AN ABSTRACT OF THE THESIS OF

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Polyhedra are geometric representations of linear systems of equations and inequalities. Since polyhedra are used to represent the iteration domains of nested loop programs, procedures for operating on polyhedra can be used for doing loop transformations and other program restructuring transformations which are needed in parallelizing compilers. Thus a need for a library of polyhedral operations has recently been recognized in the parallelizing compiler community.

Polyhedra are also used in the definition of domains of variables in systems of affine recurrence equations (SARE). ALPHA is a language which is based on the SARE formalism in which all variables are declared over polyhedral domains consisting of finite unions of polyhedra. This thesis describes a library of polyhedral functions which was developed to support the ALPHA langauge environment, and which is general enough to satisfy the needs of researchers doing parallelizing compilers.

This thesis describes the data structures used to represent domains, gives the motivations for the major design decisions that were made in creating the library, and presents the algorithms used for doing polyhedral operations. A new algorithm for recursively generating the face lattice of a polyhedron is also presented.

This library has been written and tested, and has be in use since the first quarter of 1993. It is used by research facilities in Europe and Canada which do research in parallelizing compilers and systolic array synthesis. The library is freely distributed by ftp.

# A Library for Doing Polyhedral Operations

by

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# Chapter 1

# INTRODUCTION

A large class of algorithms in linear algebra and digital signal processing can be described in terms of systems of affine recurrence equations(SARE) (see appendix A). Much work has been done recently in the development of methods to do synthesis, analysis and verification of systems of recurrence equations in order to find equivalent parallel forms of these algorithms suitable for implementation. The ultimate goal is to transform an algorithm from a mathematical type of description into an equivalent form that can be implemented either with special purpose hardware (with systolic arrays for instance) or implemented as a program which is able to run on a multiprocessor system. The ALPHA language was invented to be able to do just this kind of program transformation.

This thesis is not about ALPHA per se, however the work presented here was done in connection with the implementation of an ALPHA environment based on the commercially available MATHEMATICA system. This environment is illustrated in figure 1.1. In the following section, an informal description of the ALPHA language will be presented. It is not a complete description, but is intended only to give the reader an introduction to the language, and to provide a context and motivation for this work. A full understanding of the ALPHA language is not a prerequisite to understanding this thesis. Complete descriptions of ALPHA may be found in [Mau89, in French], and in [LMQ91, in English].

## 1.1 Introduction to ALPHA

The ALPHA language is based on the formalism of affine recurrence equations. It was originally designed for systolic array research at IRISA in Rennes, France. Using

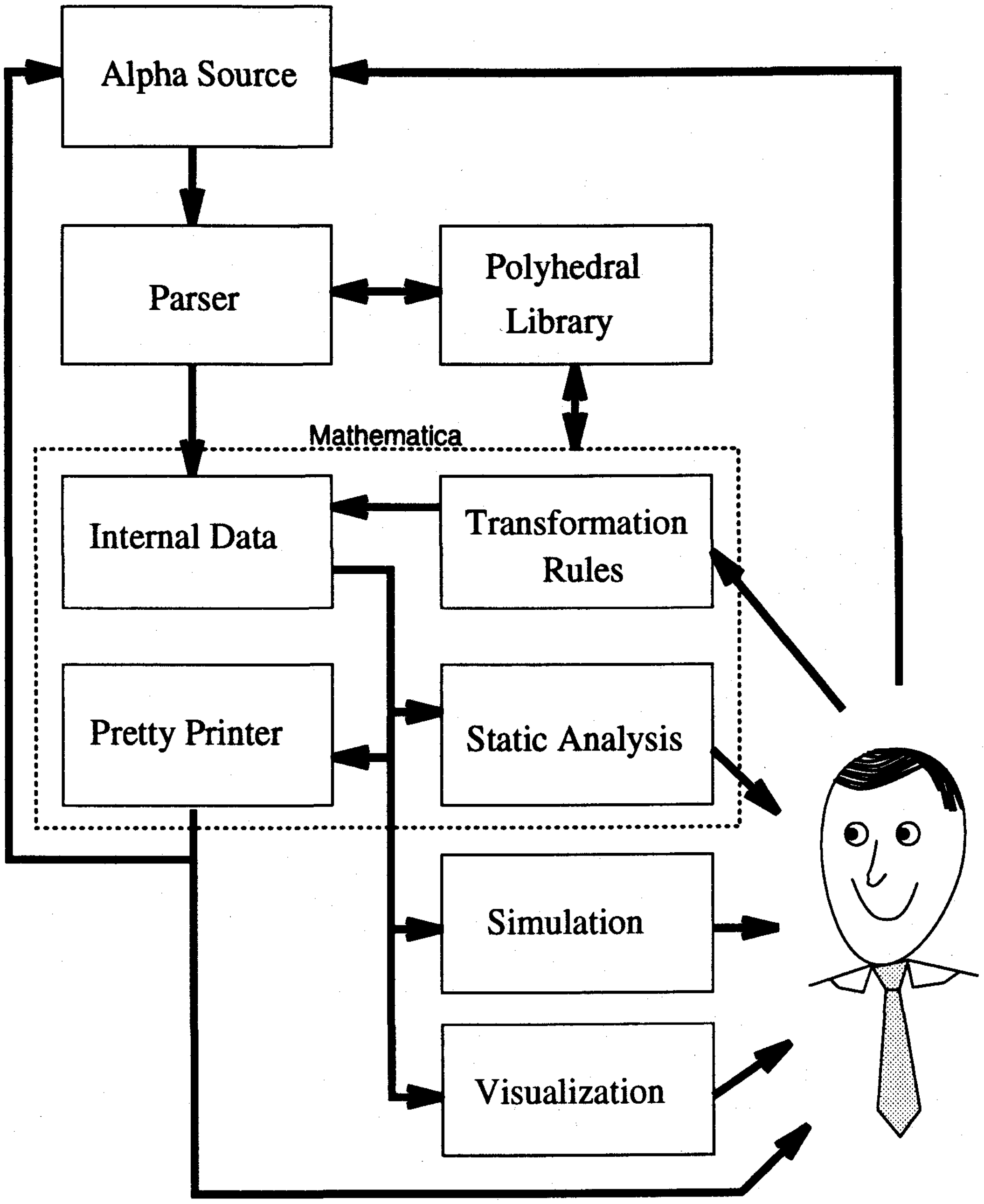


Figure 1.1: Alpha System Environment

the ALPHA language, algorithms can be specified at a mathematical level due to the expressive power of systems of recurrence equations. The ALPHA environment then provides transformation tools which can be used to transform the algorithmic program into an equivalent program suitable for implementation as a systolic array. ALPHA is similar to the CRYSTAL language [Che86] which was also designed to facilitate space-time transformations of programs, but ALPHA is more restrictive than CRYSTAL. They both are based on the substitution principle which allows program transformations to be done through syntactic rewriting. Data dependencies between variables in ALPHA are restricted to be affine functions, in the sense of affine recurrence equations defined in appendix A. ALPHA was designed to have three important properties:

1. It has the expressive power needed to represent algorithms at many levels of abstraction;
2. It is closed under a set of useful program transformations (such as Change of Basis, Variable Splitting, Variable Merging, Substitution, Pipelining, Normalization);
3. Data dependencies can be analyzed statically, since they are constant affine functions of indices, and have no "data dependent" (dynamic) components.

Systems of affine recurrence equations such as ALPHA are referentially transparent, meaning that an expression has the same meaning in every context and evaluation of a recurrence equation has no side effects. This property is shared by functional languages which makes describing algorithms with recurrence equations very similar to programming in a functional language. An algorithm written in ALPHA is described in a mathematical notation well suited for describing algorithms at a high level. Selection of an appropriate sequence of transformations will reformulate this description into a different but equivalent form which is more suitable for implementation. The static analyzability of ALPHA helps one to be able to choose which transformations to make.

Figure 1.2 shows an example ALPHA program for a simple convolution filter. Given a sequence $x_i$, where $i \geq 1$, and a set of coefficients $a_i$, where $i = 1 \cdots 4$, the convolution filter computes an output sequence $y_i = \sum_{j=1}^{4} a_j x_{i-j+1}$ for $i \geq 4$. This function is expressed as a simple set of recurrence equations in the ALPHA language.

Figure 1.3 shows the same program after having been transformed by the ALPHA system into a form amenable for implementation as a systolic array. The following

```
system convolution (a : { i | 1<=i<=4 } of integer;
                    x : { i | i>=1    } of integer)
     returns    ( y : { i | i>=4    } of integer);
  var
    sum :  { i,j | 0<=j<=4 ; i>=j }  of integer;
  let
    sum = case
          { i,j | j=0 }      : 0 .(i,j->);
          { i,j | 1<=j<=4 } : sum .(i,j->i,j-1)
                            + a .(i,j->j) * x .(i,j->i-j+1);
          esac;
    y = sum .(i->i,4);
  tel;
```

Figure 1.2: Alpha Program of a Convolution Filter

transformations are performed on the program in figure 1.2 in order to derive the program in figure 1.3:

1. The variable $a$ is pipelined in the $i$-direction creating new local variable $A$.
2. The variable $x$ is pipelined in the $j$-direction creating new local variable $X$.
3. The basis of variable $A$ is changed according to the transformation function $(i,j) \rightarrow (i+j,j)$ and the indices are renamed $(i,j) \rightarrow (t,p)$ representing time and processor number.
4. The basis of variable $X$ is changed using the same change of basis function.
5. The basis of variable *sum* is changed using the same change of basis function.

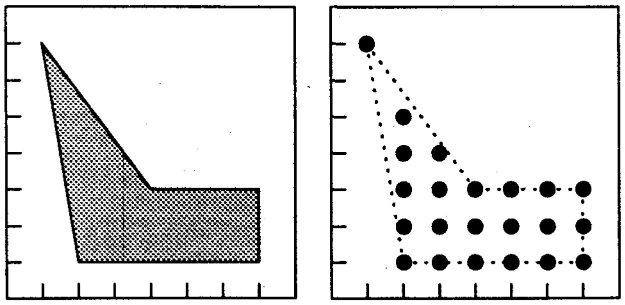
Even though the resulting program looks much more complicated, it is equivalent to the original program because it was transformed from the original (proof by construction). Furthermore, it is in a form which can be implemented using a systolic array.

A change of basis of a program variable is an example of an ALPHA program transformation. Transformations such as loop reindexing (e.g. index skewing, loop

```
system convolution (a : { i | 1<=i<=4 } of integer;
                    x : { i | i>=1 }    of integer)
       returns     (y : { i | i>=4 }    of integer);
  var
    X   : { t,p | 1<=p<=4; t>=2p } of integer;
    A   : { t,p | 1<=p<=4; t>=2p } of integer;
    sum : { t,p | 0<=p<=4; t>=2p } of integer;
  let
    A = case
        { t,p | t=2p     } : a.(t,p->p);
        { t,p | t>=2p+1 } : A.(t,p->t-1,p);
        esac;
    X = case
        { t,p | p=1  } : x.(t,p->t-2p+1);
        { t,p | p>=2 } : X.(t,p->t-1,p-1);
        esac;
  sum = case
        { t,p | p=0  } :   0.(t,p->);
        { t,p | p>=1 } : sum.(t,p->t-1,p-1) + A * X;
        esac;
    y = sum.(i->i+4,4);
  tel;
```

Figure 1.3: Alpha Program of a Convolution Filter

exchanges), uniformalization of communication and space-time mapping are all examples of doing changes of bases of program variables. An execution schedule for a variable may be found by solving a mixed linear programming problem [Dar93]. The result is a function which maps each element of a variable to a time instant. The schedule may be reflected back into the program by performing a change of basis on program variables, transforming one (or more) of the variable indices into time indices. To perform this transformation, there are certain computations involving unions of polyhedral domains that have to be performed.



| | | | |
|---|---|---|---|
| X(1,7) | X(3,1) | X(4,3) | X(6,3) |
| X(2,1) | X(3,2) | X(5,1) | X(7,1) |
| X(2,2) | X(3,3) | X(5,2) | X(7,2) |
| X(2,3) | X(3,4) | X(5,3) | X(7,3) |
| X(2,4) | X(4,1) | X(6,1) | |
| X(2,5) | X(4,2) | X(6,2) | |

Variable X

Figure 1.4: Comparison of Polyhedra, Domain, and Variable

## 1.2 The Role of the Polyhedral Library

In figure 1.1, there is a block labeled "Polyhedral Library". This is a library which operates on objects called *domains* made of unions of polyhedra. When specifying a system of affine recurrence equations, unions of polyhedra are used to describe the domains of computation of system variables. Whereas a polyhedron is a region containing an infinite number of rational[1] points, a *domain*, as the term is used in this thesis, refers to the set of integral points which are inside a polyhedron (or union of polyhedra). Figure 1.4 illustrates this difference.

**Definition 1.1** *A polyhedral* **domain** *of dimension $n$ is defined as*

$$\mathcal{D} : \{i \mid i \in \mathcal{Z}^n, i \in \mathcal{P}\} = \mathcal{Z}^n \cap \mathcal{P} \tag{1.1}$$

*where $\mathcal{P}$ is a union of polyhedra of dimension $n$.*

In affine recurrence equations of the type considered here and in the ALPHA language, every variable is declared over a domain. Elements of a variable are in a one-to-one correspondence with points in a domain. Again, figure 1.4 illustrates this. Here, we formalize the definition of a variable.

[1]Polyhedra may also be defined over the reals, however, only rationals are considered in this thesis.

**Definition 1.2** *A* **variable** $X$ *of type* "datatype" *declared over a domain* $\mathcal{D}$ *is defined as*

$$X ::= \{ \, X_i \; : \; X_i \in \text{datatype}, \; i \in \mathcal{D} \, \} \tag{1.2}$$

*where* $X_i$ *is the element of* $X$ *corresponding to the point* $i$ *in domain* $\mathcal{D}$.

$X$ can also be thought of as a function: $X : i \in \mathcal{D} \mapsto X_i \in$ *datatype.*

In order to be able to manipulate ALPHA variables, a library of "domain functions" is needed. This library is the geometric engine of the language and provides the capabilities needed for programs to be analyzed and transformed. This thesis presents the polyhedral library which was written to support the ALPHA environment. Examples of domain operations which can be performed by the library are: Image, Preimage, Intersection, Difference, Union, ConvexHull, and Simplify. The implementation of these and other library functions are described in detail in chapter 4.

Even though the library was written to support the ALPHA environment, it is also general purpose enough to be used by other applications as well.

## 1.3 Construction of Face Lattices

Closely related to the functions in the polyhedral library, but not part of the library are algorithms to generate the face lattice of a polyhedron. In this thesis, I present algorithms that I developed to compute the face lattice of a polyhedron. The computation of the face lattice is important for analyzing polyhedra which are described parametrically.

## 1.4 Summary of Chapters

Chapter 2 is necessary background information and a review of the fundamental definitions relating to polyhedra. Chapter 3 discusses issues relating to how a polyhedron is represented in memory, and the polyhedron data structure is developed and presented in detail. Chapter 4 describes the polyhedral library itself, giving the basic algorithms for all of the operations. Chapter 5 presents new algorithms developed to construct the face lattice of a polyhedron. Chapter 6 is a conclusion and summary of the thesis.

# Chapter 2

# POLYHEDRA

Polyhedra have been studied in several related fields: from the geometric point of view by computational geometrists [Gru67], from the algebraic point of view by the operations research and linear programming communities [Sch86], and from the structural/lattice point of view by the combinatorics community [Ede87]. Each community has a different view of polyhedra so the notation and terminology are sometimes different between the different disciplines.

This chapter is a review of fundamental definitions relating to polyhedra and cones. I have taken the majority of this summary from the works of Grunbaum, "Convex Polytopes" [Gru67], and of Schrijver, "Theory of Linear and Integer Programming" [Sch86], and of Edelsbrunner, "Algorithms in Combinatorial Geometry" [Ede87]. Other references used are [Weh50, KT56].

## 2.1 Notation and Prerequisites

In this presentation, polyhedra are restricted to being in the $n$-dimensional rational Cartesian space, represented by the symbol $\mathcal{Q}^n$. All matrices, vectors, and scalars are thus assumed to be rational unless otherwise specified.

**Definition 2.1** *The* **scalar product** $a \circ b$ *is defined as* $a \circ b = a^T b = \sum_{i=1}^{n} a_i b_i$ *where* $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ *and* $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$.
$a \circ b = 0$ *iff vectors* $a, b$ *are* **orthogonal**.

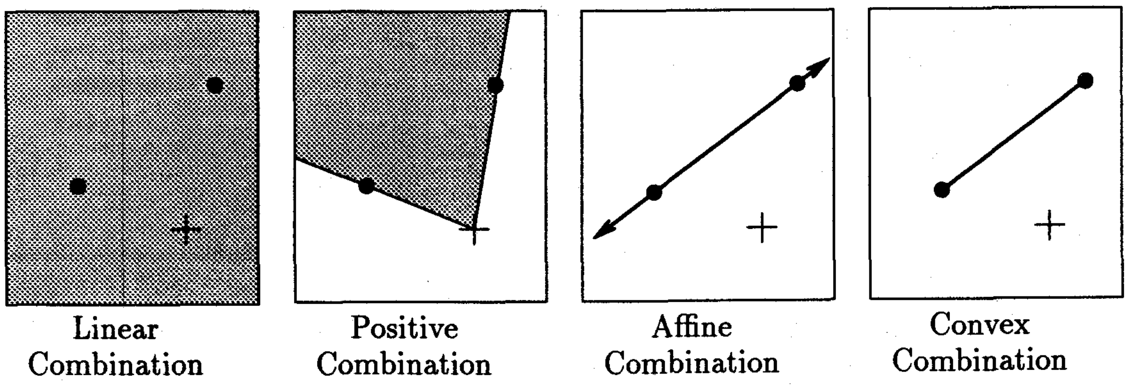


Figure 2.1: Geometric Interpretations of the Combinations of Two Points

**Definition 2.2** *Given a vector $x$ and a scalar coefficient vector $\lambda$, the following different combinations are defined:*

*A* **linear combination** $\sum \lambda_i x_i$

*A* **positive**[1] **combination** $\sum \lambda_i x_i$ *where all* $\lambda_i \geq 0$

*An* **affine combination** $\sum \lambda_i x_i$ *where* $\sum \lambda_i = 1$

*A* **convex combination** $\sum \lambda_i x_i$ *where* $\sum \lambda_i = 1$ *and all* $\lambda_i \geq 0$.

Figure 2.1 shows the geometries generated by the different combinations of two points in 2-space (with origin marked '+').

## 2.2 Sets

A set, in this context, always refers to a set of points in space $\mathcal{Q}^n$. Most definitions have meanings on any set of points (not necessarily polyhedral). These definitions are introduced in this section.

**Definition 2.3** *Given a non-zero vector $y$ and a constant $\alpha$, the following objects (sets of points) are defined:*

*A* **hyperplane** $\mathcal{H} = \{x \mid x \circ y = \alpha\}$

[1]Also called *non-negative* or *conic* combination

*A* **open half-space** $\mathcal{H} = \{x \mid x \circ y > \alpha\}$

*A* **closed half-space** $\mathcal{H} = \{x \mid x \circ y \geq \alpha\}$

**Definition 2.4** *A* **vertex** *of a set $\mathcal{K}$ is any point in $\mathcal{K}$ which cannot be expressed as a convex combination of any other distinct points in $\mathcal{K}$.*

**Definition 2.5** *A* **ray** *of $\mathcal{K}$ is a vector $r$, such that $x \in \mathcal{K}$ implies $(x + \mu r) \in \mathcal{K}$ for all $\mu \geq 0$.*

A ray is not a set of points, but a direction in which $\mathcal{K}$ is infinite. A ray may be considered as a point at infinity in the direction of $r$.

**Definition 2.6** *A ray of $\mathcal{K}$ is an* **extreme ray** *if and only if it cannot be expressed as a positive combination of any other two distinct rays of $\mathcal{K}$. The set of extreme rays form a basis which describes all directions in which the convex set is open. Extremal rays are unique up to a multiplicative constant.*

An extreme ray may be considered as a vertex at infinity in the direction of $r$.

**Definition 2.7** *A* **line** *(or* **bidirectional ray***) of $\mathcal{K}$ is a vector $l$, such that $x \in \mathcal{K}$ implies $(x + \mu l) \in \mathcal{K}$ for all $\mu$.*

Allowing $\mu$ to have both positive and negative values creates a bidirectional ray in the direction of $l$ and $-l$. Two rays in exactly opposite directions, therefore make a line. The definition of line is very much like the definition of ray (2.5), however, there is no such thing as an extreme line in general. Lines are used to describe $n$-spaces which are described in definition 2.12 and property 2.3.

**Definition 2.8** *Given two points $x, y$, the closed (line)* **segment** *$Seg(x, y)$ is defined as the set of all convex combinations of $x$ and $y$.*

**Definition 2.9** *An* **affine transformation** *is a function $T$ which maps a point $x$ to a point $x.T = Ax + b$ where $A$ is a constant matrix and $b$ is a constant vector.*

**Definition 2.10** *A set $\mathcal{K}$ is* **convex** *iff every convex combination of any two points in $\mathcal{K}$ is also a point in $\mathcal{K}$.*

Alternate definition:

*A set $\mathcal{K}$ is* **convex** *iff for each pair of points $a, b \in \mathcal{K}$, the closed segment with endpoints a,b is also included in set $\mathcal{K}$.*

Alternate definition:

*A set $\mathcal{K}$ is* **convex** *iff its intersection with any line is either empty or a connected set (line, half-line, line-segment).*

The following are important closure properties held by convex sets.

