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MTH 338 W21

Using Euclidean Geometry to Construct Objects in the Elliptic Klein Disk

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Introduction to Elliptic Geometry and the Klein Disk Model

The geometry most people are familiar with and use in their day-to-day lives is Euclidean geometry. Different geometries adhere to different sets of postulates governing the properties and structures of objects in that geometry.

Elliptic geometry shares many postulates with Euclidean geometry. Two key differences between Euclidean and double elliptic geometries are that lines in double elliptic geometry are formed by connecting two antipodal (opposite) points with a Euclidean circular arc, and there are no parallel lines in elliptic geometry. A common model of single elliptic geometry is the Klein disk. Double elliptic geometry features antipodal points, a feature that single elliptic geometry lacks, but antipodal points can still be modeled on the Klein disk as points on the perimeter of the Klein disk located half a circumference away from each other. Antipodal points are considered the same elliptic point, though different Euclidean points. An elliptic line can be formed by connecting two antipodal points using a Euclidean circular arc. Since elliptic lines correspond to Euclidean circular arcs, this begs the question: How does one construct an elliptic line using Euclidean tools, without measurement postulates? Furthermore, if given an elliptic line, how does one construct a perpendicular line using Euclidean geometry without measurement postulates?

To accomplish these two tasks without measurement postulates, only a pencil, paper, straight-edge, and compass are needed. These two questions will be explored using a GeoGebra model of a Klein disk equipped with Euclidean tools. Given that a Euclidean circular arc corresponds to an elliptic line, the first question that will be investigated is what range of radii the circle forming the arc can be, bearing in mind the fact that the arc must lie within the Klein disk and intersect two antipodal points. Thus the location of the arc is also significant. When it comes to perpendicular lines, not only must the new line be a Euclidean circular arc intersecting two antipodal points, it also must form a 90° angle with the first line. Angles in elliptic geometry are measured with respect to the Euclidean tangent lines of the elliptic lines forming the angle at the vertex of the angle.

Part 1: Constructing Elliptic Lines

Elliptical lines are arcs of Euclidean circles. The endpoints of the arc must intersect the Klein disk's perimeter at two antipodal points. The goal of this section is to describe a method of drawing an elliptical line through a given point on the perimeter of the Klein disk using Euclidean tools when the location of the center point of the Klein disk is known.

Since an elliptical line connects two antipodal points, start by picking a point somewhere on the perimeter of the Klein disk. To find its antipodal point, draw a Euclidean line segment connecting the first point and the center of the Klein disk, extending the segment to intersect the other side of the Klein disk's perimeter. The intersection of the segment and the Klein disk's perimeter is the antipodal point, as shown in Figure 1 in which the first point chosen on the Klein disk's perimeter is point A and its antipodal point is point B.

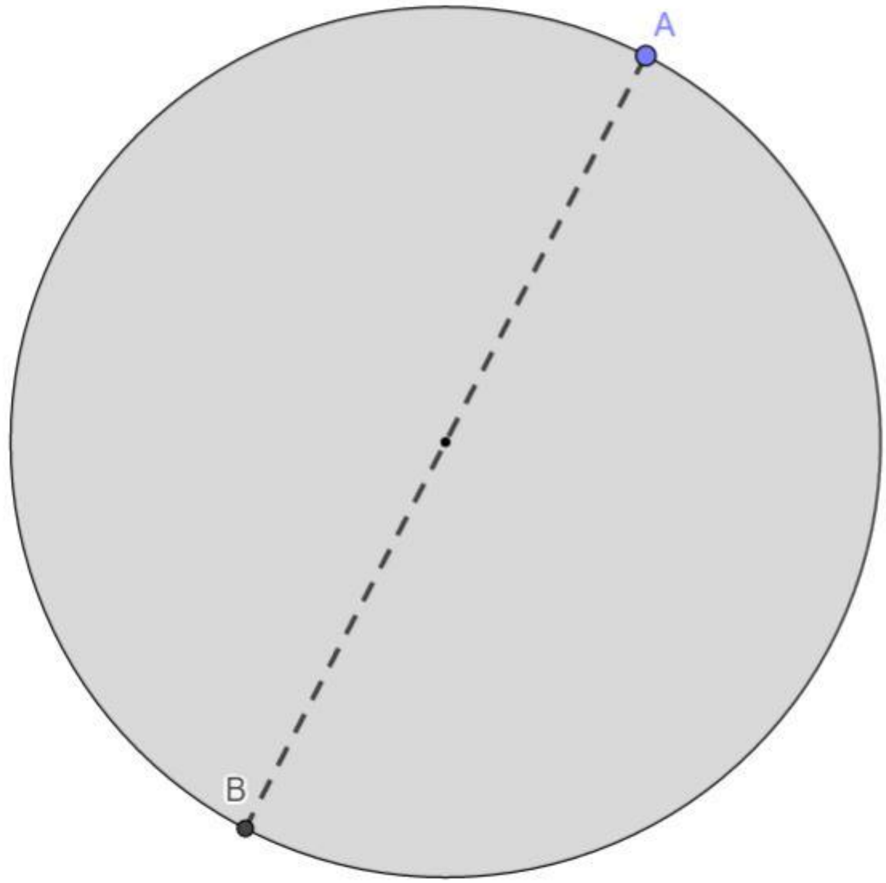

Figure 1: Points A and B are antipodal points on the Klein disk because the Euclidean segment (dotted) connecting points A and B intersects the center of the Klein disk.

Next use a Euclidean compass to draw a circle centered on one of the antipodal points with its radius equal to the distance between the two antipodal points. Repeat on the other antipodal point. The two circles intersect in two locations: points C and D shown in Figure 2. Draw a Euclidean line connecting their two intersections.

