TORSION OF RECTANGULAR SANDWICH PLATE

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TORSION OF RECTANGULAR SANDWICH PLATE[1]

By

SHUN CHENG, Engineer[2]

Forest Products Laboratory,[3] Forest Service
U. S. Department of Agriculture

I. INTRODUCTION

An elastic sandwich plate is a structural component consisting of two thin external members, called facings, separated by and bonded to a relatively thick internal member called the core. Sandwich cores are stiff in the direction perpendicular to the plane of the plate but relatively weak in the other two directions--much less stiff, in fact, than the facings in these two directions. Thus, certain stresses in the core are assumed to be negligible. These stresses are the normal stresses in the plane of the plate and the shear stresses associated with shear strains in these directions. That is, of the six components of the stress tensor, the three present in a plane stress problem are assumed to be absent here. The remaining three components, two shear stresses and one normal stress, are related to the corresponding strains by separate and unequal moduli. These assumptions have been used in many previous analyses and are known to represent actual sandwich construction very well.

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[2] The author prepared this report as a thesis for the degree of Doctor of Philosophy at the University of Wisconsin.

[3] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

The present study is concerned with the performance of sandwich plates under torsion within the linear range. McComb (1)[4] considered the torsion of shells having reinforcing cores. The shells were similar to facings, except that they enclosed the core. Moreover, he assumed the core to be isotropic and used the Saint Venant theory of torsion. Seide (2) considered the torsion of rectangular sandwich plates of the type under study here. He used the Saint Venant theory but did not make the simplifying assumption of negligible core shear stress in the plane of the plate.

In this report is presented a rather rigorous mathematical analysis of the torsion of rectangular sandwich plates, which is done to determine the limits for which the Saint Venant theory is satisfactory. Two analyses are therefore presented. In one the Saint Venant theory is used, although it does not satisfy the detail boundary conditions in regard to the applied load. In the other, a more rigorous treatment is used that satisfies all boundary conditions. The more rigorous treatment is given first.

In the rigorous analysis it is assumed that the torque to which the plate is subjected is produced by loads applied normal to the facings at or near the two corners of one end, and similarly but resisting loads applied at or near the other two corners. All other surfaces, including the edges of the core and facings, are free of boundary stress. The mathematical theory of elasticity is used with three simplifying assumptions:

(1) Core stiffness values associated with plane stress components are negligibly small, as previously explained.

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[4]Underlined numbers in parentheses refer to references at the end of the text.

(2) The facings are treated as isotropic solid membranes.

(3) One-half the load at a corner is applied to the top facing and half to the bottom facing.

Assumptions (2) and (3) were adopted to make the solution shorter, rather than appreciably simpler. If the third assumption is not made, the work is essentially doubled, because an almost exactly similar solution would need to be superimposed on the one given here. The added solution contributes to stresses by compressing the core but does not affect the distortion of the central plane and therefore has no effect on the computed torsional rigidity of the sandwich plate. Assumption (3) is therefore completely justified for this study. If the facings were not sufficiently thin to be considered membranes, they would be treated as thin plates with flexural rigidity rather than as in assumption (2). This would make only a small correction in the linear range.

The mathematical analysis given here for the torsion of a rectangular sandwich plate has many points of similarity to that used previously by Goodier and Hsu (3) and Raville (4) and (5) for bending of sandwich plates.

In the analysis using the Saint Venant theory, the mathematical theory of elasticity is used with the following two assumptions:

(1) Core stiffness values associated with plane stress components are negligibl small, as previously explained.

(2) The torque is applied by shear stresses applied at the ends of the plate. These stresses are distributed in the proper way to avoid variations in stresses with the longitudinal coordinate.

The second assumption characterizes the Saint Venant theory. It is noted that the mathematical theory of elasticity is used for the facing as well as for the core, and therefore the facings need not be thin in this analysis.

Finally, a rather simple formula is obtained from rather elementary analysis based on the Prandtl membrane analogy. This formula is found to be in good agreement with the Saint Venant analysis over the range of sandwich dimensions and properties for which the Saint Venant analysis is applicable.

II. NOTATION

x, y, z	rectangular coordinates (fig. 1).
a, b	half length and width of sandwich.
a_1, b_1	length and width of loaded area at corners along x and y. directions, respectively.
c	half thickness of core.
t	thickness of facings.
E, ν	Young's modulus of elasticity and Poisson's ratio of the facings.
G	$\frac{E}{2(1+\nu)}$, shear modulus of the facings.
E_c, G_{xz}, G_{yz}	Young's modulus and shear modulii of the core.
u, v, w	displacements of core in x, y, and z directions, respectively.
u', v', w'	displacements in the lower facing for the Saint Venant solution, displacements at the middle surface of the lower facing for the rigorous solution.

$A_{mn}, B_{mn}, C_m, D_m, K_m, F_n, H_n, L_n, A'_{mn}, C'_m, K'_m, F'_n, L'_n, A''_{mn}, C''_m, K''_m, F''_n, L''_n$	configuration parameters associated with displacements. These are sometimes written without subscripts in the general derivations.
a_{11mn}, a_{12mn}, etc.	coefficients of $A_{mn}, A'_{mn}, A''_{mn}$, in sets of three simultaneous equations.
$\sigma_z, \tau_{xz}, \tau_{yz}$	stresses in core.
τ'_{xz}, τ'_{xy}	shear stresses in lower facing.
N_x, N_{xy}, N_y	membrane forces per unit length in lower facing.
P	resultant force applied at or near a corner.
q	load per unit area
α_m	$\dfrac{(2m+1)\pi}{2a}$
β_n	$\dfrac{(2n+1)\pi}{2b}$
B_1	$\dfrac{P}{E_c ab}$
γ_m	$\left[\dfrac{G_{xz}}{G_{yz}} + \dfrac{2(1+\nu)G_{xz}}{Etc\alpha_m^2}\right]^{\frac{1}{2}}$
δ_n	$\left[\dfrac{G_{yz}}{G_{xz}} + \dfrac{2(1+\nu)G_{yz}}{Etc\beta_n^2}\right]^{\frac{1}{2}}$
λ_m	$\dfrac{2(2+\frac{t}{c})}{(1+\nu)(1-\frac{G_{xz}}{G_{yz}}) + \dfrac{2c(1-\nu^2)G_{xz}}{Et(\alpha_m c)^2}}$
θ_m	$1 + \lambda_m \dfrac{G_{xz}}{G_{yz}}$
λ_n	$\dfrac{2(2+\frac{t}{c})}{(1+\nu)(1-\frac{G_{yz}}{G_{xz}}) + \dfrac{2c(1-\nu^2)G_{yz}}{Et(\beta_n c)^2}}$

θ_n $$1 + \lambda_n \frac{G_{yz}}{G_{xz}}$$

Φ_m $$\frac{\left[\gamma_m^2\left(1 + \frac{t}{2c}\right) - \frac{t}{2c} - \frac{G_{xz}}{G_{yz}}\right]}{\left(1 - \frac{G_{xz}}{G_{yz}}\right)}$$

ϕ_m $$1 + \left(\frac{\Phi_m - 1}{\gamma_m^2}\right) \frac{G_{xz}}{G_{yz}}$$

Φ_n $$\frac{\left[\delta_n^2\left(1 + \frac{t}{2c}\right) - \frac{t}{2c} - \frac{G_{yz}}{G_{xz}}\right]}{\left(1 - \frac{G_{yz}}{G_{xz}}\right)}$$

ϕ_n $$1 + \left(\frac{\Phi_n - 1}{\delta_n^2}\right) \frac{G_{yz}}{G_{xz}}$$

σ_m $$\frac{-\left(2 + \frac{t}{c}\right)(1 - \phi_m)}{(1 - \phi_m)\left[\theta_m + \lambda_m + 1 + \frac{t}{c} + \left(2 + \frac{t}{c}\right)\alpha_m b \tanh \alpha_m b\right] + (\theta_m - 1)\left(\Phi_m + \phi_m + \frac{t}{c}\right)}$$

τ_m $$\frac{\left(2 + \frac{t}{c}\right)(1 - \theta_m)}{(1 - \phi_m)\left[\theta_m + \lambda_m + 1 + \frac{t}{c} + \left(2 + \frac{t}{c}\right)\alpha_m b \tanh \alpha_m b\right] + (\theta_m - 1)\left(\Phi_m + \phi_m + \frac{t}{c}\right)}$$

σ_n $$\frac{-\left(2 + \frac{t}{c}\right)(1 - \phi_n)}{(1 - \phi_n)\left[\theta_n + \lambda_n + 1 + \frac{t}{c} + \left(2 + \frac{t}{c}\right)\beta_n a \tanh \beta_n a\right] + (\theta_n - 1)\left(\phi_n + \Phi_n + \frac{t}{c}\right)}$$

τ_n $$\frac{\left(2 + \frac{t}{c}\right)(1 - \theta_n)}{(1 - \phi_n)\left[\theta_n + \lambda_n + 1 + \frac{t}{c} + \left(2 + \frac{t}{c}\right)\beta_n a \tanh \beta_n a\right] + (\theta_n - 1)\left(\phi_n + \Phi_n + \frac{t}{c}\right)}$$

M $2P(b - \frac{b_1}{2})$

θ angle of twist per unit length in radius

A_n, B_m parameters associated with equation (136).