**Property 2.1** *(Closure under intersection)*
*The intersection of convex sets is convex.*

**Property 2.2** *(Closure under affine transformations)*
*Affine transformations of convex sets are convex.*

**Definition 2.11** *A set of points are* **linearly independent** *iff no point in the set can be expressed as an linear combination of any other points in the set. A set of points are* **linearly dependent** *iff they are not linearly independent. A* **basis** *of a set is a linearly independent subset such that all points in the original set can be expressed as a linear combination of points in the basis. In general, the basis is not unique. The* **rank** *of a set is the size of its basis. Similary definitions for* **affinely independent** *and* **affinely dependent** *may be given in terms of affine combinations.*

**Definition 2.12** *A set $\mathcal{K}$ is called a* **linear subspace,** *(also* **subspace** *or* **space***), if it has the property: $x, y \in \mathcal{K}$ implies all linear combinations of $x, y$ are in $\mathcal{K}$. The* **dimension** *of a space is the rank of a set of lines which span the space. A space of dimension $n$ is called an $n$-***space.**

**Property 2.3** *Each $n$-space contains $n$ linearly independent lines. Any $n+1$ membered set of lines in an $n$-space is linearly dependent.*

| type ↓ | Smallest Container | Largest Contained Subset |
|---|---|---|
| Linear | Linear Hull (definition 2.16) | Lineality Space (definition 2.22) |
| Positive | Conic Hull (definition 2.17) | Characteristic Cone (definition 2.21) |
| Affine | Affine Hull (definition 2.15) | |
| Convex | Convex Hull (definition 2.14) | |

Table 2.1: Comparison of Containers

**Definition 2.13** *A set $\mathcal{K}$ is called a* **flat** *if it has the property: $x, y \in \mathcal{K}$ implies all affine combinations of $x, y$ are in $\mathcal{K}$. The* **dimension** *of a flat is the rank of a set of lines which span the flat. A flat of dimension $n$ is called an $n$-***flat***. A 0-flat, 1-flat, and 2-flat are called respectively a* **point**, **line**, *and* **plane**.

**Property 2.4** *Each $n$-flat contains $n$ affinely independent lines and $n+1$ affinely independent points. Any $n+2$ membered set of points in an $n$-flat is affinely dependent. Any $n+1$ membered set of lines in an $n$-flat is affinely dependent.*

## 2.3 Hulls

Table 2.1 summarizes the four kinds of hulls (containers) corresponding to the four kinds of combinations. Also shown in the table are the largest contained subsets.

**Definition 2.14** *The* **convex hull** *of $\mathcal{K}$ is the convex combination of all points in $\mathcal{K}$. It is the smallest convex set which contains all of $\mathcal{K}$.*

**Definition 2.15** *The* **affine hull** *of $\mathcal{K}$ is the flat consisting of the affine combination of all points in $\mathcal{K}$. It is the smallest dimensional flat which contains all of $\mathcal{K}$.*

**Definition 2.16** *The* **linear hull** *of $\mathcal{K}$ is the subspace consisting of the linear combination of all points in $\mathcal{K}$. It is the smallest dimensional linear subspace which contains all of $\mathcal{K}$.*

**Definition 2.17** *The* **conic hull** *of $\mathcal{K}$ is the cone consisting of the positive combination of all points in $\mathcal{K}$. It is the smallest cone which contains all of $\mathcal{K}$.*

**Definition 2.18** *A convex set $C$ is a* **cone** *with* **apex** *0 provided $\lambda x$ is in $C$ whenever $x$ is in $C$ and $\lambda \geq 0$. A set $C$ is a cone with apex $x_0$ provided $C - \{x_0\}$ is a cone with apex 0. A cone with apex $x_0$ is* **pointed** *provided $x_0$ is a vertex of $C$.*

**Property 2.5** *If a cone $C$ is pointed, $C$ is generated by a positive combination of its extremal rays.*

## 2.4 The Polyhedron

The following theorem was first published in 1894 by Farkas and has been sharpened through the years. It provides us the basis upon which to build a theory for polyhedra.

**Theorem 2.1 Fundamental Theorem of Linear Inequalities**
*Let $a_1, \cdots, a_m, b$ be vectors in n-dim space.*
*Then either:*
*1. $b$ is a positive combination of linearly independent vectors $a_1, \cdots, a_m$;* or,
*2. there exists a hyperplane $\{x \mid cx = 0\}$, containing $t-1$ linearly independent vectors from among $a_1, \cdots, a_m$, such that $cb < 0$ and $ca_1, ..., ca_m \geq 0$, where $t :=$ rank$\{a_1, \cdots, a_m, b\}$.*

For a proof, refer to [Sch86, page 86].

Stated in more familiar terms, given a cone generated by a set of rays $\{a_1, \cdots, a_m\}$, then given another ray $b$, either
1. $b$ is in the cone and is therefore a positive combination of rays $\{a_1, \cdots, a_m\}$, or
2. $b$ is outside the cone, and there exists a hyperplane containing $(t-1)$ extreme rays from the set $\{a_1, \cdots, a_m\}$ which separates $b$ from the cone.

### 2.4.1 The dual representations of polyhedra

**Definition 2.19** *A* **polyhedron,** *$\mathcal{P}$ is a subspace of $\mathcal{Q}^n$ bounded by a finite number of hyperplanes.*

Alternate definition:

*$\mathcal{P}$ = intersection of a finite family of closed linear halfspaces $\{x \mid ax \geq c\}$ where $a$ is a non-zero row vector, c is a scalar constant.*

**Property 2.6** *All polyhedra are convex.*

A result of the fundamental theorem is that a polyhedron $\mathcal{P}$ has a dual representation, an implicit and a parametric representation. The set of solution points which satisfy a mixed system of constraints form a polyhedron $\mathcal{P}$ and serve as the *implicit definition* of the polyhedron

$$\mathcal{P} = \{x \mid Ax = b,\ Cx \geq d\} \tag{2.3}$$

given in terms of equations (rows of $A$, $b$) and inequalities (rows of $C$, $d$), where $A$, $C$ are matrices, and $b$, $d$ and $x$ are vectors. The implicit definition corresponds definitionn 2.19 above, where the set of closed halfspaces are defined by the inequalities: $Ax \geq b$, $-Ax \geq b$, and $Cx \geq d$.

$\mathcal{P}$ has an equivalent dual *parametric representation* also called the *Minkowski characterisation* (after Minkowski— 1896) [Sch86, Page 87]:

$$\mathcal{P} = \{x \mid x = L\lambda + R\mu + V\nu, \quad \mu, \nu \geq 0, \ \textstyle\sum \nu = 1\} \tag{2.4}$$

in terms of a linear combination of lines (columns of matrix $L$), a convex combination of vertices[2] (columns of matrix $V$), and a positive combination of extreme rays (columns of matrix $R$). The parametric representation shows that a polyhedron can be generated from a set of lines, rays, and vertices. The fundamental theorem implies that two forms (eq. 2.3 and eq. 2.4) are equivalent.

[2]I am taking liberty with the term *vertices*. Here I use the term to mean the vertices of P less its lineality space.

Procedures exist to compute the dual representations of $\mathcal{P}$, that is, given $A, b, C, d$, compute $L, V, R$, and visa versa. Such a procedure is in the polyhedral library and will be described later in section 4.2.

## 2.5 The Polyhedral Cone

Polyhedral cones are a special case of polyhedra which have only a single vertex. (Without loss of generality, the vertex is at the origin.) A cone $\mathcal{C}$ (with apex at the origin) is defined parametrically as

$$\mathcal{C} = \{x \mid x = L\lambda + R\mu,\ \mu \geq 0\} \tag{2.5}$$

where $L$ and $R$ are matrices whose columns are the lines and extreme rays, respectively, which specify the cone with rays $\{R, -L, L\}$ as defined in definition 2.18 and property 2.5. If $L$ is empty, then the cone is pointed.

Since the origin is always a solution point in Eq. 2.3, the implicit description of a cone has the following form

$$\mathcal{C} = \{x \mid Ax \geq 0,\ Bx = 0\} \tag{2.6}$$

the solution of a mixed system of *homogeneous* inequalities and equations.

## 2.6 The structure of polyhedra

In this section, let $\mathcal{P}$ be a polyhedron as described in section 2.4.1.

**Definition 2.20** *A set is a (convex)* **polytope** *iff it is the convex hull of finitely many vertices. A set $\mathcal{K}$ is a polytope iff $\mathcal{K}$ is bounded (contains no rays or lines).*

**Definition 2.21** char.cone$\mathcal{P}$, *called the* **characteristic cone** *(or recession cone) of $\mathcal{P}$ is the cone $\{y \mid x + y \in \mathcal{P},\ \forall x \in \mathcal{P}\} = \{y \mid Ay \geq 0\}$.*

The following theorem was given by Motzkin in 1936.

**Theorem 2.2 Decomposition Theorem for Polyhedra** *A set $\mathcal{P}$ is a polyhedron iff $\mathcal{P} = \mathcal{V} + \mathcal{C}$, where $\mathcal{V}$ is a polytope, and $\mathcal{C} =$ char.cone$\mathcal{P}$ is a polyhedral cone.*

The proof is in [Sch86, Page 88].

**Definition 2.22** *The* **lineality space** *of $\mathcal{P}$ is defined as* lin.space$\mathcal{P}$ := char.cone$\mathcal{P} \cap -$char.cone$\mathcal{P} = \{y \mid Ay = 0\}$. *The lineality space of a polyhedron is the dimensionally largest linear subspace contained in the polyhedron. If lineality space of $\mathcal{P}$ is empty then $\mathcal{P}$ is* **pointed**.

A lineality space is represented as a fundamental set of lines which form a basis of the subspace. The lineality space of a polyhedron is unique, although it may be represented using any appropriate basis of lines. The dimension of a lineality space is the rank of a set of lines which span the space (property 2.3).

### 2.6.1 Decomposition

In 1936, Motzkin gave the decomposition theorem (2.2) for polyhedra. Any polyhedron $\mathcal{P}$ could be uniquely decomposed into a polytope $\mathcal{V} =$ conv.hull$\{v_1, \cdots, v_m\}$ generated by convex combination of the extreme vertices of $\mathcal{P}$, and a cone $\mathcal{C} =$ char.cone$\mathcal{P}$ as follows[3]

$$\mathcal{P} = \mathcal{V} + \mathcal{C} \cdot \qquad (2.7)$$

A non-pointed convex cone can in turn be partitioned into two parts,

$$\mathcal{C} = \mathcal{L} + \mathcal{R} \qquad (2.8)$$

the combination of its lineality space $\mathcal{L}$ generated by a linear combination of the lines (bidirectional-rays) of $\mathcal{P}$, and a pointed cone $\mathcal{R}$ generated by positive combination of the extreme rays of $\mathcal{P}$. Combining equations 2.7 and 2.8, a polyhedron may be

[3]The symbol '+' in the equation is called the Minkowski sum, and is defined: $R+S = \{r+s \ : \ r \in R, \ s \in S\}$.

fully decomposed into

$$\mathcal{P} = \mathcal{V} + \mathcal{R} + \mathcal{L} \cdot \tag{2.9}$$

Decomposition implies that any polyhedron may be decomposed into its vertices, rays (unidirectional rays) and lines (bidirectional rays) which can be clearly seen in the parametric description in equation 2.4.

A decomposition of a polyhedron which has a practical application in the polyhedron library, is the decomposition of a polyhedron into its *lineality space* (definition 2.22) and its *ray space* of a polyhedron. This division separates lines (bidirectional rays) from vertices and rays (unidirectional rays). In the alternate conic form of a polyhedron, developed in section 3.1, both rays and vertices are representable as unidirectional rays in the cone. In a cone, this decomposition simply separates lines and rays (equation 2.8). Table 2.2 summarizes all of the decompositions of a polyhedron.

| Decomposition | Description |
|---|---|
| $\mathcal{V}+\mathcal{R}+\mathcal{L}$ | $\mathcal{P} = \{x \mid x = L\lambda + R\mu + V\nu, \quad \mu,\nu \geq 0, \ \sum\nu = 1\}$, (eq 2.4) |
| $\mathcal{V}$ | polytope $= \{x \mid x = V\nu, \ \nu \geq 0, \ \sum\nu = 1\}$, (definition 2.20) |
| $\mathcal{R}$ | pointed cone $= \{x \mid x = R\mu, \ \mu \geq 0\}$, (definition 2.18) |
| $\mathcal{L}$ | lineality space $= \{x \mid x = L\lambda\}$, (definition 2.22) |
| $\mathcal{V}+\mathcal{R}$ | ray space of $\mathcal{P}$ |
| $\mathcal{V}+\mathcal{L}$ | set of minimal faces of $\mathcal{P}$, (definition 2.26) |
| $\mathcal{R}+\mathcal{L}$ | char.cone$\mathcal{P}$, a non-pointed cone, (definition 2.21) |

Table 2.2: Table of Decompositions

## 2.7 The faces and face lattice of a polyhedron

### 2.7.1 Supporting Hyperplanes

**Definition 2.23** *A hyperplane $\mathcal{H}$* **cuts** *a set $\mathcal{K}$ provided both open halfspaces* *determined by $\mathcal{H}$ contain points of $\mathcal{K}$, that is $\mathcal{H} = \{x \mid x \circ u = \alpha\}$* **cuts** *$\mathcal{K}$ iff there exists $x_1, x_2 \in \mathcal{K} \mid (x_1 \circ u < \alpha)$ and $(x_2 \circ u > \alpha)$*

**Definition 2.24** *A* **supporting hyperplane** *is a plane which intersects the hull of a polyhedron (close $\mathcal{P}$), but does not cut $\mathcal{P}$, or in other words, does not intersect the interior of $\mathcal{P}$.*

Alternatively:

*If $c$ is a nonzero vector, and if $\delta = \max\{c \circ x \mid Ax \leq b\}$ exists, then the affine hyperplane $\mathcal{H} = \{x \mid c \circ x = \delta\}$ is a* **supporting hyperplane** *of $\mathcal{P}$.*

The supporting hyperplane is a plane which just touches the surface of the polyhedron. The intersection of a supporting hyperplane and a polyhedron can be a point, edge, plane, or so forth.

### 2.7.2 Faces

**Definition 2.25** *A subset $\mathcal{F}$ of $\mathcal{P}$ is called a* **face** *of $\mathcal{P}$ if either:*

*(i) $\mathcal{F}$ is the intersection of $\mathcal{P}$ with a supporting hyperplane, or*

*(ii) $\mathcal{F} = \mathcal{P}$, or*

*(iii) $\mathcal{F} =$* lineality.space($\mathcal{P}$).

Case (iii) is added to force closure of the set of faces under intersection. Faces defined by cases (iii) and (ii) are called **improper faces** while faces defined by case (i) are called **proper faces.**

Every face of $\mathcal{P}$ is also a polyhedron and is called a **k-face** if it is a $k$-polyhedron. 0-faces are vertices. 1-faces are edges. The number of faces of a polyhedron is finite.

**Definition 2.26** *The $(n-1)$-faces of a $n$-polyhedron are called* **facets** *and the 0-faces of a polyhedron are called vertices and rays. A* **facet** *of $\mathcal{P}$ is a maximal face distinct from $\mathcal{P}$ (maximal relative to inclusion). A* **minimal face** *of $\mathcal{P}$ is a nonempty face not containing any other face.*

**Property 2.7** *A face $\mathcal{F}$ of $\mathcal{P}$ is a minimal face iff $\mathcal{F}$ is an affine subspace. [Hoffman and Kruskal, 1956] A set $\mathcal{F}$ is a minimal face of $\mathcal{P}$ iff $\emptyset \neq \mathcal{F} \subset \mathcal{P}$ and $\mathcal{F} = \{x \mid A'x = b'\}$ for some subsystem $A'x \geq b'$ of $Ax \geq b$.*

**Property 2.8** *Each minimal face of $\mathcal{P}$ is a translate of the lineality space of $\mathcal{P}$, and has the same dimension.*

**Property 2.9** *The set of faces of a polyhedron form a lattice with respect to inclusion which is called the* **face lattice***.*

**Definition 2.27** *$f_k(\mathcal{P})$ is defined as the number of $k$-faces of polyhedron $\mathcal{P}$.*

## 2.8 Duality of Polyhedra

In this section, the concepts of combinatorial equivalence and duality are presented. These two concepts are used in developing a memory representation of a polyhedron in chapter 3. Then the idea of the polar mapping is presented along with its properties which are used in chapter 4 of this thesis in the development of operations on polyhedra.

### 2.8.1 Combinatorial Types of Polytopes

**Definition 2.28** *Two polyhedra, $\mathcal{P}$ and $\mathcal{P}'$ are* **combinatorially equivalent** *(or* **isomorphic***) provided there exists a 1-1 mapping between the set $F$ of all faces of $\mathcal{P}$, and the set $F'$ of all faces of $\mathcal{P}'$, such that the mapping is inclusion preserving. In other words, $F_1$ is a face of $F_2$ iff map$(F_1)$ is a face of map$(F_2)$. Equivalently, the face lattices of $\mathcal{P}$ and $\mathcal{P}'$ are isomorphic. Combinatorial equivalence is an equivalence relation.*

**Definition 2.29** *Two d-polytopes, $\mathcal{P}$ and $\mathcal{P}*$ are said to be* **dual** *to each other provided there exists a 1-1 mapping between the set $F$ of all faces of $\mathcal{P}$, and the set $F*$ of all faces of $\mathcal{P}*$, such that the mapping is inclusion-reversing. in other words, $F_1$ is a face of $F_2$ iff map$(F_2)$ is a face of map$(F_1)$.*

### 2.8.2 Polar mapping

**Definition 2.30** *(Polar)*

*Given a closed convex set $P$ containing the point 0, then the* polar *$P*$ is defined as $P* = \{y \mid \forall x \in \mathcal{P} \ : \ x \circ y \geq 0\}$.*

**Property 2.10** *(duality of polars)*

*If $P*$ is the polar of $P$, then $P$ and $P*$ are duals of each other.*

Given $\mathcal{P} \overset{\text{DUAL}}{\longleftrightarrow} \mathcal{P}*$ where $\mathcal{P}$ is a closed convex set containing 0, then the following properties hold:

(i) if $\mathcal{P} = \text{conv.hull}\{0, x_1, \cdots, x_m\} + \text{cone}\{y_1, \cdots, y_t\}$ then $\mathcal{P}* = \{z \mid z^T x_i \leq 1 \text{ for } i = 1 \cdots m\} + \{z \mid z^T y_i \leq 0 \text{ for } i = 1 \cdots t\}$

(ii) $\mathcal{P}$ has dimension $k$ iff lin.space($\mathcal{P}*$) has dimension $n - k$

(iii) $\mathcal{P} * * = \mathcal{P}$

(iv) $\mathcal{A}* = \mathcal{B}*$ iff $\mathcal{A} = \mathcal{B}$

(v) $\mathcal{A}* \subseteq \mathcal{B}*$ iff $\mathcal{A} \subseteq \mathcal{B}$

(vi) $(\mathcal{A} \cup \mathcal{B})* = \mathcal{A} * \cap \mathcal{B}*$

(vii) $(\mathcal{A} \cap \mathcal{B})* = \text{convex.hull}(\mathcal{A} * \cup \mathcal{B}*)$

(viii) if $\mathcal{A}$ is a face of $\mathcal{B}$ then $\mathcal{B}*$ is a face of $\mathcal{A}*$.

(ix) there is a 1-1 correspondance between $k$-faces of $\mathcal{P}$ and $(n - k)$-faces of $\mathcal{P}*$.

The principle of duality is used in sections 3.2 and 3.3 when showing the duality between the parametric and implicit definitions of a polyhedron and in chapter 5 when discussing the lattices of dual polyhedra.

# Chapter 3

# REPRESENTATION OF POLYHEDRA

## 3.1 Equivalence of homogenous and inhomogenous systems

We want to be able to represent a *mixed inhomogeneous system* of equations as given in equations 2.3 and 2.4 and which is the most general type of constraint system. A memory representation of an $n$ dimensional mixed inhomogeneous system of $j$ equalities and $k$ inequalities would require the storage of the following arrays: $A(j \times n)$, $b(j \times 1)$, $C(k \times n)$, $d(k \times 1)$. The dual representation would require the storage of $R$, $V$, and $L$, the arrays representing the rays, vertices, and lines. The representation in memory can be simplified however, with a transformation $x \rightarrow \begin{pmatrix} \xi x \\ \xi \end{pmatrix}$, $\xi \geq 0$ that changes an inhomogeneous system $\mathcal{P}$ of dimension $n$ into a homogenous system $\mathcal{C}$ of dimension $n+1$, as shown here:

$$\begin{aligned}
\mathcal{P} &= \{x \mid Ax = b,\ Cx \geq d\} \\
&= \{x \mid Ax - b = 0,\ Cx - d \geq 0\} \\
\mathcal{C} &= \left\{ \begin{pmatrix} \xi x \\ \xi \end{pmatrix} \mid \xi Ax - \xi b = 0,\ \xi Cx - \xi d \geq 0,\ \xi \geq 0 \right\} \\
&= \left\{ \begin{pmatrix} \xi x \\ \xi \end{pmatrix} \mid \begin{pmatrix} A \mid -b \end{pmatrix} \begin{pmatrix} \xi x \\ \xi \end{pmatrix} = 0,\ \left(\begin{array}{c|c} C & -d \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} \xi x \\ \xi \end{pmatrix} \geq 0 \right\} \\
&= \{\hat{x} \mid \hat{A}\hat{x} = 0,\ \hat{C}\hat{x} \geq 0\}
\end{aligned}$$

The transformed system $\mathcal{C}$ is now an $(n+1)$ dimensional cone which contains the original $n$ dimensional polyhedron. Goldman showed that the mapping $x \rightarrow \begin{pmatrix} \xi x \\ \xi \end{pmatrix}$ is one to one and inclusion preserving [Gol56] and thus by definition 2.28 the two are combinatorially equivalent. The original polyhedron $\mathcal{P}$ is in fact the intersection of the cone $\mathcal{C}$ with the hyperplane defined by the equality $\xi = 1$. Given any $\mathcal{P}$ as defined in equation 2.3, an unique homogeneous cone form exists defined as follows:

$$\begin{aligned} \mathcal{C} &= \{\hat{x} \mid \hat{A}\hat{x} = 0,\ \hat{C}\hat{x} \geq 0\} \\ &= \text{homogoneous.cone } \mathcal{P}, \\ \text{where } \hat{x} &= \begin{pmatrix} \xi x \\ \xi \end{pmatrix},\ \hat{A} = \begin{pmatrix} A \mid -b \end{pmatrix},\ \hat{C} = \left(\begin{array}{c|c} C & -d \\ \hline 0 & 1 \end{array}\right) \end{aligned} \tag{3.10}$$

The storage requirement for the homogenous system is $\hat{A}(j \times (n+1))$, $\hat{C}((k+1) \times (n+1))$ which is about the same amount of memory needed for the original system (compare $(j+k)(n+1)+n$) words for the cone versus $(j+k)(n+1)$ words for the polyhedron) and the cone representation is simpler (two matrices versus two matrices and two vectors). Likewise the dual representation of the cone is simpler. The decomposition of a cone is $\mathcal{R}+\mathcal{L}$, and thus only rays and lines have to be represented. During the transformation process from a polyhedron to a cone, vertices get transformed into rays. The vertices and rays of an inhomogeneous polyhedron have a unified and homogenous representation as rays in a polyhedral cone. Thus the rays of the cone represent both the vertices and rays of the original polyhedron. As before, the amount of memory needed to store the dual representation is the same, however the representation itself is simpler (two matrices versus three matrices). Table 3.1 shows the equivalent forms of inhomogenous and homogenous systems, polyhedra and cones, along with their dual implicit and parametric representations. The table highlights the fundamental relationships between the polyhedron and cone.