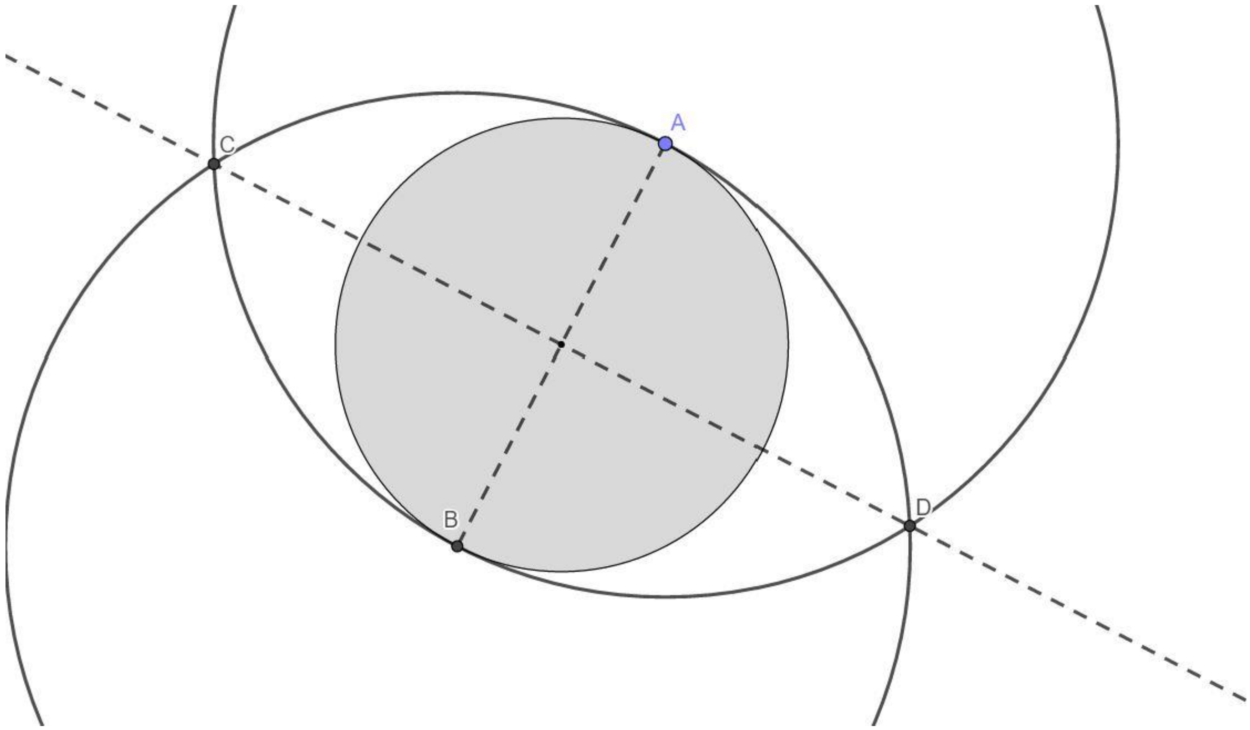

Figure 2: Since Euclidean line CD (dotted) is perpendicular to Euclidean segment AB (dotted), any elliptic line connecting points A and B is an arc of a Euclidean circle centered on line CD.

Any Euclidean circular arc drawn with its center on line CD connecting points A and B is an elliptic line. Figure 3a shows an example. If the center (point E) is located at the center of the Klein disk as shown in Figure 3b, the elliptic line is along the perimeter of the Klein disk. The farther away from the Klein disk's center point E is located, the wider the arc as shown in Figure 3c. If point E could be placed infinitely far away from the Klein disk, the elliptic line would be the Klein disk's diameter (Euclidean segment CD).

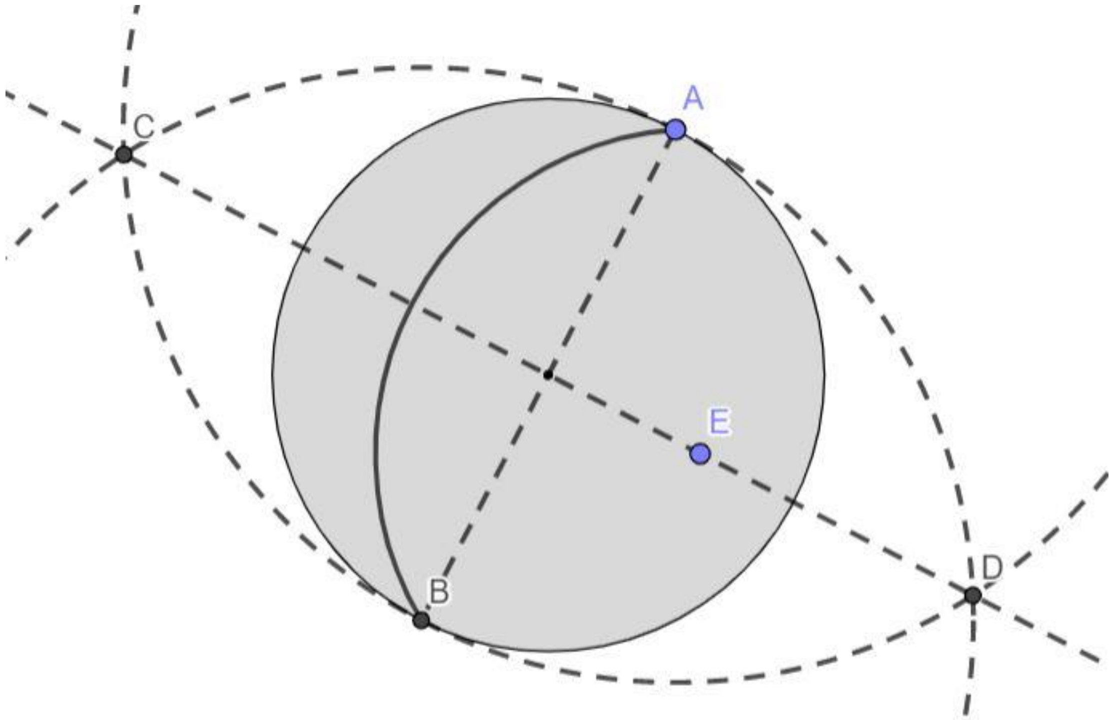

Figure 3a: The elliptic line connecting points A and B is an arc of a Euclidean circle centered at point E which lines on line CD.

