + B(\bar{z}) = U\left(r \sin\theta - \frac{a^2}{r} \sin\theta\right), \qquad (4.4.20)$$

where

$$z = re^{i\theta}.$$

Also

$$A'(z)\,B'(z) = \frac{U^2}{4r^4}\left(r^4 - 2a^2r^2 \cos 2\theta + a^4\right). \qquad (4.4.21)$$

Substituting these values as given by Equations (4.4.20) and (4.4.21) in Equation (4.4.15), we get

$$Cne^{-\frac{n}{2}\psi} = \frac{2r^2 \sin h \frac{nC}{\sqrt{2}} \left[U\left(r - \frac{a^2}{r}\right) \sin\theta\right]}{U\left(r^4 - 2a^2r^2 \cos 2\theta + a^4\right)^{1/2}}. \qquad (4.4.22)$$

In Equation (4.4.22), C is an arbitrary constant, we can take it as ic, then Equation (4.4.22) becomes

$$Cne^{-\frac{n}{2}\psi} = \frac{2r^2 \sin\{\frac{nC}{\sqrt{2}} U(r - \frac{a^2}{r}) \sin\theta\}}{U(r^4 - 2a^2 r^2 \cos 2\theta + a^4)^{1/2}} . \tag{4.4.23}$$

In particular, taking the limiting case when ψ assumes large positive values, we get (because n is a fixed number)

$$\frac{nC}{\sqrt{2}} U(r - \frac{a^2}{r}) \sin\theta = k\pi,$$

where k is an integer;

or

$$U(r - \frac{a^2}{r}) \sin\theta = \frac{\sqrt{2}k\pi}{nC} . \tag{4.4.24}$$

The stream line pattern is the same as shown in Figure 1. Thus we see that the limiting values of the stream function ψ gives the vortex formation. The above result corresponds to the existing "cylindrical vortex motion" discussed in Section 1.5. The way "cylindrical vortex" discussed here is new and different from what exists in literature. Usually two stream functions are assumed, one specifying the flow inside and the other outside. The constants involved are so chosen as to match on the boundary. But here it has been shown that such a treatment is not necessary but these phenomena can be studied with the help of the transition equation and correspond to the limiting value of the stream function.

From Equation (4, 4, 23) it is clear that the stream line pattern, in general, is given by

$$2r^2 \sin\{\frac{nC}{\sqrt{2}} U(r - \frac{a^2}{r}) \sin\theta\}$$

$$= Cne^{-\frac{n}{2}\psi} U(r^4 - 2a^2r^2\cos 2\theta - a^4), \qquad (4.4.25)$$

when ψ takes different values. Once again we take the limiting case when ψ assumes large negative values. In this case Equation (4. 4. 25) gives

$$r^4 - 2a^2r^2\cos 2\theta + a^4 = 0,$$

or

$$r^2 = a^2(\cos 2\theta \pm \sqrt{\cos^2 2\theta - 1}).$$

Hence r is real only when $\theta = 0$. Thus in this case we get just two points $(a, 0)$ and $(-a, 0)$. Further if we refer to Figure 3 these points correspond to the points D and F, and these are stagnation points. Therefore we conclude that stagnation points are also given by the transition equation and correspond to the limiting value of the stream function ψ.

4.5 Vorticity Distribution

In this section we shall try to analyze the vorticity distribution as given by the transition equation. Particular forms of functions $A(z)$ and $B(\bar{z})$, which have given spiral formations are

$$A(z) = \ln z,$$

$$B(\bar{z}) = -\ln \bar{z}.$$

With this choice of functions $A(z)$ and $B(\bar{z})$, Equation (4.4.16) gives

$$e^{n\psi} = \frac{2A'(z)B'(\bar{z})}{\{A(z)+B(\bar{z})\}^2} = \frac{1}{2r^2\theta^2}. \tag{4.5.1}$$

The vorticity is given by $\nabla^2\psi$ and from Equation (4.4.6), we get

$$\nabla^2\psi = \frac{4}{n} e^{n\psi}.$$

Using Equation (4.5.1)

$$\nabla^2 \psi = \frac{2}{n(r^2 \theta^2)} . \tag{4.5.2}$$

From Equation (4. 5. 2), it is clear that

$$\nabla^2 \psi \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty$$

Therefore at large distances from the boundary, vorticity dies out. But near the surface it becomes large and hence indicates the formation of the boundary layer. This is clear from physical considerations also because the presence of a body generates the vorticity which is large in the boundary layer. Mostly the flow outside the boundary layer is considered to be non-viscous and irrotational. Fluid may behave like non-viscous and irrotational but vorticity is still present due to non-viscous nature of the fluid. Therefore the vorticity distribution outside the boundary layer may be regarded as given by Equation (4. 5. 2). This result also agrees with the physical considerations, because near the surface vorticity becomes large and dies out rapidly away from boundary.

4. 6 Conclusion

From the above considerations it is clear that fluid flow in the presence of a body at large Reynolds number R can be divided into two regimes.

1. Boundary Layer Region: This is a thin layer in the

immediate neighborhood of the body in which viscous forces dominate inertia forces and large vorticity is present in this region.