Using the homogeneous cone form not only simplifies the data structure used to represent the polyhedron, but also simplifies computation. From practical

| | Inhomogenous System | Homogenous System |
|---|---|---|
| Structure | Polydedron $\mathcal{P}$, dimension $d$ | Cone $\mathcal{C}$, dimension $d+1$ |
| Implicit Representation using Hyperplanes | $\mathcal{P} = \{x \mid Ax = b,\ Cx \geq d\}$ | $\begin{aligned} \mathcal{C} &= \{\hat{x} \mid \hat{A}\hat{x} = 0,\ \hat{C}\hat{x} \geq 0\} \\ \hat{x} &= \begin{pmatrix} \xi x \\ \xi \end{pmatrix} \\ \hat{A} &= \left(\begin{array}{c\|c} A & -b \end{array}\right) \\ \hat{C} &= \left(\begin{array}{c\|c} C & -d \\ \hline 0 & 1 \end{array}\right) \end{aligned}$ |
| Parametric Representation using Vertices and Rays | $\mathcal{P} = \{x \mid x = L\lambda + R\mu + V\nu,\ \mu, \nu \geq 0, \sum \nu = 1\}$ | $\mathcal{C} = \{\hat{x} \mid \hat{x} = \hat{L}\lambda + \hat{R}\mu,\ \mu \geq 0\}$ |
| Vertices | $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_d \end{pmatrix}, \quad v \in V$ | $\hat{r}_v = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_d \\ \lambda \end{pmatrix}, \ \lambda > 0,\ \hat{r}_v \in \hat{R}$ |
| Rays | $r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_d \end{pmatrix}, \quad r \in R$ | $\hat{r}_r = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_d \\ 0 \end{pmatrix}, \quad \hat{r}_r \in \hat{R}$ |
| Lines | $l = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_d \end{pmatrix}, \quad l \in L$ | $\hat{l} = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_d \\ 0 \end{pmatrix}, \quad \hat{l} \in \hat{L}$ |

Table 3.1: Duality between Polyhedra and Cones

experience with the implementation of polyhedral operations, it is known that fewer array references have to be done and fewer 'end cases' have to be handled when computing with the homogeneous form. This results in slightly smaller and more efficient procedures.

A mixed system may also be transformed to a *non-mixed* systems of constraints by using the the transformation: $ax = 0 \rightarrow ax \geq 0$ and $ax \leq 0$ along with its dual: a line $l$ can be represented as two rays $l$ and $-l$. This reduces the entire representation to a non-mixed set of homogeneous inequalities (no equalities) and its dual to just an array of rays (no line or vertices). This simplification is tempting, however, it would increase the size of the memory representation (each equality and line require twice the storage). There is another advantage of keeping equalities and inequalities separate: there are different (and much more efficient— polynomial time) methods for solving equalities. Thus, by keeping equalities and inequalities distinct and separate, the memory requirement is kept at a minimum, and equalities can be treated specially using standard, efficient, and well loved methods such as Gauss elimination.

## 3.2 Dual representation of a polyhedron

A polyhedron may be fully described as either a system of constraints or by its dual form, a collection of rays and lines. Given either form, the other may be computed. However, since the duality computation is an expensive operation (see section 4.2) and since both forms are needed for computation of different operations, a decision to represent polyhedra redundantly using both forms was made. Even though the representation is redundant, keeping both forms in the data structure reduces the number of duality computations that have to be made and improves the efficiency of the polyhedral library. It is a basic memory / execution time tradeoff made in favor of execution time.

| Parametric Description | Implicit Description |
|---|---|
| Lineality Space | System of equalities |
| Ray Space | System of inequalities |
| Ray $r$ | Homogeneous Inequality $r^T x \geq 0$ |
| Vertex $r/k$ | Inhomogeneous Inequality $r^T x + k \geq 0$ |
| Line $r$ with Vertex at 0 | Homogeneous Equality $r^T x = 0$ |
| Line $r$ with Vertex not at 0 | Inhomogeneous Equality $r^T x + k = 0$ |
| Convex union of rays | Intersection of inequalities |
| Point at origin | Positivity Constraint |
| Universe Polyhedron | Empty set of Constraints |
| Empty Polyhedron | Overconstrained system |

Table 3.2: Dual Concepts

## 3.3 Saturation and the incidence matrix

After being transformed to a homogeneous coordinate system, a polyhedron is represented as a cone (equation 3.10). The dual representations of the cone are:

$$\begin{aligned} \mathcal{C} &= \{x \mid Ax = 0,\ Cx \geq 0\} && \text{(implicit form)} \\ &= \{x \mid x = L\lambda + R\mu,\ \mu \geq 0\} && \text{(parametric form)} \end{aligned}$$

Substituting the equation for $x$ in the parametric form into the equations involving $x$ in the implicit form, we obtain:

$$\forall(\mu \geq 0,\ \lambda) : \left\{ \begin{array}{l} AL\lambda + AR\mu = 0 \\ CL\lambda + CR\mu \geq 0 \end{array} \right. \Longrightarrow \left\{ \begin{array}{l} AL = 0,\ AR = 0 \\ CL = 0,\ CR \geq 0 \end{array} \right. \tag{3.11}$$

where rows of $A$ and $C$ are equalities and inequalities, respectively, and where columns of $L$ and $R$ are lines and rays, respectively.

Using the above, we can show the duality of a system of constraints with its corresponding system of lines and rays. Let $\mathcal{C}$ be a cone and $\mathcal{C}*$ be another cone created by reinterpreting the inequalities and equalities of $\mathcal{C}$ as the lines and rays, respectively, of $\mathcal{C}*$. Then the two cones are defined as:

$$\begin{aligned}
\mathcal{C} &= \{x \mid x = L\lambda + R\mu,\ \mu \geq 0\} \\
\mathcal{C}* &= \{y \mid y = A^T\alpha + C^T\gamma,\ \gamma \geq 0\}
\end{aligned}$$

then the inner product of a point $x \in \mathcal{C}$ and a point $y \in \mathcal{C}*$ is:

$$\begin{aligned}
x \circ y &= y^T x = (A^T\alpha + C^T\gamma)^T \times (L\lambda + R\mu) \\
&= (\alpha^T A + \gamma^T C) \times (L\lambda + R\mu) \\
&= \alpha^T(AL\lambda + AR\mu) + \gamma^T(CL\lambda + CR\mu) \\
&\geq 0 \quad \text{(by application of equation 3.11)} \\
\mathcal{C}* &= \{y \mid \forall x \in \mathcal{C} \ :\ x \circ y \geq 0\}
\end{aligned}$$

and thus $\mathcal{C}$ and $\mathcal{C}*$ are duals by property 2.10.

Before discussing the incidence matrix, the notion of saturation needs to be defined.

**Definition 3.1** *A ray $r$ is said to* **saturate** *an inequality $a^Tx \geq 0$ when $a^Tr = 0$, it* **verifies** *the inequality when $a^Tr > 0$, and it* **does not verify** *the inequality when $a^Tr < 0$. Likewise, a ray $r$ is said to* **saturate** *an equality $a^Tx = 0$ when $a^Tr = 0$, and it* **does not verify** *the equality when $a^Tr \neq 0$. Equalities and inequalities are collectively called* **constraints**. *A constraint is* **satisfied** *by a ray if the ray saturates or verifies the constraint.*

The incidence matrix $S$ is a boolean matrix which has a row for every constraint (rows of $A$ and $C$) and a column for every line or ray (columns of $L$ and $R$).

Each element $s_{ij}$ in $S$ is defined as follows:

$$s_{ij} = \begin{cases} 0, & \text{if constraint } c_i \text{ is saturated by ray(line) } r_j \text{, i.e. } c_i^T r_j = 0 \\ 1, & \text{otherwise, i.e. } c_i^T r_j > 0 \end{cases}$$

From the demonstrations in equation 3.11 above, we know that all rows of the $S$ matrix associated with equations ($A$) are 0, and all columns of the $S$ matrix associated with lines ($L$) are also 0. Only entries associated with inequalities ($C$) and rays ($R$) can have 1's as well as 0's. This is illustrated in the following diagram representing the saturation matrix $S$.

| S | L | R |
|---|---|---|
| A | (0) | (0) |
| C | (0) | (0 or 1) |

## 3.4 Expanding the model to unions of polyhedra

Polyhedra are closed under intersection (property 2.1), convex union (convex.hull($A \cup B$), definition 2.14), and affine transformation (property 2.2). However, they are not closed under (simple) union since the union of any two polyhedra is not necessarily convex. Likewise, polyhedra are not closed under the difference operation. To obtain closure of these two operations (union and difference), it is necessary to expand the model from a simple polyhedron to a finite union of polyhedra. The table 3.3 shows the closures of different library operations. The polyhedral library supports the extended model of a union of polyhedra. Thus, all operations in the polyhedral library are closed.

## 3.5 Data structure for unions of polyhedra

In the previous sections, the motivations for the major design decisions made in defining the data structure have been presented. The data structure should represent polyhedron in the homogeneous cone format (section 3.1), in the redundant

| Operation ↓ | Polyhedra | Finite Unions of Polyhedra |
|---|---|---|
| Intersection | Closed | Closed |
| Convex Union | Closed | Closed |
| Affine Transformation | Closed | Closed |
| Union | Not Closed | Closed |
| Difference | Not Closed | Closed |

Table 3.3: Closure of Operations

form (both constraints and rays represented)(section 3.2), and support the representation of a union of polyhedra (section 3.4). With these objectives in mind, a C-structure for a polyhedron was defined. The term "Ray" , as used in the library, needs some explaination. The term "Ray" is used to represent the vertices, rays, and lines in a polyhedron. Indeed in the homogenous cone form, vertices and rays are both representable as unidirectional rays and line is simply a bidirectional ray. Since no other good term really exists for the ensemble of geometric features of a polyhedron, the term "Ray" is used. The reader needs to differentiate it from a simple ray (definition 2.5) by context. The C-structure for a polyhedron is defined as:

```
typedef struct polyhedron
{ struct polyhedron *next;
  unsigned Dimension, NbConstraints, NbRays, NbEqualities, NbLines;
  int **Constraint;
  int **Ray;
  int *p_Init;
} Polyhedron;
```

The fields of the Polyhedron structure are described as follows:

`Dimension` the dimension of the space in which the inhomogeneous polyhedron resides.

**`NbConstraints`** the number of equalities (NbEqualities) and inequalities constraining the polyhedron.

**`NbRays`** the number of lines (NbLines), rays, and vertices in the geometry of the polyhedron.

**`NbEqualities`** the number of equalities in the constraint list.

**`NbLines`** the number of lines in the ray list.

**`Constraint[i]`** the i-th constraint (equation or inequality).

**`Ray[i]`** the i-th geometric feature (ray, vertex, or line).

**`p_Init`** for library use to do memory management.

**`next`** a link to another polyhedron, supporting domains which are finite unions of polyhedra.

The data structure is detailed in figure 3.1. Along with the main structure, three other arrays need to be allocated: an array of constraint pointers, an array of ray pointers, and finally the data array that holds the actual constraints and rays themselves. This entire data structure is created by the library function:

```
Polyhedron *Polyhedron_Alloc ( unsigned Dimension,
                               unsigned NbConstraints,
                               unsigned NbRays )
```

and is replicated by the library function:

```
Polyhedron *Polyhedron_Copy ( Polyhedron *p )
```

and is destroyed ( and memory freed ) by the library function:

```
void Polyhedron_Free ( Polyhedron *p )
```

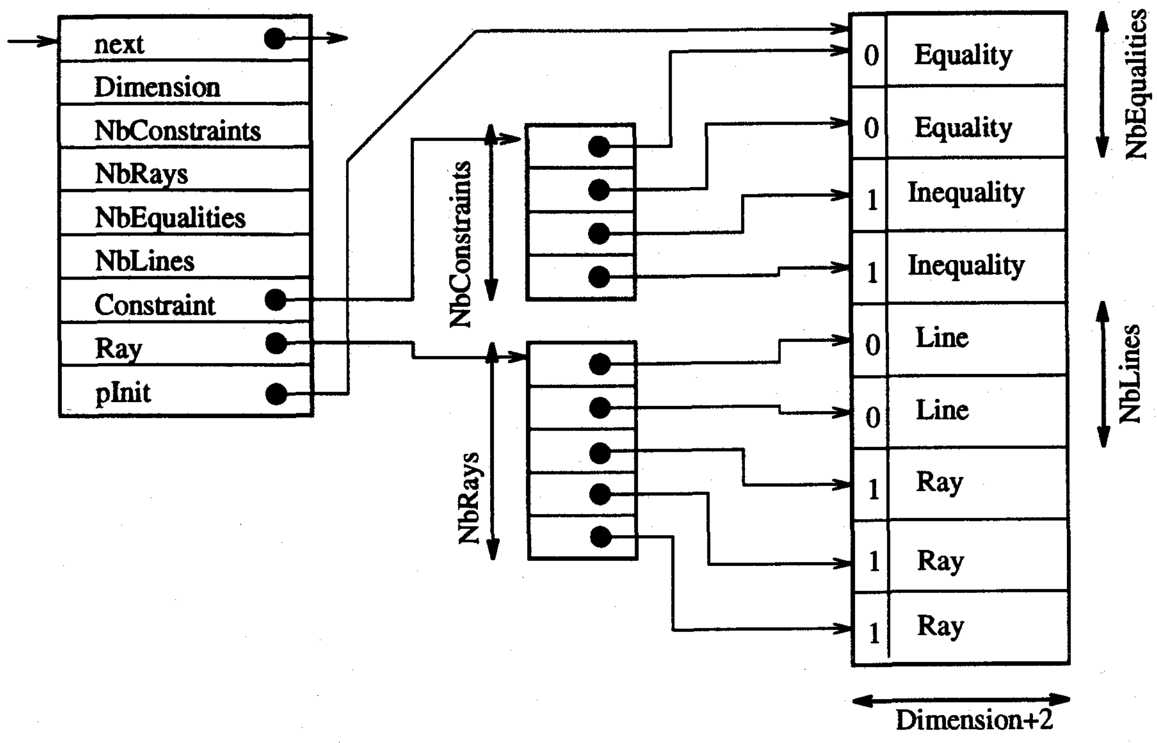


Figure 3.1: Data Structure for Polyhedron

Using the **next** pointer field, the several polyhedra whose union form a domain can be put into a single linked list structure. Thus the data structure works equally well for domains as well as for a single polyhedron. Accordingly, the procedures `Polyhedron_Copy` and `Polyhedron_Free` described above have domain equivalents which copy and free an entire linked list of polyhedra.

```
Polyhedron *DomainCopy ( Polyhedron *d )
```

returns a copy of the linked list of polyhedra (domain) pointed to by d.

```
void DomainFree ( Polyhedron *d )
```

frees memory allocated to the linked list of polyhedra (domain) pointed to by d.

## Constraint Format

Each constraint (equality or inequality) consists of a vector of `Dimension+2` elements and has the format:

$(S,\ X_1,\ X_2,\ \cdots,\ X_n,\ K)$ representing the constraint:

if $S = 0$: $X_1 i + X_2 j + ... + X_n k + K = 0$

if $S = 1$: $X_1 i + X_2 j + ... + X_n k + K \geq 0$

which are defined over the $n$-space with coordinate system $(i,\ j,\ \cdots,\ k)$. The element $S$ is a status word defined to be 0 for equalities and 1 for inequalities.

In an $n$ dimensional system, the i-th constraint $(0 \leq i < NbConstraints)$ is referenced in the following manner:

`Constraint[i][0]` $= S$

`Constraint[i][1]` $= X_1$

`Constraint[i][2]` $= X_2$

$\vdots$

`Constraint[i][Dimension]` $= X_n$

`Constraint[i][Dimension+1]` $= K$

## Ray Format

Each ray consists of a vector of `Dimension+2` elements and has the format:

$(S,\ X_1,\ X_2,\ \cdots,\ X_n,\ K)$ representing the geometric object:

if $S = 0$: *line* in direction $(X_1,\ X_2,\ \cdots,\ X_n)$

if $S = 1$: $K \neq 0$: *vertex* $(\frac{X_1}{K},\ \frac{X_2}{K},\ \cdots,\ \frac{X_n}{K})$

if $S = 1$: $K = 0$: *ray* in direction $(X_1,\ X_2,\ \cdots,\ X_n)$.

The element $S$ is a status word defined to be 0 for lines and 1 for vertices and rays.

In an $n$ dimensional system, the i-th ray $(0 \leq i < NbRays)$ is referenced in the following manner:

`Ray[i][0]` $= S$

`Ray[i][1]` $= X_1$

`Ray[i][2]` $= X_2$

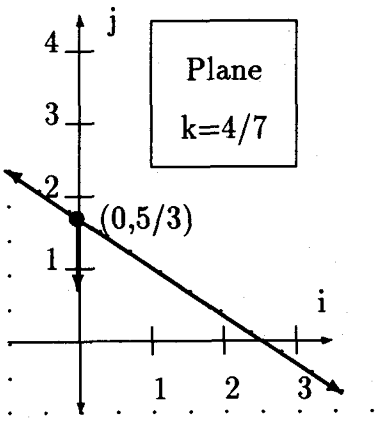
$\vdots$

`Ray[i][Dimension]` $= X_n$

`Ray[i][Dimension+1]` $= K$

The example in figure 3.2 shows the internal representation for a polyhedron.

$\{i, j, k \mid 7k = 4;\ 2i + 3j \leq 5\}$



```
Polyhedron
----------
Dimension      = 3
NbConstraints  = 3
NbRays         = 3
NbEqualities   = 1
NbLines        = 1
Constraint[0] = (  0   0   7  -4 )   Equality    7k = 4
Constraint[1] = ( -2  -3   0   5 )   Inequality 2i+3j <= 5
Constraint[2] = (  0   0   0   1 )   Inequality 1 >= 0
Ray[0]        = (  3  -2   0   0 )   Line        (3,-2, 0)
Ray[1]        = (  0  -1   0   0 )   Ray         (0,-1, 0)
Ray[2]        = (  0  35  12  21 )   Vertex      (0, 35/21, 12/21)
                                                = (0, 5/3, 4/7)
```

Figure 3.2: Example 1

## 3.6 Validity rules

All polyhedra (including empty and universe domains) generated by the polyhedral library satisfy three general rules. In this section, the consistency rules which govern the polyhedral data structure are described.

Given a polyhedron $\mathcal{P} = \mathcal{L} + \mathcal{R} + \mathcal{V}$, the following meanings of the term *dimension* are defined:

1. The **dimension** of a lineality space $\mathcal{L}$ is $n$ where $\mathcal{L}$ is an $n$-space (see definition 2.12).

2. The **dimension** of the ray space is $m$ where affine.hull($\mathcal{R} + \mathcal{V}$) is an $m$-flat (see definition 2.13).

3. The **dimension** of the polyhedron $\mathcal{P}$ is $p$ where affine.hull $\mathcal{P}$ is an $p$-flat.

**Property 3.1** *(Dimensionality Rule)*

a. *The dimension of the lineality space is the number of irredundant lines.*

b. *The dimension of the polyhedron is the dimension of the ray space plus the dimension of the lineality space.*

c. *The dimension of the ray space is the dimension of the system minus the number of irredundant lines minus the number of irredundant equalities.*

*Proof:*

Part (a). The dimension of the lineality space is the rank of a set of lines which span it (definition 2.12). The rank is the number of irredundant lines in a basis for the space. Any additional line is necessarily redundant (property 2.3).

Part (b). The dimension of a polyhedron is the dimension of the smallest flat which contains it (definition of dimension). That flat can be partitioned as follows:

$$\begin{aligned}
\text{convex.hull}(\mathcal{P}) &= \text{convex.hull}(\mathcal{L}+\mathcal{R}+\mathcal{V}) \\
&= \text{convex.hull}(\mathcal{L}) + \text{convex.hull}(\mathcal{R}+\mathcal{V}) \\
&= \text{lineality.space}(\mathcal{P}) + \text{convex.hull}(\text{ray.space}(\mathcal{P})) \\
\text{dimension}(\mathcal{P}) &= \text{dimension}(\text{lineality.space}(\mathcal{P})) + \text{dimension}(\text{ray.space}(\mathcal{P}))
\end{aligned}$$

The dimensions of the lineality space and ray space are unique and separable since no irredundant ray is equal to a linear combination of lines (else the ray is redundant) and no line is a linear combination of rays (else the basis of ray space is redundant). Thus, the lineality space and ray space of a polyhedron are dimensionally distinct and the sum of their dimensions is the dimension of the polyhedron.