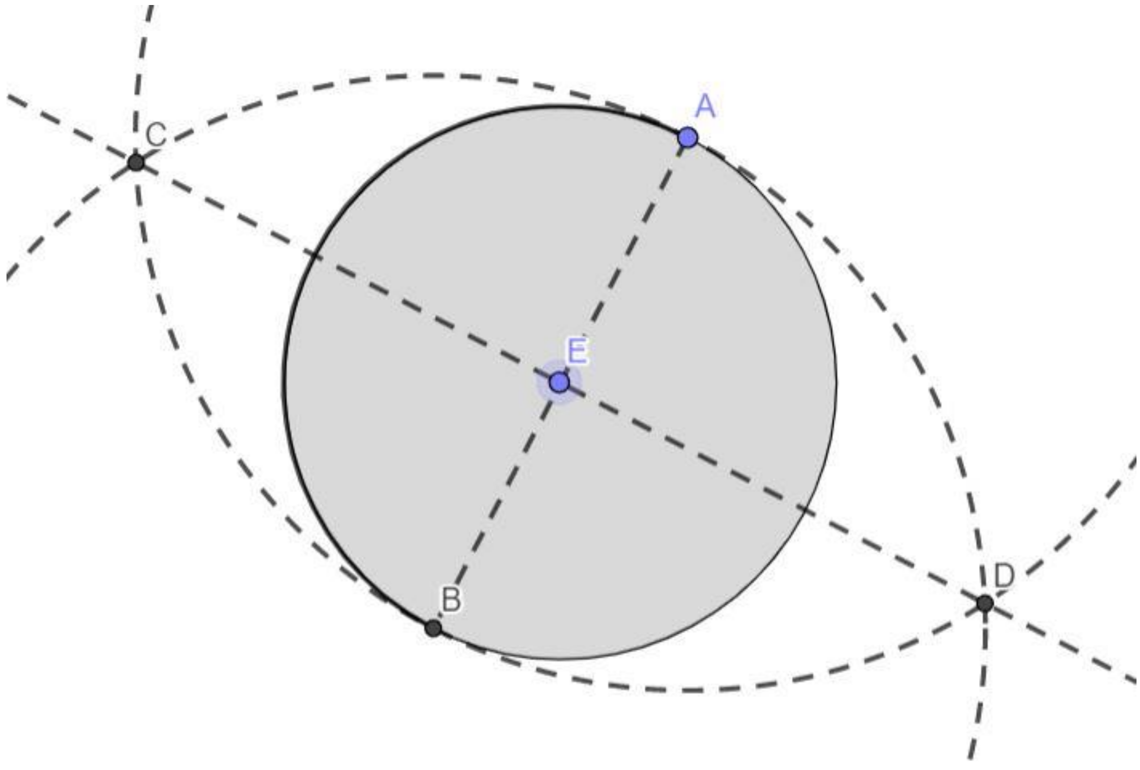

Figure 3b: If point E is located at the center of the Klein disk, then the elliptic line connecting points A and B is an arc of a Euclidean circle centered at the Klein disk's center, in which case the elliptic line is a semicircle of the Klein disk.

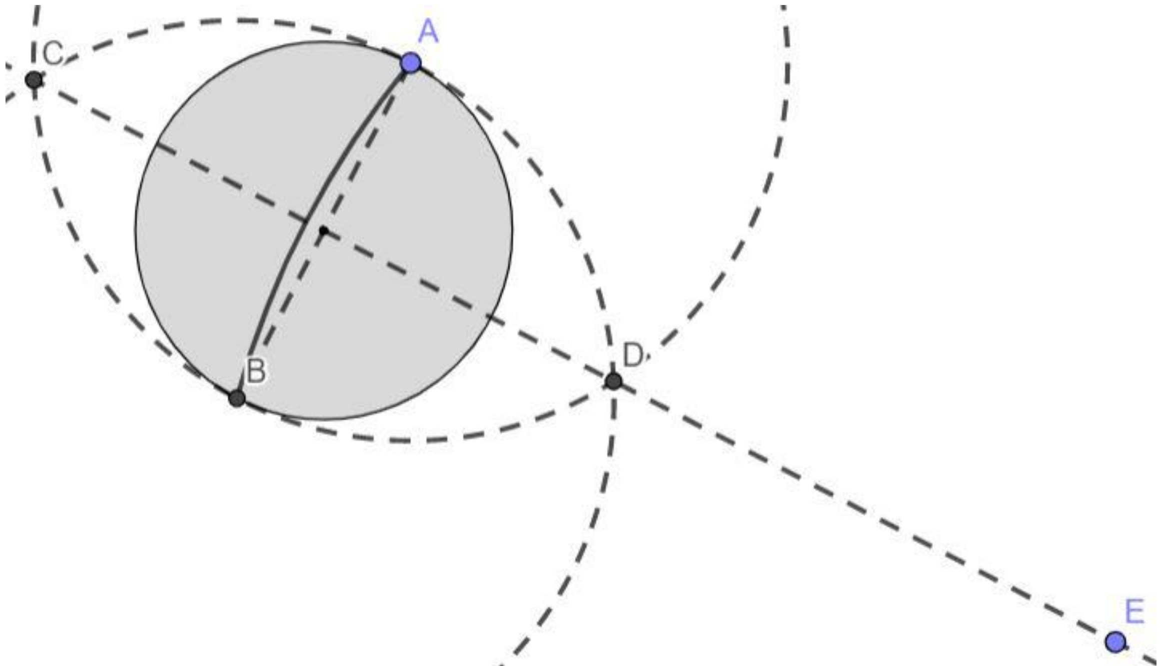

Figure 3c: The farther away point E is from the center of the Klein disk, the more the elliptic line resembles a Euclidean line. If point E were infinitely far from the Klein disk, the elliptic line would be a diameter of the Klein disk.

Part 2: Constructing Perpendicular Elliptic Lines

Angles in the Klein disk are measured with respect to the Euclidean tangent lines of the elliptic lines forming the angle. Perpendicular lines in elliptic geometry are defined the same way as in Euclidean geometry: Perpendicular lines form four 90° angles at their intersection. However in Euclidean geometry, perpendicular lines (and in fact any two distinct lines) only intersect at one point, while in elliptic geometry lines – perpendicular or not – can intersect in one or two points depending on the model used. In the spherical model of elliptic geometry, perpendicular lines intersect in two points which are antipodal points of each other as shown in Figure 4. In the Klein disk model, perpendicular lines only intersect in one point.

