SUMMATION AND TABLE OF FINITE SUMS

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ROBERT DELMER STALLEY

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APPROVED:

[redacted]

Professor of Mathematics In Charge of Major

[redacted]

Head of Department of Mathematics

[redacted]

Chairman of School Graduate Committee

[redacted]

Dean of the Graduate School

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# SUMMATION AND TABLE OF FINITE SUMS

## I. THEORY OF FINITE SUMMATION

To the knowledge of the writer, no table of finite sum formulas of value as a reference is in existence. For this reason a table of such formulas, analogous to a table of integral formulas, is presented here. The formulas for this table were selected with their utility in mind.

A finite sum formula is a relation giving the sum of a finite series. Finite summation is the evaluation of a finite series. This valuation can be accomplished by the development and application of general sum formulas. These formulas are of closed form and give the exact or approximate sum, depending on the type of series.

The theory of finite summation is presented first and is followed by the development of finite sum formulas, both general and special. Finally the table is presented.

As we will see, the subject of summation arises as a branch of the calculus of finite differences. Hence this treatment of finite summation will be through this calculus.

Throughout this paper numbers enclosed in parentheses refer to relationships in the discussion, whereas numbers not enclosed in parentheses refer to formulas in the table.

Finite calculus analogous to infinitesimal calculus.

There is a striking analogy between the finite calculus, or calculus of finite differences, and the infinitesimal calculus. Consider the function $u_t$ subjected to these calculi. The two fundamental operations of the infinitesimal calculus are differentiation,

$$\frac{d}{dt} u_t = \lim_{h \to 0} \frac{u_{t+h} - u_t}{h},$$

and its inverse, integration. The two fundamental operations of the finite calculus are the $\Delta$ - operation,

$$\frac{\Delta u_t}{h} = \frac{u_{t+h} - u_t}{h},$$

and its inverse. The analogy is not perfect because of the additional limiting operation in differentiation. Results in finite calculus may be reduced to the analogous results in infinitesimal calculus at any time if we perform this additional limiting operation. In fact by studying finite calculus we gain a greater perspective onto and insight into infinitesimal calculus.

In the finite calculus we may assume, without loss of generality, that the linear substitution, $t = hx$, has been made. Now, since $\Delta t = h\Delta x$, $\Delta x$ is unity. Hence $\Delta$ performed on $u_x$ now gives:

$$\Delta u_x = u_{x+1} - u_x. \tag{1}$$

Summing as the inverse of performing $\Delta$.

Let us investigate the inverse of the $\Delta$ operation, which

George Boole called difference integration. It is represented by the symbol $\Delta^{-1}$. Given $u_x$, if a function $V_x$ exists such that $u_x = \Delta V_x$, then $V_x = \Delta^{-1}u_x$. If we sum $u_x = \Delta V_x$ for values of x from m to n, we obtain:

$$\sum_{m}^{n} u_x = V_x \Big|_{m}^{n+1}. \tag{2}$$

Thus if $V_x = \Delta^{-1}u_x$ may be found we may sum the series whose general term is $u_x$.

<u>The constant of summation.</u>

Note that the operation $\Delta^{-1}$ is interrogative and hence that we have no assurance of the existence or uniqueness of the result from performing it on a function $u_x$.

Consider a case in which $\Delta^{-1}u_x$ exists. Now the uniqueness of $\Delta^{-1}u_x$ is not guaranteed. Hence we let $V_x$ be an expression for $\Delta^{-1}u_x$ and $V_x + \omega_x$ represent all such expressions. Now to find an expression for $\omega_x$ and thereby the relation between possible results from $\Delta^{-1}$ we consider the relation, $\Delta V_x = \Delta(V_x + \omega_x)$. Removing the parentheses we see $\Delta\omega_x = 0$. Thus the expression for $\omega_x$ is a general periodic function of period one (1, p.79-81). In finite calculus x assumes successive values differing by unity so we may consider $\omega_x$ as a constant. Thus $\Delta^{-1}u_x = V_x + C$. We use the more suggestive notation $\sum$ for $\Delta^{-1}$ and have:

$$\sum u_x = V_x + C. \tag{3}$$

If we consider this sum from m to n we have by (2):

$$\sum_{m}^{n} u_x = V_x + C \Big|_{m}^{n+1} = V_x \Big|_{m}^{n+1} . \tag{4}$$

Finite calculus as a branch of mathematics.

A look at finite calculus as a subject is in order. There is some confusion in literature about the technical terms used (1, p.81-82). Due to the close analogy with infinitesimal calculus many terms are borrowed here. Below is a list of terms from finite calculus paired with the analogous terms of infinitesimal calculus. From this and previous discussion their meaning is held to be self evident.

| Finite calculus | Infinitesimal calculus |
|---|---|
| Performing $\Delta$ | Differentiating |
| Difference integral | Indefinite integral |
| Sum | Definite integral |
| Difference calculus | Differential calculus |
| Sum calculus | Integral calculus |
| Summand | Integrand |

Infinitesimal calculus includes evaluation of definite integrals by other methods than through the indefinite integral. Likewise it is convenient to include in finite calculus summation performed by other methods than through the difference integral. Sometimes it is necessary to define and tabulate a new function in order to perform "exact" evaluation as for the integral,

$$\int_0^x \frac{\sin x}{x}\, dx = \mathrm{Si}(x),$$

and the sum,

$$\sum_m^n \frac{1}{x} = \digamma(n) - \digamma(m-1).$$

Sometimes approximate evaluation is the best we can do as in the case of the integral,

$$\int_0^{\pi} \sqrt{\sin x}\;\; dx,$$

and the sum,

$$\sum_1^n \text{ arc tan } x.$$

Summation of infinite series is analogous to evaluation of a definite integral with an infinite limit. Hence summation of infinite series is also included in the calculus of finite differences.