2. Transition Region: This is the whole region excepting the boundary layer region. The equation satisfied by the stream function ψ, in this region is

$$\nabla^2 \psi = f(\psi)$$

Besides other known results, this equation yields

a) Spiral formation.

b) Stagnation points.

These two results are not given by existing boundary layer theory. It has also been shown that the non linear vortex formation, which is observed at separation and in the wake of real fluids is given by subharmonic fields. These subharmonic fields also include stagnation points, solenoidal fields, spiral fields and vortex formation. The transition equation also explains the formation of "cylindrical vortex". The way this is obtained here, is new and different from the existing treatment. Usually two stream functions are assumed, one specifying the flow inside and the other outside the boundary. Then they are matched on the boundary. But here it has been shown that there is no need for this type of matching but these come out from the transition equation itself. In fact these phenomena are given by the limiting values of the stream function ψ.

In current literature most of these phenomena are treated as individual phenomenon and it is made out that a global representation is not possible. But it is clear from the analysis in this chapter that such a global representation is possible and for two dimensional flow it is given by the equation $\nabla^2\psi = f(\psi)$.

V. AXISYMMETRIC BOUNDARY LAYER TRANSITION

5.1 Preliminary Remarks

The transition equations for three dimensional and two dimensional flows have been obtained in the preceding two chapters. In this chapter we shall analyze the transition of steady axially symmetric flow from the boundary layer to the main stream. Axisymmetric flow has been treated separately, because the boundary layer which occurs in the case of flow past axi-symmetric bodies has practical importance such as in aerodynamics.

5.2 Axisymmetric Transition Equation

In order to obtain the transition equation, consider the non-dimensional form of Navier-Stokes equations for steady, incompressible flow with constant viscosity and no external force in non-dimensional cylindrical polar coordinates (r, θ, z).

$$u_r \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{2} = -\frac{\partial p}{\partial r} + \frac{1}{R}\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right), \tag{5.2.1}$$

$$u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{4} = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{1}{R}\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right), \tag{5.2.2}$$

$$u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = - \frac{\partial p}{\partial z} + \frac{1}{R} \nabla^2 u_z , \tag{5. 2. 3}$$

where (u_r, u_θ, u_z) are the components of velocity in radial, tangential and axial directions respectively. Assuming that the quantities involved are independent of θ, and also that $u_\theta = 0$, the above equations of motion become

$$u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z} = - \frac{\partial p}{\partial r} + \frac{1}{R} (\nabla^2 u_r - \frac{u_r}{r^2}), \tag{5. 2. 4}$$

and

$$u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z} = - \frac{\partial p}{\partial z} + \frac{1}{R} \nabla^2 u_z . \tag{5. 2. 5}$$

The equation of continuity is

$$\frac{\partial}{\partial r}(r u_r) + \frac{\partial}{\partial z}(r u_z) = 0. \tag{5. 2. 6}$$

If we choose the velocity components as,

$$u_r = \frac{1}{r} \frac{\partial \psi}{\partial z} , \quad u_z = - \frac{1}{r} \frac{\partial \psi}{\partial r} , \tag{5. 2. 7}$$

where ψ, is the Stokes stream function, Equation (5. 2. 6) is automatically satisfied. Eliminating p from Equations (5. 2. 4) and (5. 2. 5), we get

$$\frac{\partial}{\partial z}[u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z}] - \frac{\partial}{\partial r}[u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}]$$

$$= \frac{1}{R}[\frac{\partial}{\partial z}\{\nabla^2 u_r - \frac{u_r}{r^2}\} - \frac{\partial}{\partial r}\{\nabla^2 u_z\}] . \qquad (5.2.8)$$

The left hand side of Equation (5. 2. 8) can be further simplified by substituting the values of u_r and u_z from the Equation (5. 2. 7) and carrying out the indicated differentiations. Then by rearranging the terms we obtain for the left hand side of Equation (5. 2. 8) the expression

$$J(D^2\psi, \psi)$$

where

$$D^2 = \frac{1}{r^2}\frac{\partial^2}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial z^2} \qquad (5.2.9)$$

and

$$J(D^2\psi, \psi) = \begin{vmatrix} \frac{\partial^2(D^2\psi)}{\partial r} & \frac{\partial^2(D^2\psi)}{\partial z} \\ \frac{\partial\psi}{\partial r} & \frac{\partial\psi}{\partial z} \end{vmatrix}$$

Also the right hand side of Equation (5. 2. 8) can be simplified by using the values of u_r and u_z from Equation (5. 2. 7) and we obtain the expression

$$r\nabla^2(D^2\psi)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{5. 2. 10}$$

Making use of the above simplifications the Equation (5. 2. 8) may be written as

$$J(D^2\psi, \psi) = \frac{1}{R}[r\nabla^2(D^2\psi)] \tag{5. 2. 11}$$

This equation is true at all points of the region. In order to obtain the equation governing the transition region, we have to take the limiting form of Equation (5. 2. 11), when R is large, because in the transition region viscous effects are small as compared to inertial effects. Therefore the axisymmetric transition equation becomes

$$J(D^2\psi, \psi) = 0 \tag{5. 2. 12}$$

This equation has the same limitations as pointed out in Chapters III and IV.

