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In this paper, we are concerned with the very general notion of a universal algebra. A universal algebra essentially consists of a set A together with a possibly infinite set of finitary operations on A. Generally, these operations are related by means of equations, yielding different algebraic structures such as groups, groups with operators, modules, rings, lattices, etc. This theory is concerned with those theorems which are common to all these various algebraic systems. In particular, the fundamental isomorphism and homomorphism theorems are covered, as well as, the Jordan-Hölder theorem and the Zassenhaus lemma. Furthermore, new existence proofs are given for sums and free algebras in any primitive class of universal algebras.

The last part treats the theory of groups with multi-operators in a manner essentially different from that of P. J. Higgins. The approach taken here generalizes the theorems on groups with operators as found in Jacobson's "Lectures in Abstract Algebra," vol. I.

The basic language of category theory is used whenever convenient.

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UNIVERSAL ALGEBRA

INTRODUCTION

A universal algebra, in brief, consists of a set A together with an arbitrary number of finitary and totally defined operations on A.

Thus, a set is a universal algebra with no operation, and a groupoid is a universal algebra with a single binary operation. A group is a universal algebra with one binary operation, one unary operation, namely, the taking of the inverse of an element, and one nullary operation, namely, the taking of the identity.

An example of a universal algebra with an infinite number of operations is a vector space V over the real field R. Here, the scalar multiplication induces a set of unary operations f_r given by

$$f_r(v) = rv$$

for all $v \in V$, where $r \in R$.

A semi-group S is a universal algebra with a single binary operation which is associative. That is, if f denotes this binary operation, then

$$f(f(a, b), c) = f(a, f(b, c))$$

holds for all $a, b, c, \in S$. This identity distinguishes the semi-groups among the universal algebras with a single binary operation, namely, groupoids, and determines what we call a primitive class. In general, algebras in which a given set of identities hold, constitute a primitive class of universal algebras.

The concept of a universal algebra used here does not permit partially defined operations, and hence we cannot treat the theory of fields, since the unary operation $f(x) = x^{-1}$ is defined only when $x \neq 0$. It cannot be extended in such a way as to maintain the identity

$$xf(x) = 1$$

for all x in the field. There is a more general theory allowing partial operations, but we shall not consider it here.

Our subject is divided into two parts, the first on universal algebras and the second on groups with multi-operators. Part I, which comprises the first four chapters, begins with the fundamental notion of universal algebras, homomorphisms and congruence relations, and ends with the theory of primitive classes. Part II, then, concentrates on the algebraic systems called groups with multi-operators, which form a primitive class of universal algebras. It comprises Chapters V and VI.

Definitions and propositions of the fundamentals are covered in Chapter I. Chapter II, on the generalized Jordan-Hölder theorem, is devoted to the extension of the isomorphism theorems which are usually found in group theory, to those of universal algebras. In particular, it is shown that Zassehaus' lemma, Schreier's refinement theorem, and the Jordan-Hölder theorem hold in universal algebras under a very simple condition. Chapter IV introduces the notion of primitive classes, and together with this chapter, Chapter III presents existence theorem of free universal algebras, products, and sums in any primitive class of universal algebras. There, the basic language of category theory is used.

The generalized notion of normal subgroups and ideals is developed throughout the last two chapters. The theory of direct sums is covered in Chapter VI, and the uniqueness theorem of the complete decomposition of a group with multi-operators is represented as the last theorem of this paper.

As most mathematical books and papers require the concepts and methods of the theory of sets as basic to their subjects, the present paper assumes the reader's familiarity with these. As far as the theory of algebras is concerned, only a knowledge of the fundamentals of groups, homomorphisms, normal subgroups, factor groups, etc. are required.

The following list shows symbols, notations and abbreviations used in this paper:

Λ	Set
Z	Set of integers

N	Set of natural numbers
$A \cup B$, $\bigcup_{\lambda \in \Lambda} A_\lambda$	Union
$A \cap B$, $\bigcap_{\lambda \in \Lambda} A_\lambda$	Intersection
$A - B$	Difference
$A \times B$, $\prod_{\lambda \in \Lambda} A_\lambda$, $A^n (n \in N)$	Product set
$\subseteq$, $\supseteq$	Set inclusion
$\subset$, $\supset$	Proper inclusion
R^{-1}	Inverse of the relation R
RT	Composition[1] of relations R and T
$R\|A$	Restriction of R to A
1_A	Identity relation for a set A
$f : A \to B$	f maps A into B
$f(x)$	Value of f at x
$f : x \mapsto y$	$f(x) = y$
$\|A\|$	Cardinal number of A
N_0	$\|N\|$[2]
iff	"if and only if"

Sometimes we denote $A \xrightarrow{f} B$ instead of $f : A \to B$, which is used especially in diagrams to illustrate the composition of

---

[1] $RT = \{(a, b) : (a, x) \in T$ and $(x, b) \in R$ for some $x\}$.

[2] In general, the first letter "aleph" of the Hebrew alphabet is used instead of N of N_0.

mappings. The diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dr, "f_3"'] & B \arrow[d, "f_2"] \\ & C \end{tikzcd}$$

where f_1, f_2 and f_3 are mappings and A, B and C are sets, is said to be <u>commutative</u> or said to <u>commute</u> if $f_3 = f_2 f_1$. Similarly, the diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, "f_4"'] & B \arrow[d, "f_2"] \\ D \arrow[r, "f_3"'] & C \end{tikzcd}$$

is commutative if $f_2 f_1 = f_3 f_4$, and the diagram

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[dr, "f_4"'] & B \arrow[r, "f_2"] \arrow[d, "f_3"] & D \arrow[dl, "f_5"] \\ & C & \end{tikzcd}$$

is commutative iff all the triangles are commutative diagrams.

We also need a rudimentary concept of categories and functors, especially the former, which is indispensable for our construction of free algebras.

<u>Definition of Categories</u>. A category CAT consists of a collection of objects Ob(CAT) and for any pair of objects

$A, B \in Ob(CAT)$ a collection of morphisms $Hom_{CAT}(A, B)$ satisfying the following conditions:

i) If $f \in Hom_{CAT}(A, B)$ and $g \in Hom_{CAT}(B, C)$ then there is a unique morphism $gf \in Hom_{CAT}(A, C)$ called the composition of morphisms f and g.

ii) For each object $A \in Ob(CAT)$ there is a morphism 1_A in $Hom_{CAT}(A, A)$ such that if $f \in Hom_{CAT}(A, B)$ and $g \in Hom_{CAT}(B, A)$ then $f1_A = f$ and $1_A g = g$.

iii) For any morphisms f, g, h, $(hg)f = h(gf)$ whenever $(hg)f$ and $h(gf)$ are defined.

iv) Two families of morphisms $Hom_{CAT}(A, B)$ and $Hom_{CAT}(A', B')$ are disjoint if either $A \neq A'$ or $B \neq B'$.

For instance, we see immediately that all sets and mappings between the sets form a category SET. In this paper, often the objects are taken as algebras and the morphisms are taken as homomorphisms between the algebras as the symbol "Hom" indicates.

Definition of Functors. Let CAT and CAT' be categories. Then a rule

$$F : CAT \rightarrow CAT'$$

such that for each $A \in Ob(CAT)$, $F(A) \in Ob(CAT')$ and for each 1_A in $Hom_{CAT}(A, A)$, $F(1_A) = 1_{F(A)} \in Hom_{CAT'}(F(A), F(A))$, is called a functor of CAT into CAT' if it satisfies either i) or ii) below:

i) For each $f \in \mathrm{Hom}_{CAT}(A, B)$, $F(f) \in \mathrm{Hom}_{CAT'}(F(A), F(B))$, and further, if $g \in \mathrm{Hom}_{CAT}(B, C)$ then

$$F(gf) = F(g)F(f).$$

ii) For each $f \in \mathrm{Hom}_{CAT}(A, B)$, $F(f) \in \mathrm{Hom}_{CAT'}(F(B), F(A))$, and further, if $g \in \mathrm{Hom}_{CAT}(B, C)$ then

$$F(gf) = F(f)F(g).$$

The functor with the property i) is called <u>covariant</u> and the one with ii) is called <u>contravariant.</u>

A simple example is a functor $F : GRP \to SET$ where GRP is a category whose objects are groups, and whose morphisms are group homomorphisms. F maps every group onto its underlying set, and group homomorphism onto its underlying mapping. One can show at once that F is a covariant functor.

Also, let S be a fixed set, S^X a set of all mappings of a set X into S. Then we have a contravariant functor $F: SET \to SET$ such that for each set X and for each mapping $f : X \to X'$, $F(X)$ is a set S^X and $F(f)$ is a mapping of $S^{X'}$ into S^X defined by

$$F(f) : \varphi \mapsto \varphi f$$

for all $\varphi \in S^{X'}$. The proof is straightforward and is omitted.

PART I.

UNIVERSAL ALGEBRAS

I. GENERAL THEORY

1. Ω-Algebras

By an n-ary operation on a set A, we understand a mapping which assigns to each n-tuple of elements of A a single element of A, n being some finite non-negative integer. In particular, a domain of a nullary (0-nary) operation is a singleton and hence the operation selects a certain element (constant) in A. If the symbol 0 is used to denote a nullary operation which picks up a constant $c \in A$, we shall identify these two symbols 0 and c, i.e., a nullary operation is a constant of A.

Thus, if G and H are respectively a multiplicative group and an additive group, G and H are considered to have nullary operations 1 and 0, respectively. Similarly binary (2-ary) operations $\cdot$ and $+$ are defined on G and H, respectively. However, since in this paper, a distinction between the binary operations $\cdot$ and $+$ is inessential and may even cause some confusion, we shall avoid such a use of different symbols for the same kind of algebras. For this purpose we shall adopt the following definitions:

Definition 1.1. An operator domain is a set Ω whose elements are called operators such that for each $\omega \in \Omega$ a mapping $a : \omega \mapsto a(\omega) \in N$ is defined. $a(\omega)$ is called the arity of ω and if

$\alpha(\omega) = n$ we call ω an n-ary operator. Often we denote a set of all n-ary operators in Ω by Ω_n.

If A is a set such that whenever $\Omega_n \neq \emptyset$, a mapping $\Omega_n \to A^{A^n}$ is defined, then A is said to have an Ω-algebra structure, and a family of such mappings is called an Ω-algebra structure on A.

Definition 1.2 A universal algebra or an Ω-algebra is a set that has an Ω-algebra structure for a given operator domain Ω.

Hence, a set A is an Ω-algebra iff for each $\omega \in \Omega_n$ an n-ary operation $A^n \to A$ is defined. We shall denote the value of the n-ary operation at each $(a_1, a_2, \ldots, a_n) \in A^n$ by $\omega(a_1, a_2, \ldots, a_n)$. Thus, for instance if $(i_1, i_2, \ldots, i_n) \in Z^n$ then for each $\omega \in \Omega_n$ we may define the corresponding n-ary operation by

$$\omega(i_1, i_2, \ldots, i_n) = \inf\{i_1, i_2, \ldots, i_n\}. \tag{1.1.1}$$

Or more trivially if S is a set then we may define the n-ary operation as

$$\omega(s_1, s_2, \ldots, s_n) = s_1$$

for all $s_1, s_2, \ldots, s_n \in S$. Thus, Z has an Ω-algebra structure and so does any nonempty set. In particular, from (1.1.1) we see that for any $\omega \in \Omega_n$, $\omega(0, 0, \ldots, 0) = 0$. Henceforth we shall mean by the Ω-algebra Z the Ω-algebra Z given by (1.1.1).

Example 1.3. Let $+$ denote a binary operator. Then a groupoid is an Ω-algebra, Ω being a singleton $\{+\}$. If $\Omega = \{+, -, 0\}$ where $-$ is unary (1-ary) and 0 is nullary then a group is an Ω-algebra G which satisfies the following identities for all $a, b, c \in G$:

$$+(+(a, b), c) = +(a, +(b, c)),$$

$$+(a, 0) = a,$$

$$+(a, -(a)) = 0.$$

Thus, a group is an Ω-algebra for $\Omega = \{+, -, 0\}$, but the converse may not be true. The subject of Chapter IV is related to such a set of identities, and there we shall see a certain formalization of a class of Ω-algebras that satisfy the given identities.

Similarly, if Ω has an additional binary operator $\cdot$, then the Ω-algebra is a candidate for a ring. Here again, a ring is an Ω-algebra but an Ω-algebra is not necessarily a ring.

Definition 1.4. Let A be an Ω-algebra. Then a subset B of A is called an Ω-subalgebra if B is closed under all the operations on A determined by Ω, i.e., for any n-ary operation on A determined by $\omega \in \Omega_n$, $b_1, b_2, \ldots, b_n \in B$ implies $\omega(b_1, b_2, \ldots, b_n) \in B$.

Thus, a trivial example of an Ω-subalgebra of the Ω-algebra A is A itself, and in particular $\emptyset$ is an Ω-subalgebra of A iff Ω has no nullary operators. From the definition the following proposition is immediate:

Proposition 1.5. The intersection of an arbitrary number of Ω-subalgebras of an Ω-algebra is an Ω-subalgebra of the Ω-algebra.

If A is an Ω-algebra and S is a subset of A, then the intersection of all the Ω-subalgebras containing S is the smallest Ω-subalgebra B containing S. B is called the Ω-subalgebra of A generated by the set S.

2. The Fundamental Theorem of Homomorphisms

In this section, we shall start with the study of a mapping of an Ω-algebra into another Ω-algebra that preserves the Ω-algebra structure, and obtain a theorem which indicates an important characteristic of algebras in general.

Definition 1.6. Given two Ω-algebras A and B, a bijective mapping $f : A \to B$ is called an isomorphism of A onto B if for all $\omega \in \Omega_n$ and $a_1, a_2, \ldots, a_n \in A$,

$$f : \omega(a_1, a_2, \ldots, a_n) \mapsto \omega(f(a_1), f(a_2), \ldots, f(a_n)).$$

If there is an isomorphism $f: A \to B$, A and B are said to be <u>isomorphic</u> and we write $A \cong B$, or $f: A \cong B$.

If f has the property above but not necessarily bijective, f is called a <u>homomorphism</u>, and similarly the terminologies "epimorphisms," "monomorphisms," "endomorphisms" and "automorphisms" are defined in obvious manner.

<u>Proposition 1.7</u>. If $f: A \to B$ and $g: B \to C$ are homomorphisms of Ω-algebras A, B and C then the composition $gf: A \to C$ is again a homomorphism.

The proof of 1.7 is immediate from the definition. Also one can show that the identity mappings $1_A: A \to A$ and $1_B: B \to B$ are homomorphisms such that $f1_A = f$ and $1_B f = f$ for any homomorphism $f: A \to B$, and that the composition of homomorphisms is associative. Hence all Ω-algebras together with a collection of all homomorphisms of Ω-algebras form a category. This category is denoted by $[\Omega]$ and is called the <u>category of Ω-algebras</u>. A collection of objects in $[\Omega]$ is denoted by $Ob(\Omega)$ and for each pair $A, B \in Ob(\Omega)$, a collection of morphisms from A to B is denoted by $Hom_\Omega(A, B)$. In particular, a category SET is considered as a category $[\Omega]$ with $\Omega = \emptyset$.

<u>Proposition 1.8</u>. Let A and B be Ω-algebras, and f a

homomorphism of A into B. Then if A' and B' are Ω-subalgebras of A and B, respectively, the following two sets are Ω-subalgebras of B and A respectively:

$$f(A') = \{f(a') : a' \in A'\},$$

$$f^{-1}(B') = \{a \in A : f(a) \in B'\}.$$

The former is called a <u>homomorphic image</u> of A' and the latter is called an <u>inverse image</u> of B'.

<u>Proof.</u> By the definition of homomorphisms it is evident that $f(A')$ is an Ω-subalgebra of B. To prove $f^{-1}(B')$ to be an Ω-subalgebra of A, let $a_1, a_2, \ldots, a_n \in f^{-1}(B')$ and ω an arbitrary operator in Ω_n. Then for each a_i, $i=1, 2, \ldots, n$, we have $f(a_i) \in B'$, and hence

$$f(\omega(a_1, a_2, \ldots, a_n)) = \omega(f(a_1), f(a_2), \ldots, f(a_n)) \in B',$$

i.e.,

$$\omega(a_1, a_2, \ldots, a_n) \in f^{-1}(B').$$

Therefore, $f^{-1}(B')$ is an Ω-subalgebra of A. Q.E.D.

<u>Definition 1.9.</u> Let R be an equivalence relation on the Ω-algebra A such that R is closed under all the operations on A determined by Ω. It means that for all $\omega \in \Omega_n$ and $a_i, b_i \in A$,

$i=1, 2, \ldots, n$, $a_i \equiv b_i \bmod R$ implies

$$\omega(a_1, a_2, \ldots, a_n) \equiv \omega(b_1, b_2, \ldots, b_n) \bmod R.$$

Then R is called a congruence relation on A, and the R-equivalence classes are called R-congruence classes. We shall often adopt the notation:

$$\overline{a} = \{x \in A : x \equiv a \bmod R\}$$

for the R-congruence class (or R-equivalence class, as well) containing $a \in A$.

Thus, from the definition, we see that if R-congruence classes $\overline{a_i}$ and $\overline{b_i}$, $i=1, 2, \ldots, n$, are given, then $\overline{a_i} = \overline{b_i}$ implies $\overline{\omega(a_1, a_2, \ldots, a_n)} = \overline{\omega(b_1, b_2, \ldots, b_n)}$. Therefore, the n-ary operation on the family of all R-congruence classes such that

$$(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}) \mapsto \overline{\omega(a_1, a_2, \ldots, a_n)}$$

is well-defined. Now it is natural to consider the operation to be the one determined by $\omega \in \Omega_n$. That is,

$$\omega(\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}) = \overline{\omega(a_1, a_2, \ldots, a_n)}, \qquad (1.2.1)$$

and the family of all the R-congruence classes has an Ω-algebra structure.

Definition 1.10. If A is an Ω-algebra with a congruence relation R then the family of all R-congruence classes forms an Ω-algebra with respect to the n-ary operations given by (1.2.1). This Ω-algebra is called a factor Ω-algebra of A with respect to R, and is denoted by A/R.

Now let A be an Ω-algebra with a congruence relation R, and consider a mapping $\nu : A \to A/R$ defined by $\nu : a \mapsto \bar{a}$ for all $a \in A$. Evidently, ν is surjective and if $a_1, a_2, \ldots, a_n \in A$ and $\omega \in \Omega_n$ then (1.2.1) is written as

$$\omega(\nu(a_1), \nu(a_2), \ldots, \nu(a_n)) = \nu(\omega(a_1, a_2, \ldots, a_n)).$$

Hence, ν is an epimorphism, and we call it the canonical homomorphism of A onto A/R.

The following definition gives an important example of a congruence relation on the Ω-algebra.

Definition 1.11. Let f be a homomorphism of the Ω-algebra A into another Ω-algebra. Then the kernel of f is a set $\mathrm{Ker}(f)$ of the ordered pairs defined by

$$\mathrm{Ker}(f) = \{(a, b) \in A^2 : f(a) = f(b)\}.$$

Henceforth, we shall rather write $a \equiv b \bmod \mathrm{Ker}(f)$ to mean

$(a, b) \in Ker(f)$. We can show that $Ker(f)$ is obviously an equivalence relation on A. Now if $a_i \equiv b_i \mod Ker(f)$, $i = 1, 2, \ldots, n$, and $\omega \in \Omega_n$ then $f(a_i) = f(b_i)$, and it implies

$$\omega(f(a_1), f(a_2), \ldots, f(a_n)) = \omega(f(b_1), f(b_2), \ldots, f(b_n))$$

i. e.,

$$f(\omega(a_1, a_2, \ldots, a_n)) = f(\omega(b_1, b_2, \ldots, b_n)).$$

Hence, $\omega(a_1, a_2, \ldots, a_n) \equiv \omega(b_1, b_2, \ldots, b_n) \mod Ker(f)$. This proves:

Proposition 1.12. For any homomorphism f of the Ω-algebra A into another Ω-algebra, $Ker(f)$ is a congruence relation on A.

In particular, if R is a congruence relation on the Ω-algebra A, then the kernel of the canonical homomorphism of A onto A/R is the congruence relation R.

The following theorem is quite useful as a lemma for a number of theorems relative to the homomorphisms and isomorphisms of Ω-algebras.

Theorem 1.13. Let f be a homomorphism of the Ω-algebra A into the Ω-algebra B and let R be a congruence relation on A contained in $Ker(f)$. Then the assignment $\bar{f} : \bar{x} \mapsto f(x)$ for all $\bar{x} \in A/R$, is a homomorphism of A/R into B making the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\nu"'] & B \\ & A/R \arrow[u, "\bar{f}"'] \end{tikzcd}$$

where ν is the canonical homomorphism, commutative. Furthermore, $\bar{f}$ is an isomorphism iff $R = \mathrm{Ker}(f)$ and $B = f(A)$.