Applications of finite summation.

Some of the methods to be discussed in the paper permit the summation of convergent infinite series in closed form (3, p.18-23). By definition, the sum of the convergent infinite series $\sum_{i=1}^{\infty} a_i$ is $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$. Thus to sum the infinite series is to evaluate the sum of the first n terms and then the limit of this sum.

Another application of the sums and summation theory here developed is in the solution of difference equations.

We proceed to develop these methods whereby we may sum any finite series.

# II. DEVELOPMENT OF SUMMATION FORMULAS

## Methods.

The two most common methods for derivation of finite sum formulas are: (1) the inversion of a difference relation, and (2) use of the close relation between a sum and an integral. Other methods include mathematical induction, operational attacks, resolution of difference equations, use of generating functions, use of geometric considerations, and use of the calculus of probability (4, p.109-141).

Part III includes formulas, with one exception exact, developed by inversion of a $\Delta$- relation. Part IV includes approximating sum formulas, which are derived by the principle of inversion or by using the sum-integral relation.

In the table, sums appear in form (4) with the limits substituted, or in the easier written by equivalent form (3).

## Three general sum formulas.

The derivations of formulas 2, 3, and 5 of the table come under a separate heading. In each case the sums are expanded and the identity of the expansions noted (6, p.91-93). Formula 5 is called Dirichlets' sum-formula. It is useful when one of the functions $u_x$ and $V_x$ can be summed exactly, but the other not.

## III. SUMMATION FORMULAS DERIVED BY THE INVERSION OF A $\Delta$ - RELATION

To sum $u_x$ by inversion is to find $V_x$ such that $\Delta V_x = u_x$ and apply (3). For example from $\Delta cx = c$ we have formula 1, $\sum c = cx + C$. In fact any $\Delta$- relation may be inverted to give a sum.

Summation by parts.

The general formula for summation by parts, 4, is developed by this inversion. We have:

$$\Delta(u_x \overline{V}_x) = u_x \Delta \overline{V}_x + \Delta u_x \overline{V}_{x+1}, \tag{5}$$

whence by summing each side from m to n:

$$u_x \overline{V}_x \Big|_m^{n+1} = \sum_m^n u_x \Delta \overline{V}_x + \sum_m^n \Delta u_x \overline{V}_{x+1}.$$

Solving for the first sum and substituting $\Delta \overline{V}_x = V_x$, we have the first form of 5,

$$\sum_m^n u_x V_x = u_x \Sigma V_x \Big|_m^{n+1} - \sum_m^n \Delta u_x \Sigma V_{x+1}.$$

Adjustment of the limits and summand in the first form yields the second form. The third form is developed from the first by r = 1 reapplications of it to its right-hand sum. The fourth is derived from the second in the same manner. Summation by parts plays a role analogous to integration by parts.

Rational functions.

The factorial $(ax + b)^{(m)}$ is defined for positive m:

$$(ax + b)(ax + b - a)\dots(ax + b - am + a),$$

for negative m:

$$\frac{1}{(ax + b)(ax + b + a)\dots(ax + b + ma - a)}, \tag{6}$$

and for m = 0 as unity.

If we invert:

$$\Delta(ax + b)^{(m+1)} = (m + 1)a(ax + b)^{(m)}, \tag{7}$$

and divide through by a(m + 1), we obtain formula 6:

$$\sum(ax + b)^{(m)} = \frac{(ax + b)^{(m+1)}}{a(m + 1)} + C.$$

Evidently m cannot equal -1.

Formula 7 is seen to hold if we expand $\phi(x)$ as:

$$A_o u_x + A_1 u_x u_{x+1} + \dots + A_{m-1} u_x u_{x+1} \dots u_{x+m-2},$$

and reduce the resulting fractions. The A's may be determined by equating like powers of x in $\phi(x)$ and this expansion. The sums resulting from application of 7 may be summed by 6.

Formula 8 is proved by use of the relation:

$$\binom{x}{a} = \frac{x^{(a)}}{a!}, \tag{8}$$

for the binomial coefficient. Hence,

$$\sum\binom{x}{a} = \sum\frac{x^{(a)}}{a!} = \frac{x^{(a+1)}}{(a + 1)!} + C = \binom{x}{a + 1} + C.$$

Evidently a and x must be positive integers such that $0 \leqq a \leqq x$.

Formula 9 used with 6 enables us to sum the power function $x^n$ (4, p.168-173). We have as the expansion of $x^n$ by Newton's formula, (23):

$$x^n = \sum_{m=1}^{n} \left[\frac{\Delta^m x^n}{m!}\right]_{x=0} x^{(m)}. \tag{9}$$

Stirling's numbers of the second kind are defined as:

$$\mathfrak{S}_n^m = \left[\frac{\Delta^m x^n}{m!}\right]_{x=0}, \tag{10}$$

and hence we have formula 9. From (10) we have the recursion relation and initial condition:

$$\mathfrak{S}_{n+1}^m = \mathfrak{S}_n^{m-1} + m\mathfrak{S}_n^m, \tag{11}$$

$$\mathfrak{S}_0^0 = 1, \mathfrak{S}_0^m = 0 \text{ for } m \neq 0.$$

A brief table of $\mathfrak{S}$'s is included later.