Part (c). The set of equalities determine the flat in which $\mathcal{P}$ lies. Since each irredundant equality restricts the flat which contains the polyhedron by one dimension, thus

$$\begin{aligned}
\text{dimension}(\mathcal{P}) &= \text{(Dimension of system)} - \text{(Number of equalities)} \\
\text{[from part b.]} &= \text{dimension}(\text{lineality.space}(\mathcal{P})) + \text{dimension}(\text{ray.space}(\mathcal{P}))
\end{aligned}$$

and from part a. we have:

$$\text{dimension}(\text{lineality.space}(\mathcal{P})) = \text{Number of lines}$$

and combining the above three statements:

$$\begin{aligned}
\text{dimension}(\text{ray.space}(\mathcal{P})) &= \text{(Dimension of system)} \\
&\quad -\text{(Number of equalities)} - \text{(Number of lines)}
\end{aligned}$$

and thus the dimension of the ray space is the dimension of the system less the number of equalities and less the number of lines.

□

The dimension of the ray space is an important number and is used in the determination of redundant rays and inequalities. It is the key number $n$ used in the saturation rule, property 3.2. It is computed according to part–c of property 3.1, which when written in the library (C-code) is:

```
p->Dimension - p->NbLines - p->NbEqualities
```

**Property 3.2** *(Saturation Rule)*

*In an $n$-dimensional ray space,*

a. *Every inequality must be saturated by at least $n$ vertices/rays.*

b. *Every vertex must saturate at least $n$ inequalities and a ray must saturate at least $n - 1$ inequalities plus the positivity constraint.*

c. *Every equation must be saturated by all lines and vertices/rays.*

d. *Every line must saturate all equalities and inequalities.*

*Proof:*

All parts rely on the definition of saturate 3.1.

Part (a). In general, every $k$-face is the convex union of a minimum of $k + 1$ vertices/rays since each $k$-face lies on a $k$-flat which is determined by any $k+1$ affinely independent points in the flat (property 2.4) and since vertices/rays are affinely independent (property 2.6), a minimum of $k + 1$ of them can be used to determine a $k$-face. Since each inequality is associated with a $(n - 1)$-face (facet) of the polyhedron and each $(n-1)$-face is saturated by $n-1+1 = n$ vertices/rays, each inequality is also saturated by at least $n$ vertices/rays.

Part (b). Each vertex is the intersection of at least $n$ facets, and therefore saturates at least $n$ inequalities. Each ray is the intersection of at least $n - 1$ facets, and therefore saturates at least $n - 1$ inequalities plus the positivity constraint (described in section 3.6.1) which is saturated by all rays (property 3.4).

Part (c) and Part(d). Shown by the derivation of equation 3.11.

□

The independence rule is an invariant of library in which only a minimal representation of a polyhedron is stored.

**Property 3.3** *(Independence Rule)*

a. *No inequality is a positive combination of any other two inequalities or equalities.*

b. *No ray is a linear combination of any other two rays or lines.*

c. *The set of equalities must be linearly independent.*

d. *The set of lines must be linearly independent.*

*Proof:*

Part (a). Assume $a_r = a_1\alpha + a_2\beta$, with $\alpha \geq 0$, and $\beta \geq 0$. Given the inequalities $a_1^T x \geq 0$, and $a_2^T x \geq 0$, then $(a_1\alpha + a_2\beta)^T \geq 0$ and thus $a_r^T \geq 0$, and $a_r$ is a redundant inequality and may be omitted from the system.

Part (b). By the definition of extreme ray 2.6.

Part (c) and (d). From definitions of flats (2.13) and subspaces (2.12), the dimension attribute is defined in terms of the basis of the lines, and by convention redundant lines and equalities are removed to keep the basis at a minimum. The number of lines and equalities are known and have be discussed in connection with the dimensionality rule (property 3.1).

□

**Definition 3.2** *(Redundancy)*

*Inequalities that don't satisfy property 3.2.a or property 3.3.a are* **redundant.**

*Vertices/rays that don't satisfy property 3.2.b or property 3.3.b are* **redundant.**

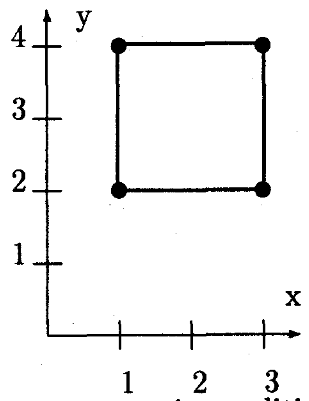
### 3.6.1 The Positivity Constraint

In the language of algebrists, the trivial constraint $1 \geq 0$ is called the "positivity[1] constraint". When true, you know that positive numbers are positive (a nice thing to know). It was generated as a side effect of converting from an inhomogenous polyhedron to a homogeneous cone representation as can be seen in equation 3.10.

[1]Also called the non-negativity constraint. Here the term *positive* is used in a non-strict way to include zero.

$\{x, y \mid 1 \leq x \leq 3;\ 2 \leq y \leq 4\}$

| | x>=1 | x<=3 | y>=2 | y<=4 |
|---|---|---|---|---|
| vertex(1,2) | sat | | sat | |
| vertex(1,4) | sat | | | sat |
| vertex(3,2) | | sat | sat | |
| vertex(3,4) | | sat | | sat |

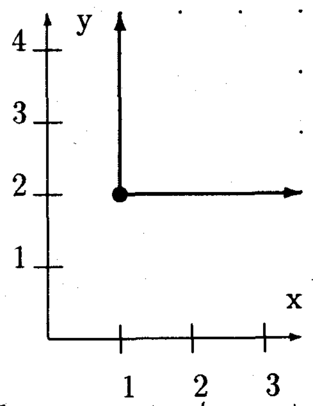


Every constraint saturates two vertices and every vertex saturates two inequalities. This is a perfectly non redundant system.

Figure 3.3: Example 2

$\{x, y \mid x \geq 1;\ y \geq 2\}$

| | x>=1 | y>=2 | 1>=0 |
|---|---|---|---|
| vertex(1,2) | sat | sat | |
| ray(1,0) | | sat | sat |
| ray(0,1) | sat | | sat |



Here, every constraint saturates two vertices/rays and every vertex/ray saturates two inequalities. This is also a non redundant system. However the positivity constraint is also irredundant... it is needed to support the presence of the two rays. Without it, the two rays are not supported and appear mistakenly to be redundant.

Figure 3.4: Example 3

As stated earlier, rays may be thought of as points at infinity. In this vein of thought, the positivity constraint generates the face that connects those points, creating a face at infinity which "closes" unbounded polyhedra. The following property gives the reasoning behind this.

**Property 3.4** *All rays are saturated by the positivity constraint and no vertex is saturated by the positivity constraint.*

$\{x \mid x \geq 1\}$

|  | x>=1 | 1>=0 |
|---|---|---|
| line(0,1) | sat | sat |
| vertex(1,2) | sat |  |
| ray(1,0) |  | sat |

A halfplane.

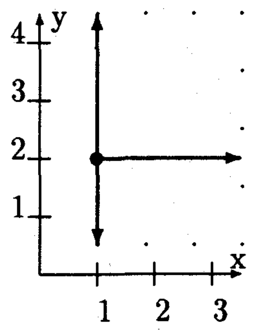


Figure 3.5: Example 4

$\{x, y \mid x = 2;\ y = 3\}$
Polyhedron consisting of a single point (2,3) is dimension 0. The dimension the system is 2, there are two equalities, and the dimension of the lineality space is 0, thus the dimension of the ray space is 2-2-0=0 (property 3.1).

Figure 3.6: Example 5

*Proof:* In the homogeneous form, the positivity constraint is $\lambda \geq 0$ represented by the vector $a = (0, \cdots, 0, 1)$, and rays are of the form $r = (r_1, \cdots, r_n, 0)$. Since $a \circ r = 0$, for all rays, all rays saturate the positivity constraint. Vertices are of the form $v = (v_1, \cdots, v_n, d)$, $d \neq 0$. Since $a \circ v = d \neq 0$, for all vertices, no vertex saturates the positivity constraint.

□

As surprising as it may seem, the positivity constraint is not always redundant, as was shown in the examples in figures 3.4 and 3.5. The following property gives a rule for when the positivity constraint will be needed.

**Property 3.5** *The positivity constraint will be irredundant iff the size of the set of rays is $\geq n$, the dimension of the ray space, and the rank of the ray set is $n$.*

*Proof:* For the positivity constraint to be irredundant, it needs at least $n$ vertices/rays which saturate it (property 3.2). Since only rays saturate the positivity constraint, at least $n$ rays are needed (property 3.4). Thus in a system with $n$ rays, the positivity constraint is irredundant.

□

Positivity constraints are included so there aren't invalid polyhedra floating around (according to properties 3.1 and 3.2). There are different strategies involving the use

$\{x, y \mid 1 = 0\}$

```
Empty Polyhedron, Dimension 2
Constraints  ( 3 equalities, 0 inequalities )
   x = 0
   y = 0
   1 = 0
Lines/Rays ( 0 lines, 0 rays )
   -none-

dim(ray space) = dimension - numlines - numequalities
        = 2   - 0         - 3
        = -1
dim(lineality space) = numlines
        = 0
```

Figure 3.7: Example 6 — Empty Polyhedron

of this constraint. One strategy is to add the positivity constraint to all polyhedra (even if it is redundant) before doing any operation– and then filter it out of the answer at the end. This works, but may not be very efficient. To add the positivity constraint may require allocating memory and then copying the polyhedron plus the positivity constraint for each polyhedral operand before doing any operation. Another alternative is to keep the positivity constraint in polyhedra where it is needed (according to property 3.5). This works well for the library. The only problem is that it usually will have to be filtered out by the user when displaying the constraints (by a pretty printer).

### 3.6.2 Empty Polyhedra

An empty domain is a polyhedron which includes no points. It is caused by overcontraining a system such that no point can satisfy all of the constraints. Empty polyhedra have the following properties:

**Property 3.6** *In an empty polyhedron*

a. *the dimension of the lineality space is 0.*

b. *the dimension of the ray space is -1.*

c. *there are no rays (vertices, to be more specific).*

*Proof:*

Part a. Since there are no points in an empty polyhedron, there are no lines, and the dimension of the lineality space is the number of lines = 0.

Part b. To overconstrain a system of dimension $n$ requires $n+1$ equalities. From property 3.1, the dimension of the ray space is (dimension of system)-(number of equalities)-(number of lines) $= n - (n+1) - 0 = -1$

Part c. Since there are no points in an empty polyhedron, there are no vertices as well.

□

A test for an empty polyhedron may be performed by either of the following C-macros:

```
#define emptyQ(P) (P->NbEqualities==(P->Dimension+1))
#define emptyQ(P) (P->NbRays==0)
```

An empty polyhedron can be created by the library by a call to the procedure:

```
Polyhedron *EmptyPolyhedron ( unsigned Dimension )
```

### 3.6.3 Universe Polyhedron

A universe polyhedron is one that encompasses all points within a certain dimensional subspace. It is therefore unbounded in all directions. It is created by not constraining a system at all (except with the positivity constraint). A universe polyhedron has the following properties:

**Property 3.7** *In an universe polyhedron*

a. *the dimension of the lineality space is the dimension of the polyhedron,*

b. *the dimension of the ray space is 0,*

c. *there are no constraints, other that the positivity constraint.*

$\{x, y \mid 1 \geq 0\}$

```
Universe Polyhedron, Dimension 2
Constraints  ( 0 equalities, 1 inequality )
   1 >= 0
Lines/Rays ( 2 lines, 0 rays)
   line   (1,0) (x-axis)
   line   (0,1) (y-axis)
   vertex (0,0) (origin)

dim(ray space) = dimension - num_lines - num_equalities
       = 2   - 2         - 0
       = 0
dim(lineality space) = num_lines
       = 2
```

Figure 3.8: Example 7 — Universe Polyhedron

*Proof:*

Part a. An unconstrained system of dimension $n$ is a $n$-space with a basis of $n$ lines.

Part b. From property 3.1, the dimension of the ray space is (dimension of system)-(number of equalities)-(number of lines) $= n - 0 - n = 0$

Part c. Any constraint other that the positivity constraint would exclude points from the system, and is therefore inadmissable.

□

A test for a universe polyhedron may be performed by the following C-macro:

```
#define universeQ(P) (P->Dimension==P->NbLines)
```

A universe polyhedron can be created using the library with a call to the procedure:

```
Polyhedron *UniversePolyhedron ( unsigned Dimension )
```

# Chapter 4

# THE POLYHEDRAL LIBRARY

The polyhedral library creates, operates on, and frees objects called *domains* (described in section 1.2) made up of unions of polyhedra. The data structure for these domains was described in section 3.5 along with operations to create and free the data structure. This chapter builds on chapter 3 and describes the operational side of the library in detail. The algorithms used to operate on domains are fully described as well.

## 4.1 Description of Operations

The polyhedral library contains a full set of operations as described in this section.

### External interface with library

In many operations there is a parameter called `NbMaxRays` which sets the size of a temporary work area. This work area is allocated by the library using a call to `malloc` at the beginning of an operation and is deallocated at the end. If the work area is not large enough, the operation will fail and a fault flag will be set. The following external domain functions are supported:

```
Polyhedron *DomainIntersection ( Polyhedron *d1, Polyhedron *d2,
                                 unsigned NbMaxRays )
```

returns the domain intersection of domains $d1$ and $d2$. The dimensions of domains $d1$ and $d2$ must be the same. Described in section 4.5.

```
Polyhedron *DomainUnion ( Polyhedron *d1, Polyhedron *d2,
                          unsigned NbMaxRays )
```

returns the domain union of domains $d1$ and $d2$. The dimensions of domains $d1$ and $d2$ must be the same. Described in section 4.6.

```
Polyhedron *DomainDifference ( Polyhedron *d1, Polyhedron *d2,
                               unsigned NbMaxRays )
```

returns the domain difference, $d1$ less $d2$. The dimensions of domains $d1$ and $d2$ must be the same. Described in section 4.7.

```
Polyhedron *DomainSimplify ( Polyhedron *d1, Polyhedron *d2,
                             unsigned NbMaxRays )
```

returns the domain equal to domain $d1$ simplified in the context of $d2$, i.e. all constraints in $d1$ that are not redundant with the constraints of $d2$. The dimensions of domains $d1$ and $d2$ must be the same. Described in section 4.8.

```
Polyhedron *DomainConvex ( Polyhedron *d, unsigned NbMaxRays )
```

returns the minimum polyhedron which encloses domain $d$. Described in section 4.9.

```
Polyhedron *DomainImage ( Polyhedron *d, Matrix *m,
                          unsigned NbMaxRays )
```

returns the image of domain $d$ under affine transformation matrix $m$. The number of rows of matrix $m$ must be equal to the dimension of the polyhedron plus one. Described in section 4.10.

```
Polyhedron *DomainPreimage ( Polyhedron *d, Matrix *m,
                             unsigned NbMaxRays )
```

returns the preimage of domain $d$ under affine transformation matrix $m$. The number of columns of matrix $m$ must be equal to the dimension of the polyhedron plus one. Described in section 4.11.

```
Polyhedron *Constraints2Polyhedron ( Matrix *m,
                                     unsigned NbMaxRays )
```

returns the largest polyhedron which satisfies all of the constraints in matrix $m$. Described in section 4.4.

```
Polyhedron *Rays2Polyhedron ( Matrix *m, unsigned NbMaxRays )
```

returns the smallest polyhedron which includes all of the vertices, rays, and lines in matrix $m$. Described in section 4.4.

```
Polyhedron *UniversePolyhedron ( unsigned Dimension )
```

return the universal polyhedron of dimension $n$. Described in section 3.6.3.

```
Polyhedron *EmptyPolyhedron ( unsigned Dimension )
```

return the empty polyhedron of dimension $n$. Described in section 3.6.2.

```
Polyhedron *DomainCopy ( Polyhedron *d )
```

returns a copy of domain $d$. Described in section 3.5.

```
void DomainFree ( Polyhedron *d )
```

frees the memory used for domain $d$. Described in section 3.5.

## 4.2 Computation of Dual Forms

A important problem in computing with polyhedral domains is being able to convert from a domain described implicitly in terms of linear equalities and inequalities (equation 2.3), to a parametric description (equation 2.4) given in terms of the geometric features of the polyhedron (lines, rays, and vertices). Inequalities and equalities are referred to collectively as *constraints.* An equivalent problem is called the *convex hull problem* which computes the facets of the convex hull surrounding given a set of points,

The algorithms to solve this problem are categorized into one of two general classes of algorithms, the pivoting and non-pivoting methods. [MR80] The pivoting methods are derivatives of the simplex method which finds new vertices located adjacent to known vertices using simplex pivot operations.

The algorithm used by the library belongs to the other class, that of the of nonpivoting methods. For convenience, the nonpivoting methods are usually stated in terms of finding the extreme rays of polyhedral *cones.* These methods rely on the principle that polyhedral cones, like polyhedra, have two dual representations, the implicit form (equation 2.6) and the parametric form (equation 2.5). These methods find a solution by first setting up a tableau in which an initial polyhedron (such as the universe or the positive orthant) is simultaneously represented in both forms. The algorithm then iterates by adding one new inequality or equality at a time and computing the new polyhedron at each step by modifying the polyhedron from the previous step. The order in which constraints are selected does not change the final solution, but may have an effect on the run time of the procedure as a whole. The complexity of this problem is known to be $\mathcal{O}(n^{\lfloor \frac{d}{2} \rfloor})$, where $n$ is the number of constraints and $d$ is the dimension. This is the best that can be done, since the size of the output (i.e. the number of rays) is of the same order.

The nonpivoting methods are based on an algorithm called *the double description method* invented by Motzkin et al. in 1953 [MRTT53]. Motzkin described a general algorithm which solves the dual-computation problem in terms of converting a mixed linear system of constraints into a non-pointed cone. In each iteration, a new constraint is added to the current cone in the tableau. Rays in the cone are divided into three groups, $R^+$ the rays which verify the constraint, $R^0$ the rays which saturate the constraint, and $R^-$ the rays which do not verify the constraint. A new cone is then constructed from the ray sets $R^+$, $R^0$, plus the convex combinations of pairs of rays, one each from sets $R^+$ and $R^-$. The main problem with the nonpivoting methods is that they can generate a non-minimal set of rays by creating non-extreme or redundant rays. If allowed to stay, the number of rays would grow exponentially and would seriously test the memory capacity of the hardware as well as degrade the performance of the procedure. Motzkin proposed a simple and rather elegant test to solve this problem. He showed that a convex combination of a pair of rays ($r^- \in R^-$, $r^+ \in R^+$) will result in an extreme ray in the new cone if and only if the minimum face which contains them both: 1) is dimension one greater than $r^-$ and $r^+$, and 2) only contains the two rays $r^-$ and $r^+$. This test inhibits the production of unwanted rays and keeps the solution in a minimal form.

Chernikova [Che65, Rub75] described a similar algorithm to solve the restricted case of the mixed constraint problem with the additional constraint that variables are all non-negative ($x \geq 0$). Chernikova's method was the same as Motzkin's method, except that she used a slightly smaller and improved tableau. Fernández and Quinton [FQ88] extended the Chernikova method by removing the restriction that $x \geq 0$ and adding a heuristic to improve speed by ordering the constraints. A large portion of the computation time is spent doing the adjacency test. Le Verge [Le 92] improved the speed of the redundancy checking procedure used in [FQ88], which is the most time consuming part of the algorithm. Seidel has an algorithm for the equivalent convex hull problem [Sei91] which executes in $\mathcal{O}(n^{\lfloor \frac{d}{2} \rfloor})$ expected running time where $n$ is the number of points and $d$ is the dimension. This is provably the best one can do, since the output of the procedure is of the same order. He solves the adjacent ray problem (the adjacent facet problem in his case) by creating and maintaining a facet graph in which facets are vertices and adjacent facets are connected by edges. It takes a little extra code to maintain the graph,

| Polyheron Type | Bound |
|---|---|
| $d$-Simplex | $f_k = \begin{pmatrix} d+1 \\ k+1 \end{pmatrix}$ |
| $d$-Simplex | $\sum_{k=-1}^{d} f_k = 2^{(d+1)}$ |
| Simplicial $d$-polytope [MR80] | $f_{d-1} \leq \begin{pmatrix} n - \lceil \frac{d+1}{2} \rceil \\ n-d \end{pmatrix} + \begin{pmatrix} n - \lceil \frac{d+2}{2} \rceil \\ n-d \end{pmatrix}$ |
| Simplicial $d$-polytope [Gru67] | $f_{d-1} \geq (d-1)n - (d-2)(d-1)$ |
| Cyclic $d$-polytope [Gru67] | $f_k = \begin{pmatrix} n \\ k+1 \end{pmatrix}, \ 0 \leq k \leq \lfloor \frac{d}{2} \rfloor$ |
| Cyclic $d$-polytope [Gru67] | $f_k = \frac{n - (d \bmod 2)(n-k-2)}{n-k-1} \times \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} \begin{pmatrix} n-1-j \\ k+1-j \end{pmatrix} \begin{pmatrix} n-k-1 \\ 2j-k-1+d \bmod 2 \end{pmatrix} = \mathcal{O}(\lfloor \frac{d}{2} \rfloor ! n^{\lfloor \frac{d}{2} \rfloor})$ |
| Cyclic $d$-polytope [Kle66] | $f_{d-1} = \begin{cases} \frac{2n}{d} \begin{pmatrix} n - \frac{d}{2} - 1 \\ \frac{d}{2} \end{pmatrix} & , d \text{ even} \\ 2 \begin{pmatrix} n - \lfloor \frac{d}{2} \rfloor - 1 \\ \lfloor \frac{d}{2} \rfloor \end{pmatrix} & , d \text{ odd} \end{cases} = \mathcal{O}(n^{\lfloor \frac{d}{2} \rfloor})$ |

Table 4.1: Compilation of results for bound on number of faces/facets

but then he does not need to do the Motzkin adjacency test on all pairs of vertices (facets).