<u>Proof</u>. Since $R \subseteq \mathrm{Ker}(f)$, the mapping $\bar{f}$ is well-defined. If $\omega \in \Omega_n$ and $a_1, a_2, \dots, a_n \in A$ then

$$\bar{f}(\omega(\overline{a_1}, \overline{a_2}, \dots, \overline{a_n})) = \bar{f}(\overline{\omega(a_1, a_2, \dots, a_n)}) = f(\omega(a_1, a_2, \dots, a_n))$$

$$= \omega(f(a_1), f(a_2), \dots, f(a_n)) = \omega(\bar{f}(\overline{a_1}), \bar{f}(\overline{a_2}), \dots, \bar{f}(\overline{a_n})),$$

and hence $\bar{f}$ is a homomorphism, making obviously the given diagram commutative. If further, $R = \mathrm{Ker}(f)$ then for all $a, b \in A$, $\bar{a} = \bar{b}$ iff $f(a) = f(b)$, which shows $\bar{f}$ is a bijective mapping. Conversely if $\bar{f}$ is an isomorphism then we obtain that $a \equiv b \bmod R$ iff $a \equiv b \bmod \mathrm{Ker}(f)$, i.e., $R = \mathrm{Ker}(f)$. Q.E.D.

This result and the previous proposition lead the following corollary.

<u>Corollary 1.14.</u> (The Fundamental Theorem of Homomorphisms). Any factor Ω-algebra of Ω-algebra A is a homomorphic image of A and conversely any homomorphic image of A is

isomorphic to a factor Ω-algebra of A.

Now, let R and T be congruence relations on the Ω-algebra A such that $R \subseteq T$. If ν and μ are the canonical homomorphisms of A onto A/R and A onto A/T, respectively, then by 1.13 we have a unique homomorphism $\bar{\mu}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[dr, "\nu"'] & A/T \\ & A/R \arrow[u, "\bar{\mu}"'] \end{tikzcd}$$

is commutative. Thus, we have a congruence relation $\mathrm{Ker}(\bar{\mu})$ on A/R, which is sometimes called the induced congruence relation from T, and is denoted by T/R. It follows then immediately from 1.13 that

$$A/T \cong (A/R)/(T/R).$$

Conversely, let R' be any congruence relation on A/R. Then if ν' is the canonical homomorphism of A/R onto $(A/R)/R'$ then $T = \mathrm{Ker}\ (\nu'\nu)$ is a congruence relation on A such that $R = \mathrm{Ker}\ (\nu) \subseteq \mathrm{Ker}\ (\nu'\nu) = T$. If μ is the canonical homomorphism of A onto A/T then clearly $\mathrm{Ker}(\bar{\mu}) = R'$ and hence $R' = T/R$. That is, any congruence relation on A/R is of the form T/R for some congruence relation T on A such that $R \subseteq T$.

3. Congruence Relations on Ω-Algebras

On the Ω-algebra A, we always have two congruence relations, namely A^2 which yields the trivial factor Ω-algebra A/A^2, and 1_A which yields the factor Ω-algebra $A/1_A$ which is isomorphic to A with respect to the canonical homomorphism. Any congruence relation on A other than A^2 is called a proper congruence relation, and the one different from 1_A is said to be non-trivial.

Proposition 1.15. The intersection of an arbitrary number of congruence relations on an Ω-algebra is a congruence relation on the Ω-algebra.

Proof. Let $\{R_\lambda\}_{\lambda\epsilon\Lambda}$ be a family of congruence relations on the Ω-algebra A. Then evidently $\bigcap_{\lambda\epsilon\Lambda} R_\lambda$ is an equivalence relation on A. If $a_i \equiv b_i \bmod \bigcap_{\lambda\epsilon\Lambda} R_\lambda$, $i=1, 2, \ldots, n$, and $\omega \epsilon \Omega_n$ then $\omega(a_1, a_2, \ldots, a_n) \equiv \omega(b_1, b_2, \ldots, b_n) \bmod R_\lambda$ for all $\lambda \epsilon \Lambda$, and hence we obtain the result. Q.E.D.

Similarly, the intersection of an arbitrary number of equivalence relations on the Ω-algebra A forms an equivalence relation on A. The intersection of all the equivalence (or congruence) relations that contain some set R of ordered pairs in A^2 is the smallest equivalence (or congruence) relation containing R, and is called the equivalence (or congruence) relation generated by R.

Theorem 1.16. If R and T are congruence relations on the Ω-algebra A, then the composition RT is the congruence relation generated by $R \cup T$ iff $RT = TR$.

Proof. Suppose first that $RT = TR$. Then $(RT)^{-1} = T^{-1}R^{-1} = TR = RT$, which shows that RT is symmetric, and also $RTRT = RRTT = RT$, which shows the transitivity of RT. Since RT is clearly reflexive, it is an equivalence relation on A. If $\omega \in \Omega_n$ and $a_i \equiv b_i \bmod RT$, $i=1, 2, \dots, n$, then there exist $x_i \in A$ such that $a_i \equiv x_i \bmod T$ and $x_i \equiv b_i \bmod R$. Hence

$$\omega(a_1, a_2, \dots, a_n) \equiv \omega(x_1, x_2, \dots, x_n) \bmod T,$$

$$\omega(x_1, x_2, \dots, x_n) \equiv \omega(b_1, b_2, \dots, b_n) \bmod R.$$

Therefore, $\omega(a_1, a_2, \dots, a_n) \equiv \omega(b_1, b_2, \dots, b_n) \bmod RT$, which shows that RT is a congruence relation on A. If $\sup\{R, T\}$ is the smallest congruence relation containing $R \cup T$, then we obtain at once

$$RT = \sup\{R, T\}.$$

Conversely, if $RT = \sup\{R, T\}$, then RT is an equivalence relation on A and hence by the symmetric property, it follows that $RT = (RT)^{-1} = T^{-1}R^{-1} = TR$. Q.E.D.

Usually in algebras like groups and rings we can show that all the congruence relations commute with respect to the composition. However, in general we are not able to assert the commutativity. To obtain the sufficient condition for all the congruence relations on the Ω-algebra A to be commutative with respect to the composition, we need some concept of a special kind of a mapping from A to A called the translation.

Let $\omega \in \Omega_n$ and fix any $n-1$ elements $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$. Then ω derives a unary operation α on A defined by

$$\alpha : x \mapsto \omega(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n)$$

for all $x \in A$. The unary operation thus defined is called an <u>elementary translation,</u> and if a mapping $\theta : A \to A$ is expressed as a product of a finite number of elementary translations, θ is called a <u>translation.</u>

In general, an Ω-subalgebra is not closed under the translation, but for congruence relations the situation is rather different. To illustrate this, let R be a congruence relation on A and $a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ any fixed elements of A. If $x \equiv y \bmod R$ then clearly for any $\omega \in \Omega_n$,

$$\omega(a_1, a_2, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_n) \equiv \omega(a_1, a_2, \ldots, a_{i-1}, y, a_{i+1}, \ldots, a_n) \bmod R,$$

i. e., $\alpha(x) \equiv \alpha(y) \bmod R$. Hence R is closed under all the elementary translations. By induction, it follows that for all translations θ, $\theta(x) \equiv \theta(y) \bmod R$.

Proposition 1.17. An equivalence relation R is a congruence relation iff R is closed under all the translations.

Proof. If R is a congruence relation, then we know that R is closed under the translation. So, suppose that R is an equivalence relation which is closed under all the translations. Then $\omega \in \Omega_n$ $a_i \equiv b_i \bmod R$, $i=1, 2, \ldots, n$, implies that $\omega(a_1, a_2, \ldots, a_n)$ $\equiv \omega(b_1, a_2, \ldots, a_n) \bmod R$ by the fact that $\alpha(a_1) \equiv \alpha(b_1) \bmod R$ for the elementary translation α such that $\alpha: x \mapsto \omega(x, a_2, \ldots, a_n)$. Repeating such an argument, we obtain

$$\begin{aligned} \omega(a_1, a_2, \ldots, a_n) &\equiv \omega(b_1, a_2, \ldots, a_n) \\ &\equiv \omega(b_1, b_2, \ldots, a_n) \\ &\cdots \\ &\equiv \omega(b_1, b_2, \ldots, b_n) \bmod R. \end{aligned}$$

Q. E. D.

Proposition 1.18. Let R and T be congruence relations on the Ω-algebra A. Then $RT = TR$ provided that for all the pairs $(a, b) \in A^2$, there exists a translation θ such that $\theta(a) = b$

and $\theta(b) = a$.

Proof. Let $a \equiv b \bmod RT$. Then there exists $x \in A$ such that $a \equiv x \bmod T$ and $x \equiv b \bmod R$. Since R and T are congruence relations, it follows that $\theta(a) \equiv \theta(x) \bmod T$ and $\theta(x) \equiv \theta(b) \bmod R$. Hence, $\theta(b) \equiv \theta(a) \bmod TR$, i.e., $a \equiv b \bmod TR$, which shows that $RT \subseteq TR$. By symmetry, we also obtain $TR \subseteq RT$. Therefore, $RT = TR$. Q. E. D.

Now let A be an Ω-algebra, R its congruence relation, B its Ω-subalgebra, and let RB denote the union of all the R-congruence classes that intersect B, in other words,

$$RB = \{a \in A : a \equiv b \bmod R, \text{ for some } b \in B\} \tag{1.3.1}$$

If $a_1, a_2, \ldots, a_n \in RB$ then there exist $b_1, b_2, \ldots, b_n \in B$ such that $a_i \equiv b_i \bmod R$, so that for any $\omega \in \Omega_n$,

$$\omega(a_1, a_2, \ldots, a_n) \equiv \omega(b_1, b_2, \ldots, b_n) \bmod R,$$

where $\omega(b_1, b_2, \ldots, b_n) \in B$. Hence, $\omega(a_1, a_2, \ldots, a_n) \in RB$, which shows that RB is an Ω-subalgebra of A. We shall call RB an Ω-subalgebra generated by the congruence relation R and the subalgebra B. In particular,

$$\bar{x} = R\{x\}$$

is an Ω-subalgebra provided that $\{x\}$ forms a trivial Ω-subalgebra of A. Similarly, we may define $RX = \{a \in A : a \equiv x \bmod R,$ for some $x \in X\}$ for any subset X of A. Of course, RX need not be an Ω-subalgebra, but it is immediate that X is an R-congruence class iff

$$X = RX. \tag{1.3.2}$$

Proposition 1.19. Let R and T be congruence relations on the Ω-algebra A, and X a subset of A. Then

$$T(RX) = (TR)X$$

provided that TR is a congruence relation on A.

Proof. If $a \in T(RX)$ then there exists $b \in RX$ such that $a \equiv b \bmod T$ and $b \equiv x \bmod R$ for some $x \in X$. Hence, $a \equiv x \bmod RT$ for some $x \in X$, which shows that $a \in (RT)X = (TR)X$. Conversely, if $a \in (RT)X$ then $a \equiv x \bmod RT$, $x \in X$, so that $a \equiv y \bmod T$ and $y \equiv x \bmod R$ for some $y \in A$. It follows that $a \equiv y \bmod T$, $y \in RX$, and thus, $a \in T(RX)$, as desired. Q. E. D.

Henceforth, if X is a subset of the Ω-algebra A and $R_1, R_2, \ldots, R_m$ are congruence relations on A we shall write

$$R_1 R_2 \ldots R_m X$$

instead of putting parentheses whenever $R_{i_1} R_{i_2} \ldots R_{i_k}$ is a congruence relation on A for every subset $\{i_1, i_2, \ldots, i_k\}$ of $\{1, 2, \ldots, m\}$.

II. THE GENERALIZED JORDAN-HÖLDER THEOREM

1. First, Second and Third Isomorphism Theorems

Let A and B be Ω-algebras, R and T congruence relations on A and B, respectively, and h an epimorphism of A onto B. Then, the homomorphic image $h(R)$ of R and the inverse image $h^{-1}(T)$ of T are respectively defined as follows:

$$h(R) = \{(h(x), h(y)) : (x, y) \in R\},$$

$$h^{-1}(T) = \{(x, y) : (h(x), h(y)) \in T\}.$$

It is trivially verified that if R contains $\mathrm{Ker}(h)$ then $h(R)$ is a congruence relation on B, and $h^{-1}(T)$ is always a congruence relation on A.

Theorem 2.1. (First Isomorphism Theorem). Let A and B be Ω-algebras and h an epimorphism of A onto B. Then, the correspondence $R \leftrightarrow h(R)$ is one-to-one between the family $\mathrm{Cong}(A)$ of all congruence relations on A that contain $\mathrm{Ker}(h)$ and the family $\mathrm{Cong}(B)$ of all congruence relations on B. Furthermore, if $T = h(R)$, i.e., R corresponds to T under this one-to-one correspondence then

$$A/R \cong B/T.$$

Proof. Since for any congruence relation T on B, $h(h^{-1}(T)) = T$, the mapping $R \mapsto h(R)$ of $\mathrm{Cong}(A)$ into $\mathrm{Cong}(B)$ is surjective. Also if $h(R) = h(R')$ for some congruence relations R and R' on A, then $x \equiv y \bmod R$ implies $h(x) \equiv h(y) \bmod h(R)$, i.e., $h(x) \equiv h(y) \bmod h(R')$. Hence $x \equiv y \bmod R'$. By symmetry, $x \equiv y \bmod R'$ implies $x \equiv y \bmod R$. It follows that $R = R'$, and hence the correspondence $R \leftrightarrow h(R)$ is one-to-one.

Now if ν' is the canonical homomorphism of B onto B/T then we have a homomorphism $\nu' h : A \to B/T$. We wish to establish the equality $\mathrm{Ker}\,(\nu' h) = R$. If $x \equiv y \bmod \mathrm{Ker}(\nu' h)$ then $h(x) \equiv h(y) \bmod T$, and hence $x \equiv y \bmod R$. This establishes $\mathrm{Ker}(\nu' h) \subseteq R$. Since $R \subseteq \mathrm{Ker}(\nu' h)$ is obvious, we obtain the desired equality. By 1.13, it follows that A/R is isomorphis to B/T. Q.E.D.

Theorem 2.2. (Second Isomorphism Theorem). Let A be an Ω-algebra, B an Ω-subalgebra of A, and R a congruence relation on A. Then $R \cap B^2$ is a congruence relation on B and

$$B/(R \cap B^2) \cong RB/R.$$

Proof. Let ν be the canonical homomorphism of A onto A/R and let $\nu' = \nu \mid B$. Then ν' is an epimorphism of B onto

$RB/R \subseteq A/R$ since for any $b \in B$, $\nu'(b) = \bar{b} \in RB/R$ and also for any $\bar{x} \in RB/R$, there exists $b \in B$ such that $b \in \bar{x}$, i.e., $\nu'(b) = \bar{x}$. Hence by 1.13,

$$B/\mathrm{Ker}(\nu') \cong RB/R.$$

But since $\mathrm{Ker}(\nu') = R \cap B^2$, the theorem follows. Q.E.D.

Lemma 2.3. Let R, T and U be congruence relations on the Ω-algebra A such that $T \subseteq U$, and suppose that all the congruence relations on A commute with respect to the composition. Then

$$(RT) \cap U = (R \cap U)T.$$

Proof. If $a \equiv b \bmod (RT) \cap U$ then $a \equiv b \bmod U$, $a \equiv x \bmod T$ and $x \equiv b \bmod R$ for some $x \in A$. Hence $b \equiv x \bmod U$, $b \equiv x \bmod R$ and $a \equiv x \bmod T$, i.e., $a \equiv b \bmod (R \cap U)T$. This shows that $(RT) \cap U \subseteq (R \cap U)T$. Now if $a \equiv b \bmod (R \cap U)T$ then $a \equiv x \bmod T$ and $x \equiv b \bmod R \cap U$ for some $x \in A$. Hence, $a \equiv b \bmod RT$ and $a \equiv b \bmod U$, i.e., $a \equiv b \bmod (RT) \cap U$. Thus, $(R \cap U)T \subseteq (RT) \cap U$, and this completes the proof. Q.E.D.

Now, we shall establish a complicated isomorphism theorem which will be used as the lemma to prove Schreier's refinement theorem.

Theorem 2.4. (Third Isomorphism Theorem, Zassenhaus' Lemma). Let A be an Ω-algebra, B and C Ω-subalgebras of A, and R, T congruence relations on B and C, respectively. Suppose that all the congruence relations on $B \cap C$ commute with respect to the composition. Then

i) $R(B^2 \cap T)R$ is a congruence relation on $R(B \cap C)$,

ii) $T(R \cap C^2)T$ is a congruence relation on $T(B \cap C)$, and

iii) $R(B \cap C)/R(B^2 \cap T)R \cong T(B \cap C)/T(R \cap C^2)T$.

Proof. First, we shall show that $R(B^2 \cap T)R$ is a congruence relation on $R(B \cap C)$. The symmetric and reflexive properties are obvious, and since $R(B^2 \cap T)RR(B^2 \cap T)R = R(B^2 \cap T)R(B^2 \cap T)R$ $= R(B^2 \cap T)(R \cap C^2)(B^2 \cap T)R = R(R \cap C^2)(B^2 \cap T)R = R(B^2 \cap T)R$, $R(B^2 \cap T)R$ is transitive. Thus, it is an equivalence relation on $R(B \cap C)$. Now for any $\omega \in \Omega_n$ and a_i, b_i in $R(B \cap C)$, $i=1,2,\ldots,n$, if $a_i \equiv b_i \bmod R(B^2 \cap T)R$ then there exist x_i, $y_i \in B \cap C$ such that $a_i \equiv x_i \bmod R$, $x_i \equiv y_i \bmod B^2 \cap T$ and $y_i \equiv b_i \bmod R$. Hence,

$$\omega(a_1, a_2, \ldots, a_n) \equiv \omega(x_1, x_2, \ldots, x_n) \bmod R$$

$$\omega(x_1, x_2, \ldots, x_n) \equiv \omega(y_1, y_2, \ldots, y_n) \bmod B^2 \cap T$$

$$\omega(y_1, y_2, \ldots, y_n) \equiv \omega(b_1, b_2, \ldots, b_n) \bmod R,$$

i.e.,
$$\omega(a_1, a_2, \ldots, a_n) \equiv \omega(b_1, b_2, \ldots, b_n) \bmod R(B^2 \cap T)R,$$

which proves that $R(B^2 \cap T)R$ is a congruence relation on $R(B \cap C)$.

This proves i) and by symmetry ii) follows.

Viewing R as a congruence relation on $B \cap C$, we have, by Lemma 2.3, that $(R(B^2 \cap T)R) \cap (B \cap C)^2 = (R(B^2 \cap T)) \cap (B \cap C)^2 = (R \cap (B \cap C)^2)(B^2 \cap T) = (R \cap C^2)(B^2 \cap T)$. Hence, by the second isomorphism theorem,

$$(B \cap C)/(R \cap C^2)(B^2 \cap T) \cong R(B^2 \cap T)R(B \cap C)/R(B^2 \cap T)R.$$

Furthermore, if a is an arbitrary element of $R(B^2 \cap T)R(B \cap C)$ then $a \equiv x \bmod R(B^2 \cap T)R$ for some $x \in B \cap C$. That is, $a \equiv y \bmod R$ for some $y \in B \cap C$, and there exists $z \in B \cap C$ such that $y \equiv z \bmod B^2 \cap T$ and $z \equiv x \bmod R$. Hence $a \in R(B \cap C)$, and we obtain $R(B^2 \cap T)R(B \cap C) \subseteq R(B \cap C)$. Since $R(B \cap C) \subseteq R(B^2 \cap T)R(B \cap C)$ is obvious, it follows that $R(B \cap C) = R(B^2 \cap T)R(B \cap C)$. Therefore,

$$(B \cap C)/(R \cap C^2)(B^2 \cap T) \cong R(B \cap C)/R(B^2 \cap T)R.$$

By symmetry, we also have

$$(B \cap C)/(R \cap C^2)(B^2 \cap T) \cong T(B \cap C)/T(R \cap C^2)T$$

and our result iii) follows. Q. E. D.

2. Normal Series and Schreier's Refinement Theorem

Definition 2.5. Let A be an Ω-algebra and I and Ω-subalgebra of A. Then by a normal series from I to A we mean a finite sequence of Ω-subalgebras of A:

$$I = A_0 \subseteq A_1 \subseteq \dots \subseteq A_s = A, \qquad (2.2.1)$$

together with congruence relations R_i on A_i, $i=1, 2, \dots, s$ such that each A_{i-1} is an R_i-congruence class. Two normal series

$$\begin{aligned} I &= A_0 \subseteq A_1 \subseteq \dots \subseteq A_s = A \\ I &= B_0 \subseteq B_1 \subseteq \dots \subseteq B_t = A \end{aligned} \qquad (2.2.2)$$

are said to be equivalent if $s = t$ and if there exists a permutation of the indices $i=1, 2, \dots, s$, written $i \mapsto i'$ such that A_i/R_i is isomorphic to $B_{i'}/T_{i'}$ where T_j are the congruence relations on B_j, $j=1, 2, \dots, t$ such that B_{j-1} are T_j-congruence classes.