III. MATHEMATICAL ANALYSIS (Rigorous Solution)

The sandwich plate consisting of two facings and a core is shown in figure 1.

1. The Core

It is assumed that

$$\sigma_x = \sigma_y = \tau_{xy} = 0 \tag{1}$$

$$\sigma_z = E_c \frac{\partial w}{\partial z} \tag{2}$$

$$\tau_{xz} = G_{xz} \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{3}$$

$$\tau_{yz} = G_{yz} \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{4}$$

where u, v, and w are displacements in the x, y, z directions respectively.

From summation of forces of a differential element of the core as shown in figure 2 the following equations of equilibrium are found

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{5}$$

$$\frac{\partial \tau_{xz}}{\partial z} = 0 \tag{6}$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \tag{7}$$

Substitution of equations (2), (3), (4) into (5), (6), (7) gives

$$E_c \frac{\partial^2 w}{\partial z^2} + G_{xz} \left(\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2}\right) + G_{yz}\left(\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2}\right) = 0 \tag{8}$$

$$\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial x} = 0 \tag{9}$$

$$\frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial z \partial y} = 0 \tag{10}$$

For the loading shown in figure 1 the displacement w is an even function of z and an odd function of x and y. With this restriction the following expressions are general solutions of equations (8), (9), (10),

$$w = c[A + B(\frac{z}{c})^2] \sin \alpha x \sin \beta y \tag{11}$$

$$u = -z[A' + \frac{1}{3}B(\frac{z}{c})^2] \alpha c \cos \alpha x \sin \beta y \tag{12}$$

$$v = -z[A'' + \frac{1}{3}B(\frac{z}{c})^2] \beta c \sin \alpha x \cos \beta y \tag{13}$$

where A', A'', B, α, β are arbitrary. As will be shown later the constants A' and A'' are for the purpose of satisfying the differential equations of equilibrium of the facings. The constant B is determined by the distribution of the applied load. The constant α is selected to make the boundary shear stress zero at $x = \pm a$ and β is selected to make the boundary shear stress zero at $y = \pm b$. Later subscripts will be attached to all of these constants. The requirements on boundary shear stress are met by setting

$$\cos \alpha a = 0 \tag{14}$$

$$\cos \beta b = 0 \tag{15}$$

2. The Facings

The simplification arising from applying half of the torsional load to the top facing and half on the bottom facing (figure 1), causes the stresses in the two facings to be equal in magnitude for all x, y coordinate points. Therefore only the lower facing is analyzed. Summation of forces of a differential element gives

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} - \tau_{xz}\Big|_{z = c} = 0 \tag{16}$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} - \tau_{yz}\Big|_{z = c} = 0 \tag{17}$$

The stresses in the facing are given by

$$N_x = \frac{Et}{1 - \nu^2}\left(\frac{\partial u'}{\partial x} + \nu \frac{\partial v'}{\partial y}\right) \tag{18}$$

$$N_y = \frac{Et}{1 - \nu^2}\left(\frac{\partial v'}{\partial y} + \nu \frac{\partial u'}{\partial x}\right) \tag{19}$$

$$N_{xy} = \frac{Et}{1 - \nu^2}\left(\frac{1 - \nu}{2}\right)\left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x}\right) \tag{20}$$

The expressions for the displacement components of points in either facing may be obtained by requiring displacement continuity between the core and facings at their interfaces (bonding surfaces), that is, the interface displacements of the facings must be equal to the interface displacements of the core.

The middle surface displacements of the facings may be expressed in terms of these interface displacements by assuming that u' and v' vary linearly through the facing thickness and that w' is constant through the facing thickness and equal to the displacements of the core at the bonding surfaces.

From these considerations

$$w' = w\Big|_{z=c} \tag{21}$$

$$u' = u\Big|_{z=c} - \frac{t}{2}\frac{\partial w'}{\partial x} \tag{22}$$

$$v' = v\Big|_{z=c} - \frac{t}{2}\frac{\partial w'}{\partial y} \tag{23}$$

3. <u>The Loading</u>

The assumption is made that the loading is transmitted directly through the facings to the core.

$$\sigma_z\Big|_{z=c} = q \tag{24}$$

$$\sigma_z\Big|_{z=-c} = -q \tag{25}$$

where q is the intensity of pressure on the top facing and the intensity of pull on the bottom facing. This is a good approximation because of the thinness of the facings and the relative small values on such nonlinear terms as $N_x \dfrac{\partial^2 w}{\partial x^2}$.

From equations (2) and (11)

$$\sigma_z\Big|_{z=c} = 2E_c B \sin \alpha x \sin \beta y \tag{26}$$

This shows that any distribution of loading q may be satisfied by superposing solutions with arbitrary values of B, α, and β. Let

$$\sigma_z \Big|_{z = c} = q = 2E_c \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} B_{mn} \sin \alpha_m x \sin \beta_n y \tag{27}$$

where in accordance with equations (14) and (15)

$$\alpha_m = \frac{(2m + 1)\pi}{2a}, \qquad \beta_n = \frac{(2n + 1)\pi}{2b}$$

By means of Fourier analysis

$$B_{mn} = \frac{1}{2abE_c} \int_{-a}^{a} \int_{-b}^{b} q \sin \alpha_m x \sin \beta_n y \, dxdy \tag{28}$$

4. <u>The Particular Solution</u>

The three constants, A, A', and A'' may be found in terms of B, α, and β by substituting the appropriate equations into equations (8), (16), and (17). Note the subscripts m and n are again omitted for this general discussion. This procedure leads to the following three equations respectively.

$$a_{11} A + a_{12} A' + a_{13} A'' = B \tag{29}$$

$$a_{21} A + a_{22} A' + a_{23} A'' = b_{11} B \tag{30}$$

$$a_{31} A + a_{32} A' + a_{33} A'' = b_{11} B \tag{31}$$

where

$$a_{11} = (\alpha c)^2 \frac{G_{xz}}{2E_c} + (\beta c)^2 \frac{G_{yz}}{2E_c} \tag{32}$$

$$a_{12} = - (\alpha c)^2 \frac{G_{xz}}{2E_c} \tag{33}$$

$$a_{13} = -(\beta c)^2 \frac{G_{yz}}{2E_c} \tag{34}$$

$$a_{21} = [(\alpha c)^2 + (\beta c)^2] \frac{t}{c} - \frac{2(1-\nu^2)c\, G_{xz}}{Et} \tag{35}$$

$$a_{22} = 2(\alpha c)^2 + (1-\nu)(\beta c)^2 + \frac{2(1-\nu^2)c\, G_{xz}}{Et} \tag{36}$$

$$a_{23} = (1+\nu)(\beta c)^2 \tag{37}$$

$$a_{31} = [(\alpha c)^2 + (\beta c)^2] \frac{t}{c} - \frac{2(1-\nu^2)c\, G_{yz}}{Et} \tag{38}$$

$$a_{32} = (1+\nu)(\alpha c)^2 \tag{39}$$

$$a_{33} = 2(\beta c)^2 + (1-\nu)(\alpha c)^2 + \frac{2(1-\nu^2)c\, G_{yz}}{Et} \tag{40}$$

$$b_{11} = -\left(\frac{2}{3} + \frac{t}{c}\right)[(\alpha c)^2 + (\beta c)^2] \tag{41}$$

