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The question of the suitability of Walsh functions or similar bi-valued functions used as a radar electromagnetic carrier waveform is considered. Walsh functions are described and effective sinusoidal steady state transfer functions are defined for transmission elements, antenna systems, and propagation media.

The argument is made that the unit impulse response is suitable in describing the responses of radar elements and the propagation media to such bi-valued excitation waveforms. A measure of the mean, or average, duration of the unit impulse response is defined and used as a measure of the degradation of rise and fall times introduced by radar system elements and the radar environment. It is shown that the mean duration of the impulse response may be easily obtained from the Laplace or Fourier transfer function of an element without resorting to time-domain evaluation of that quantity.

From transfer functions defined for transmission elements, antennas, and propagation media, corresponding unit impulse responses are obtained for many typical elements useful in the implementation of a Walsh carrier radar.

Approximate limitations are found on the maximum suitable transmission distances that may be expected when using a transmission line to transmit Walsh waves as well as the distances over which a propagation medium will have negligible effect.

The conclusion obtained is that radar system elements and the radar propagation medium are compatible to the use of Walsh functions as the functional form of the electromagnetic carrier wave if sound judgement is used.

The Use of Electromagnetic Walsh Waves in Radar

by

Alvin Schaffer Jr.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

THE USE OF ELECTROMAGNETIC WALSH WAVES IN RADAR

Chapter I Introduction and Orientation

Since the end of World War II the uses of radar have grown to become important facets of the military and civil activities of most of the developed countries of the world. In addition to the well known military uses of radar, many of which are primarily improved extensions of its World War II uses, the modern applications of radar to the civil sector and to space exploration have grown at an enormous rate to become a significant portion of the total productive effort of the electronics industry in recent years.

The worldwide use of radar in air traffic control and air navigation, its maritime counterpart in sea navigation and safety, and its meteorological applications readily establish the great importance of radar to the safe and efficient transaction of both domestic and international trade and commerce.

The expected future growth of sea and air commerce, the continued exploration of space, and meteorological and environmental sensing indicate an associated growth in the use of radar. This growth along with the discoveries of new radar applications will surely lead, in time, to intense crowding of the available radar frequency spectrum while increasing air and sea traffic will likely create the need for improvements in radar performance and capabilities far exceeding those of the present radar arts and technology.

Partial solution of these future problems may exist in the recently suggested electromagnetic spectrum of Walsh waves as a replacement for, or as a supplement to, the existing sinusoidal electromagnetic spectrum. Preliminary theoretical studies by Harmuth (1970) and Pearlman (1970) indicate that radiating electromagnetic energy in the form of Walsh functions can be generated and also formed into the narrow directional beams required of radar operations. Many researchers also believe that improved radar performance will result with their use as the electromagnetic carrier waveform.

Although the family of bi-valued, orthogonal Walsh functions was discovered and first reported by J.L. Walsh (1923) as early as 1923, they received very little attention until the early 1960's. Since then considerable effort has been expended in applying them also to other areas not directly related to the problems of radar and radio communications.

It is surprising that the universal and almost traditional practice of utilizing high frequency sinusoidal electromagnetic carrier waves in radar and radio communications has not been seriously questioned. This fact is, no doubt, explained by the past successes of this technique and by the well established technology and the industrial base of resources developed and available over the years. The ease of generating, controlling, modulating, radiating, amplifying, and filtering sinusoidally based power and signals as well as the great body of knowledge and the development of powerful and sophisticated theories for the analysis and synthesis of radar and radio systems have undoubtedly helped channel the thinking of radar and radio engineers along this line of development.

It is recognized that sinusoidal electromagnetic power was not only highly compatible with, but was also necessary for use with linear, lumped, and distributed circuits and amplifiers of early radar and radio and their subsequent developments. However, recent developments of time varying linear semiconductor integrated circuits and the rising use of high speed digital computers[1] to process and interpret radar and radio signals makes it necessary to question this universal use of sinusoidal carrier waves in radar and radio.

Although the findings of Harmuth (1970) and Pearlman (1970) show that sinusoidal time variation is not a requisite for directional radiation of electromagnetic energy, the usefullness or superiority of a nonsinusoidal carrier suchas a Walsh function does not automatically follow! In order to partially resolve this uncertainty

1) It is a well known fact that the digital computer is not highly compatible with sinusoidally based data.

this study examines the foundations of radar theory in an attempt to establish the functional requirements that an electromagnetic carrier must meet. From that study the characteristics which a function must possess in order to satisfactorily function as a carrier are deduced. An examination of the abstract and practical natures of Walsh and similar type functions are then examined to see if any of them meet the requirements of an electromagnetic carrier.

In order to be truly useful as a radar electromagnetic carrier wave, a Walsh or a similar type function must possess characteristics compatible with natural propagation media, radiation and beam forming processes, and transmission over waveguiding structures. These topics are investigated in the later chapters of this thesis.

In order to limit the scope of the study to reasonable proportions it is restricted to the use of classical electromagnetic theory and to the non-relativistic case. In addition, attention to devices and hardware for radar implementation is minimized with consideration given only where there exists a question of feasible use with Walsh functions.

A knowledge of the rudiments and fundamentals of radar and communication theory on the part of the reader is also assumed to limit the recessity of excessive background material.

The thesis is also written in three basic parts. The first, Part I consisting of Chapters I through V, is primarily of a qualitative nature. It isolates the fundamental aspects of the radar arts and sciences that must be considered in order to determine the suitability of Walsh or other nonsinusoidal functions as an electromagnetic carrier wave. Also given is an indication of the problems to be encountered if one were to proceed in a direct transient analysis of nonsinusoidal propagation and radiation phenomena necessary in the radar process.

Part II, consisting of Chapters VI and VII, proposes anddevelops techniques and methods that utilize the known and existing store of knowledge and theory based on the sinusoidal operation and characterization of radar systems, components, and the radar medium.

The third part, Chapters VIII through X, utilizes many of the principles and tools developed in the second part to analyze and characterize in the time domain those basic elements necessary for the implementation of the radar principle using a nonsinusoidal electromagnetic carrier.

Chapter II The Category of Functions Considered for Nonsinusoidal Radar Carrier Waves

The category of functions considered as possible functional forms for a radar nonsinusoidal electromagnetic carrier wave consists of those functions which assume only two values: those of equal magnitude and of opposite polarity. Ideally the transitions from one value to the other occur instantaneously in terms of their independent variable which, in the case of a time varying electrical waveform, is time, t. The transitions between the two values, or the sign changes of the function, are allowed only at a discrete set of values of the independent variable within some characteristic interval of that variable, T, hereafter called the time base of the time varying waveform. It is also desirable for the purpose of radiating such waveforms that their average values over the duration of the time base interval be zero.

For the purpose of investigating the abstract mathematical properties of these bi-valued functions, the independent variable x is treated as a dimensionless, normalized variable with the functions defined and described on a unit interval of x which corresponds to the time base T when the independent variable is time: i.e., a time varying electrical waveform with $x = t/T$. The unit interval here corresponds to the interval of 2π radians of the sinusoidal functions. The two values that the abstract functions can assume are plus and minus one.

The set of values at which transitions are allowed are those values of x separating a set of uniform subintervals into which the unit interval (or T) has been divided, and the ends of the unit interval. For the Walsh functions the number of such uniform subintervals is an integer power of two. It is important to note that the functions in question need not undergo a transition between each pair of the subintervals comprising the unit interval (of definition).

A one-to-one correspondence exists between a bi-valued function, or waveform, and a binary sequence of ones and zeroes. If either of the following transformations

$$\begin{array}{c} 1 \longrightarrow 0 \\ -1 \longrightarrow 1 \end{array} \quad \text{or} \quad \begin{array}{c} 1 \longrightarrow 1 \\ -1 \longrightarrow 0 \end{array},$$

are made, the sequence of new values assumed by the bi-valued function in each subinterval, over which it remains a constant value of plus or minus one, describes a sequence of ones and zeroes which may be interpreted as a binary sequence. The first transformation above is that often used in practice.

Many families of such bi-valued functions (or binary sequences) exist with each having diverse and useful mathematical properties. Some of these functions are:

Barker sequences, group error-correcting codes, Walsh functions also known as the Reed-Muller codes, convolutional codes, orthogonal codes, simplex codes, and cyclic codes.

Walsh Functions

Although many of the functions and binary sequences mentioned above may have suitable mathematical structure for use as an electromagnetic carrier waveform, the Walsh functions are unique in that they possess analytical properties similar to those of the sinusoids even though having little direct similarity to them. In addition, extensive recent research effort (Proceedings of Symposium and Workshop on the Applications of Walsh Functions, 1970 through 1973) has produced many relatively simple means of producing Walsh function waveforms as well as producing and manipulating them in the environment of the digital computer.

The general character of the Walsh functions are best illustrated by a graphical presentation of the first ten or fifteen such functions as shown in Figure 1. There it is noticed that they are either odd or even functions with respect to the center of the unit interval (i.e., the origin) of $-\frac{1}{2} \leqslant x < +\frac{1}{2}$. They are identified by the symbol Wal(i,x) where Wal refers to the name of their discoverer, J. Walsh, i is an integer index specifying that it is the i^{th} Walsh function, and x is it argument, or independent variable.

Regarding parity, notice that even values of index i produce even functions while odd values of i produce odd functions. In order to simplify notation the forms sal(j,x) and cal(j,x) were originated to indicate the odd and even

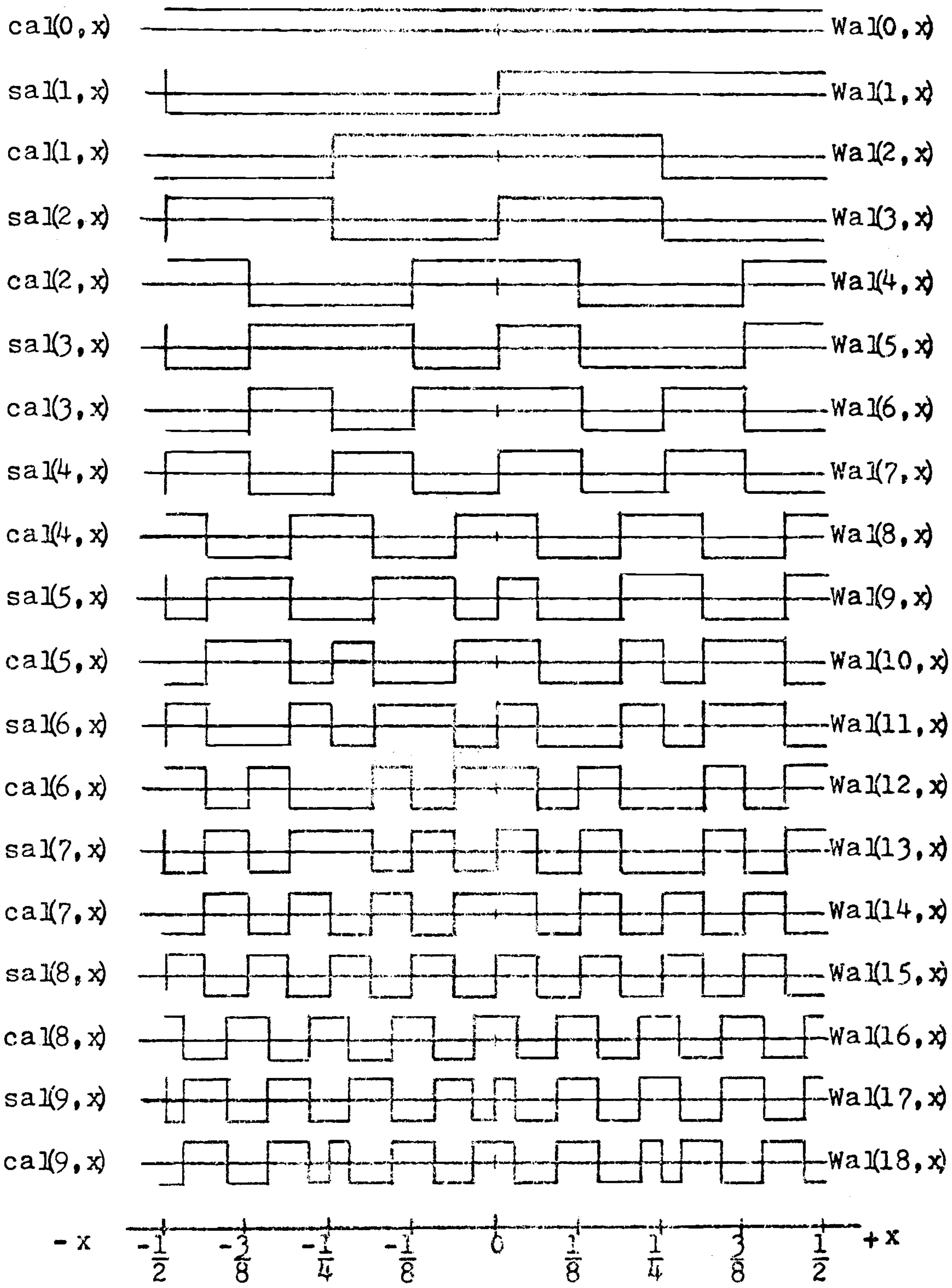

Figure 1. The first nineteen Walsh functions.

Walsh functions respectively, sal(j,x) being analogous to the odd sine function and cal(j,x) corresponding to the even cosine function of conventional harmonic analysis. In terms of the index of Wal(i,x) notation, j = ½i for even i and j = ½(i + 1) for odd values of i.

Even though the Walsh functions are defined on the unit interval centered on the origin, they may be extended periodically beyond that interval to plus or minus infinity forming periodic versions of the functions with basic period T if they are time varying waveforms.

The index i in the Wal(i,x) notation and the index j in the sal(j,x) and cal(j,x) notation are indicative of a parameter of Walsh functions analogous to the frequency variable of sinusoidal waveforms. An examination of the Walsh functions shown in Figure 1 shows the following relationship between the index i and the number of zero crossings, or transitions, in the unit interval:

i	zero crossings
0	0
1	2
2	2
3	4
4	4
5	6
6	6
7	8

Notice that when i is even the number of zero crossings is also even and equal to i:

$$i_{even} = \text{number of zero crossings} \qquad 1)$$

while for odd values of i

$$i_{odd} + 1 = \text{number of zero crossings}. \qquad 2)$$

Analogous to sinusoidal frequency we may define a Walsh function frequency as ½(number of zero crossings). This frequency-like quantity has been termed "sequency" by Harmuth (1972):

$$\text{Sequency} = \tfrac{1}{2}\text{ zero crossings} = \tfrac{1}{2}\, i_{even} \qquad 3)$$

or

$$\text{Sequency} = \tfrac{1}{2}(i_{odd} + 1). \qquad 4)$$

From the equations defining the index j of the sal(j,x) and cal(j,x) on the previous page, it is seen that

$$\text{Sequency} = \tfrac{1}{2}\, i_{even} = j \qquad 5)$$

and

$$\text{Sequency} = \tfrac{1}{2}(i_{odd} + 1) = j. \qquad 6)$$

The index of a Walsh function in the sal(j,x) or cal(j,x) notation is then seen to be a measure of its sequency, or ½(number of zero crossings) per unit interval. This sequency for the abstract Walsh function is a normalized form of sequency. In the practical case of time varying Walsh waveforms the actual sequency is equal to the normalized sequency S_n divided by the time base T:

$$\text{Absolute sequency} = S_n/T. \qquad 7)$$

An important subset of the Walsh functions are those sal(j,x) functions where j is an integer power of two: i.e., 1, 2, 4, 8, 16, 32, 64,---,. In the context of pure mathematics they are known as the Rademacher functions (Walsh, 1923) of which more will be said shortly.

Although the Walsh functions shown in Figure 1 may appear to undergo transitions in a random manner, they are specified by strict mathematical definitions. A definition due to Harmuth (1972), although mathematically clumsy and difficult to use, is a recursion equation which exhibits quite well the structure and source of Walsh functions. It is

$$\mathrm{Wal}(2i + p,x) =$$

$$(-1)^{p + \left[\frac{1}{2}i\right]}\left[\mathrm{Wal}(i,2(x+\tfrac{1}{4})) + (-1)^{i+p}\,\mathrm{Wal}(i,2(x-\tfrac{1}{4}))\right] \qquad 8)$$

where i = 0, 1, 2, 3, 4,---, $\mathrm{Wal}(0,x) = 1$ for $-\frac{1}{2} \leqslant x < \frac{1}{2}$, $\mathrm{Wal}(0,x) = 0$ for $x < -\frac{1}{2}$, $x \geqslant \frac{1}{2}$, and p assumes the values of zero and one for each value of i. The symbol $\left[\frac{1}{2}i\right]$ means the largest integer less than or equal to $\frac{1}{2}i$.

Equation 8 will yield a Walsh function for any value of index requiring only the value of Wal(0,x). However, this relationship does have the disadvantages that in order to obtain the 2ith or (2i + 1)th Walsh functions, the ith Walsh function and many of those preceeding it must be known or be stored in a computer memory before the desired Walsh function can be determined.

The structure of Wal(2i + p,x) is readily indicated by equation 8. Notice that the arguments of both Walsh functions on the right hand side of the equation are multiplied by a factor of two indicating a reduction in scale by one half. This fact means that these two functions have been "squeezed" into an interval of one half with their new values outside of that interval being zero. The terms $+\frac{1}{4}$ and $-\frac{1}{4}$ added to x serve to shift each of the component functions respectively into the left and right halves of the unit interval, each function being zero outside those half intervals. These two compressed and shifted versions of Wal(i,x) are then added or subtracted according to the factor $(-1)^{i+p}$ and the overall polarity is set by the leading coefficient $(-1)^{p + \left[\frac{1}{2}i\right]}$. Hence, the form of any Walsh function, other than Wal(0,x), is determined by a large portion of those Walsh functions preceeding it in index sequence.

Another form, or definition, much more in accord with the generation of Walsh function waveforms and their evaluation for any value of x and any index i, and their manipulation in a digital environment, is that of a product of Rademacher functions mentioned earlier. For a given index, i, this form is given by

$$\mathrm{Wal}(i,x) = R_{i_1}(x)\ R_{i_2}(x)\ R_{i_3}\ \text{---}\ R_{i_r}(x) \qquad 9)$$

where the integer index i is expressed in dyadic form as a sum of powers of two:

$$i = 2^{i1} + 2^{i2} + 2^{i3} + \text{---} + 2^{i_r} \qquad 10)$$

where the i_j are integers arranged in decending order such that $i_1 > i_2 > i_3 > \text{---} > i_r$. The $R_{i_j}(x)$ are the Rademacher functions of index i_j. In addition the set of integers i_j for any given index i are unique to that value of index. The Rademacher functions are simply the sal(n,x) functions with n an integer power of two. The relationship between the Rademacher notation index and that of the sal(n,x) notation is

$$n_j = 2^{i_j} \qquad 11)$$

so that a particular Walsh function in terms of the sal(n,x) functions is

$$\mathrm{Wal}(i,x) =$$

$$\mathrm{sal}(2^{i1},x)\ \mathrm{sal}(2^{i2},x)\ \mathrm{sal}(2^{i3},x)\ \text{---}\ \mathrm{sal}(2^{i_r},x)\ . \qquad 12)$$

The Rademacher functions are defined as

$$R_0(x) = \begin{matrix} +1 \text{ for } 0 \leq x < \frac{1}{2} \\ -1 \text{ for } -\frac{1}{2} \leq x < 0 \end{matrix}$$

and for higher index

$$R_i(x) = R_0(2^i x) = \mathrm{sal}(2^i, x) \ . \qquad 13)$$

The form indicated by equations 9 or 12, however, has the disadvantage that unit increases in the index i do not produce Walsh functions of increasing sequency. In the Walsh function form of harmonic analysis it is desirable to have increasing index correspond to increasing sequency. This need is easily accomplished by retaining equations 9 or 12 to express a product of Rademacher functions, but now express the index i in its Gray code equivalent. The Gray code of i is obtained by expressing i as a binary integer, i_b, and taking the bit-by-bit modulo 2 sum of i_b and $\frac{1}{2}$ i_b:

$$i_G = i_b \oplus \tfrac{1}{2} i_b = i_r \ i_{r-1} \ \text{---} \ i_2 \ i_1 \ i_0 \qquad 14)$$

in which the fractional part of $\frac{1}{2}i_b$ has been rejected. The sequency ordered Walsh function is then

$$\mathrm{Wal}(i,x) = \overset{r}{\underset{j=0}{\pi}} \left[R_j(x)\right]^{i_j} \qquad 15)$$

where i_j is either one or zero.

Either set of equations 9 or 10 or equations 14 and 15 may now be used to evaluate any Walsh function with only the values of the Rademacher functions being required. For use in a digital computer, memory requirements may be reduced by using the following relationships to evaluate a Rademacher function of any index at any value of x:

$$R_i(x) = \begin{cases} +1 \text{ if } m \text{ is even} \\ -1 \text{ if } m \text{ is odd} \end{cases} \qquad 16)$$

where m is a positive or negative integer, or zero, selected for the value of x such that

$$\frac{m}{2^{i+1}} \leqslant x < \frac{m+1}{2^{i+1}}. \qquad 17)$$

A property of Walsh functions making them analytically similar to the sinusoidal functions is their mutual orthogonality. Like the sinusoids the integral of the product of two unlike Walsh functions is zero while that for two like functions is unity:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{Wal}(m,x)\ \mathrm{Wal}(n,x)\ dx = \begin{cases} 0 \text{ if } m \neq n \\ 1 \text{ if } m = n \end{cases} \qquad 18)$$

It is the above property that allows a Fourier type expansion of a square integrable function over the unit interval in terms of Walsh functions in the form

$$F(x) =$$

$$a_c(0)\ \mathrm{Wal}(0,x) + \sum_{j=1}^{\infty} \left[a_c(j)\ \mathrm{cal}(j,x) + a_s(j)\ \mathrm{sal}(j,x) \right] \qquad 19)$$

for $-\frac{1}{2} \leqslant x < \frac{1}{2}$ where

$$a_c(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x)\ dx \qquad 20a)$$

$$a_c(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x)\ cal(j,x)\ dx \qquad 20b)$$

$$a_s(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} F(x)\ sal(j,x)\ dx\ . \qquad 20c)$$

If F(x) is periodic with period T, then equation 19 holds for all x.

The property of Walsh functions possibly making them of greater value as an electromagnetic radar carrier wave is their product property. Since any one Walsh function is a product of Rademacher functions, it follows that the product of any two Walsh functions is also a product of Rademacher functions, and is then another Walsh function. The resulting product of two or more Walsh functions is an exceptionally simple relationship:

$$Wal(i,x)\ Wal(j,x) = Wal(i \oplus j,x) \qquad 21)$$

where $i \oplus j$ is the bit-by-bit modulo 2 sum of i and j when expressed as binary integers. Similar relationships hold for Walsh function products in the sal(j,x) and cal(j,x) notation:

$$cal(i,x)\ cal(j,x) = cal(i \oplus j,x) \qquad 22a)$$

$$sal(i,x)\ cal(j,x) = sal\ \left[(j \oplus (i-1)) + 1,\ x\right] \qquad 22b)$$

$$\mathrm{sal}(i,x)\ \mathrm{sal}(j,x) = \mathrm{cal}\ \left[(i - 1) \oplus (j-1),\ x\right] . \qquad 22c)$$

Similar equations relate Walsh functions of the same index, but with different arguments, x and x' :

$$\mathrm{cal}(i,x)\ \mathrm{cal}(i,x') = \mathrm{cal}(i,\ x \oplus x') \qquad 23a)$$

$$\mathrm{sal}(i,x)\ \mathrm{sal}(i,x') = \mathrm{sal}(i,\ x \oplus x') . \qquad 23b)$$

Note that the relationships expressed by equations 21, 22, and 23 hold only if there is exact synchronism between the two product Walsh functions. The above product properties of Walsh functions would be quite useful in implementing a superhetrodyne Walsh receiving system.

It is worthwhile to note that subsequent work in this study applies not only to the Walsh functions, which have been described in some detail here, but also applies to any bi-valued waveform making sudden transitions from one level to the other. In this respect the study is general and need not be restricted to the Walsh functions alone. The Walsh functions have been emphasized because of their analytical similarity to the sinusoids which have been applied so successfully to radar and communications problems in the past.

The Question of Realizable Source Waveforms and Field Variations

In the context of radar applications, the sources of radiating electromagnetic fields having traveling waveforms related to the class of functions discussed here (or some form of the functions derived therefrom such as time derivatives or integrals), must ultimately be time varying currents or voltages related in some manner to the functional forms of the radiating fields.

Because of the nature of the active devices that must be used to generate or amplify such discrete and discontinuous voltage or current waveforms, and because of the unavoidable presence of parasitic inductance and capacitance in the generating circuits and transmission elements, such source quantities will always manifest switching intervals at each transition between discrete states of the waveform. These parasitic elements also force any such generated source current or voltage to be continuous and smoothly varying: any sudden change in slope or level is accompanied by rounding effects tending to make variations of the source quantity always smoothly varying. As a result, any radiating electromagnetic field generated by such sources will also be continuous and smoothly varying in time and space. In addition, antenna and medium effects may impart more rounding to the radiated field which further distort the waves.

For the purpose of this analysis it is sufficient to consider only the first order effects of generation or filtering on a discrete time varying waveform. An instantaneous change between two constant values of the idealized form of the generated waveform is realized as a change occurring at a constant rate (i.e., constant slope of di/dt or dv/dt) over the switching interval of the generating device. It is further modified by the filtering effects of its external circuit. Figures 2a, b, and c illustrate the generation of an example Walsh waveform, its first time derivative, and its first time integral, all with their naturally occurring derivatives shown below each generated waveform. This group of figures shows the difference between time derivatives and integrals of a Walsh waveform that might be purposely generated and those that can occur due to the action of a propagation medium or some transmission device.

Reasons for the above restrictions become clear in Chapter X where it is shown that the time variation of the far-field generated by arbitrarily varying sources are time derivatives of the source time variation. Hence, in order to obtain a certain field time variation, in either the near or far-zones, the required source variations may be very different from those of the desired field. For instance, it is shown in Chapter X that a simple short dipole gives rise to far-fields that have time variation proportional

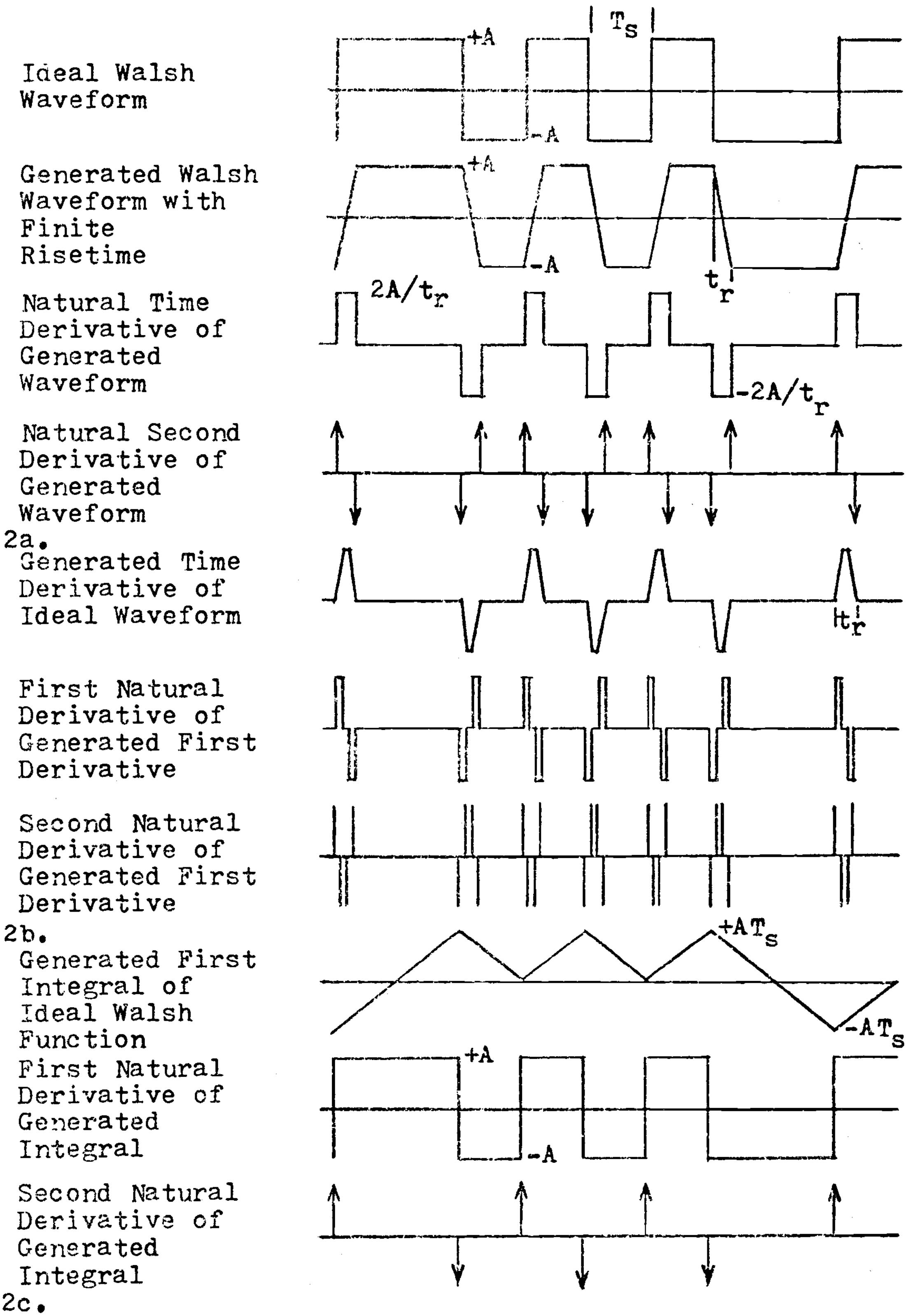

Figure 2. Time derivatives and integrals of realizable Walsh function waveforms.

to the first time derivative of the source current.

For practical interest, switching times currently achievable (Cuccia, 1972) using transistors in the common emitter mode have been as low as 150 picoseconds while step-recovery diodes have produced switching times as low 50 picoseconds.

Chapter III Reduction of the Body of Radar Theory to its Essential Features

In order to ascertain the utility of electromagnetic Walsh waves for use in radar not only is it necessary to examine their nature and their radiation properties in great detail, but it is also important to examine the fundamental concepts, principles, and practices on which the theory and arts of radar are founded. This examination is necessary in order to determine not only the need of using an electromagnetic carrier, but also the properties it must possess in order to be truly useful in the radar process.

This chapter is devoted to a brief examination of the body of radar theory in order that those features that may have a bearing on, or a dependence on, the nature of the carrier waveform might be exposed.

A survey of classical and modern texts on radar, topics and contents of several short courses on modern radar offered at several American universities by recognized radar authorities, and the numerous esoteric journal papers on radar reveal a confusing profusion of seemingly unrelated and diverse specialized topics. To offer substance to this contention the tables of contents of several radar texts and the subject content of several recent short radar courses are presented in detail in Appendix I. However, to give the reader a brief indication of the breadth of radar theory and practice a composite listing of topics is presented in Table I. Even from this brief listing it is apparent that there is great diversity in the subject matter as well as some overlap between many of the subjects while many constitute separate topics not directly related to the principles of radar.

However, in reviewing the tables of contents and the short courses of Appendix I, the topics naturally subdivide into several groupings. The first consists of those concepts and principles peculiar to radar which might be considered as the core of the principles essential to radar. They are those listed one through five in Table I.

The second grouping consists of those broad disciplines and physical theory which form what could be considered as the basic mathematical framework and the language in which the principles of radar are expressed and within which specific radar applications are most often analyzed. This is item six in the table.

Items seven through thirteen comprise the third category which describe the various modes of operation and radar applications that are possible. Each of these specific radar types are usually treated separately in the literature as each has its own particular set of problems to overcome.

A fourth major category is that set of principles, concepts, and theories from other highly specialized

Table I. A Composite Listing of Radar Theory Subject Matter.

1. Radar Equation: antenna gain and directivity; cross section; isotropic radiator.
2. Extraction of radar information: interpretation of the received signal in light of a priori knowledge of the transmitted signal; resolution and ambiguity; high resolution waveform design; fundamental resolution limitations; target parameter estimation.
3. Ambiguity functions: resolution and optimum waveforms.
4. Target properties: scattering properties; rough surface scattering; cross section density; reflectivity.
5. Principles of displaying radar information.
6. Signal analysis and representation of systems: system theory; circuit theory; electromagnetic theory; probability theory.
7. Continuous wave and phase/frequency modulation radar.
8. Bistatic and multistatic radar.
9. Moving target radar.
10. Pulse doppler techniques.
11. Tracking radar.
12. Synthetic aperture radar.
13. Special applications: solid-state radar; marine radar; airborne radar; space applications; satellite surveilance; laser radar; radar beacons.
14. Noise theory: detection of signals in noise; target statistics, signal-to-noise ratio and false alarm rate; automatic detection schemes.
15. Antenna principles: gain and directivity; noise properties.
16. Array antennas.

Table I (cont.)

17. External environment and radio frequency (EM carrier) considerations: propagation effects; weather; clutter; interference.

18. Signal processing techniques: analog (electrical, optical, etc.); digital.

19. Communication theory.

20. Hardware and technology considerations: transmitters; high frequency receivers; amplifiers; low noise and high frequency amplifiers and receivers; radio frequency components; radomes; indicators; array phase shifters.

disciplines which are necessary to refine radar performance in the presence of noise, or in other practical real world applications. This set consists of items fourteen through nineteen.

The last group, item twenty, considers the hardware and technological aspects necessary to physically implement and build operating radar equipment. The specifications such hardware must meet to allow a radar to reliably perform its task, or mission, are determined by analysis based upon application of those principles listed in the four preceeding groups.

Underlying all of the items listed in Table I are several implied assumptions of major proportions which form the framework of most analyses of the radar problem as well as the implementation of radar principles. Unfortunately, these assumption have apparently never been examined for their validity nor explored to discover any alternative premises on which to form the base of such a framework.

Of those assumptions the most important is the presupposed use of a sinusoidal high frequency carrier as the vehicle for conveying the radar energy to and from the target. This practice is certainly well fitted to the traditional practice of describing radar components, systems, and the propagation medium by their steady-state responses to excitation by a sinusoidal waveform (of infinite duration) and to the Fourier transform description of signals and systems. However, a sinusoidal carrier is not an a priori requirement for successful implementation of the radar principle.

A second assumption is that most real targets of interest form an ideal point target which produces a signal at the radar which is merely a delayed and attenuated replica of the transmitted signal. This abstraction, although applicable to the practical cases of the past where target dimensions were small compared to the smallest distance resolvable by the radar, but large with respect to the illumination wavelength, is easily extended to the general extended target which is large compared to the radar resolution capability. In this case the ideal point target is considered to be the elemental building block of which the general target consists.

A closer examination of Table I reveals two other major groupings, both of which have general applicability to all of the possible forms of radar operation making up the original third group. These two new groupings appear to be the two fundamental aspects of the radar problem whether of the traditional sinusoidal form of radar or of the non-sinusoidal form with which this thesis is concerned.

The first group, consisting of items one, two, three, five, and eighteen, can be considered as dealing with operations within the radar after receiving the wave reflected from the target, or the signal processing operations, interpretation of such signals, and presentation of the information extracted from them. The second set, consisting of items four, seven, fifteen, sixteen, and also one, concern the electromagnetic phase of the radar operation. It is to this electromagnetic aspect of a nonsinusoidal radar that the remainder of this thesis is devoted.

Chapter IV The Need for a High Frequency Sinusoidal Carrier

Discussions of radar and radar theory appearing in print from the very early to the most modern, and from the most basic to the most advanced and esoteric, presuppose the use of a modulated, narrow-band sinusoidal carrier wave which also applies to the communication case. Although the advantages are self evident it is worthwhile to question this well established practice as it can provide considerable insight into some of the desirable properties for a radar carrier wave to possess.

In the early days of radio and radar it was quickly found that some kind of high frequency vehicle was necessary to provide efficient radiation of an information bearing signal. Attempts to radiate the original (baseband) signal with reasonable efficiency would have required radiating structures (antennas) of unmanageable size as well as resulting in extreme distortion of the radiated signal through differential efficiencies in radiating the various sinusoidal constituents of the baseband signal.

For the radar situation things were more complex. Not only was efficient radiation a requirement, but the need to form and scan a narrow beam of electromagnetic energy with mechanically scanned and airborne antennas of reasonable size necessitated high frequency operation.

However, there does exist another very good reason for using a high frequency carrier in radar. For example, if it were possible to radiate a non-oscillating pulse of electromagnetic energy and to successfully detect the energy reflected back to the radar, then, in principle, the presence of a reflecting object could be detected and its range measured. But, in order to realize the full capability of radar, it must be sensitive to the motion of a potential target and to its angular position. It is doubtful if the baseband signal could be radiated in a directional manner to achieve angular discrimination since directional radiation depends upon the wave interference principle of single frequency sinusoidal radiation. This fact would make baseband signal radiation useless for many forms of radar operation.

In addition, the Doppler effect, which, in reality, is a compression or stretching of the time variable of a wave reflected from a moving target, would surely manifest itself on the non-oscillating pulse. However, even if the reflected pulse should arrive at the radar unchanged by its propagation through the radar medium, the Doppler effect on the duration of the pulse would be much too minute to be measureable by presently known techniques. And, even if the Doppler effect were measureable, other effects occurring during propagation and amplification at the receiver would distort this wideband pulse to the point of completely masking the small changes induced by the Doppler effect.

Similar effects could occur with a wideband modulated sinusoidal carrier wherein the pulse envelope can be severely distorted during propagation or during and after detection at the radar receiver.

The conclusion to be drawn is that with or without a high frequency carrier, pulse shape can be distorted beyond use for measuring target motion while the Doppler effect on pulse duration, for most practical situations, is too small to be measureable even without the presence of signal distortion.

Then, in order to provide sensitivity to target motion there must be some mechanism or parameter present in the radiated electromagnetic wave which is insensitive to waveform distortion yet which will respond to the Doppler effect. One property filling these needs is the rate of zero crossings of a periodic waveform (lacking a d.c. component). Many waveforms have this property, the sinewave being that commonly used in radio and radar. The zero crossings of a sinusoid are readily and accurately measured as a frequency by filters, wavemeters, or by use of electronic counter circuits. Hence, using present day measurement techniques, electromagnetic energy of a high repetition rate is required in order for radar to be responsive to the motions of a target as well as to its range.

From the above qualitative discussion the following may be concluded: a high frequency carrier is required to meet the very practical need of efficient radiation by antenna structures of reasonable size; a periodic carrier is also necessary in order to realize the required sensitivity to target angular position and to its motion!

Chapter V Assessment of the Direct Transient Analysis of Nonsinusoidal Propagation and Radiation Phenonema

In Chapter I mention was made of Harmuth's (1970) and Pearlman's (1970) early preliminary theoretical analyses of the radiation problem due to nonsinusoidal current sources. Harmuth's 1972 analysis is reproduced in Appendix II for the reader's convenience.

In that analysis Harmuth (1972) made use of the vector magnetic potential $\bar{A}$ and the electric scalar potential $\emptyset$ to evaluate the electric and magnetic field quantities in the region surrounding a small localized current appearing on a short Hertzian dipole antenna element. The length of the antenna element was assumed to be so short that current along its length did not vary with location while the current at any point along the element was assumed to be equal to that supplied by an electronic source exciting the device at a small gap near its center.

However, the power radiated from a very short antenna element is impractically small. In addition, those assumptions above are not realistic for mechanically realizable dipole elements nor for other forms of conductor arrangements necessary to effectively radiate the nonsinusoidal waveforms under consideration here.

In Appendix II it is shown that the vector potential $\bar{A}$ and the scalar potential $\emptyset$ are subject to the two following nonhomogeneous wave equations

$$\nabla^2 \phi - ue \frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{e} \rho(x,y,z;t), \qquad 1)$$

$$\nabla^2 \bar{A} - ue \frac{\partial^2 \bar{A}}{\partial t^2} = - u J(x,y,z;t) \qquad 2)$$

under the Lorentz gauge condition

$$\nabla \cdot \bar{A} + ue \frac{\partial \phi}{\partial t} = 0 . \qquad 3)$$

In the above wave equations $\bar{A}$ and ϕ are both functions of position and time, $\rho(x,y,z;t)$ is the charge density, and $\bar{J}(x,y,z;t)$ is the current density distribution of a time varying source. Both are functions of position and time and are related by a continuity equation

$$\nabla \cdot \bar{J} = -\frac{\partial \rho}{\partial t} . \qquad 4)$$

In the case of infinite space with no boundary conditions to be met, which is also the case relevant to the radar situation, the above two wave equations have the following integral solutions:

$$\phi(x,y,z;t) =$$

$$\underset{\text{over } \rho}{\iiiint} G(x',y',z',t';x,y,z,t)\rho(x',y',z',t')dx'dy'dz'dt' \qquad 5)$$

$$\bar{A}(x,y,z;t) =$$

$$u\underset{\text{over } \bar{J}}{\iiiint} G(x',y',z',t';x,y,z,t)\ \bar{J}(x',y',z',t')dx'dy'dz'dt' \qquad 6)$$

in which G(x',y',z',t';x,y,z,t) is a time dependent Green's function relating the potential at an arbitrary observation point, x, y, z, at time t to an infinitesimal element of the source density distribution located at x', y', z' and activated at time t'.

It can be shown (Jackson, 1967, pp 183-186) that G(x',y',z',t';x,y,z,t) reduces to the relatively simple delta function expression

$$G(x',y',z',t';x,y,z,t) = \frac{\delta(t' - t + |\bar{x} - \bar{x}'|/c)}{4\pi |\bar{x} - \bar{x}'|} \qquad 7)$$

in which $\bar{x} = ix + jy + kz$, the position vector of the observation point and $\bar{x}'$ is the position vector of an element of the source distribution producing the disturbance at $\bar{x}$.

Inserting the above Green's function into the two potential integrals and carrying out the integrations with respect to t' yields

$$\emptyset(x,y,z;t) = \iiint\limits_{\text{over } \rho} \frac{\rho(x',y',z';t - |\bar{x} - \bar{x}'|/c)}{4\pi e |\bar{x} - \bar{x}'|} \, dx'dy'dz' \qquad 8)$$

and

$$\bar{A}(x,y,z;t) = u \iiint \frac{J(x',y',z';t - |\bar{x} - \bar{x}'|/c)}{4\pi |\bar{x} - \bar{x}'|} \, dx'dy'dz'. \qquad 9)$$

The electric and magnetic field quantities are then obtained from these two potential quantites (which seem as artificial mathematical devices created as an intermediate step to simplify evaluation of the field quantities) by the following relationships:

$$\bar{B}(x,y,z;t) = \nabla X \bar{A}(x,y,z;t), \qquad 10)$$

and

$$\bar{E}(x,y,z;t) = -\nabla\phi(x,y,z;t) - \frac{\partial\bar{A}}{\partial t}(x,y,z;t) . \qquad 11)$$

If $\rho(x,y,z;t)$ and $\bar{J}(x,y,z;t)$ were both known accurately throughout the volume of these density distributions the integral expressions for equations 8 and 9 for $\phi(x,y,z;t)$ and $\bar{A}(x,y,z;t)$, and the expressions for $\bar{B}(x,y,z;t)$ and $\bar{E}(x,y,z;t)$ of equations 10 and 11 would all be exact. However, time varying charge and current density distributions actually appearing on antennas used in radar or radio communications usually result from highly localized electronic generators of high frequency sinusoidal current or voltage. It is presumed here that such will be the case for the radiation of nonsinusoidal electromagnetic energy. The exact values of $\rho(x,y,z;t)$ and $\bar{J}(x,y,z;t)$ would then be solutions to very complicated boundary value problems for both sinusoidal and nonsinusoidal radiation: the latter case would presumably be much more difficult than the sinusoidal which, itself, is only solvable for certain simple geometries of conducting or dielectric elements.