Gamma and related functions.

The gamma function $\Gamma(x)$ is the accepted generalization of the factorial $(x - 1)!$ For $x > 0$,

$$\Gamma(x) = \int_0^\infty e^{-y} y^{x-1} dy. \tag{12}$$

Integration by parts yields the recursion relation:

$$x\Gamma(x) = \Gamma(x + 1), \tag{13}$$

which defines $\Gamma(x)$ for $x \leqq 0$. The following fundamental relations may be found in almost any treatment of $\Gamma(x)$:

$$\Gamma(x + 1) = \sqrt{2\pi x}\, e^{-x}\, x^{x} \cdot$$

$$\left[1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840x^3} - \frac{571}{2488320x^4} + \dots\right] . \qquad (14)$$

$$\log \Gamma(x + 1) = \lim_{n\to\infty}\left[\log n! + (x + 1)\log n - \sum_{r=1}^{n+2} \log(x + r)\right] . \qquad (15)$$

The successive derivatives of $\log \Gamma(x)$ are defined as the digamma, trigamma, tetragamma, et cetera functions and denoted $\digamma(x)$, $\digamma'(x)$, $\digamma''(x)$, $\digamma'''(x)$, et cetera, respectively. These functions are fundamental in the theory of summation of rational fractions.

If a rational fraction is improper, division yields a rational integral function and a proper fraction. Hence we need consider only rational integral functions and proper rational fractions.

A rational integral function may be summed by formula 9 or by direct expression in factorials and use of 6 (5, p.27-28).

A proper rational fraction may be expressed as the sum of fractions in which the denominator is a power of a linear function in x and the numerator is a constant. This is affected by the theory of rational fractions where the constant term in the linear function is a complex number. Hence our problem is to sum $\frac{1}{(x - a)^r}$, $r = 1, 2, 3, \dots$, and a complex. This summation is affected by use of the above gamma and related functions.

Essentially we are inverting a $\Delta$ - relation but a more direct attack will be made on this summation problem. From the definition of the digamma function:

$$\digamma(x) = \frac{d}{dx} \log \Gamma(x + 1),$$

we have by (15):

$$\digamma(x) = \lim_{n \to \infty} \left[ \log n - \sum_{r=1}^{n+2} \frac{1}{x + r} \right] . \qquad (16)$$

Hence,

$$\digamma(n + 1) - \digamma(m) = \sum_{m}^{n} \frac{1}{r + 1} ,$$

or as $r$ is a dummy variable we have formula 10,

$$\sum_{m}^{n} \frac{1}{x + 1} = \digamma(n + 1) - \digamma(m).$$

As the trigamma, tetragamma, pentagamma, et cetera functions are defined as successive derivatives of log $\Gamma(x)$ we have from (16):

$$\digamma'(x) = \sum_{r=1}^{\infty} \frac{1}{(x + r)^2} .$$

$$\digamma''(x) = - 2 \sum_{r=1}^{\infty} \frac{1}{(x + r)^3} .$$

$$\digamma'''(x) = 6 \sum_{r=1}^{\infty} \frac{1}{(x + r)^4} .$$

- - - - - - - - - - - -

Proceeding as with $\digamma(x)$ we obtain formulas 11, 12, and 13.

From the assymptotic expansion for $\Gamma(x + 1)$, (14), we obtain the assymptotic expansion for $\digamma(x)$,

$$\digamma(x) = \frac{1}{2}\ln x(x+1) + \frac{1}{6}\left(\frac{1}{x} - \frac{1}{x+1}\right) - \frac{1}{90}\left(\frac{1}{x^3} - \frac{1}{(x+1)^3}\right)$$

$$+ \frac{1}{210}\left(\frac{1}{x^5} - \frac{1}{(x+1)^5}\right) - \ldots, \qquad (17)$$

and for $\digamma'(x)$, $\digamma''(x)$, et cetera, by the derivative relation between these functions.

A table has been prepared which gives $\digamma(x)$, $\digamma'(x)$, $\digamma''(x)$, and $\digamma'''(x)$ for real x in increments of 0.01 for $0 \leqq x \leqq 1$ and increments of 0.1 for $10 \leqq x \leqq 60$ (2, p.43-59). This table used with the following reduction formulas, (2, p. XIX-XXII), yields considerable accuracy:

$$\digamma(nx) = \log n + \frac{1}{n}\sum_{\mu=0}^{n-1}\digamma\left(x - \frac{\mu}{n}\right).$$

$$\digamma(-x) = \digamma(x-1) + \pi \operatorname{ctn} \pi x.$$

$$\digamma'(nx) = \frac{1}{n^2}\sum_{\mu=0}^{n-1}\digamma'\left(x - \frac{\mu}{n}\right).$$

$$\digamma'(-x) = \frac{\pi^2}{\sin^2 \pi x} - \digamma'(x-1). \qquad (18)$$

$$\digamma''(nx) = \frac{1}{n^3}\sum_{\mu=0}^{n-1}\digamma''\left(x - \frac{\mu}{n}\right).$$

$$\digamma'''(nx) = \frac{1}{n^4}\sum_{\mu=0}^{n-1}\digamma'''\left(x - \frac{\mu}{n}\right).$$

Tables are now being computed for gamma and digamma functions for imaginary arguments (Ref. 7).

Formula 14 is proved by reduction of the summand to type form:

$$\frac{1}{x^2+a^2} = \frac{1}{2\,ai}\left[\frac{1}{(x-ai-1)+1} - \frac{1}{(x+ai-1)+1}\right],$$

and application of formula 10.