Equations (29), (30), (31) may be solved for A, A', and A'' in terms of B, α, and β which as previously indicated may be selected to give by superposition, any desired loading. This particular solution is

$$w = c \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A_{mn} + B_{mn} (\tfrac{z}{c})^2] \sin \alpha_m x \sin \beta_n y \tag{42}$$

$$u = -z \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A'_{mn} + \tfrac{1}{3} B_{mn} (\tfrac{z}{c})^2] \alpha_m c \cos \alpha_m x \sin \beta_n y \tag{43}$$

$$v = -z \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A''_{mn} + \tfrac{1}{3} B_{mn} (\tfrac{z}{c})^2] \beta_n c \sin \alpha_m x \cos \beta_n y \tag{44}$$

$$\sigma_z = 2E_c \frac{z}{c} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} B_{mn} \sin \alpha_m x \sin \beta_n y \tag{45}$$

$$\tau_{xz} = G_{xz} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (A_{mn} - A'_{mn}) \alpha_m c \cos \alpha_m x \sin \beta_n y \tag{46}$$

$$\tau_{yz} = G_{yz} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (A_{mn} - A''_{mn}) \beta_n c \sin \alpha_m x \cos \beta_n y \tag{47}$$

This particular solution satisfies all of the boundary requirements on the core since $\cos \alpha_m a = 0$ and $\cos \beta_n b = 0$ as given previously. The constant B_{mn} is given by equation (28) and the constants A_{mn}, A'_{mn}, and A''_{mn} are found in terms of B_{mn} by means of equations (29), (30), (31).

5. <u>The Homogeneous Solutions</u>

The particular solution meets the requirements for the facings that $N_{xy} = 0$ at $x = \pm a$ and at $y = \pm b$.

However unfortunately it gives values of N_x at $x = \pm a$ and values of N_y at $y = \pm b$. Therefore other solutions must be found which if possible do not disturb those conditions already satisfied. This is accomplished by setting $B = 0$ in equations (29), (30), and (31). In order that a solution exist the determinant of the coefficients a_{11}, a_{12}, etc. must be zero. This leads to a cubic equation in α^2 and β^2. For a given α, a cubic in β^2 results. The three roots are as follows:

$$\beta^2 = - \alpha^2$$

$$\beta^2 = - \alpha^2$$

$$\beta^2 = - \alpha^2 \left[\frac{G_{xz}}{G_{yz}} + \frac{2(1 + \nu)c\, G_{xz}}{Et(\alpha c)^2} \right] = - \alpha^2 \gamma^2$$

This leads to an expression for w of the form

$$w = c \sum_{m=o}^{\infty} \sin \alpha_m x \left[\frac{C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{K_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right]$$

with corresponding expressions for u and v. These are considered homogeneous solutions because they do not contribute to the loading. Of the three terms, the first and second come from the double roots $\beta^2 = -\alpha^2$ and the third comes from the third root, $\beta^2 = -\alpha^2\gamma^2$.

The two constants D_m and K_m can be selected in terms of C_m so as not to disturb the two conditions, $\tau_{yz} = 0$ at $y = \pm b$ and $N_{xy} = 0$ at $y = \pm b$. This leaves C_m free to form a Fourier series expression for annihilating the force N_y at $y = \pm b$ given by the particular solution.

In a similar manner a given β in the cubic equation of α^2 and β^2 results in three roots for α^2 and three homogeneous solutions which for w takes the form

$$w = c \sum_{n=o}^{\infty} \sin \beta_n y \left[\frac{F_n \sinh \beta_n x + H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} + \frac{L_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right]$$

The constants H_n and L_n can be selected in terms of F_n so as not to disturb the condition $\tau_{xz} = 0$ at $x = \pm a$ and $N_{xy} = 0$ at $x = \pm a$. This leaves F_n free to form a Fourier series expression for annihilating the N_x at $x = \pm a$ produced by other solutions.

The details by which this is accomplished follow.

6. The Complete Solutions

From the foregoing analysis the complete expressions for core displacements may be written as follows:

$$w = c \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A_{mn} + B_{mn} (\frac{z}{c})^2] \sin \alpha_m x \sin \beta_n y$$

$$+ c \sum_{m=o}^{\infty} \sin \alpha_m x \left[\frac{C_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{K_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right] + c \sum_{n=o}^{\infty} \sin \beta_n y \left[\frac{F_n \sinh \beta_n x + H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{L_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \tag{48}$$

$$u = - z \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A'_{mn} + \frac{1}{3} B_{mn} (\frac{z}{c})^2] \alpha_m c \cos \alpha_m x \sin \beta_n y$$

$$- z \sum_{m=o}^{\infty} \alpha_m c \cos \alpha_m x \left[\frac{C'_m \sinh \alpha_m y + D_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{K'_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right] - z \sum_{n=o}^{\infty} \beta_n c \sin \beta_n y \left[\frac{F''_n \cosh \beta_n x + H_n \beta_n x \sinh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{L''_n \cosh \delta_n \beta_n x}{\cosh \delta_n \beta_n a} \right] \tag{49}$$

$$v = -z \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} [A''_{mn} + \frac{1}{3} B_{mn} (\frac{z}{c})^2] \beta_n c \sin \alpha_m x \cos \beta_n y$$

$$- z \sum_{m=o}^{\infty} \alpha_m c \sin \alpha_m x \left[\frac{C''_m \cosh \alpha_m y + D_m \alpha_m y \sinh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{K''_m \cosh \gamma_m \alpha_m y}{\cosh \gamma_m \alpha_m b} \right] - z \sum_{n=o}^{\infty} \beta_n c \cos \beta_n y \left[\frac{F'_n \sinh \beta_n x + H_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{L'_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \tag{50}$$

Substitution of these expressions (48), (49), (50) into equations (2), (3), (4) for core stresses and equations (18), (19), (20) for facing stresses by using equations (21), (22), (23) gives

$$\frac{\sigma_z}{E_c} = 2 \frac{z}{c} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} B_m \sin \alpha_m x \sin \beta_n y \tag{51}$$

$$\frac{\tau_{xz}}{G_{xz}} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (A_{mn} - A'_{mn}) \alpha_m c \cos \alpha_m x \sin \beta_n y$$

$$+ \sum_{m=o}^{\infty} \alpha_m c \cos \alpha_m x \left[(C_m - C'_m) \frac{\sinh \alpha_m y}{\cosh \alpha_m b} + \frac{(K_m - K'_m) \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right]$$

$$+ \sum_{n=o}^{\infty} \beta_n c \sin \beta_n y \left[(F_n + H_n - F''_n) \frac{\cosh \beta_n x}{\cosh \beta_n a} + (L_n - L''_n) \frac{\cosh \delta_n \beta_n x}{\cosh \delta_n \beta_n a} \right] \tag{52}$$

$$\frac{\tau_{yz}}{G_{yz}} = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} (A_{mn} - A''_{mn}) \beta_n c \sin \alpha_m x \cos \beta_n y$$

$$+ \sum_{m=o}^{\infty} \alpha_m c \sin \alpha_m x \left[(C_m + D_m - C''_m) \frac{\cosh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + (K_m - K''_m) \frac{\cosh \gamma_m \alpha_m y}{\cosh \gamma_m \alpha_m b} \right] + \sum_{n=o}^{\infty} \beta_n c \cos \beta_n y \left[(F_n - F'_n) \frac{\sinh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left. + \frac{(L_n - L'_n) \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \tag{53}$$

$$\frac{1 - \nu^2}{Et} N_x = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} R_{mn} \sin \alpha_m x \sin \beta_n y$$

$$+ \sum_{m=o}^{\infty} \sin \alpha_m x \left[\frac{I_m \sinh \alpha_m y + J_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{U_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right]$$

$$- \sum_{n=o}^{\infty} \sin \beta_n y \left[\frac{P_n \sinh \beta_n x + Q_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} + \frac{\nu_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \tag{54}$$

$$\frac{1 - \nu^2}{Et} N_y = \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} R'_{mn} \sin \alpha_m x \sin \beta_n y$$

$$- \sum_{m=o}^{\infty} \sin \alpha_m x \left[\frac{I'_m \sinh \alpha_m y + J'_m \alpha_m y \cosh \alpha_m y}{\cosh \alpha_m b} + \frac{U'_m \sinh \gamma_m \alpha_m y}{\gamma_m \cosh \gamma_m \alpha_m b} \right]$$