5. 3 Solutions of the Axisymmetric Transition Equation

A solution of Equation (5. 2. 12) is

$$D^2\psi = f(\psi) \tag{5. 3. 1}$$

where f is an arbitrary function of ψ.

Now we discuss a particular form of the function $f(\psi)$ which is of interest as depicted by many natural phenomena. As in Section 4.5 choose

$$f(\psi) = \frac{1}{n} e^{n\psi} .$$

With this choice of the arbitrary function $f(\psi)$, the transition equation becomes

$$D^2\psi = \frac{1}{n} e^{n\psi} . \tag{5.3.2}$$

In order to study different types of flow patterns which can exist in an axisymmetric flow we look for some solutions of the transition Equation (5.3.2).

<u>Case 1</u>: Firstly, we study the solution of the transition Equation (5.3.2) which is independent of z. In this case (5.3.2) becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) = \frac{1}{n} e^{n\psi} . \tag{5.3.3}$$

Setting $r^2 = x$, and $\frac{\partial\psi}{\partial x} = L$, Equation (5.3.3) reduces to

$$L\frac{dL}{d\psi} = \frac{1}{n} e^{n\psi} \tag{5.3.4}$$

Integrating Equation (5.3.4) twice, yields

$$\sinh^{-1}\left(C_1 e^{-\frac{n}{2}\psi}\right) = \frac{\pm C_1 x + C_2}{\sqrt{2}} , \tag{5.3.5}$$

where C_1 and C_2 are constants of integration.

Now replacing x by r^2 and rewriting (5. 3. 5) we obtain

$$C_1 e^{-\frac{n}{2}\psi} = \sin h\{\frac{1}{\sqrt{2}}\left(\frac{C_1 r^2}{2} + C_2\right)\} . \tag{5. 3. 6}$$

Now we can determine the velocity components from (5. 3. 6). By differentiating Equation (5. 3. 6) with respect to r, and with the help of (5. 3. 6) we obtain

$$u_z = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\sqrt{2}C_1}{n} \cot h \{\frac{1}{\sqrt{2}}\left(\frac{C_1 r^2}{2} + C_2\right)\} , \tag{5. 3. 7}$$

which at large distances from the body becomes $\frac{\sqrt{2}C_1}{n}$. Also a case of particular interest arises when we choose $C_1 = C_2 = 0$, then Equation (5. 3. 6) gives

$$e^{-\frac{n}{2}\psi} = \frac{r^2}{2^{3/2}} . \tag{5. 3. 8}$$

In this limiting case the stream lines are circles given by

$$r = \text{constant.}$$

Thus the solution of the transition Equation (5. 3. 2) independent of z predicts a flow pattern for which the velocity at infinity becomes uniform and the stream lines are circles $r = \text{const}$, in particular case when the constants of integration are set equal to zero. This type of stream line pattern has been observed by some workers

(Batchelor, 1967).

Case 2: Another particular solution of the transition Equation (5. 3. 2) which is of interest may be obtained by starting from the equation

$$D^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial z^2} = \frac{1}{n}e^{n\psi}, \qquad (5.3.9)$$

and setting

$$\psi_1 = n\psi + 2 \ln r.$$

Then Equation (5. 3. 9) then becomes

$$r\frac{\partial}{\partial r}\left\{\frac{1}{r}\left(\frac{\partial\psi_1}{\partial r} - \frac{2}{r}\right)\right\} + \frac{\partial^2\psi_1}{\partial z^2} = e^{\psi_1}. \qquad (5.3.10)$$

A solution of Equation (5. 3. 10), which is independent of r, is given by the equation

$$\frac{d^2\psi_1}{dz^2} = e^{\psi_1}. \qquad (5.3.11)$$

This equation can be treated exactly in the same manner as Equation (5. 3. 3). Therefore a solution of Equation (5. 3. 11) is

$$C_1 e^{-\frac{\psi_1}{2}} = \sin h\left\{\frac{\pm 1}{\sqrt{2}}(C_1 z + C_2)\right\}, \qquad (5.3.12)$$

where C_1 and C_2 are arbitrary constants.

Putting the value of ψ_1 interm of ψ and taking $C_2 = 0$, we get

$$C_1 e^{-\frac{1}{2}n\psi} = r \sin h \{\frac{\pm 1}{\sqrt{2}}(C_1 z)\}. \qquad (5.3.13)$$

The limiting form of the stream lines, when $C_1 \to 0$ is given by

$$e^{-\frac{n}{2}\psi} = \frac{\pm rz}{\sqrt{2}} ,$$

that is in this case the stream lines are given by

$$rz = \text{constant}.$$

This type of stream line pattern is observed (Schlichting, 1968) for the laminar circular jet, which leaves a small circular opening and mixes with the surrounding fluid.