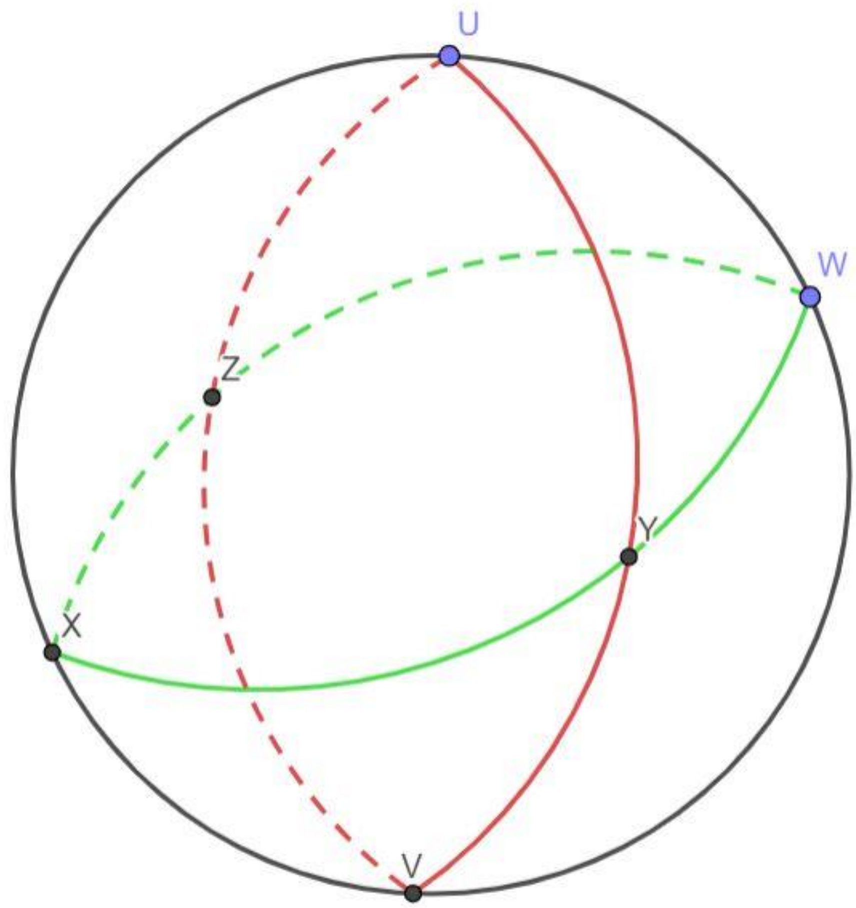

Figure 4: In the spherical model of double elliptic geometry, lines UV and XY intersect at two points: Z and Y. Point Y is on the front-facing hemisphere and point Z is on the back-facing hemisphere.

To form perpendicular elliptic lines in the Klein disk, begin with any elliptic line or segment and choose any point (point C in Figure 5) on the line as the intersection with the perpendicular line. First one must locate the center of the Euclidean circle forming the elliptic line or segment, if the line/segment was not drawn using the Euclidean method in Part 1. The center can be found by drawing two Euclidean chords (segments each intersecting a Euclidean circle at two points) on the elliptic line/segment and bisecting each chord using two Euclidean circles as shown in Figure 2. The intersection of the bisectors is the center of the Euclidean circle (point D), as shown in Figure 5. If an elliptic segment was given, it can be extended to form an elliptic line by drawing a Euclidean circle centered on point D with its perimeter on the elliptic segment, but this perpendicular line construction does not require an elliptic line to begin with – an elliptic segment is sufficient.