$$+ \sum_{n=o}^{\infty} \sin \beta_n y \left[\frac{P'_n \sinh \beta_n x + Q'_n \beta_n x \cosh \beta_n x}{\cosh \beta_n a} + \frac{\nu'_n \sinh \delta_n \beta_n x}{\delta_n \cosh \delta_n \beta_n a} \right] \tag{55}$$

$$\frac{1-\nu^2}{Et} N_{xy} = -\left(\frac{1-\nu}{2}\right)\left\{ \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \left[B_{mn}\left(\frac{2}{3} + \frac{t}{c}\right) + \frac{t}{c} A_{mn} + A'_{mn} \right.\right.$$

$$\left. + A''_{mn} \right] (\alpha_m c)(\beta_m c) \cos \alpha_m x \cos \beta_n y$$

$$+ \sum_{m=o}^{\infty} (\alpha_m c)^2 \cos \alpha_m x \left[\frac{\{C'_m + C''_m + \frac{t}{c} C_m + (1 + \frac{t}{c}) D_m\} \cosh \alpha_m y}{\cosh \alpha_m b} \right.$$

$$\left. + \frac{2(1 + \frac{t}{2c}) D_m \alpha_m y \sinh \alpha_m y}{\cosh \alpha_m b} + \frac{(K'_m + K''_m + \frac{t}{c} K_m) \cosh \gamma_m \alpha_m y}{\cosh \gamma_m \alpha_m b} \right]$$

$$+ \sum_{n=o}^{\infty} (\beta_n c)^2 \cos \beta_n y \left[\frac{\{F'_n + F''_n + \frac{t}{c} F_n + (1 + \frac{t}{c}) H_n\} \cosh \beta_n x}{\cosh \beta_n a} \right.$$

$$\left.\left. + \frac{2(1 + \frac{t}{2c}) H_n \beta_n x \sinh \beta_n x}{\cosh \beta_n a} + \frac{(L'_n + L''_n + \frac{t}{c} L_n) \cosh \delta_n \beta_n x}{\cosh \delta_n \beta_n a} \right] \right\} \quad (56)$$

where

$$\alpha_m = \frac{(2m + 1)\pi}{2a}, \qquad \beta_n = \frac{(2n + 1)\pi}{2b} \quad (57)$$

$$R_{mn} = \left[\left(\frac{1}{3} + \frac{t}{2c}\right) B_{mn} + \frac{t}{2c} A_{mn} \right] \left[(\alpha_m c)^2 + \nu (\beta_n c)^2 \right] + A'_{mn} (\alpha_m c)^2$$

$$+ A''_{mn} (\beta_n c)^2 \nu \quad (58)$$

$$R'_{mn} = \left[\left(\frac{1}{3} + \frac{t}{2c}\right) B_{mn} + \frac{t}{2c} A_{mn} \right] \left[\nu (\alpha_m c)^2 + (\beta_n c)^2 \right] + A'_{mn} \nu (\alpha_m c)^2$$

$$+ A''_{mn} (\beta_n c)^2 \quad (59)$$

$$I_m = (\alpha_m c)^2 \left[C'_m - \nu C''_m - \nu D_m \left(1 + \frac{t}{c}\right) + (1 - \nu) \frac{t}{2c} C_m \right] \tag{60}$$

$$I'_m = (\alpha_m c)^2 \left[- \nu C'_m + C''_m + D_m \left(1 + \frac{t}{c}\right) + (1 - \nu) \frac{t}{2c} C_m \right] \tag{61}$$

$$J_m = (\alpha_m c)^2 (1 - \nu)\left(1 + \frac{t}{2c}\right) D_m \tag{62}$$

$$J'_m = (\alpha_m c)^2 (1 - \nu)\left(1 + \frac{t}{2c}\right) D_m = J_m \tag{63}$$

$$U_m = (\alpha_m c)^2 [K'_m - \nu \gamma_m{}^2 K''_m + (1 - \nu \gamma_m{}^2) \frac{t}{2c} K_m] \tag{64}$$

$$U'_m = (\alpha_m c)^2 [-\nu K'_m + \gamma_m{}^2 K''_m + (-\nu + \gamma_m{}^2) \frac{t}{2c} K_m] \tag{65}$$

$$P_n = (\beta_n c)^2 [F''_n + H_n - \nu F'_n + (F_n + 2H_n - \nu F_n) \frac{t}{2c}] \tag{66}$$

$$P'_n = (\beta_n c)^2 [- \nu F''_n - \nu H_n + F'_n + (F_n - \nu F_n - 2 \nu H_n) \frac{t}{2c}] \tag{67}$$

$$Q_n = (\beta_n c)^2 (1 - \nu)\left(1 + \frac{t}{2c}\right) H_n \tag{68}$$

$$Q'_n = (\beta_n c)^2 (1 - \nu)\left(1 + \frac{t}{2c}\right) H_n = Q_n \tag{69}$$

$$V_n = (\beta_n c)^2 [L''_n \delta_n{}^2 - \nu L'_n + (\delta_n{}^2 - \nu) \frac{t}{2c} L_n] \tag{70}$$

$$V'_n = (\beta_n c)^2 [- \nu \delta_n{}^2 L''_n + L'_n + (1 - \nu \delta_n{}^2) \frac{t}{2c} L_n] \tag{71}$$

For a concentrated load at the corners B_{mn} is found from equation (28) to be

$$B_{mn} = \frac{P}{E_c ab} (-1)^{m+n} = B_1 (-1)^{m+n} \tag{72}$$

If the loading is uniformly distributed over a rectangular area a_1 by b_1 at the corners then

$$B_{mn} = \frac{P}{E_c ab} \frac{c^2}{a_1 b_1} \frac{\cos \alpha_m (a - a_1)}{\alpha_m c} \frac{\cos \beta_n (b - b_1)}{\beta_n c} \tag{73}$$

7. Determinations of Parameters in the Expressions of Displacements

By substituting equations (48), (49), (50), (52), (53), (54), (55), (56) into the equilibrium equations (8), (16), (17) the following relations are found

$$C'_m = C_m + \frac{2(2 + \frac{t}{c}) D_m}{[(1 - \nu)(\gamma_m{}^2 - 1) + 2(1 - \frac{G_{xz}}{G_{yz}})]} \tag{74}$$

$$C''_m = C_m (1 - \frac{G_{xz}}{G_{yz}}) + D_m + C'_m \frac{G_{xz}}{G_{yz}} \tag{75}$$

$$F'_n = F_n + \frac{2(2 + \frac{t}{c}) H_n}{[(1 - \nu)(\delta_n{}^2 - 1) + 2(1 - \frac{G_{yz}}{G_{xz}})]} \tag{76}$$

$$F''_n = F_n (1 - \frac{G_{yz}}{G_{xz}}) + H_n + F'_n \frac{G_{yz}}{G_{xz}} \tag{77}$$

$$K'_m = \frac{[\gamma_m{}^2 (1 + \frac{t}{2c}) - \frac{t}{2c} - \frac{G_{xz}}{G_{yz}}]}{(1 - \frac{G_{xz}}{G_{yz}})} K_m \tag{78}$$

$$K''_m = \frac{[\gamma_m{}^2 (1 + \frac{t}{2c} \frac{G_{xz}}{G_{yz}}) - \frac{G_{xz}}{G_{yz}} (1 + \frac{t}{2c})]}{\gamma_m{}^2 (1 - \frac{G_{xz}}{G_{yz}})} K_m \tag{79}$$

$$L'_n = \frac{[\delta_n{}^2 (1 + \frac{t}{2c}) - \frac{t}{2c} - \frac{G_{yz}}{G_{xz}}]}{(1 - \frac{G_{yz}}{G_{xz}})} L_n \tag{80}$$

$$L''_n = \frac{[\delta_n{}^2 (1 + \frac{t}{2c}\frac{G_{yz}}{G_{xz}}) - \frac{G_{yz}}{G_{xz}}(1 + \frac{t}{2c})]}{\delta_n{}^2 (1 - \frac{G_{yz}}{G_{xz}})} L_n \tag{81}$$