VI. EXTENSION OF TRANSITION CONCEPT TO MAGNETOHYDRODYNAMICS

6.1 Preliminary Remarks

Magnetohydrodynamics is a branch of continuum mechanics which deal with the motion of an electrically conducting fluid in the presence of a magnetic field. The motion of a conducting material across the magnetic lines of force creates potential differences which, in general, causes electric currents to flow. The magnetic fields associated with these currents modify the magnetic field which creates them. Thus there are two consequences:

1. An induced magnetic field associated with these currents appear, perturbing the original magnetic field.
2. An electromagnetic force due to the interaction of current and field appears, perturbing the original motion.

These are two basic effects of importance in magnetohydrodynamics. These two effects can be examined individually in any particular problem. In this chapter we shall be concerned with the two dimensional flow of a conducting fluid in the presence of any arbitrary magnetic field and extend the transition concepts developed so far to magnetohydrodynamics. In particular the following two aspects shall be examined:

1. Boundary layer thickness in magnetohydrodynamics.

2. Global representation of phenomena which exist in case of transition from boundary to main flow, when R_e (the Reynolds number) and R_m (the Magnetic Reynolds number) are large.

Note: In this chapter the symbol 'MHD' will be used for magnetohydrodynamics.

6.2 Magnetohydrodynamical Boundary Layer Thickness

The earliest known published works treating a problem in the flow of an electrically conducting fluid through a magnetic field are those of Hartmann (1937) and of Hartmann and Lazarus (1937). Since then a number of publications both theoretical and experimental have appeared. These publications include the following categories:

1. Flow past bodies of various shapes (Chandrasekhar, 1953; Michael, 1954; Stewartson, 1956; Rossow, 1958; Stewartson, 1960; Chawla, 1968;) etc.
2. Flow in channels (Chandrasekhar, 1951; Shercliff, 1953; Stuart, 1954) etc.
3. Astrophysical aspects of magnetohydrodynamics (Batchelor, 1950; Elasser, 1954) etc.
4. Magnetohydrodynamic waves (Alfven, 1943; Lundquist, 1949; Banos, 1955) etc.

In the present investigation we shall discuss how the magnetohydrodynamic boundary layer thickness is to be obtained on the basis of the new ideas concerning the boundary layer itself discussed in Chapter II.

The formation of the boundary layer is intimately connected with the process of diffusion and convection. As an illustration consider a fluid motion past a hot body. If the motion is slow or the conductivity high enough, diffusion is dominant and convection can be ignored, but if the motion is fast or the conductivity low, the heat diffuses with difficulty out into the main stream as it passes and the thermal disturbance is confined to a boundary layer and wake, called thermal boundary layer. More precisely, the conditions for these extremes to occur are respectively small or large values of the thermal Reynolds number, commonly known as the Peclet number, VL/α, where V is a typical velocity, L a typical length scale and α the thermal diffusivity.

The term "Reynolds number" refers to the quantities of the form $\{VL/(\text{diffusivity})\}$, which measures the extent to which a convection process prevails over a diffusion one. Besides thermal Reynolds number, there are two other commonly used Reynolds numbers; they are

$$R_e = VL/\nu \text{ (ordinary Reynolds number)},$$

$$R_m = VL/\lambda \text{ (magnetic Reynolds number)},$$

where ν is kinematic viscosity and $\lambda = \frac{1}{\mu\sigma}$ = magnetic diffusivity.

In viscous flow the viscosity causes vorticity to diffuse in the face of convection, and the ordinary Reynolds number measures the power of convection over diffusion of vorticity.

The equation for vorticity is given by

$$\frac{\partial \vec{\omega}}{\partial t} = \text{curl}\,(\vec{V} \times \vec{\omega}) + \frac{1}{R_e} \nabla^2 \vec{\omega} ,$$

where $\vec{\omega}$ is the vorticity.

In the above equation the first term on the right hand side is the convection term and the last term is the diffusion term. When convection dominates, that is when R_e is large, we may expect viscous boundary layer outside which the inviscid approximation will apply.

In MHD the equations for the motion of magnetic field corresponding to the vorticity equation mentioned above is

$$\frac{\partial \vec{B}}{\partial t} = \text{curl}\,(\vec{V} \times \vec{B}) + \frac{1}{R_m} \nabla^2 \vec{B} ,$$

where $\vec{B}$ is the magnetic field.

As before, if convection dominates diffusion that is when R_m is large, magnetic boundary layers near sources of field are to be expected; elsewhere the approximation of perfect conductivity would be valid. On a broader basis Zhigulev (1959) defines a magnetic boundary layer in the immediate vicinity of the plate a layer such that magnetic field disappears in the basic flow and remains only in a thin layer adjoining the surface of the plate. The magnetic Prandtl number is R_m / R_e. When it is small, as it is in liquid metals and low temperature plasmas, magnetic fields diffuse much more rapidly than vorticity and magnetic boundary layers are much thicker than viscous ones.