Note that formulas 10 to 13 are the one exception to exact formulas in this chapter. However, they are exact in the sense that the sums are in exact terms of a defined tabulated function.

Note that $F(x)$ plays a role in finite calculus analogous to that of $\ln(x+1)$ in infinitesimal calculus.

Exponential and logarithmic functions.

Formula 15 is proved by inversion of $\Delta a^x = (a-1)a^x$ and division by $a-1$. This division is by zero or impossible if $a=1$.

Formula 16 is proved by inversion of $\Delta \ln\Gamma(x) = \ln x$.

Trigonometric and hyperbolic functions.

Formulas 17 to 20 are proved by inversion of the following $\Delta$-relations, adjustment of the argument, and division by the constant coefficient:

$$\Delta \cos ax = -2\sin(ax+\tfrac{a}{2})\sin\tfrac{a}{2}.$$

$$\Delta \sin ax = 2\cos(ax+\tfrac{a}{2})\sin\tfrac{a}{2}. \qquad (19)$$

$$\Delta \cosh ax = 2\sinh(ax+\tfrac{a}{2})\sinh\tfrac{a}{2}.$$

$$\Delta \sinh ax = 2\cosh(ax+\tfrac{a}{2})\sinh\tfrac{a}{2}.$$

Combinations of elementary functions.

Formulas 21 to 32 are derived by summation by parts and previous results. Evidently recursion formulas 30 and 32 must be used as a pair.

Formulas 33, 34, and 35 are proved by the substitutions:

$$\sin ax \cos bx = \frac{1}{2}\left[\sin(a+b)x + \sin(a-b)x\right],$$

$$\sin ax \sin bx = -\frac{1}{2}\left[\cos(a+b)x - \cos(a-b)x\right], \quad (20)$$

and,

$$\cos ax \cos bx = \frac{1}{2}\left[\cos(a+b)x + \cos(a-b)x\right],$$

and application of formulas 17 and 18.

Formulas 36, 37, and 38 are special cases of formulas 33, 34, and 35 respectively. They are of common application in practical Fourier analysis (4, p.123-129).

Formulas 39 and 40 are proved by the substitutions,

$$\sin^2 ax = \frac{1}{2} - \frac{1}{2}\cos 2\,ax,$$

and, (21)

$$\cos^2 ax = \frac{1}{2} + \frac{1}{2}\cos 2\,ax,$$

and application of formulas 17 and 18.

Note that analogous formulas for hyperbolic functions may be easily derived by substitution through the relations,

$$\sin(iz) = i \sinh z$$

and, (22)

$$\cos(iz) = \cosh z.$$

## IV. SUMMATION BY METHODS OF APPROXIMATION

Inversion of a $\Delta$- relation or any exact evaluation of a sum is possible in relatively few cases as for the analogous inversion of a derivative relation or any exact evaluation of an integral in infinitesimal calculus. Hence as with infinitesimal calculus evaluation must often be through series expansions. In summation these series are assymptotic series so their sum is approximated by their convergent part, as in the case of the expansion of gamma and related functions. Of course convergence and the remainder term must always be investigated.

Newton's formula.

By Newton's formula we may expand $u_x$ in a series of factorials and sum term by term. It states that:

$$u_x = \sum_{k=0}^{\infty} \Delta^k u_o \frac{x^{(k)}}{k!} . \tag{23}$$

For the proof we assume $u_x$ may be expanded into a series of factorials,

$$u_x = a_o + a_1 x + a_2 x^{(2)} + a_3 x^{(3)} + \ldots,$$

and evaluate the coefficients by setting $x = 0$ in successive differences of this relation.

Extension of partial summation.

If we let $r \to \infty$ in the fourth form of formula 5 we have formula 41.

Formulas relating a sum to an integral.

The sum and integral of $u_x$ over the same interval are closely related as may be seen by geometry. The integral is the area under the curve, $y = u_x$, and the sum is an approximation to this area. Hence a sum may be expressed as the corresponding integral and correction terms. If these terms are in the form of differences, the formula is Laplace's type, and if in the form of derivatives, Euler's type (6, p.104).

Formula 42 is of Euler's type and is called the Euler-MacLaurin formula. It is derived by expanding the operational equivalent of $\sum u_x$, $(e^{d/dx}-1)^{-1}u_x$, in powers of d/dx. The numbers of the form $B_{2n+1}$ which appear in the coefficients are called Bernoulli's numbers and are of frequent occurrence in approximate sum formulas.

The recursion relation:

$$\sum_{i=1}^{n-1} (-1)^i \binom{n}{i} B_{i-1} - 1 = 0, \tag{24}$$

may be used to evaluate the Bernoullian numbers (4, p.233). This relation is consistent with the definition, $B_{2n} = 0$ for $n \geqq 1$. A brief list of Bernoulli's numbers is included later. This list is sufficient for most practical purposes because the magnitude of the numbers increases so rapidly that any assymptotic series whose coefficients contain them has a small convergent part.

Formula 43 is of Laplace's type and is called Gregory's formula. It is derived by substitution for the derivatives in formula 42 of their difference equivalents (8, p.62).

<u>Sum from every m'th term.</u>

We have a Woolhouse or Lubback type formula according as the correction terms in such a formula are in terms of derivatives or differences respectively. Formula 44, a Woolhouse formula, is derived from formula 42 by eliminating the integral between formula 42 and a formula derived from 42 by subdividing each interval into m parts. Formula 45, a Lubback formula, is derived by replacing the differences in 44 by their differential equivalents.