Rather than determine the radiation fields from realizable charge and current density distributions exactly, it has been satisfactory in past antenna design practices to assume (in the sinusoidal case) that the entire source density distribution has the same sinusoidal time variation throughout its whole volume. The charge and current density distributions may then be written as products of a space varying factor and a time varying factor:

$$\rho(x,y,z;t) = \rho(x,y,z)\, e^{j\omega t} \qquad 12)$$

and

$$\bar{J}(x,y,z;t) = \bar{J}(x,y,z)e^{j\omega t}. \qquad 13)$$

The above simplifying assumption allows the exponential factor to be written as

$$e^{j\omega(t - |\bar{x} - \bar{x}'|/c)} = e^{j\omega t}\, e^{-j\omega|\bar{x} - \bar{x}'|/c} \qquad 14)$$

which allows the exponential time function to be removed from under the integral signs of the solutions for ϕ and $\bar{A}$. Even though such an assumption makes the radiation problem tractable, it does place stringent limits on the maximum spatial extent of the source density distributions in terms of the operating (sinusoidal) frequency. In order that there be little phase difference and little distortion of the currents appearing on the antenna structure, the distance $\frac{1}{2}d$ to the outer limits of the source distribution must meet the following inequality:

$$\tfrac{1}{2}d << \lambda \text{ or } d << 2c/f. \qquad 15)$$

If a value of ten percent of 2c/f is acceptable for d and frequency is expressed in gigahertz, then

$$d_{max} \approx 0.6/f \text{ meters}. \qquad 16)$$

The practical interpretation of the above constraint is that pulse rise or fall times and pulse durations must be much longer than the time needed for propagation across the source distribution. This principle can be stated in an inequality as

$$\tfrac{1}{2}d << cT \text{ or } d << 2cT. \qquad 17)$$

Here T is the time interval of interest in the waveform (i.e., the rise or fall time or desired time resolution) to be transmitted.

If a ten percent limit is again accepted and a pulse rise time of 50 picoseconds is considered, then

$$d_{max} \approx 2 \text{ X } 3 \text{ X } 10^{8} \text{ X } 50 \text{ X } 10^{-12} = 0.03 \text{ meters}$$

$$d_{max} \approx 3.0 \text{ centimeters}.$$

The above limit means that use of the simplifying assumption on the first page of this chapter for Walsh or similar type waveforms with transition times of 50 picoseconds, or less, individual radiating elements must be of the order of a few centimeters in extent.

The above approximation is suitable for single frequency or narrow band operation. However, the natures of the sinusoidal and nonsinusoidal modes of operation are very different. It would therefore be desirable in the analysis or design of radiating elements or antennas for nonsinusoidal operation to be free of the aforementioned artificial limitation on the spatial extent of the source distribution. Accomplishment of such freedom in a direct and exact transient analysis of the nonsinusoidal radiation phenomena would, however, be very difficult and require much new effort and research. This area of endeavor has seen little activity in the past. Since the direct transient analysis is so difficult much simpler means and techniques are developed in subsequent chapters. The methods developed utilize existing knowledge and information on the sinusoidal operation of radar system components and the radar environment propagation media.

Chapter VI Radar System Components and Propagation Media Characterization

In order to adequately examine radar theory and practice and to isolate their essential features, one must also consider the means used to characterize radar system devices and components. This action is imperative to this study since many of the components and operating principles will, no doubt, prove useful or necessary for the implementation of a radar based upon a Walsh function carrier.

In this study four categories of components are examined: 1) transmission structures such as waveguides and transmission lines; 2) antennas and their radiation patterns; 3) amplifiers and/or filters; 4) the propagation medium. It is felt that these four divisions represent the indispensible components and elements that determine the successful operation of a radar based upon a sinusoidal carrier as well as a nonsinusoidal carrier wave. However, for the purposes of this chapter a detailed study of these four items is not necessary. Rather, it is important to point out certain assumptions and principles underlying their uses, specifications, and descriptions. The details of their operation pertinent to use with Walsh functions are discussed in later chapters as needed.

Although high power microwave transmitter tubes and low power sinusoidal microwave oscillators also constitute essential elements of a sinusoidal radar, they are not considered since they are unable to generate Walsh functions and cannot contribute to the implementation of a Walsh carrier radar. Knowledge of their characteristics can contribute little in determining the suitability of the four areas above for use with a Walsh carrier radar.

Transmission Elements

The various forms taken by transmission devices such as transmission lines and waveguides are characterized by their effect upon the amplitude and phase delay along the line of a hypothetical sinusoidal excitation of infinite duration.

The main concern here is on the transmission line or waveguide which is matched or else so long that reflections don't occur within the time intervals of interest. This examination is limited to these special and idealized cases since our interest is in the propagation properties of the device.

The two conductor line, which includes the open line, the coaxial cable, and stripline is described by a complex traveling voltage (or current) wave

$$E_z = E_s \, e^{-(\alpha + j\beta)z} = E_s \, e^{-\gamma z} \text{ volts} \qquad 1)$$

in which E_s is the amplitude of the excitation voltage (or current) at the sending end of the line, usually $E_s = |E_s| \, e^{j\theta}$; E_z is the resulting phasor voltage at some point z along the line; α is an attenuation factor; β is a phase factor; and γ is the symbol for the complex sum of $\alpha + j\beta$.

For the simple transmission line α and β are determined by the constants of the line: its series resistance per-unit-length, R; its series inductance per-unit-length, L; its shunt conductance per-unit-length, G; and its shunt capacitance per-unit-length, C. The dependence of $\alpha + j\beta$ on these quantities is

$$\alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = \gamma \qquad 2)$$

which is also seen to be a function of the frequency of the sinusoidal excitation voltage. In addition R, L, G, and C are generally frequency dependent.

The transmission line is also described by a characteristic impedance which is the ratio of the traveling voltage wave to traveling current wave at any point on a sinusoidally excited line having no reflections. This quantity is also a function of the line constants and frequency:

$$Z_o = \sqrt{(R + j\omega L) / (G + j\omega C)} \, . \qquad 3)$$

If a line of finite length is terminated in an impedance, Z_L, the impedance seen at the input end of the line is

$$Z_{in} = \frac{Z_o(Z_L + Z_o \tanh(\alpha + j\beta)x)}{(Z_o + Z_L \tanh(\alpha + j\beta)x)} \qquad 4)$$

where x is the length of the line. The input impedance is a function of frequency also through the quantities Z_o, Z_L, α, and β which are all frequency dependent.

Further consideration of the traveling-wave solution of the transmission line results in a useful relationship.

First, consider a line with excitation $E_s e^{j\omega t}$ at the sending end and the response at a location z_1:

$$e(t,z_1) = E_s\ e^{j\omega t}\ e^{-\gamma z_1}\ . \tag{5}$$

The response at a second location z_2 (with $z_2 > z_1$) is

$$e(t,z_2) = E_s\ e^{j\omega t}\ e^{-\gamma z_2}\ . \tag{6}$$

Now, with $z_2 = z_1 + \Delta z$,

$$e(t,z_2) = (E_s\ e^{j\omega t}\ e^{-\gamma z_1})\ e^{-\gamma \Delta z}\ . \tag{7}$$

However, the factor in parentheses is just $e(t,z_1)$ so that

$$e(t,z_2) = e(t,z_1)\ e^{-\gamma \Delta z}\ . \tag{8}$$

Since the time variation is exponential (or sinusoidal in terms of real functions) $e(t,z)$ may be written as

$$e(t,z) = e^{j\omega t}\ E(z) \tag{9}$$

so that

$$E(z_2)\ e^{j\omega t} = E(z_1)\ e^{j\omega t}\ e^{-\gamma \Delta z}\ . \tag{10}$$

With $t = 0$ the above relationship becomes

$$E(z_2) = E(z_1)\ e^{-\gamma \Delta z} \tag{11a}$$

or

$$E(z_2)/E(z_1) = e^{-\gamma \Delta z} \tag{11b}$$

which is a form of transfer function relating the amplitudes and phases of the sinusoidal time variations at two points on the line separated by a distance Δz.

Since $E(z) = E_S e^{-\gamma z}$ further simplification can result:

$$E(z_2)/E(z_1) = E(z_2 - z_1) = e^{-\gamma\Delta z} \tag{12)}$$

so that $e^{-\gamma\Delta z}$ becomes the transfer function relating amplitudes and phases on the sinusoidal traveling wave at any two points on the line separated by a distance Δz, or

$$H(\omega, \Delta z) = e^{-\gamma\Delta z} \tag{13)}$$

where $\gamma = \alpha + j\beta$, a complex function of frequency.

The above concept of a transmission line transfer function between two line locations separated by a distance Δz is of great value in characterizing the line in terms of Walsh function traveling waves.

The case of the single hollow metallic waveguide which has complicated solutions for the electric and magnetic fields within the guide as functions of time and position also allows an interpretation identical to that for the two conductor line. Consider for example, the rectangular guide with TE and TM solutions (Ramo, Whinnery, and Van Duzer, pp 421-424) for exponential excitation $e^{j\omega t}$ listed in Table II. Although the solutions in that table show no single quantity which describes the waveguide it is seen that all ten field quantities possess the same traveling-wave characteristic: i.e., $e^{(j\omega t - \gamma z)}$. Hence, they all

Table II. Electric and Magnetic Field Solutions for a Rectangular Waveguide with Propagation in the z Direction.

TM Solutions:

$$E_z(x,y,z;t) = A\, e^{(j\omega t - \gamma z)} \sin(x\, k_x) \sin(y\, k_y)$$

$$H_x(x,y,z;t) = jAk_y f\, e^{(j\omega t - \gamma z)} \sin(x\, k_x) \cos(y\, k_y)/k_c f_c Z$$

$$H_y(x,y,z;t) = -jAk_x f\, e^{(j\omega t - \gamma z)} \cos(x\, k_x) \sin(y\, k_y)/k_c f_c Z$$

$$E_x(x,y,z;t) = \sqrt{1 - f_c^2/f^2}\; H_y(x,y,z;t)\, Z$$

$$E_y(x,y,z;t) = \sqrt{1 - f_c^2/f^2}\; H_x(x,y,z;t)\, Z$$

TE Solutions

$$H_z(x,y,z;t) = B\, e^{(j\omega t - \gamma z)} \cos(x\, k_x) \cos(y\, k_y)$$

$$E_x(x,y,z;t) = jBk_y fZ\, e^{(j\omega t - \gamma z)} \cos(x\, k_x) \sin(y\, k_y)/k_c f_c$$

$$E_y(x,y,z;t) = -jBk_x fZ\, e^{(j\omega t - \gamma z)} \sin(x\, k_x) \cos(y\, k_y)/k_c f_c$$

$$H_x(x,y,z;t) = -\sqrt{1 - f_c^2/f^2}\; E_y(x,y,z;t)/Z$$

$$H_y(x,y,z;t) = \sqrt{1 - f_c^2/f^2}\; E_x(x,y,z;t)/Z$$

Legend: a= guide width in the x direction; b= guide height in the y direction; $k_x = m\pi/a$; $k_y = n\pi/b$; m, n are integers; $k_c^2 = k_x^2 + k_y^2$; Z= intrinsic wave impedance of the medium which is frequency dependent for a lossy medium; $f_c = k_c/2\pi\sqrt{ue}$, the waveguide cut-off frequency; $\gamma = \alpha + j\beta$; $\beta^2 = k_c^2 - k^2$; α and β are both complicated functions of frequency depending upon the material making up the walls of the waveguide and its filling dielectric; ω = radian operating frequency; and $\gamma = j\,\frac{\omega}{c}\,(1 - f_c^2/f^2)^{\frac{1}{2}}$.

propagate along the guide as an ensemble, or set, unchanged in their spatial relationships. As one might suspect, the same principle applies to a waveguide of arbitrary lateral cross section even though the lateral distribution (in the x and y coordinates) of the field quantities may be difficult, or even impossible, to determine analytically.

Hence, for any field component of the waveguide, an equivalent transfer function relating its amplitudes and phases for an exponential time variation at any two points along the guide separated by a distance Δz may be defined as

$$H_h(\omega, \Delta z) = e^{-\gamma \Delta z}, \qquad H_e(\omega, \Delta z) = e^{-\gamma \Delta z} \qquad 14)$$

as in the case for the two-conductor transmission line.

Since waveguides and transmission lines may often be terminated in an unmatched condition in practice, or may often possess discontinuities or irregularities in cross sectional properties along the device, all will give rise to reflections. Transmission and reflection coefficients are then of interest.

For the transmission line with a non-matching termination or some form of discontinuity, a reflected voltage wave traveling in the opposite direction of the incident wave is generated. If the amplitude of the incident (sinusoidal) wave is E_+ and that of the reflected wave E_-, then a reflection coefficient ρ is defined as

$$\rho = E_-/E_+ \qquad 15)$$

with the voltage across the terminating load impedance E_L

$$E_L = E_+ + E_- \qquad 16)$$

so that, with some algebraic manipulation,

$$\rho = E_-/E_+ = (Z_L - Z_o)/(Z_L + Z_o) = \rho(\omega) \qquad 17)$$

which can be a complex quantity.

The portion of the incident voltage wave affecting the load impedance Z_L, or, in the case of a mismatching discontinuity, the portion of the incident energy transmitted beyond the discontinuity, is related to the incident wave by a transmission coefficient T:

$$T = E_L/E_+ = 2\ Z_L/(Z_L + Z_o) = T(\omega) \qquad 18)$$

which can also be a complex quantity.

The two quantities ρ and T are similar to the relationship determined for the transmission line transfer function $H(\omega,\Delta z)$ relating amplitudes and phases of the traveling-wave at two locations along the line. The quantities ρ and T, which are both frequency dependent, constitute transfer functions relating the incident and reflected wave amplitudes and phases at the discontinuity and relating those of the incident and transmitted waves respectively. Similar quantities are used in Chapters IX and X to describe the reflection and transmission of Walsh waves on transmission lines and in waveguides as well as their reflection from objects in space.

Antennas and Radiators

Radar antennas, whether of the parabolic aperture form or a phased array, are dependent on the (monochromatic) wave interference principle for their operation and, like transmission lines, are described in terms of their response to a sinusoidal excitation of infinite duration.

Antennas for a radar application, in order to provide a reasonable degree of directional resolution, must be able to concentrate transmitted electromagnetic energy into a narrow beam (in the far field) and to respond to received energy from a very narrow solid angle in a given direction. Its ability to do this is usually expressed as a relative gain function in terms of either field strength or power density transmitted or received as a function of two orthogonal angular coordinates measured from the direction of maximum response of the antenna. This gain function is usually defined with respect to the field strength or power density that would result using a hypothetical isotropic radiator: i.e., a radiator that produces uniform power density in all directions.

For work in later chapters it is essential to determine the relative field strength or (instantaneous) power density (i.e., the radiation pattern) in the space surrounding the antenna as a function of frequency in response to a sinusoidal voltage or current excitation at some point in the antenna system. The electric field radiation pattern as a function of the angular coordinates of the radiation pattern also constitutes a form of antenna system transfer function that relates the relative field amplitudes and phases to the sinusoidally varying excitation source.

For an arbitrary aperture with general illumination distribution the Kirchhoff-Huygens diffraction integral can provide a general expression for the frequency dependence:

$$E_{far}(\theta, \phi) \approx \frac{jE_0 \; e^{-jkR}}{\lambda \; R} \iint_{aperture} A(x,y) \; e^{jk \sin \theta \; (x \cos \phi + y \sin \phi)} dx \; dy \qquad 19)$$

(Skolnik, 1970) where A(x,y) is the relative amplitude and phase distribution of the field in the aperture, E_0 is a reference value of the aperture field intensity, R the distance from the antenna origin to the observation point, $k = 2\pi f/c$ the phase constant of the medium, θ and ϕ are the angular coordinates of the standard spherical coordinate system.

A(x,y), which is generally complex, may be considered to consist of a real amplitude factor $|A(x,y)|$ and a phase factor $e^{j\lambda(x,y)}$. It is also important to realize that $|A(x,y)|$ and $\lambda(x,y)^2$ may purposely be adjusted or varied to

2) Do not confuse λ(x,y) with λ lacking the parentheses. The symbol λ is the wavelength of sinusoidal excitation.

produce some desired property in the antenna radiation pattern. Since $\lambda(x,y)$ is an electrical angle, depending on how it is produced, it can be dependent on frequency for a fixed antenna system. It is also conceivable that $A(x,y)$ could be a controlled function of frequency for a given antenna system. These two possibilities must be accounted for in any analysis for a fixed antenna system as well as in the design and development of new antenna systems.

With $k = 2\pi f/c$ and $1/\lambda = f/c$ the general expression for the far field strength becomes

$$E_{far}(\theta, \emptyset; f) \approx$$

$$\frac{jE_o f}{Rc} e^{-j2\pi fR/c} \iint_{aperture} dx\, dy\, A(x,y)\, e^{j2\pi f\left[\frac{\sin\theta}{c}(x\cos\emptyset + y\sin\emptyset) - \frac{\lambda(x,y)}{2\pi f}\right]} \qquad 20)$$

which can be an unwieldy integral to evaluate. Fortunately the greatly simplified cases of uniform amplitude and phase illumination for the rectangular and circular apertures suit the purposes of this study.

For a simple rectangular aperture with dimensions a and b with $|A(x,y;f)| = 1$ and $\lambda(x,y;f) = 0$, the far field strength becomes

$$E_{far}(\theta, \emptyset; f) \approx$$

$$\frac{jE_o f}{Rc} e^{-j2\pi fR/c} \left[\frac{\sin(\frac{\pi af}{c} \sin\theta \cos\emptyset)\, \sin(\frac{\pi bf}{c} \sin\theta \sin\emptyset)}{\frac{\pi af}{c}(\sin\theta \cos\emptyset)\frac{\pi bf}{c}(\sin\theta \sin\emptyset)}\right] \qquad 21)$$

while for the circular aperture of radius r, the far field is

$$E_{far}(\theta, \emptyset; f) \approx \frac{jE_o r}{R} e^{-j2\pi fR/c} \left[\frac{J_1(\frac{2\pi fr}{c} \sin \theta)}{\sin \theta} \right]. \qquad 22)$$

The appropriate interpretation of the far field quantities of equations 20, 21, and 22 above is critical to this study. All are actually sinusoidal field quantities occurring in response to sinusoidal excitation, or illumination, of the aperture. Even though the amplitude and phase of the aperture illumination may not be uniform, they may ultimately be referenced to a sinusoidal voltage or current source at some point in the antenna system which has a specific amplitude and phase. Considering only this reference source quantity and the far field quantity it is seen that the sinusoidal far field constitutes the response to the reference source: their relative amplitudes and phases, as functions of frequency, must be related by a transfer function. This transfer function, except for a numerical scale factor, is just the final expression for the far field strength as expressed by equations 21 and 22. In order to eliminate confusion of a scale factor we may consider the aperture field intensity E_o, at zero phase, as the excitation source yielding and antenna/far field transfer function of

$$E_{far}(\theta, \emptyset; f)/E_o = H_{ant}(\theta, \emptyset; f) \ . \qquad 23)$$

Of course, a similar expression may be derived for the magnetic field quantity B_{far} the magnitude of which is related to that of the electric far field by the intrinsic impedance of free space:

$$|B_{far}| = |E_{far}| / n_o = |E_{far}| / 120\pi \text{ .} \qquad 24)$$

Array Antennas

The general situation with the array antenna (the pattern of which may be approximated by the continuous aperture if the element spacing is less that $\frac{1}{2}\lambda$ and the number of elements is large) is complicated by the fact that the observation point field strength and associated radiation pattern are expressed as a finite summation of differently weighted and phased exponential terms.

Although the array antenna and its radiation pattern can be approximated by the continuously illuminated aperture, the fact that radiation of Walsh functions may require an array of small radiating elements necessitates consideration of the exact discrete approach. The general expression for the far field for an array of arbitrarily spaced,weighted, and phased radiating elements is

$$E_{far}(\theta,\ \phi;f) = \sum_{n=1}^{N} A_n\, e^{j\left[2\pi f(\bar{\rho}_n \cdot \frac{\bar{r}}{c} + \delta_n)\right]} \qquad 25)$$

where N is the number of elements, $\bar{r}$ a unit vector in the directions of θ and ϕ to the observation point, ρ_n the vector denoting the location of the n^{th} element with respect to the origin, A_n the relative amplitude due to the n^{th} radiator, and δ_n the delay of the n^{th} radiator. For a given set of isotropic radiators which do not interact, the above expression explicitly describes the frequency dependence and the directional properties since $\bar{\rho}_n \cdot \bar{r}$ is dependent only upon θ and ϕ of the observation point.

However, the quantities $\bar{\rho}_n$, A_n, and δ_n are usually not entirely arbitrary as they are usually selected to provide a prescribed radiation pattern.

The simple linear array of equispaced, equal amplitude, and uniformly phased elements suits our needs at this point.

The relative far field for this simple array is

$$E_{far}(\theta;f) \approx \frac{\sin(N\pi fS(\sin\theta - \sin\theta_o)/c)}{N\sin(\pi fS(\sin\theta - \sin\theta_o)/c)} \qquad 26)$$

(Skolnik, 1970, Chapter 11) where S is the element spacing, f the frequency, θ the direction of the observation point, θ_o a selected scan angle, c the speed of light in the medium, and N the number of elements.

Amplifiers and Filters

In order to be most effective the receiver of a radar based on Walsh functions must not only amplify the returned signal, but it must do so in some optimum manner in the presence of noise. It must also be able to select its own returned signal from those emanating from other nearby equipment operating on Walsh function carriers and to reject unwanted interference.

The problems of filtering and selectivity properly belong in the realm of Walsh filters which are not discussed in this study. However, the function of amplification will presumably be carried out by linear wide and/or narrow band (sinusoidal) amplifiers as presently done in radar practice. Filtering and signal processing would take place after the initial amplification phase.

Amplifying devices suitable to Walsh functions can be adequately described by their sensitivity and saturation characteristics and their linear characteristics of gain and phase as functions of frequency, their bandwidth, and center frequency. These are the same characteristics normally used to specify suitable pulse operation.

The linear characterization of the amplifier is described completely by its sinusoidal transfer function which can be obtained either from measurements or by detailed analysis of circuits and active devices comprising the amplifier.

The sensitivity of the amplifier depends on the amount of internal noise generated by the amplifier which is expressed by its noise figure. This quantity is also available by analysis or from measurement.

The saturation and nonlinear distortion properties of the amplifier depend primarily on the active devices used and the values of power supply voltages used. Although the nonlinear properties of the amplifier may generate sinusoidal components not present in the original input signal, it is very important to note that the bi-valued nature of the Walsh functions makes this property of the amplifier unimportant for amplification of a pure Walsh function waveform. For signals consisting of a superposition of several Walsh functions the nonlinear properties of the amplifier can only redistribute the signal energy among the Walsh components originally present: i.e., no extraneous Walsh components are generated.(Harmuth, 1972, pp 305-307).

Propagation Media

In the electromagnetic phase of the radar operation the radar energy propagates through the external environment essentially as a plane wave. The propagation medium, much like the transmission line and the waveguide, is characterized by its effect on the amplitude and phase of a sinusoidal plane wave as it progresses through the medium. The effect of the medium is manifested through a phase constant, k, which is analogous to the constant γ for the transmission line described earlier in this chapter.

In one dimension such a propagation constant gives rise to equations for the field strength at different locations along the propagation path identical to those for the transmission line. If the field at a given location z_1, due to a source wave at the origin, is

$$E(t;z_1) = A\ e^{j\omega t}\ e^{-jkz_1} \qquad 27)$$

and that at a second point z_2 is

$$E(t;z_2) = A\ e^{j\omega t}\ e^{-jkz_2}\ . \qquad 28)$$

The ratio

$$E(t;z_2)/E(t;z_1) = e^{-jk(\ z_2\ -\ z_1)} = e^{-jk\Delta z} \qquad 29)$$

is also a transfer function relating the amplitude and phase of the traveling-wave at point z_2 to those at an earlier point z_1. In this instance the disturbance at z_1 acts as the source, or cause, of the disturbance at z_2 even though that at z_1 is not actually a source. Notice also that since z_1 and z_2 are arbitrary the transfer function depends only upon the separation of the two points involved.

In order to characterize any propagation medium by its point-to-point transfer function all that is necessary is to determine the propagation constant k and form the quantity $e^{-jk\Delta z}$.

If the medium is lossy, k has an imaginary part leading to an exponential factor having a negative exponent giving rise to amplitude attenuation. The phase constant k is usually a function of frequency, the simplest being $k = \omega/c$ for a vacuum. For material media k is also expressible in terms of the index of refraction for the material:

$$k(\omega) = \omega\, n(\omega)/c \qquad 30)$$

where ω is the radian frequency of the sinusoidal excitation, c is the phase velocity in vacuum, and $n(\omega)$ is the index of refraction which, in general, is frequency dependent and also complex. If there are changes in its value along the path of propagation (in directions perpendicular to that of propagation) the direction of propagation will change.

Analogous to the transmission line case the medium is also described by an intrinsic impedance:

$$n = \sqrt{u/e} = E_i/H_j \qquad 31)$$

where u is the magnetic permeability of the medium, e its permittivity (both of which may be complex functions of frequency), and E_i and H_j are the electric and magnetic field strengths which are perpendicular to each other and to the direction of propagation.

Discontinuous changes in the impedance of the medium in the direction of propagation also give rise to reflection and transmission coefficients at the interface between two regions of different impedance:

$$\rho = E_{reflected}/E_{incident} = (n_2 - n_1)/(n_2 + n_1) \qquad 32)$$

and

$$T = E_{transmitted}/E_{incident} = 2n_2/(n_2 + n_1) \qquad 33)$$

which in general are complex functions of frequency and n_1 and n_2 are intrinsic impedances of the two media.

The propagation constant k of the medium through which sinusoidal electromagnetic energy is transmitted by a radar to detect and measure a target's distance, is a very important quantity since it determines signal attenuation due to the losses of the medium, and the phase and group velocities, which affect range accuracy.

For a tenuous plasma such as the terrestrial ionosphere k, neglecting the geomagnetic field, is (Ramo, Whinnery, and Van Duzer, 1965)

$$k(\omega) = \sqrt{ue_o(\omega^2 - \omega_p^2)} \qquad 34)$$

where ω_p is a frequency characteristic of the electron density of the plasma in which ionic motion has been neglected and $\omega_p^2 = 4\pi n_o e^2/m$ in which n_o is the number of free electrons per cubic centimeter, e is the electronic charge, and m is the electron mass. For radian frequencies greater than ω_p, k is real and the waves will propagate with no attenuation while, if the frequency is less than ω_p, k is imaginary giving a negative real exponent producing extreme damping of the wave in the plasma.

The propagation constant for a medium consisting of a lossy dielectric is

$$jk = j\omega\sqrt{u(e' - je'')} = \alpha + j\beta \qquad 35)$$

in which e' is the real part of the medium's permittivity and e" an imaginary part which accounts for the dielectric losses. In general both depend on frequency. The terms α and β are found to be

$$\alpha = \omega\sqrt{\tfrac{1}{2}ue'(\sqrt{1 + e''^2/e'^2} - 1)} \qquad 36)$$

$$\beta = \omega\sqrt{\tfrac{1}{2}ue'(\sqrt{1 + e''^2/e'^2} + 1)} \;. \qquad 37)$$

For a low loss material where e" is much smaller than e'

$$\alpha \approx \omega\sqrt{ue'}\ (e''/2e') \qquad 38)$$

$$\beta \approx \omega\sqrt{ue'}\ (1 + (e''/e')^2/8) \qquad 39)$$

while the intrinsic impedance is

$$n = \sqrt{u/(e'(1 - je''/e'))}$$

$$\approx \sqrt{u/e'}\ \ (1 - \frac{3}{8}(e''/e')^2 + je''/2e')\ . \qquad 40)$$

The intrinsic impedance of any material medium is also a function of frequency which may also be interpreted as a form of transfer function. It relates the amplitudes and phases of the E and H fields at a given point in the medium. Because of this frequency dependent transfer function relationship nonsinusoidal E and H fields will not be of the same time variation at a given point in the medium.

Of course, any nonsinusoidal radar application involving a planetary ionosphere must consider the planet's magnetic field if it is of significant value. This case is of much greater complexity than that for a plasma in which the geomagnetic field is negligible.

The phenomena manifested in the terrestrial ionosphere in the presence of the pervading geomagnetic field are of obvious importance in modern radar applications in which moving targets above or within the ionosphere must be detected and tracked by surface based radar equipment. Radar mapping of the Earth and neighboring planets by radar platforms orbiting the planets under scrutiny must also consider the effects of the same phenomena.

A phenomenon of great importance in the use of linearly polarized radar waves is the Faraday effect by which the plane of polarization may be rotated thereby reducing antenna sensitivity. However, of greater importance than Faraday rotation is the possibility of waveform distortion resulting from synergistic effects of the plasma and the magnetic field which is investigated in greater detail in Chapter VIII. Below the propagation functions for this case are developed.

Using the geometry shown in Figure 3 for the most general propagation arrangement with the geomagnetic field and the propagation conditions as indicated, a general solution for the propagation function from Kraus (1966) is

$$\beta = \omega\sqrt{\tfrac{1}{2}u_o / \left[(e_{11} - e_{33})\sin^2\phi + e_{33}\right]} \quad x$$
$$\Big[(e_{11}^2 - e_{12}^2 - e_{11}e_{33})\sin^2\phi + 2\, e_{11}e_{33} \pm$$
$$((e_{11}^2 - e_{12}^2 - e_{11}e_{33})^2 \sin^4\phi + 4e_{12}^2 e_{33}^2 \cos^2\phi)^{\frac{1}{2}}\Big]^{\frac{1}{2}} \qquad 41)$$

where ϕ is the angle between the direction of propagation and the static magnetic field $\bar{B}_o$ and the e_{ij} are the components of the tensor permittivity corresponding to the geometry shown. The tensor permittivity is

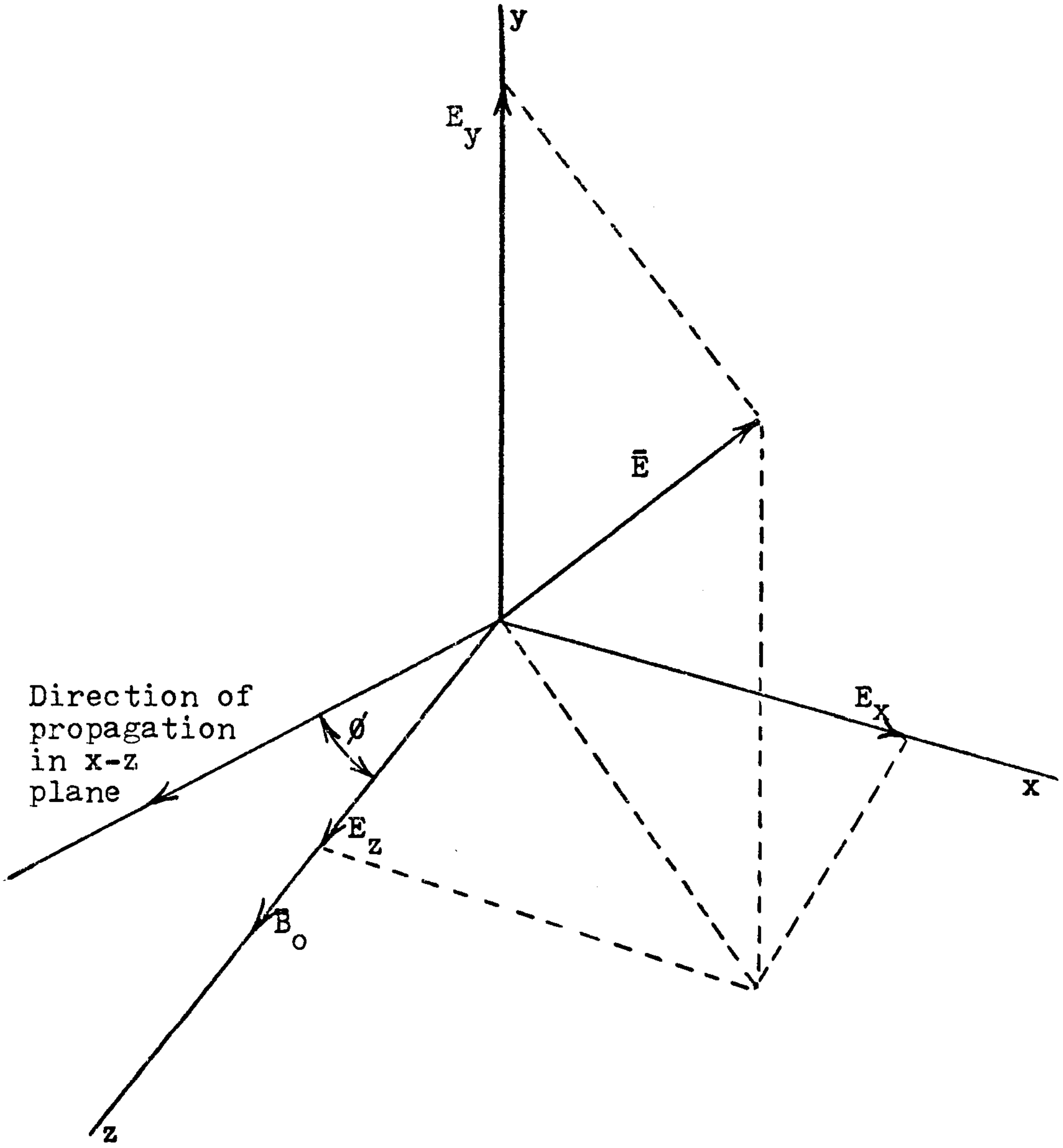

Figure 3. Geometry of electromagnetic wave traversing a plasma in a static magnetic field.

$$\bar{e} = \begin{bmatrix} e_{11} & -je_{12} & e_{13} \\ je_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} \qquad 42)$$

with

$$e_{11} = e_{22} = e_o(1 + \omega_p^2/(\omega_c^2 - \omega^2)) \qquad 43)$$

$$e_{12} = - e_{21} = e_o\omega_p^2\omega_c/\omega\ (\omega_c^2 - \omega^2) \qquad 44)$$

$$e_{33} = e_o(1 - \omega_p^2/\omega^2) \qquad 45)$$

$$e_{13} = e_{23} = e_{31} = e_{32} = 0. \qquad 46)$$

In the above quantities $\omega_p^2 = Ne^2/e_om$ = plasma frequency, e_o = permittivity of free space, and ω_c = $e\ B_z/m$ = cyclotron frequency. In both equations e = the electron charge and m its mass.

A case of interest for radar applications which provides some mathematical tractability when the direction of propagation of a linearly polarized wave is parallel, or nearly so, to the static magnetic field such that $\phi \approx 0$ making $E_z \approx 0$. This is the quasi-longitudinal case with

$$\beta = \sqrt{u_oe_o}\sqrt{\frac{(\omega^3 - \omega(\omega_p^2 + \omega_c^2) \pm \omega_p^2\omega_c \cos\phi)\omega}{(\omega + \omega_c)(\omega - \omega_c)}} \qquad 47)$$

For the purely longitudinal case with $\phi = 0$, dividing the numerator under the radical by $(\omega + \omega_c)$ and then by $(\omega - \omega_c)$ yields

$$\beta = \sqrt{u_o e_o} \sqrt{\omega(\omega^2 \pm \omega\omega_c - \omega_p^2)/(\omega \pm \omega_c)} \;. \qquad 48)$$

Both equations above are double valued due to the plus and minus signs. The interpretation of two values of β means that the wave solution contains two solutions with each having a different value of β, one with the positive sign and the other with the negative sign.

It can be shown (Ramo, Whinnery, and Van Duzer, 1965, p. 516) that the only wave solutions satisfying Maxwell's equations for the geometry shown in Figure 3 are circularly polarized waves each with oppositely sensed rotation: one corresponds to the positive sign of β while the other corresponds to the negative sign.

A linearly polarized wave in which we are interested may be considered as the superposition of two circularly polarized waves of opposite sense of rotation. The clockwise rotating component corresponds to the positive sign in β which yields

$$\bar{E}_+ = E_o\ (\bar{a}_x - j\bar{a}_y)\ e^{-j\beta_+ z} \qquad 49)$$

and, for the negative sign,

$$\bar{E}_- = E_o\ (\bar{a}_x + j\bar{a}_y)\ e^{-j\beta_- z} \qquad 50)$$

where E_o is the magnitude of the rotating field vector. The linearly polarized wave in which we are interested is then the linear superposition of two such rotating waves:

$$\bar{E}_{1in} = \tfrac{1}{2}E_o(\bar{a}_x - j\bar{a}_y)\, e^{-j\beta_+ z} + \tfrac{1}{2}E_o(\bar{a}_x + j\bar{a}_y)e^{-j\beta_- z} . \qquad 51)$$

The effect of the plasma and the magnetic field on the time domain unit impulse response is obtained, in principle, by extracting the inverse Laplace transforms of $e^{-j\beta_+ z}$ and $e^{-j\beta_- z}$ and making the appropriate vector operations in the time domain.

A second case of some interest is that of transverse propagation inwhich the direction of propagation is perpendicular to that of the magnetic field. In this situation the electric field vector may have components both parallel and perpendicular to the magnetic field. In the parallel case the effect is as if no external magnetic field were present with β the same as equation 34. The perpendicular component corresponding to $\phi \approx 90^\circ$ reduces to

$$\beta = \omega\sqrt{u_o}\sqrt{e_{11} - (1 + 1)\, e_{12}^2 \sin \phi \,/2e_{11}} \qquad 52)$$

which is known as the extraordinary wave of the quasi-transverse case which reduces to

$$\beta = \omega\sqrt{u_o}\sqrt{e_{11} - e_{12}^2 \sin \phi/e_{11}} \;. \qquad 53)$$

The purely transverse case with $\phi = 90^\circ$ reduces to

$$\beta = \omega\sqrt{u_o}\sqrt{e_{11} - e_{12}^2/e_{11}} \;. \qquad 54)$$

In terms of frequency, the plasma frequency, and the cyclotron frequency equations 53 and 54 become

$$\beta = \sqrt{u_o e_o} \quad X$$

$$\left[\frac{\omega^6 - 2\omega^4(\omega_c^2 + \omega_p^2) + \omega^2(\omega_c^2 + \omega_p^2)^2 - \omega_c^2\omega_p^4 \sin \emptyset}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_c^2 - \omega_p^2)}\right]^{\frac{1}{2}} \qquad 55)$$

for the quasi-transverse case while for the purely transverse case we have

$$\beta = \sqrt{u_o e_o} \quad X$$

$$\left[\frac{\omega^6 - 2\omega^4(\omega_c^2 + \omega_p^2) + \omega^2(\omega_c^2 + \omega_p^2)^2 - \omega_c^2\omega_p^4}{(\omega^2 - \omega_c^2)(\omega^2 - \omega_c^2 - \omega_p^2)}\right]^{\frac{1}{2}} . \qquad 56)$$

A final case of particular interest in terrestrial radar applications is the propagation factor describing the Earth's atmosphere. For the frequencies of interest the atmosphere is essentially nondispersive which also applies to weather phenomena such as precipitation and clouds which merely attenuates any electromagnetic sine-wave traversing through the medium. The phase factor is closely approximated by ω/c. However, the attenuation factor is a complicated function of frequency due to absorption by atmospheric oxygen and water-vapor. Since the quantities of oxygen and water-vapor are sensitive to atmospheric pressure and temperature as well as the absolute humidity, they are all dependent on altitude.

Bean, Dutton, and Warner (1970, p. 24- 13) have presented a summary of atmospheric attenuation factors based upon a combination of theory and empirical data. When converted to nepers per meter for a standard atmosphere[3] the oxygen attenuation factor is approximately

$$\gamma_o \approx 39.08 \times 10^{-6} \omega^2 \left[\frac{0.018}{\omega^2 + (0.34 \times 10^{10})^2} + \frac{0.05}{(\omega + 37.7 \times 10^{10})^2 + (0.94 \times 10^{10})^2} + \frac{0.05}{(\omega - 37.7 \times 10^{10})^2 + (0.94 \times 10^{10})^2} \right] \qquad 57)$$

while that for water-vapor absorption is approximately

$$\gamma_w \approx 0.403 \, \rho \, \omega^2 \times 10^{-6} \, (1 + 0.0046 \, \rho) \times$$

$$\left[\frac{0.0906}{(\omega + 13.9 \times 10^{10})^2 + (1.71 \times 10^{10})^2 (1 + 0.0046 \, \rho)^2} + \frac{0.0906}{(\omega - 13.9 \times 10^{10})^2 + (1.71 \times 10^{10})^2 (1 + 0.0046 \, \rho)^2} \right]. \qquad 58)$$

A third attenuation factor which accounts for losses

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3) Standard conditions: P = 1013.25 mb, T = 293° K. Equations 57 and 58 have been corrected to these conditions while equation 57 is independent of humidity ρ.

above a frequency of 22 GHz is

$$\gamma = 12.75\ \rho \omega^2 \times 10^{-26}\ (1 + 0.0046\ \rho)\ . \qquad 59)$$

In equations 58 and 59 ρ is the absolute humidity in gm/m^3.

The total attenuation factor is then the sum of the above three factors:

$$\gamma_t = \gamma_o + \gamma_w + \gamma\ . \qquad 60)$$

The transfer function corresponding to the phase factor ω/c and the attenuation factor γ_t is

$$H(\omega) = e^{-(\gamma_o + \gamma_w + \gamma)z}\ e^{-j\omega z/c}\ . \qquad 61)$$

It will be useful in Chapter VIII when working with the above attenuation factors to realize that they consist of the four following basic forms:

$$e^{A\omega^2/(\omega^2 + B^2)},\quad e^{C\omega^2/((\omega \pm D)^2 + E^2)},\quad e^{F\omega^2} \qquad 62)$$

in which the alphabetical coefficients are constants. As before the phase factor $e^{-j\omega z/c}$ transforms into the time domain as a time delay of z/c seconds and may be ignored in the formal inverse transform operations.

The expressions for γ_o, γ_w, and γ may easily be modified for other than standard conditions by incorporating corrective constants C_o, C_w, C, C_1, and C_2 for differing values of temperature, pressure, and humidity. They are

$$
\begin{aligned}
C_o &= (293/T)^2\ P/1013.25 \\
C_w &= (293/T)^{5/2}\ e^{-644/T} \\
C &= (293/T) \\
C_1 &= (293/T)^{3/4}\ (P/1013.25)(1 + 0.0046\ \rho) \\
C_2 &= (293/T)^{\frac{1}{2}}\ (P/1013.25)(1 + 0.0046\ \rho) \\
C_3 &= (293/T)^{3/4}\ (P/1013.25)
\end{aligned}
\qquad 63)
$$

where T is the temperature in degrees Kelvin, P is the pressure in millibars, and ρ is the absolute humidity in grams per cubic meter.

The above corrective constants are incorporated into equations 57, 58, and 59 as follows:

$$
\gamma_o \approx C_oC_1 x\ 39.08 \times 10^{-6}\omega^2 \left[\frac{0.018}{\omega^2 + (0.34C_1 x\ 10^{10})^2} + \frac{0.05}{(\omega + 37.7C_3 \times 10^{10})^2 + (0.94C_1 \times 10^{10})^2} + \frac{0.05}{(\omega - 37.7C_3 \times 10^{10})^2 + (0.94C_1 \times 10^{10})^2}\right], \qquad 64)
$$

$$
\gamma_w \approx C_wC_2 \times 3.66\ \rho \times 10^{-6}\omega^2 \times
$$

$$
\left[\frac{0.0906}{(\omega - 13.9 \times 10^{10})^2 + (1.71C_2 \times 10^{10})^2} + \right.
$$

$$\left.\frac{0.0906}{(\omega + 13.9 \times 10^{10})^2 + (1.71C_2 \times 10^{10})^2}\right] \qquad 65)$$

and

$$\gamma = C\ C_2 \times 12.75\ \rho\,\omega^2 \times 10^{-26}\ . \qquad 66)$$

The ideas and concepts developed in this chapter are utilized in Part III in the study of specific elements and components essential to the implementation of a Walsh function carrier radar.

Chapter VII Component and System Descriptions for Walsh Wave Operation

From much of the foregoing it is apparent that direct time domain analysis of antenna and electromagnetic propagation problems for nonsinusoidal excitation would require extensive new and difficult analytical effort. Such effort, however, would necessarily ignore the great wealth of theory and engineering data that exist for the special case of sinusoidal excitation. In order to avoid the former problem and yet take advantage of this existing knowledge, sinusoidal performance concepts from the disciplines of communication theory or systems theory are utilized in this study to characterize radiation systems, transmission systems, and propagation media in a manner compatible with the bi-valued functions under study.

The Unit Impulse Response

It was shown in Chapter VI that functions describing the electric and magnetic fields of radiation structures and transmission structures as well as those of propagation media, may be interpreted as steady-state sinusoidal transfer functions. If valid over a sufficiently wide band of (sinusoidal) frequencies these transfer functions may be transformed by use of the inverse Laplace or inverse Fourier transforms to yield time domain unit impulse responses which are functions of location as well as of time and other pertinent parameters of the system. This form of system description, when valid, can be more useful with the discontinuous type of functions under consideration than is the steady-state transfer function form of system description.