The requirement $N_{xy} = 0$ at $y = \pm b$ gives

$$C'_m + C''_m + \frac{t}{c} C_m + [1 + \frac{t}{c} + (2 + \frac{t}{c}) \alpha_m b \tanh \alpha_m b] D_m + K'_m$$
$$+ K''_m + \frac{t}{c} K_m = 0 \tag{82}$$

The requirement $\tau_{yz} = 0$ at $y = \pm b$ gives

$$C_m + D_m - C''_m + K_m - K''_m = 0 \tag{83}$$

The requirement $N_{xy} = 0$ at $x = \pm a$ gives

$$F'_n + F''_n + \frac{t}{c} F_n + [1 + \frac{t}{c} + (2 + \frac{t}{c}) \beta_n a \tanh \beta_n a] H_n + L'_n$$
$$+ L''_n + \frac{t}{c} L_n = 0 \tag{84}$$

The requirement $\tau_{xz} = 0$ at $x = \pm a$ gives

$$F_n + H_n - F''_n + L_n - L''_n = 0 \tag{85}$$

Solving equations (82), (83) by using equations (74), (75), (78), (79), gives

$$D_m = \frac{-(2 + \frac{t}{c})(1 - \phi_m) C_m}{(1 - \phi_m)[\theta_m + \lambda_m + 1 + \frac{t}{c} + (2 + \frac{t}{c}) \alpha_m b \tanh \alpha_m b] + (\theta_m - 1)(\Phi_m + \phi_m + \frac{t}{c})} \tag{86}$$

$$K_m = \frac{-(2 + \frac{t}{c})(\theta_m - 1)C_m}{(1 - \phi_m)[\theta_m + \lambda_m + 1 + \frac{t}{c} + (2 + \frac{t}{c})\alpha_m b \tanh \alpha_m b] + (\theta_m - 1)(\Phi_m + \phi_m + \frac{t}{c})} \tag{87}$$

Solving equations (84), (85) by using equations (76), (77), (80), (81), gives

$$H_n = \frac{-(2 + \frac{t}{c})(1 - \phi_n)F_n}{(1 - \phi_n)[\theta_n + \lambda_n + 1 + \frac{t}{c} + (2 + \frac{t}{c})\beta_n a \tanh \beta_n a] + (\theta_n - 1)(\phi_n + \Phi_n + \frac{t}{c})} \tag{88}$$

$$L_n = \frac{(2 + \frac{t}{c})(1 - \theta_n)F_n}{(1 - \phi_n)[\theta_n + \lambda_n + 1 + \frac{t}{c} + (2 + \frac{t}{c})\beta_n a \tanh \beta_n a] + (\theta_n - 1)(\phi_n + \Phi_n + \frac{t}{c})} \tag{89}$$

where the value of ϕ_m, θ_m, λ_m, Φ_m, ϕ_n, θ_n, λ_n, and Φ_n are as defined in "Notation."

By means of the Fourier sine transform of equations (54) and (55) it can be shown that the requirement, $N_x = 0$ at $x = \pm a$, would give

$$2\frac{c}{b}\sum_{m=o}^{\infty}(-1)^{m+n}(\alpha_m c)\left[\frac{-I_m - J_m(1 + \alpha_m b \tanh \alpha_m b)}{(\alpha_m c)^2 + (\beta_n c)^2} + \frac{2(\alpha_m c)^2 J_m}{[(\alpha_m c)^2 + (\beta_n c)^2]^2} - \frac{U_m}{(\beta_n c)^2 + (\gamma_m \alpha_m c)^2}\right] + \left[P_n \tanh \beta_n a + \frac{a}{c}(\beta_n c)Q_n + \nu_n \frac{\tanh \delta_n \beta_n a}{\delta_n}\right] = \sum_{m=o}^{\infty}(-1)^m R_{mn} \tag{90}$$

and the requirement, $N_y = 0$ at $y = \pm b$, would give

$$[I'_m \tanh \alpha_m b + \frac{b}{c}(\alpha_m c) J'_m + U'_m \frac{\tanh \gamma_m \alpha_m b}{\gamma_m}] + 2\frac{c}{a}\sum_{n=o}^{\infty}(-1)^{m+n}(\beta_n c)$$

$$\cdot \left[\frac{-P'_n - Q'_n(1 + \beta_n a \tanh \beta_n a)}{(\alpha_m c)^2 + (\beta_n c)^2} + \frac{2(\beta_n c)^2 Q'_n}{[(\alpha_m c)^2 + (\beta_n c)^2]^2}\right.$$

$$\left. - \frac{\nu'_n}{(\alpha_m c)^2 + (\delta_n \beta_n c)^2}\right] = \sum_{n=o}^{\infty}(-1)^n R'_{mn} \qquad (91)$$

where

$$\sum_{m=o}^{\infty}(-1)^m R_{mn} = (-1)^n \frac{P}{E_c ab}\sum_{m=o}^{\infty}\left\{\frac{1}{2}\left(\frac{2}{3} + \frac{t}{c} + \frac{t}{c}\frac{A_{mn}}{B_{mn}}\right)[(\alpha_m c)^2 + \nu(\beta_n c)^2]\right.$$

$$\left. + (\alpha_m c)^2 \frac{A'_{mn}}{B_{mn}} + \nu(\beta_n c)^2 \frac{A''_{mn}}{B_{mn}}\right\} \qquad (92)$$

$$\sum_{n=o}^{\infty}(-1)^n R'_{mn} = (-1)^m \frac{P}{E_c ab}\sum_{n=o}^{\infty}\left\{\frac{1}{2}\left(\frac{2}{3} + \frac{t}{c} + \frac{t}{c}\frac{A_{mn}}{B_{mn}}\right)[\nu(\alpha_m c)^2 + (\beta_n c)^2]\right.$$

$$\left. + \nu(\alpha_m c)^2 \frac{A'_{mn}}{B_{mn}} + (\beta_n c)^2 \frac{A''_{mn}}{B_{mn}}\right\} \qquad (93)$$

The first parts on the left side of equation (90) and (91) can be expressed in terms of C_m by means of equations (60), (61), (62), (63), (64), (65), (74), (75), (78), (79), (86), and (87).

The second parts on the left side of equations (90) and (91) can be expressed in terms of F_n by means of equations (66), (67), (68), (69), (70), (71), (76), (77), (80), (81), (88), and (89). Thus equations (90) and (91) may be solved for C_m and F_n in terms of $\frac{P}{E_c ab}$ for as many m and n as desired.

8. Determination of Torsional Rigidity $\frac{M}{\theta}$

The loads acting at the corners of the sandwich plate form a couple the magnitude of which is

$$M = 2P(b - \frac{b_1}{2}) \tag{94}$$

The angle of twist per unit length is

$$\theta = \frac{w\Big|_{z = 0,\ x = a,\ y = b}}{ab} \tag{95}$$

The displacement w is given by equation (48).

Thus the torsional rigidity can be expressed as

$$\frac{M}{\theta} = \frac{2E_c \left(\frac{a}{c}\right)^2 \left(\frac{b}{c}\right)^3 c^4}{A_t + B_t + C_t} \tag{96}$$

where

$$A_t = \frac{1}{(1 + \frac{t}{2c})} \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \frac{A_{mn}}{B_{mn}}$$

$$B_t = \sum_{m=o}^{\infty} (-1)^m \frac{C_m}{B_1} \left[\tanh \alpha_m b + \sigma_m(\alpha_m c) \frac{b}{c} + \tau_m \frac{\tanh \gamma_m \alpha_m b}{\gamma_m}\right]$$

$$C_t = \sum_{n=o}^{\infty} (-1)^n \frac{F_n}{B_1} \left[\tanh \beta_n a + \sigma_n(\beta_n c) \frac{a}{c} + \tau_n \frac{\tanh \delta_n \beta_n a}{\delta_n}\right]$$

IV. NUMERICAL COMPUTATIONS (Rigorous Solution)

For the most part the computations were made on an IBM 650 in the Numerical Analysis Laboratory of the University of Wisconsin. To facilitate the programming for these computations, the following nondimensional parameters were introduced, λ_m, σ_m, θ_m, Φ_m, τ_m, ϕ_m, λ_n, σ_n, θ_n, Φ_n, τ_n, and ϕ_n. These are all listed and defined in section II. Each of these parameters are functions of core and facing properties and the integers m and n.