Note that we may rewrite (2.2.1) as

$$I = R_1 I \subseteq R_2 I \subseteq \dots \subseteq R_s I \subseteq A$$

since $A_{i-1} = R_i I$ by the fact that $I \subseteq A_{i-1}$.

Definition 2.6. One normal series is said to be a refinement of a second normal series if its terms contain all the Ω-algebras which occur in the second series.

Theorem 2.7. (Schreier's Refinement Theorem). Let A be an Ω-algebra such that all congruence relations on any subalgebra commute with respect to the composition. If I is an Ω-subalgebra of A then any two normal series from I to A have equivalent refinements.

Proof. Let the two series be given by (2. 2. 2). We set

$$A_{ij} = R_i(A_i \cap B_j) \quad j = 0, 1, \ldots, t.$$

$$B_{ji} = T_j(A_i \cap B_j) \quad i = 0, 1, \ldots, s.$$

Then we obtain the two sequences of Ω-subalgebras of A:

$$\begin{aligned} I = A_{10} &\subseteq A_{11} \subseteq \cdots \subseteq A_{1t} = A_1 \\ = A_{20} &\subseteq A_{21} \subseteq \cdots \subseteq A_{2t} = A_2 \\ &\cdots \\ = A_{s0} &\subseteq A_{s1} \subseteq \cdots \subseteq A_{st} = A_s = A, \end{aligned} \tag{2. 2. 3}$$

$$\begin{aligned} I = B_{10} &\subseteq B_{11} \subseteq \cdots \subseteq B_{1s} = B_1 \\ = B_{20} &\subseteq B_{21} \subseteq \cdots \subseteq B_{2s} = B_2 \\ &\cdots \\ = B_{t0} &\subseteq B_{t1} \subseteq \cdots \subseteq B_{ts} = B_t = A. \end{aligned} \tag{2. 2. 4}$$

By the third isomorphism theorem, $R_i(A_i^2 \cap T_j)R_i$ is the congruence relation on $A_{ij} = R_i(A_i \cap B_j)$.

Now we wish to establish that A_{ij-1} is an $R_i(A_i^2 \cap T_j)R_i$-

congruence class so that the sequence (2.2.3) becomes a normal series from I to A. For this purpose, let $\bar{x}$ denote an $R_i(A_i^2 \cap T_j)R_i$-congruence class containing $x \in I$. If a is an arbitrary element in $\bar{x}$ then $a \equiv y \bmod R_i$, $y \equiv z \bmod A_i^2 \cap T_j$ and $z \equiv x \bmod R_i$ for some y and z in $A_i \cap B_j$. Since z and x are elements in $A_i \cap B_j$, we actually have $z \equiv x \bmod R_i \cap B_j^2$. Now both $R_i \cap B_j^2$ and $A_i^2 \cap T_j$ are congruence relations on $A_i \cap B_j$, so that they commute with respect to the composition. Thus, $y \equiv x \bmod (A_i^2 \cap T_j)(R_i \cap B_j^2)$, and since $a \equiv y \bmod R_i$, it follows that $a \equiv x \bmod (A_i^2 \cap T_j)(R_i \cap B_j^2)R_i$, i.e., $a \equiv x \bmod (A_i^2 \cap T_j)R_i$. Hence there exists $x' \in A_i \cap B_j$ such that $a \equiv x' \bmod R_i$ and $x' \equiv x \bmod A_i^2 \cap T_j$, i.e., $x' \in A_i \cap T_j I = A_i \cap B_{j-1}$. Hence, $a \in R_i(A_i \cap B_{j-1}) = A_{ij-1}$, and we obtain $\bar{x} \subseteq A_{ij-1}$.

Also, if a is an arbitrary element in $R_i(A_i \cap B_{j-1})$ then $a \equiv y \bmod R_i$ for some $y \in A_i \cap T_j I$. That is, $y \equiv z \bmod (A_i^2 \cap T_j)$ for some $z \in I$. But since I is contained in A_{i-1}, it follows that both x and z are in A_{i-1}. Hence $z \equiv x \bmod R_i$. It follows that $a \equiv x \bmod R_i(A_i^2 \cap T_j)R_i$, and we obtain $R_i(A_i \cap B_{j-1}) \subseteq \bar{x}$. Thus, the required equality $\bar{x} = R_i(A_i \cap B_{j-1}) = A_{ij-1}$ is established.

Accordingly, (2.2.3) is a normal series from I to A. Similarly, $T_j(R_i \cap B_j^2)T_j$ is a congruence relation on $B_{ji} = T_j(A_i \cap B_j)$, B_{ji-1} is a $T_j(R_i \cap B_j^2)T_j$-congruence class and (2.2.4) is another normal series from I to A.

By the third isomorphism theorem,

$$A_{ij}/R_i(A_i^2 \cap T_j)R_i \cong B_{ji}/T_j(R_i \cap B_j^2)T_j$$

and hence (2. 2. 3) and (2. 2. 4) are equivalent refinements. Q. E. D.

3. The Jordan-Hölder Theorem for Ω-Algebras

We know that any Ω-algebra A has two congruence relations A^2 and 1_A. If an Ω-algebra A has no other congruence relation, it is called a simple Ω-algebra.

Definition 2. 8. A normal series without the repetition of Ω-algebras is called a composition series, if it has no refinement without the repetitions of Ω-subalgebras, other than itself.

The factor Ω-algebras $A_1/R_1, A_2/R_2, \ldots, A_s/R_s$ are called the factors of the normal series (2. 2. 1). If (2. 2. 1) is not a composition series then by the definition there exists another term B between, say A_{i-1} and A_i which contains properly A_{i-1} and is contained properly in A_i. Hence there is a congruence relation T on A_i which contains properly R_i and is contained properly in A_i^2 such that B is a T-congruence class. By the remark after the fundamental theorem of homomorphism, it follows that A_i/R_i is not simple. The converse can be shown similarly, and thus, we obtain the following proposition:

Proposition 2.9. A normal series is a composition series iff all its factors are simple Ω-algebras.

Theorem 2.10. (Jordan-Hölder Theorem). Let A be an Ω-algebra such that all congruence relations on any subalgebra commute with respect to the composition. If I is an Ω-subalgebra of A then any two composition series from I to A are equivalent.

Proof. By Schreier's theorem the composition series have equivalent refinements, and we have a one-to-one correspondence between the factors of the refinements such that the corresponding factor Ω-algebras are isomorphic. Hence, in this one-to-one correspondence the trivial factor Ω-algebras are paired, and so, also the nontrivial factor Ω-algebras are paired. Since these nontrivial factor Ω-algebras are the factors of the given composition series, we see that the two composition series are equivalent. Q. E. D.

III. FREE UNIVERSAL ALGEBRAS

1. Direct Products

Let $\{A_\lambda\}_{\lambda\epsilon\Lambda}$ be a family of Ω-algebras. Then the set theoretical product

$$A = \prod_{\lambda\epsilon\Lambda} A_\lambda$$

forms an Ω-algebra called the direct product of Ω-algebras A_λ when an n-ary operation is defined on A suitably for each $\omega \epsilon \Omega_n$. The n-ary operation is in fact defined on A as a componentwise n-ary operation, i.e., if $(a_{1\lambda})_{\lambda\epsilon\Lambda}, (a_{2\lambda})_{\lambda\epsilon\Lambda}, \ldots, (a_{n\lambda})_{\lambda\epsilon\Lambda}$ are elements in A then for any $\omega \epsilon \Omega_n$,

$$\omega((a_{1\lambda})_{\lambda\epsilon\Lambda}, (a_{2\lambda})_{\lambda\epsilon\Lambda}, \ldots, (a_{n\lambda})_{\lambda\epsilon\Lambda}) = (\omega(a_{1\lambda}, a_{2\lambda}, \ldots, a_{n\lambda}))_{\lambda\epsilon\Lambda}. \tag{3.1.1}$$

If Λ is finite, we take A to be the product set

$$A = \prod_{j=1}^{s} A_j$$

of elements $a = (a_1, a_2, \ldots, a_s)$, $a_j \epsilon A_j$, and for any $\omega \epsilon \Omega_n$, the corresponding n-ary operation acts as

$$\omega((a_{11}, a_{12}, \ldots, a_{1s}), (a_{21}, a_{22}, \ldots, a_{2s}), \ldots, (a_{n1}, a_{n2}, \ldots, a_{ns}))$$

$$= (\omega(a_{11}, a_{21}, \ldots, a_{n1}), \omega(a_{12}, a_{22}, \ldots, a_{n2}), \ldots, \omega(a_{1s}, a_{2s}, \ldots, a_{ns})).$$

where $a_{ij} \in A_j$, $i=1, 2, \ldots, n$, $j=1, 2, \ldots, s$. Thus, in any case, A forms an Ω-algebra.

Now if the direct product $\prod_{j=1}^{s} A_j$ is given, we notice that for any permutation of $j=1, 2, \ldots, s$, written $j \mapsto j'$ we have an isomorphism

$$A_1 \times A_2 \times \ldots \times A_s \cong A_{1'} \times A_{2'} \times \ldots \times A_{s'}.$$

In fact, it is immediate from the fact that the correspondence $(a_1, a_2, \ldots, a_s) \mapsto (a_{1'}, a_{2'}, \ldots, a_{s'})$ is an isomorphism. Hence the direct product is independent of the order of the factors.

Also, if $0 = s_0 < s_1 < \ldots < s_m = s$ then we have another isomorphism

$$\prod_{j=1}^{s} A_j \cong \prod_{k=1}^{m} \prod_{j=s_{k-1}+1}^{s_k} A_j,$$

by the fact that the correspondence $(a_1, a_2, \ldots, a_s)$ $\mapsto ((a_1, a_2, \ldots, a_{s_1}), (a_{s_1+1}, a_{s_1+2}, \ldots, a_{s_2}), \ldots, (a_{s_{m-1}+1}, a_{s_{m-1}+2}, \ldots, a_{s_m}))$ is an isomorphism. In particular, it follows that $(A_1 \times A_2) \times A_3$ is isomorphic to $A_1 \times (A_2 \times A_3)$, and in this respect the binary operation "$\times$" can be considered to be associative as well as commutative.

Let us return to the more general case $A = \prod_{\lambda \in \Lambda} A_\lambda$ for a while, and let π_λ be a mapping which assigns each $x \in A$ to its

λ-th component. Hence if $(a_{1\lambda})_{\lambda\in\Lambda}, (a_{2\lambda})_{\lambda\in\Lambda}, \ldots, (a_{n\lambda})_{\lambda\in\Lambda} \in A$ and $\omega \in \Omega_n$ then

$$\pi_\lambda((a_{i\lambda})_{\lambda\in\Lambda}) = a_{i\lambda} \in A_\lambda, \quad i=1, 2, \ldots, n,$$

so that

$$\begin{aligned} &\pi_\lambda(\omega((a_{1\lambda})_{\lambda\in\Lambda}, (a_{2\lambda})_{\lambda\in\Lambda}, \ldots, (a_{n\lambda})_{\lambda\in\Lambda})) \\ &= \pi_\lambda((\omega(a_{1\lambda}, a_{2\lambda}, \ldots, a_{n\lambda}))_{\lambda\in\Lambda}) \\ &= \omega(a_{1\lambda}, a_{2\lambda}, \ldots, a_{n\lambda}) \\ &= \omega(\pi_\lambda((a_{1\lambda})_{\lambda\in\Lambda}), \pi_\lambda((a_{2\lambda})_{\lambda\in\Lambda}), \ldots, \pi_\lambda((a_{n\lambda})_{\lambda\in\Lambda})). \end{aligned}$$

Therefore, π_λ is a homomorphism, and we call it a projection homomorphism or simply a projection.

Now consider the congruence relation $\mathrm{Ker}(\pi_\lambda)$ on A determined by each projection π_λ. Since $x \equiv y \bmod \bigcap_{\lambda\in\Lambda} \mathrm{Ker}(\pi_\lambda)$ implies $x = y$, we have the following equality:

$$\bigcap_{\lambda\in\Lambda} \mathrm{Ker}(\pi_\lambda) = 1_A. \tag{3.1.2}$$

Also, for any $(x, y) \in A^2$ there exists an element z such that $\pi_{\lambda'}(x) = \pi_{\lambda'}(z)$ for some $\lambda' \in \Lambda$ and $\pi_\lambda(y) = \pi_\lambda(z)$ for all $\lambda \in \Lambda$, $\lambda \neq \lambda'$. Hence, for any $(x, y) \in A^2$ there exists $z \in A$ satisfying $x \equiv z \bmod \mathrm{Ker}(\pi_{\lambda'})$ for some $\lambda' \in \Lambda$, and $z \equiv y \bmod \bigcap_{\lambda\neq\lambda'} \mathrm{Ker}(\pi_\lambda)$, i.e., $x \equiv y \bmod (\bigcap_{\lambda\neq\lambda'} \mathrm{Ker}(\pi_\lambda))(\mathrm{Ker}(\pi_{\lambda'}))$. This gives the other equality:

$$(\bigcap_{\lambda \neq \lambda'} \mathrm{Ker}(\pi_\lambda))(\mathrm{Ker}(\pi_{\lambda'})) = A^2. \qquad (3.1.3)$$

In particular, $(\mathrm{Ker}(\pi_\lambda))(\mathrm{Ker}(\pi_{\lambda'})) = A^2$, $\lambda \neq \lambda'$, so that the composition of such congruence relations is commutative.

Now we shall determine conditions that a given Ω-algebra be isomorphic to a direct product of a finite number of Ω-algebras. In such a case, the Ω-algebra is said to be <u>representable</u> as a direct product.

<u>Theorem 3.1.</u> If an Ω-algebra A is representable as a direct product $\prod_{j=1}^{s} A_j$ of Ω-algebras then there exist congruence relations $R_1, R_2, \ldots, R_s$ on A satisfying

$$\bigcap_{j=1}^{s} R_j = 1_A \qquad (3.1.4)$$

$$(\bigcap_{j \neq k} R_j)R_k = A^2 \qquad (3.1.5)$$

$$A \cong \prod_{j=1}^{s} A/R_j. \qquad (3.1.6)$$

Conversely if there exist congruence relations $R_1, R_2, \ldots, R_s$ on A satisfying (3.1.4) and (3.1.5), then also (3.1.6) holds, and hence A is representable as a direct product of Ω-algebras.

Proof. Suppose that A is isomorphic to $\prod_{j=1}^{s} A_j$, and let us identify A and $\prod_{j=1}^{s} A_j$ for the sake of simplicity. Let π_j be the projection of A onto A_j, and set $R_j = \mathrm{Ker}(\pi_j)$. Then (3.1.4) and (3.1.5) follow from (3.1.2) and (3.1.3), and (3.1.6) follow from 1.13.

Conversely, suppose that there exist congruence relations $R_1, R_2, \ldots, R_s$ for which (3.1.4) and (3.1.5) hold. Let ν_j be the canonical homomorphism of A onto A/R_j, and define the mapping $f : A \to \prod_{j=1}^{s} A/R_j$ by

$$f : x \mapsto (\nu_1(x), \nu_2(x), \ldots, \nu_s(x))$$

for all $x \in A$. Then the homomorphic property of f is immediate, and if $f(x) = f(y)$ for some $x, y \in A$, then $\nu_j(x) = \nu_j(y)$ for all j so that by (3.1.4), $x = y$. Hence, f is injective. Now we wish to show by induction that for any $r \leq s$ there exists $x \in A$ such that

$$(\nu_1(y_1), \nu_2(y_2), \ldots, \nu_s(y_s))$$
$$= (\nu_1(x), \nu_2(x), \ldots, \nu_r(x), \nu_{r+1}(y_{r+1}), \ldots, \nu_s(y_s))$$

for all

$$(\nu_1(y_1), \nu_2(y_2), \ldots, \nu_s(y_s)) \in \prod_{j=1}^{s} A/R_j.$$

If this is true then by (3.1.5), $y_{r+1} \equiv x \bmod (\bigcap_{j \neq r+1} \mathrm{Ker}(\nu_j)) \cdot (\mathrm{Ker}(\nu_{r+1}))$. Hence, there exists $z \in A$ such that

$y_{r+1} \equiv z \bmod \mathrm{Ker}(\nu_{r+1})$ and $z \equiv x \bmod \mathrm{Ker}(\nu_j)$ for all $j \neq r+1$,

i.e., $\nu_{r+1}(z) = \nu_{r+1}(y_{r+1})$, and

$$\nu_1(z) = \nu_1(x) = \nu_1(y_1)$$

$$\nu_2(z) = \nu_2(x) = \nu_2(y)$$

$$\cdots$$

$$\nu_r(z) = \nu_r(x) = \nu_r(y_r).$$

Thus, the induction is completed, and accordingly f is surjective. This completes the proof. Q. E. D.

2. Free Ω-Algebras Generated by Sets

Let A be an Ω-algebra and S its subset. We say that A is the Ω-algebra generated by a set S if A coincides with its Ω-subalgebra generated by S. Further, if S is finite, A is said to be finitely generated, and in any case, elements of S are called generators of A.

Theorem 3. 2. If an Ω-algebra A is generated by a set S then every element of A is completely determined by the generators and a finite number of n-ary operations defined by Ω.

Proof. Let $\Omega = \{\omega_\lambda\}_{\lambda \in \Lambda}$, and define the sequence $S_0, S_1, S_2, \ldots$ of subsets of A by the following manner:

$$S_0 = S$$

$$S_1 = S_0 \cup (\bigcup_{\lambda \in \Lambda} \omega_\lambda (S_0))$$

$$\cdots$$

$$S_{m+1} = S_m \cup (\bigcup_{\lambda \in \Lambda} \omega_\lambda (S_m))$$

$$\cdots$$

where $\omega_\lambda(S_m)$ denotes the totality of elements of the form $\omega_\lambda(x_1, x_2, \ldots, x_n)$, $x_i \in S_m$, $i=1, 2, \ldots, n$ if $\omega_\lambda \in \Omega_n$. Let

$$X = \bigcup_{j \in N} S_j.$$

Then we claim that X is an Ω-subalgebra of A. For, let $\omega \in \Omega_n$ and $a_1, a_2, \ldots, a_n \in X$. Now every a_i lies in some S_{m_i} for some $m_i \in N$. Hence, $a_1, a_2, \ldots, a_n \in S_{m_0}$ where $m_0 = \max\{m_1, m_2, \ldots, m_n\}$. It follows that

$$\omega(a_1, a_2, \ldots, a_n) \in \bigcup_{\lambda \in \Lambda} \omega_\lambda (S_{m_0}) \subseteq S_{m_0+1} \subseteq X.$$

But since $S = S_0 \subseteq X$, we must have $A = X$.

Now every element in A is an element of X, i.e., an element of S_m for some $m \in N$, and therefore the theorem follows. Q. E. D.

The proof of 3. 2 can be extended to another important theorem in our subject, namely:

Theorem 3.3. If an Ω-algebra A is generated by a set S then

$$|A| \leq |\Omega|\ |S|\aleph_0 = \max\{|\Omega|, |S|, \aleph_0\}.$$

Proof. The proof is continued from that of 3.2. Since all ω_λ are finitary, $|\omega_\lambda(S_m)| \leq |S_m|\aleph_0$, so that $|\bigcup_{\lambda\in\Lambda}\omega_\lambda(S_m)| \leq |\Omega|\,|S_m|\aleph_0$. Hence

$$|S_{m+1}| \leq \max\{|S_m|, |\Omega|\,|S_m|\aleph_0\} = |\Omega|\,|S_m|\aleph_0.$$

Now $|S_0| = |S|$ and $|S_1| \leq |\Omega|\,|S|\aleph_0$. If $|S_m| \leq |\Omega|\,|S|\aleph_0$ then $|S_{m+1}| \leq |\Omega|\,|S_m|\aleph_0 \leq |\Omega|^2|S|\aleph_0^2 = |\Omega|\,|S|\aleph_0$, and therefore for all $m \in N$,

$$|S_m| \leq |\Omega|\,|S|\aleph_0,$$

i.e.,

$$|A| = |X| = |\bigcup_{j\in N} S_j| \leq \aleph_0|\Omega|\,|S|\aleph_0 = |\Omega|\,|S|\aleph_0$$

Q.E.D.

Lemma 3.4. Let A be an Ω-algebra with a set S of generators of A. Then any homomorphism of A into another Ω-algebra is completely determined by its restriction to S.