The resulting unit impulse response could be used in the superposition integral with the excitation waveform to directly yield the field or voltage and current quantities of interest. Or, if desired, the product of the system transfer function and the excitation transform could be inverted to yield the time domain response to a given excitation. However, for the general Walsh function, or any other bi-valued function, either approach is difficult. For our purposes, neither operation is necessary to aquire the information needed. The important factor in the potential use of discrete bi-valued functions is the "spreading" of the signal at its transitions caused by the finite duration of the unit impulse response of the system. The nature of a bi-valued function is such that only its transitions (i.e., its zero crossings) and/or its impulse-like time derivatives or its integrals are of concern since it is the spreading of these phenomena which limits the rate at which the transitions may be generated.

The increases in the rise and fall times of the transitions and the increase in the pulse width of the time derivatives of the transitions caused by the system are of primary concern rather than the detailed structure of the resulting waveform. Hence, for bi-valued waveforms, the unit impulse response is a more useful form of system description than that of its sinusoidal steady-state performance description.

Since our interest is primarily in the distortion that a system or component imparts to a bi-valued waveform, it would be convenient if the terms, or factors, in the impulse response of the system causing the distortion could be isolated in order to see which system parameters cause it. If this could be done it is feasible that action could be taken to reduce the distortion. In several cases, as shown in Part III, quite often the transfer functions of many of the systems and components examined may be separated into several additive terms one of which is often a constant term. This constant term, when transformed to the time domain, represents an ideal impulse which reproduces a component of the signal having no distortion. The remaining terms in this additive decomposition of the transfer function (or equivalently, the impulse response) represent distortion terms. If the effect of these distortion terms are negligible compared to that of the ideal impulse term, or if their temporal durations are very small compared to the smallest time interval of interest in the excitation waveform, their effects may be ignored. This method of decomposing the unit impulse response in order to isolate the distortion terms is utilized on several of the systems and components examined in Part III.

Waveform Parameters

Since the duration of the unit impulse response, or that of the distortion terms that result when it is decomposed, is so important in determining the distortion caused by a system, it is important to define the term "duration" more precisely, if possible. Because of the variety of waveform shapes that are possible it is obvious that any criterion selected to specify the duration of a pulse-like waveform (of finite duration) may not be applicable to all situations or waveforms. For example, the duration of a signal could arbitrarily be defined as that time interval containing a prespecified amount of the total area under the waveform curve or a prespecified amount of the energy contained in the waveform. Another definition that might be suitable is that time interval over which the magnitude of the waveform exceeds some prespecified value.

Although many such definitions are possible and would surely be of value, they require that the detailed temporal structure of the waveform be known or be determined from the readily available transfer function. With some of the transfer functions encountered in Part III such evaluation of the unit impulse responses can be very difficult. Quite often the resulting impulse responses found in Part III are very complicated functions of time from which evaluation of the above mentioned measures of signal duration could also be very difficult.

Another useful measure of waveform duration is possible which, in certain cases, may be calculated directly from its Laplace or Fourier transforms. This measure is the root-mean-square duration which is the square root of the second time moment of the waveform about its own average time which is also its first temporal moment about the origin ($t = 0$). These average quantities are defined quantitatively in following paragraphs by equations 1 and 2. The root-mean-square duration is preferable for this study since the investigation of a system or component starts most often with the Laplace or Fourier transform of the waveforms involved.

Fortunately these mean value time quantities are easily determined from the system transfer function without resorting to time domain considerations. It is also shown (at the end of this section) that a measure of the increase in rise and fall times resulting with pulse excitation can be estimated very easily from the root-mean-square duration of the unit impulse response.

Borrowing techniques from probability theory[4] the mean value of the temporal duration of a waveform $h(t)$ and its

---

4) Since many of the waveforms considered may assume positive and negative values, the theory developed departs somewhat from probability theory. The analogy is exact for waveforms of only one polarity.

mean-square duration (or variance) about its mean value are defined by the following expressions:

$$t_{ave} = \bar{t} = t_o = \frac{\int_0^\infty t\ h(t)\ dt}{\int_0^\infty h(t)\ dt} \qquad 1)$$

and

$$\sigma^2 = \frac{\int_0^\infty (t - t_o)^2\ h(t)\ dt}{\int_0^\infty h(t)\ dt} = \overline{t^2} - t_o^2 \ . \qquad 2)$$

In the above equations h(t), when divided by the area under its time graph (i.e., the integral in the above denominators) is analogous to a probability density function which allows a valid and suitable means of computing the above moments.

In principle it is possible to substitute the impulse response as determined from the system transfer function into the above integrals to obtain the indicated averaged quantities. However, the resulting impulse responses are usually difficult functions to integrate or manipulate mathematically. Fortunately Laplace transform theory allows evaluation of the above quantities in many instances from the original transfer function or from the terms in its additive decomposition. For instance, the integral in the denominators of equations 1 and 2 is a normalizing constant which may be evaluated from the system transfer function as follows:

$$L\left[h(t)\right] = H(S) = \int_{0}^{\infty} h(t)\ e^{-St} dt \quad . \qquad 3)$$

Taking the limit as S approaches zero yields

$$\lim_{S \to 0} H(S) = \int_{0}^{\infty} h(t)\ dt \qquad 4)$$

if such a limit exists. Hence, the normalizing constant is equal to the original system transfer function in the limit as S approaches zero.

In a similar manner the temporal moments of h(t) may be evaluated from related transform domain quantities. Again taking the Laplace transform of h(t), or using the original system transfer function, differentiating with respect to S and taking the limit as S approaches zero yields t_0:

$$\frac{dH}{dS} = \frac{d}{dS} \int_{0}^{\infty} h(t)\ e^{-St}\ dt = - \int_{0}^{\infty} t\ h(t)\ e^{-St}\ dt \qquad 5)$$

and as S approaches zero equation 5 becomes

$$\lim_{S \to 0} \frac{dH}{dS} = - \int_{0}^{\infty} t\ h(t)\ dt = - t_0\ H(0) \qquad 6)$$

or

$$t_0 = - \lim_{S \to 0} \frac{1}{H(S)} \quad \frac{dH}{dS} \quad . \qquad 7)$$

Differentiating H(S) again with respect to S and letting S approach zero yields the mean-square moment of time as follows:

$$\frac{d^2H}{dS^2} = \int_0^\infty t^2\, h(t)\, e^{-St}\, dt \qquad 8)$$

and letting S approach zero

$$\lim_{S\to 0} \frac{d^2H}{dS^2} = \int_0^\infty t^2\, h(t)\, dt = \overline{t^2}\, H(0) \qquad 9)$$

or

$$\overline{t^2} = \lim_{S\to 0} \frac{1}{H(S)} \frac{d^2H}{dS^2} \quad . \qquad 10)$$

Using equation 2 the variance, or mean-square duration, is:

$$\sigma^2 = \overline{t^2} - t_o^2 = \lim_{S\to 0} \left[\frac{1}{H(S)} \frac{d^2H}{dS^2} - \left[\frac{1}{H(S)} \frac{dH}{dS}\right]^2 \right]. \qquad 11)$$

Some simple algebraic manipulation and repeated differentiation of the system transfer function yields the following general form for the higher temporal moments of the unit impulse response of the system:

$$\overline{t^n} = \lim_{S\to 0} \frac{(-1)^n}{H(S)} \frac{d^nH}{dS^n} \quad . \qquad 12a)$$

In Part III the simple relationships of equations 2 through 11 are used in several cases to evaluate these important waveform parameters.

It is useful to note that similar techniques with the

Fourier transform form of the system transfer function yields a similar expression for the higher temporal moments of a waveform:

$$\overline{t^n} = \lim_{\omega \to 0} \frac{(-j)^n}{H(\omega)} \frac{d^n H}{d\omega^n} \quad . \qquad 12b)$$

Methods similar to those above may be applied to the instantaneous power of a waveform to yield temporal moments of that form of signal expression.

The conditions under which differentiation of the definite integrals of equations 5, 7, 12a, and 12b (with respect to S) is valid must be specified. Normally, in order that the Laplace transform of h(t) exist, it is sufficient that h(t) be piecewise continuous and of exponential order; i.e., be Laplace transformable. Under these conditions the Laplace transform integral is uniformly convergent and may be differentiated under the integral sign. However, in this situation we are starting with a known Laplace transform of h(t) which requires that the transfer functions with which we are dealing be analytic in their regions of convergence.

Impulse Response Effects on Rise and Fall Times

An upper bound on the increase in rise and fall times of a unit step or linear ramp excitation which is followed by a unit step may easily be found in terms of the variance of the duration of the system unit impulse response. That for an ideal unit step excitation is found first followed by that for a ramp function of arbitrary duration T followed by a unit step of infinite duration.

The unit step response of a system is merely the time integral of its unit impulse response, or the area under the unit impulse response as a function of time. From equation 2 the variance of the duration of the impulse response is

$$\sigma^2 = \lim_{S \to 0} \frac{\int_0^{\infty} (t - t_o)^2 \; h(t)\, dt}{H(S)} \quad . \qquad 13)$$

If now a small arbitrary portion of the area under the unit impulse response curve $k\sigma$ units on either side of t_o is rejected, the following inequality is obtained:

$$\sigma^2 \geqslant \lim_{S \to 0} \frac{1}{H(S)} \left[\int_0^{t_o - k\sigma} (t - t_o)^2 \; h(t)\, dt + \int_{t_o + k\sigma}^{\infty} (t - t_o)^2 \; h(t)\, dt \right] \quad . \qquad 14)$$

The smallest magnitude that $(t - t_o)$ can assume in equation 14 is $k\sigma$ so that substitution of this constant value into the integrand will create an even stronger inequality:

$$\sigma^2 \geqslant \frac{k^2\sigma^2}{H(0)} \left[\int_0^{t_0 - k\sigma} h(t)\,dt + \int_{t_0 + k\sigma}^{\infty} h(t)\,dt \right] . \qquad 15)$$

The integrals in the brackets constitute the area under the unit impulse response curve outside the interval of ± kσ centered on t_0. This area may also be viewed as (1 - area within the region bounded by $t_0 \pm k\sigma$). The area within that bounded region also corresponds to the change in the integral from $t_0 - k\sigma$ to $t_0 + k\sigma$ which in turn corresponds to the change in the unit step response over the same interval of time. We may then select any symmetrical portion of the change in the amplitude of the unit step response as a criterion to define its rise time. The reference points usually selected are the 10% and 90% levels on the waveform which will also be used in this study. This change in amplitude corresponds to an overall change of 80%, or equivalently, 80% of the area in the center of the impulse response curve. This change in amplitude results in the following inequality:

$$\sigma^2 \geqslant k^2\sigma^2 (1 - 0.8) = k^2\sigma^2 \times 0.2 . \qquad 16)$$

Cancelling σ^2 on either side of the inequality and solving for k^2 yields

$$k^2 \leqslant 1/0.2 = 5.0 \qquad 17)$$

or

$$k \leqslant 2.236 \quad . \qquad 18)$$

The meaning of equation 18 is that the 10% to 90% rise time of the unit step response of the system takes place within $\pm$ 2.236 standard deviations of the system unit impulse response from t_o. Hence,

$$10\% \text{ to } 90\% \text{ rise time} \leqslant 4.472\sigma \quad . \qquad 19)$$

The response of a system to a ramp of finite duration followed by a unit step of infinite duration, as depicted in Figure 4a, may easily be determined from the system's response to a short pulse of arbitrary duration T shown in part b of the same figure.

The transform domain response of the pulse r(t) (of Figure 4b) is

$$G_1(S) = R(S)\ H(S) = \frac{1}{TS}(1 - e^{-ST})\ H(S) \qquad 20)$$

while the response corresponding to the ramp is

$$G_2(S) = \frac{R(S)\ H(S)}{S} \quad . \qquad 21)$$

A simple modification of equation 21 yields

$$G_2(S) = \frac{1}{S}\ R(S)\ H(S) \qquad 22)$$

in which R(S) H(S) corresponds to an effective transfer

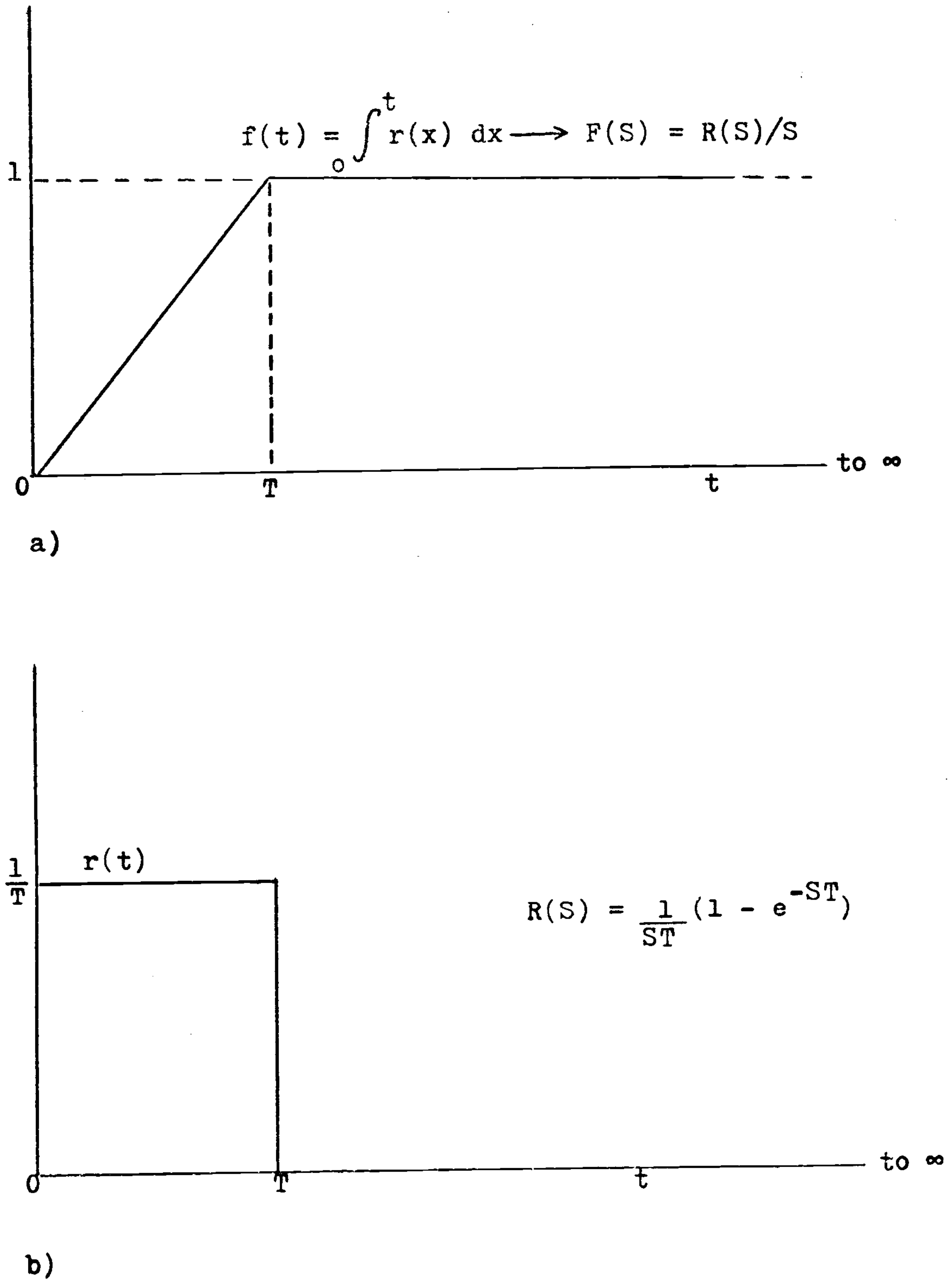

Figure 4. Ramp and pulse excitation waveforms.

function and 1/S represents a unit step excitation. R(S) H(S) may then be used as the effective transfer function in equation 11 which establishes an effective mean-square duration σ^2 for a fictitious unit impulse response corresponding to R(S) H(S). This value of variance may now be used in equation 19 to determine the 10% to 90% rise time which is equal to the overall rise time in response to the linear ramp excitation of Figure 4a.

Since antenna and transmission systems often consist of cascaded elements it is of interest to determine the effects of component interaction. Recall that in the transfer function domain that the cascading of elements corresponds to multiplying the individual transfer functions of the elements involved.

The principles inherent in equations 1, 2, 3, 4, 7, 10, and 11 may be applied to the product of two transfer functions, $H_1(S)$ $H_2(S)$. By using some simple algebraic manipulations it can be shown (Appendix III) that the overall mean-square duration of the cascaded impulse responses is

$$\sigma_t^2 = \sigma_1^2 + \sigma_2^2 \quad . \qquad 23)$$

From equation 23 it is seen that the overall duration cannot be less than the sum of the individual variances of the individual impulse responses.

Another item of possible value in the analyses to follow is the cumulative effects of adding, or superimposing, unit impulse responses that overlap in time. It is desired to have this effect expressed in terms of the temporal moments of the individual impulse responses. As above, the principles inherent in equations 1, 2, 3, 4, 7, 10, and 11 may be applied to a superposition of unit impulse responses to yield the following expression for the overall variance of the combined signal duration:

$$\sigma^2 = \frac{1}{H_t(0)} \sum_{i=1}^{n} \left[H_i(0) \left[\sigma_i^2 + \bar{t}_i^2 (1 - H_i(0)/H_t(0)) \right] \right] - \sum^{n} \sum^{n}_{i \neq j} \frac{\frac{dH_i}{dS} \frac{dH_j}{dS}}{H_t(0)} \qquad 24)$$

where $H_t(0)$ is the sum of the individual transforms evaluated as S⟶0; $H_i(0)$ the individual transforms also evaluated as S⟶0; σ_i^2 the mean-square duration of each impulse response; and $\bar{t}_i^2$ the square of the mean-time of each impulse response. Of course, many other variations of this expression may be obtained by further algebraic manipulation. Derivation of equation 24 appears in Appendix IV.

A last topic of importance is evaluation of t_o (or $\bar{t}$)

and σ^2 of unit impulse responses for the special class of transfer functions which consist of pure exponential functions of the following form:

$$H(S) = A\ e^{F(S)} \tag{25}$$

where A is a real or complex constant. The first and second derivatives of the above form of transfer function are

$$\frac{dH}{dS} = A\ e^{F(S)}\ \frac{dF}{dS} = H(S)\ \frac{dF}{dS} \tag{26}$$

and

$$\begin{aligned}\frac{d^2H}{dS^2} &= A\ e^{F(S)} \left[\frac{dF}{dS}\right]^2 + A\ e^{F(S)}\ \frac{d^2F}{dS^2} \\ &= H(S) \left[\frac{d^2F}{dS^2} + \left(\frac{dF}{dS}\right)^2\right].\end{aligned} \tag{27}$$

The values of t_o and σ^2 from equations 7 and 11, in terms of the above derivatives of H(S) are

$$t_o = \lim_{S \longrightarrow 0} - \frac{dF}{dS} \tag{28}$$

and

$$\sigma^2 = \lim_{S \longrightarrow 0} \frac{d^2F}{dS^2} . \tag{29}$$

Equations 28 and 29 are used in Chapter VIII to estimate unit impulse duration for atmospheric and for ionospheric propagation media which have no easily obtained inverse Laplace transforms.

An Example of the Proposed Technique

The techniques described in this chapter are applied in the following chapters to several elements and components of importance not only to conventional radar, but especially to those of probable importance to a non-sinusoidal radar.

At this point in the development, however, it is felt that an example of the transfer function/unit impulse response technique applied to the short Hertzian dipole would be useful. This example will not only illustrate the method, but will also lend credence to it when compared to the results of Harmuth's (1972) direct transient analysis.

The instantaneous electric and magnetic fields associated with the short dipole undergoing steady-state sinusoidal excitation in standard spherical coordinates and in conventional complex notation (Ramo, Whinnery, Van Duzer, 1965) and in which the complex time factor $e^{j\omega t}$ is omitted, and with the amplitude of the driving current set to unity, are shown below:

$$H_\phi(\omega) = \frac{h \sin \theta}{4\pi} \left[\frac{j\beta}{r} + \frac{1}{r^2} \right] e^{-j\beta r} , \qquad 30)$$

$$E_r(\omega) = \frac{h \cos \theta}{4\pi} \left[\frac{2n}{r^2} - \frac{2j}{\omega e r^3} \right] e^{-j\beta r} , \qquad 31)$$

and

$$E_\theta(\omega) = \frac{h \sin \theta}{4\pi} \left[\frac{j\omega u}{r} + \frac{n}{r^2} - \frac{j}{\omega e r^3}\right] e^{-j\beta r}. \qquad 32)$$

In the above set of equations h is the dipole length, θ the colatitude angle of the spherical coordinate system, r the distance from the center of the antenna to the field point, ω the operating frequency, e the permittivity and u the permeability of the medium, and n its intrinsic impedance, and $\beta = \omega/c$, with c the speed of light in the medium.

Equations 30, 31, and 32 are valid only over a range of frequencies in which the length h is very small compared to the shortest wavelength that will be present in any excitation current. The condition leading to this limitation is the fact that the antenna element is so short that the instantaneous current along its length is uniform and changing in time in unison. This situation requires that the time needed for a disturbance at the driving terminals to propagate to the ends of the antenna and back is a very small portion of the shortest time interval of interest in the excitation current. Quantitatively, for a period of 50 pico seconds, which appears to be the present lower limit on switching times, the above limitation corresponds to a dipole length of $h << 2cT = 600 \times 10^6 \times 50 \times 10^{-12} = 3$ cm. This fact means that practical antenna elements would be limited to lengths of a fraction of a centimeter or less which could be difficult to achieve mechanically. However, when considered as an ideal element of current or current density of infinitesimal dimensions, equations 30 through 32 are applicable since the length of the element can always be made small enough to meet the above inequality. In this abstract case, the inverse transform, which is integrated from $-\infty$ to $+\infty$ over the frequency variable will be valid.

In terms of the transfer functions discussed earlier in Chapter VI , the field quantities of equations 30, 31, and 32 may be considered as such functions. They relate the sinusoidal amplitudes and phases of these field quantities at an observation point to those of a sinusoidal current exciting the dipole element.

Note also the exponential factor $e^{-j\beta r}$. The β factor in the exponent is a function of frequency describing the medium surrounding the antenna element. For a lossless, nondispersive medium $\beta = \omega/c$. This is the case considered in this example while the dispersive case is considered in Part III. The effects of the medium are further reflected in equations 31 and 32 through the intrinsic impedance of the medium n, the permittivity e, and the permeability u.

It is also important to notice that extraction of the inverse Fourier or Laplace transforms of equations 30, 31, and 32 in no way affect the directional nature of these field quantities. For a lossless and nondispersive medium (i.e., $\beta = \omega/c$ and n is a constant) the only direct involvement of the distance r in the inverse transform is in the exponential factor $e^{j\omega r/c}$. This factor transforms to the time domain as a simple time retardation (t - r/c) of the wave due to the finite speed of the radiation.

Because the systems under consideration are causal their impulse responses must vanish for negative time, hence their Fourier and Laplace transforms are identical with S substituted for $j\omega$. The extensive tables of Laplace transforms may then be used to evaluate the unit impulse responses desired.

Equations 30, 31, and 32 may be written in terms of the Laplace transform variable S. For the nondispersive case with $\beta = \omega/c$ they become

$$H_{\phi}(S) = \frac{h \sin \theta}{4\pi} \left[\frac{S}{rc} + \frac{1}{r^2} \right] e^{-Sr/c} \tag{33}$$

$$E_{r}(S) = \frac{h \cos \theta}{4\pi} \left[\frac{2n}{r^2} + \frac{2}{Ser^3} \right] e^{-Sr/c} \tag{34}$$

$$E_{\theta}(S) = \frac{h \sin \theta}{4\pi} \left[\frac{Su}{r} + \frac{n}{r^2} + \frac{1}{Ser^3} \right] e^{-Sr/c} \tag{35}$$

Realizing that the exponential factors merely represent a time retardation of -r/c in the time domain, it is only necessary to look up the quantites in brackets in a suitable table of Laplace transforms. This act yields, for a unit impulse of current driving the dipole, the following:

$$h_{\phi}(t') = \frac{h \sin \theta}{4\pi} \left[\frac{\delta'(t')}{rc} + \frac{\delta(t')}{r^2}\right] \qquad 36)$$

$$e_r(t') = \frac{h \cos \theta}{4\pi} \left[\frac{2n\ \delta(t')}{r^2} + \frac{2\ u(t')}{er^3}\right] \qquad 37)$$

$$e_\theta(t') = \frac{h \sin \theta}{4\pi} \left[\frac{u\ \delta'(t')}{r} + \frac{n\ \delta(t')}{r^2} + \frac{u(t')}{er^3}\right] \qquad 38)$$

where $\delta'(t')$ indicates the time derivative of the unit impulse $\delta(t')$, t' = retarded time argument = $t - r/c$, and $u(t')$ is the unit step function.

The response of the short dipole to any arbitrary current excitation is provided by the superposition, or convolution integral:

$$\int_{-\infty}^{\infty} i(x)\ h(t' - x)\ dx = \int_{-\infty}^{\infty} i(t' - x)\ h(x)\ dx \qquad 39)$$

where $i(t')$ is the arbitrary excitation current and $h(t')$ represents $h_{\phi}(t')$, $e_r(t')$, or $e_\theta(t')$.

The limits of integration on the above superposition integral may be modified to fit a causal system with the excitation current starting at $t' = 0$:

$$\int_{0}^{t'} i(x)\ h(t' - x)\ dx = \int_{0}^{t'} i(t' - x)\ h(x)\ dx\ . \qquad 40)$$

Recall that t' is the retarded time variable $t - r/c$.

This extremely simple case is readily evaluated by the use of the following relationships:

$$\int_{-\infty}^{\infty} i(x)\ \delta(x - a)\ dx = \int_{-\infty}^{\infty} i(x)\ \delta(a - x)\ dx = i(a),\ a>0, \qquad 41)$$

$$\int_{-\infty}^{\infty} i(x)\ \delta'(x - a)\ dx = \int_{-\infty}^{\infty} i(x)\ \delta'(a - x)\ dx = - \frac{di}{dx}\bigg|_{x=a}, \qquad 42)$$

and

$$\int_{0}^{t'} i(x)\ u(t' - x)\ dx = \int_{0}^{t'} i(x)\ dx\ . \qquad 43)$$

The field quantities due to arbitrary current excitation become

$$H_{\phi}(t') = \frac{h \sin \theta}{4\pi} \left[\frac{-1}{rc} \frac{di}{dt'} + \frac{i(t')}{r^2}\right] \qquad 44)$$

$$E_r(t') = \frac{h \cos \theta}{4\pi} \left[\frac{2n}{r^2} i(t') + \frac{2}{er^3} \int_{0}^{t'} i(x)\ dx\right] \qquad 45)$$

$$E_{\theta}(t') = \frac{h \sin \theta}{4\pi} \left[\frac{u}{r} \frac{di}{dt'} + \frac{n}{r^2} i(t') + \frac{1}{er^3} \int_{0}^{t'} i(x)\ dx\right]. \qquad 46)$$

With slight manipulation equations 44, 45, and 46 may be shown to be identical to the field quantities derived by Harmuth (1972). Notice also that the dependence upon θ is identical to that for sinusoidal excitation and that it is independent of the functional time variation of the excitation current.

The presence of field components inversely proportional to the distance r, its square, r^2, and its cube, r^3 indicates the generation of near, intermediate, and far zones. It is of great importance also to notice the relative absolute magnitudes between the near and far zone components (1/r versus $1/r^2$ or $1/r^3$) depends also upon the instantaneous values of the excitation current, its time derivative, and its time integral. Hence, it is seen that for arbitrary time variation, division of the space around the dipole element into near, intermediate, and far zones depends on the temporal partitioning of the waveform and its related time derivatives and integrals. This fact is in contrast to the frequency partitioning criterion when sinusoidal excitation is used. The functional forms of the field quantities resulting from a current having the time variation of a Walsh function, one of its time derivatives, or its time integral (or those of similar type bi-valued waveforms) are then easily obtained by use of equations 44, 45, and 46.

From the practical view the short dipole has little value except as an elemental portion, or building block, for antenna structures of finite dimensions. The preceding analysis has value in that the equivalence of its result to that of Harmuth's (1972) direct analysis lends credence to the method proposed.

Chapter VIII Propagation Characteristics of Nonsinusoidal Waves in Physical Media

A very important consideration in the application of a nonsinusoidal carrier wave in radar is the effect of the terrestrial atmospheric environment on its propagation. Two general categories of media must be considered in Earth-bound radar applications: the lower, unionized regions[5]; and the upper ionosphere. The lower atmosphere, although not ionized and essentially nondispersive in the range of electronically generated (sinusoidal) frequencies, does cause varying degrees of attenuation depending on frequency, local pressure, temperature, and humidity. Because of the extreme mathematical difficulty the impulse response of the lower atmosphere is not determined in this study. Rather, the techniques of Chapter VII are exploited to estimate its root-mean-square duration.

The upper ionized regions of the atmosphere present varying dispersive conditions which depend upon the direction of propagation with respect to the direction and strength of the local geomagnetic field, and the operating frequency. Because of the extreme mathematical difficulty the general case with the geomagnetic field is also not considered here. The simplest case offerring mathematical ease is that of the ionospheric plasma with the geomagnetic

5) Below approximately 50 kilometers.

field weak enough to have negligible effect. The propagation constant corresponding to this case is that of equation 34 of Chapter VI:

$$j\,k(S) = \frac{1}{c_o}\sqrt{S^2 + \omega_p^2} \qquad 1)$$

in which c_o is the free space speed of light and ω_p is the plasma radian frequency of the free electrons in the plasma[6]. In the MKS units $\omega_p^2 \approx 3{,}200\ N$ in which N is the density of the electrons per cubic meter.

The main interest here is in how the electric and magnetic fields change as the wave progresses through the medium from one point to another. In the sinusoidal steady-state case this effect is described by

$$\frac{E_2(S)}{E_1(S)} = \frac{H_2(S)}{H_1(S)} = e^{-jkz} = e^{-z\sqrt{S^2 + \omega_p^2}\,/c_o} \quad . \qquad 2)$$

The direct inverse Laplace transform (Roberts and Kaufman, 1966, Item 48, p. 251) yields for the unit impulse response

$$h(t) = \delta(t - z/c_o) - \frac{z}{c_o}\,\omega_p^2\, u(t - z/c_o)\,\frac{J_1(\omega_p\sqrt{t^2 - z^2/c_o^2})}{\omega_p\sqrt{t^2 - z^2/c_o^2}} \, . \qquad 3)$$

Fortunately equation 3 decomposes in a manner which isolates the distortion terms from the ideal impulse term.

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[6] The motion of the positive ions has been neglected.

This ideal term is a unit impulse delayed by the speed of light (in vacuum) between any two points separated by a distance z. It reproduces a nondistorted component of the wave. The second term is a subtractive distortion component accompanying the disturbance and, to a first approximation, it is proportional to the separation z and to the square of the plasma frequency.

A measure of the distortion caused by the second term of equation 3 above is given by its total area as compared to the unit area of the first ideal impulse term. The area of the second term is

$$\text{Area} = \frac{z}{c_o}\omega_p^2 \int_{z/c_o}^{\infty} \frac{J_1(\omega_p\sqrt{t^2 - z^2/c_o^2})\,dt}{\omega_p\sqrt{t^2 - z^2/c_o^2}} \qquad 4)$$

or, in terms of $T = \sqrt{t^2 - z^2/c_o^2}$, the area is

$$\text{Area} = \frac{z}{c_o}\omega_p \int_{o}^{\infty} \frac{J_1(\omega_p T)\,dT}{\sqrt{T^2 - z^2/c_o^2}} \quad . \qquad 5)$$

Equation 5 above is a standard form (Abramowitz and Stegun, 1965, item 11.4.48, p. 488) which yields

$$\text{area} = \frac{z}{c_o}\omega_p \; I_{\frac{1}{2}}(\tfrac{1}{2}z\,\omega_p/c_o)\; K_{\frac{1}{2}}(\tfrac{1}{2}z\,\omega_p/c_o) = 1 - e^{-z\,\omega_p/c_o} \qquad 6)$$

where $I_{\frac{1}{2}}(x) = (2/\pi x)^{\frac{1}{2}} \sinh(x)$ and $K_{\frac{1}{2}}(x) = (\pi/2x)^{\frac{1}{2}} e^{-x}$ are the modified Bessel functions of half order of the first and second kinds respectively.

Depending on the distortion criterion selected the quantity $z\omega_p/c_o$ must be such that the right-hand side of equation 6 be much less than unity. For example, if it is desired that the distortion terms contribute no more than 10% area to the progressing wave, then

$$1 - e^{-z\omega_p/c_o} \leqslant 0.1 \longrightarrow z \leqslant 0.1\, c_o/\omega_p \qquad 7)$$

or

$$z \leqslant 5.0 \times 10^5/N^{\frac{1}{2}} \;. \qquad 8)$$

For typical ionospheric electron densitites of 10^{10} electrons per cubic meter the maximum allowable separation would be

$$z \leqslant 5.0 \times 10^5/10^5 \approx 5 \text{ meters} \qquad 9)$$

which shows that low-distortion propagation of a very short pulse or a sudden step through typical electron densities will be limited to very short distances.

It is also important to realize that at a given single point in the plasma that propagating electric and magnetic fields do not have the same time variation because they are related by the frequency dependent intrinsic impedance of the medium. In Laplace operational form we have

$$\frac{H(S)}{E(S)} = \frac{1}{n} = \frac{1}{n_o}\sqrt{S^2 + \omega_p^2}\,/S \;. \qquad 10)$$

which yields for the unit impulse response relationship between the instantaneous values of the two fields (Roberts and Kaufman, 1966, item 85, p. 215)

$$h(t) = \frac{1}{n_o} \left[\delta(t) + \omega_p \, u(t) \int_o^t \frac{J_1(\omega_p x) \, dx}{x} \right] . \qquad 11)$$

Equation 11 shows clearly that there is a distortion term in this situation also. The area of the integral term has a maximum value of unity (Abramowitz and Stegun, 1965, item 9.1.27, p. 361), hence, in order that the distortion term have small effect the plasma frequency ω_p (or equivalently, N) must be small.

It is also of interest to notice that after a time interval of $4/\omega_p$ (the first zero of $J_1(x)$ is approximately 4) the integral is essentially unity. This has the effect of producing a distortion component of the magnetic field proportional to the time integral of the electric field.

Also of interest in airborne or space borne radar applications is the effect of immersing a short dipole antenna element in a plasma or in the terrestrial ionosphere. Referring to equations 30 and 32 of Chapter VII the steady-state sinusoidal far fields in spherical coordinates are

$$\bar{H} = H_\phi \, \bar{a}_\phi = \bar{a}_\phi \, \frac{h \sin \theta}{4\pi r} \, jk \, e^{-jkr} \qquad 12)$$

$$\bar{E} = E_\theta \, \bar{a}_\theta = \bar{a}_\theta \, \frac{uh \sin \theta}{4\pi r} \, j \, \omega \, e^{-jkr} \, . \qquad 13)$$

Only the frequency sensitive portions of equations 12 and 13 need be considered. For a plasma jk(S) is given by

$$jk(S) = \frac{1}{c_o} \sqrt{S^2 + \omega_p^2} \tag{14)}$$

so that

$$H_\phi(S) = \frac{1}{c_o} \sqrt{S^2 + \omega_p^2} \quad e^{-r \sqrt{S^2 + \omega_p^2}/c_o} \tag{15)}$$

and

$$E_\theta(S) = S \, e^{-r \sqrt{S^2 + \omega_p^2}/c_o} \tag{16)}$$

The inverse transforms of the above expressions are (Roberts and Kaufman, 1966, item 48, p. 251 and item 39, p. 172)

$$\begin{aligned} e_\theta(t) = {} & \delta'(t - r/c_o) \\ & + \frac{\omega_p^4 r}{c_o} \, t \, \frac{J_2(\omega_p \sqrt{t^2 - r^2/c_o^2})}{\omega_p^2 \sqrt{t^2 - r^2/c_o^2}} \, u(t - r/c_o) \\ & - \frac{1}{2} \frac{\omega_p^2 r}{c_o} \, \delta(t - r/c_o) \end{aligned} \tag{17)}$$

and

$$\begin{aligned} h_\phi(t) = {} & \frac{1}{c_o} \delta'(t - r/c_o) \\ & + \frac{u(t - r/c_o)\, \omega_p^2}{c_o} \int_\sigma^t \delta'(t - r/c_o) \; T \, \frac{J_1(\omega_p \sqrt{t^2 - T^2}) dT}{\omega_p \sqrt{t^2 - T^2}} \end{aligned} \tag{18)}$$

or

$$h_{\phi}(t) = \frac{1}{c_o}\,\delta'(t - r/c_o)$$

$$- \frac{\omega_p^2}{c_o}\left[\frac{J_1(\omega_p \sqrt{t^2 - r^2/c_o^2})}{\omega_p \sqrt{t^2 - r^2/c_o^2}}\right.$$

$$\left. + \frac{r\omega_p^2}{c_o}\,\frac{J_2(\omega_p \sqrt{t^2 - r^2/c_o^2})}{\omega_p^2\,\left[t^2 - r^2/c_o^2\right]}\right] u(t - r/c_o) \qquad 19)$$

Fortunately equations 17 and 19 separate the distortion terms from the ideal wave. The first terms each represent undistorted propagation of the first time derivative of the excitation source current as expected while the remaining terms represent distortion terms propagating with the original disturbance.

The distortion terms are all seen to increase with the distance from the antenna and to increase sharply with plasma frequency ω_p. In order to have little effect on the propagating waves, the time derivative of the excitation current must be such as to make the term containing the derivative of the unit impulse the dominant quantity when equations 17 and 19 are convolved with the excitation current waveform to obtain the resultant fields. It is obvious that satisfactory performance depends upon the temporal form of the excitation current as well as upon the characteristics of the plasma medium.

Lower Atmosphere Impulse Response

The propagation transfer function of the lower atmosphere is discussed in the latter portion of Chapter VI and is given by an exponential expression similar to equation 29 of that chapter:

$$H(\omega) = e^{-z\gamma(\omega)} \qquad 20)$$

where $\gamma(\omega)$ consists of a component due to oxygen absorption, one due to water vapor absorption, and one due to high frequency losses giving

$$H(\omega) = e^{-z(\gamma_o(\omega) + \gamma_w(\omega) + \gamma\ (\omega))} \qquad 21a)$$

or, equivalently

$$H(\omega) = e^{-z\gamma_o(\omega)} \; e^{-z\gamma_w(\omega)} \; e^{-z\gamma(\omega)} \, . \qquad 21b)$$

The quantities $\gamma_o(\omega)$, $\gamma_w(\omega)$, and $\gamma(\omega)$ are all complicated functions of frequency, pressure, temperature, and absolute humidity so that any detailed analysis of a given situation must include these atmospheric parameters along the path of propagation.

Unfortunately the three exponential factors above are too complicated to easily invert either individually or as a single exponential factor. However, we may still estimate the root-mean-square duration of the unit impulse response due to each factor. From these quantities we may then obtain the overall impulse response root-mean-square duration by equation 23 of Chapter VII. Equation 29 of that chapter provides the mean-square duration, σ^2, contributed by each exponential factor in equation 21 above. Note that $\gamma_o(\omega)$ and $\gamma_w(\omega)$ further decompose into additive components yielding

$$\gamma_o(\omega) = \gamma_{o1} + \gamma_{o2} + \gamma_{o3} \qquad 22a)$$

$$\gamma_w(\omega) = \gamma_{w1} + \gamma_{w2} \quad . \qquad 22b)$$

Use of equations 57, 58, and 59 of Chapter VI then gives for each component, each of which has been corrected to a temperature of 293° K, an atmospheric pressure of 1013.25 millibars, and an absolute humidity of 7.5 grams per cubic meter, in operational form

$$\gamma_{o1} = 0.781 \times 10^{-7} \left[\frac{S^2}{S^2 - (0.3517\text{x}10^{10})^2}\right] \qquad 23a)$$

$$\gamma_{o2} = 0.217 \times 10^{-6} \left[\frac{S^2}{(S + j\ 39.0 \times 10^{10})^2 - (0.972\text{x}10^{10})^2}\right] \qquad 23b)$$

$$\gamma_{o3} = 0.217 \times 10^{-6} \left[\frac{S^2}{(S - j\ 39.0 \times 10^{10})^2 - (0.972\text{x}10^{10})^2}\right] \qquad 23c)$$

$$\gamma_{w1} = 0.286 \times 10^{-6} \left[\frac{S^2}{(S - j\ 13.9 \times 10^{10})^2 - (1.79\text{x}10^{10})^2}\right] \qquad 23d)$$

$$\gamma_{w2} = 0.286 \times 10^{-6} \left[\frac{S^2}{(S + j\ 13.9 \times 10^{10})^2 - (1.79 \times 10^{10})^2} \right]$$

23e)

$$\gamma = - 98.92 \times 10^{-26} \times S^2 \ .$$ 23f)

Differentiating the above set of equations to obtain the second derivatives as required by equation 29 of Chapter VII yields

$$\frac{d^2\gamma_{o1}}{dS^2} = 5.796 \times 10^{12} \left[\frac{S^2 + (0.203 \times 10^{10})^2}{\left[S^2 - (0.351 \times 10^{10})^2\right]^3} \right]$$ 24a)

$$\frac{d^2\gamma_{o2}}{dS^2} = -j\ 5.281 \times 10^5 \times$$

$$\left[\frac{S^3 + j\ 5.854 \times 10^{11}\ S^2 + j\ 2.969 \times 10^{34}}{\left[(S + j\ 39.0 \times 10^{10})^2 - (0.972 \times 10^{10})^2\right]^3} \right]$$ 24b)

$$\frac{d^2\gamma_{o3}}{dS^2} = +j\ 5.281 \times 10^5 \times$$

$$\left[\frac{S^3 - j\ 5.854 \times 10^{11}\ S^2 - j\ 2.969 \times 10^{34}}{\left[(S - j\ 39.0 \times 10^{10})^2 - (0.972 \times 10^{10})^2\right]^3} \right]$$ 24c)

$$\frac{d^2\gamma_{w1}}{dS^2} = +j\ 1.590 \times 10^5 \times$$

$$\left[\frac{S^3 - j\ 21.19 \times 10^{10}\ S^2 - j\ 1.388 \times 10^{33}}{\left[(S - j\ 13.9 \times 10^{10})^2 - (1.79 \times 10^{10})^2\right]^3} \right]$$ 24d)

$$\frac{d^2\gamma_{w2}}{dS^2} = -j\ 1.590 \times 10^5 \times$$

$$\left[\frac{S^3 + j\ 21.19 \times 10^{10}\ S^2 + j\ 1.388 \times 10^{33}}{\left[(S + j\ 13.9 \times 10^{10})^2 - (1.79 \times 10^{10})^2\right]^3}\right] \qquad 24e)$$

$$\frac{d^2\gamma}{dS^2} = -\ 1.980 \times 10^{-24}\ . \qquad 24f)$$

As S approaches zero the six second derivatives above become

$$\lim_{S \to 0} \frac{d^2\gamma_{o1}}{dS^2} = -\ 126.2 \times 10^{-28}$$

$$\lim_{S \to 0} \frac{d^2\gamma_{o2}}{dS^2} = -\ 0.0445 \times 10^{-28}$$

$$\lim_{S \to 0} \frac{d^2\gamma_{o3}}{dS^2} = -\ 0.0445 \times 10^{-28}$$

$$\lim_{S \to 0} \frac{d^2\gamma_{w1}}{dS^2} = -\ 0.2913 \times 10^{-28}$$

$$\lim_{S \to 0} \frac{d^2\gamma_{w2}}{dS^2} = -\ 0.2913 \times 10^{-28}$$

$$\lim_{S \to 0} \frac{d^2\gamma}{dS^2} = -\ 19\ 800 \times 10^{-28}\ .$$

The mean-square durations due to each of the above factors are

$$\sigma^2_{o1} = 126.2 \times 10^{-28}\ z\ \text{seconds}^2 \qquad 25a)$$

$$\sigma^2_{o2} = 0.0445 \times 10^{-28} \text{ z seconds}^2 \qquad 25b)$$

$$\sigma^2_{o3} = 0.0445 \times 10^{-28} \text{ z seconds}^2 \qquad 25c)$$

$$\sigma^2_{w1} = 0.2913 \times 10^{-28} \text{ z seconds}^2 \qquad 25d)$$

$$\sigma^2_{w2} = 0.2913 \times 10^{-28} \text{ z seconds}^2 \qquad 25e)$$

$$\sigma^2_{\gamma} = 19\ 800 \times 10^{-28} \text{ z seconds}^2 . \qquad 25f)$$

The total mean-square duration by equation 23 of Chapter VII is the sum of the quantities in equations 25 above:

$$\sigma^2_{total} = 19\ 926.9 \times 10^{-28} \text{ z seconds}^2 \qquad 26)$$

or

$$\sigma_{total} = 141.16 \times 10^{-14} z^{\frac{1}{2}} \text{ seconds}$$

$$= 1.4116\ z^{\frac{1}{2}} \text{ picoseconds.} \qquad 27)$$

Two important facts are to be concluded from equations 25 and 26 above. First, the root-mean-square duration of the response of the atmospheric medium to an ideal impulse increases directly with the square root of the distance z as the wave propagates through the medium. This fact puts a limit on the maximum useful distance, z. If twenty percent of the pulse width, T_w, of a propagating Walsh wave is selected as the maximum permissible value of σ_{total} below which suitable operation takes place, then

$$z_{max} \approx (0.2)^2 \; T_w^2 \,/(1.4116)^2 = 0.02 \; T_w^2 \qquad 28)$$

where T_w is in picoseconds. As an example, for $T_w = 50$ picoseconds,

$$z_{max} \approx 0.02 \times 50^2 = 50 \text{ meters} \qquad 29a)$$

while for $T_m = 10^{-9}$ seconds yields

$$z_{max} \approx 0.02 \times 10^6 = 20\ 000 \text{ meters.} \qquad 29b)$$

Equations 29 mean that operation of a Walsh wave radar in the terrestrial atmosphere must be limited to sequencies for which the increase of rise and fall times are negligible portions of pulse duration T_w for the range performance required of a particular radar application.