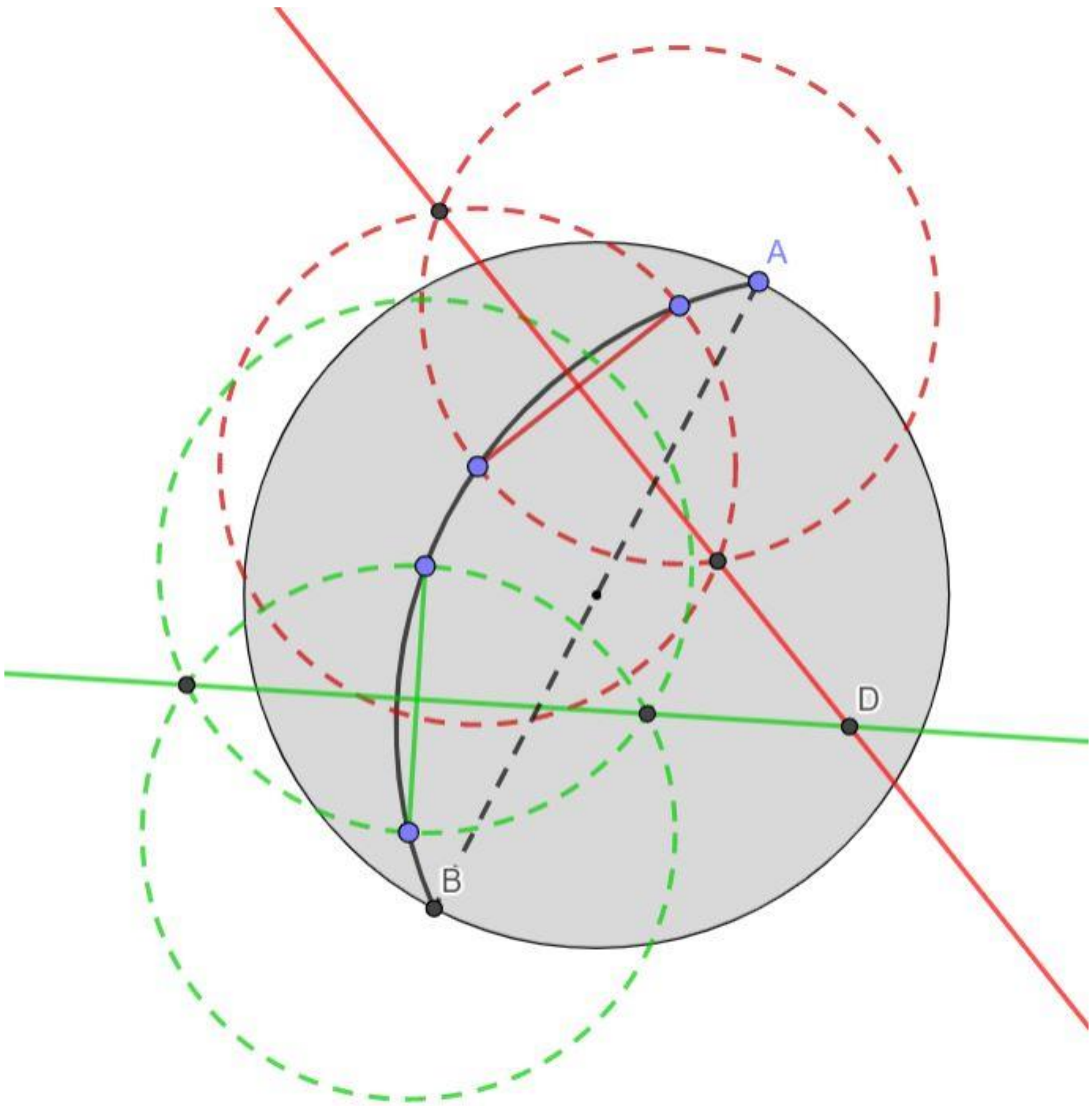

Figure 5: If given an elliptic line which is a Euclidean circular arc of unknown center such as elliptic line AB, the center can be found using two chords (solid red and green) and finding the intersection of their perpendicular bisectors, point D. Point D is the center of the Euclidean circle of which elliptic line AB is an arc.

Next, draw a Euclidean segment connecting points C and D. Since elliptic angles are measured using the Euclidean tangent lines to the elliptic lines forming the angle, segment CD must be tangent to the perpendicular elliptic line at point C. Thus, the center of the Euclidean circle of which the elliptic line is an arc must lie along a Euclidean line perpendicular to segment CD at point C. To draw a Euclidean line perpendicular to segment CD at point C, first draw a Euclidean circle centered at point C intersecting point D. Next extend segment CD so that it intersects the other side of the circle as shown in Figure 6, where point E is the intersection.

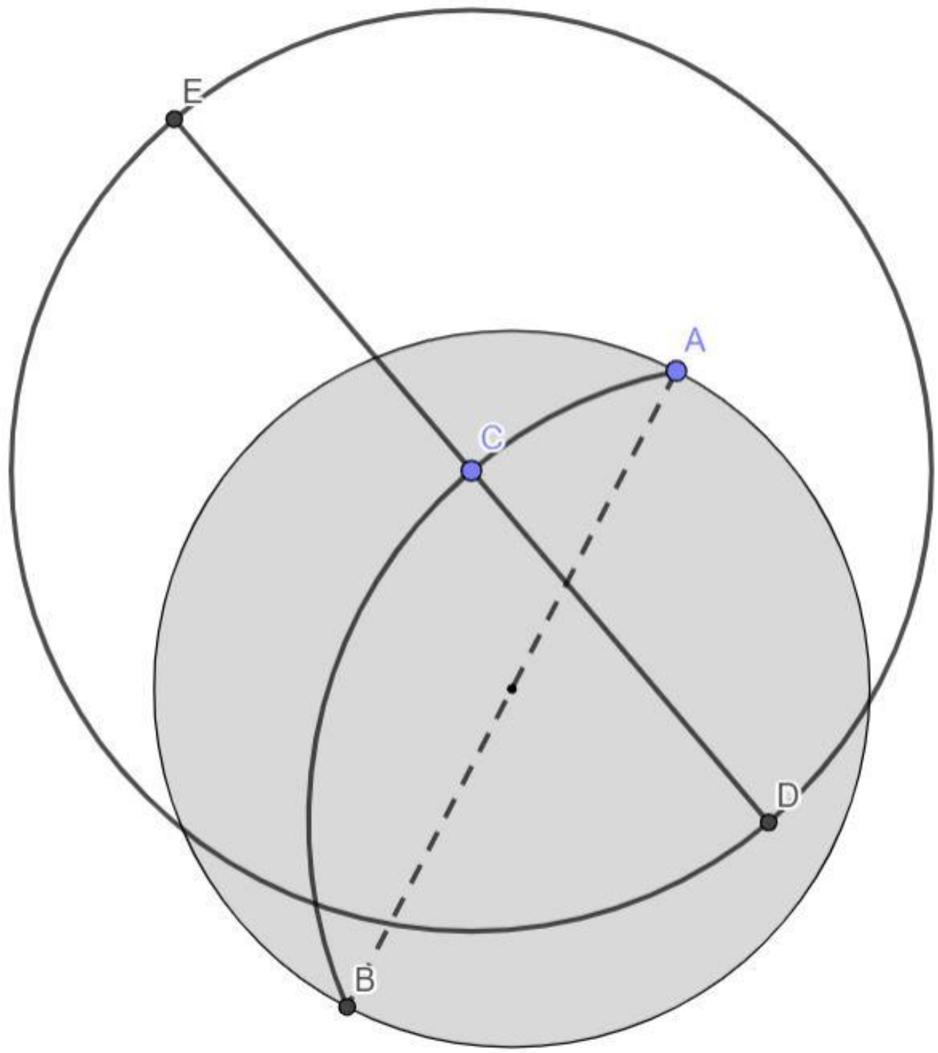

Figure 6: The Euclidean circle centered at point C has segment CD as its radius, so by extending the radius to find point E, the perpendicular bisector of segment ED will be found.

Draw a Euclidean circle centered at point E intersecting point D, as well as a Euclidean circle centered at point D intersecting point E. These two circles intersect at two points (points F and G in Figure 7). Draw a line connecting their intersections as shown in Figure 7.