Proof. Suppose that h_1 and h_2 are two homomorphisms from A to an Ω-algebra such that $h_1|S = h_2|S$. Let A' be a subset of A which contains all the elements x for which

$h_1(x) = h_2(x)$. Then for all $a_1, a_2, \ldots, a_n \in A'$ and $\omega \in \Omega_n$,

$$\begin{aligned} h_1(\omega(a_1, a_2, \ldots, a_n)) &= \omega(h_1(a_1), h_1(a_2), \ldots, h_1(a_n)) \\ &= \omega(h_2(a_1), h_2(a_2), \ldots, h_2(a_n)) \\ &= h_2(\omega(a_1, a_2, \ldots, a_n)), \end{aligned}$$

and hence A' is an Ω-subalgebra that contains S. Thus, $A' = A$ and it follows that $h_1(a) = h_2(a)$ for all $a \in A$. Q. E. D.

Now let us consider an arbitrary category CAT. An object A of CAT is called <u>terminal</u> if there exists a unique morphism of each object of CAT into A, and is called <u>initial</u> if for each object of CAT there exists a unique morphism of A into this object. With this preliminary remark, we shall define a free Ω-algebra.

<u>Definition 3. 5.</u> Let $[\Omega]$ be the category of Ω-algebras, and let S be a nonempty set. Also let CAT be a category whose objects are mappings of S into Ω-algebras and whose morphisms are defined as follows:

If $f : S \to A$ and $g : S \to B$ are two objects in CAT then a morphism from f to g is a homomorphism $\varphi \in \mathrm{Hom}_\Omega(A, B)$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, "\varphi"] \\ & B \end{tikzcd}$$

is commutative.

Then a pair (F, f_0) or often F is called a <u>free Ω-algebra generated by S</u> if the mapping $f_0 : S \to F$ is an initial object of CAT.

Thus, if (F, f_0) is a free Ω-algebra generated by a set S, then for any object $A \in [\Omega]$ with a mapping $f : S \to A$, we have a unique homomorphism $f^* : F \to A$ such that $f^* f_0 = f$. Throughout this paper we shall use the notation f^* for such a uniquely determined homomorphism by a mapping f.

<u>Theorem 3.6.</u> (Existence Theorem). For any set $S \neq \emptyset$, there exists a free Ω-algebra (F, f_0) generated by S. Furthermore, f_0 is injective and $f_0(S)$ is a set of generators of F.

<u>Proof.</u> Let $T = S \cup Z \cup \Omega$ so that T is either infinite or denumerable, and let Γ be a set of all Ω-algebra structures on T. Using Γ as an index set, we denote by T_γ the corresponding Ω-algebra whose underlying set is T, and by Φ_γ the family of all mappings of S into T_γ. Also for each $\varphi \in \Phi_\gamma$, we set

$T_{\gamma,\varphi} = T_\gamma$. Now let

$$F_0 = \prod_{\gamma \in \Gamma} \prod_{\varphi \in \Phi_\gamma} T_{\gamma,\varphi}$$

and define the mapping $f_0 : S \to F_0$ by $f_0(s) = ((\varphi(s))_{\varphi \in \Phi_\gamma})_{\gamma \in \Gamma}$ for all $s \in S$. Further, suppose that A is an arbitrary object in $[\Omega]$ with a mapping $f : S \to A$. We may assume that $f(S)$ generates A simply by restricting our attention to the Ω-subalgebra of A generated by $f(S)$. So, $|A| \leq |S|\,|\Omega| N_0 = |T|$.

For all $t \in T$, let $Z_t = Z$ and set $P = \prod_{t \in T} Z_t$. Also we set $\overline{A} = A \times P$ so that $|\overline{A}| = |A|\,|T| N_0 = |T|$. Therefore, for some Ω-algebra structure $\gamma_0 \in \Gamma$, there exists an isomorphism $g : \overline{A} \to T_{\gamma_0}$. If τ denotes the injection[3] of A into $\overline{A}$, then we have a mapping $\varphi_0 \in \Phi_{\gamma_0}$ such that $\varphi_0 = g\tau f : S \to T_{\gamma_0} = T_{\gamma_0, \varphi_0}$.

Let $\pi_{\gamma_0, \varphi_0}$ be a projection of F_0 onto T_{γ_0} such that

$$\pi_{\gamma_0, \varphi_0}(((\varphi(s))_{\varphi \in \Phi_\gamma})_{\gamma \in \Gamma}) = \varphi_0(s),$$

and let π be a projection of $\overline{A}$ onto A. Set $h = \pi g^{-1} \pi_{\gamma_0, \varphi_0} : F_0 \to A$. Then for any $s \in S$,

---

[3] τ maps $a \in A$ onto $(a, (0)_{t \in T}) \in \overline{A}$. About $0 \in Z$, see Chapter 1, Section 1.

$$\begin{aligned}
hf_0(s) &= \pi g^{-1}\pi_{\gamma_0,\varphi_0}((((\varphi(s))_{\varphi\in\Phi_\gamma})_{\gamma\in\Gamma}) \\
&= \pi g^{-1}\varphi_0(s) \\
&= \pi g^{-1} g\tau f(s) \\
&= f(s)
\end{aligned}$$

and hence the following diagram is commutative:

$$\begin{tikzcd}
S \arrow[r, "f_0"] \arrow[rddd, "f"'] & F_0 \arrow[d, "\pi_{\gamma_0,\varphi_0}"] \arrow[ddd, bend left, "h"] \\
& T_{\gamma_0} \arrow[d, "g^{-1}"'] \\
& \overline{A} \arrow[d, "\pi"] \\
& A
\end{tikzcd}$$

Since $|S| \leq |T_\gamma|$, there exists an injective mapping $\varphi' \in \Phi_\gamma$. Hence, for any $s, s' \in S$, $((\varphi(s))_{\varphi\in\Phi_\gamma})_{\gamma\in\Gamma} = ((\varphi(s'))_{\varphi\in\Phi_\gamma})_{\gamma\in\Gamma}$ implies $s = s'$, i.e., f_0 is injective.

Finally, let F be an Ω-subalgebra of F_0 generated by $f_0(S)$ and let $f^* = h|F$. Using Lemma 3.4, we conclude that f^* is a unique homomorphism which makes the diagram

$$\begin{tikzcd}
S \arrow[r, "f_0"] \arrow[rd, "f"'] & F \arrow[d, "f^*"] \\
& A
\end{tikzcd}$$

commutative, and (F, f_0) is the desired free Ω-algebra generated by S. Q. E. D.

Thus, we always associate with a nonempty set S, a free Ω-algebra generated by S, which is often denoted by $(F(S), f_S)$ or simply $F(S)$ when we disregard the mapping $f_S : S \to F(S)$.

<u>Corollary 3.7.</u> Given any Ω-algebra A, there is a free Ω-algebra (F, f_0) with an epimorphism $h : F \to A$. If A is finitely generated by m elements then F can be chosen as an Ω-algebra generated by m elements.

<u>Proof.</u> Let S be a set of generators of A and let F be a free Ω-algebra generated by S. Note that we always have a set of generators for an Ω-algebra A, for instance, A itself. Then the corollary follows from the commutativity of the diagram:

$$\begin{tikzcd} S \arrow[r, "f_0"] \arrow[dr, "inc"'] & F \arrow[d, "inc^* = h"] \\ & A \end{tikzcd}$$

where inc is an inclusion mapping. Q. E. D.

Accordingly, every Ω-algebra is a homomorphic image of some free Ω-algebra. That is, that every Ω-algebra is isomorphic to a factor Ω-algebra of some free Ω-algebra by the fundamental theorem of homomorphisms.

<u>Corollary 3.8.</u> Let F be a free Ω-algebra with a homomorphism f of F into an Ω-algebra B. If there exists an Ω-algebra A with an epimorphism $g : A \to B$ then there exists a unique homomorphism $h : F \to A$ such that the following diagram is commutative:

$$\begin{tikzcd} F \arrow[r, "h"] \arrow[dr, "f"'] & A \arrow[d, "g"] \\ & B \end{tikzcd}$$

<u>Proof.</u> Let $S = \{x_\lambda\}_{\lambda \in \Lambda}$ be a set of generators of F. Then for each $\lambda \in \Lambda$, $f(x_\lambda)$ is an element in B and by the surjective property of g there exists a subset $\{a_\lambda\}_{\lambda \in \Lambda}$ in A such that $g(a_\lambda) = f(x_\lambda)$. This gives a mapping $h_1 : S \to A$ defined by $h_1(x_\lambda) = a_\lambda$. Hence, by the property of free Ω-algebras, there exists a unique homomorphism $h = h_1^*$ of F into A such that $h|S = h_1$. Clearly $gh|S = gh_1 = f|S$, and hence by Lemma 3.4, it follows that $gh = f$. Q.E.D.

Corollary 3.9. Let S and T be sets with a mapping $f : S \to T$, and let $(F(S), f_S)$ and $(F(T), f_T)$ be the corresponding free Ω-algebras. Then f induces a unique homomorphism $F(f) : F(S) \to F(T)$ such that the following diagram is commutative:

$$\begin{tikzcd}
S \arrow[r, "f_S"] \arrow[d, "f"'] & F(S) \arrow[d, "F(f)"] \\
T \arrow[r, "f_T"'] & F(T)
\end{tikzcd}$$

<u>Proof.</u> Put $F(f) = (f_T f)^*$. Q. E. D.

In fact, the Corollary 3. 9 gives rise to the covariant functor F of SET into $[\Omega]$ such that for each object $S, T \in$ SET, $F(S)$ and $F(T)$ are free Ω-algebras, and for each morphism $f: S \to T$ in SET, $F(f)$ is the induced homomorphism of $F(S)$ into $F(T)$. Using the following commutative diagram, the properties of the covariant functor are immediately verified:

$$\begin{tikzcd}
S \arrow[r, "f"] \arrow[d, "f_S"'] & T \arrow[r, "g"] \arrow[d, "f_T"'] & U \arrow[d, "f_U"'] \\
F(S) \arrow[r, "F(f)"'] & F(T) \arrow[r, "F(g)"'] & F(U)
\end{tikzcd}$$

In particular, if S is a set with an equivalence relation R then the canonical mapping $\nu : S \to S/R$, S/R being a set of R-equivalence classes, induces a unique homomorphism $F(\nu) : F(S) \to F(S/R)$. In other words, the equivalence relation R on S induces a unique congruence relation $R^* = \mathrm{Ker}(F(\nu))$ with the properties: $R = R^* \cap S^2$ and

$$F(S/R) \cong F(S)/R^{*}.$$

Theorem 3.10. Let S and T be sets with a mapping $f: S \to T$. Then the homomorphism $F(f): F(S) \to F(T)$ is i) injective iff f is injective, and ii) surjective iff f is surjective.

Proof. If f is injective then there exists a mapping g such that $gf = 1_S$ and hence there exists a homomorphism $F(g)$ such that $F(g)F(f) = F(gf) = F(1_S) = 1_{F(S)}$. Conversely, if $F(f)$ is injective then for any $s, s' \in S$, $f(s) = f(s')$ implies $F(f)f_S(s) = f_T f(s) = f_T f(s') = F(f)f_S(s')$. Hence, by injectiveness of $F(f)f_S$, it follows that $s = s'$.

If f is surjective then for any $f_T(t) \in f_T(T)$, there exists $s \in S$ such that $f_T f(s) = f_T(t)$. Since $F(f)f_S(s) = f_T f(s) = f_T(t)$, $F(T)$ is generated by the elements of the form $F(f)f_S(s)$. Hence by the homomorphic property of $F(f)$, every element in $F(T)$ is in the image of $F(f)$. Conversely, if f is not surjective then $f(S)$ is a proper subset of T. For the sake of simplicity, we identify T and $f_T(T)$. Assume that $F(f(S)) = F(T)$, i.e., for $t \in T$ such that $t \notin f(S)$, $t \in F(f(S))$. Hence, by Theorem 3.2, t is a consequence of a finite number of operations determined by Ω on the set $f(S)$ of generators of $F(f(S))$. Since these operations are finitary, it allows us to write $t = \gamma(f(s_1), f(s_2), \ldots, f(s_r))$ for some function[4/] γ on $f(S)^r$, i.e.,

$$t = \gamma(t_1, t_2, \ldots, t_r),$$

where $t_j = f(s_j) \in T$, $j=1, 2, \ldots, r$. But since T is an arbitrary set, such an identity does not hold in T. Thus, we have a contradiction. Therefore, $F(f(S))$ is properly contained in $F(T)$, and since the surjective mapping $f : S \to f(S)$ induces an epimorphism $F(f) : F(S) \to F(f(S))$, it follows that $F(f)(F(S)) = F(f(S))$ is properly contained in $F(T)$. This completes the proof. Q.E.D.

Corollary 3.11. If S and T are sets then $|S| = |T|$ iff $F(S)$ is isomorphic to $F(T)$.

3. Products and Sums

Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of Ω-algebras, and consider the direct product $\prod_{\lambda\in\Lambda} A_\lambda$ and a family of projections $\{\pi_\lambda : \prod_{\lambda\in\Lambda} A_\lambda \to A_\lambda\}_{\lambda\in\Lambda}$. If A is an arbitrary Ω-algebra with a family $\{f_\lambda : A \to A_\lambda\}_{\lambda\in\Lambda}$ of homomorphisms, then there always exists a unique homomorphisms $h : A \to \Pi A_\lambda$ such that the diagram

[4] Such a function is called a compound operation composed of a finite number of operations determined by Ω. We shall give the details in Chapter IV.

$$\begin{tikzcd} A \arrow[d, "h"'] \arrow[dr, "f_\lambda"] & \\ \Pi A_\lambda \arrow[r, "\pi_\lambda"'] & A_\lambda \end{tikzcd}$$

is commutative for all $\lambda \in \Lambda$. In fact, h is defined by $h(x) = (f_\lambda(x))_{\lambda \in \Lambda}$ and one can show that h has the property asserted above. This property is now extended into the more general situation:

<u>Definition 3.12.</u> Let CAT_1 be any category and $\{A_\lambda\}_{\lambda\in\Lambda}$ a family of objects in CAT_1. We define a category CAT_2 as follows:

The objects of CAT_2 are the families $\{f_\lambda : B \to A_\lambda\}_{\lambda\in\Lambda}$ of morphisms in CAT_1, and given two such families $\{f_\lambda : B \to A_\lambda\}_{\lambda\in\Lambda}$ and $\{f'_\lambda : B' \to A_\lambda\}_{\lambda\in\Lambda}$, a morphism in CAT_2 from the first to the second is a morphism $h : B \to B'$ in CAT_1 making the diagram

$$\begin{tikzcd} B \arrow[d, "h"'] \arrow[dr, "f'_\lambda"] & \\ B' \arrow[r, "f'_\lambda"'] & A_\lambda \end{tikzcd}$$

commutative for all $\lambda \in \Lambda$.

Then if a family $\{f_{0\lambda} : A_0 \to A_\lambda\}_{\lambda\in\Lambda}$ is a terminal object in CAT_2 we call a pair $(A_0, \{f_{0\lambda}\}_{\lambda\in\Lambda})$ or often A_0 a <u>product</u> of

$\{A_\lambda\}_{\lambda\in\Lambda}$, and denote A_0 by $\prod\limits_{\lambda\in\Lambda} A_\lambda$.

As we have seen, the direct product of Ω-algebras together with projections forms a product, and hence, we have the following theorem:

<u>Theorem 3.13.</u> The product exists in the category $[\Omega]$.

The following definition indicates the dual to the notion of products.

<u>Definition 3.14.</u> Let CAT_1 be any category and $\{A_\lambda\}_{\lambda\in\Lambda}$ a family of objects in CAT_1. Also let CAT_2 be a category defined as follows:

The objects of CAT_2 are the families $\{f_\lambda : A_\lambda \to B\}_{\lambda\in\Lambda}$ of morphisms in CAT_1, and given two such families $\{f_\lambda : A_\lambda \to B\}_{\lambda\in\Lambda}$ and $\{f'_\lambda : A_\lambda \to B'\}_{\lambda\in\Lambda}$, a morphism in CAT_2 from the first to the second is a morphism $h : B \to B'$ in CAT_1 making the diagram

$$\begin{tikzcd} A_\lambda \arrow[r, "f_\lambda"] \arrow[dr, "f'_\lambda"'] & B \arrow[d, "h"] \\ & B' \end{tikzcd}$$

commutative for all $\lambda \in \Lambda$. Then if a family $\{f_{0\lambda} : A_\lambda \to A_0\}_{\lambda\in\Lambda}$ is an initial object in CAT_2 we call a pair $(A_0, \{f_{0\lambda}\}_{\lambda\in\Lambda})$ or often A_0 a <u>sum</u> of $\{A_\lambda\}_{\lambda\in\Lambda}$, and denote A_0 by $\sum\limits_{\lambda\in\Lambda} A_\lambda$.

As an example of a sum, consider the direct product $\prod_{\lambda\in\Lambda} G_\lambda$ of additive abelian groups. Let G be the subgroup of $\prod_{\lambda\in\Lambda} G_\lambda$ consisting of all elements of the form

$$(x_\lambda)_{\lambda\in\Lambda}, \quad x_\lambda = 0 \quad \text{for all but a finite number of} \quad \lambda \in \Lambda.$$

Then G together with a family of injections forms a sum of $\{G_\lambda\}_{\lambda\in\Lambda}$. The proof is found in most materials on the group theory or the module theory. Thus, the category ABE of abelian groups has both products and sums, and in particular, the sum and the product of a finite family of abelian groups coincide with each other.

For Ω-algebras, we have the following existence theorem:

<u>Theorem 3.15.</u> The sum exists in the category $[\Omega]$.

<u>Proof.</u> Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of Ω-algebras and we set for all $\lambda \in \Lambda$, $S_\lambda = A_\lambda \cup Z \cup \Omega$ so that S_λ is either denumerable or infinite. Also, we set $\overline{S_\lambda} = S_\lambda \times \{\lambda\}$ and $T = \bigcup_{\lambda\in\Lambda} \overline{S_\lambda}$, and hence, $|T| = \sum_{\lambda\in\Lambda} |S_\lambda|$. Let Γ be a family of all Ω-algebra structures on T. Using Γ as an index set, let T_γ be an Ω-algebra corresponding to $\gamma \in \Gamma$ whose underlying set is T, and Φ_γ a collection of all families

$$\varphi^{(\gamma)} = \{\varphi_\lambda^{(\gamma)} : A_\lambda \to T\}_{\lambda\in\Lambda}$$

of homomorphisms. For all $\varphi^{(\gamma)} \in \Phi_\gamma$, we set $T_{\gamma, \varphi^{(\gamma)}} = T_\gamma$.

Now we define an Ω-algebra A'_0 by

$$A'_0 = \prod_{\gamma \in \Gamma} \prod_{\varphi^{(\gamma)} \in \Phi_\gamma} T_{\gamma, \varphi^{(\gamma)}},$$

and a mapping $f_{0\lambda} : A_\lambda \to A'_0$ by

$$f_{0\lambda}(x) = ((\varphi_\lambda^{(\gamma)}(x))_{\varphi^{(\gamma)} \in \Phi_\gamma})_{\gamma \in \Gamma}$$

for all $x \in A_\lambda$. $f_{0\lambda}$ is a homomorphism since for any $\omega \in \Omega_n$ and $a_1, a_2, \dots, a_n$ in A_λ.

$$f_{0\lambda}(\omega(a_1, a_2, \dots, a_n))$$

$$= \left((\varphi_\lambda^{(\gamma)}(\omega(a_1, a_2, \dots, a_n)))_{\varphi^{(\gamma)} \in \Phi_\gamma}\right)_{\gamma \in \Gamma}$$

$$= \left((\omega(\varphi_\lambda^{(\gamma)}(a_1), \varphi_\lambda^{(\gamma)}(a_2), \dots, \varphi_\lambda^{(\gamma)}(a_n)))_{\varphi^{(\gamma)} \in \Phi_\gamma}\right)_{\gamma \in \Gamma}$$

$$= \left(\omega((\varphi_\lambda^{(\gamma)}(a_1))_{\varphi^{(\gamma)} \in \Phi_\gamma}, (\varphi_\lambda^{(\gamma)}(a_2))_{\varphi^{(\gamma)} \in \Phi_\gamma}, \dots, (\varphi_\lambda^{(\gamma)}(a_n))_{\varphi^{(\gamma)} \in \Phi_\gamma})\right)_{\gamma \in \Gamma}$$

$$= \omega\left(\left((\varphi_\lambda^{(\gamma)}(a_1))_{\varphi^{(\gamma)} \in \Phi_\gamma}\right)_{\gamma \in \Gamma}, \left((\varphi_\lambda^{(\gamma)}(a_2))_{\varphi^{(\gamma)} \in \Phi_\gamma}\right)_{\gamma \in \Gamma}, \dots, \left((\varphi_\lambda^{(\gamma)}(a_n))_{\varphi^{(\gamma)} \in \Phi_\gamma}\right)_{\gamma \in \Gamma}\right)$$

$$= \omega(f_{0\lambda}(a_1), f_{0\lambda}(a_2), \dots, f_{0\lambda}(a_n)).$$

Let A be an arbitrary object in $[\Omega]$ with a family of homomorphisms $\{f_\lambda : A_\lambda \to A\}_{\lambda\in\Lambda}$. Restricting our attention to the images of f_λ's we may assume that A is generaged by $\bigcup_{\lambda\in\Lambda} f_\lambda(A_\lambda)$. Hence,

$$|A| \le N_0|\Omega| \sum_{\lambda\in\Lambda} |f_\lambda(A_\lambda)| \le N_0|\Omega| \sum_{\lambda\in\Lambda} |A_\lambda|$$

$$\le N_0|\Omega| \sum_{\lambda\in\Lambda} |\overline{S_\lambda}| = N_0|\Omega|\,|T| = |T|.$$

Let $Z_t = Z$ for all $t \in T$, and we set $\overline{A} = A \times \prod_{t\in T} Z_t$, so that $|\overline{A}| = |T|$. Hence, we have a bijective mapping $g : \overline{A} \to T$ such that for some Ω-algebra structure $\gamma_0 \in \Gamma$, g is an isomorphism of $\overline{A}$ onto T_{γ_0}. Since we have an injection $\tau : A \to \overline{A}$, there exists a family $\varphi^{(\gamma_0)} \in \Phi_{\gamma_0}$ such that $\varphi_\lambda^{(\gamma_0)} \in \varphi^{(\gamma_0)}$ implies $\varphi_\lambda^{(\gamma_0)} = g\tau f_\lambda : A_\lambda \to T_{\gamma_0}$.