The second important observation, which comes from equation 25, is that the major contribution to increases in rise and fall times comes from the σ_γ term. Contributions from the other terms may safely be ignored.

From equations 63, 64, 65, and 66 of Chapter VI it is evident that the magnitude of σ_{total} depends upon atmospheric pressure, temperature, and absolute humidity. At higher altitudes σ_γ remains the dominant contribution to σ_{total}. With correction factors C and C_2, the value of σ_{total}, to a good approximation, becomes

$$\sigma^2_{total} \approx \frac{1.98\ z\ (293/T)^{3/2}\ P\ (1 + 0.0046\varrho)}{1013.25 \times 10^{24}} \qquad 30)$$

or

$$\sigma_{total} \approx \frac{1.4116\ z^{\frac{1}{2}}}{10^{12}} \left[\frac{(293/T)^{3/2}\ P(1 + 0.0046\varrho)}{1013.25}\right]^{\frac{1}{2}} \qquad 31)$$

If temperature, pressure, and absolute humidity vary with position along the propagation path equation 31 would have to be applied and integrated over the entire path with these three quantities expressed as functions of location, or z.

The above discussion indicates that, since σ_{γ} is the dominant contributor to waveform distortion, the operational form of the atmospheric medium transfer function is essentially

$$H(S) \approx e^{0.989\ z\ S^2 \times 10^{-24}} \quad . \qquad 32)$$

In the frequency domain of the Fourier transform it becomes, with $S = j\omega$,

$$H(\omega) \approx e^{-0.989\ z\,\omega^2 \times 10^{-24}} \quad . \qquad 33)$$

The inverse Fourier transform of equation 33 (Thomas, 1969), including the propagation delay z/c_0, yields a good approximation to the atmospheric unit impulse response:

$$h(t) \approx \frac{8.04 \times 10^{22}}{z} e^{-2.53 \times 10^{23}(t - z/c_o)^2/z}. \qquad 34)$$

The form of the above impulse response is that of a delayed Gaussian waveform which stretches out and decreases in peak amplitude as it propagates into the medium.

Chapter IX Transmission Structures

A very important category of radar system element, either for sinusoidal or nonsinusoidal operation, is that consisting of the two-conductor transmission line and the hollow, metallic waveguide. It is anticipated that these elements will be required or useful in the implementation of a nonsinusoidal radar much as they are in conventional radar: i.e., to transmit energy or signals between generating elements and transmitting elements (antennas), or to signal processing elements within the radar system.

The specific transmission elements examined here are the coaxial cable, microwave stripline, and the rectangular and circular closed waveguides. The microwave stripline is especially important since it also constitutes the primary means of interconnecting the individual solid state devices comprising integrated circuits. Such integrated circuits surely will find extensive use in the implementation of a nonsinusoidal (Walsh) radar.

In all cases only the traveling-wave transmission characteristics of these devices are considered. The termination problem and reflections are not treated here as it is felt by the writer that distortion imparted by the device to a signal as it propagates along the line or guide is the more important problem to investigate in this study.

Transmission Lines

In this section general relationships for transmission lines are developed and later specialized to the different types under study.

The expressions relating sinusoidal voltage or current at two different points along a transmission line and separated by a distance z in the operational form of the Laplace transform are

$$\frac{E_2}{E_1} = e^{-z\gamma}, \qquad \frac{I_2}{I_1} = e^{-z\gamma} \qquad 1)$$

with $\gamma = (ZY)^{\frac{1}{2}}$ where Z = series impedance of the line per unit length and Y = shunt admittance per unit length. These impedance and admittance quantities are in turn expressible in terms of the geometry of the line and the electrical properties of the materials used to construct the line.

In general the impedance and admittance functions in complex and operational forms are:

$$Z(\omega) = r + j\omega L, \quad Z(S) = r + SL \qquad 2)$$

$$Y(\omega) = g + j\omega C, \quad Y(S) = g + SC \qquad 3)$$

where r, L, g, and C are all functions of geometry, material characteristics, and operating frequency. In addition, in a realizable system Z(S) and Y(S) must be analytic functions of S (Murakami and Corrington, 1948).

The voltage and current at any location along the line are related by

$$E(z,S) = I(z,S)\ Z_o(S) \qquad 4)$$

where $Z_o(S)$ is the characteristic impedance of the line which is a function independent of z for a uniform line:

$$Z_o(S) = (Z/Y)^{\frac{1}{2}}\ . \qquad 5)$$

The conductance and susceptance quantities comprising the line admittance Y may be expressed in terms of the properties of the dielectric material separating the conductors making up the line:

$$g = \sigma_d\ F_g\ , \quad C = e'\ F_c\ . \qquad 6)$$

In equation 6 above the quantity σ_d is the loss factor, or conductivity of the dielectric usually expressed as

$$\sigma_d = \omega e''\ . \qquad 7)$$

The quantities e' and e" are the real and imaginary parts of the permittivity of the dielectric material in complex form:

$$e = e' - j\ e''\ . \qquad 8)$$

In general e' and e" are frequency dependent and are related by the Hilbert transform (Ramo, Whinnery, and Van

Duzer, 1965). The quantities F_g and F_c are dimensionless quantities entirely dependent upon the line geometry and, in the International System of units,

$$F_g = F_c \quad . \qquad 9)$$

The series impedance of the line is similar with the exception that the geometric factors of the resistive and inductive terms are not equal. The resistance due to the bulk conductivity of the conductors and the external inductance due to the flux linkages existing between the conductors are both augmented by the skin effect. This effect adds frequency dependent components. The general form of the line series impedance is then

$$z = r + z_{sk} + SL \qquad 10)$$

where z_{sk}= high frequency asymptote of the skin effect impedance,

$$z_{sk} = F_{sk} \, (Su/\sigma_c)^{\frac{1}{2}} \qquad 11)$$

where u is the permeability of the conductor material, σ_c is its conductivity, and F_{sk} is a skin effect geometric factor.

The inductance term is

$$L = u \, F_L \qquad 12)$$

where F_L is a dimensionless geometric factor which, in the

International System of units is the reciprocal of F_c: $F_L = 1/F_c$. The resistive term in equation 10 is a static d.c. resistance which would be included in z_{sk} if that impedance term were exact. In approximate terms

$$r = F_r/\sigma_c \qquad 13)$$

where F_r is also a geometrical factor. However, for the band of frequencies over which Walsh function pulse trains are likely to be used the r term will be insignificant. Then

$$z \approx z_{sk} + SL = F_{sk}\, S^{\frac{1}{2}}(u/\sigma_c)^{\frac{1}{2}} + SL\ . \qquad 14)$$

Also, for the purposes of this study the permeability of all materials is assumed to be that of free space, or u_o.

Before proceeding further it is necessary to examine the properties of the conductor and insulating materials normally used in the construction of transmission lines.

The conductivity of conductors normally used falls in the range of 3.8×10^7 to 6.8×10^7 mhos per meter with that for copper being 5.8×10^7. Conductivity is also frequency dependent, its general frequency dependence being given by Kittel (1968) as

$$\sigma_c(S) = \sigma_o/(1 + S/v_c) \qquad 15)$$

in which σ_o is the low frequency, or d.c. value of the conductance and v_c is the collision frequency[7] of the electrons within the conductor material. For the family of conductor materials used this frequency varies from 2.40×10^{13} radians per second for silver to 4.106×10^{13} radians per second for copper. These quantities and the plasma frequency, ω_p, are tabulated in Table III for commonly used metallic conducting materials. The values of plasma frequencies are from Sze (1969) while the collision frequencies were calculated using methods discussed in Appendix V.

Taking copper as the conducting material most likely to be used, for a collision frequency of 4.106×10^{13} radians per second, little effect will be encountered for operating frequencies up to one tenth of that value. This condition corresponds to a useable frequency limit of about 653 GHz. If the shortest switching time to be expected in operational systems is 50 picoseconds, the highest frequency component to be found in such a wavefront is approximately $1/50\mathrm{x}10^{-12}$ seconds $\approx$ 20 GHz. This value is well below the 653 GHz limit at which the conductivity of copper is no longer purely real. For the purposes of this study, then, frequency dependence of the conductivity of

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7) As used here, v_c is the collision frequency which is the reciprocal of the relaxation time τ as used by Kittel. A plus sign is also used for S/v_c since in engineering practice $\exp(j\omega t)$ is used rather than $\exp(-j\omega t)$ as used in physics.

Table III. Conductance and Collision Frequencies of Typical Transmission Line Conducting Materials.

Materials	Conductivity, σ_o mhos/meter	Collision Frequency, v_c, radians/second	Plasma Frequency, ω_p, radians/second
Copper	5.80×10^7	4.106×10^{13}	1.64×10^{16}
Gold	4.15×10^7	4.063×10^{13}	1.38×10^{16}
Silver	6.80×10^7	2.444×10^{13}	1.37×10^{16}
Aluminum	3.80×10^7	13.42×10^{13}	2.40×10^{16}

typical metallic materials used (see Table III) may safely be ignored. Also, in order to reduce computing effort, a value of 6.0 x 10^7 mhos per meter is used for the conductivity of copper in subsequent analyses.

Past measurements of the dielectric constants and loss tangents of several commonly used dielectric materials have been summarized recently by Breeden and Sheppard (1967) and Balanis (1971). These investigators also presented their own findings at frequencies of 60 GHz., 71 GHz, 90 GHz., and in the 250 to 450 GHz. band. Measurements by earlier investigators range from 10 GHz. to 1000 GHz. These data are summarized in Tables IV and V. In Table IV it is apparent that, up to a frequency of 25 GHz., the relative dielectric constants e'/e_0 for all the materials shown are constant, while the changes occurring up to 450 GHz. are all less than four percent.

The loss tangents, although varying much more over the entire frequency range shown, change very little over the 20 GHz. band expected for a 50 picosecond rise time. In addition to their small variations, their values are all less than 10^{-3}. Because of the small value of e'' in comparision with e', it will be ignored in the following discussions.

Using equations 3, 12, and 14 the product of ZY becomes

Table IV. Relative Dielectric Constant, $e'/e_o = k'$. Measurements at 71 GHz. and in the 250 to 450 GHz. band due to Breeden and Sheppard.

Material	Frequency, GHz.								
	10	25	70	71	139	343	250-450	890	% of change up to 450 GHz.
Fiberglas				4.38			4.34		-0.9
Polyethylene	2.25	2.24		2.28		2.31	2.27		3.1
Polystyrene	2.54	2.54	2.53	2.57	2.53	2.57	2.48		-3.6
Rexolite	2.54			2.58		2.54	2.52		-2.3
Teflon	2.08	2.08	2.10	2.10	2.07	2.07	1.99	1.94	-2.5

Table V. Loss Tangent, 10^3 x e"/e' for Typical Transmission Line Dielectric Materials

Material	Frequency, GHz.						
	10	25	70	139	400	600	1000
Teflon	0.37	0.60	< 2	< 2	1.0	2.0	2.0
Rexolite	0.47				1.0	3.0	5.0
Polystyrene	0.30	0.53	0.90	2 to 4	3.0	5.0	7.0

$$ZY = (S^{\frac{1}{2}} F_{sk}\sqrt{u_o/\sigma_c} + SL)(-jSe'' F_g + Se' F_c) \qquad 16)$$

$$= (Su_oF_L + S^{\frac{1}{2}}F_{sk}\sqrt{u_o/\sigma_c}\ ) SF_ce' (1 - je''/e') \qquad 17)$$

$$\approx S(S + \frac{S^{\frac{1}{2}}F_{sk}}{F_L}\sqrt{1/u_o\sigma_c}) F_cF_L e' u_o . \qquad 18)$$

Recalling that $F_c = 1/F_L$ the above yields

$$ZY \approx S(S + \frac{S^{\frac{1}{2}} F_{sk}}{F_L}\sqrt{1/u_o\sigma_c}\ ) e' u_o \qquad 18a)$$

and

$$e' u_o = 1/c^2 = k'/c_o^2$$

where c is the speed of light in the dielectric medium while c_o is the speed of light in free space and k' is the relative dielectric constant of the medium. Then

$$\gamma = \sqrt{ZY} \approx \sqrt{e'u_o}\ \ S^{\frac{1}{2}}\sqrt{S + S^{\frac{1}{2}} F_{sk} /(F_L\sqrt{u_o\sigma_c})} \qquad 19)$$

$$\gamma \approx \frac{1}{c_o}(k')^{\frac{1}{2}} S^{\frac{1}{2}}\sqrt{S + S^{\frac{1}{2}} F_{sk} /(F_L\sqrt{u_o\sigma_c})} \qquad 20)$$

$$\gamma \approx \frac{1}{c_o}(k')^{\frac{1}{2}} S\sqrt{1 + F_{sk} /(S^{\frac{1}{2}}F_L\sqrt{u_o\sigma_c}\ )} \qquad 21)$$

and

$$Z_o = \sqrt{Z/Y} \approx \sqrt{\frac{S^{\frac{1}{2}} F_{sk}\sqrt{u_o/\sigma_c} + S\ u_oF_L}{F_c\ e'\ S}} \qquad 22)$$

$$\approx \sqrt{u_oF_L/ F_c\ e'}\sqrt{1 + u_o^{\frac{1}{2}} F_{sk}/(u_oF_LS^{\frac{1}{2}}\ \sigma_c^{\frac{1}{2}})} . \qquad 23)$$

But $F_L = 1/F_c$ so that

$$Z_o \approx F_L\sqrt{u_o/e'}\sqrt{1 + F_{sk}/(S^{\frac{1}{2}} F_L\sqrt{u_o\sigma_c})} \qquad 24)$$

and $\sqrt{u_o/e'} = n'$, the intrinsic impedance of the dielectric medium. Therefore,

$$Z_o \approx F_L\, n'\sqrt{1 + F_{sk}/(S^{\frac{1}{2}} F_L\sqrt{u_o\sigma_c})}\ . \qquad 25)$$

$$Z_o \approx F_L\, n_o\sqrt{1/k'}\sqrt{1 + F_{sk}/(S^{\frac{1}{2}} F_L\sqrt{u_o\sigma_c})}. \qquad 26)$$

Expressions for the quantities F_{sk}, F_L, and F_c are tabulated in Table VI for the various transmission line configurations that might be encountered. Table VII lists the same quantities for several commercially available coaxial transmission lines.

The quantity of primary concern is the transfer function for a section of transmission line which is

$$\begin{aligned} H(S) &= V_2(S)/V_1(S) \\ &= e^{-z\gamma} \\ &\approx e^{-z\sqrt{k'}\, S\sqrt{1 + F_{sk}/(S^{\frac{1}{2}}F_L\sqrt{u_o\sigma_c})}/c_o}\ . \end{aligned} \qquad 27)$$

Since large values of S will have the dominant effect in expected applications, the following approximation may be used for the radical in the exponent:

$$\sqrt{1 + F_{sk}/(S^{\frac{1}{2}} F_L\sqrt{u_o\sigma_c})} \approx 1 + F_{sk}/(2S^{\frac{1}{2}} F_L\sqrt{u_o\sigma_c}). \qquad 28)$$

Table VI. Transmission Line Geometric Factors.

Line Type	F_{sk}, meter^{-1}	F_L	F_c	$\frac{F_{sk}}{F_L}$, meter^{-1}
Coaxial	$\frac{1}{2\pi}\left[\frac{1}{r_o}+\frac{1}{r_i}\right]$	$\frac{1}{2\pi}\ln\frac{r_o}{r_i}$	$\frac{2\pi}{\ln(r_o/r_i)}$	$\frac{(1/r_o)+(1/r_i)}{\ln(r_o/r_i)}$
Twin Lead	$\frac{2s}{\pi d^2\sqrt{s^2/d^2 - 1}}$	$\frac{\cosh^{-1}s/d}{\pi}$	$\frac{\pi}{\cosh^{-1}s/d}$	$\frac{2s/d}{(s^2-d^2)^{\frac{1}{2}}\cosh^{-1}s/d}$
Parallel slab	2/b	a/b	b/a	2/a
Stripline, 1 gnd plane	1/2b	a/b	b/a	1/2a
Stripline, 2 gnd planes	1/4b	a/2b	2b/a	1/2a

Legend: All dimensions in meters. s= spacing; d= diameter of twin lead conductor; r_o= inner radius of coaxial outer conductor; r_i= radius of coaxial inner conductor; b= conductor width; a= conductor spacing. For the parallel slab and stripline configurations F_{sk}, F_L, and F_c are approximations that ignore fringing effects at conductor edges. For a typical integrated circuit a≈ 0.001 inch.

Table VII. Coaxial Transmission Line Geometric Factors.

Line Type	Outside diameter, in.	F_{sk}, meter^{-1}	F_L	F_{sk}/F_L, meter^{-1}
RG-55A/U	0.116	233	0.193	1 210
RG-63B/U	0.285	269	0.386	698
RG-62A/U	0.146	290	0.279	1 040
RG-59A/U	0.146	316	0.294	1 075
RG-141/U	0.116	230	0.185	1 240
Mini coax	0.0042	6 700	0.200	33 500

Then

$$H(S) \approx e^{-zS(k')^{\frac{1}{2}}/c_o} \; e^{-\frac{1}{2}zS^{\frac{1}{2}} F_{sk}(k')^{\frac{1}{2}}/c_o F_L (u_o \sigma_c)^{\frac{1}{2}}} . \qquad 29)$$

In accordance with the shifting theorem, the first factor of equation 29 has the effect of introducing a uniform delay of $z(k')^{\frac{1}{2}}/c_o$ in the time domain. With

$$t' = t - z(k')^{\frac{1}{2}}/c_o \qquad 30)$$

the delayed unit impulse response corresponding to the remaining factor is (Roberts and Kaufman, 1966, item 14, p. 246)

$$L^{-1}\left[e^{-aS^{\frac{1}{2}}}\right] = \frac{a\, e^{-a^2/4t'}}{2\, \pi^{\frac{1}{2}}\, (t')^{3/2}} = h(t') \qquad 31)$$

where

$$a = \tfrac{1}{2}\, z\, F_{sk}(k')^{\frac{1}{2}}/c_o F_L (u_o \sigma_c)^{\frac{1}{2}} = z\, a' \ . \qquad 32)$$

The function h(t') may be rearranged to make its time behavior stand out more clearly:

$$h(t') = \frac{4}{a^2\, \pi^{\frac{1}{2}}} (a^2/4t')^{3/2}\, e^{-a^2/4t'} \qquad 33)$$

which is a continuous function having a value of zero for $t' \leqslant 0$ and a single maximum, or peak value at $t' = a^2/6$. These conclusions are easily arrived at by examination of the function

$$(1/T^{3/2})\; e^{-1/T}$$

in which T is substituted for $4t'/a^2$. It is plotted in Figure 5.

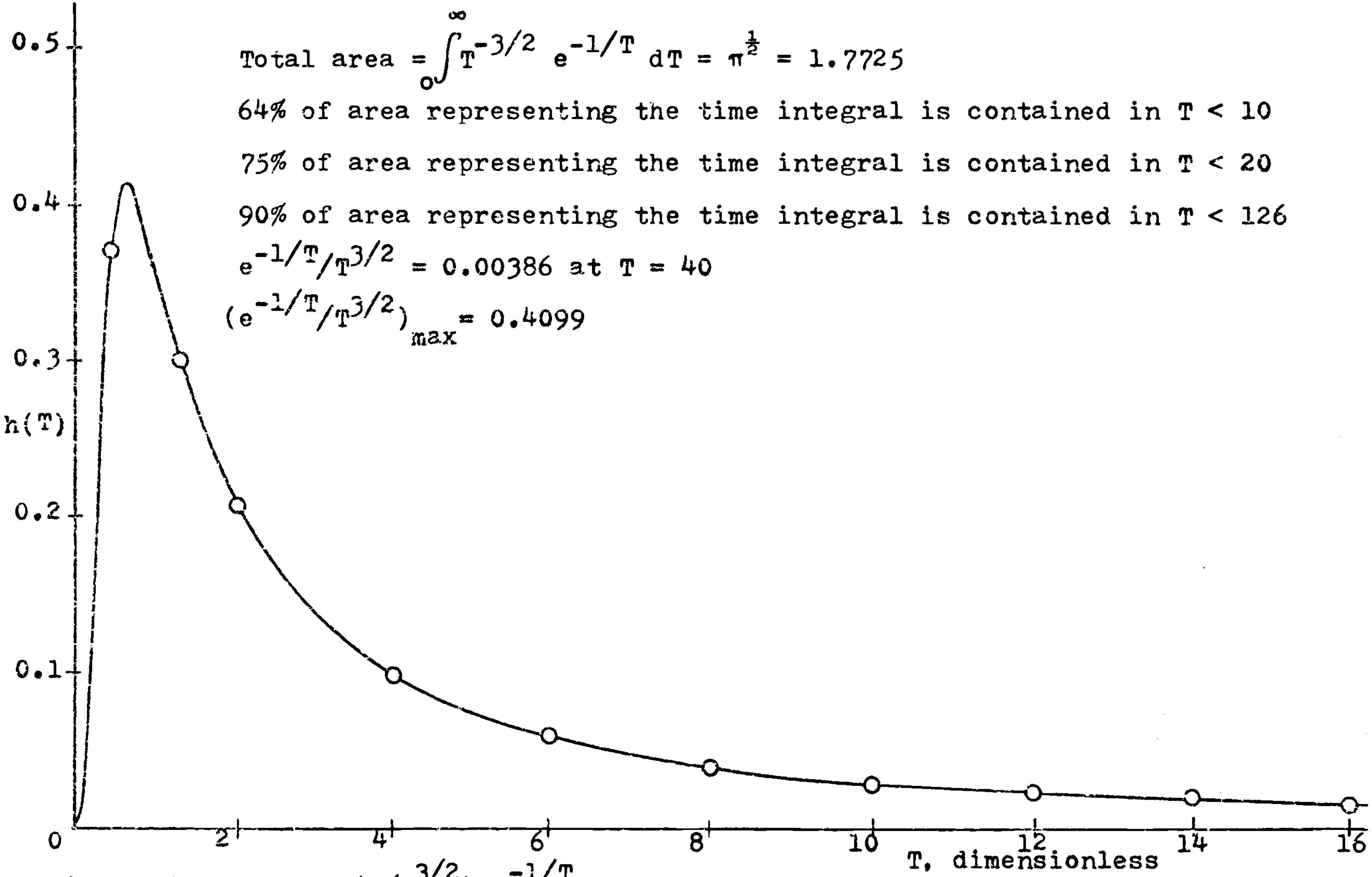
Unfortunately, the principles discussed in Chapter VII which allow evaluation of the temporal parameters of a waveform from the frequency domain transfer function do not apply to the above unit impulse response. It must then be considered in the time domain. Writing equation 33 in terms of a normalized variable $T = 4t'/a^2$ yields

$$h(T) = \frac{4}{a^2 \pi^{\frac{1}{2}}} (1/T^{3/2}) e^{-1/T} . \qquad 34)$$

The factor $a^2/4$ is a scale factor which stretches or shrinks the normalized function proportional to $a^2/4$. Hence, the smaller a^2 may be made, the shorter will be the duration of the unit impulse response and the greater will be its amplitude making it approach an ideal impulse.

It is obvious from equation 32 that better performance with a transmission line is obtained with a medium having low dielectric constant k', a geometric form or structure providing a large inductance factor F_L (or equivalently, a low capacitance factor F_c), high conductivity conductor material, and a low value of skin effect resistance geometric factor, F_{sk}. Making the permeability of the conductor material greater will also improve performance if it can be done without increasing losses.

The plot in Figure 5 shows what can be considered to



Total area $= \int_0^\infty T^{-3/2}\ e^{-1/T}\ dT = \pi^{\frac{1}{2}} = 1.7725$

64% of area representing the time integral is contained in $T < 10$

75% of area representing the time integral is contained in $T < 20$

90% of area representing the time integral is contained in $T < 126$

$e^{-1/T}/T^{3/2} = 0.00386$ at $T = 40$

$(e^{-1/T}/T^{3/2})_{max} = 0.4099$

Figure 5. Plot of $(1/T^{3/2})\ e^{-1/T}$.

be a universal impulse response of a transmission line in terms of the normalized time variable $T = 4t'/a^2$.

The quantity $a^2/4$ is also linearly dependent on z^2, the effect of which is to stretch the time scale of the impulse response as it propagates along the transmission line. The overall effect of this phenomenon is that at some particular distance the duration of the impulse response can become great enough to cause excessive distortion of a waveform that may be transmitted along the line. The distance beyond which the distortion is considered excessive depends upon the particular waveform and the minimum time interval associated with it over which such distortion can be tolerated.

Examination of Figure 5 shows that the impulse response of a line has fallen to about seven percent of its peak value by the normalized instant of time T = 10. Hence, if the value of t' which corresponds to that value of T is considerably smaller than the minimum time interval, T_m, associated with the transmitted waveform, acceptable distortion occurs for

$$t' = 10\ (a')^2\ z^2/4 << T_m\ . \qquad 35)$$

The useable length of the line becomes

$$z << \frac{2}{a'}\ (T_m/10)^{\frac{1}{2}}\ . \qquad 36)$$

If T_m is expressed in nanoseconds the above inequality becomes, in terms of the quantity $(a')^2$ in the fifth column of Table VIII,

$$z << 2\sqrt{T_m \times 10^{-9}/(10(a')^2)} = \frac{2 \times 10^{-5}}{a'}\sqrt{T_m} \qquad 37)$$

which puts an upper limit on the useable length of a transmission line. Arbitrarily accepting a maximum value for z as one tenth of the quantity on the right hand side of equation 37, the maximum useable length of transmission line is

$$z_{max} \approx \frac{2 \times 10^{-6}}{a'}\sqrt{T_m} \qquad 38)$$

in which $a' = \sqrt{k' F_{sk}^2 /(4c_o^2 u_o \sigma_c F_L^2)}$. Values for z_{max} are also listed in Table VIII for several commercial coaxial lines and for several stripline configurations.

Equations 32 and 34 also indicate that the amplitude of the impulse response varies inversely as z^2. This fact means that the magnitude of the waveform decreases very quickly with distance.

The remaining quantity necessary in the description of a transmission line is its characteristic impedance, Z_o, of equations 22 through 26. In the frequency domain Z_o relates the amplitude and phase of the sinusoidal traveling-wave voltage on the line at a given point to those of the accompanying sinusoidal current at the same point. In this respect Z_o can be considered as a transfer function relating these two quantities. Its inverse transform will

Table VIII. Commercial Transmission Line Time Domain Propagation Characteristics.

Line Type	Insulating Material	k'	F_{sk}/F_L, $meter^{-1}$	$a' = \left[\frac{k' F_{sk}^2}{4c_o^2 u_o \sigma_c F_L^2}\right]^{\frac{1}{2}}$	z_{max}, meters, T_m in nanoseconds
(Coaxial cable)					
RG-55A/U	Polyethylene	2.3	1210	3.536×10^{-7}	$5.66\sqrt{T_m}$
RG-59A/U	Polyethylene	2.3	1070	3.098×10^{-7}	$6.46\sqrt{T_m}$
RG-141/U	Teflon	2.1	1240	3.450×10^{-7}	$5.80\sqrt{T_m}$
Minicoax	FEP Teflon	2.1	33500	93.00×10^{-7}	$0.22\sqrt{T_m}$
Stripline					
MPC-062-2 a=0.0564"	Polyethylene	2.3	350	1.015×10^{-7}	$19.7\sqrt{T_m}$
MPC-125-2 a=0.1194"	Polyethylene	2.3	165	0.477×10^{-7}	$41.9\sqrt{T_m}$
MPC-187-2 a=0.1814"	Polyethylene	2.3	108.5	0.315×10^{-7}	$63.5\sqrt{T_m}$
MPC-250-2 a=0.2444"	Polyethylene	2.3	80.9	0.235×10^{-7}	$85.1\sqrt{T_m}$
Integrated circuits a=0.001"	Silicon	11.7	19700	128.8×10^{-7}	$0.16\sqrt{T_m}$

represent the unit impulse response of the voltage to an ideal unit impulse of current in the time domain.

As shown above, the unit impulse response relating arbitrary voltage and current on a lossy transmission line is not an ideal impulse function. This fact means that a traveling-wave voltage and accompanying current on a line do not have the same time variation. The same principle applies to the reciprocal of the the characteristic impedance, $Y_o = 1/Z_o$ also.

From equation 26

$$Z_o = \frac{F_L\ n_o}{\sqrt{k'}} \sqrt{1 + F_{sk} /(S^{\frac{1}{2}}\ F_L \sqrt{u_o \sigma_c})} \qquad 39)$$

and

$$Y_o = \sqrt{k'} /(F_L\ n_o \sqrt{1 + F_{sk} /(S^{\frac{1}{2}}\ F_L \sqrt{u_o \sigma_c}\ )}\ ) \qquad 40)$$

both of which, for the high frequency case wherein the second term under the radical is very small, reduce to

$$Z_o \approx \frac{F_L\ n_o}{\sqrt{k'}} (1 + \tfrac{1}{2} F_{sk} /(S^{\frac{1}{2}}\ F_L \sqrt{u_o \sigma_c})) \qquad 41)$$

and

$$Y_o \approx \frac{\sqrt{k'}}{F_L n_o} (1 - \tfrac{1}{2} F_{sk} /(S^{\frac{1}{2}}\ F_L \sqrt{u_o \sigma_c}))\ . \qquad 42)$$

The unit impulse responses corresponding to Z_o and Y_o are

$$z_o(t) \approx \frac{F_L\ n_o}{\sqrt{k'}} \left[\delta(t) + \frac{F_{sk}}{8F_L(\pi\sigma_c u_o)^{\frac{1}{2}}}(t^{-\frac{1}{2}})^3 u(t)\right] \qquad 43)$$

and

$$y_o(t) \approx \frac{\sqrt{k'}}{F_L\ n_o} \left[\delta(t) - \frac{F_{sk}}{8F_L(\pi\sigma_c u_o)^{\frac{1}{2}}}(t^{-\frac{1}{2}})^3 u(t)\right] \qquad 44)$$

both of which exhibit distortion terms. It is obvious also that distortion due to these quantities may be reduced by minimizing the F_{sk}/F_L ratio as well as using conducting materials having higher conductivity, σ_c.

Closed, Hollow Metallic Waveguide.

Much like the two conductor transmission line, the closed single conductor waveguide has potential use in a nonsinusoidal carrier radar just as in radar of the conventional type. The properties of circular and rectangular waveguide are investigated in this section for the lossless case (i.e., an air filled space with ideally conducting walls) which is mathematically tractable. Because of the mathematical complexity, the effect due to conductor wall loss is not considered.

The electrical and magnetic fields in a waveguide due to sinusoidal excitation are available in any number of texts on the topic. In Table IX are listed the TM and TE fields in circular and rectangular waveguide from Ramo, Whinnery, and Van Duzer (1965, pp 421, 422, 430, and 431). The quantity of principle interest related to these fields is the waveguide transfer function relating the amplitudes and phases of a field component at two places along the guide. This function is the same for all components of a given mode of the TM or TE type. It is

$$H(S,z) = e^{-z\gamma} . \qquad 45)$$

Once γ is found the waveguide transfer function is known.

Another quantity of interest is that relating the transverse magnetic and transverse electric fields in the

Table IX. Electric and Magnetic Field Expressions in Circular and Rectangular Waveguide.

Cross Section	Mode	
	TM	TE
Circular	$E_z=AJ_n(k_c r)\ e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$	$H_z=AJ_n(k_c r)\ e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$
Circular	$H_r=-\frac{nSAJ_n(k_c r)}{k_c n_o r \omega_c} e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$	$E_r=\frac{n\ n_o SA}{k_c r \omega_c} J_n(k_c r) e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$
Circular	$H_\phi=-\frac{SA}{n_o \omega_c} J_n'(k_c r) e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$	$E_\phi=\frac{n_o SA}{\omega_c}\ J_n'(k_c r)\ e^{-z\gamma} \begin{matrix}\cos n\phi \\ \sin n\phi\end{matrix}$
Circular	$E_\phi=-H_r\ Z_{tm},\ E_r= H_\phi\ Z_{tm}$ $k_c= p_{nl}/a$ A= arbitrary amplitude factor	$H_\phi= E_r/Z_{te},\ H_r=-E_\phi/Z_{te}$ $k_c= p'_{nl}/a$
Rectangular	$E_z=A \sin k_x x \sin k_y y\ e^{-z\gamma}$	$H_z=A \cos k_x x \cos k_y y\ e^{-z\gamma}$
Rectangular	$H_x=\frac{SA\ \ k_y}{k_c n_o \omega_c} \sin k_x x \cos k_y y e^{-z\gamma}$	$E_x=\frac{n_o SA\ k_y}{k_c \omega_c} \cos k_x x \sin k_y y e^{-z\gamma}$
Rectangular	$H_y=-\frac{SA\ k_x}{k_c n_o \omega_c} \cos k_x x \sin k_y y e^{-z\gamma}$	$E_y=-\frac{n_o SA\ k_x}{k_c \omega_c} \sin k_x x \cos k_y y e^{-z\gamma}$
Rectangular	$E_x=H_y\ Z_{tm},\ E_y=-H_x\ Z_{tm}$ $k_x=m\pi/a,\ k_y=n\pi/b,\quad k_c^2=k_x^2 + k_y^2,$	$H_x=-E_y/Z_{te},\ H_y=E_x/Z_{te}$ $k_x=m\pi/a,\ k_y=n\pi/b$

waveguide. This quantity is

$$E_r/H_\phi = Z_{tm} \quad \text{or} \quad H_\phi/E_r = 1/Z_{tm} \qquad 46a)$$

and

$$E_r/H_\phi = Z_{te} \quad \text{or} \quad H_\phi/E_r = 1/Z_{te} \quad , \qquad 46b)$$

or the wave impedances of the waveguide and their reciprocals.

The inverse Laplace transform of the waveguide transfer function H(S,z) yields the unit impulse response of a section of the waveguide which shows explicitly the form of the distortion produced as a function of distance along the guide.

The inverse transforms of the wave impedances and their reciprocals (i.e., admittances) likewise show that the transverse electric and magnetic fields at a given transverse plane do not retain the same temporal form.

For the ideal lossless conductor case it can be shown (Ramo, Whinnery, and Van Duzer, 1965, p. 421) that a waveguide propagation constant in operational form is

$$\gamma = \frac{1}{c_o}\sqrt{S^2 + \omega_c^2} \qquad 47)$$

where ω_c is the waveguide cut-off frequency (in radians per second) determined by the waveguide cross section, size, and mode of the fields propagating down the guide.

It turns out that nonsinusoidal fields are described by the same modal descriptions as are the sinusoidal waves.

The waveguide wave impedances in operational form (Ramo, Whinnery, and Van Duzer,1965) are

$$Z_{tm} = \frac{n_o}{S}\sqrt{S^2 + \omega_c^2} \ , \qquad \text{48a)}$$

$$Z_{te} = n_o S \ (S^2 + \omega_c^2)^{-\frac{1}{2}} \ , \qquad \text{48b)}$$

and

$$Y_{tm} = \frac{S}{n_o} \ (S^2 + \omega_c^2)^{-\frac{1}{2}} \ , \qquad \text{49a)}$$

$$Y_{te} = \frac{1}{n_o S}\sqrt{S^2 + \omega_c^2} \qquad \text{49b)}$$

where n_o is the wave impedance of free space. The cut-off frequencies for the two waveguide cross section types are shown in Table X.

The inverse transforms of the quantities in equations 45, 48, and 49 are readily available in Roberts and Kaufman (1966, items 48, p. 251; 35, p. 225; and 56, p. 212 respectively) as

$$L^{-1}\left[e^{-z\sqrt{S^2 + \omega_c^2}/c_o}\right] =$$

$$\delta(t - z/c_o) \ - \frac{\omega_c z}{c_o} \ u(t - z/c_o) \ \frac{J_1(\omega_c\sqrt{t^2 - z^2/c_o^2})}{\sqrt{t^2 - z^2/c_o^2}}, \qquad \text{50)}$$

Table X. Radian Cut-off Frequencies of Rectangular and Circular Waveguides

Cross Section	Wave Mode	Cut-off Frequency, ω_c, radians per second
Circular	TM	$p_{nl}\ c_o/r$
Circular	TE	$p'_{nl}\ c_o/r$
Rectangular	TM	$\pi\ c_o \sqrt{m^2/a^2 + n^2/b^2}$
Rectangular	TE	$\pi\ c_o \sqrt{m^2/a^2 + n^2/b^2}$

Legend: r= inner radius, meters; c_o= speed of light in free space, meters per second; p_{nl} = l^{th} root of Bessel function $J_n(x)=0$; p'_{nl}= l^{th} root of $J'_n(x) = 0$; a = x dimension in meters; b= y dimension in meters; m, n = integers describing the transverse wave modal structure.

$$L^{-1}\left[Z_{tm}\right] = L^{-1}\left[\frac{n_o}{S}\sqrt{S^2 + \omega_c^2}\right] =$$

$$n_o\left[\delta(t) + \omega_c\, u(t)\int_0^t \frac{J_1(\omega_c x)\, dx}{x}\right] = z_{tm}(t) \quad , \qquad 51)$$

and

$$L^{-1}\left[Z_{te}\right] = L^{-1}\left[n_o S \,/\sqrt{S^2 + \omega_c^2}\right] =$$

$$n_o\left[\delta(t) + \omega_c\, u(t)\, J_1(\omega_c t)\right] = z_{te}(t) \;. \qquad 52)$$

Equation 50 is the unit impulse response of the waveguide. This equation is very revealing in that the first term, the ideal impulse, reproduces the transmitted field with no distortion while the second terms is an error term. Its effect in the convolution integral is dependent upon its area with respect to the unit area of the ideal impulse term. The net area of the error term is

$$\text{Area} = \frac{\omega_c^2}{c_o} z \int_{z/c_o}^{\infty} \frac{J_1(\omega_c\sqrt{t^2 - z^2/c_o^2}\,)}{\omega_c\sqrt{t^2 - z^2/c_o^2}}\, dt \;. \qquad 53)$$

The above integral is evaluated by making the following change of variables:

$$T = \sqrt{t^2 - z^2/c_o^2} \text{ or } \quad t = \sqrt{T^2 + z^2/c_o^2}, \; dt = \frac{T\, dT}{\sqrt{T^2 + z^2/c_o^2}}$$

and

$$t = z/c_o \longrightarrow T = 0.$$

The area is then

$$\text{Area} = \frac{\omega_c^2}{c_o} z \int_0^{\infty} \frac{J_1(\omega_c T)\, dT}{\omega_c \sqrt{T^2 + z^2/c_o^2}}$$

$$= \frac{\omega_c}{c_o} z \int_0^{\infty} \frac{J_1(T)\, dT}{\sqrt{T^2 + z^2 \omega_c^2/c_o^2}} \qquad 54)$$

where the change of variable $T \longrightarrow T\omega_c$ has been made. From Abramowitz and Stegun (1965, item 11.4.48, p.488) the above area is

$$\text{Area}^8 = \frac{\omega_c}{c_o} z \, I_{\frac{1}{2}}(\omega_c z/2c_o)\, K_{\frac{1}{2}}(\omega_c z/2c_o) \quad . \qquad 55)$$

Fortunately equation 55 reduces to a very simple form:

$$\text{Area} = 1 - e^{-\omega_c z/c_o} \; . \qquad 56)$$

For values of z and ω_c such that

$$1 - e^{-\omega_c z/c_o} << 1 \qquad 57)$$

the error term should produce little distortion in the propagating nonsinusoidal field. This fact suggests a maximum value of distance z which, for a given cut-off frequency, distortion would be negligible. Depending upon the particular waveform transmitted, an acceptable level

8) $I_{\frac{1}{2}}(x) = (2/\pi x)^{\frac{1}{2}} \sinh(x)$ and $K_{\frac{1}{2}}(x) = (\pi/2x)^{\frac{1}{2}} e^{-x}$ are the half order Bessel functions of the first and second kinds respectively.

of distortion might be met when $(1 - e^{-\omega_c z/c_o}) < 0.2$. This inequality yields a maximum value of z for a given waveguide mode:

$$z_{max} = 0.223\ c_o/\omega_c \ . \qquad 58)$$

For a circular waveguide this relationship becomes

$$(z_{max})_{tm} = 0.223\ r/p_{nl} \qquad 59a)$$

and

$$(z_{max})_{te} = 0.223\ r/p'_{nl} \qquad 59b)$$

while for rectangular waveguide the maximum useable length is

$$\begin{aligned} z_{max} &= \frac{0.223}{\pi}\sqrt{(m^2/a^2) + (n^2/b^2)} \\ &= 0.071\sqrt{(m^2/a^2) + (n^2/b^2)} \ . \end{aligned} \qquad 60)$$

The largest value of z_{max} occurs for a circular waveguide at the minimum values of p_{nl} and p'_{nl} which are

$$p_{nl}(\min) = p_{01} = 2.405 \text{ and } p'_{nl}(\min) = p'_{11} = 1.841$$

producing values of z_{max} of

$$(z_{max})_{01} = 0.0927\ r \text{ meters}$$

and

$$(z_{max})_{11} = 0.1211\ r \text{ meters} \ .$$

The largest value of z_{max} for the rectangular waveguide occurs for the minimum values of m and n which correspond to the TE_{01} mode:

$$(z_{max})_{01} = 0.071 \text{ b meters} .$$

Equations 59 and 60 indicate that the useable length of a waveguide is much less than its cross sectional dimensions! This observation will limit the use of a waveguide to impractically short lengths. The validity of this finding, of course, depends greatly upon the particular waveform to be transmitted and its intended application. Detailed study of the individual case would be necessary.