From the above two considerations it is clear that if R is large as compared to one, then convection dominates over diffusion and viscous boundary layer forms. Similarly if R_m is large as compared to one, then convection of magnetic field prevails over diffusion and a magnetic boundary layer is formed. Thus if both R_e and R_m are large as compared to one, then both the magnetic and viscous effects will be predominant near the boundary. Thus it will be of interest to consider the MHD boundary layer, which is the viscous boundary layer as affected by magnetic effects. To be specific, MHD boundary layer is the viscous boundary layer in case of a conducting fluid in the presence of a magnetic field. This magnetohydrodynamic boundary layer is definitely different from viscous boundary layer because due to finite conductivity of the fluid, the induced currents within the boundary layer tend to spread away from the wall, pushing the vorticity out of the magnetic boundary layer. This results in the thickening of the boundary layer (Chawla, 1967).

We shall estimate the MHD boundary layer thickness on the basis of magnitude analysis as discussed in Chapter II. In order to do that, we sum up the basic differential equations in the case of incompressible MHD flow. Under the usual MHD approximations the equations are (Shercliff, 1965, p. 24)

$$\operatorname{curl} \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{(Faraday's law)}, \tag{6. 2. 1}$$

$$\operatorname{curl} \vec{B} = \mu \vec{j} \quad \text{(Ampere's law)}, \tag{6. 2. 2}$$

$$\operatorname{div} \vec{B} = 0, \tag{6. 2. 3}$$

$$\operatorname{div} \vec{j} = 0 \quad \text{(Kirchhoff's first law)}, \tag{6. 2. 4}$$

$$\vec{j} = \sigma(\vec{E} + \vec{V} \times \vec{B}) \quad \text{(Ohm's law, without Hall effect)}, \tag{6. 2. 5}$$

$$\operatorname{div}(\rho \vec{V}) = -\frac{\partial \rho}{\partial t} \quad \text{(continuity equation)}, \tag{6. 2. 6}$$

$$\rho \frac{D\vec{V}}{Dt} + \operatorname{grad} p = \vec{j} \times \vec{B} + \eta \nabla^2 \vec{V} + \frac{1}{3}\eta \operatorname{grad}(\operatorname{div} \vec{V}), \tag{6. 2. 7}$$

where

$\vec{V}$ = velocity vector,

ρ = density of the material,

σ = electrical conductivity of the material,

$\vec{B}$ = magnetic field,

$\vec{E}$ = the electric field,

$\vec{j}$ = current density vector,

μ = permeability of the material,

η = viscosity of the fluid.

After some simplifications, the equations governing the vectors $\vec{V}$ and $\vec{B}$ can be obtained from the equation (6. 2. 1) to (6. 2. 7) as

$$\operatorname{div} \vec{V} = 0 \quad \text{and} \quad \operatorname{div} \vec{B} = 0, \tag{6. 2. 8}$$

$$\frac{\partial \vec{B}}{\partial t} + (\vec{V} \cdot \operatorname{grad}) \vec{B} = (\vec{B} \cdot \operatorname{grad}) \vec{V} + \lambda \nabla^2 \vec{B}, \tag{6. 2. 9}$$

and

$$\rho\left(\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \operatorname{grad}) \vec{V}\right) + \operatorname{grad} p = \frac{1}{\mu} (\vec{B} \cdot \operatorname{grad}) \vec{B} + \eta \nabla^2 \vec{V} \tag{6. 2. 10}$$

where

$$\lambda = \frac{1}{\mu \sigma}.$$

In order to estimate the magnetohydrodynamic boundary layer thickness for a two dimensional flow, consider a two dimensional problem then the components of the vectors $\vec{V}$ and $\vec{H}$ may be taken in rectangular cartesian coordinates as

$$\vec{V} = (u, v, 0)$$

and

$$\vec{H} = (H_x, H_y, 0).$$

With this choice of vectors $\vec{V}$ and $\vec{H}$ the Equations (6. 2. 8) to (6. 2. 10) for a steady case may be written in non-dimensional form as

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - R_H(H_x\frac{\partial H_x}{\partial x} + H_y\frac{\partial H_x}{\partial y}) = -\frac{\partial}{\partial x}(p + R_H\frac{H^2}{2}) + \frac{1}{R_e}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{6.2.11}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - R_H(H_x\frac{\partial H_y}{\partial x} + H_y\frac{\partial H_y}{\partial y}) = -\frac{\partial}{\partial y}(p + R_H\frac{H^2}{2}) + \frac{1}{R_e}(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{6.2.12}$$

$$u\frac{\partial H_x}{\partial x} + v\frac{\partial H_x}{\partial y} - (H_x\frac{\partial u}{\partial x} + H_y\frac{\partial u}{\partial y}) = \frac{1}{R_m}(\frac{\partial^2 H_x}{\partial x^2} + \frac{\partial^2 H_x}{\partial y^2}) \tag{6.2.13}$$

$$u\frac{\partial H_y}{\partial x} + v\frac{\partial H_y}{\partial y} - (H_x\frac{\partial v}{\partial x} + H_y\frac{\partial v}{\partial y}) = \frac{1}{R_m}(\frac{\partial^2 H_y}{\partial x^2} + \frac{\partial^2 H_y}{\partial y^2}) \tag{6.2.14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6.2.15}$$

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0 \tag{6.2.16}$$

where the same symbols have been retained for the non-dimensional variables as for their dimensional counterparts and

$R_e = \frac{VL}{\nu}$ (Reynolds number).