Now put $h' = \pi g^{-1}\pi_{\gamma_0, \varphi^{(\gamma_0)}}$ where π is a projection from $\overline{A}$ to A and $\pi_{\gamma, \varphi^{(\gamma_0)}}$ is a projection from A'_0 to $T_{\gamma_0, \varphi^{(\gamma_0)}}$ such that for all $x \in A'_0$,

$$\pi_{\gamma_0, \varphi^{(\gamma_0)}}\left(\left(\left(\varphi_\lambda^{(\gamma)}(x)\right)_{\varphi^{(\gamma)}\in\Phi_\gamma}\right)_{\gamma\in\Gamma}\right) = \varphi_\lambda^{(\gamma_0)}(x).$$

Therefore, for any $x \in A_\lambda$,

$$h'f_{0\lambda}(x) = \pi g^{-1} \pi_{\gamma_0, \varphi}^{(\gamma_0)} \left(\left(\left(\varphi_\lambda^{(\gamma)}(x) \right)_{\varphi^{(\gamma)} \in \Phi_\gamma} \right)_{\gamma \in \Gamma} \right)$$

$$= \pi g^{-1} \varphi_\lambda^{(\gamma_0)}(x) = \pi g^{-1} g \tau f_\lambda(x) = f_\lambda(x),$$

which shows that the diagram

$$\begin{tikzcd}
A_\lambda \arrow[r, "f_{0\lambda}"] \arrow[dddr, "f_\lambda"'] & A_0' \arrow[d, "\pi_{\gamma_0, \varphi}^{(\gamma_0)}"] \arrow[ddd, bend left=60, "h'"] \\
& T_{\gamma_0} \arrow[d, "g^{-1}"'] \\
& \bar{A} \arrow[d, "\pi"] \\
& A
\end{tikzcd}$$

is commutative.

Finally let A_0 be the Ω-subalgebra generated by $\bigcup_{\lambda \in \Lambda} f_{0\lambda}(A_\lambda)$ and $h = h'|A_0$. Then h is a unique homomorphism such that for all $\lambda \in \Lambda$, the following diagram is commutative:

$$\begin{tikzcd}
A_\lambda \arrow[r, "f_{0\lambda}"] \arrow[dr, "f_\lambda"'] & A_0 \arrow[d, "h"] \\
& A
\end{tikzcd}$$

Therefore the pair $(A_0, \{f_{0\lambda}\}_{\lambda \in \Lambda})$ is the required sum. Q.E.D.

IV. PRIMITIVE CLASSES

Throughout this chapter, we shall identify a set S and the set of generators of a free Ω-algebra $F(S)$ whenever a free Ω-algebra appears.

1. Notes on Definitions of an Algebraic System

In definitions of most algebras such as groups and rings, we always encounter statements like "G satisfies $(ab)c = a(bc)$, $aa^{-1} = 1, \ldots$ for all $a, b, c \in G$." Therefore, in our subject, too, we must have some convenient way to define the Ω-algebras that satisfy certain identities.

In this section, we shall state a conventional definition of a familiar algebraic system, and then introduce a new definition for the system which is practically equivalent to the conventional one. Although the new definition may look rather messy, we shall find it convenient to use it especially when we attempt to formalize the entire class of algebras that satisfy the given identities in the conventional definition.

This is also a good opportunity to show an example of a connection between Ω-algebras and the familiar algebraic systems, other than groups and rings which have already been shown, and therefore, throughout this section, we consider a Boolean algebra as an

Ω-algebra. Let us now set an operator domain Ω as

$$\Omega = \{+, \cdot, -, 0, 1\}$$

where $+$ and $\cdot$ are binary, $-$ is unary and 0 and 1 are nullary operators. If B is an Ω-algebra then B contains $0, 1,$ and for any $a, b \in B$, the elements $a + b$, ab, $\overline{a}$ where each denotes $+(a, b)$, $\cdot(a, b)$ and $-(a)$, respectively.

The following definition is the one given by Huntington in 1904.

Definition 1. An Ω-algebra B is a Boolean algebra if it satisfies the following identities for all $a, b, c \in B$:

$$a + b = b + a, \qquad ab = ba,$$

$$a + 0 = a, \qquad a1 = a,$$

$$a(b + c) = ab + ac, \qquad a + (bc) = (a + b)(a + c),$$

$$a + \overline{a} = 1, \qquad a\overline{a} = 0.$$

Now consider the following definition:

Definition 2. Let X be a set with at least three elements, and $F(X)$ a free Ω-algebra generated by X. Further, let Γ be the equivalence relation generated by the eight pairs:

$$(x + y, y + x), \qquad (xy, yx),$$

$$(x + 0, x), \qquad (x1, x),$$

$$(x(y+z),\ xy+xz), \qquad (x+(yz),\ (x+y)(x+z)),$$

$$(x+\bar{x},\ 1), \qquad (x\bar{x},\ 0),$$

where x, y, z are distinct elements in X.

Then, an Ω-algebra B is a Boolean algebra if for all $(u, v) \in \Gamma$ and for any mapping $f: X \to B$,

$$f^*(u) = f^*(v)$$

holds.

Accordingly, since for instance $(xy, yx) \in \Gamma$, we have $f^*(xy) = f^*(yx)$, i.e., $f(x)f(y) = f(y)f(x)$ for any mapping $f: X \to B$. In other words, $ab = ba$ for all $a, b \in B$, and hence Definition 2 implies Definition 1. Since the converse is obvious, they are in fact, equivalent.

Let us now consider the class of all Boolean algebras, i.e., the class of all Ω-algebras satisfying Definition 2. Although this class is an example of the primitive classes of Ω-algebras, we shall not, in this section, refer to it by this terminology. If we fix a set X in Definition 2, we see that the class depends only on the operator domain and the equivalence relation Γ on $F(X)$. This allows us to designate this class simply as $P(\Omega, \Gamma)$ to express fully the class of all Boolean algebras.

Similarly, giving a definition of the same kind as Definition 2, one can obtain a class $P(\Omega, \Gamma')$ of all rings with identities, and again we see that this class is associated with an equivalence relation Γ' on $F(X)$.

The question is now whether or not among a family of all equivalence relations on $F(X)$, the equivalence relation Γ is the unique one which determines the class of all Boolean algebras. However, we cannot assert this to be true unless we restrict the family (cf. 4. 8) of equivalence relations in some way. For, although the use of an additional generator such as $(x+x, x)$ or $((x+y)+z, x+(y+z))$ in Definition 2 would not change its equivalence with Definition 1, it could change the equivalence relation Γ. We shall meet this problem later.

Finally, we shall state a definition which will be used in the following section.

Definition 4. 1. Let A be an Ω-algebra for an arbitrary operator domain Ω. Then for any natural number r, a function $\gamma : A^r \to A$ is called a compound operation of arity r composed of $\omega_1, \omega_2, \ldots, \omega_m$ if for any $(a_1, a_2, \ldots, a_r) \in A^r$, $\gamma(a_1, a_2, \ldots, a_r)$ is obtained from $a_1, a_2, \ldots, a_r \in A$ and $\omega_1, \omega_2, \ldots, \omega_m \in \Omega$ in some order. When we disregard $\omega_1, \omega_2, \ldots, \omega_m \in \Omega$, we simply say that γ is a compound operation derived from Ω.

Thus, functions $\gamma_1, \gamma_2 : B^2 \to B$ defined by $\gamma_1(a, b) = a + b$

and $\gamma_2(a, b) = a\bar{b} + \bar{a}b$ for all $a, b \in B$ are compound operations. Note that γ_1 is nothing but the binary operation determined by $+ \in \Omega$, and γ_2 is composed of $+, -, \cdot \in \Omega$ in certain way. But if γ is defined by

$$\gamma(a, b, c) = a(b+c) \quad \text{if} \quad a, b, c \in S \subset B$$

$$\gamma(a, b, c) = a + (bc) \quad \text{if} \quad a, b, c \in B - S$$

then it is not a compound operation since not all $\gamma(a, b, c)$ are obtained by applying $+$ and $\cdot$ in the same order.

2. Primitive Classes of Ω-Algebras

In the preceding section, we have used X for a set with at least three elements. The size of X was in fact, determined by the maximum arity of compound operations which appeared in Definition 2.

In this section, we deal with Ω-algebras with an arbitrary operator domain Ω. And since all the compound operations derived from Ω are of at most finite arities, we fix X as a denumerable set, which is called a standard set.

As preliminary remarks for this section, note that, by Theorem 3.2, every element in $F(X)$ is of the form

$$\gamma(x_1, x_2, \ldots, x_r)$$

for some compound operation γ derived from Ω where $x_1, x_2, \ldots, x_r \in X$. We shall use the symbols γ and γ' as compound operations throughout this section. Also, if γ_1 and γ_2 are compound operations composed of the same operators in the same order, then we shall use the same symbol for them even though they have distinct domains. Thus, for example, if $\gamma_1 : B^3 \to B$ and $\gamma_2 : C^3 \to C$ are defined by $\gamma_1(a, b, c) = a(b+c)$ and $\gamma_2(a', b', c') = a'(b'+c')$ for all $a, b, c \in B$ and $a', b', c' \in C$ where B and C are Boolean algebras, the same symbol is used for γ_1 and γ_2.

Definition 4.2. A primitive relation Γ is an equivalence relation on $F(X)$, and the elements in Γ are called laws in Γ over Ω.

Definition 4.3. Let A be an Ω-algebra. Then we say that A satisfies the law

$$(\gamma(x_1, x_2, \ldots, x_r), \gamma'(x_1', x_2', \ldots, x_t')) \tag{4.2.1}$$

in the primitive relation Γ if for any mapping $f : X \to A$

$$f^*(\gamma(x_1, x_2, \ldots, x_r)) = f^*(\gamma'(x_1', x_2', \ldots, x_t')),$$

i.e.,

$$\gamma(f(x_1), f(x_2), \ldots, f(x_r)) = \gamma'(f(x_1'), f(x_2'), \ldots, f(x_t')).$$

If A satisfies all the laws in Γ, then we say that A satisfies the primitive relation Γ.

Definition 4.4. If Γ is a primitive relation then a class of all Ω-algebras that satisfy Γ is called a primitive class of Ω-algebras defined by Γ. We denote the class by $P(\Omega, \Gamma)$. Often Ω-algebras in $P(\Omega, \Gamma)$ are called Γ-algebras to emphasize the situation.

Γ is not an operator domain of course, and the notation "Γ-algebra" should not be confused with the notation "Ω-algebra." The operator domain Ω is always associated with Γ-algebras.

Proposition 4.5. Let L be any set of generators of a primitive relation Γ. Then an Ω-algebra A satisfies all the laws in L iff A satisfies Γ.

Proof. Let A be an arbitrary Ω-algebra satisfying all the laws in L, and let M be a set of all mappings of X into A. Then we set

$$R = \bigcap_{f \in M} \{(u, v) : f^*(u) = f^*(v), u, v \in F(X)\}.$$

R is evidently an equivalence (even congruence) relation on $F(X)$ which contains L. Hence, $\Gamma \subseteq R$ and since A satisfies all the laws in R, it satisfies all the laws in Γ. The converse is trivial. Q. E. D.

Theorem 4. 6. Let Γ be a primitive relation on $F(X)$ and let Γ_0 denote a set of all elements of the form

$$(\gamma(y_1, y_2, \ldots, y_r), \gamma'(y_1', y_2', \ldots, y_t'))$$

where $y_1, y_2, \ldots, y_r, y_1', y_2', \ldots, y_t' \in X$ and $(\gamma(x_1, x_2, \ldots, x_r), \gamma'(x_1', x_2', \ldots, x_t')) \in \Gamma$. Further, let Δ be the congruence relation generated by Γ_0. Thus, we have three primitive relations Γ, Γ_0 and Δ. Then i) A is a Γ-algebra iff ii) A is a Γ_0-algebra iff iii) A is a Δ-algebra.

Remark. If for each $\omega \in \Omega$, we define the corresponding n-ary operation componentwisely on the set of ordered pairs, then we have a free Ω-algebra $F(\Gamma_0)$ generated by Γ_0. $F(\Gamma_0)$ is a congruence relation, and in fact it coincides with Δ. The proof is simple, and we shall omit it.

Proof of 4. 6. Let (4. 2. 1) be an arbitrary law in Γ. Then it follows from the definition of Γ-algebra that any Γ-algebra A satisfies the identity

$$\gamma(a_1, a_2, \ldots, a_r) = \gamma'(a_1', a_2', \ldots, a_t')$$

for all $a_1, a_2, \ldots, a_r, a_1', a_2', \ldots, a_t' \in A$. Hence, we see at once that

i) implies ii). Also since $\Gamma \subseteq \Delta$, we have that iii) implies i).

Now if (u, v) is an arbitrary element in Δ then by Theorem 3.2, it has the form

$$(\delta(u_1, u_2, \ldots, u_k), \quad \delta(u'_1, u'_2, \ldots, u'_k))$$

where $(u_j, u'_j) \in \Gamma_0$, $j=1, 2, \ldots, k$ for some compound operation δ. Hence, if we assume ii) then for any mapping $f: X \to A$, $f^*(u_j) = f^*(u'_j)$ and since f^* is a homomorphism, $f^*(\delta(u_1, u_2, \ldots, u_k)) = f^*(\delta'(u'_1, u'_2, \ldots, u'_k))$. This proves the theorem. Q. E. D.

Now consider a factor Ω-algebra $F(X)/\Delta$. If $f: X \to F(X)/\Delta$ is an arbitrary mapping, then there exists a mapping $g: X \to F(X)$ such that $f = \nu g$ for the canonical homomorphism ν of $F(X)$ onto $F(X)/\Delta$. If (u, v) is an arbitrary law in Δ then $(g^*(u), g^*(v)) \in \Delta$, and hence we have $\nu g^*(u) = \nu g^*(v)$. It follows that $f^*(u) = f^*(v)$, and $F(X)/\Delta$ is a Γ-algebra. This proves:

Proposition 4.7. For any primitive relation Γ there exists a Γ-algebra, i.e., $P(\Omega, \Gamma) \neq \emptyset$.

We have seen in the proof of 4.7 that for the endomorphism g^* of $F(X)$, $u \equiv v \bmod \Delta$ implies $g^*(u) \equiv g^*(v) \bmod \Delta$. One can show that this is true for all endomorphisms of $F(X)$.

In general, a congruence relation R on the Ω-algebra A

is said to be <u>fully invariant</u> if $a \equiv b \bmod R$ implies $f(a) \equiv f(b) \bmod R$ for every endomorphism f of A. Thus, Δ in 4.6 is an example of a fully invariant congruence relation.

<u>Proposition 4.8.</u> If Δ is a primitive relation which is a fully invariant congruence relation, then there is no other fully invariant congruence relation Δ' on $F(X)$ such that $P(\Omega, \Delta) = P(\Omega, \Delta')$. Hence, every primitive class is associated with a unique fully invariant congruence relation.

<u>Proof.</u> Suppose that (u, v) is a law in, say Δ such that (u, v) is not in Δ'. Hence $\nu(u) \neq \nu(v)$ for the canonical homomorphism ν of $F(X)$ onto $F(X)/\Delta'$. Thus, we have a mapping $f = \nu | X : X \rightarrow F(X)/\Delta'$ such that $f^*(u) = \nu(u) \neq \nu(v) = f^*(v)$, and $F(X)/\Delta'$ is not in $P(\Omega, \Delta)$. Since $F(X)/\Delta' \in P(\Omega, \Delta')$, we have $P(\Omega, \Delta') \neq P(\Omega, \Delta)$. Q. E. D.

The following two propositions are immediate from the definitions, and indicate the characteristics of primitive classes.

<u>Proposition 4.9.</u> If $A \in P(\Omega, \Gamma)$ and B is any Ω-subalgebra of A then $B \in P(\Omega, \Gamma)$.

<u>Proposition 4.10.</u> If h is any homomorphism of $A \in P(\Omega, \Gamma)$ into another Ω-algebra which is not necessarily a Γ-algebra, then

$h(A)$ is a Γ-algebra.

Let $[\Gamma]$ denote a category of all Γ-algebras and homomorphisms between Γ-algebras.

By the following three theorems we shall show that a primitive class contains free Ω-algebras, products and sums.

Theorem 4.11. The product exists in the category $[\Gamma]$.

Proof. We shall show that for any family $\{A_\lambda\}_{\lambda\epsilon\Lambda}$ of Γ-algebras $\prod_{\lambda\epsilon\Lambda} A_\lambda$ forms again a Γ-algebra. The rest of the proof is similar to the existence of the product in the category $[\Omega]$.

Let π_λ denote a projection, and for any mapping $f: X \to \prod_{\lambda\epsilon\Lambda} A_\lambda$ we write $f(x) = (f_\lambda(x))_{\lambda\epsilon\Lambda}$ where $f_\lambda = \pi_\lambda f$. Hence if (4.2.1) is an arbitrary law in Γ then for each $\lambda \epsilon \Lambda$,

$$\gamma(f_\lambda(x_1), f_\lambda(x_2), \ldots, f_\lambda(x_r)) = \gamma'(f_\lambda(x_1'), f_\lambda(x_2'), \ldots, f_\lambda(x_t')).$$

i.e.,

$$(\gamma(f_\lambda(x_1), f_\lambda(x_2), \ldots, f_\lambda(x_r)))_{\lambda\epsilon\Lambda}$$

$$= (\gamma'(f_\lambda(x_1'), f_\lambda(x_2'), \ldots, f_\lambda(x_t')))_{\lambda\epsilon\Lambda}.$$

It follows that

$$\gamma((f_\lambda(x_1))_{\lambda\epsilon\Lambda}, (f_\lambda(x_2))_{\lambda\epsilon\Lambda}, \ldots, (f_\lambda(x_r))_{\lambda\epsilon\Lambda})$$

$$= \gamma'((f_\lambda(x_1'))_{\lambda\epsilon\Lambda}, (f_\lambda(x_2'))_{\lambda\epsilon\Lambda}, \ldots, (f_\lambda(x_t'))_{\lambda\epsilon\Lambda}),$$

i.e.,

$$\gamma(f(x_1), f(x_2), \ldots, f(x_r)) = \gamma'(f(x_1'), f(x_2'), \ldots, f(x_t')).$$

Q.E.D.

Definition 4.12. Let S be a nonempty set and let CAT be a category whose objects are mappings of S into Γ-algebras and whose morphisms are defined as follows:

If $f : S \to A$ and $g : S \to B$ are two objects in CAT then a morphism from f to g is a homomorphism h such that the diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "g"'] & A \arrow[d, "h"] \\ & B \end{tikzcd}$$

is commutative.

Then a pair (G, f_0) or often G is called a *free Γ-algebra generated by S* if a mapping $f_0 : S \to G$ is an initial object in CAT.

Theorem 4.13. For any nonempty set S, there exists a free Γ-algebra (G, f_0) generated by S. Moreover, if $[\Gamma]$ contains a Γ-algebra with more than one element, then f_0 is injective and G is generated by $f_0(S)$.