Equation 51 implicitly shows the time domain relationship between the transverse electric and magnetic fields in the TM mode at any point in the waveguide. The electric field corresponding to a given magnetic field results from convolving it with $z_{tm}(t)$. That operation yields

$$e_{tm}(t) = h_{tm}(t) * z_{tm}(t) \qquad \text{61a)}$$

$$n_o \left[h_{tm}(t) * \delta(t) \right.$$

$$\left. + \omega_c h_{tm}(t) * \left[u(t) \int_0^t \frac{J_1(\omega_c x)\ dx}{x} \right] \right] \qquad \text{61b)}$$

$$e_{tm}(t) = n_o \left[h_{tm}(t) + \omega_c \, h_{tm}(t) * \int_0^t \frac{J_1(\omega_c x) \, dx}{x} \right]. \qquad 61c)$$

The first term of equation 61c produces an undistorted component of the electric field having the same time variation as the magnetic field. The integrand in the integral of the second term is a pulse-like function having an initial value of $\omega_c/2$ and quickly diminishes to zero at the first zero of the Bessel function in the numerator. From that instant it undergoes negative and positive excursions of rapidly decreasing amplitude. The end of the first main pulse occurs at $\omega_c t = 3.83$, the first zero of the first order Bessel function (of the first kind). For practical purposes the time integral of $J_1(\omega_c x)/x$ then appears as a step-like function that reaches its maximum and final value at the first zero of $J_1(x)$. This zero yields an interval of

$$t_o = 3.83/\omega_c \text{ seconds.}$$

For magnetic waves having pulse durations or transition times much greater than $3.83/\omega_c$ seconds, the convolution of the integral of the second term with the magnetic field is approximately

$$\omega_c \, h_{tm}(t) * \; u(t) \int_0^t \frac{J_1(\omega_c x) \, dx}{x} \approx \omega_c \int_0^t h_{tm}(x) \, dx \; . \qquad 62)$$

The total field is then

$$e_{tm}(t) \approx n_o\, h_{tm}(t) \quad + \quad n_o \omega_c \int_0^t h_{tm}(x)\, dx \quad . \qquad 63)$$

From equation 63 it is seen that if

$$\omega_c \int_0^t h_{tm}(x) dx \approx h_{tm}(t) \qquad 64)$$

the electric field will have a sizable component that is proportional to the time integral of the magnetic field resulting in a badly distorted version of the electric field. Conversely, if the time variation of the magnetic field is such that

$$\omega_c \int_0^t h_{tm}(x)\, dx >> h_{tm}(t) \qquad 65)$$

then the resulting electric field is

$$e_{tm}(t) \approx n_o \omega_c \int_0^t h_{tm}(x)\, dx \quad . \qquad 66)$$

The relationship between the electric and magnetic fields in the TE mode is implied by $z_{te}(t)$ of equation 52. As with equation 51 the electric field is the result of convolving the magnetic field $h_{te}(t)$ with $z_{te}(t)$. The electric field is then

$$e_{te}(t) = h_{te}(t) * z_{te}(t)$$

$$= n_o\, h_{te}(t) * \delta(t)$$

$$+ n_o \omega_c\, h_{te}(t) * (u(t)\, J_1(\omega_c t)) \qquad 67b)$$

$$e_{te}(t) = n_o\, h_{te}(t) + n_o\, h_{te}(t) * (u(t)\, J_1(\omega_c t))\omega_c. \quad 67c)$$

As in the TM mode the electric field consists of a nondistorted component $n_o\, h_{te}(t)$ and a distorted component resulting from the convolution of the magnetic field and and $\omega_c\, u(t)\, J_1(\omega_c t)$. The magnitude of the contribution of the second term is approximately equal to the total time integral of $\omega_c\, u(t)\, J_1(\omega_c t)$. The value of that integral is

$$\omega_c \int_0^\infty u(t)\, J_1(\omega_c t)\, dt = \int_0^\infty J_1(\omega_c t)\, d(\omega_c t) = 1. \quad 68)$$

The contribution of the second term is then comparable to the ideal term, hence the electric field is a distorted version of the magnetic field. However, the duration of $\omega_c\, u(t)\, J_1(\omega_c t)$ is approximately equal to the first zero of $J_1(x)$ which is 3.83. Therefore, if the time scale of interest, T, is much less than $3.83/\omega_c$ seconds, the Bessel function $J_1(\omega_c t)$ will appear as an ideal impulse to the magnetic field. The resultant electric field in that case is

$$e_{te} \approx 2\, n_o\, h_{te}(t)\ .$$

Chapter X. Radiation and Reflection Considerations of Nonsinusoidal Electromagnetic Waves

An aspect of great importance in the operation of a nonsinusoidal radar is the characteristic of nonsinusoidal radiation from charge and current distributions on mechanically realizable arrangements of conducting elements. The charge and current distributions under consideration are those generated by localized electronic power sources. Accordingly, this chapter investigates the radiation of nonsinusoidal carrier waveforms from several radiating and antenna systems used so successfully in the past.

Traditionally antennas have been characterized by their directional responses to a single frequency sinusoidal excitation. Frequency is also often a variable, or controllable, system parameter. These responses to single frequency excitation are often slowly varying functions of operating frequency such that use of the antenna over a band of frequencies (i.e., resulting from modulation of a high frequency sinusoidal carrier) is fairly uniform.

Quantities most often used in describing antenna performance have been: 1) the steady-state impedance presented at the excitation terminals of the device; 2) its polarization properties; 3) its radiation resistance which is a measure of how effectively the antenna converts the power available at its terminals into radiating electromagnetic energy; 4) the directional properties of the antenna which are characterized by the relative intensities of the electric and magnetic fields as functions of direction with respect to a suitable reference direction associated with the antenna. These quantites are all expressed in terms of a response to a single frequency excitation. The directional property of the device is often expressed in terms of the relative time averaged power density of the radiated sinusoidal electromagnetic field at one frequency as a function of direction. This quantity, often termed the radiation pattern of the device, is also a frequency dependent quantity.

Of the above quantities, the more important for radar applications is the directional property of the antenna which is utilized to provide target direction information. It is this property which is emphasized in this study.

In order to establish the suitability of a family of nonsinusoidal functions as the electromagnetic carrier waveform in radar applications it is necessary to determine their radiation characteristics when launched into a propagation medium by various antenna structures. It is also desirable, where possible, to determine principles and criteria by which more effective and more directive arrays of radiating elements may be devised: i.e., solution of the synthesis, or design, problem as well as the analysis problem.

The family of nonsinusoidal functions under study are such that the directional, time averaged power radiation patterns now used to characterize antennas having sinusoidal excitation constitute unsatisfactory means of describing an antenna response to general excitation. In the sinusoidal case the field components arising from different portions of the radiating system merely superimpose to form sinusoidal traveling waves throughout the surrounding propagation medium. These fields have the same temporal form and frequency as the generating sinusoidal source with their amplitudes and phases being functions of location with respect to the antenna system.

The only distortion the propagating field can suffer from the differential time delays is in the modulation that might be imposed upon it. However, for the narrow band modulation usually used in radar (and communication) applications, the distortion arising from this cause is insignificant and undetectable.

In the general nonsinusoidal and wideband case, however, the situation is much different. In the following sections it is shown that superposition of the field components resulting from different portions of the antenna system generally give rise to field time variations which are greatly distorted versions of the original source time function. In some cases they are completely different functions of time. This is the principal factor which distinguishes most nonsinusoidal waves from purely sinusoidal waves. Because of this distinction this chapter is devoted to developing means of establishing the radiation and directional characteristics of the class of discrete functions under study.

Because of the distortion that a nonsinusoidal wave can suffer it is readily apparent that the instantaneous values of the electric and magnetic field or the instantaneous power density are the quantities of primary concern. In the sinusoidal case time-averaged power density radiation patterns are the significant factors. The transmitted electric and magnetic fields undergo various transformations, or distortions, during propagation through the radar medium and on reflection from the radar target. The voltage or current waveforms that the reflected fields subsequently generate at the receiving antenna are ultimately the quantities of primary concern.

Although the distortion of the instantaneous values of the fields seems a detrimental feature of their propagation, the distortion is predictable and, most often, is angle dependent. This fact may allow this distortion to be used as an added (and perhaps the only) measure of the directional position of a radar target.

Rather than attack the problem of nonsinusoidal waves directly as done by Harmuth (1970) and Pearlman (1970), an indirect technique which utilizes existing sinusoidal antenna theory and the vast store of engineering data on antenna system performance is used in the following study. The linear dipole of arbitrary length is also investigated to determine its suitability as a basic radiating element. Such an antenna element might be used in larger arrays to form antenna systems or to illuminate large parabolic reflectors. The parbolic reflector is studied through examination of the aperture diffraction integral model of such antennas when illuminated by an ideal source of nonsinusoidal radiation.

Since successful use of the parabolic reflector as an antenna depends upon the reflection properties of the nonsinusoidal electromagnetic wave from a smooth and highly conductive metallic surface, the parbolic antenna study is prefaced by an analysis of the general reflection process.

The Long Dipole Antenna

The field quantities of the long dipole antenna (of finite length), being of great importance to conventional radar and communications, may also prove useful with nonsinusoidal waves. It is important, then, to investigate it to determine its suitability for use with bi-valued waveforms.

By analysis which is almost traditional (Ramo, Whinnery, and Van Duzer, 1965, pp. 644-647) the differential elements of the field quantites of a long dipole can be shown to be

$$dH_{\phi}(\omega) = \frac{I(z)\ dz}{4\pi}\left[\frac{j\omega}{cr'} + \frac{1}{r'^2}\right]e^{-j\beta r'}\sin\theta' \qquad 1)$$

$$dE_r(\omega) = \frac{I(z)dz}{4\pi}\left[\frac{2n}{r'^2} - \frac{2j}{\omega e r'^3}\right]e^{-j\beta r'}\cos\theta' \qquad 2)$$

$$dE_{\theta}(\omega) = \frac{I(z)\ dz}{4\pi}\left[\frac{j\omega u}{r'} + \frac{n}{r'^2} - \frac{j}{\omega e r'^3}\right]e^{-j\beta r'}\ \sin\theta' \qquad 3)$$

where θ' is the angle between the z axis and the radius vector from the current element at z to the field observation point, and r' is the distance from the same current element to the observation point. Since the dimensional extent of the antenna is much smaller than the distance r from the origin to the observation point, the quantity r' in the denominator of the above equations 1, 2, and 3 may be replaced by the distance r from the origin to the observation point with little error while θ' may be replaced by θ, the angle between $\bar{r}$ and the z axis. These quantities are illustrated in Figure 6.

In order to account for the phase differences from various portions of the antenna, $(r - z \cos \theta)$ is substituted for r' in the exponential term which is sufficiently accurate for $r >> z$.

The total field quantities at point 0 of Figure 6 are obtained by summing contributions due to all current elements by integration over z from $-\frac{1}{2}L$ to $+\frac{1}{2}L$ which gives

$$H_{\not{o}}(\omega) \approx \frac{\sin \theta}{4\pi}\left[\frac{j\omega}{cr} + \frac{1}{r^2}\right]e^{-j\beta r}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} I(z)\ e^{j\beta z \cos \theta}\,dz \qquad 4)$$

$$E_r(\omega) \approx \frac{\cos \theta}{4\pi}\left[\frac{2n}{r^2} - \frac{2j}{\omega e r^3}\right]e^{-j\beta r}\int_{-\frac{1}{2}L}^{\frac{1}{2}L} I(z)\ e^{j\beta z \cos \theta}\,dz \qquad 5)$$

$$E_\theta(\omega) \approx \frac{\sin \theta}{4\pi}\left[\frac{j\omega u}{r} + \frac{n}{r^2} - \frac{j}{\omega e r^3}\right]e^{-j\beta r}\ X$$

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} I(z)\ e^{j\beta z \cos \theta}\,dz\ . \qquad 6)$$

With sinusoidal time variation the distribution of the current along the antenna is very close to being sinusoidally distributed in z (Ramo, Whinnery, and Van Duzer, 1965). For antennas greater than one half of a wavelength in length the maximum standing wave current

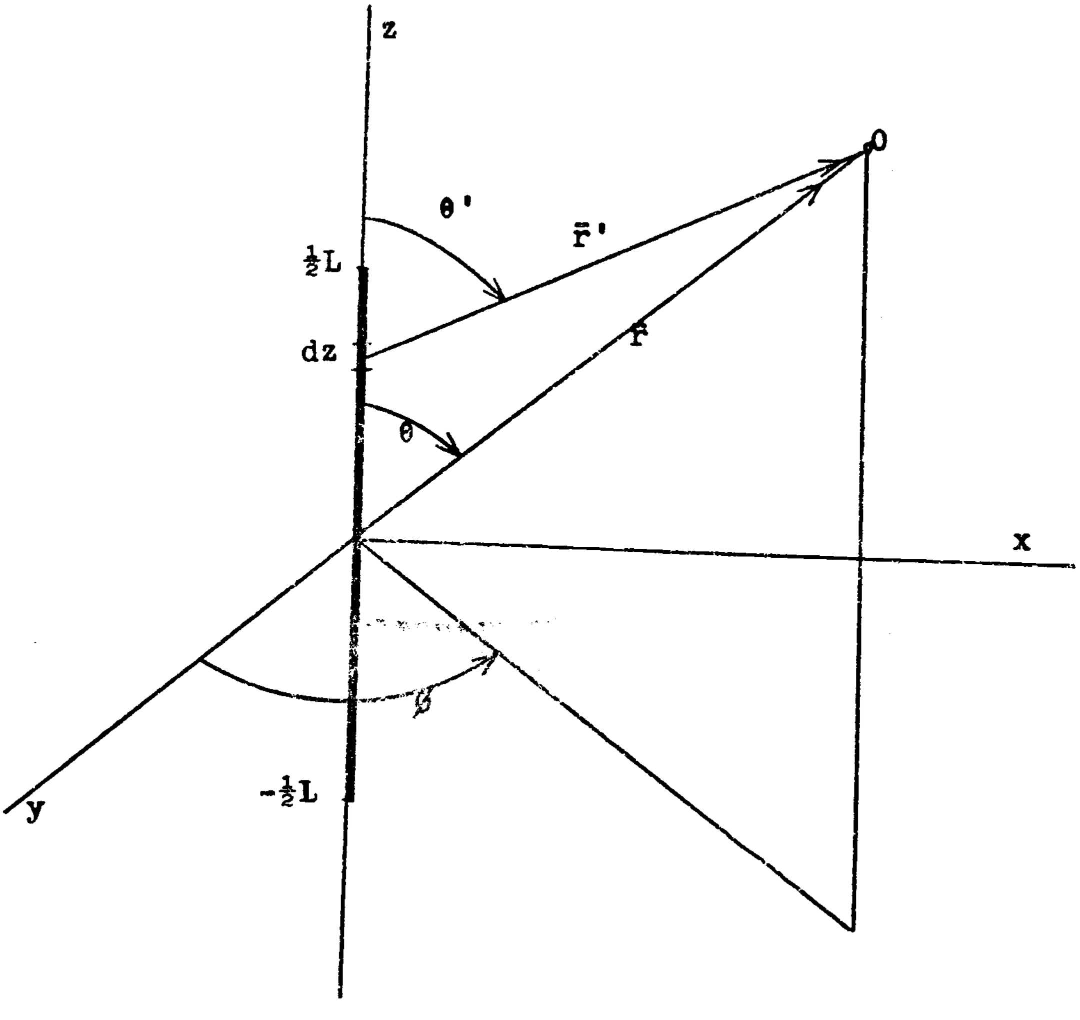


Figure 6. Arrangement of the long dipole with the standard spherical coordinate system.

amplitude (I_m) attained is not that at the input terminals. The terminal current may be expressed in terms of the maximum current amplitude I_m as

$$I_{ter} = I_m \sin(\omega L/2c), \text{ for } \omega L/2c > \tfrac{1}{2}\pi \qquad 7)$$

from which

$$I_m = I_{ter} / \sin(\omega L/2c) \ . \qquad 8)$$

It is important to realize that I_m is the maximum current amplitude occurring on an antenna longer than one half of a wavelength of the excitation frequency.

The current amplitude distribution along the antenna is now expressed as

$$I(\pm z) = I_m \sin\left[\omega(\tfrac{1}{2}L \mp z)/c\right] , \quad 0 \leqslant z \leqslant \tfrac{1}{2}L \qquad 9)$$

Substitution of equation 9 into the integrals of equations 4 through 6 yields for those integrals

$$\int_{-\frac{1}{2}L}^{\frac{1}{2}L} I(z)\, e^{j\beta z \cos\theta}\, dz =$$

$$\frac{2\, I_m\, c}{\sin^2\theta} \left[\cos(\frac{\frac{1}{2}\omega L}{c} \cos\theta) - \cos(\tfrac{1}{2}\omega L/c)\right] . \qquad 10)$$

Equation 10 may be substituted directly into equations 4 through 6 to yield the phasor forms of the field quantities. Although this step is valid the equations are not in the form of the transfer functions discussed in

Chapter VI. In order to satisfy this criterion the field quantities must be expressed in terms of the terminal current I_{ter}. This may be effected by replacing I(z) in equations 4 through 6 by the right side of equation 8. The field quantities are now in terms of an independent sinusoidal input current. This step is necessary since a constant value of the terminal current amplitude will produce different values of I_m at different frequencies. The inverse transforms of the field quantities will now yield valid impulse responses.

Making the above mentioned substitutions and converting to the Laplace transform variable gives

$$H_{\phi}(S) =$$

$$\frac{I_{ter}\, e^{-Sr/c}}{2\pi \sin\theta} \left[- \left[\frac{\cosh(\frac{1}{2}\frac{S\ L \cos\theta}{c}) - \cosh(\frac{1}{2}\ SL/c)}{r \sinh(\frac{1}{2}SL/c)} \right] - c \left[\frac{\cosh(\frac{1}{2}\frac{S\ L \cos\theta}{c}) - \cosh(\frac{1}{2}SL/c)}{r^2\ S \sinh(\frac{1}{2}LS/c)} \right] \right] \qquad 11)$$

$$E_r(S) =$$

$$\frac{I_{ter} \cot\theta\ e^{-Sr/c}}{2\pi \sin\theta} \left[cn \left[\frac{\cosh(\frac{1}{2}\frac{S\ L \cos\theta}{c}) - \cosh(\frac{1}{2}SL/c)}{r^2\ S \sinh(\frac{1}{2}SL/c)} \right] + \frac{\cosh(\frac{1}{2}\frac{S\ L \cos\theta}{c}) - \cosh(\frac{1}{2}SL/c)}{e\ r^3\ S^2 \sinh(\frac{1}{2}SL/c)} \right] \qquad 12)$$

$$E_\theta(S) =$$

$$\frac{I_{ter}\, e^{-Sr/c}}{2\pi \sin\theta}\left[\frac{uc\left[\cosh\left(\frac{\frac{1}{2}S\ L\cos\theta}{c}\right) - \cosh(\tfrac{1}{2}SL/c)\right]}{r\sinh(\frac{1}{2}SL/c)}\right.$$

$$+\ nc\left[\frac{\cosh\left(\frac{\frac{1}{2}S\ L\cos\theta}{c}\right) - \cosh(\frac{1}{2}SL/c)}{r^2\ S\sinh(\frac{1}{2}SL/c)}\right]$$

$$\left. +\ c\left[\frac{\cosh\left(\frac{\frac{1}{2}S\ L\cos\theta}{c}\right) - \cosh(\frac{1}{2}SL/c)}{e\ r^3\ S^2\sinh(\frac{1}{2}SL/c)}\right]\right]. \qquad 13)$$

On first sight the above equations appear to be very cumbersome transform expressions to invert, but writing the hyperbolic functions in terms of exponentials and factoring $\sinh(\frac{1}{2}SL/c)$ in the denominator as

$$\sinh(\tfrac{1}{2}SL/c) = \tfrac{1}{2}\ e^{\frac{1}{2}SL/c}(1 - e^{-SL/c}) \qquad 14)$$

simplifies them greatly. The term in parentheses in equation 14 produces periodic time functions of period L/c in the time domain. The numerators of equations 11, 12, and 13 become sums of exponential terms such as e^{-Sx} producing delayed, or retarded, time arguments in the corresponding time functions. The only inverse Laplace transforms to be determined are those for a constant, and those for such terms as $1/S$, and $1/S^2$. The inverse transforms are, respectively, $\delta(t)$, $u(t)$, and $t\ u(t)$.

Applying these inverse transforms to equations 11, 12, and 13 yields the following field quantities all of which are periodic with period L/c:

$$H_\phi(t) = \frac{1}{2\pi \sin \theta} \sum_{n=0}^{\infty} \left[\frac{1}{r} \left[\delta(t - \frac{r}{c} - \frac{nL}{c}) \right.\right.$$

$$- \delta(t - b + a - \frac{r}{c} - \frac{nL}{c}) - \delta(t - b - a - \frac{r}{c} - \frac{nL}{c})$$

$$\left. + \delta(t - 2b - \frac{r}{c} - \frac{nL}{c}) \right] + \frac{c}{r^2} \left[-u(t - \frac{r}{c} - \frac{nL}{c}) \right.$$

$$+ u(t - b + a - \frac{r}{c} - \frac{nL}{c}) + u(t - b - a - \frac{r}{c} - \frac{nL}{c})$$

$$\left.\left. - u(t - 2b - \frac{r}{c} - \frac{nL}{c}) \right]\right], \qquad 15)$$

$$E_r(t) = \frac{\cot \theta}{2\pi \sin \theta} \sum_{n=0}^{\infty} \left[\frac{c\eta}{r^2} \left[-u(t - \frac{r}{c} - \frac{nL}{c}) \right.\right.$$

$$+ u(t - b + a - \frac{r}{c} - \frac{nL}{c}) + u(t - b - a - \frac{r}{c} - \frac{nL}{c})$$

$$\left. - u(t - 2b - \frac{r}{c} - \frac{nL}{c}) \right]$$

$$+ \frac{1}{er^3} \left[-(t - \frac{r}{c} - \frac{nL}{c})\, u(t - \frac{r}{c} - \frac{nL}{c}) \right.$$

$$+ (t - b + a - \frac{r}{c} - \frac{nL}{c})\, u(t - b + a - \frac{r}{c} - \frac{nL}{c})$$

$$+ (t - b - a - \frac{r}{c} - \frac{nL}{c})\, u(t - b - a - \frac{r}{c} - \frac{nL}{c})$$

$$\left.\left. - (t - 2b - \frac{r}{c} - \frac{nL}{c})\, u(t - 2b - \frac{r}{c} - \frac{nL}{c}) \right]\right], \qquad 16)$$

and

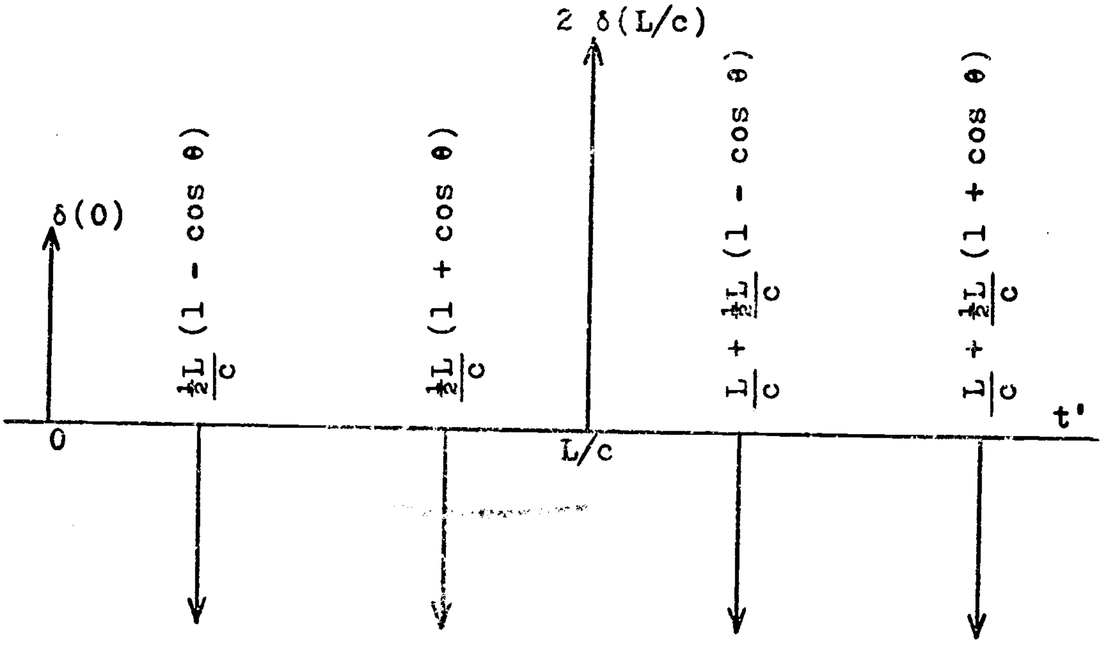
$$
\begin{aligned}
E_{\theta}(t) = \frac{1}{2\pi \sin \theta} \sum_{n=0}^{\infty} \Bigg[\frac{n}{r} \Big[& - \delta(t - \frac{r}{c} - \frac{nL}{c}) \\
& + \delta(t - b + a - \frac{r}{c} - \frac{nL}{c}) + \delta(t - b - a - \frac{r}{c} - \frac{nL}{c}) \\
& - \delta(t - 2b - \frac{r}{c} - \frac{nL}{c}) \Big] + \frac{cn}{r^2} \Big[-u(t - \frac{r}{c} - \frac{nL}{c}) \\
& + u(t - b + a - \frac{r}{c} - \frac{nL}{c}) + u(t - b - a - \frac{r}{c} - \frac{nL}{c}) \\
& - u(t - 2b - \frac{r}{c} - \frac{nL}{c}) \Big] + \\
& + \frac{c}{er^3} \Big[-(t - \frac{r}{c} - \frac{nL}{c})\, u(t - \frac{r}{c} - \frac{nL}{c}) \\
& + (t - b + a - \frac{r}{c} - \frac{nL}{c})\, u(t - b + a - \frac{r}{c} - \frac{nL}{c}) \\
& + (t - b - a - \frac{r}{c} - \frac{nL}{c})\, u(t - b - a - \frac{r}{c} - \frac{nL}{c}) \\
& - (t - 2b - \frac{r}{c} - \frac{nL}{c})\, u(t - 2b - \frac{r}{c} - \frac{nL}{c}) \Big] \Bigg] \qquad 17)
\end{aligned}
$$

where $a = \frac{1}{2} L (\cos \theta)/c$ and $b = \frac{1}{2} L/c$.

A surprising feature of the above field quantities is their periodicity even though excited by a single impulse. The analysis, however, was based upon an ideal situation wherein the ohmic resistance of the dipole was neglected. In actual circumstances the resistive losses of the dipole element would cause the periodic variations to dampen out with time and would also cause broadening of the individual impulses comprising the periodic response.

The form of the far field impulses (those terms proportional to $1/r$), shown in Figure 7, indicates that the long dipole antenna would likely be unsatisfactory for use with any carrier waveform except that of a sinusoidal carrier. Substitution of the this series of impulses into the superposition, or convolution, integral would show that, in general, the temporal form of the far field would consist of the superposition of progressively delayed replicas of the excitation current waveform, although not all of the same polarity. The resulting field variations would definitely not be a duplication of the excitation waveform! Rather, it would be a greatly distorted version of the excitation current waveform. A steady-state, unmodulated sinusoid is the only waveform that would be reproduced in the far field without distortion.

At small angles of θ the delay between the first and second impulses and that between the third and fourth impulses becomes small enough that these two pairs of impulses both approach ideal unit doublets. In the convolution integral they would produce delayed replicas of the time derivative of the excitation waveform with each of opposite polarity. In theory the transverse field would vanish at $\theta = 0$: the far as well as the intermediate and near fields. These facts are evident from equations 11, 12, and 13. In equations 11 and 13 the only angle dependent



$t' = t - r/c$

Figure 7. The first cycle of the far field impulse response of a long dipole antenna.

factor is

$$F(\theta) = \frac{\cosh(\frac{\frac{1}{2}SL\cos\theta}{c}) - \cosh(\frac{1}{2}SL/c)}{\sin\theta} \qquad 18)$$

Application of l'Hospital's rule yields

$$\lim_{\theta \to 0} F(\theta) =$$

$$\lim_{\theta \to 0} \frac{-\frac{1}{2}(SL/c)\sin\theta \sinh(\frac{\frac{1}{2}SL\cos\theta}{c})}{\cos\theta} = 0 \; . \qquad 19)$$

Therefore, $H_\phi(S)$ and $E_\theta(S)$, and hence $H_\phi(t)$ and $E_\theta(t)$ vanish at $\theta = 0$. Also in equation 12 the angle dependent factor is

$$G(\theta) = \frac{\cos\theta}{\sin^2\theta}\left[\cosh(\frac{\frac{1}{2}SL\cos\theta}{c}) - \cosh(\tfrac{1}{2}SL/c)\right] . \qquad 20)$$

At angles near $\theta = 0$ the $\cos\theta$ factor is approximately unity and can be ignored. The rule of l'Hospital can be applied to the remaining factor to yield

$$\lim_{\theta \to 0} G(\theta) = \lim_{\theta \to 0} \frac{\frac{1}{2}(SL/c)\sin\theta \sinh(\frac{\frac{1}{2}SL\cos\theta}{c})}{2\sin\theta\cos\theta} \qquad 21)$$

$$= \lim_{\theta \to 0} \frac{\frac{1}{4}(SL/c)\sinh(\frac{1}{2}SL\cos\theta)}{c} \neq 0 \; . \qquad 22)$$

Hence, $E_r(S)$ and $E_r(t)$ don't vanish at $\theta = 0$. That $H_\phi(t)$ and $E_\theta(t)$ vanish at $\theta = 0$ is also heuristically obvious in the time domain: as the relative delay between two ideal impulses of equal strength but of opposite polarity approaches zero, the two impulses will cancel.

The near and intermediate field components produce equally distorted versions of the excitation current waveform. They would consist of the superposition of of progressively delayed replicas of the first and second time integrals of the excitation current waveform except near $\theta = 0$ where they form the superposition of components proportional to the excitation current and its first time integral.

Reflection off a Smooth, Flat, Lossy Conducting Plane

Another phenomenon of great importance to the suitability of nonsinusoidal, bi-valued functions and related functions (time derivatives and integrals) to radar applications is their reflection from smooth metallic surfaces. The importance stems from the fact that many radar targets of interest consist of structures of smooth metallic surfaces of a planar nature or having a fairly large radius of curvature, approaching a plane surface over small portions of the surface.

The reflection process is also part of the antenna problem if it is desired to use an appropriately shaped metallic surface to either focus or to form directional beams of nonsinusoidal electromagnetic waves.

The reflection process is similar to the terminated transmission line problem in that a similar reflection coefficient completely describes the process. In order to establish the reflection properties of nonsinusoidal waves at arbitrary angles of incidence and at arbitrary polarization with respect to the plane of incidence, it is only necessary to study two special cases: polarization of the electric vector perpendicular to the plane of incidence and that parallel to it. Any other case of general linear polarization may be treated as a linear superposition of these two special cases.

As discussed in Chapter VI the quantity that describes, or characterizes, the reflection properties of nonsinusoidal waves is a form of impulse response derived from a form of reflection transfer function which turns out to be the sinusoidal reflection coefficient of the reflecting surface. This fact is established in the ensuing development.

In both cases of polarization the frequency dependence of the conductance of the metallic reflecting surface is included since this still yields a mathematically tractable situation even though adding considerable complexity to the analysis. In both cases of polarization that factor could have been ignored in order to arrive at a solution of more reasonable proportions. However, the more complex solution does lend greater insight into the limiting factors of the reflection process.

The conventional textbook approach to the (sinusoidal) reflection problem is to determine the total electric and magnetic field consisting of the superposition of the incident and reflected fields outside of the reflecting surface. The interest here, however, lies in the effect of the reflection process at the surface on the reflected portion of an incident nonsinusoidal wave. It is shown that the reflection coefficient ϱ is the desired surface sinusoidal transfer function which relates the amplitudes and phases of the incident and reflected plane sinusoidal waves. The inverse Fourier or Laplace transforms then yield analogous quantities in the time domain.

Polarization of $\bar{E}$ Perpendicular to the Plane of Incidence

Fortunately elementary and standard techniques may be used to attack the problem of the nonsinusoidal reflection process. Those of Chapter 6 of Ramo, Whinnery, and Van Duzer (1965) are utilized in this section.

Figure 8 depicts the geometrical situation of interest. The vector quantities $\bar{k}_i$ and $\bar{k}_r$ represent the propagation vectors of the incident and reflected waves respectively. Both lie in the xz plane forming the plane of incidence. The angle of incidence, θ, is equal to the angle of reflection, θ'. The incident electric field vector is $\bar{E}_i$ while the reflected quantity is $\bar{E}_r$. Both are perpendicular to the plane of incidence.

The quantity of interest is the ratio E_r/E_i at the surface with $z = 0$. This ratio is the surface sinusoidal transfer function. It may be expressed in terms of the angle of incidence θ, the operating (sinusoidal) frequency, and the parameters of the conducting interface, and by examining the electric field equation that results from equation 22, page 361 of Ramo, Whinnery, and Van Duzer (1965). There the only field component is in the y direction which yields for the tangential (to the surface) component

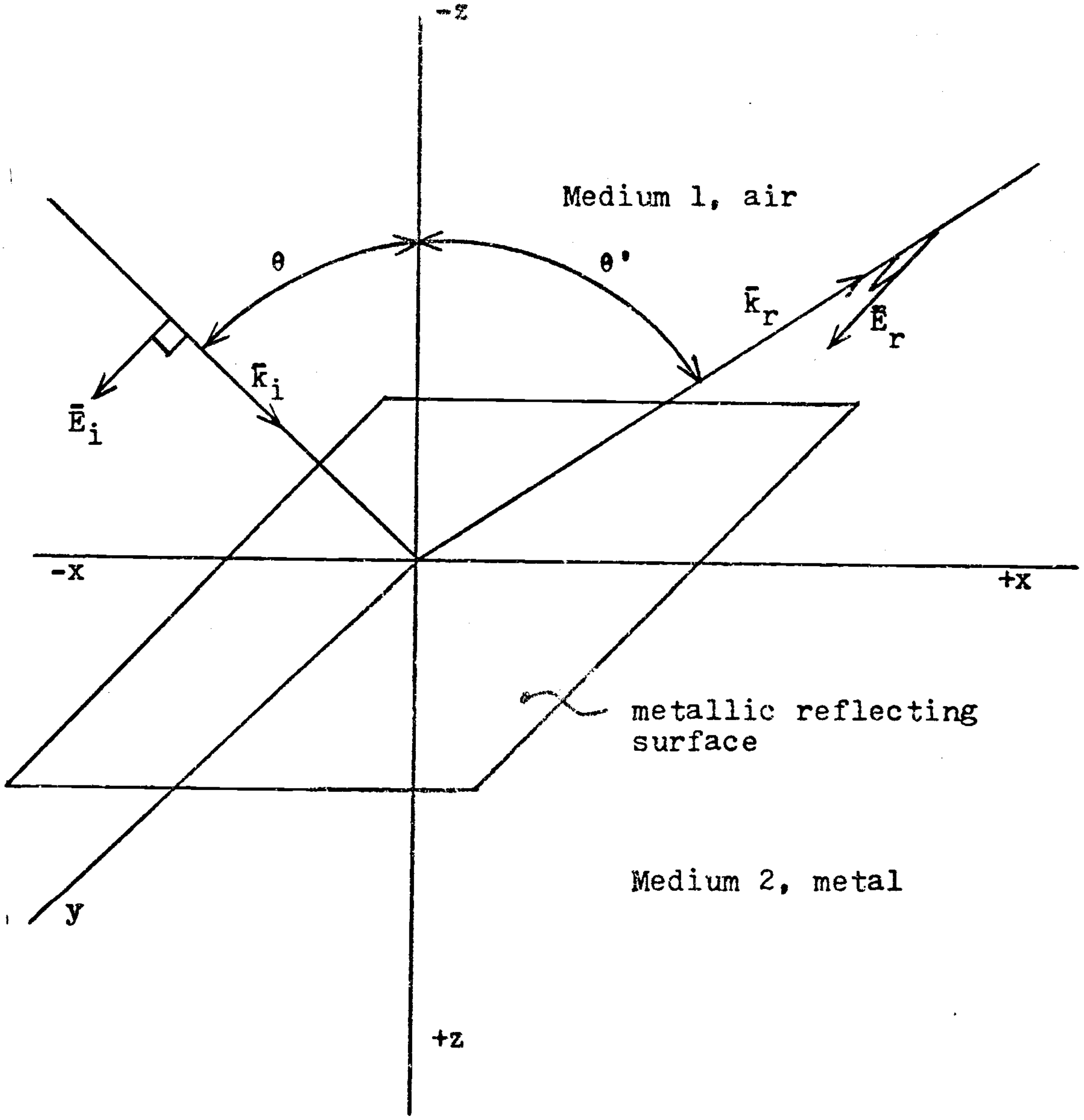


Figure 8. Geometry of the reflection process.

$\bar{E}_t =$

$$\bar{a}_y \left[E_i \, e^{-jkx \sin \theta - jkz \cos \theta} + \rho E_i \, e^{-jkx \sin \theta + jkz \cos \theta} \right] \qquad 23)$$

where ρ is the ratio E_r/E_i which is defined in terms of the parameters of the system by

$$\rho = (Z_L - Z_{z1})/(Z_L + Z_{z1}) \qquad 24)$$

with

$$Z_{z1} = n_1 \sec \theta \qquad 25)$$

and

$$Z_L = n_2/\sqrt{1 - \frac{e_1}{e_2} \sin^2\theta} \ . \qquad 26)$$

The quantites n_1 and n_2 are the intrinsic wave impedances of media one and two respectively and e_1 and e_2 are their respective permittivities. As simple as these equations appear, n and e represent complex functions of frequency which greatly complicate the analysis.

Examination of equation 23 shows that the first term represents the incident wave while the second term represents the reflected wave. Hence

$$E_r/E_i = \frac{\rho E_i \, e^{-jkx \sin \theta + jkz \cos \theta}}{E_i \, e^{-jkx \sin \theta - jkz \cos \theta}} \qquad 27)$$

$$= \rho \, e^{j2kz \cos \theta} \ . \qquad 28)$$

Our interest is in this quantity at the surface at $z = 0$ so that ρ is indeed the required surface transfer function.

Since medium one is air, n_1 is essentially the intrinsic wave impedance of free space:

$$n_1 \approx n_o = \sqrt{u_o/e_o} \qquad 29)$$

where u_o is the permeability of vacuum and e_o permittivity, both being independent of frequency.

The intrinsic wave impedance of the second medium, the metal, is

$$n_2 = \sqrt{u_2/e_2} \quad . \qquad 30)$$

Since only a non-permeable conductor is being considered

$$n_2 \approx \sqrt{u_o/e_2} \quad . \qquad 31)$$

It now remains to establish the nature of e_2, the permittivity of the conducting material. The general expression for a dielectric with high conductance, or a metal, is

$$e_2 = e_m = (e' + \sigma/j\omega) = e'(1 + \sigma/j\omega e') \qquad 32)$$

with e' the dielectric of the metal and σ its conductivity. For all frequencies of interest the second term in equation 32 normally far exceeds the first which could be neglected without greatly affecting the outcome. However, leaving e' in the expression yields a tractable solution.

The value of the permittivity, e', for metals is not measurable, but it is felt by many authorities (Adler, Chu, and Fano, 1960 and Bromwell and Beam, 1947) to be of the same order as that for free space, hence

$$e_2 \approx e_o + \sigma/j\omega = e_o(1 + \sigma/j\omega e_o) \qquad 33)$$

which, in terms of the Laplace transform variable, becomes

$$e_2 \approx e_o(1 + \sigma/ Se_o) . \qquad 34)$$

The conductivity of a metal is also known to be frequency dependent. In terms of the Laplace transform variable it is given by Kittel (1968) as

$$\sigma(S) = \sigma_o/(1 + S/v_c) = \sigma_o\, v_c/(S + v_c) \qquad 35)$$

where σ_o is the low frequency conductivity and v_c is the collision frequency of the free electron gas within the conducting material. Table III of Chapter IX lists these parameters and the plasma frequency of four widely used conducting materials.

Substituting equation 35 into equation 34 yields an expression for the detailed frequency dependence of the complex permittivity of a metal. Then, substituting that modified form of equation 34 into 31 and 26 then 30 into 26 and 24 gives the operational expression for the surface transfer function by the following sequence of steps:

$$e_2 = e_o(1 + \sigma_o\, v_c/(e_o S(S + v_c))) \;,$$

$$= e_o \left[\frac{S(S + v_c) + \sigma_o\, v_c/e_o}{S(S + v_c)}\right] , \qquad 36)$$

$$n_2 = \left[\frac{u_o\; S(S + v_c)}{e_o(S(S + v_c) + \sigma_o\, v_c/e_o)}\right]^{\frac{1}{2}} ,$$

$$= n_o\left[\frac{S(\,S \;+\; v_c)}{S(S + v_c) + \sigma_o\, v_c/e_o}\right]^{\frac{1}{2}} , \qquad 37)$$

$$Z_L = \left[\frac{S(S + v_c)}{S(S + v_c) + \sigma_o\, v_c/e_o}\right]^{\frac{1}{2}} n_o \left[1 - \frac{\sin^2\theta\; S(S + v_c)}{S(S + v_c) + \sigma_o v_c/e_o}\right]^{-\frac{1}{2}}$$

$$= n_o\left[\frac{S(S + v_c)}{S(S + v_c)\,\cos^2\theta + \sigma_o\, v_c/e_o}\right]^{\frac{1}{2}} , \qquad 38)$$

and

$$Z_L = n_o\;(A/B)^{\frac{1}{2}}$$

where $A = S(S + v_c)$ and $B = S(S + v_c)\,\cos^2\theta + \sigma_o v_c/e_o$.

Also

$$Z_{z1} = n_o \sec\theta = n_o \,/\, \cos\theta \qquad 39)$$

and

$$\rho(S) = \frac{Z_L - Z_{z1}}{Z_L + Z_{z1}} = \frac{n_o\; A^{\frac{1}{2}} - n_o B^{\frac{1}{2}}/\cos\theta}{n_o\; A^{\frac{1}{2}} + n_o B^{\frac{1}{2}}/\cos\theta} , \qquad 40a)$$

or

$$\varrho(S) = \frac{A^{\frac{1}{2}} - B^{\frac{1}{2}}/\cos\theta}{A^{\frac{1}{2}} + B^{\frac{1}{2}}/\cos\theta} \qquad 40b)$$

where Z_{z1} is n_1 sec θ. Equation 40b then reduces to

$$\varrho(S) = \frac{(S(S + v_c))^{\frac{1}{2}} - (S(S + v_c) + \sigma_0 v_c/e_0 \cos^2\theta)^{\frac{1}{2}}}{(S(S + v_c))^{\frac{1}{2}} + (S(S + v_c) + \sigma_0 v_c/e_0 \cos^2\theta)^{\frac{1}{2}}} \qquad 41)$$

The direct inverse transform of equation 41 is not available in published tables. This fact, however, is of no great consequence since our interest lies in a form that will isolate, or show, the distortion effects of the reflection process. By the expedient of clearing the denominator of radicals by multiplying both numerator and denominator by $(S(S + v_c))^{\frac{1}{2}} - (S(S + v_c) + \alpha)^{\frac{1}{2}}$ a form results which clearly shows the distorting effect:

$$\varrho(S) = -1 - \frac{2S^2}{\alpha} - \frac{2\,v_c S}{\alpha} + \frac{2}{\alpha}\left[S(S + v_c)(S(S + v_c) + \alpha)\right]^{\frac{1}{2}} \qquad 42)$$

where α has been substituted for $\sigma_0 v_c/e_0 \cos^2\theta$ in order to clarify and facilitate notation.

The corresponding time domain unit impulse response is

$$\varrho(t) = -\delta(t) - \frac{2v_c}{\alpha}\,\delta'(t) - \frac{2}{\alpha}\,\delta''(t) + \frac{2}{\alpha}\,L^{-1}\left[\left[S(S + v_c)(S(S + v_c) + \alpha)\right]^{\frac{1}{2}}\right]. \qquad 43)$$

Equation 43 is a very interesting and meaningful result. Examining each term individually we see that the first, $-\delta(t)$, when convolved with a given nonsinusoidal incident wave yields an exact, but inverted, replica of the incident wave on reflection while the remaining terms produce additive distortion components.

The first time derivative of the impulse function in the second term, and the second time derivative in the third, respectively yield the first and second time derivatives of the incident wave as additive distortion terms. However, notice the numerical values of their multiplying coefficients. That of the first, for copper, is

$$2v_c/\alpha = 2e_o\cos^2\theta/\sigma_o \approx 2x8.85 \cos^2\theta \text{ x } 10^{-12}/6x10^7$$

$$\approx 0.3 \text{ x } 10^{-18} \cos^2\theta$$

while that for the second is

$$2/\alpha = 2e_o \cos^2\theta / \sigma_o v_c \approx 2x8.85x10^{-12}\cos^2\theta /6x4x10^{20}$$

$$\approx 0.74 \text{ x } 10^{-32} \cos^2\theta \text{ .}$$

Corresponding values for gold and silver are close to those above.

The relative magnitudes of the reflected components corresponding to the terms of equation 43 depend upon the the temporal form of the incident wave, $E_i(t)$, and its first and second time derivatives. These quantities are relatively easy to estimate for a bi-valued wave such as a Walsh function or one of its time derivatives. The relevant parameters of such pulse-like waveforms are, for the first term, the peak value attained, E_i. The relevant parameter in the second term is the peak of its time derivative which is approximately $2E_i/T_r$ where T_r is the transition time changing from $-E_i$ to $+E_i$. For the third term the peak of the second time derivative which is $2E_i/aT_r^2$ is the important parameter and the factor a is the fractional part of the transition time, T_r, required for the first time derivative to reach its peak value. The value of the quantity (a) might vary from one tenth to slightly less than one half for typical applications.