By means of these new dimensionless parameters equations (74) to (81) and (86) to (89) were written as follows:

$$C'_m = (1 + \lambda_m \sigma_m) C_m \tag{97}$$

$$C''_m = (1 + \theta_m \sigma_m) C_m \tag{98}$$

$$F'_n = (1 + \lambda_n \sigma_n) F_n \tag{99}$$

$$F''_n = (1 + \theta_n \sigma_n) F_n \tag{100}$$

$$K'_m = \Phi_m \tau_m C_m \tag{101}$$

$$K''_m = \phi_m \tau_m C_m \tag{102}$$

$$L'_n = \Phi_n \tau_n F_n \tag{103}$$

$$L''_n = \phi_n \tau_n F_n \tag{104}$$

$$D_m = \sigma_m C_m \tag{105}$$

$$K_m = \tau_m C_m \tag{106}$$

$$H_n = \sigma_n F_n \tag{107}$$

$$L_n = \tau_n F_n \tag{108}$$

Then the equations (60) to (71) may be expressed as

$$I_m = (\alpha_m c)^2[(1 - \nu)(1 + \frac{t}{2c}) + \sigma_m(\lambda_m - \nu\theta_m - \nu - \nu\frac{t}{c})]C_m \tag{109}$$

$$I'_m = (\alpha_m c)^2[(1 - \nu)(1 + \frac{t}{2c}) + \sigma_m(\theta_m - \nu\lambda_m + 1 + \frac{t}{c})]C_m \tag{110}$$

$$J_m = (\alpha_m c)^2 (1 - \nu)(1 + \frac{t}{2c})\, \sigma_m C_m \tag{111}$$

$$J'_m = J_m \tag{112}$$

$$U_m = (\alpha_m c)^2 \left[\frac{t}{2c} + \Phi_m - \nu\gamma_m^2 (\phi_m + \frac{t}{2c})\right] \tau_m C_m \tag{113}$$

$$U'_m = (\alpha_m c)^2 \left[\gamma_m^2(\phi_m + \frac{t}{2c}) - \nu(\Phi_m + \frac{t}{2c})\right] \tau_m C_m \tag{114}$$

$$P_n = (\beta_n c)^2 \left[(1 - \nu)(1 + \frac{t}{2c}) + \sigma_n(\theta_n - \nu\lambda_n + 1 + \frac{t}{c})\right] F_n \tag{115}$$

$$P'_n = (\beta_n c)^2 \left[(1 - \nu)(1 + \frac{t}{2c}) + \sigma_n(-\nu\theta_n + \lambda_n - \nu - \nu\frac{t}{c})\right] F_n \tag{116}$$

$$Q_n = (\beta_n c)^2(1 - \nu)(1 + \frac{t}{2c})\, \sigma_n F_n \tag{117}$$

$$Q'_n = Q_n \tag{118}$$

$$V_n = (\beta_n c)^2 \left[\delta_n^2\phi_n - \nu\Phi_n + (\delta_n^2 - \nu)\frac{t}{2c}\right] \tau_n F_n \tag{119}$$

$$V'_n = (\beta_n c)^2 \left[- \nu\delta_n^2\phi_n + \Phi_n + (1 - \nu\delta_n^2)\frac{t}{2c}\right] \tau_n F_n \tag{120}$$

The procedure of numerical computations for torsional rigidity $\frac{M}{\theta}$ thus can be outlined as follows:

(1) Compute α_m, γ_m, λ_m, σ_m, θ_m, Φ_m, τ_m, ϕ_m, β_n, δ_n, λ_n, σ_n, θ_n, Φ_n, τ_n, and ϕ_n for the properties assumed for and as many m and n as considered desirable.

(2) Substitute these values into equations (97) to (120) to obtain all parameters C'_m, etc., occurring in the homogeneous solutions, in terms of C_m and F_n.

(3) Compute b_{11mn}, a_{11mn}, a_{12mn}, etc. as given by equations (32) to (41) and B_{mn} from equation (72) or (73) for each combination of m and n used.

(4) Solve equations (29) to (31) A_{mn}, A'_{mn}, A''_{mn} in terms of $B_1\left(= \frac{P}{E_c ab}\right)$. Note the subscript m and n have been omitted in the general expressions of equations (29) to (41).

(5) Compute the expressions $\sum\limits_{m=o} (-1)^m R_{mn}$ and $\sum\limits_{n=o} (-1)^n R'_{mn}$ as given by equations (92) and (93).

(6) Prepare the matrix of equations represented by equations (90) and (91). These are of the form

$$\sum_{m=o} c_{mn} C_m + f_n F_n = R_n$$

$$c_m C_m + \sum_{n=o} f_{mn} F_n = R_m$$

(7) Solve these equations for C_m and F_n.

(8) Compute $\frac{M}{\theta}$ by means of equation (96).

The principal results for the rigorous solution are given in table 1. The first four columns give assumed values of the core rigidities E_c and G_{yz} and the dimensions a and b. The last column gives computed values of the non-dimensional parameter $\frac{M}{\theta G t c^2 b}$. The remaining properties of the sandwich were assumed as follows:

$G_{xz} = 25,000$ p. s. i. $= 25$ k. s. i.

$E = \frac{32}{3} \times 10^6$ p. s. i.

$\nu = \frac{1}{3}$

$G = 4 \times 10^6$ p.s.i.

$c = 0.25$ inch

$t = 0.0125$ inch

Except as noted the loads were taken to be concentrated at corners.

All values shown are the result of summing over m from 0 to 10 and over n from 0 to 10. Other computations were made in which fewer terms were used. The results showed that very little change resulted from using terms with m or n in excess of 7. However, the indications were that round off errors were a greater source of error leading to the final results being accurate to perhaps only two significant digits.

Increases in the rigidities E_c and G_{yz} result in increased torsional rigidity. However, increasing E_c from 100 k.s.i. to ∞ only increased the rigidity factor from 14.9 to 15.23 and increasing G_{yz} from 5 k.s.i. to 10 k.s.i. increased the rigidity factor from 14.4 to 14.9. Increases in either the length 2a or the width 2b resulted in an increase in the rigidity factor.

V. SAINT VENANT SOLUTION

The derivations based on the Saint Venant theory are as follows:

Let the core displacements be

$$u = \theta \frac{z}{c} F(c, y) \tag{121}$$

$$v = -\theta xz \tag{122}$$

$$w = \theta xy \tag{123}$$

and the lower facing displacements be

$$u' = \theta\, F(z, y) \tag{124}$$

$$v' = -\,\theta xz \tag{125}$$

$$w' = \theta xy \tag{126}$$

Upon solving for stresses all are found to be zero except the core stress τ_{xz} and the facing stresses τ'_{xz} and τ'_{xy}. These are as follows:

$$\tau_{xz} = G_{xz}\, \theta \left[\frac{F(c, y)}{c} + y \right] \tag{127}$$

$$\tau'_{xz} = G\, \theta \left[\frac{\partial F}{\partial z} + y \right] \tag{128}$$

$$\tau'_{xy} = G\, \theta \left[\frac{\partial F}{\partial y} - z \right] \tag{129}$$

All equilibrium equations are identically satisfied except the following for the facing

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} = 0 \tag{130}$$

This requires that

$$\nabla^2 F = 0 \tag{131}$$

The boundary conditions are

$$\tau'_{xy} = 0 \qquad \text{at } y = \pm b \tag{132}$$

$$\tau'_{xz} = 0 \qquad \text{at } z = c + t \tag{133}$$

$$\tau_{xz} = \tau'_{xz} \qquad \text{at } z = c \tag{134}$$

$$u = u' \qquad \text{at } z = c \tag{135}$$

Condition (135) is satisfied by any function F. Equation (131) and condition (133) are satisfied by the following expression

$$F = -yz + c\sum_{n=o}^{\infty} \frac{A_n \sin \frac{(2n+1)\pi(z-c)}{2t} \sinh \frac{(2n+1)\pi y}{2t}}{\frac{(2n+1)\pi}{2t} \cosh \frac{(2n+1)\pi b}{2t}}$$

$$- c\sum_{m=o}^{\infty} B_m \sin \frac{(2m+1)\pi y}{2b} \left\{ \frac{\left[\cosh \frac{(2m+1)\pi(z-c)}{2b}\right]}{\frac{(2m+1)\pi}{2b}} \right.$$

$$\left. - \frac{\left[\tanh \frac{(2m+1)\pi t}{2b} \sinh \frac{(2m+1)\pi(z-c)}{2b}\right]}{\frac{(2m+1)\pi}{2b}} \right\} \qquad (136)$$

where the A's are determined from the condition (132) which requires that

$$\sum_{n=o}^{\infty} A_n \sin \frac{(2n+1)\pi(z-c)}{2t} = \frac{2z}{c} \quad \text{for } 0 \leq z - c \leq t \qquad (137)$$