Proof. Let S be a nonempty set, $F(S)$ a free Ω-algebra generated by S, and M a set of all homomorphisms of $F(X)$ into $F(S)$. Also let R be the congruence relation on $F(S)$ generated by all the pairs of the form $(h(u), h(v))$ where $h \in M$ and (u, v) is an Γ. If A is an arbitrary Γ-algebra with a mapping $f : S \to A$ then by the definition of Γ-algebra, $f^*h(u) = f^*h(v)$ for all $h \in M$ and $(u, v) \in \Gamma$. Hence,

$$R \subseteq \mathrm{Ker}(f^*). \tag{4.2.2}$$

So if ν is the canonical homomorphism of $F(S)$ onto $F(S)/R$ there exists a unique homomorphism $\overline{f^*}$ such that the diagram

$$\begin{tikzcd} S \arrow[d, "f"'] \arrow[rr, phantom, "\subseteq"] & & F(S) \arrow[d, "\nu"] \arrow[dll, "f^*"] \\ A & & F(S)/R \arrow[ll, "\overline{f^*}"] \end{tikzcd}$$

is commutative. Since it is immediate that $F(S)/R$ is a Γ-algebra, $(F(S)/R, \nu|S)$ is the required free Γ-algebra.

Now let $f_0 = \nu|S$, and let A be a Γ-algebra with more than one element. Assume that f_0 is not injective, i. e., for some distinct $s, s' \in S$, $s \equiv s' \bmod R$. Since A has at least two elements there is a mapping $f : S \to A$ such that $f(s) \neq f(s')$. Hence, $s \not\equiv s' \bmod \mathrm{Ker}(f^*)$. Thus, we have a desired contradiction to (4.2.2). Q. E. D.

As a result of 4. 13, we may identify the set S and the generators of G unless the category $[\Gamma]$ has only trivial objects. If $[\Gamma]$ has only trivial objects it means that Γ contains a law of the form (x, x'), $x \neq x'$ where $x, x' \in X$, which yields the identities $a = a'$ for all $a, a' \in A$ for each object $A \in [\Gamma]$. Hence our free Γ-algebra (G, f_0) is also trivial even though the mapping f_0 is an initial object in the category CAT in the Definition 4.12. In any case, $f_0(S)$ is in fact a set of generators of G since all the elements of G are R-congruence classes of the form $\overline{\gamma(s_1, s_2, \ldots, s_r)}$ for some compound operation γ and $s_1, s_2, \ldots, s_r \in S$, which is rewritten as $\gamma(\overline{s_1}, \overline{s_2}, \ldots, \overline{s_r})$.

<u>Theorem 4.14.</u> The sum exists in the category $[\Gamma]$.

<u>Proof</u>. Let $(A_0, \{f_{0\lambda}\}_{\lambda\in\Lambda})$ be the sum of a family $\{A_\lambda\}_{\lambda\in\Lambda}$ of Γ-algebras in the category $[\Omega]$ and let R be the congruence relation on A_0 which is generated by all the pairs $(h(u), h(v))$ for all laws $(u, v) \in \Gamma$ and for all homomorphisms h of $F(X)$ into A_0. If A is an arbitrary Γ-algebra with a family of homomorphisms $\{f_\lambda : A_\lambda \to A\}_{\lambda\in\Lambda}$ then there exists a unique homomorphism g of A_0 into A such that for all $\lambda \in \Lambda$, $gf_{0\lambda} = f_\lambda$. Since A is a Γ-algebra, $gh(u) = gh(v)$ for all laws $(u, v) \in \Gamma$ and for all homomorphisms h of $F(X)$ into A_0, and hence $R \subseteq \mathrm{Ker}(g)$. Thus if ν is the canonical homomorphism of A_0 onto A_0/R there exists a unique homomorphism $\overline{g}$ such that the diagram

$$\begin{tikzcd}
A_\lambda \arrow[r, "f_{0\lambda}"] \arrow[dr, "f_\lambda"'] & A_0 \arrow[r, "\nu"] \arrow[d, "g"] & A_0/R \arrow[dl, "\bar{g}"] \\
& A &
\end{tikzcd}$$

is commutative for all $\lambda \in \Lambda$. Evidently A_0/R is a Γ-algebra, and hence $(A_0/R, \{\nu f_{0\lambda}\}_{\lambda\in\Lambda})$ is the required sum of the family $\{A_\lambda\}_{\lambda\in\Lambda}$. Q. E. D.

PART II

GROUPS WITH MULTI-OPERATORS

V. Ω-GROUPS

So far, we have studied a completely generalized algebraic system, namely, an Ω-algebra which is a set with an arbitrary algebraic structure (Ω-algebra structure). In Part II, however, instead of taking a set with an Ω-algebra structure, we shall take a group with an Ω-algebra structure.[5] As a result, this new algebraic system does not apply to all algebras but only to the ones that are somehow related to groups. Even though we have such a restriction, our subject still represents a wide generalization in the sense that there are many algebras with group structures, say, groups, abelian groups, rings, modules, vector spaces and so forth, to which it is applicable. Thus, the unification of the theories of groups, rings, etc. will be one of our results in Part II.

Also, we shall see that groups with Ω-algebra structures may be considered as algebras in a primitive class of Ω*-algebras for some restricted operator domain Ω*. Therefore, all the theories developed so far perfectly apply to our new subject.

1. A Construction of Ω-Groups

Let G be an Ω^*-algebra, where Ω^* is an operator domain which contains at least a binary operator $+$, a unary operator $-$

[5] Replace the word "set" by "group" in Definitions 1.1 and 1.2.

and a nullary operator 0. Suppose further that G satisfies the primitive relation generated by the following three laws:

$$(+(+(x, y), z),\ +(x, +(y, z)))$$
$$(+(x, -(x)), 0)$$
$$(+(x, 0), x) \tag{5.1.1}$$

where x, y and z are distinct elements in the standard set X. Thus, G is an element of the primitive class of Ω^*-algebras defined by these three laws. We shall always use the symbol Ω^* to denote such an operator domain and set

$$\Omega = \Omega^* - \{+, -, 0\}.$$

If $\Omega = \emptyset$ then G is simply an additive group which need not be abelian, and if Ω is arbitrary then G is a group that has an Ω-algebra structure.

<u>Definition 5.1</u>. A group with an Ω-algebra structure is called a <u>group with multi-operators Ω</u> or <u>Ω-group</u> if it satisfies all the laws

$$(\omega(0, 0, \ldots, 0), 0), \quad \omega \in \Omega. \tag{5.1.2}$$

A subset H of G is called an <u>Ω-subgroup</u> of G if H is an Ω^*-subalgebra of G.

Accordingly, a class of all Ω-groups coincides with the primitive class of Ω^*-algebras defined by the set of laws (5.1.1) and (5.1.2), and by Proposition 4.9, Ω-subgroups of Ω-groups are in the same primitive class.

Henceforth, we shall write for $a, b \in G$, $a + b$ instead of $+(a, b)$ and $-a$ instead of $-(a)$. The nullary operation 0 is an element of G called the zero (rather than the identity) of G, and is unique. Also whenever we speak of homomorphisms, isomorphisms, congruence relations, etc. in the rest of this paper, it should be understood that they are homomorphisms (or isomorphisms, etc.) of Ω^*-algebras.

If G and G' are Ω-groups with a homomorphism $f: G \to G'$ then for all $\omega \in \Omega_n^*$,

$$\omega(f(0), f(0), \ldots, f(0)) = f(\omega(0, 0, \ldots, 0)) = f(0)$$

and hence $f(0)$ is the zero of G'.

2. Ω-Ideals

One of the most remarkable differences between Ω-algebras and Ω-groups is that always the latter are assured to contain trivial subalgebras, namely $O = \{0\}$. As indicated in Section 3 of Chapter I, this property gives us some possibility to express each congruence relation on the Ω-group conveniently in terms of an Ω-subgroup.

Let G be an Ω-group, R any congruence relation on G, and let $\bar{x}$ denote an R-congruence class containing $x \in G$. Since $\{0\}$ is an Ω-subgroup, it follows that $\bar{0}$ is an Ω-subgroup of G (cf. 3, Chapter I).

If x is an arbitrary element in G, then $a \equiv x \bmod R$ implies $a - x \equiv 0 \bmod R$, and hence $a - x \in \bar{0}$. It means that, $\bar{x}$ is contained in the coset $\bar{0} + x$. Also, if $a \in \bar{0} + x$ then $a \equiv x \bmod R$, and at once we obtain $\bar{0} + x \subseteq \bar{x}$, i.e., $\bar{x} = \bar{0} + x$. Similarly, we have $x + \bar{0} = \bar{x}$, and thus every R-congruence class is a coset of $\bar{0}$.

Let $x_1, x_2, \ldots, x_n$ be arbitrary elements in G and consider n cosets $x_1 + \bar{0}$, $x_2 + \bar{0}, \ldots, x_n + \bar{0}$. If a_i is any element in $\bar{0}$, $(i=1, 2, \ldots, n)$ then $x_i + a_i \equiv x_i \bmod R$, and it follows that

$$\omega(x_1+a_1, x_2+a_2, \ldots, x_n+a_n) \equiv \omega(x_1, x_2, \ldots, x_n) \bmod R$$

i.e.,

$$\omega(x_1+a_1, x_2+a_2, \ldots, x_n+a_n) \in \omega(x_1, x_2, \ldots, x_n) + \bar{0}$$

for any $\omega \in \Omega_n^*$. This property of $\bar{0}$ is precisely the property of Ω-ideals which are defined as follows:

<u>Definition 5.2.</u> Let H be a subset of the Ω-group G such that for any $x_1, x_2, \ldots, x_n \in G$, $a_1, a_2, \ldots, a_n \in H$ and $\omega \in \Omega_n^*$

$$\omega(x_1+a_1, x_2+a_2, \ldots, x_n+a_n) \in \omega(x_1, x_2, \ldots, x_n) + H. \tag{5.2.1}$$

Then H is called an <u>Ω-ideal</u> of G.

In particular, if $x_1 = x_2 = \ldots = x_n = 0$ in (5.2.1) then it shows the property of Ω-subgroups. Accordingly, every Ω-ideal is an Ω-subgroup. An example of the Ω-ideal is a normal subgroup of a group or an ideal of a ring. For instance, if we consider ω in (5.2.1) to be a multiplication of a ring then $(x_1+a_1)(x_2+a_2) \in x_1x_2 + H$. Thus, $x_1 = 0$ gives us the familiar membership $a_1x_2 \in H$ for any $a_1 \in H$ and $x_2 \in G$.

As we have seen, for any congruence relation R on G, R-congruence class $\overline{0}$ is an Ω-ideal. Now, let H be an Ω-ideal in the Ω-group G, and consider the relation of all elements (x, y) in G^2 for which $y \in x + H$ holds. This relation is obviously an equivalence relation, and furthermore, for all $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ in the relation and for any $\omega \in \Omega_n^*$

$$\begin{aligned}\omega(y_1, y_2, \ldots, y_n) &= \omega(x_1+a_1, x_2+a_2, \ldots, x_n+a_n)\\ &\in \omega(x_1, x_2, \ldots, x_n) + H\end{aligned}$$

for some $a_1, a_2, \ldots, a_n \in H$. That is, $(\omega(x_1, x_2, \ldots, x_n), \omega(y_1, y_2, \ldots, y_n))$ is also in the relation, and hence it is a congruence relation.

Thus, we have a one-to-one correspondence between the family of all congruence relations on G and the family of all Ω-ideals of G such that for each congruence relation R on G the corresponding Ω-ideal H is an R-congruence class $\overline{0}$. Symbolically, we shall write this correspondence as

$$R \sim H.$$

Proposition 5. 3. Let f be a homomorphism of an Ω-group G into an Ω-group G'. Then

$$\mathrm{Ker}(f) \sim f^{-1}(0).$$

Proof. Let $\overline{0}$ be a congruence class containing $0 \in G$ defined by $\mathrm{Ker}(f)$. Then since $a \equiv 0 \bmod \mathrm{Ker}(f)$ is equivalent to saying $f(a) = 0$, or $f(a) = f(0)$, we obtain the result at once. Q. E. D.

In particular, $f^{-1}(0)$ is an Ω-ideal of the Ω-group G. Now for each pair $(x, y) \in G^2$, define a translation θ by $\theta(z) = x - z + y$ for all $z \in G$. Evidently, it has a property that $\theta(x) = y$ and $\theta(y) = x$. Therefore, by 1. 18, all congruence relations on the Ω-group G commute with respect to the composition, and hence a composition of the congruence relations is again a congruence relation on G.

Proposition 5.4 Let H and K be Ω-ideals in the Ω-group G. Then $H \cap K$ and $H + K = \{x+y : x \in H, y \in K\}$ are both Ω-ideals of G.

Proof. Let R and T be congruence relations on G such that $R \sim H$ and $T \sim K$. Then the correspondences

$$R \cap T \sim H \cap K,$$

$$RT \sim H + K \tag{5.2.2}$$

are immediate. Q. E. D.

Thus, the intersection of two Ω-ideals of the Ω-group G is an Ω-ideal, and since for any two congruence relations R and T on G, RT is the congruence relation generated by $R \cup T$, the correspondence (5.2.2) shows that $H + K$ is the smallest Ω-ideal containing $H \cup K$. In other words, $H + K$ is the intersection of all Ω-ideals containing $H \cup K$. Similarly, if B is any Ω-subgroup of G then one can show that $H + B$ is the Ω-subgroup generated by $H \cup B$. In fact,

$$H + B = RB \tag{5.2.3}$$

(cf. (1.3.1)) where R is a congruence relation on G such that $R \sim H$.

In the theory of groups, an inner automorphism of the group G

is defined to be a translation θ_x such that for all $z \in G$

$$\theta_x : z \mapsto -x + z + x \tag{5.2.4}$$

for a fixed element $x \in G$. However, θ_x need not be an automorphism of the Ω-group. Now we extend the notion of inner automorphisms of groups for our Ω-groups.

Definition 5.5. Let G be an Ω-group. Then for each $(x_1, x_2, \ldots, x_n) \in G^n$, $\omega \in \Omega_n^*$ and $i=1, 2, \ldots, n$, we have a translation $\theta^{(\omega)}_{i,(x_1, x_2, \ldots, x_n)} : G \to G$ defined by

$$\theta^{(\omega)}_{i,(x_1, x_2, \ldots, x_n)} : z \mapsto -\omega(x_1, x_2, \ldots, x_n)$$
$$+ \omega(x_1, x_2, \ldots, x_{i-1}, x_i+z, x_{i+1}, \ldots, x_n).$$

The translation of this type is called an inner translation of the Ω-group G.

In particular, a translation θ_x of the Ω-group G defined by (5.2.4) coincides with an inner translation $\theta^{(+)}_{1,(0,x)}$.

If an Ω-subgroup H of the Ω-group G has the property that $\theta(a) \in H$ for all $a \in H$ and for all inner translations θ of G, then we shall say that H is closed under all inner translations of G.

Lemma 5.6. An Ω-subgroup H of the Ω-group G is an Ω-ideal iff H contains all the elements of the form

$$-\omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_{i-1}, x_i+a, x_{i+1}, \ldots, x_n)$$

where $\omega \in \Omega_n^*$, $x_1, x_2, \ldots, x_n \in G$ and $a \in H$.

Proof. If H contains all the elements stated above, then $\omega(x_1+a_1, x_2, \ldots, x_n) \in \omega(x_1, x_2, \ldots, x_n) + H$ and $\omega(x_1+a_1, x_2+a_2, x_3, \ldots, x_n) \in \omega(x_1+a_1, x_2, \ldots, x_n) + H$ for all $x_1, x_2, \ldots, x_n \in G$ and $\omega \in \Omega_n^*$. From these two, it follows that $\omega(x_1+a_1, x_2+a_2, x_3, \ldots, x_n) \in \omega(x_1, x_2, \ldots, x_n) + H$ for all $a_1, a_2 \in H$. Thus, by induction, we obtain

$$\omega(x_1+a_1, x_2+a_2, \ldots, x_n+a_n) \in \omega(x_1, x_2, \ldots, x_n) + H$$

for all $a_1, a_2, \ldots, a_n \in H$. The converse is obvious. Q.E.D.

From this lemma, we immediately obtain the following proposition.

Proposition 5.7. An Ω-subgroup H of the Ω-group G is an Ω-ideal iff H is closed under all inner translations of G.

Now if H and R are respectively an Ω-ideal of the Ω-group G and a congruence relation on G such that $R \sim H$ then the set of all the cosets of H coincides with the set of all R-congruence classes. Hence, the set of all the cosets is an Ω^*-algebra G/R, and since G/R is, of course, a homomorphic image of G, it follows from 4.10 that G/R is an Ω-group. The action of each $\omega \in \Omega_n^*$ on the set of the cosets is obtained directly from (1.2.1) as

$$\omega(x_1+H, x_2+H, \ldots, x_n+H) = \omega(x_1, x_2, \ldots, x_n) + H$$

for all $x_1, x_2, \ldots, x_n \in G$.

The Ω-group of the cosets is called a <u>factor Ω-group</u> of G by H, and henceforth denoted by G/H instead of G/R.

<u>3. The Composition Series and the Chain Conditions</u>

As we have seen in the previous section, all the congruence relations appeared in Part I are replaced by the suitable Ω-ideals, and by the establishment of the equalities and the correspondences like (5.2.2) and (5.2.3), we may rewrite the isomorphism theorems in Chapter II in terms of Ω-subgroups and Ω-ideals, often in much more simple forms. For example, without repeating the proof, we may state Zassenhaus' lemma for Ω-groups as follows:

<u>Zassenhaus' Lemma (Restatement of 2.4).</u> Let G be an Ω-group, B and C Ω-subgroups of G, and H and K

Ω-ideals in B and C, respectively. Then $H + (B \cap K)$ is an Ω-ideal in $H + (B \cap C)$, $K + (H \cap C)$ is an Ω-ideal in $K + (B \cap C)$, and

$$(H + (B \cap C))/(H + (B \cap K)) \cong (K + (B \cap C))/(K + (H \cap C)).$$

In Chapter II, we defined a normal series as the series from I to A (cf. 2. 5). However, the most important normal series is obtained if I is fixed as a trivial Ω-subalgebra (if possible), and it can be shown that such a selection of I does not break a generality. Since we have a definite Ω-ideal $\{0\}$ in the Ω-group, the normal series of the Ω-group will have a slightly simplified form:

Let $O = \{0\}$. Then a normal series of the Ω-group G is a sequence of Ω-subgroups of G:

$$O = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_s = G \tag{5. 3. 1}$$

such that each G_{i-1} is an Ω-ideal in G_i, $i=1, 2, \ldots, s$.

We say that an Ω-ideal H of the Ω-group G is maximal in G if $H \subset G$ and there is no Ω-ideal K such that $H \subset K \subset G$. In other words, there is no Ω-ideal in G/H other than G/H and H/H, i. e., G/H is simple.

A normal series (5. 3. 1) is called a composition series of the Ω-group G if each G_{i-1} is a maximal Ω-ideal in G_i.

Jordan-Hölder Theorem for Ω-Groups (Restatement of 2.10). Any two composition series of the Ω-group G are equivalent.

Now we shall state two conditions that together constitute a necessary and sufficient condition that an Ω-group G have a composition series.

Definition 5.8. Let G be an Ω-group.

i) The first descending chain condition. If $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ is a sequence of Ω-subgroups such that G_1 is an Ω-ideal in G and each G_{i+1} is an Ω-ideal in the preceding, then there exists a positive integer r such that $G_r = G_{r+1} = \ldots$.

ii) The first ascending chain condition. If $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ is a sequence of Ω-ideals in H which is any term of a normal series then there exists a positive integer r such that $G_r = G_{r+1} = \ldots$.

The two chain conditions can be stated equivalently for an arbitrary Ω-algebra A, and the two conditions are sufficient that an Ω-algebra A possess a composition series.

iii) Descending chain condition for Ω-algebra. If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a sequence of Ω-subalgebras of A together with the congruence relations R_i on A_i such that A_1 is a congruence class in A for some congruence relation on A and each A_{i+1} is an R_i-congruence class in the preceding then there exists a positive integer r such that $A_r = A_{r+1} = \ldots$.

iv) Ascending chain condition for Ω-algebra. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ is an ascending sequence of Ω-subalgebras of A each of which is a congruence class in B defined by some congruence relation on B, B being any term of a normal series, then there exists a positive integer r such that $A_r = A_{r+1} = \ldots$.

Theorem 5.9. If an Ω-algebra A which possesses a nontrivial normal series [6] from its Ω-subalgebra I to A, satisfies the two chain conditions iii) and iv), then A has a composition series from I to A.