The above quantities applied to equation 43 yield relative magnitudes of reflection components corresponding to the first three terms. For the fastest rise times that might be electronically produced in the near future, say ten picoseconds, we have

first term: E_i
second term: $6 \times 10^{-4} E_i \cos^2\theta$
third term: $1.48 \times 10^{-8} E_i \cos^2\theta$ with $a = 0.01$.

The above estimates indicate that the distortion components due to the second and third terms of equation 43 are negligible for the bi-valued waveforms under study.

Notice also that as the angle of incidence increases, $\cos^2\theta$ decreases causing the magnitudes of all the distortion terms to decrease also. It is obvious that normal incidence is the worst case. Also note that all distortion terms would vanish if the conductivity of the reflecting material were infinite.

The last term is also extremely small although its detailed nature is not known. Although the inverse transform of the last term of equation 42 is not easily obtained, some information about it can be obtained by factoring it as follows:

$$\left[S(S + v_c)(S(S + v_c) + \alpha)\right]^{\frac{1}{2}} =$$

$$(S(S + v_c))^{\frac{1}{2}} \times \left[(S(S + v_c) + \alpha)\right]^{\frac{1}{2}} =$$

$$(S(S + v_c))^{\frac{1}{2}} \times \left[(S + a)(S + b)\right]^{\frac{1}{2}} \qquad 44)$$

where a and b result from factoring $(S(S + v_c) + \alpha)$. The corresponding time domain quantities may be convolved to form the resultant component of the overall unit impulse response of equation 43.

Applying the transform domain shifting theorem to item 15 of page 246 of Roberts and Kaufman (1966) and setting $a^2 = 0$ in that item gives

$$L^{-1}\left[\left[(S + a)(S + b)\right]^{\frac{1}{2}}\right] = \frac{e^{-at}}{2\,\pi^{\frac{1}{2}}\,t^{3/2}} * \frac{e^{-bt}}{2\,\pi^{\frac{1}{2}}\,t^{3/2}}. \qquad 45)$$

Direct convolution of the two factors above (developed in Appendix VI) yields

$$L^{-1}\left[\left[(S + a)(S + b)\right]^{\frac{1}{2}}\right] =$$

$$\tfrac{1}{4}\,(a - b)^{4}\, e^{-\frac{1}{2}(a + b)t}\, J_1(\tfrac{1}{2}(a - b)t) \qquad 46)$$

for (a - b) imaginary or, for (a - b) real,

$$L^{-1}\left[\left[(S + a)(S + b)\right]^{\frac{1}{2}}\right] =$$

$$\tfrac{1}{4}\,(a - b)\, e^{-\frac{1}{2}(a + b)t}\, I_1(\tfrac{1}{2}(a - b)t) \qquad 47)$$

which are both finite, pulse-like functions having decay time constants of the order of 2/(a + b) or less due to the Bessel function factors. Similar expressions result for $(S(S + v_c))^{\frac{1}{2}}$ of equation 44 with b set to zero and $a = v_c$.

In terms of the constants of equation 44, its time domain equivalent is the convolution of two quantities similar to those of equation 46. Making the appropriate substitutions, the last term of equation 43 becomes

$$I_4 = v_c^2\frac{(a - b)^2}{2\alpha}\left[e^{-\frac{1}{2}v_c t}\,\frac{I_1(\frac{1}{2}v_c\ t)}{\frac{1}{2}v_c\ t}\right] * \left[e^{-\frac{1}{2}(a+b)t}\,\frac{J_1(\frac{1}{2}\,|a-b|\,t)}{\frac{1}{2}\,v_c t}\right] \qquad 48)$$

Numerically the multiplying coefficient is very large being on the order of 3×10^{27}. Substituting the quantities of equation 47 into a convolution integral with $(a + b) = v_c$ and for copper as the conducting material, I_4 becomes

$$I_4 = 3.46 \times 10^{27} \cos\theta \, e^{-2.08\text{x}10^{13}t} \times$$

$$\int_0^t \frac{I_1(2.08\text{x}10^{13}\ x)\ J_1(3.30\text{x}10^{16}(t-x)/\cos\theta)\ dx}{(2.08\text{x}10^{13}\ x)(3.30\text{x}10^{16}(t-x))} \,.$$

49)

Because of the large difference between the coefficients in the arguments of the two Bessel functions, the following approximation may be made:

$$I_4 \approx \frac{3.46 \times 10^{27} \cos\theta}{3.30 \times 10^{16}} \times$$

$$\left[\frac{e^{-2.08\text{x}10^{13}t}\ I_1(2.08 \times 10^{13}t)}{(2.08 \times 10^{13}\ t)}\right] \int_0^{3.30\text{x}10^{16}t} \frac{J_1(X)\ dX}{X} \qquad 50)$$

which applies for $t > 10^{-16}$ seconds. For these values of time the value of the integral factor approaches a constant value near unity. The value of I_4 is governed principally by the time functions outside of the integral sign. For smaller values of time, the time functions preceding the integral are close to the constant value of 0.5 with the value of I_4 then determined by the integral. Since for this case the value of X in the integrand is small, $J_1(X)/X \approx 0.5 - X^2/16$ the integral of which is $0.5X - X^3/48$. For values of $t << 10^{-16}$ seconds the integral is a linear function of time t. Its amplitude, therefore, starts at zero and increases very rapidly to larger values.

However, the peak amplitude of the quantity in either equation 48 or 49 is not of great significance. Since all of the quantities discussed here are components of a system unit impulse response, it is their relative areas that are of significance when convolved with an appropriate excitation function.

The area of the quantity in equation 48 or 49 is approximated by

$$A_4 \approx \frac{3.46 \times 10^{27} \cos \theta}{3.30 \text{x} 2.08 \text{x} 10^{29}} \int_0^\infty \frac{e^{-T} I_1(T)\, dT}{T} = 5.04 \text{x} 10^{-3} \cos \theta$$

since the value of the integral is unity. Therefore, when convolved with a suitable excitation function, or wave, this term contributes less than one half of one percent to the resulting wave. Also note that this error term becomes less as the angle of incidence is increased.

It is then concluded that a typical smooth conducting surface will reflect nonsinusoidal electromagnetic waves when polarized normal to the plane of incidence as long as its first and second time derivatives are small enough to make the second and third terms of equation 43 much less than unity. Specific conditions on excitation time derivatives are

$$\left| dE_i/dt \right| << \alpha/2v_c = 10^{19} \text{ x } 0.328/ \cos^2\theta$$

$$\left| d^2E_i/dt^2 \right| << \tfrac{1}{2}\alpha = 1.36 \times 10^{32}/\cos^2\theta \ .$$

Notice also that all the distortion components are maximum at normal incidence.

Reflection off a Smooth, Flat, Lossy Conducting Plane with Polarization in the Plane of Incidence

As in the case of polarization normal to the plane of incidence the same elementary techniques may be used in the case of polarization in the plane of incidence. The reflection process in this case is also completely described by a reflection coefficient, ρ . This fact is established by equations 14 and 16, page 360 of Ramo, Whinnery, and Van Duzer (1965) which yield for the incident and reflected waves

$$\bar{E}_i = E_+(\bar{a}_x \cos \theta - \bar{a}_z \sin \theta)e^{-jk(x \sin \theta + z \cos \theta)} \quad 51a)$$

$$\bar{E}_r = E_+\rho(\bar{a}_x \cos \theta + \bar{a}_z \sin \theta)e^{-jk(x \sin \theta - z \cos \theta)} \quad 51b)$$

where the factors in parentheses are vector quantities of unity magnitude. Taking the ratio of $|\bar{E}_r| / |\bar{E}_i|$ at the surface z = 0 yields

$$|\bar{E}_r| / |\bar{E}_i| = \rho \ . \quad 52)$$

In this case the reflection coefficient is also described in terms of frequency and the parameters of the conductor as

$$\rho(S) = (Z_L - Z_{z1})/(Z_L + Z_{z1}) \quad 53)$$

where

$$Z_{z1} = n_1 \cos \theta = n_o \cos \theta \ , \quad 54)$$

and

$$Z_L = n_2 \cos \theta'' = n_2 \left[1 - \frac{e_1}{e_2} \sin^2\theta\right]^{\frac{1}{2}} \qquad 55)$$

and n_2 is the intrinsic wave impedance of the metallic medium and θ'' = angle of refraction:

$$n_2 = (u_o/e_2)^{\frac{1}{2}} \approx (u_o/e_o(1 + \sigma_o v_c/e_o S(S + v_c)))^{\frac{1}{2}} \qquad 56)$$

$$n_2 \approx n_o/(1 + \sigma_o v_c/e_o S(S + v_c))^{\frac{1}{2}} \qquad 57)$$

where the relationship $e_2 \approx e_o + \sigma_o v_c/S(S + v_c)$ has been used for the permittivity of the metallic medium.

Rearranging equation 53 yields

$$\rho(S) =$$

$$\frac{\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}} - (S(S + v_c) + A)}{\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}} + (S(S + v_c) + A)} \qquad 59)$$

where A has been substituted for $\sigma_o v_c/e_o$ to simplify notation. The value of v_c for copper is

$$v_c = 4.16 \times 10^{13} \text{ and } A = 2.725 \times 10^{32} .$$

Equation 59 may be decomposed in many ways to reveal the distortion components explicitly. Techniques similar to those in the case of normal polarization are used in the following by adding and subtracting the quantity $\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}}$ to and from the numerator of equation 59 to yield

$$\rho(S) =$$

$$-1 + 2 \frac{\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}}}{\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}} (S(S + v_c) + A)} \quad . 60)$$

Equation 60 clearly separates the ideal term, -1, from the second distortion term. The second term may be further simplified by clearing the denominator of radicals

$$\rho(S) = - 1$$

$$+ 2 \frac{\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}}}{A \left[S(S + v_c)\left(\frac{1 - 2\cos^2\theta}{\cos^2\theta}\right) - A\right]} \; X$$

$$\left[\left[S(S + v_c)(S(S + v_c) + A/\cos^2\theta)\right]^{\frac{1}{2}} - (S(S + v_c) + A)\right] . 61a)$$

Carrying out the indicated multiplication in equation 61a and noting that $(1 - 2\cos^2\theta)/\cos^2\theta = -2 + \sec^2\theta = -2 + 1 + \tan^2\theta = \tan^2\theta - 1$ by use of standard trigonometric identities, the equation reduces to

$$\rho(S) = -1 + \frac{2\, S(S + v_c)(S(S + v_c) + A/\cos^2\theta)}{A(\tan^2\theta - 1)(S(S + v_c) - A/(\tan^2\theta - 1))}$$

$$- \frac{2(S(S + v_c) + A) \left[S(S + v_c)(S(S + v_c) + \frac{A}{\cos^2\theta})\right]^{\frac{1}{2}}}{A(\tan^2\theta - 1)(S(S + v_c) - A/(\tan^2\theta - 1))}$$

61b)

Notation may be greatly simplified by factoring the quadratic quantities above giving

$$S^2 + Sv_c + A/\cos^2\theta \approx (S + c)(S + d) \qquad 61c)$$

and

$$S^2 + Sv_c - A/(\tan^2\theta - 1) = (S + a)(S + b) \qquad 61d)$$

where

$$a = \tfrac{1}{2}v_c + j\left[-\tfrac{1}{4}v_c^2 + A/(1 - \tan^2\theta)\right]^{\frac{1}{2}},$$

$$b = \tfrac{1}{2}v_c - j\left[-\tfrac{1}{4}v_c^2 + A/(1 - \tan^2\theta)\right]^{\frac{1}{2}}$$

$$c = \tfrac{1}{2}v_c + j\left[-\tfrac{1}{4}v_c^2 + A/\cos^2\theta\right]^{\frac{1}{2}} \approx \tfrac{1}{2}v_c + j\,A^{\frac{1}{2}}/\cos\theta$$

$$d \approx \tfrac{1}{2}v_c - j\,A^{\frac{1}{2}}/\cos\theta$$

After considerable algebraic manipulation equation 61b (see Appendix VII) may be reduced to

$$\begin{aligned}\rho(S) = -1 &+ \frac{2v_c\,S}{A(\tan^2\theta - 1)} + \frac{2\,S^2}{A(\tan^2\theta - 1)} \\ &+ \frac{2\,S(S + v_c)}{(1 - \cot^2\theta)^2(S + a)(S + b)} \\ &- \frac{2\left[S(S + v_c)(S + c)(S + d)\right]^{\frac{1}{2}}}{A(\tan^2\theta - 1)} \\ &- \frac{\tfrac{1}{2}\tan^2(2\theta)\left[S(S + v_c)(S + c)(S + d)\right]^{\frac{1}{2}}}{(S + a)(S + b)}\,, \qquad 62)\end{aligned}$$

Notice that two critical values of θ appear to exist for the quantities a and b above. First, the term $A/(1 - \tan^2\theta)$ under the radical is positive for θ less than 45°, and it is much greater than $\frac{1}{4}v_c^2$ making the radical term completely imaginary. At $\theta = 45^\circ$ a and b are both imaginary and of infinite magnitude. However, for angles $45^\circ < \theta < 90^\circ$, $A/(1 - \tan^2\theta)$ changes sign and starts decreasing in magnitude and the radical is also real so that

$$a \approx \tfrac{1}{2}v_c + j\left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}} \qquad 63a)$$

$$0 < \theta < 45^\circ$$

$$b \approx \tfrac{1}{2}v_c - j\left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}} \qquad 63b)$$

and

$$a \approx \tfrac{1}{2}v_c + \left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}} \qquad 63c)$$

$$45^\circ < \theta < 89.5^\circ.$$

$$b \approx \tfrac{1}{2}v_c - \left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}} \qquad 63d)$$

Also, for θ between about 72° and 89.5° the denominator $(\tan^2\theta - 1)$ may be approximated by $\tan^2\theta$ so that

$$a \approx \tfrac{1}{2}v_c + A^{\frac{1}{2}}/\tan\theta \qquad 63e)$$

$$72^\circ < \theta < 89.5^\circ .$$

$$b \approx \tfrac{1}{2}v_c - A^{\frac{1}{2}}/\tan\theta \qquad 63f)$$

Another special condition, although of no particular interest in the study of the reflection process, is the case where the angle of incidence is very near 90°. In this case $\frac{1}{4}v_c^2 >> A/\tan^2\theta$, so that

$$a \approx v_c - A/(v_c \tan^2\theta)$$

$$b \approx A/(v_c \tan^2\theta)$$

$$c \approx j\, A^{\frac{1}{2}}/\cos\theta$$

$$d \approx -j\, A^{\frac{1}{2}}/\cos\theta \quad .$$

Taking the direct inverse transform of equation 62 would provide the desired unit impulse response of this reflection process. However, it is not necessary to obtain the complete inversion of that equation in order to gain useful information about this impulse response. The following suffices:

$$\begin{aligned}
\rho(t) = &- \delta(t) + \frac{2\, v_c}{(\tan^2\theta - 1)A}\, \delta'(t) + \frac{2}{(\tan^2\theta - 1)A}\, \delta''(t) \\
&+ \frac{2}{(1 - \cot^2\theta)^2}\left[\frac{a(a-v_c)}{(a-b)}\, e^{-at} - \frac{b(b-v_c)}{(a-b)}\, e^{-bt} - u(t)\right] \\
&- \frac{2}{(\tan^2\theta - 1)A}\, L^{-1}\left[\left[S(S + v_c)(S + c)(S + d)\right]^{\frac{1}{2}}\right] \\
&- \tfrac{1}{2}\tan^2(2\theta)\left[e^{-at} * e^{-bt} * L^{-1}\left[S(S + v_c)(S + c)(S + d)\right]\right]
\end{aligned} \qquad 64)$$

As in the case for normal polarization, the first term of equation 64 above represents the distortion-free reflected component while all the remaining terms represent distortion components. The second and third terms again represent first and second derivative distortion components which are small for all angles of incidence except those near 45° if the time derivatives are sufficiently small. As θ approaches 45° both terms approach infinity and can badly distort the reflected wave. For θ greater than 45°, but less than 90°, both terms again become small and also change sign.

The remaining terms also suffer similar amplitude increases as θ appoaches 45°. They undergo changes in form because of the dependence of a, b, c, and d on θ shown explitly in the equations preceding equation 62. Those cases are examined individually below.

For the case that θ is less than 45° the fourth term in equation 64 is

$$T_4 = \frac{2}{(1 - \cot^2\theta)^2} X$$

$$\left[\left[\tfrac{1}{4}v_c^2 + A/(1 - \tan^2 \theta) \right] e^{-\frac{1}{2}v_c t} \frac{\sin(t\, A/(1 - \tan^2\theta)^{\frac{1}{2}})}{(A/(1 - \tan^2\theta)^{\frac{1}{2}}} - u(t) \right] \qquad 65)$$

The above expression has large values near $t = 0$ and is of very short duration. However, it is the total time integral that is of importance when used in the convolution integral with a given excitation function.

The integral of the bracketed term in equation 65 is

$$\text{area} = \frac{\tfrac{1}{4}v_c + A/(1 - \tan^2\theta)}{A/(1 - \tan^2\theta)} \text{ X}$$

$$\int_0^\infty e^{-\frac{1}{2}v_c \left[(1 - \tan^2\theta)/A\right]^{\frac{1}{2}} T} \sin(T)\, dT = 1 \qquad 66)$$

since the value of the integral is the reciprocal of its multiplying coefficient. Note that the unit step function term has been neglected for the moment.

Taking into account the coefficient $2/(1 - \cot^2\theta)^2$, the integral of the fourth term becomes simply

$$I_4 = 2/(1 - \cot^2\theta)^2 \ .$$

The fourth term has negligible effect only for those values of θ for which $\left|2/(1 - \cot^2\theta)^2\right| << 1$. If ten percent is selected as a criterion, angles of incidence such that

$$\left|2/(1 - \cot^2\theta)^2\right| < 0.1,$$

then angles of incidence of 10° or greater will produce noticible distortion of the reflected wave. The same argument applies to the unit step function term since it it multiplied by the same coefficient. Its effect is to produce the time integral of the incident wave.

For the case where θ is greater than 45° but less than about 89.5°, the sign of $A/(1 - \tan^2\theta)$ changes to negative which changes equation 65 to

$$T_4 = \frac{2}{(1 - \cot^2\theta)^2} \quad X$$

$$\left[\left[\tfrac{1}{4}v_c^2 - A/(\tan^2\theta - 1)\right] e^{-\frac{1}{2}v_c t} \; \frac{\sinh(t\left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}})}{\left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}}}\right.$$

$$\left. - \; u(t)\right] \; . \qquad 67)$$

A curious phenomenon occurs over the range of θ such that $|A/(\tan^2\theta - 1)| > \frac{1}{4}v_c^2$ which corresponds to 45° to about 89.9°. Recall the exponential form of the hyperbolic sine:

$$e^{-\frac{1}{2}v_c t}\sinh(t(A^{\frac{1}{2}}/(\tan^2\theta -)^{\frac{1}{2}}) =$$

$$\tfrac{1}{2}e^{t((A^{\frac{1}{2}}/(\tan^2\theta - 1)^{\frac{1}{2}} - \frac{1}{2}v_c)}$$

$$-\tfrac{1}{2}e^{-t((A^{\frac{1}{2}}/(\tan^2\theta - 1)^{\frac{1}{2}} - \frac{1}{2}v_c)} \qquad 68)$$

When $|A/(\tan^2\theta - 1)| > \frac{1}{4}v_c^2$ notice that the first term in equation 68 has a positive exponent producing a component of the impulse response which grows exponentially with time! Considering the energy required for such growth, it is obvious that some important energy limiting process has been neglected in the analysis. However, it must be expected that severe distortion occurs over this range of θ.

For angles of incidence between 89.9° and 90° both exponential terms are well behaved: i.e. a pulse-like waveform of unit area. However, for that condition the coefficient $2/(1 - \cot^2\theta)^2$ is quickly approaching zero so that both terms in equation 65 become negligibly small.

The fifth term in equation 64, except for becoming infinite at $\theta = 45^\circ$, is identical to the radical term in equation 43 for the normally polarized case. It is well behaved except near 45°.

The last term in equation 64 is similar to the fifth term in that its coefficient also becomes infinite as θ approaches 45°. Except for its angular behavior it is the result of convolving $e^{-at} * e^{-bt}$ with the inverse transform of the radical factor. The convolution of e^{-at} and e^{-bt} is

$$e^{-at} * e^{-bt} = (e^{-at} - e^{-bt})/(b - a) \ . \qquad 69)$$

The total area of this convolution function is

$$\text{Area} = 1/|ab| \ . \qquad 70)$$

The convolution of e^{-at} and e^{-bt} is also very dependent on the angle of incidence, θ. For $0 < \theta < 45^\circ$

$$e^{-at} * e^{-bt} =$$

$$-\left[(1 - \tan^2\theta)/A\right]^{\frac{1}{2}} e^{-\frac{1}{2}v_c t} \sin\left(t\left[A/(1 - \tan^2\theta)\right]^{\frac{1}{2}}\right) \qquad 71)$$

which is well behaved. When convolved with

$$L^{-1}\left[\left[S(S + v_c)(S + c)(S + d)\right]^{\frac{1}{2}}\right]$$

it produces a convolution quantity on the order of $\left[A/(1 - \tan^2\theta)\right]^{\frac{1}{2}}$. For θ such that $2/(1 - \cot^2\theta)^2$ is small, this term can be neglected. However, for $45^\circ < \theta < 89.9^\circ$ the convolution of e^{-at} and e^{-bt} becomes

$$e^{-at} * e^{-bt} =$$

$$\left[(\tan^2\theta - 1)/A\right]^{\frac{1}{2}} e^{-\frac{1}{2}v_c t} \sinh \left(t \left[A/(\tan^2\theta - 1)\right]^{\frac{1}{2}}\right) \qquad 72)$$

which grows exponentially with time as does the fourth term of equation 64. For angles of incidence between 89.9° and 90° it is again well behaved being exponentially damped with time.

It may be concluded that for polarization in the plane of incidence, reflection of nonsinusoidal waves will occur with sever distortion at angles of incidence of 45° or more, but will reflect with little distortion for angles less than 45°.

Reflection from Lossy Dielectric Materials

A case of reflection of great interest in radar mapping is the reflection of electromagnetic waves off of imperfect dielectric materials near normal incidence. Reflection back in the direction of the incident wave would be the situation of primary interest to the monostatic radar case. Although not studied in detail, the general form of the solution is briefly outlined.

The reflection coefficient at normal incidence off a lossy dielectric is especially simple being

$$\rho(S) = \frac{n_2 - n_1}{n_2 + n_1} = \frac{e_o^{\frac{1}{2}} - e_2^{\frac{1}{2}}}{e_o^{\frac{1}{2}} + e_2^{\frac{1}{2}}} \qquad 73)$$

which may be modified to yield a form which explicitly reveals the distortion characteristics as additive terms:

$$\rho(S) = -1 + \frac{2e_o^{\frac{1}{2}}\,(e_2^{\frac{1}{2}} - e_o^{\frac{1}{2}})}{e_2 - e_o}\,. \qquad 74)$$

The above reflection coefficient may also be expressed in terms of a relative dielectric constant, $k' = e_2'/e_o$ and a relative loss factor $k'' = e_2''/e_o$ as follows:

$$\rho(S) = -1 + \frac{2((k' - jk'')^{\frac{1}{2}} - 1)}{k' - jk'' - 1} \qquad 75)$$

$$\rho(S) = -1 - \frac{2}{k' - jk'' - 1} + \frac{2(k' - jk'')^{\frac{1}{2}}}{k' - jk'' - 1} \qquad 76)$$

where $e_2 = e_2' - je_2''$.

In the time domain equation 76 is

$$\rho(t) = - \delta(t) - 2\, L^{-1}\left[1/(k' - jk'' - 1)\right] + 2\, L^{-1}\left[\frac{(k' - jk'')^{\frac{1}{2}}}{k' - jk'' -1}\right] \qquad 77)$$

which also explicitly reveals the second and third terms as additive distortion terms. In order to use equation 77 the frequency dependence of k' and k" must be known. Since these two quantities are complicated functions of frequency which are different for each dielectric material, this case does not lead to a generalized form of analysis as do the metallic conductors. Because of this added complexity each dielectric is a special case which must be analyzed individually. Therefore, it will be discussed no further in this work.

General Considerations of Nonsinusoidal Electromagnetic Fields Generated by Aperture Devices

Because of the importance of such devices and systems as parabolic antennas, phased arrays, horns, slots, and open waveguides, etc., to conventional radar, the question of their suitability for use with nonsinusoidal excitation is naturally of interest.

In the sinusoidal case, under certain conditions specified in later sections, the active areas of such devices may be treated as if they were illuminated openings, or apertures, in a highly conductive screen of infinite extent. In these cases the field in the aperture, by Huygen's principle, is a source of radiated fields. As such the aperture field is equivalent to a distributed current source.

If the smallest dimension of the aperture is considerably larger than the wavelength of the source excitation, then the vector form of Kirchhoff's diffraction integral may be applied to determine the intermediate and far zone fields produced by the excited aperture. At distances within a few wavelengths from the aperture, or for aperture dimensions on the order of a few wavelengths or less, the assumed boundary conditions on the conducting screen break down badly. This fact makes use of the diffraction integral invalid under those conditions. However, for the high frequency case in which the aperture dimensions are large compared to the operating wavelength, the diffraction integral produces accurate results.

As with the other systems examined so far, the expressions for the radiating electric and magnetic fields produced by an aperture type device constitutes a form of steady-state transfer function. It describes the amplitudes and phases of the fields at various points in the space around the device in terms of the amplitude and phase of the steady-state sinusoidal excitation.

Examination of the Diffraction Integral

Before evaluating the unit impulse responses of apertures (parabolic antennas) from their far field transfer functions, it is first necessary to consider any limitations that may result from the ultimate source of those functions. Such transfer functions are derived from the sinusoidal far field quantities expressed by the vector diffraction integral of the fields existing over an aperture.

For sinusoidal operation (at a single frequency) the far field (Fraunhofer) diffraction integral is an approximation which is only valid for distances much greater than the dimensions of the aperture and at frequencies where the operating wavelength is much less than the aperture size a, or the conditions

$$\lambda << a << r.$$

The second limitation, $a << r$, offers no great difficulty since most conceivable radar situations, whether using a conventional sinusoidal carrier waveform or using a nonsinusoidal waveform, correspond to distances much greater than the aperture.

The first limitation, $\lambda << a$, however, stems from the fact that the (approximate) Kirchhoff boundary conditions used to evaluate the diffraction integral break down for wavelengths close to, or greater than, the aperture dimensions.

The second limitation may offer difficulties or even render invalid the evaluation of the unit impulse response which requires integration of the transfer function with respect to the frequency variable from $-\infty$ to $+\infty$. The path of integration necessarily passes through the low frequency, or long wavelength, region where the Kirchhoff boundary condition approximation breaks down. This fact makes it necessary to examine the transfer function for each aperture shape and each field distribution considered. It is shown in subsequent sections that neglecting the long wavelength breakdown of the boundary conditions contribute little error to the resulting impulse response functions.

The classical Kirchhoff diffraction integral may be derived for an opening, or aperture, in a sheet of conducting material of infinite extent and of infinitesimal thickness as illustrated in Figure 9. This arrangement of conducting sheet and aperture with a propagating sinusoidal field progressing from left to right in the general direction of the positive z axis represents the aperture presented by a large parabolic reflector antenna, or array, as used in modern radar applications.

The $\bar{E}$ and $\bar{B}$ fields at an arbitrary observation point x_0, y_0, and z_0, may be expressed as an integral of the

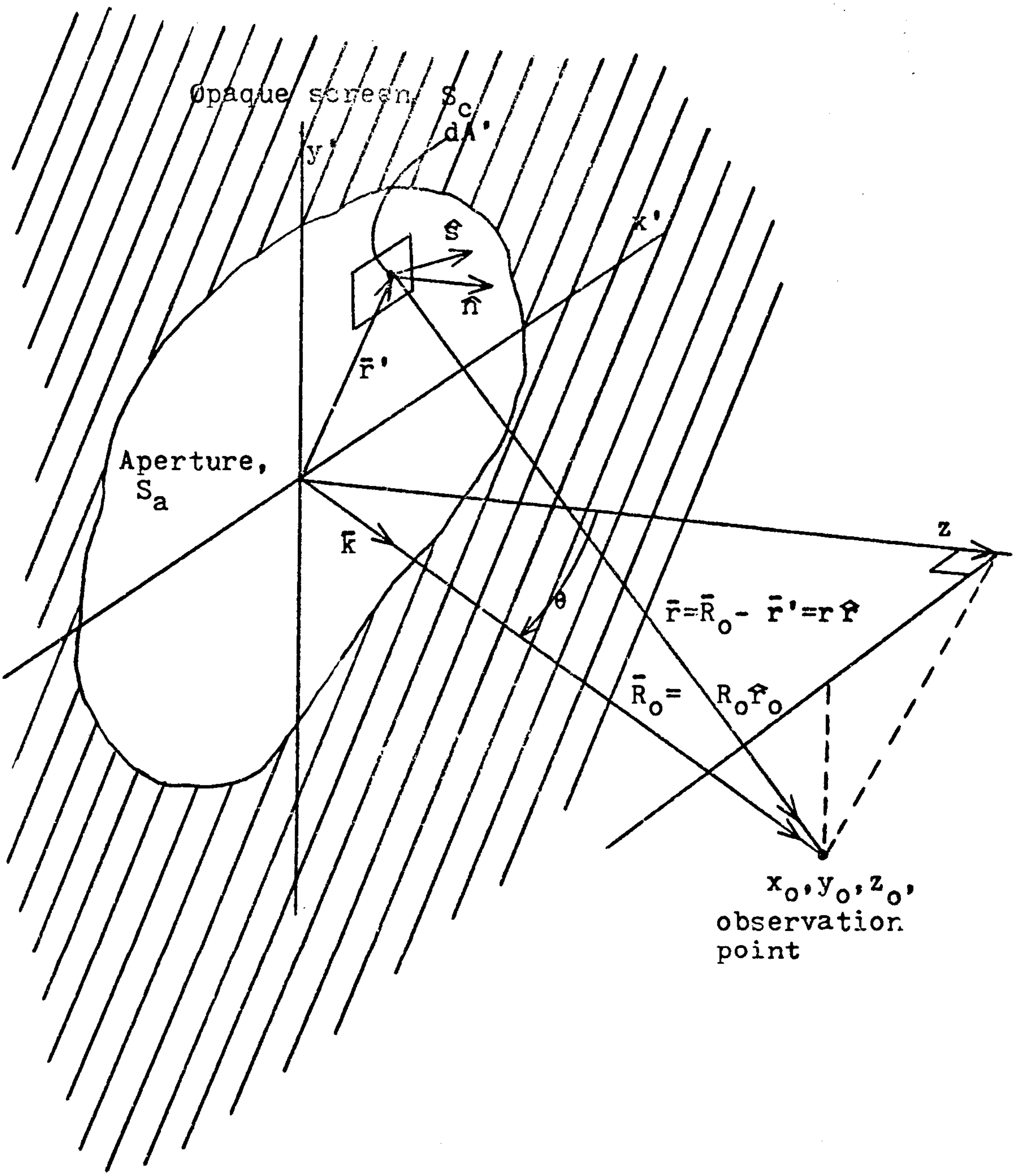


Figure 9. Details of the Kirchhoff-Huygens Diffraction integral.

fields appearing in the aperture and on the surface of the conducting screen. The classical diffraction integral expressing the far electric field due to one rectangular component of the $\bar{E}$ (in this case the x component) field in the aperture from Silver (1963) is

$$E_x(x_o,y_o,z_o) =$$

$$\frac{1}{4\pi} \iint_{S_a+S_c} E_x(x',y',0) \frac{e^{-j\omega r/c}}{r} \left[(\hat{n}\cdot\hat{r} + \hat{n}\cdot\hat{s}) \frac{j\omega}{c} + \frac{\hat{n}\cdot\hat{r}}{r} \right] dx'\, dy' \qquad 78)$$

where the quantities involved are depicted in Figure 9 with the following definitions.

$\hat{n}$ = unit vector perpendicular to differential area, dA'.

$\bar{r}'$= position vector of dA' in aperture S_a.

$\hat{r}$ = unit vector in the direction of $\bar{r}'$.

$\hat{s}$ = unit vector in the direction of propagation at each point in the aperture = $\hat{n}$ in the cases to follow.

$\bar{R}_o$= position vector of the observation point (x_o,y_o,z_o) with respect to the origin.

$\hat{r}_o$= unit vector in the direction of $\bar{R}_o$.

Notice that the field $E_x(x',y',0)$ on S_a and S_c includes not only the amplitude distribution of the field over that surface, but also any phase distribution that may be associated with it. In normal practice the third term in the brackets of the integrand is usually dropped as being negligible compared to the other two terms. For our case it is advisable to retain that term and determine its contribution after the impulse responses have been determined.

To obtain the entire field at the observation point such an integral must be evaluated for each rectangular component of the field in the aperture and on the conducting surface.

Evaluation of these diffraction integrals is made mathematically tractable by use of the Kirchhoff approximations for the fields on the right side of the conducting screen: the field components and their derivatives normal to the conductor surface are assumed to vanish on the screen and the fields and their normal derivatives in the aperture are assumed to be unchanged from their values in the absence of the screen.

Although the above Kirchhoff approximations to the conditions on the screen and in the aperture are actually inconsistent,[9] the diffraction integral gives satisfactory solutions to the far fields when the wavelength is much smaller than the aperture dimensions. For the purposes here the diffraction integral may then be evaluated only over the field in the aperture.

9) Since both E_x and dE_x/dn cannot both be specified on the surface $S_a + S_c$.

For the special case to be considered, many simplifications may be imposed on the diffraction integral. Only aperture fields that lie entirely in the plane of the aperture and that are linearly polarized in the x or y directions are considered. This restriction reduces the effort to the evaluation of only one such diffraction integral. This restriction also causes the unit vector $\hat{s}$, which is a vector in the direction of propagation of the field at each point in the aperture, to be the normal direction (i.e., $\hat{n}$) over the entire aperture.

Note also the decomposition of $\bar{R}_o$, the distance vector from the origin to the observation point, and $\bar{r}$, the vector from the area element in the aperture to the observation point, into products of a scalar magnitude and a unit vector in the direction of the corresponding vector. The following simplifications and approximations then ensue:

$$\hat{n}.\hat{s} = 1 ,$$

$$\hat{n}.\hat{r} = \cos \theta \ .$$

For the condition that R_o is much greater than the aperture size, the distance r in the denominator may readily be approximated by R_o. This quantity may be removed from under the integral sign since it is not a function of the aperture coordinates. This approximation reduces the diffraction integral to

$$E_x(x_o,y_o,z_o) =$$

$$\frac{1}{4\pi R_o} \iint_{S_a} E_x(x',y',0)\, e^{-j\omega r/c} \left[(\cos\theta + 1)\frac{j\omega}{c} + \frac{1}{R_o}\cos\theta\right] dx'\, dy' \quad . \tag{79}$$

The value for r in the exponent is not so easily managed. Values of R_o much greater than the aperture size, however, allow suitable approximations to be made. The following expression for r result:

$$r = \left[z_o^2 + (x_o - x')^2 + (y_o - y')^2\right]^{\frac{1}{2}} \tag{80a}$$

$$= z_o\left[1 + \frac{(x_o - x')^2}{z_o^2} + \frac{(y_o - y')^2}{z_o^2}\right]^{\frac{1}{2}} \tag{80b}$$

which may be approximated by

$$r \approx z_o + \frac{(x_o - x')^2}{2\, z_o} + \frac{(y_o - y')^2}{2\, z_o} + --- \tag{81a}$$

$$\approx z_o - \frac{x'\, x_o}{z_o} - \frac{y'\, y_o}{z_o} + \frac{(x_o^2 + y_o^2 + z_o^2) - z_o^2}{2\, z_o} \frac{}{2z_o}$$

$$+ \frac{(x')^2 + (y')^2}{2\, z_o} + --- \quad . \tag{81b}$$

It is convenient to express the subscripted quantities in spherical coordinates R_o, θ_o, and ϕ_o. With

$$x_o = R_o \sin \theta_o \cos \emptyset_o = R_o \alpha$$

$$y_o = R_o \sin \theta_o \sin \emptyset_o = R_o \beta$$

$$z_o = R_o \cos \theta_o$$

the expression for r becomes

$$r \approx R_o \cos \theta_o + \tfrac{1}{2} R_o(-\cos \theta_o + 1/\cos \theta_o) - \frac{(\alpha x' + \beta y')}{\cos \theta_o} + \frac{(x')^2 + (y')^2}{2R_o \cos \theta_o} + --- \qquad 82)$$

For $R_o >> |x'|$ and $R_o >> |y'|$ the last term containing $(x')^2$ and $(y')^2$ is negligible while the first two terms with R_o and $\cos \theta_o$ may be combined by trigonometric identities and series representations of the trigonometric functions to give

$$R_o(\cos \theta_o - \tfrac{1}{2} \cos \theta_o + \tfrac{1}{2} \sec \theta_o) \approx$$

$$R_o(\tfrac{1}{2} - \tfrac{1}{4} \theta_o^2 + \frac{\theta_o^4}{48} - \frac{\theta_o^6}{1440} + --- + \tfrac{1}{2} + \tfrac{1}{4} \theta_o^2 + \frac{5}{48} \theta_o^4 + \frac{61 \theta_o^6}{1440} + --- \qquad 83a)$$

so that

$$r \approx R_o(1 + \frac{\theta_o^4}{8} + \frac{\theta_o^6}{24}) - \frac{(\alpha x' + \beta y')}{\cos \theta_o} \; . \qquad 83b)$$

Since the angles for which an antenna pattern are of of interest are usually less than 10°, or 0.2 radian, the higher powers of θ_o in the first parentheses may be neglected leaving

$$r \approx R_o - \frac{(\alpha x' + \beta y')}{\cos \theta_o} \,. \qquad 84)$$

The diffraction integral then reduces to

$$E_x(x_o, y_o, z_o) = E_x(R_o, \theta_o, \phi_o) = \frac{e^{-j\omega R_o/c}}{4\pi R_o} \quad X$$

$$\left[(\cos \theta_o + 1) \frac{j\omega}{c} \iint_{S_a} E_x(x', y', 0)\; e^{j\omega(\alpha x' + \beta y')/c \cos \theta_o} dx' dy' \right.$$

$$\left. + \frac{\cos \theta_o}{R_o} \iint_{S_a} E_x(x', y', 0)\; e^{j\omega(\alpha x' + \beta y')/c \cos \theta_o} \; dx' \; dy' \right] . \qquad 85)$$

The integrals in both terms of equation 85 above are identical which reduces the task of determining the field transfer function to the simple evaluation of the following integral for any aperture shape and field distribution:

$$I(\theta_o, \phi_o, \omega) =$$

$$\iint_{S_a} E_x(x', y', 0) \;\; e^{j\omega(\alpha x' + \beta y')/c \cos \theta_o} \; dx' \; dy' \, . \qquad 86)$$

Recall that $\alpha = \sin \theta_o \cos \phi_o$ and $\beta = \sin \theta_o \sin \phi_o$.

In order to keep this study of aperture antennas to reasonable proportions it is limited to the rectangular and the circular apertures having uniform intensity and phase distributions. These two forms of aperture are basic to the sinusoidal steady-state mode of operation. Other intensity and phase distributions have been used in the past (Skolnik, 1970) in order to optimize performance in some manner such as maximizing gain, increasing directivity, or reducing side lobe levels. Any change from uniform intensity and/or phase distributions usually degrades antenna performance in some other manner.

Rectangular Aperture with Uniform Intensity and Phase Distributions

For a rectangular aperture of dimensions 2a and 2b in the x and y directions respectively, the integral $I(\theta_o, \phi_o, \omega)$ becomes

$$I(\theta_o, \phi_o, \omega) =$$

$$E_o \int_{-a}^{a} e^{j\omega\alpha x'/c \cos \theta_o} dx' \int_{-b}^{b} e^{j\omega\beta y'/c \cos \theta_o} dy' =$$

$$4ab\, E_o \frac{\sin(\omega Aa)}{\omega Aa} \frac{\sin(\omega Bb)}{\omega Bb} \qquad 87)$$

where $A = \alpha/(c \cos \theta_o)$, $B = \beta/(c \cos \theta_o)$, and E_o is the peak value of the electric field in the aperture. In terms of the Laplace transform variable, S, the above integral is

$$I(\theta_o, \phi_o, S) = 4\, ab\, E_o \frac{\sinh(SAa)}{SAa} \frac{\sinh(SBb)}{SBb} \qquad 88)$$

or, in terms of exponentials,

$$I(\theta_o, \phi_o, S) =$$

$$\frac{4\, ab\, E_o}{SABab} \left[\frac{e^{S(Aa + Bb)} - e^{S(Aa - Bb)}}{S} \right.$$

$$\left. - \frac{e^{-S(Aa - Bb)} - e^{-S(Aa + Bb)}}{S} \right] \qquad 89)$$

The first term in the diffraction integral of equation

85 is multiplied by $j\omega = S$ which cancels the first S in the denominator of equation 89 above. The inverse Laplace transforms required are

$$L^{-1}\left[S\ I(\theta_0,\emptyset_0,S) \right] =$$

$$\frac{4\ E_0}{AB}\left[u(t + Aa + Bb) - u(t + Aa - Bb)\right.$$

$$\left. - u(t - Aa + Bb) + u(t - Aa - Bb)\right] \qquad 90)$$

$$= I(\theta_0,\emptyset_0,t)$$

and

$$L^{-1}\left[I(\theta_0,\emptyset_0,S) \right] =$$

$$\frac{4\ E_0}{AB}\left[(t + Aa + Bb)\ u(t + Aa + Bb)\right.$$

$$- (t + Aa - Bb)\ u(t + Aa - Bb)$$

$$- (t - Aa + Bb)\ u(t - Aa + Bb)$$

$$\left. + (t - Aa - Bb)\ u(t - Aa - Bb)\right] = \int I(\theta_0,\emptyset_0,T)\ dT\ . \qquad 91)$$

Inserting the above transformed quantities into the diffraction integral of equation 85 yields the overall unit impulse response of a rectangular aperture. The exponential factor $e^{-SR_0/c}$ merely represents a time delay of $-R_0/c$ in the time domain.

The total unit impulse response is

$$E_x(R_o, \theta_o, \phi_o; \ t - R_o/c) =$$

$$\frac{E_o}{cABR_o\pi}(\cos \theta_o + 1) \ X$$

$$\Big[u(t + Aa + Bb - R_o/c) - u(t + Aa - Bb - R_o/c)$$

$$- u(t - Aa + Bb - R_o/c) + u(t - Aa - Bb - R_o/c)\Big]$$

$$+ \ \frac{E_o \cos \theta_o}{AB \ R_o^2 \ \pi} \ X$$

$$\Big[(t + Aa + Bb - R_o/c) \ u(t + Aa + Bb - R_o/c)$$

$$- (t + Aa - Bb - R_o/c) \ u(t + Aa - Bb - R_o/c)$$

$$- (t - Aa + Bb - R_o/c) \ u(t - Aa + Bb - R_o/c)$$

$$+ (t - Aa - Bb - R_o/c) \ u(t - Aa - Bb - R_o/c)\Big] \qquad 92)$$

where A and B are functions of θ_o.