This gives

$$A_n = \frac{8}{(2n+1)\pi} \left[1 + \frac{(-1)^n}{\frac{(2n+1)\pi c}{2t}} \right] \qquad (138)$$

The B's are determined from condition (134) which requires that

$$\sum_{n=o}^{\infty} \frac{A_n \sinh \frac{(2n+1)\pi y}{2t}}{\cosh \frac{(2n+1)\pi b}{2t}} + \sum_{m=o}^{\infty} B_m \sin \frac{(2m+1)\pi y}{2b} \left[\tanh \frac{(2m+1)\pi t}{2b} \right.$$

$$\left. + \frac{G_{xz}}{\frac{(2m+1)\pi c}{2b} G} \right] = 0 \qquad (139)$$

This gives

$$B_m = \frac{-(-1)^m \sum_{n=o}^{\infty} \dfrac{A_n \dfrac{(2n+1)t}{b}}{\left(\dfrac{2n+1}{2}\right)^2 + \left[\dfrac{(2m+1)t}{2b}\right]^2}}{\pi\left[\tanh\dfrac{(2m+1)\pi t}{2b} + \dfrac{G_{xz}}{\dfrac{(2m+1)\pi c}{2b}G}\right]} \tag{140}$$

The moment M is given by

$$M = \int_{-b}^{b}\int_{-c}^{c} \tau_{xz}\, y\, dy\, dz + 2\int_{-b}^{b}\int_{c}^{c+t} (\tau'_{xz}\, y - \tau'_{xy}\, z)\, dy\, dz \tag{141}$$

From which the torsional rigidity is found to be

$$\frac{M}{\theta} = 4Gc^4 \frac{b}{c}\frac{t}{c}\Bigg\{2\left[1 + \frac{t}{c} + \frac{1}{3}\left(\frac{t}{c}\right)^2\right] + \frac{t}{b}\sum_{n=o}^{\infty}\frac{A_n}{\left[\dfrac{(2n+1)\pi}{2}\right]^2}\left[\left(\frac{b}{c} - \frac{4t}{(2n+1)\pi c}\right)(-1)^n - 1\right] + \frac{b}{t}\sum_{m=o}^{\infty}\frac{B_m(-1)^m}{\left[\dfrac{\pi(2m+1)}{2}\right]^2}\left[\frac{2b}{(2m+1)\pi c}\left(1 - \frac{G_{xz}}{G} - \frac{1}{\cosh\dfrac{(2m+1)\pi t}{2b}}\right) + \tanh\frac{(2m+1)\pi t}{2b}\right]\Bigg\} \tag{142}$$

It is seen that equation (142), unlike equation (96), is independent of the plate length a, Young's modulus E_c and shear modulus G_{yz} of the core.

Table 2 gives results using the same values of G_{xz}, G, c, and t as previously used and for a wide range of b. The summations were taken over both m and n from 0 to 29.

The results given in the last column are of interest in connection with the elementary analysis to be discussed later. This column was suggested by figure 3 where the torsional rigidity $\frac{M}{\theta}$ is plotted against b.

The rigidity factors shown by the third column are higher than those given in table 1 for the same values of b. It is of interest that the value 15.23 for b = 10 inches in table 2 is also found in table 1 for $E_c = \infty$. This is considered to be a coincidence since the values in table 1 are only accurate to perhaps two significant digits.

For the core rigidities assumed, which are believed to be within the practical range, the Saint Venant theory gives values in excess of the rigorous theory by the following percentages depending on $\frac{a}{c}$ and $\frac{b}{c}$ ratios

$\frac{a}{c}$	$\frac{b}{c}$	percentage excess
80	40	3%
40	40	10%
20	20	28%
40	12	5%
20	12	28%
12	12	56%

VI. ELEMENTARY ANALYSIS

The lack of core stiffness causes the core stresses to be very small except for the edge regions where the major portion of shear forces are transmitted from one facing to the other. Therefore a rectangular sandwich in torsion performs similarly to a hollow tube in torsion. A well known approximate formula for the torsional stiffness of a thin hollow tube is the following

$$\frac{M}{\theta} = \frac{4GA^2}{\int \frac{ds}{t}}$$

where A is approximately the gross cross sectional area.

For the sandwich one of the A's in A^2 would be the gross area

$$A = 4b(c + t)$$

and the other A would be approximately

$$A = 4c(b - \delta)$$

where δ is a correction to the width b similar to that used in the torsion of solid rectangular shafts (7). δ would be a function of t, c, and b and the ratio of G_{xz} to G.

The integral $\int \frac{ds}{t}$ should be approximately that found from integrating along the facings only or $\frac{4b}{t}$.

From the foregoing one finds

$$\frac{M}{\theta Gtc(c + t)b} = 16\left(1 - \frac{\delta}{b}\right) \qquad (143)$$

or

$$\frac{M}{\theta G t c^2 b} = \frac{c + t}{c} 16(1 - \frac{\delta}{b}) \tag{144}$$

For the dimensions given this becomes $16.8(1 - \frac{\delta}{b})$ which is to be compared with $16.37(1 - \frac{.69}{b})$ given by table 2 for all except very small values of b. This is remarkable agreement but still better agreement would be found if $c + t$ were replaced by $c + \frac{t}{2}$.

VII. CONCLUSIONS

The primary conclusions reached from the foregoing analysis are:

1. The computed torsional rigidity obtained with the more rigorous solution is smaller than that based on the Saint Venant Theory. There is little difference, however, for relatively long sandwich plates. These results are reasonable, since the local bending deflections near the applied loads are taken into account by the more rigorous theory.

2. The effects of the rigidities E_c and G_{yz} are shown by the more rigorous theory. For loads applied to the facing surfaces, some stiffness, E_c, is required to keep the facings apart, and some stiffness, G_{yz}, is required to transmit the shear associated with bending. Increasing E_c from a reasonable value to infinity resulted in a small increase in overall torsional rigidity. Increasing G_{yz} also caused a small increase in torsional rigidity.

3. The more rigorous method shows that torsional rigidity per unit length increases as the length or width of the sandwich plate is increased, and approaches the value found by the Saint Venant method. Except for length-to-thickness or width-to-thickness ratios of less than about 40, the Saint Venant theory gives the torsional rigidity with sufficient accuracy.

4. The simple formula based on elementary theory and containing an empirical correction factor gives excellent agreement with the results from the Saint Venant theory.

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Table 1. --Torsional Rigidity of Sandwich Plate from Rigorous Solution

E_c	G_{yz}	a	b	$\frac{M}{\theta G t c^2 b}$
K. s. i.	K. s. i.	In.	In.	
100	10	10	10	13.9
100	10	15	10	14.5
100	10	20	10	14.9
100	10	25	10	14.8
100	10	28	10	15.2
100	10	32.5	10	14.3
100	10	40	10	14.9
100	10	60	10	14.9
100	10	5	5	11.0
100	10	8	5	12.2
100	10	12	5	13.0
100	10	16	5	13.5
100	10	3	3	8.05
100	10	5	3	9.82
100	10	8	3	11.45
100	10	10	3	11.92
100	5	20	10	14.4
100	1	20	10	12.26
∞	10	20	10	15.23
100	10	20	10	15.5*

*Load is uniformly distributed over an area 2.5" by 2" at each corner of the sandwich plate.