Proof. In any nontrivial normal series from I to A, let $B \neq I$ be an arbitrary term and let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ be any ascending sequence given in iv). Then we necessarily have an Ω-algebra A_r such that $A_r = A_{r+1} = \ldots$. This means that there exists a congruence relation T which defines a T-congruence class B' in the sequence such that B/T is simple. Since B is arbitrary we set

$$B = A_{0i} \quad \text{and} \quad B' = A_{0i+1}, \quad i=1, 2, \ldots . \tag{5.3.2}$$

[6] A normal series is nontrivial if it differs from the series: $A \subseteq A$.

This gives rise to a descending sequence $A = A_{00} \supseteq A_{01} \supseteq A_{02} \supseteq \cdots$ together with a congruence relation T_i on A_{0i} such that A_{0i}/T_i is simple and A_{0i+1} is a T_i-congruence class. By the descending chain condition iii) we meet $B' = I$ after a finite number of iterations (5. 3. 2), and hence we obtain the composition series

$$A = A_{00} \supset A_{01} \supset A_{02} \supset \cdots \supset I .$$

Q. E. D.

The converse of 5. 9 is not true in general since if A is infinite and does not contain a trivial Ω-subalgebra then it is possible for A to possess an infinitely descending sequence of Ω-subalgebras even though A have a composition series from I to A for some Ω-subalgebra I. However, if A contains a trivial Ω-subalgebra, the situation becomes different. We shall state this with the Ω-group G rather than with the Ω-algebra that has a trivial Ω-subalgebra.

<u>Theorem 5. 10.</u> A necessary and sufficient condition that an Ω-group G have a composition series is that G satisfies chain conditions i) and ii).

<u>Proof.</u> The sufficiency follows from 5. 9. To prove the necessity, let $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ be any descending sequence of Ω-subgroups such that G_1 is an Ω-ideal in G and G_{i+1} is an

Ω-ideal in the preceding. Then for any integer k

$$G \supseteq G_1 \supseteq G_2 \supseteq \cdots \supseteq G_k \supseteq O \qquad (5.3.3)$$

where $O = \{0\}$, is a normal series of G. If we delete the repititions of Ω-subgroups from (5.3.3) then there is a composition series which is a refinement of the resultant of the deletion. Hence, the number of terms in the resultant is finite, and the descending chain condition i) holds.

Let $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ be an ascending sequence such that all G_i are Ω-ideals in H which is any term of a normal series of G. Then obviously G_i is an Ω-ideal in G_{i+1} and hence

$$O \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_k \subseteq H \subseteq \cdots \subseteq G$$

is a normal series for any integer k. Therefore, the same argument as we used for the descending sequence shows that $G_1 \subseteq G_2 \subseteq G_3 \subseteq \cdots$ cannot be an infinitely ascending sequence.
Q. E. D.

Definition 5.11. A normal series (5.3.1) of the Ω-group G is called an invariant series if all G_i, $i=1, 2, \ldots, s$, are Ω-ideals in G. Also an invariant series without repetitions of Ω-subgroups which has no refinement without repetitions is called a chief series.

Let Ω' be an operator domain such that $\Omega' = \Omega \cup \Omega_1$ where Ω_1 is an operator domain of unary operators such that $|\Omega_1|$ coincides with the number of inner translations of the Ω-group G. If we consider every inner translation as a unary operation on G determined by the operator in Ω_1, then G becomes an Ω'-group as well as an Ω-group. And it is immediate from 5.7 that a composition (or normal) series of the Ω'-group G is a chief (or invariant) series of the Ω-group G.

Of course, all the theorems for the composition (or normal) series hold for the chief (or invariant) series.

VI. DIRECT SUMS OF Ω-GROUPS

1. Direct Products and Direct Sums

If a family of Ω-groups $\{G_\lambda\}_{\lambda\in\Lambda}$ are given then by 4.11, the direct product

$$G = \prod_{\lambda\in\Lambda} G_\lambda$$

forms an Ω-group called the direct product of Ω-groups G_λ. For each $\omega \in \Omega_n^*$, the corresponding n-ary operation is defined by the same way as (3.1.1), and hence it is immediate that for the unary operator $- \in \Omega^*$,

$$-((a_\lambda)_{\lambda\in\Lambda}) = (-a_\lambda)_{\lambda\in\Lambda},$$

and the element $0 = (0)_{\lambda\in\Lambda}$ is the zero of G.

Now let $G_1, G_2, \ldots, G_s$ be Ω-groups and let

$$G = \prod_{j=1}^{s} G_j.$$

Then for each $k=1, 2, \ldots, s$, G contains the Ω-subgroup

$$H_k = \{(0, 0, \ldots, 0, a_k, 0, \ldots, 0) \in G : a_k \in G_k\}$$

which is isomorphic to G_k under the correspondence

$(0, 0, \ldots, 0, a_k, 0, \ldots, 0) \mapsto a_k$.

Let π_j be the projection of G onto G_j and

$$K_j = \pi_j^{-1}(0) = \{(a_1, a_2, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_s) \in G\}.$$

Then by 5.3, K_j is an Ω-ideal of G, and at once we obtain the following equalities:

$$H_k = \bigcap_{j \neq k} K_j = K_1 \cap K_2 \cap \ldots \cap K_{k-1} \cap K_{k+1} \cap \ldots \cap K_s$$

$$K_k = \sum_{j \neq k} H_j = H_1 + H_2 + \ldots + H_{k-1} + H_{k+1} + \ldots + H_s$$

$$G = \sum_{j=1}^{s} H_j = H_1 + H_2 + \ldots + H_s . \tag{6.1.1}$$ [7]

It follows from 5.4 that H_k is an Ω-ideal of G and

$$H_k \cap \Big(\sum_{j \neq k} H_j \Big) = O, \tag{6.1.2}$$

[7] For the rest of the paper, the symbol $\sum$ does not represent the sum of $H_1, H_2, \ldots, H_s$ in the category theoretical sense (cf. 3.14), i.e., $\sum_{j=1}^{s} H_j$ is only the totality of the elements of the form $\sum_{j=1}^{s} a_j = a_1 + a_2 + \ldots + a_s$, where $a_j \in H_j$ (cf. 5.4).

where O denotes the trivial Ω-subgroup throughout this chapter. Also, (6.1.1) and (6.1.2) may be written as

$$G = (\bigcap_{j \neq k} K_j) + K_k \qquad (6.1.3)$$

$$O = \bigcap_{j=1}^{s} K_j, \qquad (6.1.4)$$

respectively, which correspond to the equalities (3.1.5) and (3.1.4).

Lemma 6.1. If any Ω-group G has Ω-ideals $H_1, H_2, \ldots, H_s$ satisfying (6.1.1) and (6.1.2) then every element of G has a unique expression

$$\sum_{j=1}^{s} a_j = a_1 + a_2 + \ldots + a_s$$

where $a_j \in H_j$.

Proof. If $\sum_{j=1}^{s} a_j = \sum_{j=1}^{s} a_j'$ then $-a_1' + a_1 = \sum_{j=2}^{s} a_j' - (\sum_{j=2}^{s} a_j)$.

Hence, $-a_1' + a_1 \in H_1 \cap \sum_{j=2}^{s} H_j = O$. This gives $a_1 = a_1'$, and the similar argument will show that $a_j = a_j'$ for all j. Q.E.D.

Proposition 6.2. In order that an Ω-group G be representable as a direct product of the form $\prod_{j=1}^{s} G_j$ where G_j is an Ω-group, it is necessary and sufficient that there exist Ω-ideals

$H_1, H_2, \ldots, H_s$ in G satisfying (6.1.1) and (6.1.2).

Proof. The necessity follows from 3.1 and the equalities (6.1.3) and (6.1.4). To show the sufficiency of the condition, let $H_1, H_2, \ldots, H_s$ be the Ω-ideals of G satisfying (6.1.1) and (6.1.2). Then we set

$$K_k = \sum_{j \neq k} H_j.$$

From this and from 6.1, we obtain the equalities (6.1.3) and (6.1.4), and hence the sufficiency follows from 3.1. Q.E.D.

Definition 6.3. Let G be an Ω-group which contains the Ω-ideals $H_1, H_2, \ldots, H_s$ satisfying (6.1.1) and (6.1.2). Then G is called the direct sum of $H_1, H_2, \ldots, H_s$ and is written

$$G = \bigoplus_{j=1}^{s} H_j = H_1 \oplus H_2 \oplus \ldots \oplus H_s. \tag{6.1.5}$$

Also, each H_j is called the direct summand of G.

As we know, if G is the direct sum (6.1.5) then every element of G has the unique expression $\sum_{j=1}^{s} a_j$, $a_j \in H_j$. Now note that

$$H_1 \oplus (H_2 \oplus H_3) = (H_1 \oplus H_2) \oplus H_3,$$

$$H_1 \oplus H_2 = H_2 \oplus H_1.$$

Hence, if $G = H_1 \oplus H_2 \oplus \ldots \oplus H_s$ then for any permutation of $j=1, 2, \ldots, s$, written $j \mapsto j'$, $G = H_{1'} \oplus H_{2'} \oplus \ldots \oplus H_{s'}$. Let t be an integer such that $i \leq t \leq s$, and let

$$H = H_{1'} \oplus H_{2'} \oplus \ldots \oplus H_{t'} ,$$

$$K = H_{(t+1)'} \oplus H_{(t+2)'} \oplus \ldots \oplus H_{s'} .$$

Further, define a mapping $f : H \to G/K$ by $f(x) = x + K$ for all $x \in H$. f is obviously surjective, and using the uniqueness of the expression of the element in $H \oplus K$ we also obtain the injectiveness of f. Now if $x_1, x_2, \ldots, x_n \in H$ and $\omega \in \Omega_n^*$, then

$$\begin{aligned} \omega(f(x_1), f(x_2), \ldots, f(x_n)) &= \omega(x_1+K, x_2+K, \ldots, x_n+K) \\ &= \omega(x_1, x_2, \ldots, x_n) + K \\ &= f(\omega(x_1, x_2, \ldots, x_n)), \end{aligned}$$

and it follows that f is an isomorphism. This gives us

$$H_{1'} \oplus H_{2'} \oplus \ldots \oplus H_{t'} \cong G/(H_{(t+1)'} \oplus H_{(t+2)'} \oplus \ldots \oplus H_{s'}).$$

In particular, if

$$K_k = \sum_{j \neq k} H_j ,$$

then

$$H_k \cong G/K_k , \qquad (6.1.6)$$

and from 3.1, we obtain the isomorphism

$$\bigoplus_{j=1}^{s} H_j = G \cong \prod_{j=1}^{s} (G/K_j) \tag{6.1.7}$$

defined by

$$\sum_{j=1}^{s} a_j \mapsto (a_1+K_1, a_2+K_2, \ldots, a_s+K_s).$$

By (6.1.6) and (6.1.7), the following theorem is now immediate.

<u>Theorem 6.4.</u> If an Ω-group G is a direct sum of Ω-ideals $H_1, H_2, \ldots, H_s$ then

$$\prod_{j=1}^{s} H_j \cong \bigoplus_{j=1}^{s} H_j = G$$

under the correspondence

$$(a_1, a_2, \ldots, a_s) \mapsto a_1 + a_2 + \ldots + a_s. \tag{6.1.8}$$

<u>Corollary 6.5.</u> If $G = \bigoplus_{j=1}^{s} H_j$ for its Ω-ideals H_j then for all $\omega \in \Omega_n^*$ and $\sum a_{1j}, \sum a_{2j}, \ldots, \sum a_{nj}$ in $\bigoplus_{j=1}^{s} H_j$

$$\omega(\sum a_{1j}, \sum a_{2j}, \ldots, \sum a_{nj}) = \sum \omega(a_{1j}, a_{2j}, \ldots, a_{nj}) \tag{6.1.9}$$

Proof. Let f be the isomorphism (6.1.8). Then

$$\omega(\sum a_{1j}, \sum a_{2j}, \dots, \sum a_{nj})$$

$$= \omega(f(a_{11}, a_{12}, \dots, a_{1s}), f(a_{21}, a_{22}, \dots, a_{2s}), \dots, f(a_{n1}, a_{n2}, \dots, a_{ns}))$$

$$= f(\omega((a_{11}, a_{12}, \dots, a_{1s}), (a_{21}, a_{22}, \dots, a_{2s}), \dots, (a_{n1}, a_{n2}, \dots, a_{ns})))$$

$$= f(\omega(a_{11}, a_{21}, \dots, a_{n1}), \omega(a_{12}, a_{22}, \dots, a_{n2}), \dots, \omega(a_{1s}, a_{2s}, \dots, a_{ns}))$$

$$= \omega(a_{11}, a_{21}, \dots, a_{n1}) + \omega(a_{12}, a_{22}, \dots, a_{n2}) + \dots + \omega(a_{1s}, a_{2s}, \dots, a_{ns})$$

$$= \sum \omega(a_{1j}, a_{2j}, \dots, a_{nj}). \qquad \text{Q.E.D.}$$

2. Projective Endomorphisms

Let M be a family of all mappings of the Ω-group G into itself, and for each $\omega \in \Omega_n^*$, we define the corresponding n-ary operation on M as

$$(f_1, f_2, \dots, f_n) \mapsto \omega(f_1, f_2, \dots, f_n)$$

for all $f_1, f_2, \dots, f_n \in M$ where $\omega(f_1, f_2, \dots, f_n)$ is an element of M such that

$$(\omega(f_1, f_2, \dots, f_n))(x) = \omega(f_1(x), f_2(x), \dots, f_n(x))$$

for all $x \in G$. Hence M is an Ω*-algebra. In particular, for all

$f_1, f_2 \in M$ and $x \in G$,

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$

$$(-f_1)(x) = -(f_1(x))$$

$$0(x) = 0$$

where 0 denotes the zero in M called the zero mapping. Evidently,

$$(f_1+f_2) + f_3 = f_1 + (f_2+f_3)$$

for all $f_1, f_2, f_3 \in M$, but the addition need not be commutative. Also we have the additional properties:

$$f_1 + (-f_1) = 0 = -f_1 + f_1$$

$$f_1 + 0 = f_1 = 0 + f_1,$$

and thus, $\underline{M \text{ is an } \Omega\text{-group}}$. Furthermore, the composition is left distributive relative to addition, i.e.,

$$(f_1+f_2)f_3 = f_1f_3 + f_2f_3$$

for all $f_1, f_2, f_3 \in M$. However, the other distributive law need not hold, since we can <u>not</u> assert

$$f_3(f_1(x)+f_2(x)) = f_3f_1(x) + f_3f_2(x)$$

unless f is an endomorphism of G.

Let $H_1, H_2, \ldots, H_s$ be the Ω-ideals of the Ω-group G such that $G = \bigoplus_{j=1}^{s} H_j$. Since every element in G has a unique expression $\sum_{j=1}^{s} a_j$, a_j being an element of H_j, we have a mapping $\sigma_j : G \to G$ defined by

$$\sigma_j : \sum_{j=1}^{s} a_j \mapsto a_j \tag{6.2.1}$$

for each $j=1, 2, \ldots, s$. If $\sum a_{1j}, \sum a_{2j}, \ldots, \sum a_{nj} \in G$ and $\omega \in \Omega_n^*$ then it follows from (6.1.9) that

$$\sigma_j(\omega(\sum a_{1j}, \sum a_{2j}, \ldots, \sum a_{nj})) = \omega(\sigma_j(\sum a_{1j}), \sigma_j(\sum a_{2j}), \ldots, \sigma_j(\sum a_{nj})),$$

which shows that σ_j is an endomorphism of G. From (6.2.1), we also obtain

$$\sigma_j \sigma_k = 0 \quad (j \neq k) \tag{6.2.2}$$

$$\sigma_j^2 = \sigma_j \tag{6.2.3}$$

$$\sigma_1 + \sigma_2 + \ldots + \sigma_s = 1_G. \tag{6.2.4}$$

Definition 6.6. An endomorphism of the Ω-group G is said to be normal if it commutes with all the inner translations of G with respect to composition.

Proposition 6.7. Let G be an Ω-group and $\sigma_1, \sigma_2, \ldots, \sigma_s$ endomorphisms of G satisfying (6.2.2), (6.2.3) and (6.2.4). Then each σ_j is a normal endomorphism.

Proof. Let $\theta = \theta^{(\omega)}_{i, (x_1, x_2, \ldots, x_n)}$ (cf. 5.5) be an arbitrary inner translation of G. Then for any $a \in G$

$$\theta\sigma_j(a) = -\omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_{i-1}, x_i + \sigma_j(a), x_{i+1}, \ldots, x_n),$$

and hence for any $k \neq j$, $\sigma_k(x_i) = y_i$ implies

$$\sigma_k\theta\sigma_j(a) = -\omega(y_1, y_2, \ldots, y_n)$$
$$+ \omega(y_1, y_2, \ldots, y_{i-1}, y_i + 0, y_{i+1}, \ldots, y_n) = 0$$

and

$$\sigma_j\theta\sigma_j(a) = -\omega(\sigma_j(x_1), \sigma_j(x_2), \ldots, \sigma_j(x_n))$$
$$+ (\sigma_j(x_1), \sigma_j(x_2), \ldots, \sigma_j(x_{i-1}), \sigma_j(x_i)$$
$$+ \sigma_j(a), \sigma_j(x_{i+1}), \ldots, \sigma_j(x_n))$$
$$= \sigma_j\theta(a).$$

Hence, by (6.2.4), we obtain

$$\theta\sigma_j(a) = 1_G\theta\sigma_j(a) = (\sigma_1 + \sigma_2 + \ldots + \sigma_s)\theta\sigma_j(a) = \sigma_1\theta\sigma_j(a) + \sigma_2\theta\sigma_j(a)$$
$$+ \ldots + \sigma_s\theta\sigma_j(a) = \sigma_j\theta\sigma_j(a) = \sigma_j\theta(a).$$

Q. E. D.

Thus, the endomorphism σ_j obtained from the direct sum $\bigoplus_{j=1}^{s} H_j$ of Ω-ideals by (6.2.1) is a normal endomorphism.

Lemma 6.8. If f is a normal endomorphism of the Ω-group G then for any Ω-ideal H of G, $f(H)$ is again an Ω-ideal.

Proof. For all $f(a) \in f(H)$ and for all inner translation θ, $\theta(f(a)) = \theta f(a) = f\theta(a) = f(\theta(a)) \in f(H)$, so that $f(H)$ is closed under all the inner translations of G. Q.E.D.

Proposition 6.9. Let f and g be normal endomorphisms of the Ω-group G. Then i) fg is again a normal endomorphism of G, and ii) if $f(G) \cap g(G) = O$ then $f + g = g + f$, and moreover $f + g$ is a normal endomorphism of G.

Proof. i) is obvious, and so, we consider ii). By Lemma 6.8, both $f(G)$ and $g(G)$ are Ω-ideals of G and hence they are normal subgroups of the group G. Therefore,
$-f(x) - g(x) + f(x) + g(x) \in f(G) \cap g(G) = O$, i.e.,
$-f(x) - g(x) + f(x) + g(x) = 0$ for all $x \in G$. This proves

$$f + g = g + f \tag{6.2.5}$$

Let $\omega \in \Omega_n^*$ and $x_1, x_2, \ldots, x_n \in G$. Using (6.2.5) and the definition of Ω-ideals we see that

$$-\omega(f(x_1), f(x_2), \ldots, f(x_n)) - \omega(g(x_1), g(x_2), \ldots, g(x_n))$$

$$+ \omega(f(x_1)+g(x_1), f(x_2)+g(x_2), \ldots, f(x_n)+g(x_n)) = 0$$

since the left side of the equality is an element of $f(G) \cap g(G)$. It follows that

$$(f+g)(\omega(x_1, x_2, \ldots, x_n)) = \omega((f+g)(x_1), (f+g)(x_2), \ldots, (f+g)(x_n)),$$

which shows that $f + g$ is an endomorphism of G.

Now, let $\theta = \theta^{(\omega)}_{i, (x_1, x_2, \ldots, x_n)}$ (cf. 5. 5). Then for all $a \in G$,

$$\begin{aligned}
&(f+g)\theta(a) - \theta(f+g)(a) \\
&= f\theta(a) + g\theta(a) - \theta(f+g)(a) \\
&= \theta f(a) + \theta g(a) - \theta(g+f)(a) \\
&= -\omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_i + f(a), \ldots, x_n) \\
&\quad -\omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_i + g(a), \ldots, x_n) \\
&\quad -\omega(x_1, x_2, \ldots, x_i + g(a) + f(a), \ldots, x_n) + \omega(x_1, x_2, \ldots, x_n).
\end{aligned}$$

Since $f(G)$ is an Ω-ideal,

$$y = \omega(x_1, x_2, \ldots, x_i+g(a), \ldots, x_n) - \omega(x_1, x_2, \ldots, x_i+g(a)+f(a), \ldots, x_n) \in f(G),$$

i. e.,

$$-\omega(x_1, x_2, \ldots, x_n) + y + \omega(x_1, x_2, \ldots, x_n) \in f(G).$$

It follows that $(f+g)\theta(a) - \theta(f+g)(a) \in f(G)$. Similarly, using the commutativity (6.2.5), we also have $(f+g)\theta(a) - \theta(f+g)(a) \in g(G)$. Hence,

$$(f+g)\theta(a) - \theta(f+g)(a) \in f(G) \cap g(G) = O.$$

Since θ is arbitrary this is precisely saying that $f + g$ is a normal endomorphism of G. Q.E.D.