Pictorially the far and intermediate field impulse responses are as shown in Figure 10. This result has some very interesting interpretations. First, the ratio of the absolute magnitudes of the intermediate field and the far field is

$$\frac{2 \ b \sin \theta_o \ \sin \phi_o}{R_o \ (\cos \theta_o + 1)}$$

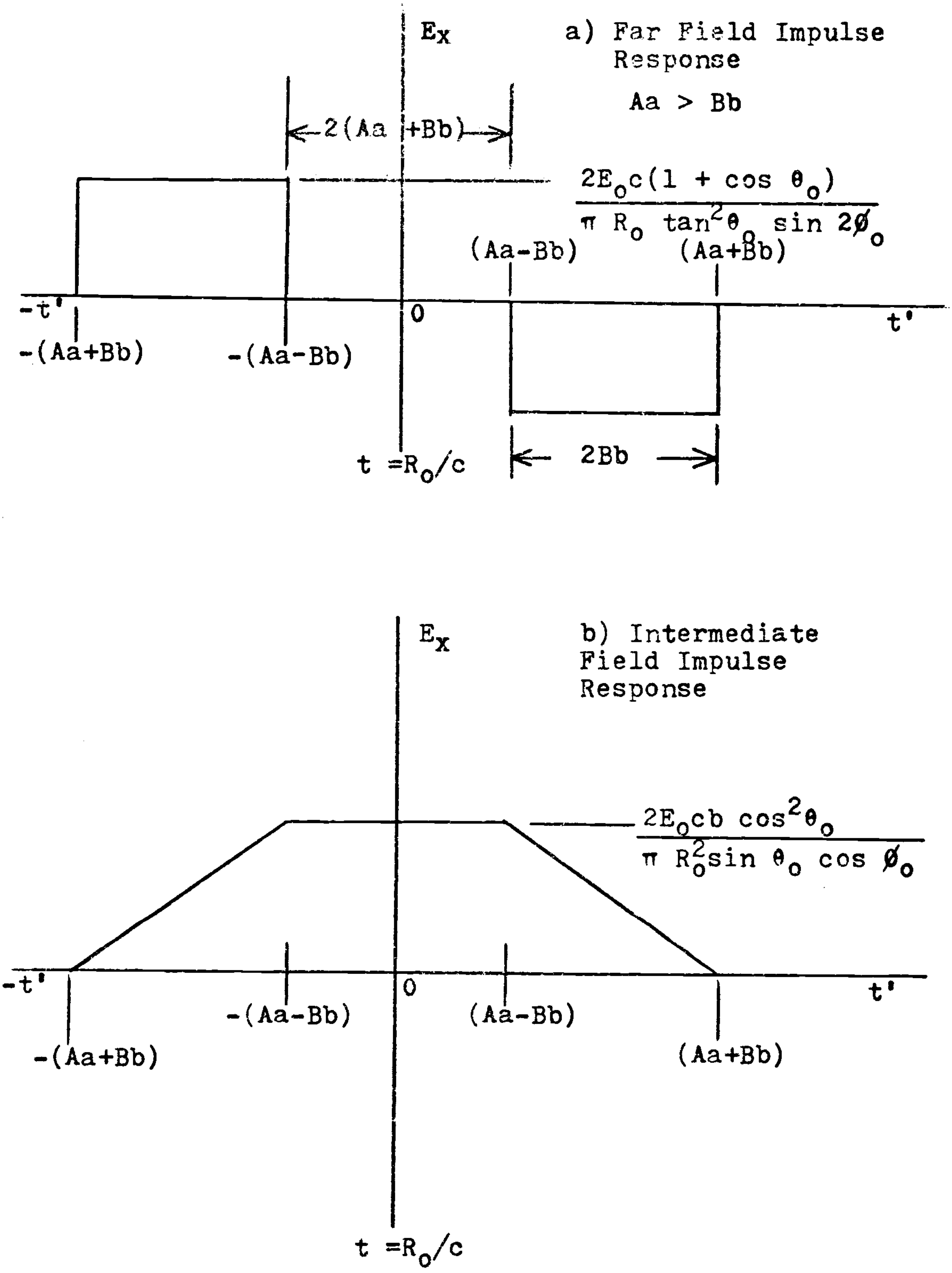


Figure 10. Rectangular aperture unit impulse response.

which shows that the intermediate field is negligible for $R_o >> b$. In addition, at $\theta_o = 0$, on the z axis, the ratio is zero.

Next, the form of the dominant far field is such that for waveforms for which the time interval of interest is much greater than the duration of this impulse response, $2(Aa + Bb)$, it appears as a unit doublet, or the first time derivative of the ideal impulse function. Therefore, for such excitation waveforms, the far field of the rectangular aperture will be the first time derivative of the fields in the aperture. This relationship may be put on a more quantitative basis:

$$t_i >> 2(Aa + Bb) = \frac{\tan \theta_o}{c} (a^2 + b^2)^{\frac{1}{2}} \cos(\phi_o - \phi_p) \tag{93}$$

where t_i is the time interval of interest of the field in the aperture and $\phi_p = \tan^{-1}(b/a)$.

Further study of the far field impulse response shows more interesting features. For instance, the duration of one of the pulses is

$$2Bb = \frac{2b}{c} \tan \theta_o \sin \phi_o \ . \tag{94}$$

The above relationship shows that near the z axis where $\tan \theta_o$ approaches zero, the duration of the pulse also approaches zero. In addition, the separation between the positive and negative portions is

$$2(Aa - Bb) = \frac{\tan \theta_o}{c} (a^2 + b^2)^{\frac{1}{2}} \cos(\phi_o + \phi_p) \qquad 95)$$

which is smaller than the pulse duration and which also approaches zero on the z axis. The amplitude of the pulse is

$$\frac{2 E_o (\cos \theta_o + 1) c}{\pi R_o \tan^2\theta_o \sin(2\phi_o)}$$

which approaches infinity as θ_o approaches zero. All of the preceding discussion means that on and near the z axis the impulse response approaches an ideal doublet which produces a far field proportional to the first time derivative of the field in the aperture. At locations off the z axis poorer replicas of the time derivative are formed.

The above result is very significant. It was mentioned earlier that other indicators of target angular location might exist. This means that if an aperture is excited by a very short duration impulse-like field, a far field response similar to Figure 10a is produced at a given angular location. A small point target at that location will reflect a doublet-like waveform peculiar to the coordinates θ_o and ϕ_o having a fixed relationship to signal duration, pulse separation, and pulse duration. For example,

$$\frac{\text{Total signal duration}}{\text{Pulse separation}} = K = \frac{a \cos \phi_o + b \sin \phi_o}{a \cos \phi_o - b \sin \phi_o} \qquad 96)$$

from which

$$\tan \phi_o = \frac{a(K - 1)}{b(K + 1)} \qquad 97)$$

where K is a measurable quantity from which ϕ_o, one coordinate of the angular location of the target, could be surmised. Once ϕ_o is known the value of θ_o may be calculated from the pulse duration:

$$T_d = \text{pulse duration} = 2 \frac{b}{c} \tan \theta_o \sin \phi_o \; , \qquad 98a)$$

or

$$\tan \theta_o = \frac{c \; T_d}{2 \; b \sin \phi_o} \; . \qquad 98b)$$

The above technique would be very important for measuring the angular position of a target since the relative amplitudes of the signal with respect to direction does not provide nonambiguous indications of the two angular coordinates. Recall that the far field amplitude is proportional to

$$\frac{(\cos \theta_o + 1)}{\tan^2\theta_o \sin(2\phi_o)} \; . \qquad 99)$$

It is also of interest to note that the intermediate field pulse approximates an ideal impulse function near the z axis, but its contribution is insignificant.

As mentioned earlier it is also necessary to determine if the long wavelength (i.e., low frequency) breakdown of the Kirchhoff boundary conditions can safely be ignored in this analysis. It is shown in Jackson (1967, Section 9.9 and problems 9.10 and 9.11) that in the long wavelength limit, where

$$2\,a\omega/c << 1 \text{ or } \omega << c/2a\ ,$$

that the far field varies as $(\omega a/c)^2$ for a small circular aperture. This fact indicates that for frequencies satisfying the above inequality, the aperture transfer function may be considered to be zero at these lower frequencies. The physical aperture then appears as the ideal aperture we have considered here in cascade with a high-pass filter which rejects the lower band of frequencies. We may then determine the overall effect of rejecting the low frequency band by recalculating the impulse response from the transfer function cascaded with a suitable high-pass filter. A suitable high-pass filter offering mathematical tractability is the traditional RC high-pass filter with the following transfer function:

$$H_{hp}(S) = S/(S + \omega_o) \qquad 100)$$

with $\omega_o \approx c/2a$.

Applying the above transfer function to that of the rectangular aperture (equation 89 or 90) yields

$$\frac{S}{S+\omega_o}\left[S\ I(\theta_o,\phi_o,S)\right] =$$

$$\frac{4\ E_o}{AB}\left[\frac{e^{S(Aa + Bb)}}{S + \omega_o} - \frac{e^{S(Aa - Bb)}}{S + \omega_o}\right.$$

$$\left. - \frac{e^{-S(Aa - Bb)}}{S + \omega_o} + \frac{e^{-S(Aa + Bb)}}{S + \omega_o}\right] . \qquad 101)$$

The inverse transform of equation 101 is a summation of appropriately delayed exponential functions. Neglecting propagation delay this is

$$L^{-1}\left[\frac{S}{S + \omega_o}\ S\ I(\theta_o,\phi_o,S)\right] =$$

$$\frac{4E_o}{AB}\left[u(t + Aa + Bb)\ e^{-\omega o(t + Aa + Bb)}\right.$$

$$- u(t + Aa - Bb)\ e^{-\omega_o(t + Aa - Bb)}$$

$$- u(t - Aa + Bb)\ e^{-\omega_o(t - Aa + Bb)}$$

$$\left. + u(t - Aa - Bb)\ e^{-\omega_o(t - Aa - Bb)}\right] . \qquad 102)$$

The corrected waveform of equation 102 is shown in Figure 11. The amounts of droop, Δ_1 and Δ_2, can be calculated from

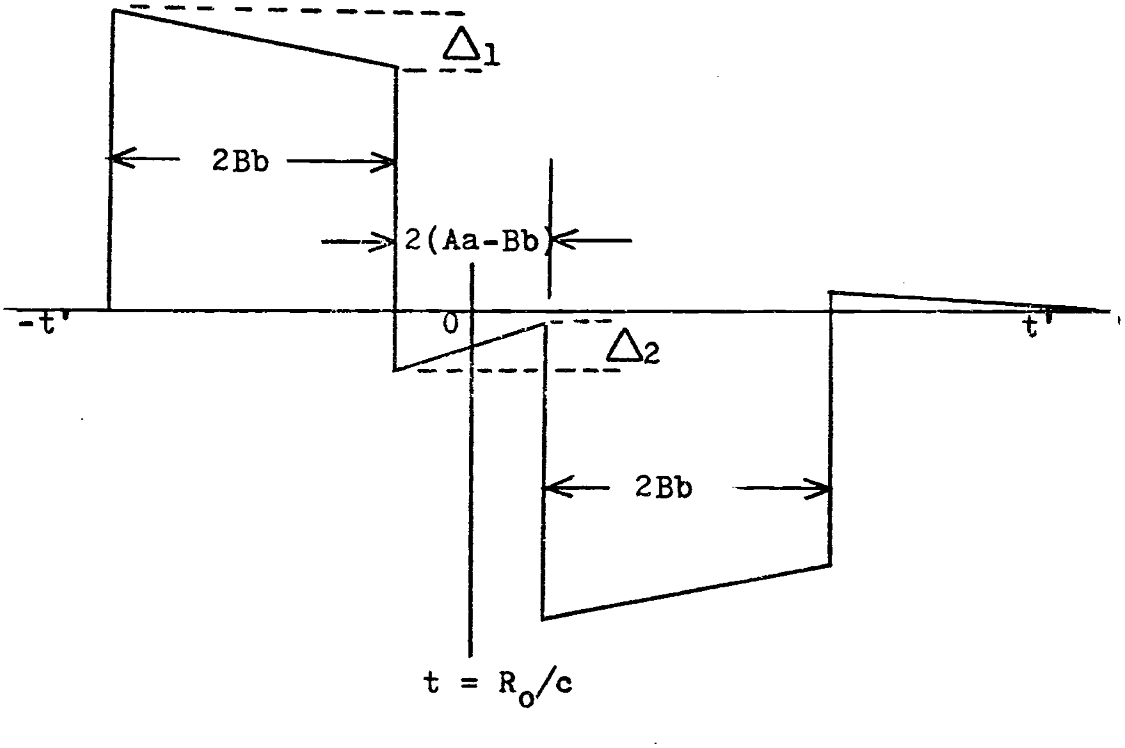


Figure 11. Distortion of a rectangular aperture unit impulse response due to low frequency attenuation effects of the aperture.

$$\triangle_1 = 1 - e^{-b(\tan \theta_o \sin \phi_o)/a} \qquad 103a)$$

$$\triangle_2 = 1 - e^{-(a + b)^{\frac{1}{2}}(\tan \theta_o \cos(\phi_o + \phi_p)/2a} \qquad 103b)$$

Typical worst case considerations for $\theta_o \approx 5^o$, sin $\phi_o = 1$, $a \approx b$, $\phi_p \approx 45^o$ yields

$$\triangle_1 = 1 - e^{-0.1} \approx 1 - 0.9 = 0.1$$

and

$$\triangle_2 = 1 - e^{-0.1/2^{\frac{1}{2}}} \approx 1 - 0.93 = 0.07 .$$

The above approximate values of droop show that the far field impulse response may be distorted by as much as 10% droop in the individual pulses. The droop would have little effect when convolved with a field appearing in the aperture: the relative durations, pulse separations, amplitudes, and areas of the pulses would be changed very little by this distortion. As an approximation to the unit doublet, the exact shape of the two separated pulses is of no great concern.

Circular Aperture with Uniform Intensity and Phase Distributions

The diffraction integral may be evaluated for a circular aperture by changing the variables x' and y' to polar coordinates in the aperture. In terms of polar coordinates x' and y' are

$$x' = r' \cos \phi'$$

$$y' = r' \sin \phi'$$

while the area element is

$$dA = dx'\, dy' = r'\, dr'\, d\phi' \ .$$

Substituting the above expressions into the integral of equation 87 gives, with a = radius of aperture,

$$I(\theta_o, \phi_o, \omega) =$$

$$E_x \int_o^a r'\, dr' \int_o^{2\pi} e^{j\omega(\alpha \cos \phi' + \beta \sin \phi')r'/(c \cos \theta_o)} d\phi' \ . \qquad 104)$$

The exponent in the integrand may be written as

$$\frac{j\,\omega r' \ (\alpha^2 + \beta^2)^{\frac{1}{2}} \cos(\phi' - \phi'_p)}{c \cos \theta_o} =$$

$$j\ F \cos(\phi' - \phi'_p) \qquad 105)$$

where

$$\phi_p = \tan^{-1}(\beta/\alpha) = \tan^{-1}(\tan \phi_o) = \phi_o \ . \qquad 106)$$

The integral in terms of ϕ' is a standard form for the zero order Bessel function of the first kind so that

$$\int_{o}^{2\pi} e^{j F \cos(\phi' - \phi_o)} d\phi' = \frac{2}{\pi} J_o(F) \ . \qquad 107)$$

Since $(\alpha^2 + \beta^2)^{\frac{1}{2}} = \sin \theta_o$, the quantity F simplifies to

$$F = \frac{\omega r' \tan \theta_o}{c} \ . \qquad 108)$$

The second integration with respect to r' is

$$I_{r'} = \frac{E_o \, 2}{\pi} \int_o^a r' \ J_o(\omega r' \tan \theta_o /c) \, dr'$$

$$= \frac{2 \ E_o \ c^2}{\pi \omega^2 \ \tan^2\theta_o} \int_o^{a \omega \tan \theta_o/c} R' \ J_o(R') \, dR'$$

$$= \frac{2 \ E_o \ c^2 \ a \omega \tan \theta_o}{\pi \ c \ \omega^2 \ \tan^2\theta_o} \ J_1(a \omega \tan \theta_o /c)$$

$$= \frac{2 \ a \ E_o \ c}{\pi \omega \tan \theta_o} \ J_1(a \omega \tan \theta_o /c) \ . \qquad 109)$$

The far field transfer function then becomes

$$E_x(\theta_o, \phi_o, \omega) =$$

$$\frac{j \ e^{-j \omega R_o/c}}{2 \ \pi^2 \ R_o} \ \frac{(\cos \theta_o + 1) \ E_o a}{\tan \theta_o} \ J_1(a \omega \tan \theta_o /c) \qquad 110)$$

which in operational form is

$$E_x(\theta_o, \emptyset_o, S) =$$

$$\frac{e^{-SR_o/c}}{2\,\pi^2\,R_o} \; \frac{(\cos\theta_o + 1)\,E_o\,a}{\tan\theta_o} \; I_1(aS\tan\theta_o\,/c)\,. \qquad 111)$$

If equation 111 is written in the form

$$E_x(\theta_o, \emptyset_o, S) = \frac{E_o\,a^2\,(\cos\theta_o + 1)}{2\,\pi^2\,R_o\,c}\, e^{-SR_o/c} \quad X$$

$$\frac{I_1(aS\tan\theta_o\,/c)}{a\tan\theta_o\,/c}\,, \qquad 112)$$

the portion of equation 112 which is a function of S is

$$e^{-SR_o/c}\;\frac{I_1(A\,S)}{A}$$

where $A = \frac{a}{c}\tan\theta_o$.

The exponential factor just contributes an overall delay term of R_o/c to the corresponding time domain quantity and may be ignored in inverting the factor $I_1(A\,S)$ to the time domain. From Roberts and Kaufman (item 12.3-1, page 297, 1966) the inverse Laplace transform of the modified Bessel function is

$$L^{-1}\left[\frac{I_1(A\,S)}{A}\right] = \frac{-t}{\pi\,A^2(A^2 - t^2)^{\frac{1}{2}}} \quad \text{for } -A < t < A. \qquad 113)$$

Recall that t actually incorporates a delay of R_o/c . The form of the above function of time is illustrated in Figure 12.

As A approaches zero, or as θ_c approaches zero, the waveform approaches an ideal doublet. For angles away from the z axis the impulse response appears as if it were an ideal unit doublet to an aperture field for which the interval of interest of the waveform is much greater than the value of A.

The above fact is verified and made clear by a series expansion of $I_1(A\ S)/A$:

$$\frac{I_1(A\ S)}{A} = \tfrac{1}{2}\, S + \frac{S^3\ A^2}{2^4} + \frac{S^5\ A^4}{3 \times 2^6} + --- $$

$$= \tfrac{1}{2}\, S + \tfrac{1}{2} \sum_{i=1}^{\infty} \frac{S^{1\ +\ 2i}\ A^{2i}}{2^{2i}\ i!\ (i\ +\ 1)!} \qquad 114)$$

Taking the inverse Laplace transform of this series form of $I_1(A\ S)/A$ gives the following

$$L^{-1}\left[\frac{I_1(A\ S)}{A}\right] =$$

$$\tfrac{1}{2}\ \delta'(t) + \frac{A^2}{2^4}\ \delta'''(t) + \frac{A^4}{3 \times 2^6}\ \delta^{(5)}(t) + --- \ . \qquad 115)$$

where $\delta^{(5)}(t)$ indicates the fifth derivative of the impulse function. Since $A = \frac{a}{c} \tan \theta_0$ and useful values of θ_0 are on the order of 10° or less, it is seen that all terms after

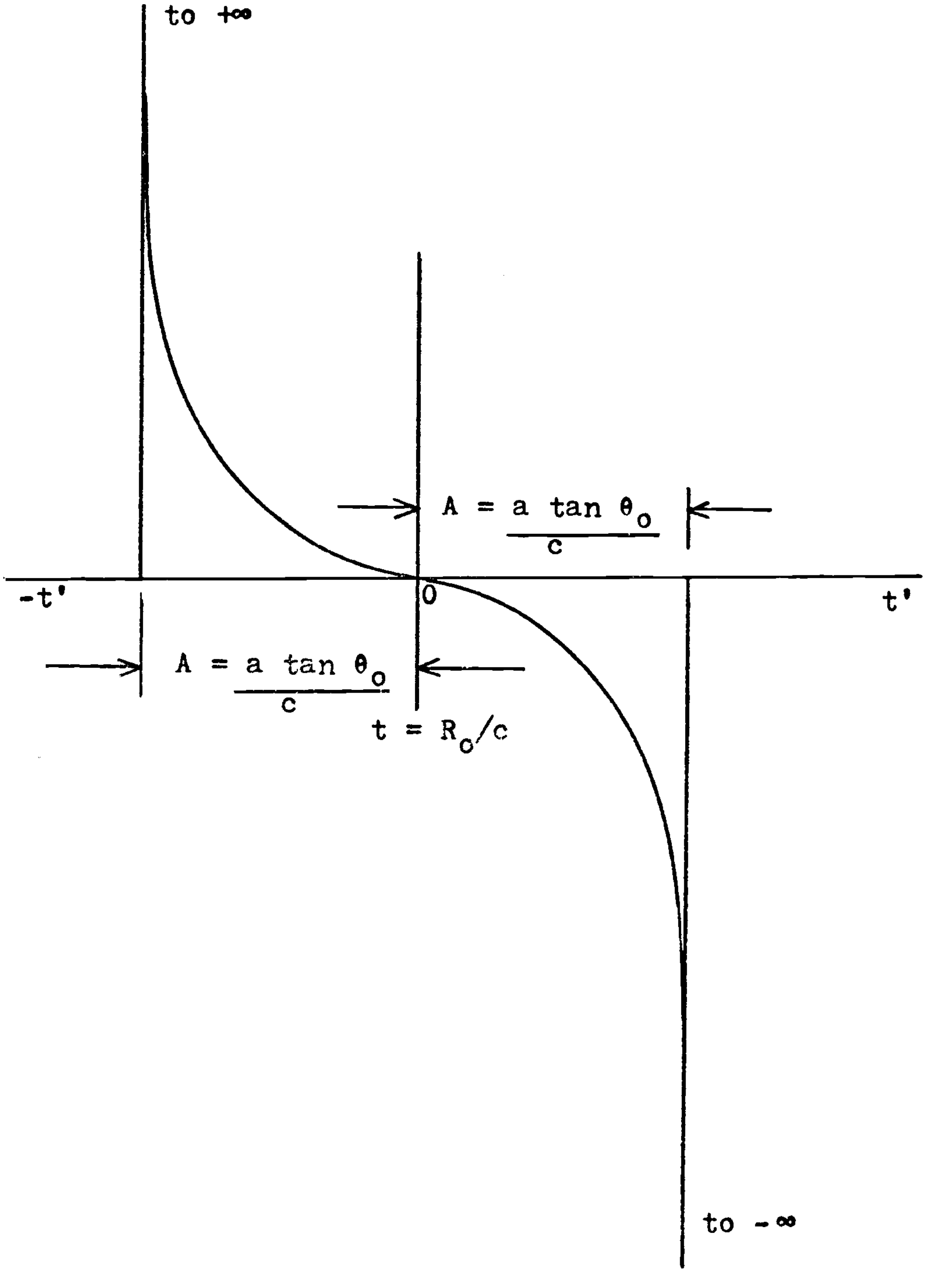


Figure 12. Idealized unit impulse response for a circular aperture.

the first diminish inversely with increasing even powers of c (= 3 x 10^8 meters per second) so that they are negligible. It is safe to assume that a circular aperture produces far fields that are the first time derivative of the field in the aperture.

The complete impulse response of the circular aperture is

$$E_x(\theta_o,\phi_o,t) =$$

$$\frac{E_o\ c\ (\cos\theta_o + 1)}{2\ \pi^3\ R_o}\ \frac{(t - R_o/c)}{\tan^2\theta_o \left[\frac{a^2\ \tan^2\theta_o}{c^2} - (t - R_o/c)^2\right]^{\frac{1}{2}}} \qquad 116)$$

or, as an approximation from equation 115,

$$E_x(\theta_o,\phi_o,t) \approx$$

$$\frac{E_o\ a^2\ c\ (\cos\theta_o + 1)}{2\ \pi^2\ R_o}\ \delta'(t - R_o/c)\ . \qquad 117)$$

The above equations 116 and 117 are interesting in that there is very little angular variation of the far field amplitude and it is completely independent of ϕ_o! This fact indicates that some phenomenon other than amplitude variation with angle is required to indicate target direction.

The effect of the low frequency cut-off can be estimated by subtracting the low frequency components contributed by this analysis. This is done in the frequency domain using the transfer function of the aperture. The frequency dependent portion of the transfer function is

$$\frac{j\ J_1(a\omega \tan \theta_0 /c)}{a \tan \theta_0 /c} .$$

For $\omega = \omega_0 \approx c/2a$ the argument of the transfer function is $\frac{1}{2} \tan \theta_0$. For small angles the Bessel function can be

$$\frac{j\ J_1(a\ \omega_0 \tan \theta_0 /c)}{a \tan \theta_0 /c} \approx \tfrac{1}{2}\, j\omega_0 . \qquad 118)$$

The contribution to the impulse response by these low frequency components is

$$\tfrac{1}{2}\ j \int_{-\omega_0}^{\omega_0} e^{j\omega t}\omega d\omega = \frac{d}{dt}\left[\frac{\sin \omega_0 t}{t}\right]$$
$$= \left[\frac{\omega_0 \cos \omega_0 t}{t} - \frac{\sin \omega_0 t}{t^2}\right] . \qquad 119)$$

The overall time dependent factor of the impulse response is

$$f(t) = \frac{-\ t\ \left[u(t + A) - u(t - A)\right]}{\pi A^2 (A^2 - t^2)^{\frac{1}{2}}} - \frac{d}{dt}\left[\frac{\sin \omega_0 t}{t}\right] . \qquad 120)$$

The form that this corrected impulse response might take is sketched in Figure 13. It is obvious from this figure that some distortion can occur.

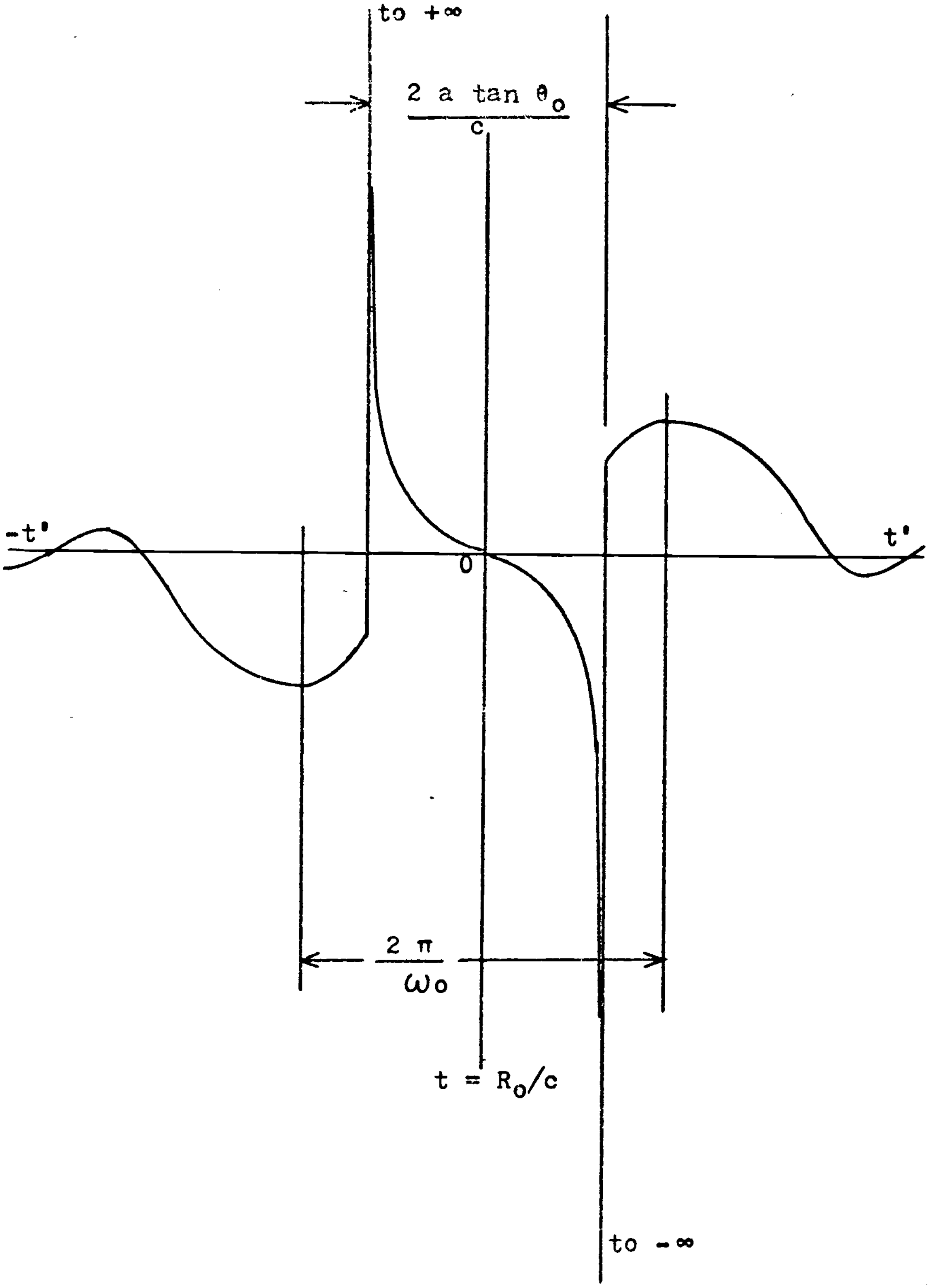


Figure 13. Form of the corrected impulse response for the circular aperture.

Array Antennas

The general effect of an array of several radiating elements may be quickly demonstrated by use of the fundamental relationship of equation 25 of Chapter VI. It is repeated here for the reader's convenience:

$$E_{far}(\theta, \emptyset, \omega) = \sum_{n=1}^{N} A_n \, e^{j\omega(\bar{\rho}_n \cdot \bar{r}/c + \delta_n)} \qquad 121)$$

or, in terms of the Laplace transform variable,

$$E_{far}(\theta, \emptyset, S) = \sum_{n=1}^{N} A_n \, e^{S(\bar{\rho}_n \cdot \bar{r}/c + \delta_n)} \qquad 122)$$

with A_n the relative strength of the n^{th} radiator, δ_n the relative delay of the excitation for the n^{th} radiator, $\bar{\rho}_n$ a vector denoting its location with respect to the origin, and $\bar{r}$ a unit vector in the direction of the observation point.

Inverting equation 122 to the time domain yields

$$E_{far}(\theta, \emptyset, t) = \sum_{n=1}^{N} A_n \, \delta(t - \bar{\rho}_n \cdot \bar{r}/c - \delta_n) \, . \qquad 123)$$

Since the overall far field transform domain response is simply the product of the aperture, or array, response and that of the radiating elements making up the array, the overall time domain unit impulse response is obtained from the convolution of equation 123 with the impulse response of the radiating element. If the far field unit impulse response of the radiating element is h(t), then the overall unit impulse response in terms of h(t) is simply

$$E_{far}(\theta, \emptyset, t) = \sum_{n=1}^{N} A_n h(\theta, \emptyset, t - \bar{\rho}_n \cdot \bar{r}/c - \delta_n) \qquad 124)$$

which is nothing more than the linear superposition of the several relatively delayed impulse responses due to the individual radiating elements.

As an example of the effect of an array, consider a one dimensional linear array having N evenly spaced elements with uniform amplitude and phase distributions. Its normalized response is

$$E_{far} = \frac{\sinh(N\ S\ B)}{N \sinh(S\ B)} \qquad 125)$$

where $B = \frac{d}{c} \cos \theta$ and d= element spacing, and θ is the direction measured from a direction perpendicular to the array. The above equation may be written as

$$E_{far} = \frac{1}{N}(1 + e^{SB} + e^{2SB} + e^{3SB} + \text{---} + e^{(N-1)SB}). \qquad 126)$$

Assuming ideal isotropic radiating elements, equation 126 constitutes the overall response of the array.

Equation 11 of Chapter VII provides the value of σ^2 of the overall unit impulse response of the array. In order to exploit that equation the first and second derivatives of E_{far} with respect to S are required:

$$\frac{dE_{far}}{dS} =$$

$$\frac{B}{N}(e^{SB} + 2e^{2SB} + 3e^{3SB} + --- + (N - 1)e^{(N-1)SB}) , \quad 127)$$

$$\left.\frac{dE_{far}}{dS}\right|_{S=0} =$$

$$\frac{B}{N}(1 + 2 + 3 + 4 + --- + (N - 1)) = \tfrac{1}{2} B(N - 1) , \quad 128)$$

$$\frac{d^2E_{far}}{dS^2} =$$

$$\frac{B^2}{N}(e^{SB} + 4e^{2SB} + 9e^{3SB} + --- + (N - 1)^2 e^{(N-1)SB}) , \quad 129)$$

$$\left.\frac{d^2E_{far}}{dS^2}\right|_{S=0} =$$

$$\frac{B^2}{N}(1 + 4 + 9 + --- + (N - 1)^2) = \frac{B^2(N - 1)(2N - 1)}{6} . \quad 130)$$

Then

$$\sigma^2 = \overline{t^2} - \bar{t}^2 =$$

$$\frac{B^2(N - 1)(2N - 1)}{6} - \tfrac{1}{4} B^2(N - 1)^2 . \quad 131)$$

With $B = \frac{d}{2c} \sin\theta$, we have

$$\sigma^2 = \frac{(N - 1)(N + 1)\ d^2 \sin^2 \theta}{48\ c^2}$$

$$= \frac{(N^2 - 1)\ d^2 \sin^2\theta}{48\ c^2} \qquad 132)$$

and for $N \geqslant 4$,

$$\sigma \approx \frac{N\ d \sin \theta}{3 \times 10^8 (48)^{\frac{1}{2}}} = 0.48\ N\ d \sin \theta \text{ nanoseconds.} \qquad 133)$$

The value of σ is also the amount by which the mean-square duration of a pulse-like excitation is increased:

$$\sigma_{\text{far field}} = (\sigma^2_{\text{signal}} + \sigma^2)^{\frac{1}{2}} \quad . \qquad 134)$$

Equation 19 of Chapter VII may be used to estimate the rise time in response to an ideal unit step function. That equation yields, for the 10% to 90% rise time

$$T_r \leqslant 4.472 \times 0.48\ N\ d \sin \theta \qquad 135)$$

or

$$T_r \leqslant 2.14\ N\ d \sin \theta \quad \text{nanoseconds .} \qquad 136)$$

It is obvious from equation 136 that closer element spacing and a smaller number of elements will improve the time domain unit impulse response of an array antenna.

Two-way Antenna Performance

To this point discussion has centered on the electric intensity of the far fields produced by various antenna configurations. However, the ultimate concern in radar (or communications) applications is the form of, and the distortion impressed on a signal after it has been transmitted, reflected, then received by the antenna system of the radar. The two-way time domain response is quite easily established by considering the two-way frequency domain response which is just the square of the complex antenna one-way response (Harger, 1970, page 63). Note that this assumption neglects any effects induced by the radar target or the intervening radar medium. This being the case, the effect in the time domain is then simply the convolution of the antenna unit impulse response with itself. This fact is highly significant. It means for those combinations of excitation time scale and aperture which produce far fields proportional to the first time derivative of the aperture source field, that the final received signal will appear as the second time derivative of the aperture source field.

For those combinations of excitation source time scale and aperture size not producing the time derivative of the source, the self convolution of the antenna unit impulse response must be accounted for in conjunction with the excitation waveform.

Chapter XI Summary and Conclusions

Before stating the conclusions to be drawn from this study it is worthwhile to restate the original aims and purposes in making it.

As stated in Chapter I crowding of the available radar electromagnetic spectrum is likely to occur in the future making the possible availability of another form of electromagnetic spectrum for radar applications very attractive. It was also noted in Chapter III that radar theory and its applications consists of two major divisions: the electronic and signal processing aspects of the radar operation; the electromagnetic and propagation principles on which the radar operation itself depends. Since a great deal is already known about the electronic processing and amplifying of bi-valued and pulse-like wave trains it was felt that little contribution could be made in that area. However, little investigative effort has apparently been made into the nature of the transmission, radiation, and propagation characteristics of bi-valued waveforms such as Walsh waves. The aim of this thesis, then, is to answer the question of the compatibility of the radar environment and several standard high frequency (sinusoidal) electromagnetic devices with bi-valued waveforms and whether useful radar information (target direction, velocity, etc.) might be provided by their use as a carrier waveform. No consideration is given regarding gains to be made in signal processing using a Walsh carrier wave rather than a sinusoidal carrier.

Contributions and Findings

In Part I several qualitative conclusions were made regarding the nature of the problems to be encountered in using Walsh wave carriers in radar operations. First it was concluded that detectable and useable target speed and direction information can only be provided by an electromagnetic wave possessing a high periodic rate of one of its fundamental parameters such as its rate of zero crossings. This property is inherent in the bi-valued functions considered here.

Another important conclusion is that direct transient analysis of transmission, radiation, and propagation phenomena would be very difficult and would require much new research effort. In addition, within the body of knowledge of such phenomena the properties of devices and materials have traditionally been described in terms of their responses to steady-state sinusoidal excitation; i.e., suitable time domain descriptions of the items and materials relevant to the problem are lacking.

Contributions of Part II

Part II offers concepts and techniques by which the desired transient or pulse-like responses may be obtained by other means. In Chapter VI it is pointed out that transmission elements, radiating devices, and propagation media may all be described by effective transfer functions which relate the relative amplitudes and phases of steady-state sinusoidal electromagnetic fields at two different locations either along a transmission device or in free space.

In Chapter VII the argument is made that the unit impulse responses associated with transmission and radiation devices and propagation media are sufficient to characterize them when responding to bi-valued waves such as Walsh functions. If this is true then such impulse responses are readily available from the aforementioned transfer function descriptions of these elements if valid over a sufficiently wide band of (sinusoidal) frequencies.

However, in Chapter VII it is also shown that the effect of the three classes of elements considered upon a bi-valued wave is only of interest at its transition points in either affecting rise and fall times or in increasing the duration of and changing the shape of a pulse-like waveform. A direct measure of these effects is the root-mean-square duration of the related unit impulse response, σ, which was shown to be easily obtainable from the transform domain transfer function of an element or device.

It is then not necessary to resort to time domain evaluation of the root-mean-square duration, σ, quite a difficult task for many of the complicated unit impulse responses obtained in Part III. The increase in rise or fall time in response to a waveform having a finite rise time is also established in Chapter VII. It is felt that the simple but fundamental concepts introduced in that chapter constitute a major contribution of this thesis.

Another important contribution arose in the work for Part III in which the ideas of Chapter VII are applied to actual transmission and radiation devices and antennas. In this work it was noticed that very often a transfer function can be decomposed into a set of additive terms of which one is a constant. In the time domain the corresponding unit impulse response then contains an ideal impulse term plus others that are functions of time. The ideal impulse term reproduces the original excitation without distortion (other than an amplitude or delay change) while the remaining terms describe the distortion that the device or medium imparts to a signal or waveform passing through it. The distortion terms are then isolated and explicitly expose the parameters of the system responsible for distortion as well as their relationships to each other. Their effects can be minimized by keeping their amplitudes or areas small with respect to the weight of the ideal impulse term.

Findings of Part III

Chapters VIII, IX, and X provide the quantitative results of the study pertinent to the aims stated earlier while the material of Chapters VI and VII provide the mathematical framework within which the analyses of Part III are carried out and interpreted. Chapter VIII deals principally with the radar environment. Its major conclusions are few, but clear:

> A planetary ionosphere can reduce useable transmission distances to only a few meters. The electric and magnetic fields are also found to undergo relative distortion as they progress through a plasma, meaning that they don't possess the same temporal functional form. Because of this fact, care must be taken in devising circuits that extract energy from the fields.
>
> The lower atmosphere is found to produce rather low distortion, allowing useful transmission distances of several thousands of meters. These distances are proportional to the square of the smallest time interval of interest in the progressing wave: i.e., the time resolution desired.

In Chapter IX it was found that closed, hollow metallic waveguide appear to be useless for transmitting nonsinusoidal waveforms within the radar system or over any considerable distance from the electronic signal processing units to antenna units. The useable transmission distances are found to of the same order as the lateral dimensions of the waveguide. Since this is the case, it appears that a more detailed analysis which takes wall and dielectric losses into account is unnecessary.

Coaxial and strip transmission lines are found to operate with useable transmission distances with the following breakdown by the classes of transmission line considered:

Coaxial line: $0.2\sqrt{T_m}$ to $6.5\sqrt{T_m}$ meters

Stripline: $20\sqrt{T_m}$ to $85\sqrt{T_m}$ meters

Integrated circuit interconnection: $0.1\sqrt{T_m}$ to $0.2\sqrt{T_m}$ meters

where T_m is the smallest time interval of interest in the transmitted waveform in nanoseconds.

In addition a simple criterion was established as a by product of the analysis in that the transmission distance may be maximized for a given amount of distortion, or pulse distortion may be minimized by maximizing the quantity

$$\frac{F_L\sqrt{u_o\sigma}}{F_{sk}\sqrt{k'}}$$

in the design of a transmission line. It is also found that the current and voltage on a transmission line don't retain the same temporal forms as they progress along the line.

Chapter X provided many interesting results. As expected it is shown that a dipole antenna of finite length is unsatisfactory as a radiating element if its length is comparable to the product of the speed of light and the duration of one Walsh function pulse interval.

Reflection yielded to analysis showing that for angles of incidence much smaller than 45 degrees waveforms suffer little distortion on reflection. This fact means that for angles of incidence much less than 45 degrees parabolic reflectors would be suitable to focus bi-valued waves.

When considering apertures (or parabolic reflectors) as means of forming beams of Walsh waves it is found that the far radiation field takes the form of the first time derivative of the field within the aperture. In addition, little unambiguous amplitude (or power) variation with direction occurs in the far fields. However, a detectable change in the shape of the impulse response occurs for variations in direction.

In addition, a qualitative analysis shows the two-way performance of an aperture antenna produces a received voltage or current which has the temporal form of the second time derivative of the field in the aperture.

General Conclusion

Walsh functions or similar type functions and their time derivatives and integrals may be used as a radar carrier waveform while conventional coaxial or strip transmission line may be used to interconnect the elements comprising the system. The lower atmosphere will offer little hindrance while applications involving a planetary ionosphere must be approached with some caution. Parabolic reflectors are suitable for producing angle sensitive parameters in the far field and target reflected fields, although it appears that they don't produce a high concentration of electromagnetic energy in a small solid angle as is done in the sinusoidal case. Beam forming properties of bi-valued waveforms do not allow the same interpretation as found in the sinusoidal steady-state response.

Areas for Further Research and Consideration

This work is admittedly just the start of research in a very neglected area of study. Because of the breadth of the work, it was often limited to approximate or lossless cases while the more mathematically difficult topics had to be neglected. However, several topics for further, or more intensive, research are apparent. Some of these are: 1) a more exact analysis of the transmission line and its termination problem; 2) inclusion of the effect of the geomagnetic field with the ionosphere, or plasma; 3) obtain the time domain unit impulse response of the lower atmosphere; 4) obtain a more exact analysis of the low frequency response of an aperture. These problems and the general area of pulse, or transient, responses of propagation media and transmission and radiation devices offers not only the promise of an interesting and fruitful area of research, but should also become very useful as the pulsed mode, or bi-valued waveforms, come into greater use.

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APPENDIX

Appendix I: Subject Content of Classical Radar Text Books and Specialized Short Courses in Modern Radar Theory and Applications.

Below are listed the tables of contents of several recent text books and the generalized subject matter of several specialized short courses offered in recent years by several universities and recognized authorities in the field of modern radar.

Text Books

"Radar System Engineering". Volume I of the Massachusetts Institute of Technology Radiation Laboratory Series. Edited by Louis N. Ridenour, 1963.

Chapter	Topic
1	Introduction
2	Radar Equation
3	Properties of Radar Targets
4	Limitations of Pulse Radar
5	CW Radar
6	The Gathering and Presentation of Radar Data
7	The Employment of Radar Data
8	Radar Beacons
9	Antennas, Scanners, and Stabilization
10	The Magnetron and the Pulser
11	RF Components

12	The Receiving System- Radar Receivers
13	The Receiving System- Indicators
14	Prime Power Supplies for Radar
15	Examples of Radar System Design
16	Moving Target Indication
17	Radar Relay

"Introduction to Radar Systems" by M. I. Skolnik, McGraw-Hill, 1962.

1	The Nature of Radar
2	The Radar Equation
3	CW and Frequency Modulated Radar
4	MTI and Pulse Doppler Radar
5	Tracking Radar
6	Radar Transmitters
7	Antennas
8	Receivers
9	Detection of Radar Signals in Noise
10	Extraction of Information from Radar Signals
11	Propagation of Radar Waves
12	Clutter, Weather, and Interference
13	Systems Engineering and Design
14	Radar Detection of Extraterrestrial Objects

"Modern Radar- Analysis, Evaluation and System Design" by R. S. Berkowitz (ed.), John Wiley & Sons, 1965.