Table 2.--Torsional Rigidity of Sandwich Plate From Saint Venant Solution

b in.	$\frac{M}{\theta}$ Lb.-in.^{2}	$\frac{M}{\theta Gtc^2 b}$	$\frac{M}{\theta Gtc^2 b\,(1 - \frac{.69}{b})}$
.5	3,360	2.15	
1.0	19,100	6.13	19.7
1.5	41,900	8.94	16.55
2.0	66,800	10.69	16.32
2.5	92,300	11.81	16.31
3.0	118,000	12.59	16.35
4.0	169,000	13.52	16.34
5.0	220,000	14.08	16.33
6.0	270,000	14.40	16.27
7.0	322,000	14.72	16.33
8.0	374,000	14.96	16.37
9.0	425,000	15.11	16.37
10.0	476,000	15.23	16.36
15.0	732,000	15.62	16.37
20.0	988,000	15.81	16.37
30.0	1,500,000	16.00	16.37
40.0	2,000,000	16.00	16.28
80.0	4,050,000	16.20	16.34

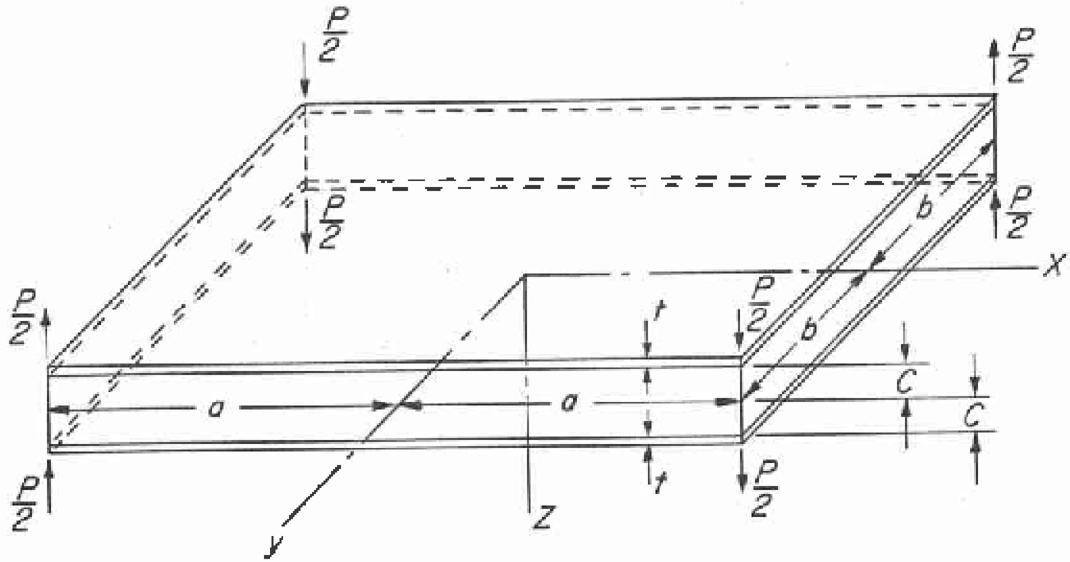

Figure 1. --Sketch of sandwich plate.

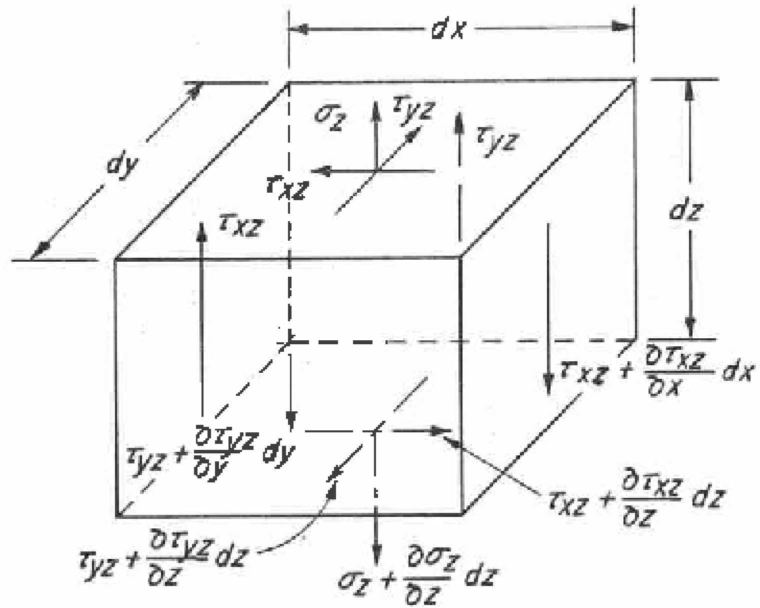

Figure 2. --Differential element of core.

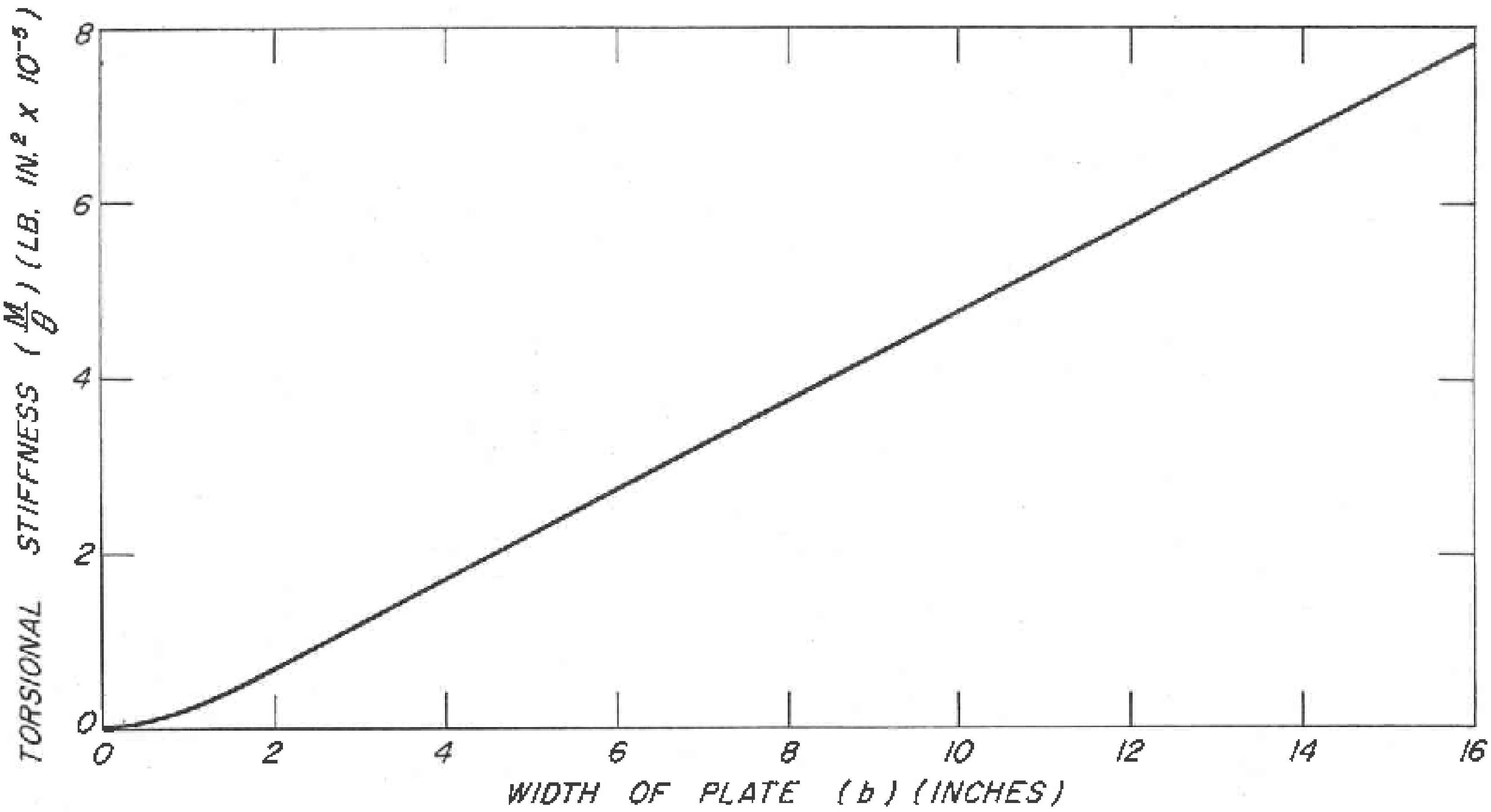

Figure 3.--Plot of torsional stiffness against width of panel as given by the Saint Venant Solution.

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