McMullen [McM70, MS71] showed that for any $d$-polytope with $n$ vertices, the number of $k$-faces, $f_k$ is upper bounded by the number of $k$-faces of a cyclic $d$-polytope with the same number of vertices. One of the implications of this is that the number of facets, $f_{d-1} = \mathcal{O}(n^{\lfloor \frac{d}{2} \rfloor})$. The bound on the number of faces, facets, vertices, etc. in a polyhedron is a difficult and well studied problem. Table 4.1 summarizes some of the work that has been done.

### 4.2.1 The general algorithm

The nonpivoting solvers successively refine their solution by adding one constraint at a time and modifying the solution polyhedron from the previous step to reflect the new constraint. An inequality $a^T x \geq 0$ is co-represented by the closed halfspace $H^+$ which is the set of points $\{x \mid a^T x \geq 0\}$. Likewise the equality $a^T x = 0$ is co-represented by the hyperplane $H$ which is the set of points $\{x \mid a^T x = 0\}$. At each step of the algorithm, a new inequality or equality represented by either $H^+$ or $H$, respectively, is intersected with the cone $C$ represented by $C = L + R$ (equation 2.8), the combination of its lineality space and ray space, to produce a modified cone $C' = L' + R'$. The algorithm **Dual** in figure 4.1 gives the generalized algorithm given by Motzkin.

In the algorithm **Dual**, there are three procedures which alter the polyhedron. They are **ConstrainL** which constrains the lineality space, **AugmentR** which augments the dimension of the ray space, and finally **ConstrainR** which constrains the ray space. These procedures are discussed below in greater detail.

The **ConstrainL** procedure shown in figure 4.3 constrains the lineality space $L$ by slicing it with a new constraint, and if the new constraint cuts $L$, then $L$'s dimension is reduced by one and a new ray $r_{new}$ is generated which is added to the ray space. It is fairly straightforward and runs in $\mathcal{O}(n)$ time where $n$ is the dimension of the lineality space.

There are two procedures which perform transformations on the ray space. The first one is **AugmentR** shown in figure 4.4 which adds a new ray $r_{new}$ created by **ConstrainL** to the ray space. When $r_{new}$ is added to the ray space $R$, it increases the dimension of $R$ by one. It is of complexity $\mathcal{O}(r)$ time, where $r$ is the number of rays.

The second operation **ConstrainR** shown in figure 4.5 constrains the ray space by slicing it with the hyperplane $H$, discarding the part of the ray space which lies outside of constraint. For inequalities, the part of the polyhedron which lies outside of the halfspace $H^+$ is removed. For equalities, the part of the polyhedron which lies outside of the hyperplane $H$ is removed. In either case, the new face lying on the cutting hyperplane surface needs to be computed.

The two functions **AugmentR** and **ConstrainR** follow Motzkin's algorithm closely. **ConstrainR** computes a new pointed cone by adding a new constraint. Rays which verify and saturate the constraint are added. Rays which do not verify the constraint are combined with *adjacent* rays which verify the constraint to create new rays which saturate the constraint. Motzkins adjacency test is used to find adjacent pairs of rays. The Motzkin adjacency test is used to test every pair of rays to determine if that pair will combine to produce an extreme ray or not. This is done by computing what constraints the pair of rays have in common and making sure that no other ray also saturates that same set of constraints. Thus the list of rays produced by **ConstrainR** is always extreme (non-redundant). The entire **ConstrainR** procedure has an $\mathcal{O}(n^3k)$ complexity where $n$ is the number of rays and $k$ is the number of constraints. Much of this time is spent in performing the adjacency tests.

The procedure **Combine** shown in figure 4.2 is where all of the actual computation takes place. It uses as input two rays, $r^+$ and $r^-$, as well as a constraint $a$. It then computes the ray $r^=$ which firstly is a linear combination of $r^+$ and $r^-$ ($r^= = \lambda_1 r^+ + \lambda_2 r^-$), and secondly, saturates constraint $a$, ($a^T r^= = 0$).

```
Procedure Dual($A$), returns $L$, $R$

$L$ := basis for $d$-dimensional lineality space.
$R$ := point at the origin.
For each constraint $a \in A$ Do
    $r_{new}$ := ConstrainL ($L$, $a$)
    If $r_{new} \neq 0$ Then AugmentR ($R$, $a$, $r_{new}$) Else ConstrainR ($R$, $a$)
End
Return $L$ and $R$.
```

Figure 4.1: Procedure to compute **Dual**($A$)

```
Procedure **Combine**($r_1$, $r_2$, $a$), returns $r_3$

$D = \text{GCD}(a^T r_1, a^T r_2)$
$\lambda_1 = a^T r_2 / D$
$\lambda_2 = -a^T r_1 / D$
$r_3 = \lambda_1 r_1 + \lambda_2 r_2$
```

Figure 4.2: Procedure to compute **Combine**($r_1$, $r_2$, $a$)

```
Procedure ConstrainL ($L$, $a$), modifies $L$, returns $r_{new}$

Find an $l_1 \in L$ such that $a^T l_1 \neq 0$, ($l_1$ does not saturate constraint $a$)
If $l_1$ does not exist Then ($L \cap H$ is $L$ itself and $r_{new}$ is empty) Return 0.
$L' :=$ empty.
For each line $l_2 \in L$ such that $l_2 \neq l_1$ Do
    $L' := L' +$ Combine ($l_1$, $l_2$, $a$)
End
If $a^T l_1 > 0$, ($l_1$ verifies constraint $a$) Then Create ray $r_{new}$ equal to $l_1$
Else ($a^T l_1 < 0$) Create ray $r_{new}$ equal to $-l_1$
$L := L'$
Return $r_{new}$
```

Figure 4.3: Procedure to compute **ConstrainL**($L$, $a$)

```
Procedure AugmentR ($R$, $a$, $r_{new}$), modifies $R$

Set $R' :=$ empty.
For each ray $r \in R$ do
    If $a^T r = 0$ Then $R' := R' + r$
    If $a^T r > 0$ Then $R' := R' +$ Combine ($r$, $-r_{new}$, $a$)
    If $a^T r < 0$ Then $R' := R' +$ Combine ($r$, $r_{new}$, $a$)
End
If $a$ is an inequality Then $R' := R' + r_{new}$
$R := R'$
```

Figure 4.4: Procedure to compute **AugmentR**($R$, $a$, $r_{new}$)

```
Procedure **ConstrainR** ($R$, $a$), modifies $R$

Partition $R = R^+ + R^0 + R^-$.
    $R^0 := \{r \mid r \in R, a^T r = 0\}$, the rays which saturate constraint $a$.
    $R^+ := \{r \mid r \in R, a^T r > 0\}$, the rays which verify constraint $a$.
    $R^- := \{r \mid r \in R, a^T r < 0\}$, the rays which do not verify constraint $a$.
If constraint $a$ is an inequality, Then set $R' := R^+ + R^0$.
Else (constraint $a$ is an equality) set $R' := R^0$.
For each ray $r^+ \in R^+$ do
    For each ray $r^- \in R^-$ do
        Adjacency test on $(r^+, r^-)$
        c := set of common constraints saturated by both $(r^+, r^-)$
        For each ray $r \in R \mid r \neq r^+,\ r \neq r^-$ Do
            If $r$ also saturates all of the contraints in set $c$ Then
                ($r^+$ and $r^-$ are not adjacent.) Continue to next ray $r^-$.
        End
        ( $r^+$ and $r^-$ are adjacent.) $R' := R' +$ Combine $(r^+,\ r^-,\ a)$
    End
End
$R := R'$
```

Figure 4.5: Procedure to compute **ConstrainR**($R$, $a$)

### 4.2.2 Implementation

The procedure **Dual** is implemented in the polyhedral library as the procedure:

```
static int Chernikova ( Matrix *Constraints,
                        Matrix *Rays,
                        Matrix *Sat,
                        unsigned NbLines,
                        unsigned NbMaxRays,
                        unsigned FirstConstraint )
```

It is named "Chernikova" for historical reasons, however, a more suitable name would have been "Motzkin", since the procedure is primarily due to him. It is somewhat different than the basic one described in section 4.2.1 in that it allows a new set of constraints to be added to an already existing polyhedron (there may be preexisting constraints and rays). The entire list of constraints (both the old and the new) is passed as the parameter `Matrix *Constraints`, with the parameter `unsigned FirstConstraint` indicating which is the first "new" constraint. The preexisting lines and rays are passed in as the parameter `Matrix *Rays`. The lines must be grouped together at the beginning of the matrix, and the parameter `unsigned NbLines` indicates how many lines are in the `Rays` matrix. The parameter `unsigned NbMaxRays` is the allocated dimension of the `Rays` matrix and limits the number of lines and rays that can be stored at any one time. And finally, the parameter `Matrix *Sat` contains the incidence matrix (defined in section 3.3) between the original constraints and rays of the input polyhedron. The procedure adds the new constraints to the polyhedron, updating the `Rays` and `Sat` matrices. The updated matrices are returned by the procedure. This procedure has been benchmarked and times published in [Le 92].

## 4.3 Producing a minimal representation

After computing the dual of a set of constraints, the set of rays produced is guaranteed to be non-redundant, by virtue of the adjacency test which is done when each ray was produced. However, the constraints are still possibly redundant. There remain a number of simplifications which can still be done on the resulting polyhedron, among which are:

1. Detection of implicit lines such as line(1,2) given that there exist rays (1,2) and (-1,-2).
2. Finding a reduced basis for the lines.
3. Removing redundant positivity constraints $1 \geq 0$.
4. Detection of trivial redundant inequalities such as $y \geq 4$ given $y \geq 3$, or $x \geq 2$ given $x = 1$.

5. Detection of redundant inequalities such as $x + y \geq 5$ given $x \geq 3$ and $y \geq 2$.

6. Solving the system of equalities and eliminating as many variables as possible.

The algorithm to do all of these reductions is sketched out below. In the procedure, each constraint and each ray needs a status word which is provided for in the polyhedron structure (see section 3.5).

**Reduce**(Constraints, Rays, Sat), returns a Polyhedron structure.

Step 0 Count the number of vertices among the rays while initializing the ray status counts to 0. If no vertices are found, quit the prodedure and return an empty polyhedron as the result.

Step 1 Compute status counts for both rays and inequalities. For each constraint, count the number of vertices/rays saturated by that constraint, and put the result in the status words. At the same time, for each vertex/ray, count the number of constaints saturated by it.
Delete any positivity constraints, but give rays credit in their status counts for saturating the positivity constraint.

Step 2 Sort equalities out from among the constraints, leaving only inequalities. Equalities are constraints which saturate all of the rays. (Status count = number of rays)

Step 3 Perform gaussian elimination on the list of equalities. Obtain a minimal basis by solving for as many variables as possible. Use this solution to reduce the inequalities by elimating as many variables as possible. Set NbEq2 to the rank of the system of equalities.

Step 4 Sort lines out from among the rays, leaving only unidirectional rays. Lines are rays which saturate all of the constraints (status count = number of constraints + 1(for positivity constraint) ).

Step 5 Perform gaussian elimination of on the lineality space to obtain a minimal basis of lines.
Use this basis to reduce the representation of the unidirectional rays. Set NbBid2 to the rank of the system of lines.

Step 6 Do a first pass filter of inequalities and equality identification.
New positivity constraints may have been created by step 3. Check for and elimate them.
Count the irredundant inequalities and store count in NbIneq.
if (Status==0) Constraint is redundant.
else if (Status<Dim) Constraint is redundant.

```
        else if (Status==NbRays) Constraint is an equality.
        else Constraint is a irredundant inequality.

Step 7  Do first pass filter of rays and identification of lines.
        Count the irredundant Rays and store count in NbUni.
        if (Status<Dim) Ray is redundant.
        else if (Status==(NbConstraints+1)) Ray is a line.
        else Ray is an irredundant unidirectional ray.

Step 8  Create the polyhedron. Allocate the polyhedron (using approximate sizes).
        Number of constraints = NbIneq+NbEq2+1
        Number of rays = NbUni+NbBid2
        Partially fill the Polyhedron structure with the lines computed in step 3 and
        the equalities computed in step 5.

Step 9  Final pass filter of inequalities.
        Every 'good' inequality must saturate at least Dimension rays and be unique.
        The final list of inequalities is written to polyhedron.

Step 10 Final pass filter of rays and detection of redundants rays.
        The final list of rays is written to polyhedron.

Step 11 Return polyhedron.
```

In the polyhedral library, the **Reduce** algorithm described above is implemented as an internal library procedure defined as:

```
static Polyhedron *Remove_Redundants ( Matrix *Constraints,
                                       Matrix *Rays,
                                       Matrix *Sat,
                                       unsigned *Filter )
```

It takes the list of constraints and rays as generated by the `Chernikova` procedure, as well as the incidence matrix `Sat` relating the two. It assumes that either the unidirectional rays are non redundant, *or* that the inequalities are non redundant. This is guaranteed by the `Chernikova` procedure. The procedure performs the reductions on the lists of constraints and rays, then builds a `Polyhedron` structure from the results. The parameter `unsigned *Filter`, if non-zero, points to a bit vector with one bit for each ray. The `Remove_Redundants` procedure sets the bits corresponding to the rays which it finds non-redundant. The `Filter` vector is used in the implementation of the `DomSimplify` function.

## 4.4 Conversion of rays/constraints to polyhedron

Given a set of rays, the corresponding polyhedron is computed by simply running the **Chernikova** procedure to get the dual list of constraints and then the **Remove_Redundants** procedure to reduce the ray/constraint lists and create a polyhedron.

Likewise, starting from a list of constraints, the corresponding polyhedron is computed by running the **Chernikova** procedure to get the dual list of rays, and then the **Remove_Redundants** procedure to reduce the ray/constraint lists and create a polyhedron. The conversion of a list of rays or a list of contraints to a polyhedron is the most basic application of the **Chernikova** and **Remove_Redundants** procedures.

## 4.5 Intersection

Intersection is performed by concatenating the lists of constraints from two (or more) polyhedra into one list, and finding the polyhedron which satisfies all of the combined constraints. This is done by finding the extremal rays which satisfy the combined constraints, (finding the dual of the list of constraints), and then reducing both the constraints and rays into one polyhedron. This procedure is illustrated in figure 4.6.

To intersect two domains, $A$ and $B$, which are unions of polyhedra, $A = \cup_i A_i$ and $B = \cup_j B_j$, the pairwise intersection of the component polyhedra from $A$ and $B$ must be computed, and the union of the results is the resulting domain of intersection, as shown below:

$$\begin{aligned} A \cap B &= (\cup_i A_i) \cap (\cup_j B_j) \\ &= \cup_{i,j}(A_i \cap B_j) \end{aligned}$$

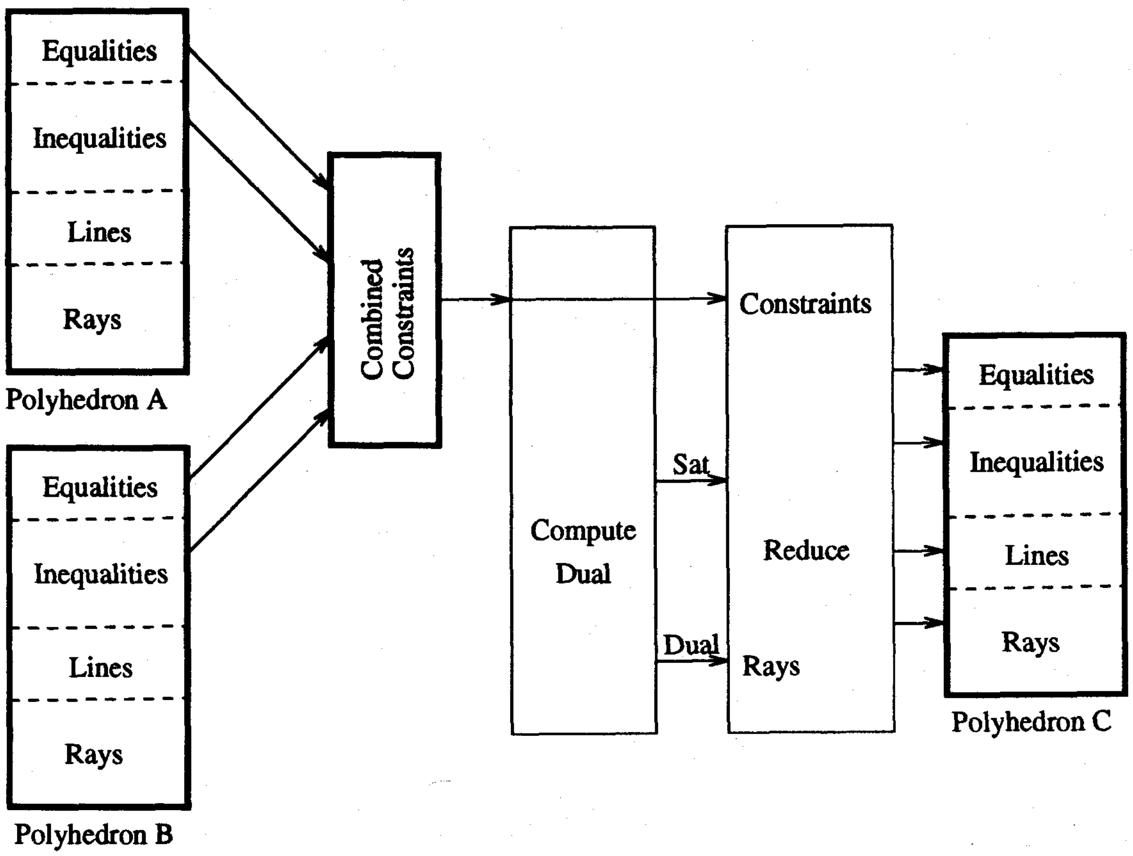


Figure 4.6: Computation of Intersection

## 4.6 Union

The domain (non-convex) union operation simply combines two domains into one. The lists of polyhedra associated with the domains are combined into a single list. However, combining the two lists blindly may create non-minimal representations. For instance, if in forming the union of domains $A = \{i \mid i \geq 1\}$ and $B = \{i \mid i \geq 2\}$, the fact that $A \supset B$ is taken into consideration, then the union can be reduced to simply $A$. The algorithm used in the library performs this kind of simplification during the union operation. Before adding any new polyhedron to an existing list of polyhedra, it first checks to see if that polyhedron is covered by some polyhedron already in the domain. If it *is* covered, then the new polyhedron is not added to the domain. Likewise, polyhedra in the existing list may be deleted if they are covered by the new polyhedron. In the new combined list, no polyhedron is a subset of any other polyhedron.

The test for when one polyhedron covers another is performed by the library procedure:

```
int  PolyhedronIncludes(p1, p2)
     Polyhedron *p1;
     Polyhedron *p2;
```

which returns a 1 if $p1 \supseteq p2$, (polyhedron **p1** includes (covers) polyhedron **p2**), and returns a 0 otherwise. The test for when a polyhedron $p1$ covers or includes another polyhedron $p2$ is straight forward using the dual representation of polyhedra in the library: $p1 \supseteq p2$ if all of the rays of $p2$ *satisfy* (see definition 3.1) all of the constraints of $p1$. This is a case in point of when the dual representation comes in handy. The constraint representation of $p1$ and the dual ray representation of $p2$ are used to determine $p1 \supseteq p2$. Since both representations are kept in the data structure, the dual does not need to be (re)computed in order to do this test.

## 4.7 Difference

Domain difference $A - B$ computes the domain which is part of $A$ but not part of $B$. It is equivalent to $A \cap \sim B$, where $\sim B$ is the *complement* domain of $B$. If $B$ is the intersection of a set of hyperplanes (representing the equalities) and halfspaces (representing the inequalities), then the inverse of $B$ is computed as follows:

$$\begin{aligned}
\sim B &= \sim (\cap_i H_i) \\
&= \cup_i (\sim H_i) \\
&\phantom{=}\ \text{where} \\
\sim H_i &= \begin{cases} \{x \mid a^T x < 0\} & \text{when } \ H_i = \{x \mid a^T x \geq 0\} \\ \{x \mid (a^T x < 0 \ \cup \ a^T x > 0)\} & \text{when } \ H_i = \{x \mid a^T x = 0\} \end{cases} \\
&\phantom{=}\ \text{and normalizing for integer lattice domains :} \\
&= \begin{cases} \{x \mid \ -a^T x + 1 \geq 0\} & \text{when } \ H_i = \{x \mid a^T x \geq 0\} \\ \{x \mid (-a^T x + 1 \geq 0 \ \cup \ a^T x - 1 \geq 0)\} & \text{when } \ H_i = \{x \mid a^T x = 0\} \end{cases}.
\end{aligned}$$

The ALPHA program fragment:

```
var  A:{t,p | 0<=t; 1<=p<=4};
A = case
      {t,p | t=0; 1<=p<=4} ... ;
      {t,p | t>0; 1<=p<=3} ... ;
      {t,p | t>0; p=4}     ... ;
    esac;
```

is operationally equivalent to the following fragment:

```
var  A:{t,p | 0<=t; 1<=p<=4};
A = case
      {t,p | t=0}        ... ;
      {t,p | t>0; p<=3} ... ;
      {t,p | t>0; p=4}  ... ;
    esac;
```

which has simplified case conditions on the domain $A$. The above simplifications can be found using the simplify operation using the domain of $A$ as the context, and simplifying the case condition domains.