$R_m = \frac{VL}{\lambda} = \mu\sigma VL$ (magnetic Reynold number).

$R_H = \frac{\mu H_0^2}{\rho V^2}$ (magnetic pressure number).

$R_h = \mu H_0 L(\frac{\sigma}{\eta})^{1/2}$ (Hartmann number).

Here V, L, H_0 are some characteristic velocity, length and magnetic field. The order of magnitude of different terms involved may be taken as

$$y \sim \delta, \quad v \sim \delta^{\frac{1}{k}}, \quad u \sim 1, \quad x \sim \delta^{1-\frac{1}{k}}$$

and

$$H_x \sim 1, \quad H_y \sim \delta^{\frac{1}{k}}, \quad \text{where} \quad 0 < k < 1.$$

With the help of these estimates, the orders of magnitudes of different terms in Equations (6. 2. 11) and (6. 2. 12) can be estimated as discussed in Chapter II, Section 4. From Equation (6. 2. 11) the order of the ratio of the magnitude of viscous forces and magnetic forces is

$$\frac{1}{R_e R_H} \delta^{-(1+\frac{1}{k})}. \tag{6. 2. 15}$$

Also the Hartmann number is

$$R_h = \mu H_0 L (\frac{\sigma}{\eta})^{1/2} \doteq \sqrt{\frac{\text{magnetic forces}}{\text{viscous forces}}} = \sqrt{R_e R_H R_m} \tag{6. 2. 16}$$

In view of Equation (6. 2. 16), Equation (6. 2. 15) may be rewritten as

$$\frac{\text{viscous forces}}{\text{magnetic forces}} \sim \frac{1}{R_e R_H} \delta^{-(1+\frac{1}{k})} \sim (\frac{1}{R_h})^p, \tag{6. 2. 17}$$

where p is some rational number. The last estimate in (6. 2. 17) follows from the consideration that the ratio of viscous to magnetic forces is of the order of some power of the Hartmann number which characterizes the MHD flow. On simplification Equation (6. 2. 17) gives

$$\delta \sim R_m^{-s} R_h^{-st} , \tag{6. 2. 18}$$

where

$$\frac{k}{k+1} = s \quad \text{and} \quad p - 2 = t.$$

Equation (6. 2. 18) gives the magnetohydrodynamic boundary layer thickness in terms of two parameters s and t. These two parameters are, however, to be determined in an actual physical situation by experiments.

6. 3 Two Dimensional Transition Equation in Magnetohydrodynamics

In the MHD case, the idea of transition and transition region is basically the same as discussed in Chapter IV for two dimensional ordinary hydrodynamic flow, the only difference being that in MHD two non-dimensional numbers R_e and R_m are involved instead of one (R_e). Here the transition equation will be the limiting form of the governing differential system for large values of R_e and R_m and the transition region will be the whole of the region excluding the MHD boundary layer.

Two dimensional, incompressible, steady MHD flow is governed by Equations (6. 2. 11) to (6. 2. 16). If we write

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{6. 3. 1}$$

and

$$H_x = \frac{\partial \phi}{\partial y}, \; H_y = -\frac{\partial \phi}{\partial x}, \tag{6. 3. 2}$$

then the Equations (6. 2. 15) and (6. 2. 16) are automatically satisfied.

Eliminating the pressure term from Equations (6. 2. 11) and (6. 2. 12) and making use of Equations (6. 3. 1) and (6. 3. 2), we get

$$J_{xy}(\nabla^2\psi, \psi) - R_H J_{xy}(\nabla^2\phi, \phi) = \frac{1}{R_e} \nabla^4\psi, \tag{6. 3. 3}$$

where

$$J_{xy}(\nabla^2\psi, \psi) = \begin{vmatrix} \frac{\partial}{\partial x}(\nabla^2\psi) & \frac{\partial}{\partial y}(\nabla^2\psi) \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \end{vmatrix} .$$

Therefore the transition equation is

$$J_{xy}(\nabla^2\psi, \psi) - R_H J_{xy}(\nabla^2\phi, \phi) = 0, \tag{6. 3. 4}$$

which may be rewritten as

$$J_{\phi\psi}(\nabla^2\psi, \psi) - R_H J_{\phi\psi}(\nabla^2\phi, \phi) = 0. \tag{6. 3. 5}$$

A solution of the transition Equation (6. 3. 5) can be readily written as

$$\nabla^2 \psi = f_\psi(\phi, \psi), \tag{6.3.6}$$

and

$$\nabla^2 \phi = - \frac{1}{R_H} f_\phi(\phi, \psi), \tag{6.3.7}$$

where f is an arbitrary function,

$$f_\psi = \frac{\partial f}{\partial \psi} \quad \text{and} \quad f_\phi = \frac{\partial f}{\partial \phi} .$$

Similarly, the transition equation for large values of the magnetic Reynolds number R_m can be obtained from Equations (6. 2. 12) and (6. 2. 13) in vector form as

$$\text{curl } (\vec{V} \times \vec{H}) = 0. \tag{6.3.8}$$

Making use of the values of $\vec{V}$ and $\vec{H}$ as given by Equations (6.3.1) and (6. 3. 2) in Equation (6. 3. 8) and after some simplification, we get

$$\frac{\partial}{\partial y} (J_{xy}(\psi, \phi)) = 0, \tag{6.3.9}$$

and

$$\frac{\partial}{\partial x} (J_{xy}(\psi, \phi)) = 0. \tag{6.3.10}$$

Obviously Equations (6. 3. 9) and (6. 3. 10) will be satisfied if

$$J_{xy}(\psi, \phi) = k, \tag{6.3.11}$$

where k is a constant.