Part I Radar Basics

1 Basic Radar Concepts

2 The Radar Equation

Part II Basic Signal Analysis Techniques

1 Linear System Analysis Fundamentals

2 Theory of Noise

3 Response of Devices to Noise

4 Noise Plus Signal Situations in Radar

5 Complex Signal Analysis Concepts

Part III Radar Target Detection and Parameter Estimation

1 Statistical Decision Theory and Detection of Signals in Noise

2 Target Parameter Estimation

3 Probability Density and Distribution Functions

Part IV Resolution, Ambiguity, Pulse Compression Techniques

1 Ambiguity and Resolution

2 Linear FM Pulse Compression

3 Optical Correlation

4 Pseudo-random Binary Coded Waveforms

Part V Radio Frequency Considerations

1 Atmospheric Effects on Radio Wave Propagation

2 Factors in Antenna Design

3 Radar System Sensitivity

4 Modern Low Noise Devices

Part VI Radar System Analysis and Design Techniques

1 Target, Clutter, and Noise Spectra

2 MTI Radar Filters

3 Radar Feedback Filters

4 Predetection Integration

5 Radar Cross Section Target Models

6 Illustrative Problems in Radar Detection Analysis

7 Tracking Radars

8 Satellite Tracking

"Principles of High Resolution-Radar" by A. W. Rihaczek, McGraw-Hill, 1969.

1 Introduction

2 Fundamentals of Waveform Analysis

3 Single Target Measurements

4 Resolution in a Matched Filter Radar

5 Resolution Theory for Targets with Constant Range Rate

6 Pulse Compression Waveforms

7 Linear FM Waveforms

8 Coherent Pulse Trains

9	Radar Mapping of Distributed Targets
10	Target Detection in Clutter
11	Extensions of Resolution Theory
12	Waveforms for Simplified Doppler Processing
13	Synthetic Aperture Radar

"Radar Design Principles: Signal Processing and the Environment" by F. E. Nathanson, McGraw-Hill, 1969.

1	Radar and Its Composite Environment
2	Review of Radar Range Performance Computations
3	Statistical Relationships for Various Detection Processes
4	Automatic Detection by Nonlinear, Sequential, and Adaptive Processes
5	Radar Targets
6	Atmospheric Effects, Weather, and Chaff
7	Sea and Land Backscatter
8	Signal Processing Concepts and Waveform Design
9	Moving Target Indicators
10	Environmental Limitations of CW Radar
11	Pulse Doppler and Burst Waveforms
12	Phase Coding Techniques
13	Linear Frequency Modulation and Frequency Coding
14	Hybrid Processors, Correlators, and Incoherent Techniques

"Radar Handbook" by M. Skolnik (ed.), McGraw-Hill, 1970.

1	An Introduction to Radar
2	Prediction of Radar Range
3	Waveform Design
4	Radar Measurement Accuracy
5	Receivers
6	Radar Indicators and Displays
7	Transmitters
8	Transmission Lines, Components, Devices
9	Aperture Antenna Analysis
10	Reflectors and Lenses
11	Array Antennas
12	Phase Shifters for Arrays
13	Frequency Scanned Arrays
14	Radomes
15	Automatic Detection Theory
16	CW and FM Radar
17	MTI Radar
18	Airborne MTI
19	Pulse-doppler Radar
20	Pulse-compression Radar
21	Tracking Radar
22	Radar Height Finding
23	Synthetic Aperture Radar
24	Weather Effects on Radar

25	Ground Echo
26	Sea Echo
27	Radar Cross Section of Targets
28	Target Noise
29	Electromagnetic Compatibility
30	Solid-state Radar
31	Civil Marine Radar
32	Satellite Surveillance Radar
33	Radar Astronomy
34	Spaceborne-radar Applications
35	Digital Signal Processing
36	Bistatic and Monostatic Radar
37	Laser Radars
38	Beacons
39	Passive Detection

Radar Short Courses

"Principles of Modern Radar" - Georgia Institute of Technology, Atlanta, Georgia, October 21-25, 1974.

1 Radar System Fundamentals- Range Equation

2 Radar Cross Section

3 Propagation Effects

4 Radar Detection Problem

5 Elements of Radar Systems

6 Mechanical Aspect of Radar Design

7 Radar Measurement and Tracking

8 Special Signal Processing Techniques

9 Electronic Countermeasures

10 Basic Systems Analysis Approach

11 Laboratory Demonstration

"Radar Systems and Technology" - The George Washington University, Washington, D.C., May 20-24, 1974.

1 Introduction to Radar; Performance and Capabilities

2 Detection of Targets in Clutter; MTI Radar

3 Signal Processing; Pulse Compression

4 Data Processing; Computer Control

5 Phased Array Radar; Solid-state Devices in Array Radar; Adaptive Antennas

6 3-D Radar; Low Angle Tracking; Frequency Agility

7 System Design Considerations

8	Synthetic Aperture Radar; Optical Processing and Holography
9	Remote Sensing of the Environment; Ice Detection; Earth Resources Detection
10	Clear Air Turbulence; Over-the-horizon Radar
11	Millimeter Wave Radar; Future Trends

"Introduction to Radar" - University of Missouri-Rolla, Missouri, May 25-29, 1970 and repeated January 11-15, 1971.

1	Introduction to Pulse Radar, Search Radar Coverage, and Radar Range Equation
2	Radar Measurement Problem, Classification of Radars, Ambiguities
3	Minimum Detectable Signal, Probability Density Functions, S/N, False Alarm Rates, Pulse Integration
4	System Losses
5	Tracking Losses, Monopulse, Angular Glint, Multiple Targets
6	MTI and Pulse Radar
7	Radar Transmitters
8	Antennas
9	Phased Array Antennas
10	Antenna Temperature, Noise Figure, Effect on Radar Range
11	Estimation of Signal Parameters
12	Digital Signal Processing
13	Ambiguities, Pulse Compression
14	Propagation, Refractivity, Effect on Range and Elevation Data

15	Weather Effects
16	Radar Astronomy

"Radar Systems Design" - University of Southern California, July 24 to August 4, 1972.

1	Introduction, Radar Equation
2	Matched Filters
3	Radar Detection
4	Antennas
5	Nonfluctuating and Fluctuating Target Detection
6	Target Characteristics and the Radar Channel
7	Transmitter and RF Hardware Constraints
8	Receivers and Noise Sources
9	Ambiguity Functions and Radar Signal Design
10	Pulse Compression
11	Scan-to-scan Performance
12	Fast Fourier Transform
13	Clutter Rejection
14	Binary Coded Waveforms
15	Electronic Scanning and Phased Arrays
16	Angle Tracking
17	Synthetic Arrays
18	Range and Doppler Trackers
19	Digital Signal Processing
20	Radar Astronomy

21 Countermeasures

"Advanced Methods of Modern Radar Systems" - Technology Service Corporation (Santa Monica, California), Los Angeles, California, June 27-30, 1972 and at Washington, D.C., June 5-8, 1973.

1 Course Overview

2 Target Detection and Target Models in Modern Radar Systems

3 High Resolution Waveform Design

4 Adaptive Antenna Processing and MTI Techniques

5 Technology Advances in Radar Transmitter-Receiver and Array Antennas

6 Advanced Signal Processing Techniques

7 Environmental Modeling

"Radar Signal Processing and Clutter" - Technology Service Corporation (Santa Monica, California), Silver Spring, Maryland, October 18-22, 1971 and October 16-20, 1972.

1 Introduction and Types of Radar

2 Principles of Waveform Design

3 Radar Equation Review, Clutter and Jamming Equations

4 Probability and Detection Theory, Minimum Detectable Signal

5 Effects of Limiting, Pulse Integration in Noise and Clutter, Automatic Detection

6 Fourier Analysis and Power Spectral Density, Correlation Processes, Matched Filter Design, and Quadrature Detection

7 Radar Target Properties, Amplitude Distributions, Fluctuation Spectrum,

7(cont.)	Weather and Chaff Clutter-Reflectivity and Spectrum, Frequency Decorrelation Effect
8	Land and Sea Clutter, Amplitude and Spatial Distributions, Grazing Angle Effects, Doppler Spectrum, Short Pulse Effects
9	Ambiguity Functions, Pulse Compression, Choosing Optimum Waveforms, Subclutter Visibility, Airborne Pulse Doppler (ICW)
10	Pulse Doppler Techniques, FFT Techniques, Phase Coding Techniques, Digital Implementations, Quantization Noise
11	MTI, Variable Interpulse Period,I and Q Implementation, Scanning Losses, Linear FM and Chirp
12	Comparision of Processing Techniques, Equipment Limitations, Pulse Compression and MTI Hybrids, Adaptive Techniques

"Prediction of Radar Detection Range" - Technology Service Corporation (Santa Monica, California), Silver Spring, Maryland, October 23-25, 1973.

1	Introduction, Range Equations
2	Signal/Noise Relationships
3	Wave Propagation Phenomena
4	System Losses; Range Calculation Techniques

"Applied Theory of Radar Resolution" - Technology Service Corporation (Santa Monica, California), Los Angeles, California, November 6-10, 1972.

1	The Problem of Resolution
2	Signal Notation and Waveform Analysis
3	Single Target Measurements

4 The Role of Resolution

5 Basic Theory of Resolution and Waveform Design

6 Radar Waveforms

7 The Linear FM Signal

8 Coherent Pulse Trains

"Radar Meteorology" - Technology Service Corporation (Santa Monica, California), Silver Spring, Maryland, November 9-12, 1971.

1 Fundamentals of Radar

2 Attenuation and Fluctuation of Precipitation Echoes

3 Weather Radar Equations

4 Doppler Radar

5 The Relationship of Radar Reflectivity Factor, Z, to Other Meteorological Parameters

6 Clear-air Radar Echoes

7 Calibration, Measurements, and Data Handling Techniques

"Microwave Sensing of the Earth" - Technology Service Corporation (Santa Monica, California), Silver Spring, Maryland, September 11-14, 1973.

1 Introduction and Overview

2 Environmental Considerations

3 Spatial and Temporal Resolution

4 Radar Mapping

5 Synthetic Aperture Radar (SAR)

6 Satellite Radiometry

7 Radar Satellite Altimetry

"Radar Simulation" - Technology Service Corporation (Santa Monica, California), Silver Spring, Maryland, September 26-29, 1972.

1 Introduction and Overview

2 Signals, Filters, and Noise

3 Radar Receiver Response

4 Radar Environment Models

5 Video Signal Simulation Techniques

6 Functional Simulation Examples

7 Video Signal Simulation Examples

8 Real-time Applications

"Radar Simulation" - Technology Service Corporation, (Santa Monica, California), Los Angeles, California, May 8-11, 1973.

1 Introduction

2 Functional Simulation Examples

3 Radar Data Processing and Real-time Simulation

4 Signals, Filters, and Noise

5 Radar Receiver Responses

6 Radar Environment Models

7 Video Signal Simulation Techniques

8 Video Signal Simulation Examples

Appendix II: Derivation of the $\bar{E}$ and $\bar{B}$ Fields for the Short Hertzian Dipole with Arbitrary Excitation Using Harmuth's Direct Technique

Analysis of two simplified, but fundamental, cases as done by Harmuth, do much to expose the nature of the radiation of electromagnetic waves produced by sources having arbitrary (i.e., nonsinusoidal) time variation. These cases are the short Hertzian dipole and the small magnetic moment dipole radiating elements. However, the radiation characteristic of the magnetic dipole is the dual of that of the short electric dipole so that only the latter need be considered in great detail.

In both cases the spatial dimensions of the source distribution is considered to be so small that at any instant of time the current density distribution due to any electronically produced localized current or voltage source is constant over its entire extent and proportional, or equal, to the electronically produced quantity. Under these conditions the current density distribution over a short dipole element oriented along the z axis at the origin of a rectangular coordinate system is

$$\bar{J}(x,y,z;t) = \bar{k}\, i(t), \qquad -\tfrac{1}{2}s \leqslant z \leqslant +\tfrac{1}{2}s \qquad 1)$$

with s the length of the short dipole element. Substituting this quantity into equation 6 of Chapter V and using the Lorentz gauge condition, both ϕ and $\bar{A}$ may easily be determined. The integral for $\bar{A}$ becomes

$\bar{A}(x,y,z;t) =$

$$\bar{k} \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{\infty} \int_{-\frac{1}{2}s}^{\frac{1}{2}s} \frac{u\ i(t - |\bar{x} - \bar{x}'|/c)\ \delta(x')\ \delta(y')}{4\pi\ |\bar{x} - \bar{x}'|}\, dx'dy'dz' =$$

$$\bar{k}\ u \int_{-\frac{1}{2}s}^{\frac{1}{2}s} \frac{i(t - |\bar{x} - \bar{k}z'|/c)}{4\pi\ |\bar{x} - \bar{k}z'|}\, dz' . \qquad 2)$$

Since $|\bar{x}| \geqslant z'$, the denominator of the integrand may be set equal to $|\bar{x}|$ and since $\frac{1}{2}s/c <<$ (the time interval over which significant changes in $i(t)$ occur) we have

$\bar{A}(x,y,z:t) =$

$$\frac{\bar{k}\ u\ i(t - |\bar{x}|/c)}{4\pi\ |\bar{x}|} \int_{-\frac{1}{2}s}^{\frac{1}{2}s} dz' = \frac{\bar{k}\ u\ s\ i(t - |\bar{x}|/c)}{4\pi\ |\bar{x}|} . \qquad 3)$$

Since $\bar{A}$ and $i(t)$ are vectors in the same direction, $\bar{A}$ may be expressed in terms of current vectors having arbitrary orientation:

$$\bar{A}(x,y,z;t) = \frac{\bar{s}\ u\ i(t - |\bar{x}|/c)}{4\pi\ |\bar{x}|} \qquad 4)$$

in which $\bar{s}$ is a fixed vector in the direction of the current element and of magnitude equal to the length of the dipole element.

The magnetic induction $\bar{B}$ is obtained as the curl of $\bar{A}$ which reduces to

$$\bar{B}(x,y,z;t) = \nabla \times \bar{A} = u\ \bar{H}(x,y,z;t)$$

or

$\bar{B}(x,y,z;t) =$

$$\frac{us}{4\pi|\bar{x}|c}\,\frac{di}{dt}(t-|\bar{x}|/c)\,\frac{\bar{s} \times \bar{x}}{s|\bar{x}|} + \frac{us}{4\pi|\bar{x}|^2}\,i(t-|\bar{x}|/c)\,\frac{\bar{s} \times \bar{x}}{s|\bar{x}|}. \qquad 5)$$

Notice that both terms of $\bar{B}$ are perpendicular to $\bar{s}$ and $\bar{x}$. The first term is normally called the far zone component as it is inversely proportional to the distance $|\bar{x}|$ while the second, inversely proportional to $|\bar{x}|^2$, is the near zone component since it will dominate at distances close to the antenna. However, the relative significance of either term at a given distance $|\bar{x}|$ depends also upon the instantaneous relative magnitudes of the excitation current and its time derivatives.

The scalar potential, in principle, is easily obtained from the Lorentz gauge condition:

$$\nabla\cdot\bar{A} + ue\frac{\partial \phi}{\partial t} = 0. \qquad 6)$$

From this equation we obtain

$$\begin{aligned}\phi(x,y,z;t) &= \frac{-1}{ue}\int_0^t \nabla\cdot\bar{A}(x,y,z;T)\,dT \\ &= \frac{-1}{e}\nabla\cdot\int_0^t \frac{i(T-|\bar{x}|/c)}{4\pi|\bar{x}|}\,dT\,. \qquad 7)\end{aligned}$$

The electric field $\bar{E}$ is much more difficult to obtain from

$$\begin{aligned}\bar{E}(x,y,z;t) &= -\frac{\partial\bar{A}}{\partial t} - \nabla\phi \\ &= -\frac{\partial\bar{A}}{\partial t} + \frac{\nabla(\nabla\cdot\bar{s}\int_0^t i(T-|\bar{x}|/c)\,dT)}{4\pi e|\bar{x}|}. \qquad 8)\end{aligned}$$

After considerable manipulation the above equation may be reduced to the following complicated expression

$$
\begin{aligned}
\bar{E}(x,y,z;t) = {} & \frac{u}{4\pi|\bar{x}|}\frac{di}{dt}(t-|\bar{x}|/c)\,\frac{\bar{x}\ X\ (\bar{x}\ X\ \bar{s})}{s\ |\bar{x}|^2} \\
& + \frac{u^{\frac{1}{2}}}{4\pi\ e^{\frac{1}{2}}|\bar{x}|^2}\, i(t-|\bar{x}|/c)\left[\frac{3(\bar{s}\cdot\bar{x})\bar{x}}{|\bar{x}|^2}-\bar{s}\right] \\
& + \frac{1}{4\pi\ e\ |\bar{x}|^3}\int_0^t i(T-|\bar{x}|/c)\,dT\left[\frac{3(\bar{s}\cdot\bar{x})\bar{x}}{|\bar{x}|^3}-\bar{s}\right] \qquad 9)
\end{aligned}
$$

which consists of three basic components. The first is usually termed the far zone component inversely proportional to distance $|\bar{x}|$ from the dipole to the point of observation. The third, inversely proportional to the cube of $|\bar{x}|$, is normally termed the near field since it predominates at distances close to the antenna. The second term, inversely proportional to the square of $|\bar{x}|$, is intermediate to the near and far zone components. Notice that the far zone component is perpendicular to both B and $\bar{x}$.

As in the case with $B(x,y,z;t)$ the relative significance of the three terms in $E(x,y,z;t)$ depends also upon the instantaneous magnitudes of the excitation current, its time derivative, and its time integral.

The small loop antenna element has been treated by Harmuth (1972) by making use of Babinet's duality principle which states that if E and $\bar{B}$ are solutions to Maxwell's equations in a charge and current free region then the transformed quantities

$$\bar{E} \longrightarrow \bar{H}\sqrt{u/e} = \bar{H}\,Z_0, \qquad \bar{H} \longrightarrow -\bar{E}\sqrt{u/e} = -\bar{E}/Z_0 \qquad 10)$$

are also solutions since they also satisfy Maxwell's equations. If the expressions for $\bar{E}$ and $\bar{H}$ contain source quantities they must also be transformed to their dual quantities by the following transformation

$$\dot{\bar{p}}(t) = i(t)\,\bar{s} \longrightarrow \bar{M}\sqrt{u/e} = i(t)\,\bar{a}\,\sqrt{u/e} = \bar{M}/Z_0 \qquad 11)$$

where $\dot{\bar{p}}(t)$ is the original electric dipole source, and $\bar{M}$ the magnetic moment of a current i(t) flowing around a loop of area a with its normal in the direction of the original dipole.

Applying these principles to Harmuth's short electric dipole element yields the dual solution for a small loop antenna element:

$$\bar{E}(x,y,z;t) = \frac{u}{4\pi\,Z_0|\bar{x}|}\,\frac{\dot{\bar{M}} \times \bar{x}}{|\bar{x}|} + \frac{uc}{4\pi Z_0|\bar{x}|^2}\,\frac{\bar{M} \times \bar{x}}{|\bar{x}|}$$

$$= \frac{1}{4\pi|\bar{x}|c}\,\frac{di(t - |\bar{x}|/c)}{dt}\,\frac{\bar{a} \times \bar{x}}{|\bar{x}|}$$

$$+ \frac{1}{4\pi|\bar{x}|^2}\,i(t - |\bar{x}|/c)\,\frac{\bar{a} \times \bar{x}}{|\bar{x}|} \qquad 12)$$

while for the magnetic induction we have the following expression

$$B(x,y,z;t) = \frac{u}{4\pi Z_0 c\,|\bar{x}|} \frac{\bar{x} \times (\bar{x} \times \bar{M})}{|\bar{x}|^2}$$

$$+ \frac{u}{4\pi Z_0 |\bar{x}|^2} \left[\frac{3(\bar{x} \cdot \bar{M})\bar{x}}{|\bar{x}|^2} - \bar{M} \right]$$

$$+ \frac{u}{Z_0 |\bar{x}|^3} \left[\frac{3(\bar{x} \cdot {}_0\int^t \bar{M}\, dT)\bar{x}}{|\bar{x}|^2} - {}_0\int^t \bar{M}\, dT \right] . \qquad 13)$$

The magnetic induction may also be written in terms of the excitation current and its time derivative and integral:

$$\bar{B}(x,y,z;t) = \frac{1}{4\pi c^2 |\bar{x}|} \frac{di}{dt}(t - |\bar{x}|/c) \frac{\bar{x} \times (\bar{x} \times \bar{a})}{|\bar{x}|^2}$$

$$+ \frac{1}{4\pi c |\bar{x}|^2} i(t - |\bar{x}|/c) \left[\frac{3(\bar{x} \cdot \bar{a})\bar{x}}{|\bar{x}|^2} - \bar{a} \right]$$

$$+ \frac{1}{c\,|\bar{x}|^3} \int_0^t i(T - |\bar{x}|/c)\, dT \left[\frac{3(\bar{x} \cdot \bar{a})\bar{x}}{|\bar{x}|^2} - \bar{a} \right] . \quad 14)$$

As expected from the electric dipole soutions, the far field terms for the current loop element are found to be proportional to the time derivative of the excitation current in this case also.

Appendix III: Derivation of the Mean-square Duration of the Unit Impulse Response of Cascaded Elements

The effect of cascading two or more elements, each being individually characterized by its unit impulse response or by its transfer function, is easily determined. Let F(S) and G(S) be the individual transfer functions of two cascaded elements such that the overall transfer function, H(S), is their product:

$$H(S) = F(S)\ G(S)\ . \qquad 1)$$

By use of Equation 11 of Chapter VII, the mean-square duration of the pair of cascaded elements is

$$\sigma_h^2 = \overline{t_h^2} - \bar{t}_h^2 = \lim_{S \longrightarrow 0} \left[\frac{1}{H(S)} \frac{d^2H}{dS^2} - \left(\frac{1}{H(S)} \frac{dH^2}{dS^2} \right)^2 \right] \ . \qquad 2)$$

From Equation 1 above

$$\frac{dH}{dS} = F \frac{dG}{dS} + G \frac{dF}{dS}, \qquad 3)$$

so that

$$\frac{1}{H(S)} \frac{dH}{dS} = \frac{1}{G(S)} \frac{dG}{dS} + \frac{1}{F(S)} \frac{dF}{dS}, \qquad 4)$$

and

$$\frac{d^2H}{dS^2} = F \frac{d^2G}{dS^2} + G \frac{d^2F}{dS^2} + 2 \frac{dG}{dS} \times \frac{dF}{dS}\ . \qquad 5)$$

Then,

$$\frac{1}{H(S)}\frac{d^2H}{dS^2} = \frac{1}{G}\frac{d^2G}{dS^2} + \frac{1}{F}\frac{d^2F}{dS^2} + \frac{2}{FG}\frac{dF}{dS} \times \frac{dG}{dS} \; . \qquad 6)$$

Considering the equations above the overall mean-square duration of the cascade is then

$$\sigma_h^2 =$$

$$\lim_{S \to 0}\left[\frac{1}{G}\frac{d^2G}{dS^2} + \frac{1}{F}\frac{d^2F}{dS^2} + \frac{2}{FG}\frac{dF}{dS} \times \frac{dG}{dS} + \frac{1}{(FG)^2}\left(F\frac{dG}{dS} + G\frac{dF}{dS}\right)^2\right]$$

7a)

Completing the indicated multiplication and combining terms gives

$$\sigma_h^2 = \lim_{S \to 0}\left[\frac{1}{G}\frac{d^2G}{dS^2} + \frac{1}{F}\frac{d^2F}{dS^2} - \frac{1}{G^2}\left(\frac{dG}{dS}\right)^2 - \frac{1}{F^2}\left(\frac{dF}{dS}\right)^2\right] \qquad 7b)$$

or

$$\sigma_h^2 = \lim_{S \to 0}\left[\frac{1}{G}\frac{d^2G}{dS^2} - \left(\frac{dG}{dS}/G\right)^2 + \frac{1}{F}\frac{d^2F}{dS^2} - \left(\frac{dF}{dS}/F\right)^2\right] \; . \qquad 7c)$$

It is noted in equation 7c above that the first two terms constitute the mean-square duration, σ_g^2, of the unit impulse response of the element G(S) while the second pair of terms constitute σ_f^2 of element F(S). Then,

$$\sigma_h^2 = \sigma_f^2 + \sigma_g^2 \; . \qquad 8)$$

By simple mathematical induction, the mean-square duration of the unit impulse response of an arbitrary number of elements is simply

$$\sigma_h^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2 + --- + \sigma_N^2 \quad . \qquad 9)$$

Appendix IV: The Mean-square Duration of a Signal Consisting of the Superposition of an Arbitrary Number of Individual Signals

First note that the Laplace transform of a signal consisting of the superposition of several terms is

$$H_t(S) = H_1(S) + H_2(S) + H_3(S) + --- + H_N(S) \qquad \text{1a)}$$

$$= \sum_{i=1}^{N} H_i(S) \; . \qquad \text{1b)}$$

The overall average duration, $\bar{t}$, is then

$$\bar{t} = \lim_{S \to 0} \frac{-1}{H_t(S)} \frac{dH_t}{dS} = \lim_{S \to 0} \frac{-1}{H_t(S)} \sum_{i=1}^{N} \frac{dH_i}{dS} \; . \qquad \text{2)}$$

The square of $\bar{t}$ is

$$\bar{t}^2 = \lim_{S \to 0} \frac{\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{dH_i}{dS} \frac{dH_j}{dS}}{\sum_{m=1}^{N} \sum_{n=1}^{N} H_m(S)\, H_n(S)} \; . \qquad \text{3)}$$

The mean-square value of t_t is

$$\overline{t_t^2} = \lim_{S \to 0} \frac{1}{H_t(S)} \frac{d^2H_t}{dS^2} = \lim_{S \to 0} \frac{\sum_{p=1}^{N} \frac{d^2H_p}{dS^2}}{\sum_{q=1}^{N} H_q(S)} \; . \qquad \text{4)}$$

The mean-square duration is then

$$\sigma_t^2 = \lim_{S \to 0} (\overline{t_t^2} - \bar{t}^2) \; . \qquad \text{5)}$$

Making the appropriate substitutions in equation 5 gives

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \left(\frac{\sum\limits_{p=1}^{N} \frac{d^2 H_p}{dS^2}}{H_t(S)} - \frac{\sum\limits_{i=1}^{N} \sum\limits_{j=1}^{N} \frac{dH_i}{dS} \frac{dH_j}{dS}}{H_t^2(S)} \right) \qquad 6)$$

which may be factored to give

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \frac{1}{H_t(S)} \left(\sum_{p=1}^{N} \frac{d^2 H_p}{dS^2} - \frac{1}{H_t(S)} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{dH_i}{dS} \frac{dH_j}{dS} \right) . \qquad 7)$$

Notice that d^2H_p/dS^2 may be written as

$$\frac{d^2 H_p}{dS^2} = H_p(S) \left(\frac{1}{H_p(S)} \frac{d^2 H_p}{dS^2} \right) = H_p(S)\, \overline{t_p^2} \qquad 8a)$$

and

$$H_p(S)\, \overline{t_p^2} = H_p(S)\, (\overline{t_p^2} - \bar{t}_p^2) + H_p(S)\, \bar{t}_p^2 \ . \qquad 8b)$$

Substituting equations 8a and 8b into equation 7 and combining summation indices in the first and second terms yields

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \frac{1}{H_t(S)} \left(\sum_{i=1}^{N} \left[H_i(S)(\overline{t_i^2} - \bar{t}_i^2) + H_i(S)\, \bar{t}_i^2 - \frac{1}{H_t(S)} \left(\frac{dH_i}{dS} \right)^2 \right] - \frac{1}{H_t(S)} \sum_{i \neq j}^{N} \sum^{N} \frac{dH_i}{dS} \frac{dH_j}{dS} \right) . \qquad 9)$$

If now $(dH_i/dS)^2/H_t(S)$ is written as

$$\frac{1}{H_t(S)} \left(\frac{dH_i}{dS} \right)^2 = \frac{H_i^2(S)}{H_t(S)} \left(\frac{dH_i}{dS} \right)^2 / H_i^2(S) = H_i^2(S)\, \bar{t}_i^2 / H_t(S) \ . \qquad 10)$$

Equation 9 then becomes

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \frac{1}{H_t(S)} \left[\sum_{i=1}^{N} \left(H_i(S) \; (\overline{t_i^2} - \bar{t}_i^2) \right.\right.$$

$$\left. + \quad H_i(S) \; \bar{t}_i^2 (1 - H_i^2(S)/H_t(S)) \right) \; - \; \frac{1}{H_t(S)} \sum_{i \neq j}^{N} \sum^{N} \frac{dH_i}{dS} \frac{dH_j}{dS} \quad . \qquad 11)$$

With $\overline{t_i^2} - \bar{t}_i^2 = \sigma_i^2$ equation 11 becomes

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \frac{1}{H_t(S)} \left[\sum_{i=1}^{N} \left(H_i(S) \; \sigma_i^2 \right.\right.$$

$$\left. + \quad H_i(S) \; \bar{t}_i^2 \; (1 - H_i^2(S)/H_t(S)) \right)$$

$$\left. - \; \frac{1}{H_t(S)} \; \sum_{i \neq j}^{N} \sum^{N} \; \frac{dH_i}{dS} \frac{dH_j}{dS} \right] \qquad 12)$$

which may be factored to yield

$$\sigma_t^2 = \lim_{S \longrightarrow 0} \frac{1}{H_t(S)} \left[\sum_{i=1}^{N} \quad H_i(S) \left(\sigma_i^2 \right.\right.$$

$$\left. + \; \bar{t}_i^2 (1 - H_i^2(S)/H_t(S)) \right)$$

$$\left. - \; \frac{1}{H_t(S)} \sum_{i \neq j}^{N} \sum^{N} \quad \frac{dH_i}{dS} \frac{dH_j}{dS} \right] \quad . \qquad 13)$$

Due to the complexity of equation 13 it is obvious that it could be manipulated into many different forms.

Appendix V: Calculation of Collision Frequency, v_c, for Metallic Conducting Materials

Values of collision frequency for Chapter IX are calculated from the frequency dependent form of conductivity given by (Kittel, 1968)

$$\sigma(S) = N e^2/m(S + v_c) \qquad 1)$$

where N = electron density per cubic meter, e the electron charge in coulombs, m the electron mass in kilograms, and $v_c = 1/\tau$ of Kittel's development of 1968. Setting S=0 yields the known static value of conductivity, σ_o, in mhos per meter. Then

$$v_c = N e^2/m \sigma_o . \qquad 2)$$

The quantity v_c may also be expressed in terms of plasma frequency, ω_p, given by Kittlel (1968) as

$$\omega_p^2 = N e^2/e_o m \qquad 3)$$

from which

$$e_o \omega_p^2 = N e^2/m . \qquad 4)$$

Setting $e_o \omega_p^2$ for Ne^2/m in equation 2 gives

$$v_c = \omega_p^2 e_o/ \sigma_o . \qquad 5)$$

The quantity ω_p is the free electron plasma frequency in radians per second and e_o is the permittivity of free space which is equal to 8.85×10^{-12} farads per meter.

Appendix VI: Convolution of $\frac{e^{-at}}{t}$ and $\frac{e^{-bt}}{t}$

It is shown below that the following convolution relationship holds:

$$L^{-1}\left[\sqrt{(S + a)(S + b)}\right] = \frac{e^{-at}}{2\pi^{\frac{1}{2}}\,t} * \frac{e^{-bt}}{2\pi^{\frac{1}{2}}\,t}$$

$$= \frac{(a - b)^2}{4}\, e^{-\frac{1}{2}(a+b)t}\, \frac{J_1(\frac{1}{2}(a-b)t)}{\frac{1}{2}(a-b)t} \qquad 1)$$

or

$$L^{-1}\left[\sqrt{(S + a)(S + b)}\right] = \frac{(a - b)^2}{4}\, e^{-\frac{1}{2}(a+b)t}\, \frac{I_1(\frac{1}{2}(a-b)t)}{\frac{1}{2}(a-b)t} \qquad 2)$$

as (a - b) is real or imaginary.

Substitution of the exponential functions into the convolution integral yields

$$I_c = \frac{e^{-bt}}{4\pi} \int_0^t \frac{e^{-(a - b)x}\,dx}{x^{3/2}(t - x)^{3/2}} \;. \qquad 3)$$

Making the following change of variable

$$x \longrightarrow yt$$

yields

$$I_c = \frac{e^{-bt}}{4\pi} \int_0^1 \frac{t\, e^{-t(a - b)y}\,dy}{t^3\, y^{3/2}(1 - y)^{3/2}} \qquad 4)$$

or

$$I_c = \frac{e^{-bt}}{4\pi\ t^2} \int_0^1 \frac{e^{-t(a\ -\ b)y}\ dy}{y^{3/2}(1\ -\ y)^{3/2}} \ . \qquad 5)$$

Next add and subtract $\frac{1}{2}$ to y to obtain

$$I_c = \frac{e^{-bt}}{4\pi\ t^2} \int_0^1 \frac{e^{-t(a\ -\ b)(y\ -\frac{1}{2}\ +\ \frac{1}{2})}\ d(y\ -\ \frac{1}{2}\ +\ \frac{1}{2})}{(y\ -\ \frac{1}{2}\ +\ \frac{1}{2})^{3/2}(\frac{1}{2}\ -\ (y\ -\ \frac{1}{2}))^{3/2}} \qquad 6)$$

and change the variable $y - \frac{1}{2}$ to Y to obtain

$$I_c = \frac{e^{-t(b\ +\ \frac{1}{2}a\ -\ \frac{1}{2}b)}}{4\pi\ t^2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{-\frac{1}{2}t(a\ -\ b)2Y}\ dY}{(\frac{1}{2}\ +\ Y)^{3/2}\ (\frac{1}{2}\ -\ Y)^{3/2}} \ . \qquad 7)$$

Making another change of variable $Y \longrightarrow \frac{1}{2}z$ provides

$$I_c = \frac{e^{-\frac{1}{2}t(a\ +\ b)}}{\pi\ t^2} \int_{-1}^{1} \frac{e^{-\frac{1}{2}t(a\ -\ b)z}\ dz}{(1\ -\ z^2)^{3/2}} \qquad 8)$$

or

$$I_c =$$

$$\frac{e^{-\frac{1}{2}t(a\ +\ b)}}{\pi\ t} \left[\frac{2\ \pi^{\frac{1}{2}}(a-b)(-3/2)!}{4\ x\ \frac{1}{2}t\ \pi^{\frac{1}{2}}(a-b)((-3/2)!)} \int_0^1 \frac{e^{-\frac{1}{2}t(a-b)z}}{(1-z^2)^{3/2}}\ dz\right] \qquad 9)$$

where $(-3/2)! = -\ 2\pi^{\frac{1}{2}}$. Collecting all factors outside of the brackets gives

$$I_c = \frac{e^{-\frac{1}{2}t(a\ +\ b)}\ (a\ -\ b)}{2t} \left[\int_0^1 \frac{e^{-\frac{1}{2}t(a\ -\ b)z}}{(1\ -\ z^2)^{3/2}}\ dz\right] \ . \qquad 10)$$

The factor in brackets is a standard form for $J_1(\frac{1}{2}t(a - b))$ if $(a - b)$ is imaginary or $I_1(\frac{1}{2}t(a - b))$ if $(a - b)$ is real.

Therefore

$$I_c = \tfrac{1}{4}(a - b)^2 \, e^{-\frac{1}{2}t(a + b)} \, \frac{J_1(\frac{1}{2}t(a - b))}{\frac{1}{2}t(a - b)} \qquad 11)$$

when (a - b) is imaginary and

$$I_c = \tfrac{1}{4}(a - b)^2 \, e^{-\frac{1}{2}t(a + b)} \, \frac{I_1(\frac{1}{2}t(a - b))}{\frac{1}{2}t(a - b)} \qquad 12)$$

when (a - b) is real.

Appendix VII: Derivation of Equation 62 of Chapter X from Equation 61 b

It is shown that equation 61 b,

$$\begin{aligned}\rho(S) = &- 1 \\ &+ \frac{2S(S + v_c)(S(S + v_c) + A/\cos^2\theta)}{A(\tan^2\theta - 1)(S(S + v_c) - A/(\tan^2\theta - 1))} \\ &- \frac{2(S(S+v_c)+A)\sqrt{S(S+v_c)(S(S+v_c)+A/\cos^2\theta)}}{A(\tan^2\theta - 1)(S(S+v_c) - A/(\tan^2\theta - 1))}, \qquad 1)\end{aligned}$$

may be decomposed into the following additive components shown below:

$$\begin{aligned}\rho(S) = &- 1 + \frac{2\, v_c\, S}{A(\tan^2\theta - 1)} + \frac{2\, S^2}{A(\tan^2\theta - 1)} \\ &+ \frac{2\, S(S + v_c)}{(1 - \cot^2\theta)^2\, (S + a)(S + b)} \\ &- \frac{2\sqrt{S(S + v_c)(S + c)(S + d)}}{A\, (\tan^2\theta - 1)} \\ &- \frac{\frac{1}{2} \tan^2(2\theta)\sqrt{S(S + v_c)(S + c)(S + d)}}{(S + a)(S + b)} \qquad 2)\end{aligned}$$

where

$$a = \tfrac{1}{2}v_c + j\sqrt{-\tfrac{1}{4}v_c^2 + A/(1 - \tan^2\theta)}$$

$$b = \tfrac{1}{2}v_c - j\sqrt{-\tfrac{1}{4}v_c^2 + A/(1 - \tan^2\theta)}$$

$$c = \tfrac{1}{2}v_c + j\sqrt{-\tfrac{1}{4}v_c^2 + A/\cos^2\theta}\ , \qquad 3)$$

$$d = \tfrac{1}{2}v_c - j\sqrt{-\tfrac{1}{4}v_c^2 + A/\cos^2\theta}\ .$$

First consider the second term of equation 1 above. Add and subtract $A/(\tan^2\theta - 1)$ to the numerator factor $(S(S + v_c) + A/\cos^2\theta)$. This procedure yields for the second term

$$\frac{2S(S + v_c)(S(S + v_c) - \frac{A}{\tan^2\theta - 1} + \frac{A}{\tan^2\theta - 1} + A/\cos^2\theta}{A(\tan^2\theta - 1)(S(S + v_c) - A/(\tan^2\theta - 1))} =$$

$$\frac{2S(S + v_c)}{A(\tan^2\theta - 1)}\left[\frac{S(S + v_c) - A/(\tan^2\theta - 1)}{S(S + vc) - A/(\tan^2\theta - 1)}\right.$$

$$\left.- \frac{A\ (\frac{1}{\cos^2\theta} + \frac{1}{\tan^2\theta - 1})}{S(S + v_c) - A/(\tan^2\theta - 1)}\right] =$$

$$\frac{2S(S + v_c)}{A(\tan^2\theta - 1)} + \frac{2S(S + v_c)}{(S + a)(S + b)}\left[\frac{\frac{1}{\cos^2\theta} + \frac{1}{\tan^2\theta - 1}}{\tan^2\theta - 1}\right]. \qquad 4)$$

The trigonometric factor of the second term in equation 4 may be reduced to $1/(1 - \cot^2\theta)^2$ by use of standard trigonometric identities as follows:

$$\left[\frac{1}{\cos^2\theta} + \frac{1}{\tan^2\theta - 1}\right]\frac{1}{\tan^2\theta - 1} =$$

$$\frac{\tan^2\theta - 1 + \cos^2\theta}{\cos^2\theta(\tan^2\theta - 1)^2} = \frac{\tan^2\theta - \sin^2\theta}{\cos^2\theta\ \tan^4\theta(1 - \cot^2\theta)^2} =$$

$$\left[\frac{\tan^2\theta}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}\right] x \, \frac{1}{\tan^4\theta(1 - \cot^2\theta)^2} =$$

$$\frac{(\tan^2\theta/\cos^2\theta) - \tan^2\theta}{\tan^4\theta(1 - \cot^2\theta)^2} = \frac{\cos^2\theta(-1 + 1/\cos^2\theta)}{\sin^2\theta(1 - \cot^2\theta)^2} =$$

$$\frac{1 - \cos^2\theta}{\sin^2\theta(1 - \cot^2\theta)^2} = \frac{\sin^2\theta}{\sin^2\theta(1 - \cot^2\theta)^2} = \frac{1}{(1 - \cot^2\theta)^2} . \quad 5)$$

Equation 4 is then

$$\frac{2(S^2 + Sv_c)}{A(\tan^2\theta - 1)} + \frac{2S(S + v_c)}{(1 - \cot^2\theta)^2(S + a)(S + b)} =$$

$$\frac{2\ S^2}{A(\tan^2\theta - 1)} + \frac{2\ S\ v_c}{A(\tan^2\theta - 1)}$$

$$+ \frac{2\ S(S + v_c)}{(1 - \cot^2\theta)^2(S + a)(S + b)} . \qquad 6)$$

The third term of equation 1 is treated also by adding and subtracting $A/(\tan^2\theta - 1)$ to the factor $(S(S + v_c)+A)$. This operation yields

$$\frac{-2\left[S(S + v_c) - \frac{A}{\tan^2\theta - 1} + \frac{A}{\tan^2\theta - 1} + A\right]\sqrt{S(S+v_c)(S+c)(S+d)}}{A(\tan^2\theta - 1)\ (S(S + v_c) - A/(\tan^2\theta - 1))}$$

$$= -2\left[1 + \frac{A + A/(\tan^2\theta-1)}{(S+a)(S+b)}\right]\frac{\sqrt{S(S+v_c)(S+c)(S+d)}}{A(\tan^2\theta - 1)} =$$

$$\frac{-2\sqrt{S(S + v_c)(S + c)(S + d)}}{A(\tan^2\theta - 1)}$$

$$- \frac{2(\tan^2\theta - 1 + 1)\sqrt{S(S + v_c)(S + c)(S + d)}}{(\tan^2\theta - 1)(S + a)(S + b)} \; . \qquad 7)$$

The trigonometric coefficient of the second term above is

$$\frac{2(\tan^2\theta - 1 + 1)}{(\tan^2\theta - 1)^2} = \frac{1}{2}\,\frac{4\tan^2\theta}{(\tan^2\theta - 1)^2} = \frac{1}{2}\left[\frac{2\tan\theta}{\tan^2\theta - 1}\right]^2$$

$$= \frac{1}{2}\tan^2(2\theta) \; . \qquad 8)$$

The third term of equation 1 is then

$$- \frac{2\sqrt{S(S + v_c)(S + c)(S + d)}}{A(\tan^2\theta - 1)}$$

$$- \frac{\frac{1}{2}\tan^2(2\theta)\sqrt{S(S + v_c)(S + c)(S + d)}}{(S + a)(S + b)} \; . \qquad 9)$$

Substituting equation 6 and expression 9 into equation 1 yields equation 2.

It is noted that equation 2 appears to possess singularities at $\theta = 45^\circ$ whereby the second, third, fourth, fifth, and sixth terms become infinite. The overall behavior of $\varrho(S)$ as θ approaches 45° is easily determined by setting $x = \tan^2\theta - 1$, $(x + 1)^2/x^2 = 1/(1 - \cot^2\theta)^2$, and $2(x + 1)/x^2 = \tan^2(2\theta)$ and then taking the limit of $\varrho(S)$

as x approaches zero. Making the above substitutions gives

$$\rho(S) = - 1 + \frac{2\ S(S + v_c)}{A\ x} + \frac{2\ S(S + v_c)(x + 1)^2}{x^2\ (S(S + v_c) - A/x)}$$

$$- \frac{2\sqrt{S(S + v_c)(S + c)(S + d)}}{A\ x}$$

$$- \frac{(x + 1)\sqrt{S(S + v_c)(S + c)(S + d)}}{x^2(S(S + v_c) - A/x)}$$

$$\rho(S) = - 1 + 2\ S(S + v_c)\left[\frac{1}{Ax} + \frac{(x + 1)^2}{x(S(S + v_c)x - A)}\right]$$

$$- \frac{2\sqrt{S(S + v_c)(S + c)(S + d)}}{(S + a)(S + b)}\left[\frac{1}{A\ x}\right.$$

$$\left. + \frac{(x + 1)}{x(S(S + v_c)x - A)}\right] . \qquad 10)$$

As x approaches zero (or equivalently, as θ approaches 45°) the limits of the two bracketed factors are as follows: the first is

$$\lim_{x \longrightarrow 0} \frac{S(S + v_c)x - A + A(x^2 + 2x + 1)}{Ax(S(S + v_c)x - A)} =$$

$$\lim_{x \longrightarrow 0} \frac{x(S(S + v_c) + Ax + 2A)}{x\ A(S(S + v_c)x - A)} = \frac{S(S + v_c) + 2A}{- A^2} \qquad 11)$$

while the second bracketed term is

$$\lim_{x \longrightarrow 0} \frac{S(S + v_c)x - A + Ax + A}{Ax(S(S + v_c)x - A)} = \lim_{x \longrightarrow 0} \frac{S(S + v_c) + A}{- A^2}$$

$$= \frac{S(S + v_c) + A}{- A^2} . \qquad 12)$$

The above equations 11 and 12 show that the entire expression for $\rho(S)$ remains finite as θ approaches 45^o even though the individual terms in the decomposition of equation 2 tend to infinity.