Definition 6.10. A projective endomorphism of the Ω-group G is an endomorphism of G which is normal and idempotent. A set of projective endomorphisms is called orthogonal if the composition of any two distinct ones in the set is 0. Finally, we say that the orthogonal set of projective endomorphisms is complementary if their sum is the identity mapping.

Accordingly, the endomorphisms $\sigma_1, \sigma_2, \ldots, \sigma_s$ defined by (6.2.1) give an orthogonal and complementary set of projective endomorphisms, i.e., if the Ω-group G is a direct sum $\bigoplus_{j=1}^{s} H_j$ of Ω-ideals it always possesses an orthogonal and complementary set P of projective endomorphisms. Now conversely, if there is such a set $P = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ for the Ω-group G then by Lemma 6.8, each σ_j determines an Ω-ideal $\sigma_j(G)$. Let $H_j = \sigma_j(G)$. Then for any $x \in G$, $x = 1_G(x) = \sigma_1(x) + \sigma_2(x) + \ldots + \sigma_s(x) \in \sum_{j=1}^{s} H_j$, which shows that $G = \sum_{j=1}^{s} H_j$. Also if y is an element in $H_k \cap (\sum_{j \neq k} H_j)$

then $y = \sigma_k(x_k)$ and $y = \sum_{j \neq k} \sigma_j(x_j)$ for some elements $x_1, x_2, \ldots, x_s \in G$. Hence

$$\begin{aligned} y = 1_G(y) &= \sigma_1(y) + \sigma_2(y) + \ldots + \sigma_s(y) \\ &= \sigma_1\sigma_k(x_k) + \sigma_2\sigma_k(x_k) + \ldots + \sigma_{k-1}\sigma_k(x_k) + \sigma_k(\sum_{j \neq k} \sigma_j(x_j)) \\ &\quad + \sigma_{k+1}\sigma_k(x_k) + \ldots + \sigma_s\sigma_k(x_k) \\ &= 0. \end{aligned}$$

Therefore, $H_k \cap (\sum_{j \neq k} H_j) = O$ and it follows that $G = \bigoplus_{j=1}^{s} H_j$. This proves the following theorem:

<u>Theorem 6.11.</u> An Ω-group G is a direct sum of its Ω-ideals iff there is an orthogonal and complementary set of projective endomorphisms of G.

From this it is immediate that for any Ω-group G there exists a one-to-one correspondence between a family of the orthogonal and complementary sets of projective endomorphisms of G and a family of the sets of Ω-ideals each of which determines a direct sum G.

3. The Krull-Schmidt Theorem for Ω-Groups

In the rest of the paper, we shall call $\overset{s}{\underset{j=1}{\oplus}} H_j$ a direct decomposition of the Ω-group G if $G = \overset{s}{\underset{j=1}{\oplus}} H_j$.

Definition 6.12. An Ω-group G is said to be decomposable if $G = H_1 \oplus H_2$ for some proper Ω-ideals H_1 and H_2.

Hence, if an Ω-group is decomposable, its decomposition $H_1 \oplus H_2$ determines projective endomorphisms σ_1 and σ_2 defined by (6.2.1), which are distinct from 1_G and 0. This shows the necessity of the condition in the following theorem.

Theorem 6.13. In order that an Ω-group G be decomposable it is necessary and sufficient that there exists a projective endomorphism of G other than 1_G and 0.

Proof. To prove the sufficiency of the condition, let σ be a projective endomorphism distinct from 1_G and 0. Also, set $H_1 = \sigma(G)$ and $H_2 = \sigma^{-1}(0)$ so that H_1 and H_2 are Ω-ideals of G.

Now for any $x \in G$, $x = \sigma(x) - \sigma(x) + x$, and since $\sigma(-\sigma(x)+x) = 0$, it follows that $x \in H_1 + H_2$. This proves that $G = H_1 + H_2$. But since for any $y \in H_1 \cap H_2$, $y = \sigma(x) = \sigma^2(x) = \sigma(y) = 0$ for some $x \in G$, we have $H_1 \cap H_2 = O$. Thus, $G = H_1 \oplus H_2$.

Of course, both H_1 and H_2 are proper Ω-ideals. Q. E. D.

In 5. 8, we had the two chain conditions for Ω-subgroups that together are necessary and sufficient that an Ω-group have a composition series. In this section, we similarly define the following two chain conditions.

Definition 6. 14. Let G be an Ω-group.

i) The second descending chain condition. If $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$ is a descending sequence of Ω-ideals of G, then there exists an integer r such that $H_r = H_{r+1} = \cdots$.

ii) The second ascending chain condition. If $H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ is an ascending sequence of Ω-ideals of G, then there exists an integer r such that $H_r = H_{r+1} = \cdots$.

Lemma 6. 15. If an Ω-group G is a direct sum of Ω-ideals then any Ω-ideal of the direct summand is again an Ω-ideal of G.

Proof. We shall prove the lemma for the particular case $G = H_1 \oplus H_2$. The proof for $G = \bigoplus_{j=1}^{s} H_j$ for an arbitrary integer s is similar. Let K be an Ω-ideal of, say H_1. If σ_1 and σ_2 are projective endomorphisms defined by (6. 2. 1) then for any $x_1, x_2, \ldots, x_n \in G$, $\omega \in \Omega_n^*$ and $a \in K$,

$$-\omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n)$$

$$= -\omega(\sigma_1(x_1)+\sigma_2(x_1), \sigma_1(x_2)+\sigma_2(x_2), \ldots, \sigma_1(x_n)+\sigma_2(x_n))$$

$$+ \omega(\sigma_1(x_1)+\sigma_2(x_1), \sigma_1(x_2)+\sigma_2(x_2), \ldots, \sigma_1(x_{i-1})+\sigma_2(x_{i-1}), \sigma_1(x_i)$$

$$+ a+\sigma_2(x_i), \sigma_1(x_{i+1})+\sigma_2(x_{i+1}), \ldots, \sigma_1(x_n)+\sigma_2(x_n))$$

$$= -\omega(\sigma_1(x_1), \sigma_1(x_2), \ldots, \sigma_1(x_n)) - \omega(\sigma_2(x_1), \sigma_2(x_2), \ldots, \sigma_2(x_n))$$

$$+ \omega(\sigma_1(x_1), \sigma_1(x_2), \ldots, \sigma_1(x_{i-1}), \sigma_1(x_i)+ a, \sigma_1(x_{i+1}), \ldots, \sigma_1(x_n))$$

$$+ \omega(\sigma_2(x_1), \sigma_2(x_2), \ldots, \sigma_2(x_n)).$$

Now note that for any direct sum $\bigoplus_{j=1}^{s} H_j$, $a_j \in H_j$ implies $a_j + a_k = a_k + a_j$ if $j \neq k$, since $-a_k + a_j + a_k - a_j \in H_j \cap H_k = O$. Hence it follows that

$$- \omega(x_1, x_2, \ldots, x_n) + \omega(x_1, x_2, \ldots, x_{i-1}, x_i + a, x_{i+1}, \ldots, x_n)$$

$$= - \omega(\sigma_1(x_1), \sigma_1(x_2), \ldots, \sigma_1(x_n)) + \omega(\sigma_1(x_1), \sigma_1(x_2), \ldots, \sigma_1(x_{i-1}), \sigma_1(x_i)$$

$$+ a, \sigma_1(x_{i+1}), \ldots, \sigma_1(x_n)) \in K.$$

By 5. 6, the lemma follows. Q. E. D.

<u>Theorem 6. 16.</u> Any nontrivial Ω-group that satisfies the second descending chain condition is either indecomposable or a direct sum of indecomposable nontrivial Ω-ideals.

<u>Proof.</u> Suppose that G is decomposable, i. e., $G = K_1 \oplus K_2$,

where K_1 and K_2 are proper Ω-ideals of G. If K_1 is indecomposable we set $H_1 = K_1$. If K_1 is decomposable then $K_1 = K_{11} \oplus K_{12}$ where K_{11} and K_{12} are proper Ω-ideals of G (cf. Lemma 6.15). Again if K_{11} is indecomposable we set $H_1 = K_{11}$. If K_{11} is decomposable then $K_{11} = K_{111} \oplus K_{112}$.

The lemma just proved assures that every term in the descending sequence

$$K_1 \supseteq K_{11} \supseteq K_{111} \supseteq \cdots$$

is an Ω-ideal of G. Hence by hypothesis, we must have an indecomposable direct summand H_1 after a finite number of the process.

Now we can write $G = H_1 \oplus K'$ for some nontrivial Ω-ideal K' of G. By the same process by which we have obtained H_1, we obtain the indecomposable direct summand H_2 of K', so that we can write $G = H_1 \oplus H_2 \oplus K''$ for some Ω-ideal K'' of K'. If $K'' \neq O$ we repeat the process and obtain $G = H_1 \oplus H_2 \oplus H_3 \oplus K'''$. Applying the given chain condition again to the descending sequence

$$K' \supseteq K'' \supseteq K''' \supseteq \cdots$$

we conclude that $K^{(s)} = O$ for some finite number s, i.e.,

$$G = H_1 \oplus H_2 \oplus \dots \oplus H_s$$

where each H_j is an indecomposable and nontrivial Ω-ideal.

Q. E. D.

If $\bigoplus_{j=1}^{s} H_j$ is a direct decomposition of the Ω-group G such that every H_j is indecomposable, then we shall call it a complete decomposition of G.

Lemma 6. 17. Let G be an Ω-group satisfying the descending and ascending chain conditions in 6. 14. Then a normal endomorphism f of G is an automorphism provided that f is either injective or surjective.

Proof. First, assume that f is injective but not surjective. We shall show by induction that for the descending sequence

$$G \supseteq f(G) \supseteq f^2(G) \supseteq \dots \supseteq f^r(G) \supseteq \dots$$

of Ω-ideals, we have $f^{r-1}(G) \neq f^r(G)$ for all integer $r=1, 2, \dots$ so that it contradicts the hypothesis. Since f is not surjective, $G \neq f(G)$. Now if $f^{r-1}(G) \neq f^r(G)$ then there exists $x \in f^{r-1}(G)$ such that $x \notin f^r(G)$. Hence $f(x) \in f^r(G)$ but $f(x) \notin f^{r+1}(G)$, i. e., $f^r(G) \neq f^{r+1}(G)$. This completes the induction, and f must be surjective as well as injective.

Next, assume that f is surjective but not injective. This time, we shall show by induction that for the ascending sequence

$$f^{-1}(0) \subseteq f^{-2}(0) \subseteq \ldots \subseteq f^{-r}(0) \subseteq \ldots$$

of Ω-ideals, we have no integer r such that $f^{-r}(0) = f^{-(r+1)}(0)$. If $r = 1$, $O \neq f^{-1}(0)$ since f is not injective. If $f^{-(r-1)}(0) \neq f^{-r}(0)$ then there exists $x \in G$ such that $f^{r}(x) = 0$ and $f^{r-1}(x) \neq 0$, and it follows that $f^{r+1}(x') = 0$ and $f^{r}(x') \neq 0$ where $x = f(x')$. That is, $f^{-r}(0) \neq f^{-(r+1)}(0)$. Thus we have obtained a contradiction to the second ascending chain condition, and f must be injective as well as surjective. Q. E. D.

If an Ω-group G satisfies the second descending chain condition and if f is a normal endomorphism of G, then it is clear that

$$H = \bigcap_{k \in N} f^{k}(G)$$

is an Ω-ideal of G. Also if an Ω-group G satisfies the ascending chain condition in 6. 14 and if f is any endomorphism of G

$$K = \bigcup_{k \in N} f^{-k}(0)$$

is an Ω-ideal of G. In general, such a subset K of the Ω-group G (regardless of the second ascending chain condition) is called the <u>radical</u> of f.

<u>Theorem 6.18.</u> Let G be an Ω-group satisfying the two chain conditions in 6. 14. Then $G = H \oplus K$ where $H = \bigcap_{k\in N} f^k(G)$ and $K = \bigcup_{k\in N} f^{-k}(0)$ provided that f is a normal endomorphism of G.

<u>Proof.</u> By hypothesis, there exists an integer r such that $H = f^r(G)$ and $K = f^{-r}(0)$. Since for any $x \in G$, $f^r(x) \in f^r(G) = f^{2r}(G)$, we have $f^r(x) = f^{2r}(y)$ for some $y \in G$. Hence $x = f^r(y) - f^r(y) + x$ and $f^r(-f^r(y)+x) = -f^{2r}(y)+f^r(x) = 0$. It follows that $x \in H + K$, i. e., $G = H + K$. Also, for any $x' \in H \cap K$, $x' = f^r(y')$ for some $y' \in G$, and $f^{2r}(y') = f^r(x') = 0$. This implies $y' \in K$ so that $x' = f^r(y') = 0$. Thus, $H \cap K = O$ and we obtain the result. Q. E. D.

<u>Corollary 6.19.</u> If G is an indecomposable Ω-group which satisfies the two chain conditions in 6. 14 then any normal endomorphism of G is either an automorphism or nilpotent.

<u>Lemma 6. 20.</u> Let G be an indecomposable Ω-group that satisfies the second descending and ascending chain conditions, and let f, g be normal endomorphisms of G such that $f(G) \cap g(G) = O$. Then $f + g$ is a nilpotent endomorphism provided that both f and g are nilpotent.

<u>Proof.</u> By 6. 9, $h = f + g$ is a normal endomorphism of G. Assume that h is not nilpotent, i. e., h is an automorphism of G.

Then h^{-1} is also an automorphism of G since for any $x_1, x_2, \ldots, x_n \in G$ there are $y_1, y_2, \ldots, y_n$ such that $h(y_i) = x_i$ so that for any $\omega \in \Omega_n^*$,

$$\omega(y_1, y_2, \ldots, y_n) = h^{-1}(\omega(h(y_1), h(y_2), \ldots, h(y_n)),$$

i.e.,

$$\omega(h^{-1}(x_1), h^{-1}(x_2), \ldots, h^{-1}(x_n)) = h^{-1}(\omega(x_1, x_2, \ldots, x_n)).$$

Now, if θ is any inner translation of G then $h\theta = \theta h$, and hence $\theta h^{-1} = h^{-1}\theta$, which shows that h^{-1} is normal. Therefore, both $\varphi = h^{-1}f$ and $\psi = h^{-1}g$ are normal endomorphisms of G. From

$$\varphi + \psi = 1_G,$$

it follows that $\varphi\psi = -\varphi^2 + \varphi^2 + \varphi\psi = -\varphi^2 + \varphi(\varphi+\psi) = -\varphi^2 + (\varphi+\psi)\varphi = -\varphi^2 + \varphi^2 + \psi\varphi = \psi\varphi$. Since φ and ψ are nilpotent, an application of the binomial formula to $(\varphi+\psi)^k$ for a sufficiently large integer k, gives

$$(\varphi+\psi)^k = 0.$$

Thus, we have a contradiction. That is $h = f + g$ must be nilpotent. Q. E. D.

<u>Theorem 6.21.</u> (The Krull-Schmidt Theorem for Ω-Groups). Let G be an Ω-group which satisfies the second ascending and

descending chain conditions, and let

$$G = H_1 \oplus H_2 \oplus \ldots \oplus H_s \tag{6.3.1}$$

$$G = K_1 \oplus K_2 \oplus \ldots \oplus K_t \tag{6.3.2}$$

be two complete decompositions of G. Then $s = t$ and for a suitable ordering of K_j there exists a normal automorphism f such that $f(K_j) = H_j$.

Proof. Assume that $s \geq t$. We make the following induction hypothesis: For each integer r, $1 \leq r \leq t$, there is a suitable ordering of $K_1, K_2, \ldots, K_{r-1}$ so that there exist normal endomorphisms $f_1, f_2, \ldots, f_{r-1}$ of G with the property $f_k : K_k \cong H_k$, $k < r$, and

$$G = K_1 \oplus K_2 \oplus \ldots \oplus K_{r-1} \oplus H_r \oplus \ldots \oplus H_s. \tag{6.3.3}$$

We wish to show that this also holds when $k = r$. Let $\varphi_1, \varphi_2, \ldots, \varphi_s$ and $\psi_1, \psi_2, \ldots, \psi_t$ be the projective endomorphisms obtained from (6.3.3) and (6.3.2), respectively (cf. (6.2.1)). Since $\sum_{j=1}^{t} \psi_j = 1_G$,

$$\varphi_r = \varphi_r\psi_1 + \varphi_r\psi_2 + \ldots + \varphi_r\psi_t.$$

But for all $x \in G$, $\varphi_k\psi_k(x) = \varphi_k(\psi_k(x)) = \psi_k(x)$ for $k < r$, so that

$\varphi_r\psi_k(x) = \varphi_r\varphi_k\psi_k(x) = 0$, i.e., $\varphi_r\psi_k = 0$. Hence,

$$\varphi_r = \varphi_r\psi_r + \varphi_r\psi_{r+1} + \dots + \varphi_r\psi_t$$

i.e.,

$$\varphi_r\psi_r + \varphi_r\psi_{r+1} + \dots + \varphi_r\psi_t = 1_{H_r}$$

on H_r. By Lemma 6.20, it follows that for some $q = r, r+1, \dots, t$, $\varphi_r\psi_q$ is not nilpotent. Thus, $\varphi_r\psi_q$ is an automorphism of H_r, and ψ_q is a monomorphism of H_r into K_q. Let

$$H = \varphi_r^{-1}(0) \cap K_q$$

$$K = \psi_q(H_r).$$

Clearly, H and K are Ω-ideals of G, and furthermore $K_q = H \oplus K$. For, if $z \in H \cap K$ then $\varphi_r(z) = 0$ and $z = \psi_q(y)$ for some $y \in H_r$, and hence $\varphi_r\psi_q(y) = 0$, which shows that $y = 0$, i.e., $z = 0$. Also, if $z \in K_q$ then $\varphi_r(z) \in H_r = \varphi_r\psi_q(H_r)$, so that $\varphi_r(z) = \varphi_r\psi_q(y)$ for some $y \in H_r$. This shows that $z = (z-\psi_q(y)) + \psi_q(y) \in H + K$.

It follows that $K_q = H$ or $K_q = K$, and since evidently $K \neq O$, we must have $K_q = K = \psi_q(H_r)$. Thus, ψ_q is an isomorphism of H_r onto K_q. This also implies that φ_r is an isomorphism of K_q onto H_r.

Let us reorder $K_r, K_{r+1}, \dots, K_t$ so that K_q becomes

K_r, i.e., φ_r is an isomorphism of K_r onto H_r, and ψ_r is an isomorphism of H_r onto K_r. We set

$$f_r = \varphi_r.$$

If $x \in K_r$ and $x \in K_1 + K_2 + \ldots + K_{r-1} + H_{r+1} + \ldots + H_s$ then $\varphi_r(x) \in H_r$ and $\varphi_r(x) = 0$. Since φ_r is an isomorphism of K_r onto H_r, it follows that $x = 0$, i.e.,

$$K_r \cap (K_1 + K_2 + \ldots + K_{r-1} + H_{r+1} + \ldots + H_s) = O.$$

Now, this equality implies $\psi_r \varphi_r(G) \cap \varphi_j(G) = O$, $j=1, 2, \ldots, s, j \neq r$, so that the mapping $g = \varphi_1 + \varphi_2 + \ldots + \varphi_{r-1} + \psi_r \varphi_r + \ldots + \varphi_s$ is a normal endomorphism of G. Since g is written as

$$g : b_1 + b_2 + \ldots + b_{r-1} + a_r + \ldots + a_s \mapsto b_1 + b_2 + \ldots + b_{r-1} + \psi_r(a_r) + \ldots + a_s$$

where $b_j \in K_j$ and $a_j \in H_j$, and is obviously injective, g is an automorphism of G (cf. 6.17). Therefore,

$$G = g(G) \subseteq K_1 \oplus K_2 \oplus \ldots \oplus K_r \oplus H_{r+1} \oplus \ldots \oplus H_s \subseteq G$$

or

$$G = K_1 \oplus K_2 \oplus \ldots \oplus K_r \oplus H_{r+1} \oplus \ldots \oplus H_s,$$

and this completes the induction.

Thus, we have $s = t$. Now we set

$$f = f_1\psi_1 + f_2\psi_2 + \ldots + f_s\psi_s.$$

f is a normal automorphism of G, and has the property $f(K_j) = H_j$. This completes the proof. Q. E. D.

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