A general solution of the differential Equation (6. 3. 11) can be written as

$$\psi = g(\phi) + Cx + Dy, \tag{6. 3. 12}$$

where C and D are constants and ϕ satisfies the differential equation

$$C\frac{\partial\phi}{\partial y} - D\frac{\partial\phi}{\partial x} = k. \tag{6. 3. 13}$$

The solution of the differential Equation (6. 3. 13) is

$$\phi_1(kx + D\phi, ky - C\phi) = 0,$$

where ϕ_1 is an arbitrary function.

Making use of Equations (6. 3. 12) and (6. 3. 13), a solution of the differential Equation (6. 3. 11) may be written as

$$\psi = g(Ax + By + E) + Cx + Dy, \tag{6. 3. 14}$$

where

$$BC - AD = k.$$

In like manner ϕ can be obtained as a function of x and y.

With this choice of solutions for ϕ and ψ, it is clear that interaction between $\vec{H}$ and $\vec{V}$ may be neglected in the transition region as can be expected. With this in mind, the solution of Equation

(6.3.5) as given by (6.3.6) and (6.3.7) becomes,

$$\nabla^2\psi = f_\psi(\psi), \tag{6.3.15}$$

and

$$\nabla^2\phi = f_\phi(\phi). \tag{6.3.16}$$

It is clear from the above analysis that in the case of MHD the transition equations are given by Equations (6.3.5) and (6.3.8) and our object is to find solutions for these equations. These equations will be satisfied if ψ and ϕ have forms similar to the one given by Equation (6.3.14) and are solutions of Equations (6.3.15) and (6.3.16) respectively. In the remainder of this chapter, we shall discuss some cases of special interest.

As discussed in Chapter IV, Section 4, the function $f_\psi(\psi)$ may be chosen as

$$f_\psi(\psi) = \frac{4}{n} e^{n\psi}. \tag{6.3.17}$$

By making the substitutions

$$z = x + iy,$$

$$\bar{z} = x - iy,$$

Equation (6.3.15), with the help of Equation (6.3.17) becomes

$$\frac{\partial^2\psi}{\partial z\partial\bar{z}} = \frac{1}{n} e^{n\psi} \tag{6.3.18}$$

and ψ may now be taken as

$$\psi = F[A(z) + B(\bar{z}) + E] + C(z) + D(\bar{z})$$

where A, C and B, D are arbitrary function of z and $\bar{z}$ respectively and E is a constant.

Equation (6. 3. 18) is exactly of the same form as Equation (4. 4. 6) and hence its solution may be written down readily from Equation (4. 4. 16), that is

$$e^{n\psi} = \frac{2A'(z)B'(\bar{z})}{\{A(z)+B(\bar{z})\}^2}.$$

Here $A(z)$ and $B(\bar{z})$ are arbitrary functions, therefore an infinite number of particular solutions can be built up for different choices of these functions. In particular the following two choices of $A(z)$ and $B(\bar{z})$

$$A(z) = \ln z, \qquad B(\bar{z}) = -\ln \bar{z}$$

and

$$A(z) = z^{\pm m}, \qquad B(\bar{z}) = (\bar{z})^{\pm m}$$

give spiral formation, as discussed in Chapter 4, Section 4.

From the above analysis it is clear that, in general, spiral formation can exist in MHD cases also. Besides spirals, all those phenomena which are discussed in Chapter 4, for ordinary viscous flow can also take place in MHD, because in these two cases the transition equation is the same. But it should be clear in our minds that it is not necessary that these spirals should exist in each MHD case, because they depend on the type of the magnetic field affecting the flow (Sears, 1960). In some cases the effect of the MHD forces produce vorticity and in other cases they may supress the vorticity depending upon the nature of the magnetic field.

A similar analysis can be carried out for the case of the function ϕ also.