Figure 4.7: Application of DomSimplify

The computation of difference is the same as the computation of intersection after taking the inverse of $B$. Since the inverse of $B$ is a union of polyhedra, the difference of two polyhedra can be a union of polyhedra. Thus, polyhedra are not closed under the operation difference, where as unions of polyhedra are closed under this operation.

## 4.8 Simplify

The operation *simplify* is defined as follows:

Given domains $A$ and $B$, then Simplify$(A, B) = C$, when $C \cap B = A \cap B$, $C \supseteq A$ and there does not exist any other domain $C' \supset C$ such that $C' \cap B = A \cap B$.

The domain $B$ is called the *context*. The simplify operation therefore finds the largest domain set (or smallest list of constraints) that, when intersected with the context $B$ is equal to $A \cap B$. This operation is used in ALPHA to simplify case statements, as shown in the example in figure 4.7.

The simplify operation is done by computing the intersection $A \cap B$ and while doing the **Remove_Redundants** procedure, recording which constraints of $A$ are "redundant" with the intersection. The result of the simplify operation is then the domain $A$ with the "redundant" constraints removed.

An interesting subproblem in the simplify operation occurs when the intersection of $A$ and its context $B$ are empty. In this case, simplify should find the minimal set of constraints of $A$ which contradict all of the constraints of $B$. This is believed to be an NP-hard problem and a heuristic is employed to solve it in the library.

## 4.9 Convex Union

Convex union is performed by concatenating the lists of rays and lines of the two (or more) polyhedra in a domain into one combined list, and finding the set of constraints which tightly bound all of those objects. This is done by finding the dual of the list of rays and lines, and then reducing both the constraints and rays into one polyhedron. This procedure is illustrated in figure 4.8. This procedure is very similar to the intersection procedure which has already be described in section 4.5. Convex union finds the polyhedron generated from the union of the lines and rays of the two input polyhedra. Intersection finds the polyhedron generated from the union of the equalities and inequalities of the inputs.

## 4.10 Image

The function *image* transforms a domain $\mathcal{D}$ into another domain $\mathcal{D}'$ according to a given affine mapping function, $Tx + t$ (see definition 2.9 and property 2.2). The

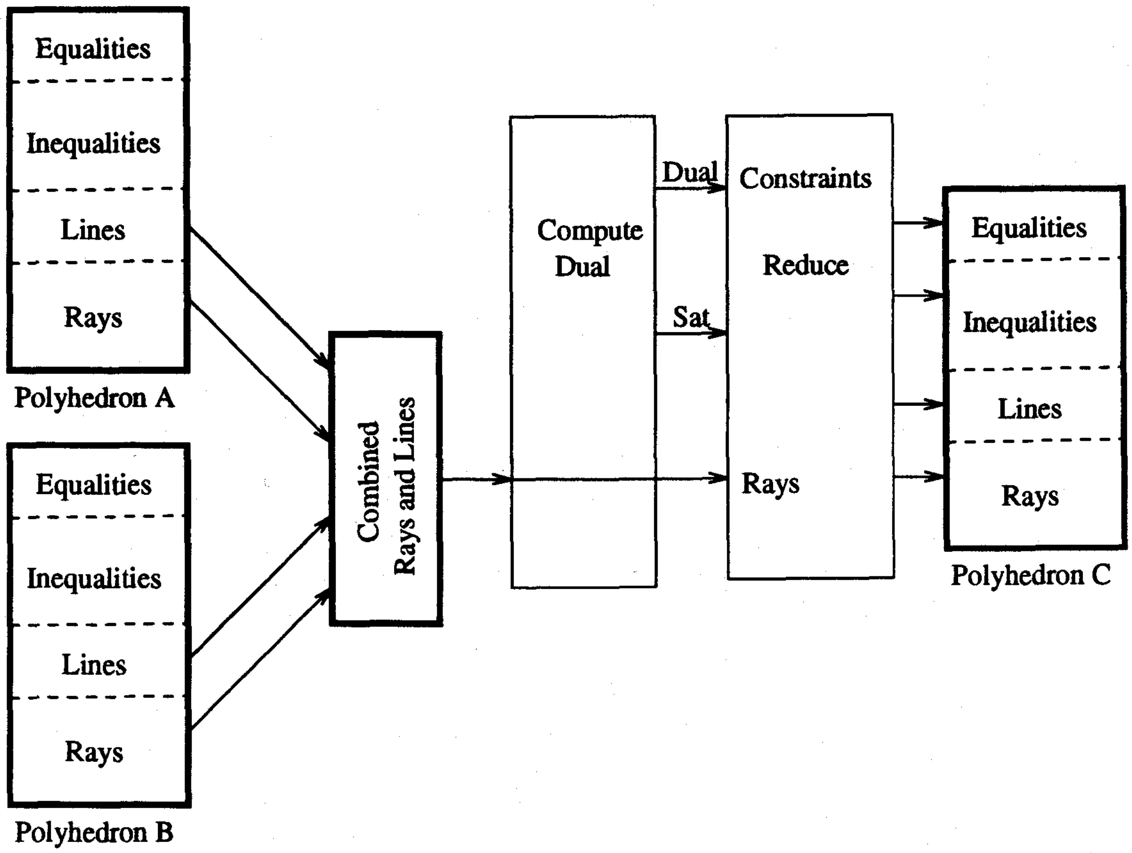


Figure 4.8: Computation of Convex Union

resulting domain $\mathcal{D}'$ is defined as:

$$\mathcal{D}' = \{x' \mid x' = Tx + t,\ x \in \mathcal{D}\}$$

In homogeneous terms, the transformation is expressed as

$$\mathcal{C}' = \{\begin{pmatrix} \xi x' \\ \xi \end{pmatrix} \mid \begin{pmatrix} \xi x' \\ \xi \end{pmatrix} = \left(\begin{array}{c|c} T & t \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} \xi x \\ \xi \end{pmatrix},\ \begin{pmatrix} \xi x \\ \xi \end{pmatrix} \in \mathcal{C}\}$$

Thus in the homogeneous representation, an affine transfer function becomes a linear transfer function (no constant added in). In the analysis that follows, we will treat the transfer function from the linear point of view. The transformation

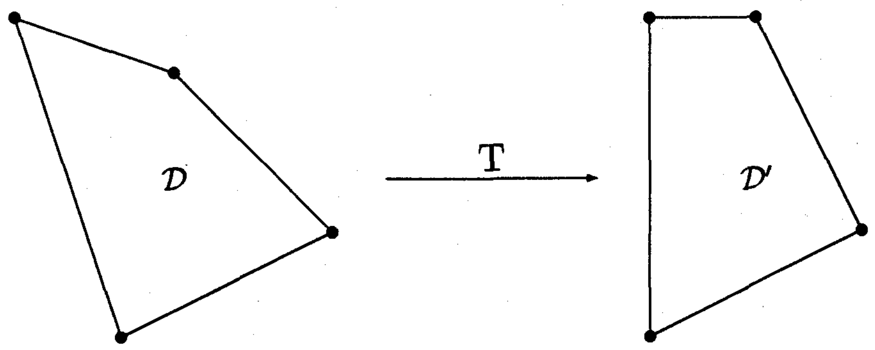


Figure 4.9: Affine transformation of $\mathcal{D}$ to $\mathcal{D}'$

function $\left(\begin{array}{c|c} T & t \\ \hline 0 & 1 \end{array}\right)$ is a matrix dimensioned by $(n+1)\times(m+1)$, where $n$ and $m$ are the dimensions of $x$ and $x'$, respectively. This transformation matrix is passed as a parameter to the image procedure. If $n = m$, $x$ and $x'$ are the same dimension. If $n \neq m$, the transformed space is of a larger (or smaller) dimension. The transformation does not have to be one-to-one, and therefore may not be invertable. Also, if $\det T \neq 1$, then the volume of the domain (the number of points in the domain) will be scaled by the determinant. To compute an image of $D$, given the full redundant representation:

$$D = \{x \mid Ax \geq 0,\ x = \mu R,\ AR \geq 0,\ \mu \geq 0\}$$

and given the transformation $x' = Tx$, the result $D'$ is

$$\begin{aligned} D' &= \{x' \mid A'x' \geq 0,\ x' = TR\mu,\ A'TR \geq 0,\ \mu \geq 0\} \\ &= \{x' \mid A'x' \geq 0,\ x' = R'\mu,\ A'R' \geq 0,\ \mu \geq 0\} \end{aligned}$$

$A'$ can be computed as the dual of $R'$. Thus, $R' = TR$ and $A' = \text{dual}(R')$. This computation is illustrated in figure 4.10 The image of a domain is simply the union of the images of the component polyhedra contained in the domain, as follows:

$$T.\mathcal{D} \;=\; T.(\cup_i \mathcal{P}_i)$$

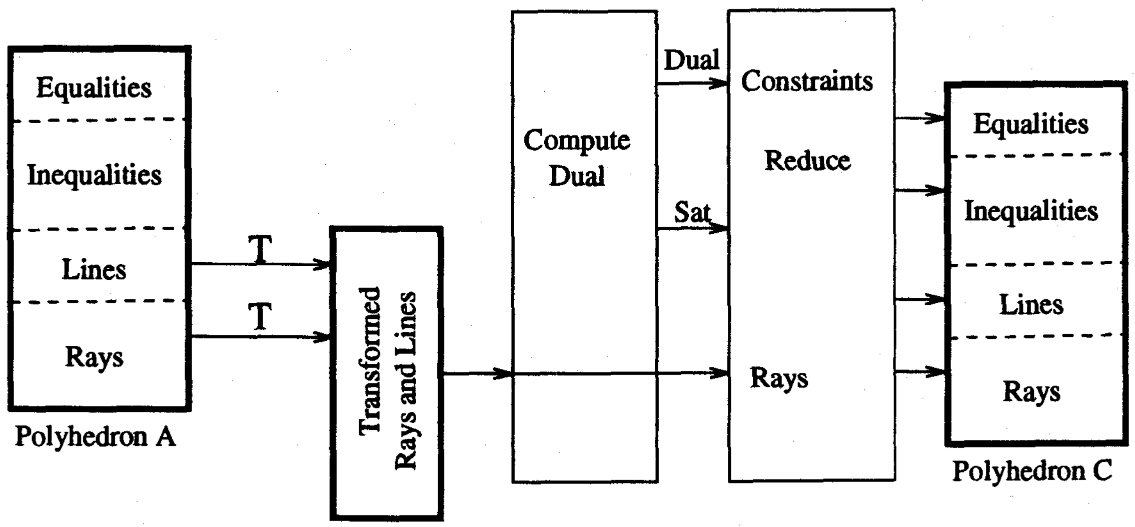


Figure 4.10: Computation of Image

$$= \cup_i(T.\mathcal{P}_i)$$

## 4.11 Preimage

Preimage is the inverse operation of image. That is given a domain $D'$ defined as

$$D' = \{x' \mid A'x' \geq 0,\ x' = R'\mu,\ A'R' \geq 0,\ \mu \geq 0\}$$

and a transformation $T$, find the domain $D$ which when transformed by $T$ gives $D'$. The relation $x = Tx'$ still holds. (Refer again to figure 4.9.) The result $D$ is

$$\begin{aligned} D &= \{x \mid A'Tx \geq 0,\ x = R\mu,\ A'TR \geq 0,\ \mu \geq 0\} \\ &= \{x \mid Ax \geq 0,\ x = R\mu,\ AR \geq 0,\ \mu \geq 0\} \end{aligned}$$

In the result, $A = A'T$ and $R = \text{dual}(A)$. This procedure is illustrated in figure 4.11. The preimage of a domain is simply the union of the preimages of the component polyhedra contained in the domain, as follows:

$$T^{-1}.\mathcal{D} = T^{-1}.(\cup_i\mathcal{P}_i)$$

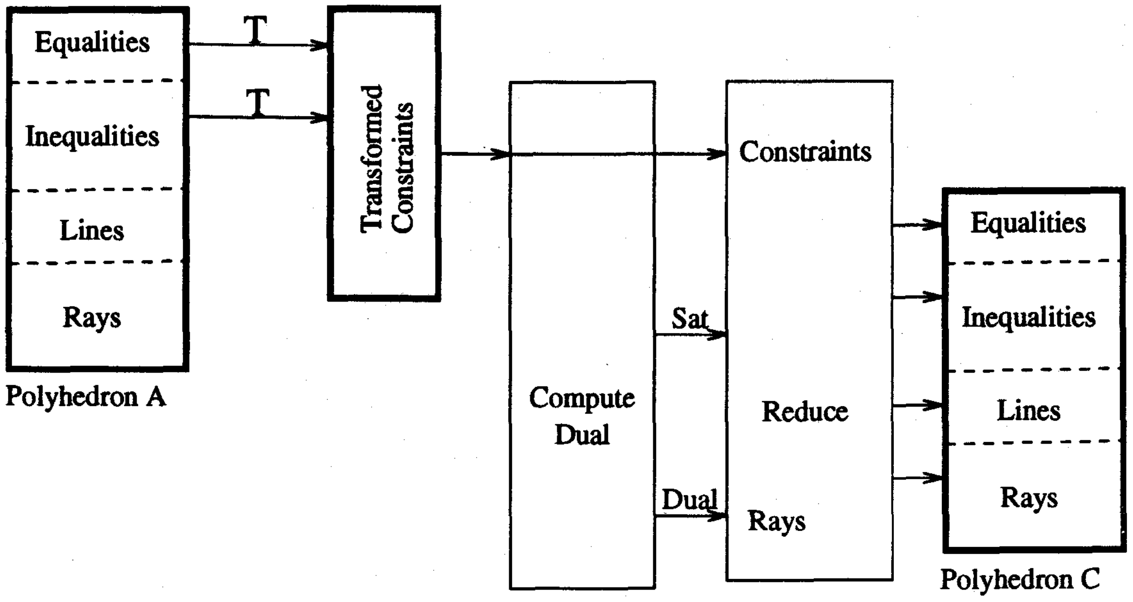


Figure 4.11: Computation of Preimage

$$= \cup_i(T^{-1}.\mathcal{P}_i)$$

$T^{-1}$ here is simple a notation for the preimage operation and does not mean to imply that $T$ is invertable.

## 4.12 The complexity of domain operations

Each of the domain operations described in this chapter make a call to the **Dual** procedure. That procedure dominates both the time and space complexity of each operation to be of order $\mathcal{O}(k^{\lfloor \frac{d}{2} \rfloor})$ where $k$ is the number of constraints and $d$ is the dimension.

## 4.13 The implementation of the polyhedral library

The polyhedral library has been implemented in the C-language and is currently in use in the ALPHA environment, as well as at other sites. The library code is composed of five files: three headers and two code files as follows:

`types.h` The header file defining the data structures used in the library.

`vector.h` The header file containing the forward definitions of the external vector operation procedures.

`vector.c` The code file containing the code for vector operations.

`polyhedron.h` The header file containing the forward definitions of the external vector polyhedral and domain procedures.

`polyhedron.c` The code file containing the code for polyhedral and domain operations.

The code itself is too large to be included as part of this thesis, but is available by ftp at host `ftp.irisa.fr`. It has been nicknamed the Chernikova library by its users.

# Chapter 5

# CONSTRUCTION OF THE FACE LATTICE

In this chapter, a new algorithm to construct the full face lattice of a polyhedron is presented. Throughout this chapter, I take advantage of the simplifications afforded by work in previous chapters:

1. Conversion to homogenous systems (section 3.1). Polyhedra are given in their "cone form", and all constraints are homogeneous. The cone form preserves incidences between faces and therefore the lattice structure of the polyhedron.

2. Decomposition (section 2.6.1). The polyhedron is decomposed and separated into its lineality and ray spaces. The lineality space has no effect on the lattice structure other than a 'dimensional displacement' of the entire lattice. The structure of the lattice is exclusively contained in the ray space.

Thus even though we are talking about polyhedra, we are really computing with the pointed cones derived from the polyhedra. We will also use the notation "$f_k(\mathcal{P})$" to mean the number of $k$-faces of a polyhedron (definition 2.27) [Gru67].

## 5.1 Foundation

Given a polyhedron $\mathcal{P} = \{x \mid Ax \geq 0, \;\; Bx = 0\}$, there is a one-to-one correspondance between each nonredundant inequality $a_i x \geq 0$ that bounds the polyhedron and the corresponding facet $\mathcal{F}_i$ which is formed by intersecting the hyperplane $\mathcal{H}_i = \{x \mid a_i x = 0\}$ and $\mathcal{P}$ as stated in the following theorem:

**Theorem 5.1** *(relating non redundant inequalities and facets)*

*There is a one to one correspondance between the facets of a polyhedron $\mathcal{P}$ and the* *irredundant inequalities of $\mathcal{P}$. Given $\mathcal{P} = \{x \mid Ax \leq 0,\ Bx = 0\}$, facet $\mathcal{F}_i = \{x \in \mathcal{P} \mid a_i x = 0\}$ (where $a_i$ is the $i^{\text{th}}$ row of $A$) is in one-to-one correspondance with the inequality $a_i x \leq 0$. The constraint $a_i x \leq 0$* **defines** *or* **determines** *facet $\mathcal{F}_i$.*

*Proof:* Given a $a_i$, row $i$ of matrix $A$, define hyperplane $\mathcal{H}_i = \{x \mid a_i x = 0\}$. $\mathcal{H}_i$ does not cut $\mathcal{P}$ since no $x \in \mathcal{P}$ exists such that $a_i x < 0$ (definition 2.23). Since $a_i$ is a non-redundant row, some point in $\mathcal{H}$ is also in $\mathcal{P}$, thus $\mathcal{H}_i$ is a supporting hyperplane (definition 2.24) and $\mathcal{P} \cap \mathcal{H}_i$ is a face of $\mathcal{P}$ (definition 2.25). Calling that face $\mathcal{F}_i$ we have:

$$\begin{aligned} \mathcal{F}_i &= \mathcal{P} \cap \mathcal{H}_i \\ &= \{x \in \mathcal{P}\} \cap \{x \mid a_i x = 0\} \\ &= \{x \in \mathcal{P} \mid a_i x = 0\} \quad \cdot \end{aligned}$$

Since $\mathcal{F}_i$ is the result of the intersection of $\mathcal{P}$ with a non-redundant equality, the face is a polyhedron of dimension one less than $\mathcal{P}$ and is thus a facet of $\mathcal{P}$ (definition 2.26).

□

Assuming $A$ does not contain any redundant constraints (rows), the number of facets is equal to the number of rows in $A$ since each *facet* of $\mathcal{P}$ is *defined* or *determined* by a unique row of $A$.

In general, any face $\mathcal{F}$ of polyhedron $\mathcal{P}$ can be determined by a unique subset of rows of $A$. For each face $\mathcal{F}$, there exists a row submatrix $A'$ of $A$, such that $\mathcal{F}$ can be described as:

$$\mathcal{F} = \{x \in \mathcal{P} \mid A'x = 0\} \quad \cdot$$

## 5.2 The Face-Lattice

The relation $f \vdash g$, "$f$ is a subface of $g$", is transitive and anti-symmetric and hence can be used to define a partial order among the faces of a polyhedron.

**Property 5.1** *(Transitive Property of $\vdash$ relation)*
*If $f \vdash g$ and $g \vdash h$ then $f \vdash h$.*

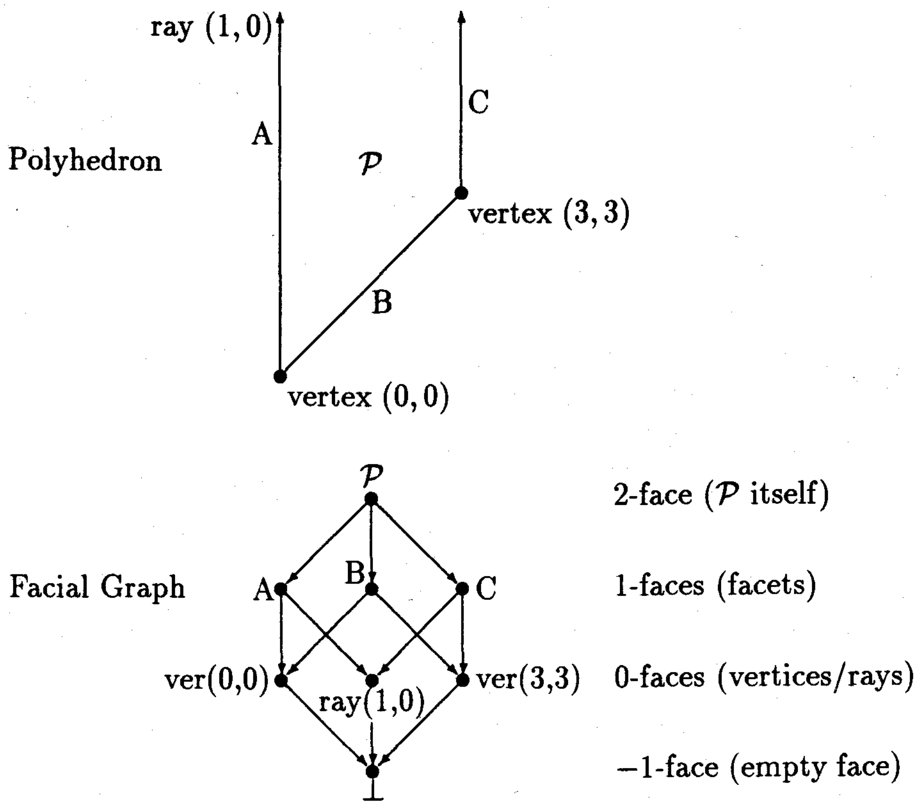


Figure 5.1: Example of a Facial Graph

**Property 5.2** *(Anti-symmetry property of $\vdash$ relation)*
*If $f \vdash g$ and $f \neq g$ then $g \not\vdash f$.*

The $\vdash$ relation, along with the partially ordered set of all of the faces of a polyhedron, form a lattice called the **face lattice** with the $n$-dimensional polyhedron at the top, and the empty set (called the $-1$-face) at the bottom (figure 5.1).