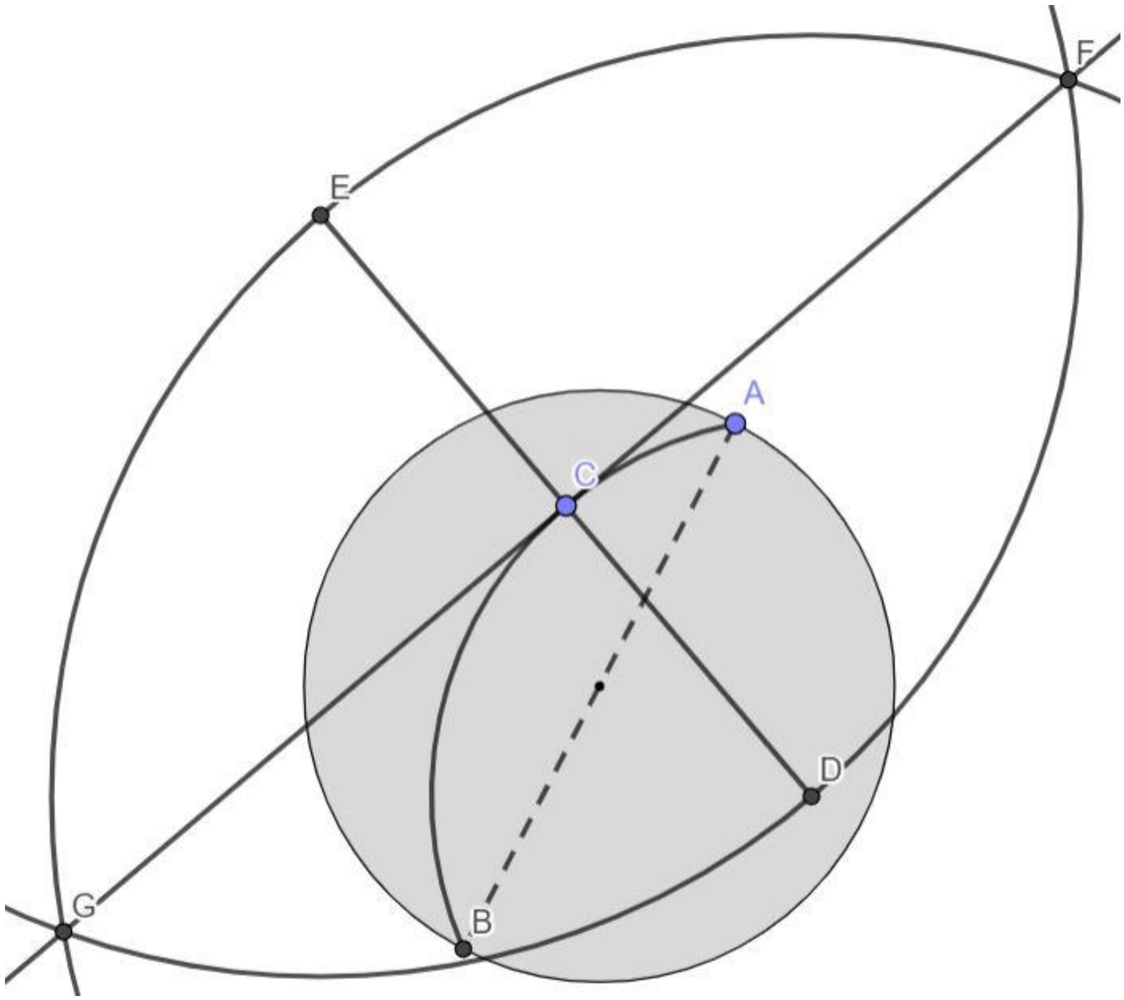

Figure 7: Line FG is perpendicular to segment ED, so an elliptic line intersecting point C perpendicular to elliptic line ACB must also be perpendicular to line FG.

The center of the Euclidean circle of which the elliptic line perpendicular to line AB is an arc must lie along line FG. To determine where the center of the circle is located, we will exploit a property of the Klein disk model: Any elliptic line intersecting a given point is an arc of a Euclidean circle with its center on one Euclidean line. In other words, all Euclidean arcs intersecting point C are arcs of circles with their center points located on a single Euclidean line. The line must intersect point D, since elliptic line AB, which contains point C, is a Euclidean arc of a circle centered on point D. To determine the Euclidean line along which the center of the perpendicular Euclidean arc's circle must lie, we must find the center of the Euclidean circle forming one other elliptic line that intersects point C.

To draw an elliptic line intersecting any given point in the Klein disk (in this case point C), pick any point on the perimeter of the Klein disk and locate its antipodal point by drawing a Euclidean segment from the chosen point to the opposite side of the perimeter of the Klein disk. The two antipodal points are points H and I in Figure 8. Then draw two Euclidean segments, one connecting point H to point C and the other connecting point C to point I as shown in Figure 8.

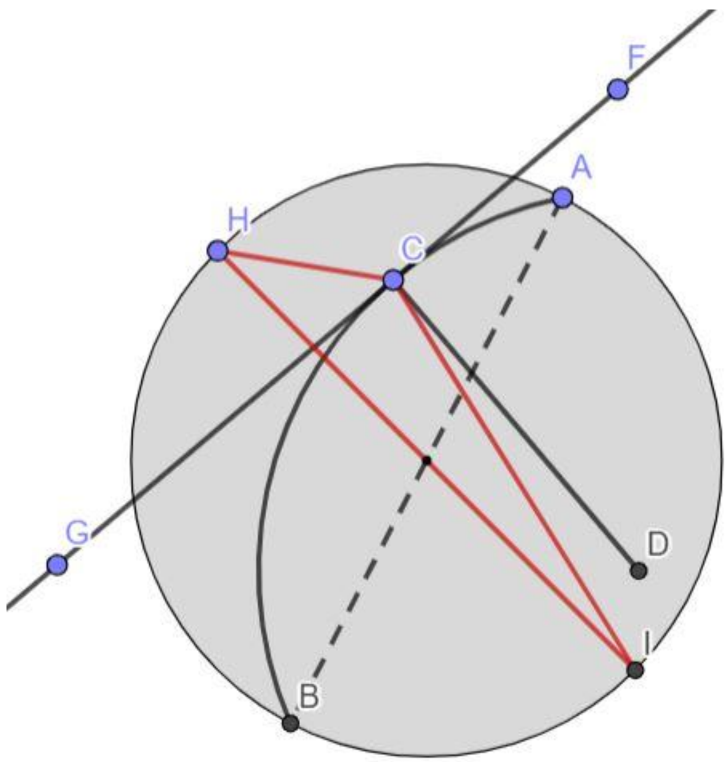

Figure 8: Points F and G have been moved along line FG to accommodate the size of the figures. Segments HC and CI are chords of the Euclidean circle of which an elliptic line through points H, C, and I would be an arc.