Formulas 46, 47, and 48 are derived by applying formula 42 to $x^n$, $1/x$, and $1/x^{2n}$ respectively.

Formula 49 is discussed in Chapter VI.

## V. TABLE OF SUMS

1. $\sum c = cx + c.$

2. $\sum_{m}^{n} c\, f(x) = c \sum_{m}^{n} f(x).$

3. $\sum_{m}^{n} \left[f(x) \pm g(x) \pm \cdots\right] = \sum_{m}^{n} f(x) \pm \sum_{m}^{n} g(x) \pm \cdots$

4. $$\sum_{m}^{n} u_x V_x = u_x \Sigma V_x \Big|_{m}^{n+1} - \sum_{m+1}^{n+1} \Delta u_{x-1} \Sigma V_x$$

$$= u_x \Sigma V_x \Big|_{m}^{n+1} - \sum_{m}^{n} \Delta u_x \Sigma V_{x+1}$$

$$= u_x \Sigma V_x \Big|_{m}^{n+1} - \Delta u_{x-1} \Sigma^2 V_x \Big|_{m+1}^{n+2} + \Delta^2 u_{x-2} \Sigma^3 V_x \Big|_{m+2}^{n+3} - \cdots$$

$$\cdots + (-1)^r \sum_{m+r}^{n+r} \Delta^r u_{x-r} \Sigma^r V_x$$

$$= u_x \Sigma V_x - \Delta u_x \Sigma^2 V_{x+1} + \Delta^2 u_x \Sigma^3 V_{x+2} - \cdots \Big|_{m}^{n+1}$$

$$+ (-1)^r \sum_{m}^{n} \Delta^r u_x \Sigma^r V_{x+r}.$$

5. $\sum_{m}^{n} V_x \sum_{m}^{x} u_x = \sum_{m}^{n} u_x \sum_{x}^{n} V_x.$

6. $\sum (ax + b)^{(m)} = \dfrac{(ax + b)^{(m+1)}}{a(m + 1)} + C$; m zero or any integer $\neq -1$.

7. $\sum \frac{\phi(x)}{u_x u_{x+1} \cdots u_{x+m}} = A_o \sum \frac{1}{u_x u_{x+1} \cdots u_{x+m}} + A_1 \sum \frac{1}{u_{x+1} u_{x+2} \cdots u_{x+m}}$

$+ \dots + A_{m-2} \sum \frac{1}{u_{x+m-1} u_{x+m}}$ ; $u_x = ax + b$, $\phi(x)$ a rational

integral function of degree $\leqq m - 1$, A's from expansion

of $\phi(x)$, $\phi(x) = A_o u_x + A_1 u_x u_{x+1} + \dots + A_{m-2} u_x u_{x+1} \cdots u_{x+m-2}$.

8. $\sum \binom{x}{a} = \binom{x}{a+1} + C$; a, x integers such that $0 \leqq a \leqq x$.

9. $\sum x^n = \sum \left[ \sum_{m=1}^{n} G_n^m \, x^{(m)} \right]$ ; Values of $G_n^m$ on page 25.

10. $\sum_m^n \frac{1}{x+1} = F(n+1) - F(m)$; $F(x) = D_x \ln \Gamma(x+1)$.

11. $\sum_m^n \frac{1}{(x+1)^2} = -\not{F}(n+1) + \not{F}(m)$; $\not{F}(x) = D_x F(x)$.

12. $\sum_m^n \frac{1}{(x+1)^3} = \frac{1}{2}\not{\not{F}}(n+1) - \frac{1}{2}\not{\not{F}}(m)$; $\not{\not{F}}(x) = D_x \not{F}(x)$.

13. $\sum_m^n \frac{1}{(x+1)^4} = -\frac{1}{6}\not{\not{\not{F}}}(n+1) + \frac{1}{6}\not{\not{\not{F}}}(m)$; $\not{\not{\not{F}}}(x) = D_x \not{\not{F}}(x)$.

14. $\sum \frac{1}{x^2 + a^2} = \frac{1}{2\,ai} F(x - 1 - ai) \Big|_m^{n+1} - \frac{1}{2\,ai} F(x - 1 + ai) \Big|_m^{n+1}$.

15. $\sum a^x = \frac{a^x}{a-1} + C$; $a \neq +1$.

16. $\sum \ln(x) = \ln\Gamma(x) + C.$

17. $\sum \sin(ax + b) = \dfrac{\cos(ax + b - a/2)}{-2 \sin a/2} + C.$

18. $\sum \cos(ax + b) = \dfrac{\sin(ax + b - a/2)}{2 \sin a/2} + C.$

19. $\sum \sinh(ax + b) = \dfrac{\cosh(ax + b - a/2)}{2 \sinh a/2} + C.$

20. $\sum \cosh(ax + b) = \dfrac{\sinh(ax + b - a/2)}{2 \sinh a/2} + C.$

21. $\sum xa^x = x \dfrac{a^x}{a - 1} - \dfrac{a^{x+1}}{(a - 1)^2} + C.$

22. $\sum x^{(m)} a^x = x^{(m)} \dfrac{a^x}{a - 1} - \dfrac{ma}{a - 1} \sum x^{(m-1)} a^x.$

23. $\sum x \sin ax = - \dfrac{x \cos(ax - a/2)}{2 \sin a/2} + \dfrac{\sin ax}{4 \sin^2 a/2} + C.$