This lattice induces a directed graph called the **facial graph** in which the nodes are the faces of $\mathcal{P}$ and a directed edge exists between nodes $f$ and $g$ if and only if $g$ is a facet of $f$. The size of the facial graph of $\mathcal{P}$ is the number of nodes and arcs and is denoted by $L(\mathcal{P})$. The number of vertices and extremal rays of a polyhedron (the 0-faces) is written as $f_0(\mathcal{P})$. Furthermore, since there is a one to one correspondance between the non-redundant constraints in the implicit description of a polyhedron and the facets of that polyhedron (theorem 5.1), the number of non-redundant constraints which is the number of facets can be written as $f_{d-1}(\mathcal{P})$. It has been shown that for a $d$-polyhedron $\mathcal{P}$ that both $L(\mathcal{P})$ and the number of

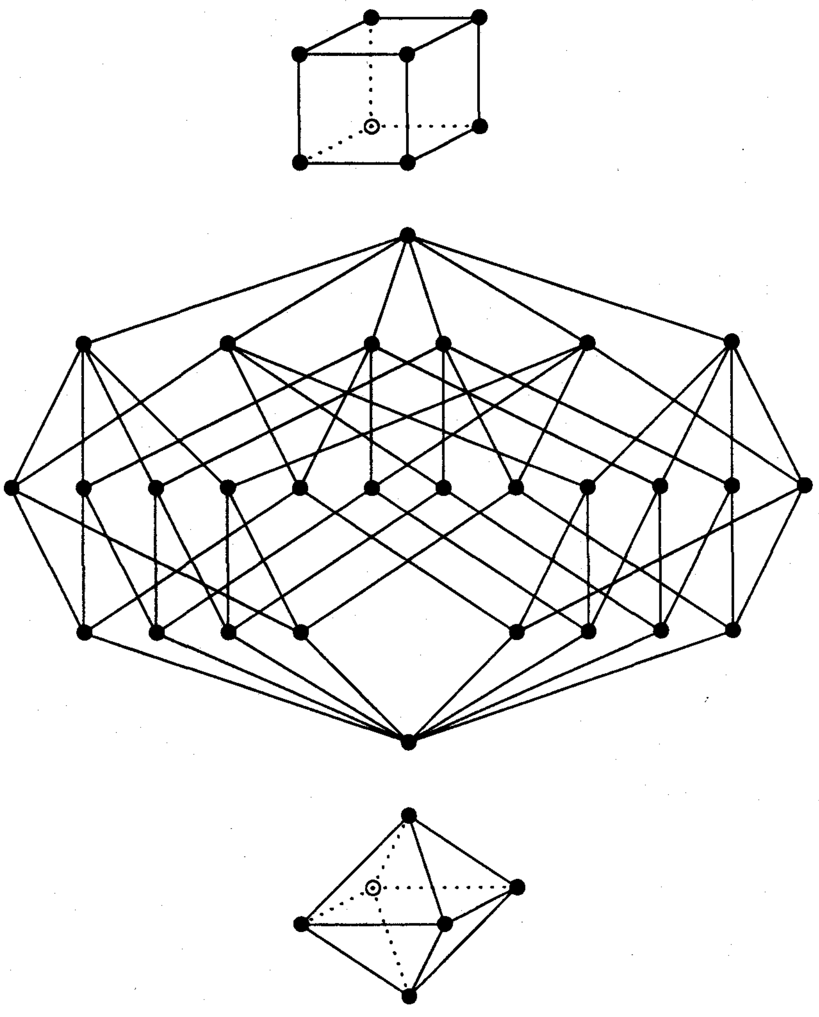


Figure 5.2: Face Lattice of Dual Polyhedra

vertices and rays $f_0(\mathcal{P})$ are $\mathcal{O}(k^{\lfloor \frac{d}{2} \rfloor})$ where $k = f_{d-1}(\mathcal{P})$, the number of constraints [Ede87].

### 5.2.1 Lattices of dual polyhedra

The definition of dual polyhedra (definition 2.29) stated that two polyhedra are dual to each other when there is a 1-1 mapping from faces of one to the faces of the other which is inclusion reversing. Let $M$ be such a mapping between polyhedra $\mathcal{P}$

and $\mathcal{Q}$. then

(i) for each face $f$ in $\mathcal{P}$, $M(f)$ is a face of $\mathcal{Q}$.

(ii) for each incidence $f \vdash g$ in $\mathcal{P}$, $M(g) \vdash M(f)$ is an incidence in $\mathcal{Q}$.

This implies that the face lattice of $\mathcal{P}$ and $\mathcal{Q}$ are exact inversions of each other. Figure 5.2 shows two such polyhedra. The face graph interpreted from top to bottom represents the face lattice of the polyhedron on the top. The face graph interpreted from bottom to top represents the polyhedron on the bottom. There is a 1-1 correspondance between facets of one polyhedron and vertices of the other, and visa versa. The reason that duality is important is that (for instance) every thing proven for facets, by duality, is proven for rays. Duality can be looked at from the point of view of two dual polyhedra, or from the point of view of the dual representation (constraint representation vs. ray representation) of a single polyhedron. The two points of view were proved equivalent in section 3.3. Here, we make the most advantage of the second point of view.

## 5.3 Previous art

Grunbaum stated the basic theorem which, given a set of vertices, generates all faces of the minimal polytope containing those vertices. He did this in an iterative fashion, adding one vertex at a time to an existing polyhedron $\mathcal{P}$, and computing the faces of the new polyhedron $\mathcal{P}*$ given the new point and the faces of the old polyhedron $\mathcal{P}$. This generates the faces of the facial graph but not the arcs (incidences) between the faces.

Before giving the theorem, two definitions are needed. Letting $P$ be a $n$-polytope, $H$ be a hyperplane such that $H$ does not cut $P$, and $V$ be a point, then the following definitions are given:

**Definition 5.1** *$V$ is* beneath *$H$ (with respect to $P$) provided $V$ belongs to the open halfspace determined by $H$ which contains* internal *$P$. [inside]*

**Definition 5.2** *$V$ is* beyond *$H$ (with respect to $P$) provided $V$ belongs to the open halfspace determined by $H$ which does not meet $P$. [outside]*

**Definition 5.3** *$V$ is* on *$H$ provided $V$ belongs to the hyperplane $H$. [on]*

The relation between the set of faces of a polytope $P$ and that of the convex hull of $P$ and one additional point $V$ is given by the following theorem:

**Theorem 5.2** *(Theorem by Grunbaum)*

*Let $P$ and $P*$ be two $n$-polytopes in $Q^n$, and let $V$ be a vertex of $P*$ but not of $P$, such that $P* = \text{convex.hull}(V \cup P)$. Then,*

*(i) a face $F$ of $P$ is also a face of $P*$ iff there exists a facet $F'$ of $P$ such that $F$ in $F'$ and $V$ is beneath $F'$;*

*(ii) if $F$ is a face of $P$ then $F* = \text{conv}(V \cup F)$ is a face of $P*$ iff either*

*(a) $V$ is in affine.hull $F$, or*

*(b) among the facets of $P$ containing $F$, there is at least one such that $V$ is beneath it and at least one such that $V$ is beyond it.*

*(iii) each face of $P^*$ is generated by either rule (i) or (ii) above.*

### 5.3.1 Seidel's method

Seidel added the generation of incidences to the procedure of Grunbaum, and thus was able to generate the full face lattice (faces and incidences) of a polytope surrounding a given set of points. Like the Grunbaum procedure, the Seidel procedure adds one point $p$ at a time to $P$ to get $P'$, iteratively building up the lattice. The procedure **AddPoint** which updates a list of faces and incidences, given a new point, is presented on the next page.

When adding a new point to an existing polytope, the algorithm differentiates two cases:

(1) the point is not in the affine hull of $P$, and

(2) the point is in the affine hull of $P$.

In the first case, the polytope will grow a dimension. In the second case, the dimension of the polytope stays the same.

```
Procedure **AddPoint**($P$, $p$), Returns $P' =$ convex.hull($P \cup p$)

If $p$ not in affine.hull($P$) Then
    ($P'$ is created in which dim($P'$) = dim($P$)+1).
    For every face $f$ of $P$ Do
        (Find faces of $P'$ when $p$ is not in affine.hull($P$))
        $f' = f$ is a face of $P'$
        $f'' =$ convex.hull($f \cup \{p\}$) is a face of $P'$
    For pairs of faces $f$ and $g$ of $P$ Do
        (Find incidences of $P'$ when $p$ is not in affine.hull($P$))
        Let $f'$,$f''$,$g'$,$g''$ be faces of $P'$ induced by faces $f$ and $g$ in $P$
        $f' \vdash g'$ in $P'$ iff $f \vdash g$ in $P$.
        $f'' \vdash g''$ in $P'$ iff $f \vdash g$ in $P$.
        $f' \vdash f''$ in $P'$ iff $f$ in $P$.
Else ($p$ is in affine.hull($P$))
    ($P'$ is created in which dim($P'$) = dim($P$)).
    **Classification of facets of $P$**
        For each facet $f$ of $P$ Do
            Let hyperplane $h$ = affine.hull($f$).
            $f$ is [out] if $p$ is beyond $h$.
            $f$ is [on] if $p$ is contained in $h$.
            $f$ is [in] if $p$ is beneath $h$.
        End
    **Classification of other faces of $P$**
        For each $k$-face $e$ of $P$ which is not a facet ($k <$dim($P$) $- 1$) Do
            $e$ is [out,on] if $e$ is bounded by [out] and [on] faces.
            $e$ is [in,on] if $e$ is bounded by [in] and [on] faces.
            $e$ is [in,out] if $e$ is bounded by [in] and [out] faces.
            $e$ is [in,on,out] if $e$ is bounded by [in], [on], and [out] faces.
        End
    For every face $f$ of $P$ Do
        (Find faces of $P'$ when $p$ is in affine.hull($P$))
        $f' = f$ is a face of $P'$ if $f$ has a [in] component.
        $f''$ =convex.hull($f \cup \{p\}$) is a face of $P'$ if $f$ has [in] and [out] components.
        $f'''$ =convex.hull($f \cup \{p\}$) is a face of $P'$ if $f$ is [on] or if $f = P$
    For all pairs of faces $f$ and $g$ of $P$ Do
        (Find incidences of $P'$ when $p$ is in affine.hull($P$))
        Let $f'$, $f''$, $f'''$, $g'$, $g''$, $g'''$ be faces induced in $P'$ by faces $f$ and $g$ in $P$
        $f' \vdash g'$ iff $f \vdash g$
        $f' \vdash g''$ iff $f = g$
        $f' \vdash g'''$ iff $f \vdash g$
        $f'' \vdash g''$ iff $f \vdash g$
        $f'' \vdash g'''$ iff there exists subface $x$ of $g$ where $f \vdash x \vdash g$
        $f''' \vdash g'''$ iff $f \vdash g$
    End
End
```

## 5.4 The Inductive Face Lattice Algorithm

A modification of Motzkin's **Dual** algorithm (presented in section 4.2.1) can be used to produce the entire face lattice of a polyhedron using an inductive constructive method.

The algorithm presented here constructs the face lattice of the ray space partition of a polyhedron from a list of constraints. A polyhedron $P$ is stored as a homogenous cone represented by the union of the lineality space $L$ and the ray space $R$ with an added data structure to represent the lattice. The lattice is represented by a hierarchy of faces with incident faces connected with pointers. The top of the face lattice is the whole polyhedron and at the bottom are the extreme rays of $R$. Each face in the lattice has a dimension corresponding to its level in the lattice. Thus, a lattice for a $d$-dimensional ray space would have $d+1$ levels, the levels having dimensions $d, d-1, \cdots, 0$, from top to bottom. The computation of the lattice is not directly a function of the lineality space. Thus the Motzkin procedure to constrain the lineality space (i.e. to compute **ConstrainL** ) can remain the same. The differences in the method presented here over previous methods are found in the computation on the ray space $R$ by procedures **AugmentR** and **ConstrainR**.

### 5.4.1 Data structure for face lattice

A $k$-face is composed of a set of $(k-1)$-facets (subfaces of dimension $k-1$) as shown in figure 5.4.

$$F^k = \{f_0^{k-1}, f_1^{k-1}, \cdots, f_n^{k-1}\} \ .$$

A facial graph consists of nodes representing faces and edges representing incidences from a face to its facets. Each face is a node in the face lattice and is represented by a data structure with the following fields

**dimension** The dimension of this face.

**flags** A set of attributes of this face in the context of the current constraint. Attributes defined are:

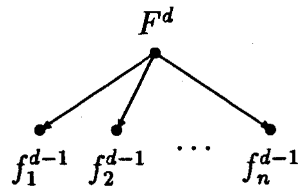


Figure 5.4: Facial Graphs for a Face and its Facets

*in*: set when this face verifies the constraint,

*on*: set when this face saturates the constraint,

*out*: set when this face does not verify the constraint. A face can have any non-empty combination of these attributes in its flag set.

**facets** The set of facets (of dimension one less than the dimension of this face) that are incident to this face. This field is the head of a linked list of pointers to subface nodes in the face lattice (figure 5.4).

**ray** (Only used for dimension 0 faces). The extreme ray in $R$ corresponding to this face.

Using a standard "dot notation", these fields will be referred to as $F$.dim, $F$.flags, $F$.facets, $F$.ray in the remainder of this chapter.

### 5.4.2 Modifications to the Dual procedure

The main procedure for converting constraints to lines, vertices and rays, is the same as the procedure **Dual** given in section 4.2.1. However, the subprocedures **AugmentR** and **ConstrainR** which determine the way ray space $R$ is constrained and augmented, have been rewritten in order to generate the lattice. The procedure to compute **ConstrainL** is not changed and is given in section 4.2.1. The procedures **AugmentR** and **ConstrainF** which recursively construct the lattice are the primary contributions of this chapter. The procedure **AugmentR** adds a new basis ray $r_{new}$ which is not in the affine hull of $R$ to the lattice— increasing the

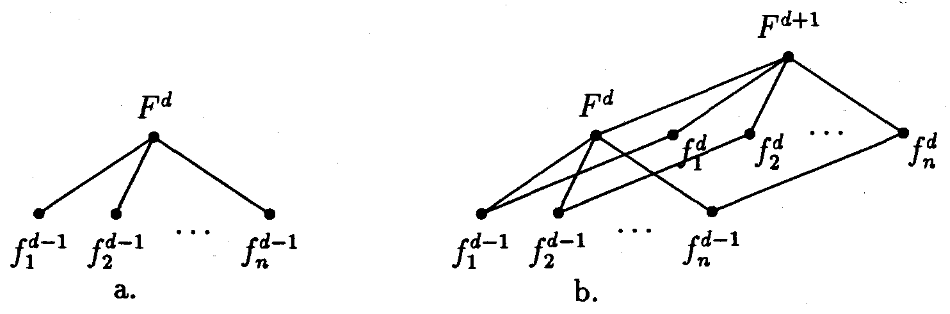


Figure 5.5: Facial Graphs for **AugmentR** Algorithm

dimension of the lattice by one. The procedure **ConstrainF** (which is called by **ConstrainR**) recursively constrains the lattice by slicing it with a new constraint. The resulting lattice is of the same dimension, however parts of the lattice outside the new constraint are removed and replaced with new faces created by the cut.

### 5.4.3 AugmentR

The procedure **AugmentR** adds a ray $r_{new}$ which is not in the affine hull of $R$ to the lattice $F^d$ representing a face of dimension $d$, and returns an augmented face $F^{d+1}$ of dimension $d+1$. It is computed recursively as follows:

Given $F^d = \{\ f_1^{d-1},\ f_2^{d-1},\ \cdots,\ f_n^{d-1}\}$

then $F^{d+1} = \text{convex.hull}(F^d \cup \{r_{new}\}) = \{F^d,\ f_1^d,\ f_2^d,\ \cdots,\ f_n^d\}$

where $f_1^d = \text{convex.hull}(f_1^{d-1} \cup r_{new})$

$f_2^d = \text{convex.hull}(f_1^{d-1} \cup r_{new})$

$\vdots$

$f_n^d = \text{convex.hull}(f_1^{d-1} \cup r_{new})$.

Figure 5.5a. shows a piece of the Hasse diagram of the lattice $F^d$. Figure 5.5b. shows the same piece of the diagram after it has been augmented by adding a new basis ray using this procedure. A new face $F^{d+1}$ is created and assigned the following subfaces: (1) $F^d$ (the original face) and (2) the new faces $f_1^d, f_2^d, \cdots, f_n^d$ which are computed by recursively calling this procedure on $f_1^{d-1},\ f_2^{d-1}, \cdots, f_n^{d-1}$, respectively ( the facets of the original face $F^d$). This procedure is given below.

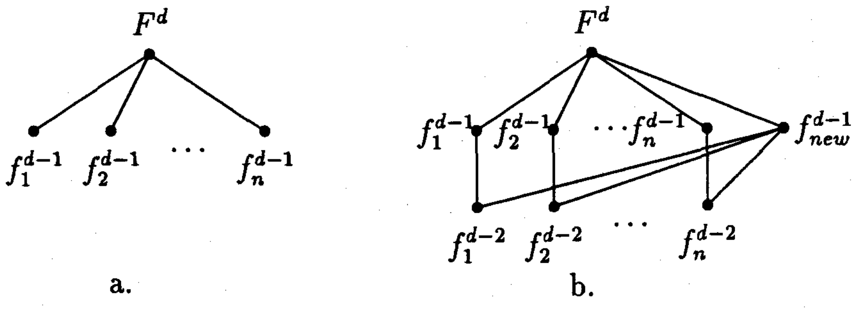
```
**AugmentR**($F$, $a$, $r_{new}$), Modifies $F$, returns $F_{new}$ =convex.hull($F \cup r_{new}$)

If $F_{new}$ =**AugmentR**($F$, $a$, $r_{new}$) has already been computed
  Then return $F_{new}$
$F_{new}$ := null
If $F_{.\mathrm{dim}} == 0$ Then (face $F$ is a ray)
    If $a^T F_{.\mathrm{ray}} \neq 0$ Then (constraint not saturated)
        Allocate a new face $r_1$
        If $a^T F_{.\mathrm{ray}} > 0$ $r_{1.\mathrm{ray}}$ =**Combine**($F_{.\mathrm{ray}}, -r_{new.\mathrm{ray}}, a$)
        Else ($a^T F_{.\mathrm{ray}} < 0$) $r_{1.\mathrm{ray}}$ =**Combine**($F_{.\mathrm{ray}}, r_{new.\mathrm{ray}}, a$)
        If constraint $a$ is an inequality Then Create $F_{new}$ with subfaces $\{F, r_1\}$
Else ( $F_{.\mathrm{dim}} > 0$ )
    For each facet $f$ of face $F$ Do
        $g$=**AugmentR**($f$, $a$, $r_{new}$)     (recursive call)
        If $g$ is not null
            If $F_{new}$ is null
                Create new face $F_{new}$ (with no facets)
                Add $F$ as a facet to augmented face $F_{new}$
            Add face $g$ as a facet to augmented face $F_{new}$
    End
Return augmented face $F_{new}$
```

Figure 5.6: Procedure to compute **AugmentR**($F$, $a$, $r_{new}$)

### 5.4.4 ConstrainR

A recursive procedure **Evaluate** is first called which evaluates constraint $a$ on each of the faces of polyhedron $R$ and marks status flags in each face, saying whether each face verifies, saturates, or does not verify the constraint, or a combination of the three for non trivial faces. After evaluating the faces in light of the constraint, **ConstrainR** either returns an empty lattice if none of the polyhedron verifies or saturates the constraint, or returns the saturating faces if only a part of the polyhedron saturates the constraint, or returns the constrained lattice if the polyhedron partially verifies and partially does not verify the constraint. The last two cases are done by calling the **SelectF** and **ConstrainF** procedures respectively.



$F^d$
$f_1^{d-1}$ $f_2^{d-1}$ $\cdots$ $f_n^{d-1}$
a.
$F^d$
$f_1^{d-1}$ $f_2^{d-1}$ $\cdots f_n^{d-1}$ $f_{new}^{d-1}$
$f_1^{d-2}$ $f_2^{d-2}$ $\cdots$ $f_n^{d-2}$
b.

Figure 5.7: Facial Graphs for **ConstrainF** Algorithm

## Evaluate

The procedure to evalute a constraint $a$ on a face lattice $F$ is also a recursive procedure, setting the flags of a face $F$ as follows:

$$F_{.\mathbf{flags}} = \begin{cases} \bigcup_i^n (f_i)_{.\mathbf{flags}} & F_{.\mathbf{dim}} > 0,\ F = \{f_1, f_2, \cdots, f_n\} \\ \{\text{in}\} & F_{.\mathbf{dim}} = 0 \text{ and } a^T F_{.\mathbf{ray}} > 0 \\ \{\text{on}\} & F_{.\mathbf{dim}} = 0 \text{ and } a^T F_{.\mathbf{ray}} = 0 \\ \{\text{out}\} & F_{.\mathbf{dim}} = 0 \text{ and } a^T F_{.\mathbf{ray}} < 0 \end{cases}$$

The procedure performs a depth first traversal of the face lattice. Details of this procedure are presented at the end of this section.

## SelectF

The procedure **SelectF** is simply a breadth first search of the face lattice looking for the highest dimensioned face in the lattice which entirely saturates the constraint $a$, or in other words, the highest dimensioned face with $F_{.\mathbf{flags}}$ ={on}. This procedure is straightforward and its details are not presented here.