Segments HC and CI are chords of the Euclidean circle of which an elliptic line intersecting points H, C, and I is an arc. To locate the center of the circle, bisect each of the segments (HC and CI) by drawing a Euclidean circle centered at each endpoint intersecting the other endpoint, then connect the intersections of the circles using Euclidean lines. The intersection of the two resulting Euclidean lines, shown in Figure 9, is the center of the desired Euclidean circle (point J).

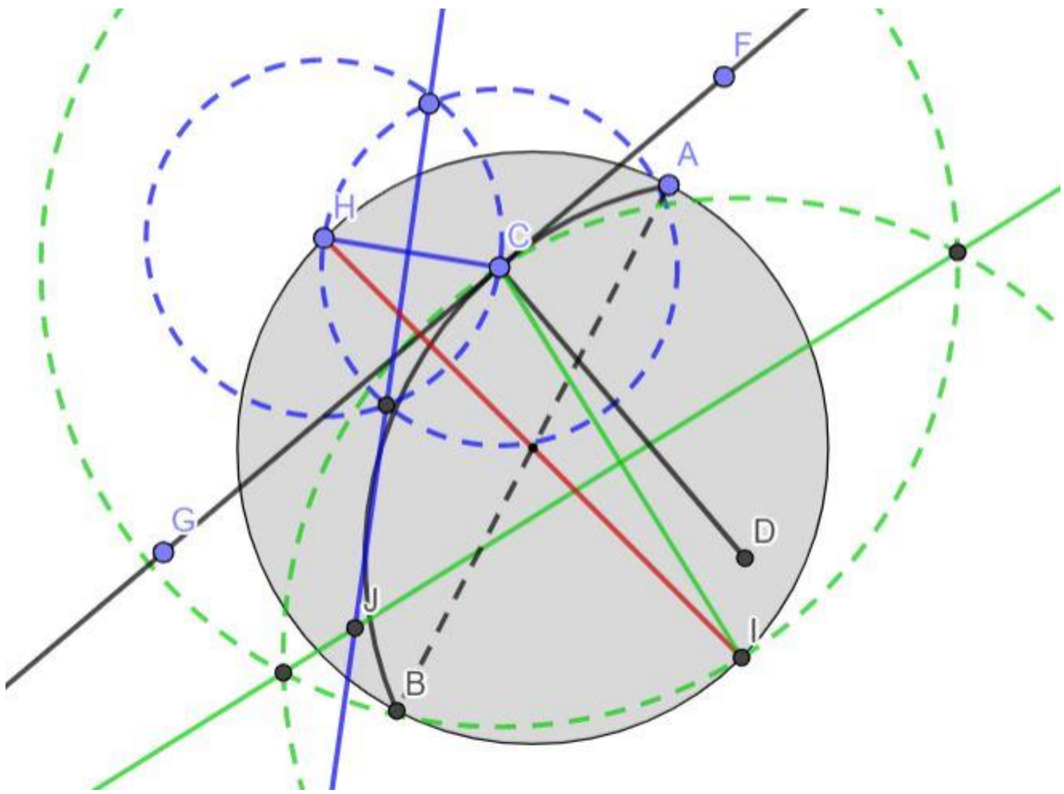

Figure 9: The blue and green constructions show the perpendicular bisectors of segments HC and CI, found using the intersections of circles centered on the endpoints with radius equal to the lengths of the segments. The intersection of the two perpendicular bisectors (solid blue and green lines) is point J, which is the center of the Euclidean circle of which an elliptic line through points H, C, and I must be an arc.

Connect points J and D using a Euclidean line. The intersection of line JD and line FG (point K) is the location of the center of the Euclidean circle of which the elliptic line perpendicular to elliptic line ACB is an arc. To draw the perpendicular elliptic line, draw a Euclidean arc centered on point K intersecting point C as shown in Figure 10, with the endpoints of the arc intersecting the perimeter of the Klein disk. The two intersections are points L and M.