24. $\sum x^{(m)} \sin ax = - \dfrac{x^{(m)} \cos(ax - a/2)}{2 \sin a/2} + \dfrac{m\, x^{(m-1)} \sin ax}{4 \sin^2 a/2}$

$$- \frac{m(m - 1)}{4 \sin^2 a/2} \sum x^{(m-2)} \sin(ax + a).$$

25. $\sum x \cos ax = \dfrac{x \sin(ax - a/2)}{2 \sin a/2} + \dfrac{\cos ax}{4 \sin^2 a/2} + C.$

26. $$\sum x^{(m)}\cos ax = \frac{x^{(m)}\sin(ax - a/2)}{2\sin a/2} + \frac{m\,x^{(m-1)}\cos ax}{4\sin^2 a/2}$$

$$- \frac{m(m-1)}{4\sin^2 a/2}\sum x^{(m-2)}\cos(ax + a).$$

In formulas 27-32:

$$K_1 = \frac{a\cos b - 1}{a^2 + 1 - 2a\cos b},$$

$$K_2 = \frac{a\cos b}{a^2 + 1 - 2a\cos b}.$$

27. $$\sum a^x \sin bx = K_1 a^x \sin bx - K_2 a^x \cos bx + C.$$

28. $$\sum a^x \cos bx = K_2 a^x \sin bx + K_1 a^x \cos bx + C.$$

29. $$\sum x a^x \sin bx = (K_1 - K_2) x a^x (\sin bx + \cos bx) + (K_2^2 - K_1^2) a^{x+1} \sin(bx + b)$$

$$+ 2K_1 K_2 a^{x+1}\cos(bx + b) + C.$$

30. $$\sum x^{(m)} a^x \sin bx = K_1 x^{(m)} a^x \sin bx - K_2 x^{(m)} a^x \cos bx$$

$$- m K_1 \sum x^{(m-1)} a^{x+1} \sin(bx + b) + m K_2 \sum x^{(m-1)} a^{x+1} \cos(bx + b).$$

31. $$\sum x a^x \cos bx = (K_1 + K_2) x a^x (\sin bx + \cos bx) + (K_1^2 + K_2^2)\cdot$$

$$a^{x+1}\cos(bx + b) + C.$$

32. $$\sum x^{(m)} a^x \cos bx = K_2 x^{(m)} a^x \sin bx + K_1 x^{(m)} a^x \cos bx$$

$$- mK_2 \sum x^{(m-1)} a^{x+1} \sin(bx + b) - mK_1 \sum x^{(m-1)} a^{x+1} \cos(bx + b).$$

33. $$\sum \sin ax \cos bx = -\frac{\cos\left[(a+b)x - \frac{a+b}{2}\right]}{4 \sin \frac{a+b}{2}} - \frac{\cos\left[(a-b)x - \frac{a-b}{2}\right]}{4 \sin \frac{a-b}{2}} + C.$$

34. $$\sum \sin ax \sin bx = -\frac{\sin\left[(a+b)x - \frac{a+b}{2}\right]}{4 \sin \frac{a+b}{2}} + \frac{\sin\left[(a-b)x - \frac{a-b}{2}\right]}{4 \sin \frac{a-b}{2}} + C.$$

35. $$\sum \cos ax \cos bx = \frac{\sin\left[(a+b)x - \frac{a+b}{2}\right]}{4 \sin \frac{a+b}{2}} + \frac{\sin\left[(a-b)x - \frac{a-b}{2}\right]}{4 \sin \frac{a+b}{2}} + C.$$

36. $$\sum_{i=0}^{k} \sin ax_i \cos bx_i = 0; \text{ a, b integers} < k, \; x_i = -\pi + \frac{2\pi i}{k}.$$

37. $$\sum_{i=0}^{k} \sin ax_i \sin bx_i = 0.$$

38. $$\sum_{i=0}^{k} \cos ax_i \cos bx_i = 0.$$

39. $$\sum \sin^2 ax = \frac{1}{2}x - \frac{\sin(2ax - a)}{4 \sin a} + C.$$

40. $$\sum \cos^2 ax = \frac{1}{2}x + \frac{\sin(2ax-a)}{4 \sin a} + C.$$

41. $$\sum_{1}^{\infty} u_x V_x = u_x \Sigma V_x - \Delta u_x \Sigma^2 V_{x+1} + \Delta^2 u_x \Sigma^3 V_{x+2} - \cdots$$

$$\cdots + (-1)^{r+1} \Delta^{r-1} u_x \Sigma^r V_{x+r-1} + \cdots \Big|_1^{\infty}.$$

42. $$\sum u_x = C + \int u_x dx - \frac{1}{2} u_x + \frac{B_1}{\underline{|2}} \frac{du_x}{dx} - \frac{B_3}{\underline{|4}} \frac{d^3 u_x}{dx^3} + \cdots$$

$$= C + \int u_x dx - \frac{1}{2} u_x + \frac{1}{12} \frac{du_x}{dx} - \frac{1}{720} \frac{d^3 u_x}{dx^3} + \frac{1}{30,240} \frac{d^5 u_x}{dx^5} - \cdots;$$

Values of $B_{2n+1}$ on page 25.