VII. SUMMARY, DISCUSSION AND SCOPE OF FURTHER RESEARCH

7.1 Summary and Discussion

The problem involved is the investigation of interaction or border fields, which may be called the transition fields, without assuming the ad-hoc laws. The presence of spin, rotation or vorticity in the transition makes it a non-linear, irreversible and non-conservative phenomenon and hence cannot be treated satisfactorily by perturbation techniques. It seems very possible that while the basic mechanisms may differ greatly in their nature, the properties of the aggregate take only restricted forms. Thus, though the formulation and explanation of isolated processes is very important, a full interpretation makes the study of combining these processes in terms of aggregate structures no less urgent. An attempt has been made in this thesis, to reexamine the underlying concepts in the case of transition from a boundary layer flow to the main stream, where a laminar flow exists. The classical boundary layer theory due to Prandtl is based on his main assumptions that (1) in the boundary layer the viscous and inertial forces are of the same order, (2) the transverse velocity in the case of a flat plate is taken of the same order as that of the transverse coordinate, (3) the variation of pressure in the boundary layer is negligible. On careful examination, it becomes clear that the above assumptions are not quite reasonable. In the present investigation the boundary layer thickness is estimated without making any of these assumptions since the ratio of the viscous to the inertial forces varies continuously from infinity near the boundary to zero at the outer edge of the boundary layer. Also the order of the transverse velocity need not be the same as that of the transverse coordinate and the continuity of the pressure across the boundary layer comes out from the transition analysis and therefore it is not necessary to assume it.

By an order of magnitude analysis of different terms in the Navier-Stokes equation, an estimation of the boundary layer thickness for two dimensional flow is obtained in terms of two parameters. One of these parameters depends upon the relative order of magnitude of viscous and inertia forces at the outer edge of the boundary layer and the second depends upon the order of vorticity allowable at the outer edge of the boundary layer. This general result includes all the known estimations for boundary layer thickness as particular cases, and Prandtl's result as a very strong limiting case. This result shows that usually the boundary layer thickness is greater than that given by Prandtl. This is otherwise also clear, because Prandtl's result does not take into account the vorticity which may exist outside the boundary layer, whereas our result includes this effect. Also the ratio of the viscous to the inertia forces varies from infinity near the boundary to zero at the outer edge of the boundary layer, it assumes the value unity somewhere within the boundary layer. Thus Prandtl's boundary layer thickness is much smaller than what the actual thickness should be.

In order to study the transition region, limiting form of the Navier-Stokes equations is obtained for the large values of the Reynolds number. The equation so obtained is called the transition equation. This equation holds good at all points of the region excepting the boundary layer and thus includes the main flow and the stagnation points. Owing to the importance of the vorticity in the transition region, the transition equation is solved for the vorticity. The form of the vorticity shows that, in general, the transition functions are either subharmonic or superharmonic functions. Heuristically this is otherwise also clear, since any transition in harmonic or biharmonic fields, which permeates the natural phenomenon, may be expected to exhibit itself in terms of the allied subharmonic or superharmonic fields. The transition equation in three dimensions is very complicated, but its general solution in terms of vorticity, includes the Beltrami's flow, which can occur in three dimensional flows.

In particular the transition equation for the axisymmetric and two dimensional flow is also obtained. Some particular solutions of axisymmetric transition equation are obtained including as special cases the flow patterns obtained by earlier workers. The prime object of obtaining the transition equation in this case is to give a global representation to the different phenomena which exist in an axisymmetric flow.

As has been pointed out earlier that the vorticity plays a dominant role in the transition, therefore the transition equation for two dimensional flow is solved for vorticity. The form of vorticity again confirms the previous general result that the transition fields are either subharmonic or superharmonic fields. The transition equation so obtained is solved by noting that the vorticity in the transition region may take the Gaussian distribution form after sufficient lapse of time as is evidenced by available informations concerning natural phenomena. This solution is obtained in terms of two arbitrary functions. An infinite number of solutions can be built up for different choices of these arbitrary functions. One particular choice of this function has given the spiral formations which exists for a real flow past any body, and for which no treatment exists in current literature. Another choice of these functions has given the formation of "cylindrical vortices." This indicates that it is not necessary to study the cylindrical vortices by assuming two stream functions and then matching them on the bounday, as is currently done. But these can be studied with the help of transition equations without any such matching technique. These cylindrical vortices correspond to the limiting values of the stream function. The two dimensional transition equation also includes the main flow and the stagnation points. Therefore it can be summed up that the transition equation is a global representation of all the phenomena which exist in the transition region for flow past a body at large Reynolds number.

The transition concept has been extended for the MHD case. In this case, the MHD boundary layer is defined as the viscous boundary layer in the case of a conducting fluid in the presence of a magnetic field. The MHD boundary layer thickness is also obtained on the basis of a magnitude analysis, in terms of two parameters, which can be obtained for any particular flow on experimental basis. Also the transition equations have been obtained for MHD case, which are the limiting forms of the usual MHD equations, for large values of the Reynold and magnetic Reynold numbers. The solutions of these equations shows that in general, spirals can exist in MHD cases also.

7.2 Scope of Further Research

An example of shockwave transition for one dimensional compressible flow as done by Seth is discussed in Chapter I. This result can be extended on the basis of transition concept to two and three dimensions and for various types of shocks.

As has been pointed out earlier, non-linear vortex formation which is observed at the separation and in the wake of the real fluid is represented by superharmonic or subharmonic functions. This is a theoretical result and this result could be identified with experimental data.

The present concept of transition analysis of boundary layer can be extended to wake, separation and turbulence and the results so obtained can be confirmed with experimental results.

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