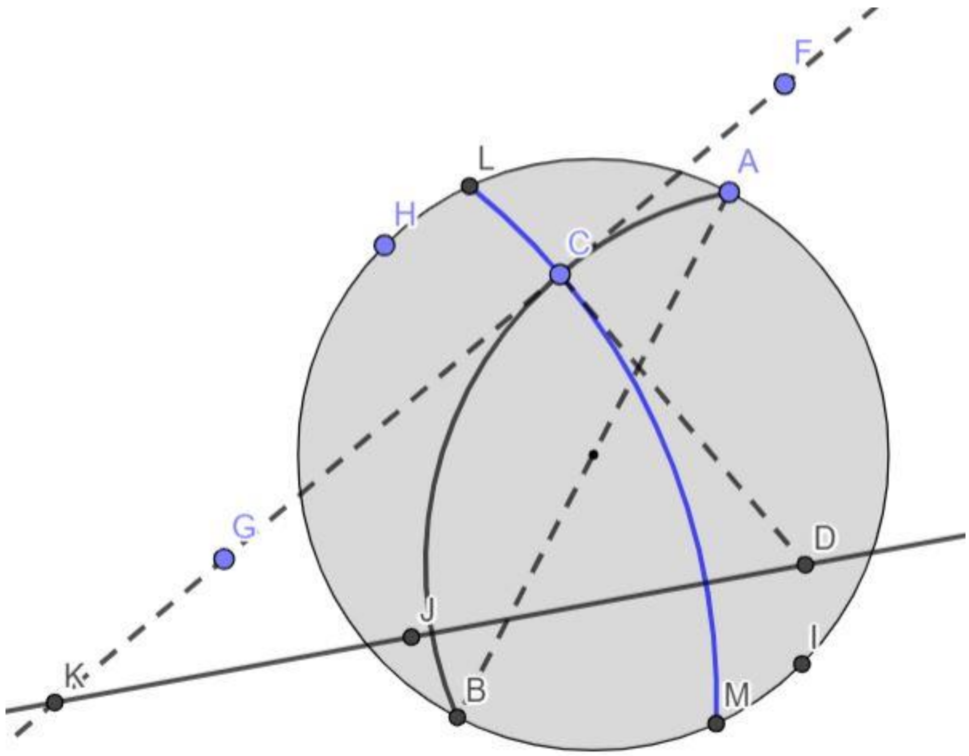

Figure 10: Since any elliptic line through point C must be a Euclidean circular arc with its center lying on a distinct Euclidean line, and points D and J are the centers of two Euclidean circular arcs which are elliptic lines through point C, any elliptic line through point C must be a Euclidean circular arc with its center located on line JD. Since JD intersects line FG at point K, point K is the location of the center of a Euclidean circular arc which is an elliptic line perpendicular to elliptic line ACB at point C. By drawing an arc through point C centered at point K, the blue elliptic line LCM was constructed.

To verify that arc KL is an elliptic line, draw a Euclidean line through point K and the center of the Klein disk. Since the line intersects point L, arc KL is an elliptic line as shown in Figure 11.

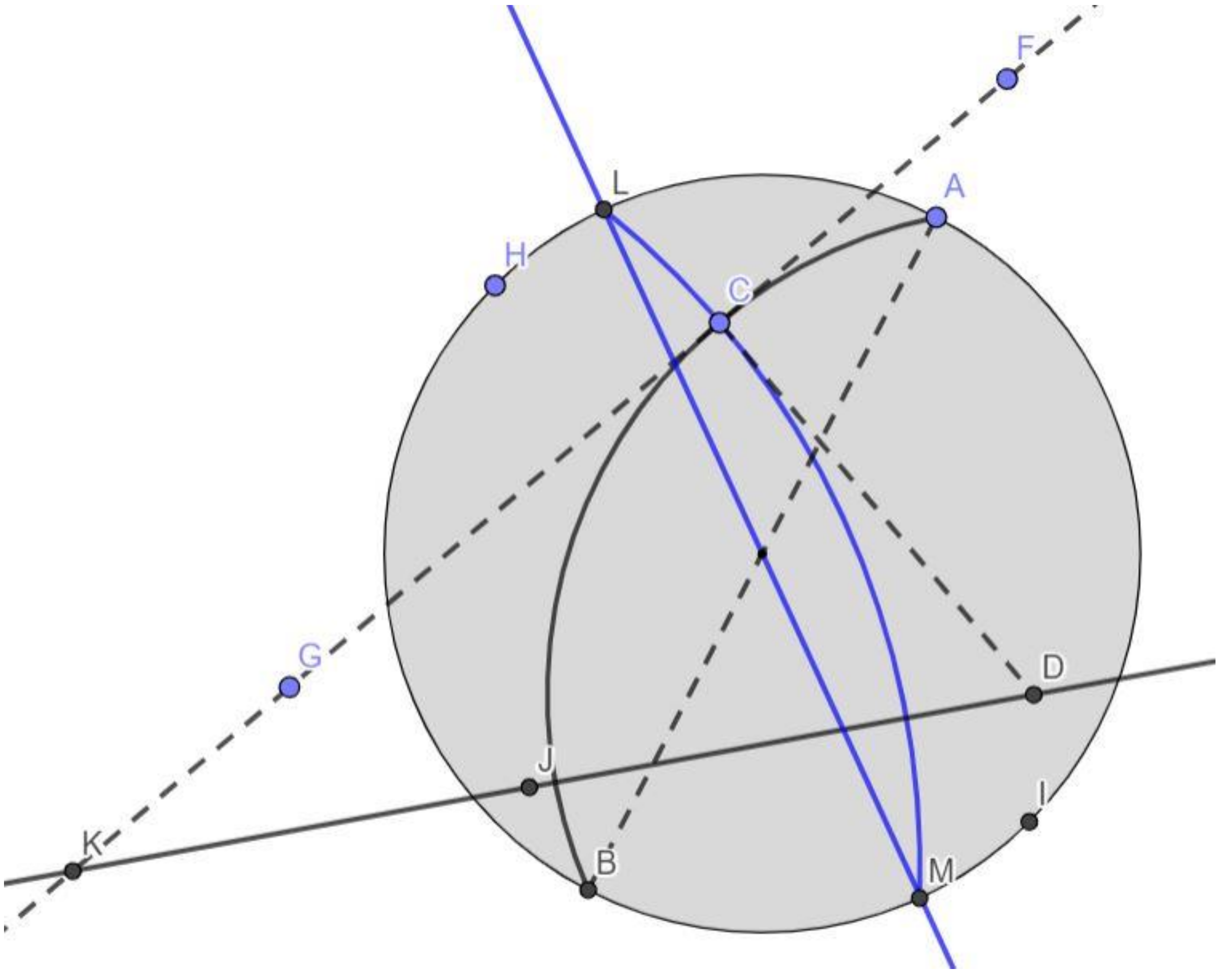

Figure 11: Euclidean line LM passes through the center of the Klein disk, so Euclidean circular arc LM is an elliptic line.

Conclusion

Using Euclidean tools but no measurement postulates, it was shown in Part 1 that it is possible to construct elliptic lines in the Klein. The construction of elliptic lines is accomplished by placing a point on the Klein disk's perimeter, locating the antipodal point, and using Euclidean circles to perpendicularly bisect the Euclidean line connecting the two antipodal points. An elliptic line through the two antipodal points is any Euclidean arc connecting the two antipodal points that is an arc of a circle centered on a any point lying on the perpendicular bisector. In Part 2 it was shown that is it possible to construct an elliptic line perpendicular to a given elliptic line or segment using Euclidean tools and no measurement postulates. The construction of perpendicular elliptic lines is accomplished by exploiting the property that all elliptic lines through a given point are arcs of Euclidean circles whose centers lie along a distinct Euclidean line. Both of these constructions are possible with only a Euclidean compass and straightedge.