43. $$\sum_{m}^{n} u_x = C + \int_m^{n+1} u_x dx - \frac{1}{2} u_x \Big|_m^{n+1} + \frac{1}{12} \Delta u_x \Big|_m^{n} + \frac{1}{24} \left[\Delta^2 u_{n-1} + \Delta^2 u_m\right]$$

$$+ \frac{19}{720} \Delta^3 u_x \Big|_m^{n-2} + \cdots$$

44. $$\sum_{o}^{n} u_x = m \sum_{o}^{n/m} u_{mx} - \frac{m-1}{2} \left[u_n + u_o\right] - \frac{m^2-1}{12m} \frac{du_x}{dx} \Big|_o^n$$

$$+ \frac{m^4-1}{720m^3} \frac{d^3 u_x}{dx^3} \Big|_o^n + \cdots$$

45. $$\sum_{o}^{n} u_x = m \sum_{o}^{n/m} u_{mx} - \frac{m-1}{2} \left[u_n + u_o\right] - \frac{m^2-1}{12m} \Delta u_x \Big|_o^{n-1}$$

$$- \frac{m^2-1}{24m} \left[\Delta^2 u_{n-2} + \Delta^2 u_o\right] - \frac{(m^2-1)(19m^2-1)}{720m^3} \Delta^3 u_x \Big|_o^{n-3}$$

$$- \frac{(m^2-1)(9m^2-1)}{480m^3} \left[\Delta^4 u_{n-4} + \Delta^4 u_o\right] - \cdots$$

46. $$\sum x^n = C + \frac{x^{n+1}}{n+1} - \frac{x^n}{2} + \frac{B_1}{\lfloor\underline{2}} n x^{n-1} - \frac{B_3}{\lfloor\underline{4}} n(n-1)(n-2)\, x^{n-2} - \dots$$

47. $$\sum \frac{1}{x} = C + \log x - \frac{1}{2x} - \frac{1}{2x^3} - \dots$$

48. $$\sum \frac{1}{x^{2n}} = C - \frac{1}{(2n-1)x^{2n-1}} + \frac{1}{2x^{2n}} - \frac{2n}{12x^{2n-1}} + \frac{2n(2n+1)(2n+2)}{720x^{2n+3}} - \dots$$

49. $$\sum_{k=1}^{\infty} k^n x^k = \frac{\sum_{r=1}^{n} \left[ \sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n \right] x^r}{(1-x)^{n+1}} .$$

## Stirling's Numbers of the Second Kind, $\mathfrak{S}_m^n$

| n \ m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | | | | | | | |
| 2 | 1 | 1 | | | | | | |
| 3 | 1 | 3 | 1 | | | | | |
| 4 | 1 | 7 | 6 | 1 | | | | |
| 5 | 1 | 15 | 25 | 10 | 1 | | | |
| 6 | 1 | 31 | 90 | 65 | 15 | 1 | | |
| 7 | 1 | 63 | 301 | 350 | 140 | 21 | 1 | |
| 8 | 1 | 127 | 966 | 1701 | 1050 | 266 | 28 | 1 |

## Bernoulli's Numbers

$B_0 = \frac{1}{2}$

$B_1 = \frac{1}{6}$

$B_3 = \frac{1}{30}$

$B_5 = \frac{1}{42}$

$B_7 = \frac{1}{30}$

$B_9 = \frac{5}{66}$

$B_{11} = \frac{691}{2,730}$

$B_{13} = \frac{7}{6}$

$B_{15} = \frac{3,617}{510}$

$B_{17} = \frac{43,867}{798}$

$B_{19} = \frac{174,611}{330}$

$B_{21} = \frac{854,513}{138}$

# VI. SUMMATION OF A SPECIAL TYPE POWER SERIES

The summation of a power series in which the coefficient of the k'th term is a rational integral function of k is studied on the following pages. By summation we mean, here, the replacement of the power series or infinite sum by a finite expression, identical for values of the variable in the interval of convergence of the power series. In other words the problem of approximate evaluation of the infinite series is reduced to exact evaluation by substitution in an identical finite form.

We proceed to develop this finite expression. Also we develop three recursion relationships which are of more practical use and which yield identical results for this type of summation. Finally we present a triangle, the summation triangle, which, with a simple formula, not only provides the most rapid method for such summation, but is also easy to remember.

The general form of such a series being written:

$$S(x) = \sum_{k=1}^{\infty} (C_0 k^n + C_1 k^{n-1} + \dots + C_n) x^k, \tag{25}$$

where the C's are constants and n is a non-negative integer, we have immediately the form:

$$S(x) = C_0 \sum_{k=1}^{\infty} k^n x^k + C_1 \sum_{k=1}^{\infty} k^{n-1} x^k + \dots + C_n \sum_{k=1}^{\infty} x^k .$$

Thus the stated problem of summation may be reduced to the problem of evaluating sums of the form:

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k, \tag{26}$$

where n is a non-negative integer. In other words rather than studying the original power series (25), we study the power series (26).

The interval of convergence of $K_n(x)$, $(-1, +1)$, may be readily determined by the ratio test.

When $n = 0$, we have:

$$K_o(x) = \sum_{k=1}^{\infty} x^k,$$

which is a geometric series. Hence,

$$K_o(x) = \frac{x}{1 - x}, \ /x/<1. \tag{27}$$

We see $K_n(x)$ is a particular generalization of the geometric series.