## ConstrainF

The procedure **ConstrainF** constrains the lattice $F$ by cutting it with the constraint $a$ represented by the halfspace $H^+ = \{x \mid a^T x \geq 0\}$. The the resulting lattice will be of the same dimension, however the parts of the lattice outside of the slicing hyperplane will be removed and a new face created by the cut plane will be added. The procedure builds the new face $f_{new}$ which consists of the new faces of dimension $(d-2)$ created by cutting each of the original facets of $F^d$: $f_1^{d-1},\ f_2^{d-1}, \cdots, f_n^{d-1}$ and which lies on the cutting hyperplane. The procedure then links $f_{new}$ as a new subface to $F$, and then returns the new face $f_{new}$ to the caller. When procedure **ConstrainF** is called, face $F^d$ is modified to be $F^d \cap H^+$, which is computed as follows:

Given $F^d = \{f_1^{d-1}, f_2^{d-1}, \cdots, f_n^{d-1}\}$

$H^+ = \{x \mid a^T x \geq 0\}, \quad H^0 = \{x \mid a^T x = 0\}$

then $F^d \cap H^+ = \{f_1^{d-1} \cap H^+, f_2^{d-1} \cap H^+, \cdots, f_n^{d-1} \cap H^+, f_{new}^{d-1}\}$

where $f_{new}^{d-1} = F^d \cap H^0 = \{f_1^{d-2},\ f_2^{d-2}, \cdots, f_n^{d-2}\}$

$f_1^{d-2} = f_1^{d-1} \cap H^0$

$f_2^{d-2} = f_2^{d-1} \cap H^0$

$\vdots$

$f_n^{d-2} = f_n^{d-1} \cap H^0$

Figure 5.7a. shows a piece of the Hasse diagram of the lattice $F^d$. Figure 5.7b. shows the same piece of the diagram after it has been constrained by this procedure. This procedure is presented in detail below.

```
ConstrainR($R$,$a$) Modifies $R$

Call Evaluate($R$, $a$)
If $R_{.flags}$ = { in, on } or { in } or { on } then no change.
Else if $R_{.flags}$ = { out } then set $R$ := null ray space.
Else if $R_{.flags}$ = { on, out } then set $R$ := SelectF($R$, $a$)
Else ($R_{.flags}$ = { in, out } or { in, out, on }) set $R$ := ConstrainF($R$, $a$)
```

Figure 5.8: Procedure for computing **ConstrainR**($R$,$a$)

```
**Evaluate**$(F, a)$ Modifies the flags in $F$

If $F_{.\text{dim}} = 0$ then
    $x := a^T F_{.\text{ray}}$
    If $x = 0$ then $F_{.\textbf{flags}} := \{\text{on}\}$.
    Else if $x > 0$ then $F_{.\textbf{flags}} := \{\text{in}\}$.
    Else $(x < 0)$ $F_{.\textbf{flags}} := \{\text{out}\}$.
Else $(F_{.\text{dim}} > 0)$
    $F_{.\textbf{flags}} := \{\}$
    For each subface $f$ of $F$, do
        Call Evaluate$(f, a)$; (recursive call)
        $F_{.\textbf{flags}} := F_{.\textbf{flags}} \cup f_{.\textbf{flags}}$
    End
```

Figure 5.9: Procedure for computing **Evaluate**$(F, a)$

```
**ConstrainF**$(F,\ a)$ Modifies $F$, returns facet $f_{new} = F \bigcap H^0$
Invariant: $F_{.\mathbf{flags}}$ is either {in,out}, {in,on,out}, or {in,on}.

If $f_{new}$ =**ConstrainF**$(F,a)$ has already been computed Then return $f_{new}$
If $F_{.\mathbf{dim}} = 1$ Then ($F$ is an edge with endpoints $f_1$ and $f_2$)
    Let $F ==$ subfaces$\{f_1, f_2\}$ s.t. $f_{1.\mathbf{flags}}$ ={in} and $f_{2.\mathbf{flags}}$ ={out} or {on}
    If $F_{.\mathbf{flags}}$ = {in,on} Then Return $f_2$
    Else ($F_{.\mathbf{flags}}$ = {in,out})
        Allocate a new face $f_{new}$ (of dimension 0)
        $f_{new.\mathbf{ray}}$ =**Combine**$(a, f_{1.\mathbf{ray}}, f_{2.\mathbf{ray}})$
        Remove $f_2$ from subfaces of $F$
        Add $f_{new}$ to subfaces of $F$
Else ($F_{.\mathbf{dim}} > 1$)
    If $F_{.\mathbf{flags}}$ ={in,out}, or {in,on,out} Then
        Allocate a new face $f_{new}$
        For each facet $f$ of face $F$ Do
            If $f_{.\mathbf{flags}}$={in} Then (do nothing- $f$ continues to be a subface)
            Else if $f_{.\mathbf{flags}}$={out} or {on,out} Then Remove $f$ from subfaces of $F$
            Else ( $f_{.\mathbf{flags}}$ ={in,on},{in,out},or {in,on,out})
                $g$ = **ConstrainF**$(f,\ a)$ (recursive call)
                If $g \neq$ null Then Add $g$ to subfaces of $f_{new}$
        End
        Add $f_{new}$ to subfaces of $F$
    Else ( $F_{.\mathbf{flags}}$ ={in,on})
        For each facet $f$ of face $F$ Do
            If $f_{.\mathbf{flags}}$ ={in} or {in,on} Then (do nothing- $f$ continues to be a subface)
            Else if $f_{.\mathbf{flags}}$ ={on} Then $f_{new} = f$ ($f$ continues to be a subface)
        End
Return $f_{new}$
```

Figure 5.10: Procedure for computing **ConstrainF**$(F,\ a)$

## 5.5 Example

This algorithm has been implemented in C and an example is shown here. The lattice shown in figure 5.2 was generated from the system of constraints $\{\ i, j, k \mid 0 \leq i \leq 1;\ 0 \leq j \leq 1;\ 0 \leq k \leq 1\ \}$. The following is a printed representation of the data structure which was created by the algorithm.

```
face(3) ed30
|
+-- face(2) ebb0
|   |
|   +-- face(1) eb10
|   |   |
|   |   +-- face(0) ca90 = point[0] [0,0,0]
|   |   |
|   |   +-- face(0) eeb0 = point[4] [1,0,0]
|   |
|   +-- face(1) eb70
|   |   |
|   |   +-- face(0) ca90 = point[0] [0,0,0] (LINK)
|   |   |
|   |   +-- face(0) ebf0 = point[1] [0,1,0]
|   |
|   +-- face(1) ee90
|   |   |
|   |   +-- face(0) eeb0 = point[4] [1,0,0] (LINK)
|   |   |
|   |   +-- face(0) ee10 = point[5] [1,1,0]
|   |
|   +-- face(1) ed00
|       |
|       +-- face(0) ebf0 = point[1] [0,1,0] (LINK)
|       |
|       +-- face(0) ee10 = point[5] [1,1,0] (LINK)
|
+-- face(2) eca0
|   |
|   +-- face(1) eb10 (LINK)
|   |
|   +-- face(1) ec60
|   |   |
|   |   +-- face(0) ca90 = point[0] [0,0,0] (LINK)
|   |   |
|   |   +-- face(0) f010 = point[2] [0,0,1]
|   |
|   +-- face(1) ef00
|   |   |
|   |   +-- face(0) eeb0 = point[4] [1,0,0] (LINK)
|   |   |
|   |   +-- face(0) f040 = point[6] [1,0,1]
|   |
|   +-- face(1) eff0
|       |
|       +-- face(0) f010 = point[2] [0,0,1] (LINK)
|       |
|       +-- face(0) f040 = point[6] [1,0,1] (LINK)
```

```
|
+-- face(2) ed70
|   |
|   +-- face(1) eb70 (LINK)
|   |
|   +-- face(1) ec60 (LINK)
|   |
|   +-- face(1) ef60
|   |   |
|   |   +-- face(0) ebf0 = point[1] [0,1,0] (LINK)
|   |   |
|   |   +-- face(0) f0c0 = point[7] [0,1,1]
|   |
|   +-- face(1) f090
|       |
|       +-- face(0) f010 = point[2] [0,0,1] (LINK)
|       |
|       +-- face(0) f0c0 = point[7] [0,1,1] (LINK)
|
+-- face(2) ee70
|   |
|   +-- face(1) ee90 (LINK)
|   |
|   +-- face(1) ef00 (LINK)
|   |
|   +-- face(1) efb0
|   |   |
|   |   +-- face(0) ee10 = point[5] [1,1,0] (LINK)
|   |   |
|   |   +-- face(0) f140 = point[8] [1,1,1]
|   |
|   +-- face(1) f110
|       |
|       +-- face(0) f040 = point[6] [1,0,1] (LINK)
|       |
|       +-- face(0) f140 = point[8] [1,1,1] (LINK)
|
+-- face(2)ece0
|   |
|   +-- face(1) ed00 (LINK)
|   |
|   +-- face(1) ef60 (LINK)
|   |
|   +-- face(1) efb0 (LINK)
|   |
|   +-- face(1) f180
|       |
|       +-- face(0) f0c0 = point[7] [0,1,1] (LINK)
|       |
|       +-- face(0) f140 = point[8] [1,1,1] (LINK)
|
+-- face(2) edc0
    |
    +-- face(1) eff0 (LINK)
    |
    +-- face(1) f090 (LINK)
    |
    +-- face(1) f110 (LINK)
    |
    +-- face(1) f180 (LINK)
```

## 5.6 Summary

The inductive method for constructing the face lattice which has been presented in this chapter differs from the Seidel method in its inductive approach but shares the same order of execution time. The inductive approach is a natural consequence of the recursive structure of the face lattice and is an extension of the Motzkin algorithm to compute the dual of a polyhedron. The procedure starts with a mixed system of constraints and produces an interlinked data structure representing the lattice.

Execution time for this algorithm is lower bounded by the size of the output, which is $\mathcal{O}(k^{\lfloor \frac{d}{2} \rfloor})$ where $k$ is the number of inequalities and $d$ is the dimension.

# Chapter 6

# CONCLUSION

The polyhedral library described in this thesis implements basic geometrical operations on stuctures called domains which consist of finite unions of convex polyhedra. The domains are described in a dual representation consisting of:

1. a set of constraints— inequalities and equalities, and

2. a set of geometric features— lines, rays, and vertices.

It is convenient to represent polyhedra in their pointed cone form, which results from the transformation $x \rightarrow \begin{pmatrix} \xi x \\ \xi \end{pmatrix}$. This transformation maps inhomogeneous constraints to homogeneous constraints and maps both the vertices and rays of a polyhedron to rays in a pointed cone. The pointed cone is thus stored as a list of rays, and the lineality space as a list of basis lines.

Domains can be created starting with a list of constraints or a list of geometric features. Given one representation, the other is computed by the procedure **Dual** which has been described in section 4.2.1. The library is written in the C–language and may be linked in with an application to provide capability for doing geometric operations, such as union, intersection, difference, and simplification, on domains.

The library was originally written to support the ALPHA environment in which all variables are based on domains. Subsequently, the library was placed in the public domain (`ftp.irisa.fr`) and has been used by several other research groups. For example, one group which is doing parallel code generation, uses it to simplify loop bounds and other expressions involving loop variables [CFR93]. Other groups have also found it useful in removing redundant inequalities from a system of constraints. The algorithms in the library work very well for problems with relatively small dimension and a small number of constraints. Such problems occur in the generation of loop bound expressions.

In the ALPHA environment, the library is used extensively in carrying out program transformations and in doing static analysis of ALPHA programs. To support ALPHA development, the library has been interfaced to the MATHEMATICA symbolic mathematics package.

The library is easy to interface to and use. Appendix B contains an example C–program which demonstrates the use of the library. This sample program was written to show how a C–program may be interfaced with the library.

The library code is composed of five files: three headers and two code files as follows:

| | |
|---|---|
| `types.h` | The header file defining the data structures used in the library. |
| `vector.h` | The header file containing the forward definitions of the external vector operation procedures. |
| `vector.c` | The code file containing the code for vector operations. |
| `polyhedron.h` | The header file containing the forward definitions of the external vector polyhedral and domain procedures. |
| `polyhedron.c` | The code file containing the code for polyhedral and domain operations. |

The library is made available by ftp from `ftp.irisa.fr`.

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## APPENDICES

# Appendix A

# SYSTEMS OF AFFINE RECURRENCE EQUATIONS

The definitions in this appendix review the basic concepts of systems of affine recurrence equations and are taken primarily from the works of [RF90], Yaacoby and Cappello [YC89], Delosme and Ipsen [DI86], Quinton and Van Dongen [QV89].

**Definition A.1** (Recurrence Equation)
A **Recurrence Equation** over a domain $\mathcal{D}$ is defined to be an equation of the form

$$f(z) = g(\ f_1(I_1(z)), f_2(I_2(z)), \cdots, f_k(I_k(z))\ )$$

where

- $f(z)$ is a variable indexed by $z$ and the *left hand side* of the equation.
- $z \in \mathcal{D}$
- $\mathcal{D}$ is the (possibly parameterized) domain of variable $f$.
- $I_i$ are index mapping functions which map $z \in \mathcal{D}$ to $I_i(z) \in \mathcal{D}_i$.
- $\mathcal{D}_1, \cdots \mathcal{D}_k$ are the (possibly parameterized) domains of variables $f_1, \cdots, f_k$, respectively.
- $g$ is a strict single-valued function whose complexity is $\mathcal{O}(1)$ defining the *right hand side* of the equation.

A variation of an equation allows $f$ to be defined in a finite number of disjoint "cases" consisting of convex subdomains each having its own left hand side as follows:

$$f(z) = \begin{cases} z \in D_1 & \Rightarrow \quad g_1(\ldots f_1(I_1(z))\cdots) \\ z \in D_2 & \Rightarrow \quad g_2(\ldots f_2(I_2(z))\cdots) \\ \quad\vdots & \end{cases} \tag{A.12}$$

where the domain of variable $f$ is $\mathcal{D} = \cup_i D_i$ and $(i \neq j) \rightarrow (D_i \cap D_j = \{\})$

**Definition A.2** (Affine Recurrence Equation)
A recurrence equation of the form defined above is called a **Uniform Recurrence Equation** (URE) if all of the index mapping functions $I_i$ are of the form $I(z) = z{+}\mathrm{b}$, where b is a (possibly parameterized) constant n-dimensional vector. It is called an **Affine Recurrence Equation** (ARE) if $I(z) = \mathrm{A}z + \mathrm{b}$, where A is a constant matrix, and b is a (possibly parameterized) constant $n$-vector.

**Definition A.3** (System of Affine Recurrence Equations, or SARE)
A **system** of recurrence equations is a set of $m$ such equations, defining the functions $f_1 \ldots f_m$ over domains $\mathcal{D}_1 \cdots \mathcal{D}_m$ respectively. The equations may be mutually recursive, self recursive, or combinations of the two. Variables are designated as either input, output, or local variables of the system. Each variable (which is not a system input) appears on the left hand side of an equation once and only once. Variables may appear on the right hand sides of equations as often as needed.

Since there is a one to one correspondance between (non-input) variables and equations, the two terms are often used interchangeably.

Such equations serve as a purely functional definition of a computation, and are usually in the form of a **static program**— a program whose dependency graph can be determined and analyzed statically (for any given instance of the parameters). Static programs require that all $g_i$ be strict functions and that any conditional expressions be limited to linear inequalities involving the indices of the left hand side variable. By convention, it is assumed that, boundary values (or input values) are all specified whenever needed for any function evaluation.

**Definition A.4** *(Dependency)*

*For a system of recurrences, we say that a variable $f_i$ at a point $p \in D_i$ (directly)* **depends on** *variable $f_j$ at $q$, (denoted by $p_i \mapsto q_j$), whenever $f_j(q)$ occurs on the right hand side of the equation defining $f_i(p)$. The transitive closure of this is called the* **dependency relation**, *denoted by $p_i \rightarrow q_j$.*

**Definition A.5** (Schedule)

For any system of recurrence equations, $f_1 \ldots f_m$, defined over (possibly parameterized) domains $\mathcal{D}_1 \cdots \mathcal{D}_m$, a **schedule** is a set of $m$ (non-negative) integer valued functions $t_i : \mathcal{Z}^n \mapsto \mathcal{N}$, which satisfy the condition that $t_i(p) > t_j(q) \geq 0$, whenever $p_i \rightarrow q_j$.

**Definition A.6** (Affine Schedule)

An **affine schedule** is of the form $t(p) = \pi p + \alpha$, where $\pi$ is a $n$-vector and $\alpha$ is a scalar.

$t_i(p)$ may be interpreted as the time instant at which $f_i(p)$ is computed, under the assumption that each of the functions $g_i$ take a unit time to compute.

# Appendix B

# EXAMPLE C–PROGRAM

The following is an example program written to demonstrate how the library is called from a C-program. The first section has the code itself, the second section has the input for the program, and the last section has the output.

## B.1 Program Code

```
/* main.c
  This file along with polyhedron.c and vector.c do the following functions:
   – Extraction of a minimal set of constraints from some set of constraints
   – Intersection of two convexes
   – Application of a linear function to some convex
   – Verification that a convex is included in some other convex */

#include "types.h"
#include "vector.h"
#include "polyhedron.h"

int main()
{  Matrix *a, *b, *t;
   Polyhedron *A, *B, *C, *D;

   /* read in a matrix containing your equations */
   /* for example, run this program and type in these five lines:
       4 4
       1 0 1 −1
       1 −1 0 6
       1 0 −1 7
       1 1 0 −2
      This is a matrix for the following inequalities
       1 = inequality,  0x +  1y −1 ≥0  −−>   y ≥ 1
       1 = inequality, −1x +  0y +6 ≥0  −−>   x ≤ 6
       1 = inequality,  0x + −1y +7 ≥0  −−>   y ≤ 7
       1 = inequality,  1x +  0y −2 ≥0  −−>   x ≥ 2
      If the first number is a 0 instead of a 1, then that constraint
   is an 'equality' instead of an 'inequality'.
   */
```

```
a = Matrix_Read();

/* read in a second matrix containing a second set of constraints:
   for example :
    4 4
    1 1 0 -1
    1 -1 0 3
    1 0 -1 5
    1 0 1 -2
*/
b = Matrix_Read();

/* Convert the constraints to a Polyhedron.
This operation 1. Computes the dual ray/vertex form of the
system, and 2. Eliminates redundant constraints and reduces
them to a minimal form.  */
A = Constraints2Polyhedron(a, 200);
B = Constraints2Polyhedron(b, 200);

/* the 200 is the size of the working space (in terms of number
of rays) that is allocated temporarily
-- you can enlarge or reduce it as needed */

/* There is likewise a rays to polyhedron procedure */

/* Since you are done with the matrices a and b, be a good citizen
and clean up your garbage */
Matrix_Free(a);
Matrix_Free(b);

/* If you want the the reduced set of equations back, you can
get the matrix back in the same format it started in... */
a = Polyhedron2Constraints(A);
b = Polyhedron2Constraints(B);

/* Take a look at them if you want */
printf("\na␣=");  Matrix_Print("%4d", a);
printf("\nb␣=");  Matrix_Print("%4d", b);

/* To intersect the two systems, use the polyhedron formats...
   Again, the 200 is the size of the working space.  */
C = DomainIntersection(A, B, 200);

/* This time, lets look a the polyhedron itself... */
printf("\nC␣=␣A␣and␣B␣=");    Polyhedron_Print("%4d", C);

/* The operations DomainUnion, DomainDifference, DomainConvex,
and DomainSimplify are also available */

/* read in a third matrix containing a transformation matrix,
        this one swaps the indices (x,y --> y,x):
         3 3
         0 1 0
         1 0 0
         0 0 1
    */
    t = Matrix_Read();

    /* Take the preimage (transform the equations) of the domain C to
    get D.  */
    D = Polyhedron_Preimage(C, t, 200);

    /* cleanup */
    Matrix_Free(t);

    printf("\nD = transformed C =");
    Polyhedron_Print("%4d", D);    Domain_Free(D);

    /* in a similar way, Polyhedron_Image(dom, mat, 200), takes the image
    of dom under matrix mat  (transforms the vertices/rays) */

    /* The function PolyhedronIncludes(Pol1, Pol2) returns 1 if Pol1
    includes (covers) Pol2, and a 0 otherwise */

    if (PolyhedronIncludes(A,C))
    printf("\nWe expected A to cover C since C = A intersect B\n");
    if (!PolyhedronIncludes(C,B))
    printf("and C does not cover B...\n");

    Domain_Free(A);
    Domain_Free(B);
    Domain_Free(C);

    return 0;
}
```

## B.2 Program Input

```
4 4
1 0 1 -1
1 -1 0 6
1 0 -1 7
1 1 0 -2
4 4
1 1 0 -1
1 -1 0 3
1 0 -1 5
1 0 1 -2
3 3
0 1 0
```

```
1 0 0
0 0 1
```

## B.3 Program Output

```
a =4 4
   1   0   1  -1
   1  -1   0   6
   1   0  -1   7
   1   1   0  -2

b =4 4
   1   1   0  -1
   1  -1   0   3
   1   0  -1   5
   1   0   1  -2

C = A and B =POLYHEDRON Dimension:2
           Constraints:4  Equations:0  Rays:4  Lines:0
Constraints 4 4
Inequality: [   1   0  -2 ]
Inequality: [  -1   0   3 ]
Inequality: [   0  -1   5 ]
Inequality: [   0   1  -2 ]
Rays 4 4
Vertex: [   3   5 ]/1
Vertex: [   2   5 ]/1
Vertex: [   2   2 ]/1
Vertex: [   3   2 ]/1

D = transformed C =POLYHEDRON Dimension:2
           Constraints:4  Equations:0  Rays:4  Lines:0
Constraints 4 4
Inequality: [   0   1  -2 ]
Inequality: [   0  -1   3 ]
Inequality: [  -1   0   5 ]
Inequality: [   1   0  -2 ]
Rays 4 4
Vertex: [   5   3 ]/1
Vertex: [   5   2 ]/1
Vertex: [   2   2 ]/1
Vertex: [   2   3 ]/1

We expected A to cover C since C = A intersect B
and C does not cover B...
```