Now proceeding from (27) we have:

$$K_1(x) = \sum_{k=1}^{\infty} kx^k = xK_o^1(x) = \frac{x}{(1 - x)^2},$$

$$K_2(x) = \sum_{k=1}^{\infty} k^2 x^k = xK_1^1(x) = \frac{x(1 + x)}{(1 - x)^3}, \tag{28}$$

and in general:

$$K_n(x) = \sum_{k=1}^{\infty} k^n x^k = x\, K_{n-1}^1(x) = \frac{x\, P_n(x)}{(1 - x)^{n+1}}.$$

$P_n(x)$ may be seen to be a polynomial of degree $n - 1$ by induction. Note that $P_2(x)$ is a polynomial of degree one and that if $P_n(x)$ is a polynomial of degree $n - 1$, then $P_{n+1}(x)$ will be a polynomial of degree n.

Since $K_n(x)$ satisfies the recursion relation,

$$K_{n+1}(x) = x\, K_n^1\,(x), \tag{29}$$

and since,

$$K_n(x) = \frac{x\, P_n(x)}{(1-x)^{n+1}}, \tag{30}$$

we have,

$$\frac{x\, P_{n+1}(x)}{(1-x)^{n+2}} = x\,\frac{d}{dx}\left[\frac{x\, P_n(x)}{(1-x)^{n+1}}\right],$$

whence we obtain, by performing the differentiation and removing the factor, $\dfrac{x}{(1-x)^{n+2}}$,

$$P_{n+1}(x) = (nx+1)P_n(x) + x(1-x)P_n^1\,(x). \tag{31}$$

The successive polynomials, $P_1$, $P_2$, $P_3$, ... , can be built up easily by repeated use of (31), starting from the known fact that, $P_1(x) = 1$. Thus we have:

$$\begin{aligned} P_1(x) &= 1 \\ P_2(x) &= 1 + x \\ P_3(x) &= 1 + 4x + x^2 \\ P_4(x) &= 1 + 11x + 11x^2 + x^3 \\ P_5(x) &= 1 + 26x + 66x^2 + 26x^3 + x^4 \end{aligned} \tag{32}$$

It may be noted that the coefficients in these polynomials are symmetrical, positive, integral, and that the first and last coefficients are unity.

If we assume that:

$$P_n(x) = \sum_{r=1}^{n} f_r(n) x^{r-1}, \tag{33}$$

and substitute in (31) we obtain the recursion relation for the coefficients $f_r(n)$ as follows:

$$f_r(n) = (n - r + 1)f_{r-1}(n - 1) + r\, f_r(n - 1). \qquad (34)$$

Since $f_o(n) = 0$, we see that $f_1(n) = 1$ for all n. Since $f_{n+1}(n) = 0$, we see that $f_n(n) = 1$ for all n. Hence we see that the first and last coefficients are always unity. The other properties noted for the coefficients follow easily from (34). Their symmetry is shown later, (47).

Using (34) we build up a triangular array of coefficients for the polynomials $P_n(x)$, shown on page 30.

Looking at the geometrical meaning of (34) with respect to the triangle we find a device whereby the law of formation may be easily applied and remembered. This law is similar to that of Pascal's triangle. Rather than adding two elements to obtain the one below we add multiples of the elements. These multiples are determined by the number of steps we must take from the sides of the triangle to reach the elements. The method is evident from (34).

Development of the Summation Triangle for the First Nine Lines

1

1 1

1 4 1

1 11 11 1

1 26 66 26 1

1 57 302 302 57 1

1 120 1191 2416 1191 120 1

1 247 4293 15,619 15,619 4293 247 1

1 502 14,608 88,234 156,190 88,234 14,608 502 1

The next step is to find a finite expression for $f_r(n)$. We have by repeated application of (34):

$$f_r(n) = (n - r + 1)f_{r-1}(n - 1) + r\, f_r(n - 1)$$
$$= (n - r + 1)f_{r-1}(n - 1) + r(n - r)f_{r-1}(n - 2) + r^2 f_r(n - 2)$$
$$= (n - r + 1)f_{r-1}(n - 1) + r(n - r)f_{r-1}(n - 2)$$
$$+ r^2(n - r - 1)f_{r-1}(n - 3) + \dots$$
$$\dots + r^{i-1}(n-r-i+2)f_{r-1}(n - i) + r^i f_r(n - i);\ i \leqq n - 1.$$

Now since $f_r(r) = 1$, $f_r(n - i) = 1$ when $n - i = r$ or when $i = n - r$. Hence,

$$f_r(n) = \sum_{i=1}^{n-r} r^{i-1}(n - r - i + 2)f_{r-1}(n - i) + r^{n-r}f_r(r),$$

or since $f_r(r) = f_{r-1}(r - 1)$,

$$f_r(n) = \sum_{i=1}^{n-r+1} r^{i-1}(n - r - i + 2)f_{r-1}(n - i) \qquad (35)$$

Now we substitute $r = 2$ in (35) and note $f_1(n - i) = 1$ for $i < n$ to obtain,

$$f_2(n) = \sum_{i=1}^{n-1} 2^{i-1}(n - i).$$

Summing by parts, formula 4, form 3, in the table, we obtain $f_2(n)$, $f_3(n)$, $f_4(n)$, and $f_5(n)$ successively, each time depending on the previous result and noting $f_1(n) = 1$, to obtain:

$$\begin{aligned}
f_1(n) &= 1^n \\
f_2(n) &= 2^n-(n+1)\,1^n \\
f_3(n) &= 3^n-(n+1)\,2^n+\frac{(n+1)n}{2!}\,1^n \\
f_4(n) &= 4^n-(n+1)\,3^n+\frac{(n+1)n}{2!}\,2^n-\frac{(n+1)n(n-1)}{3!}\,1^n \\
f_5(n) &= 5^n-(n+1)\,4^n+\frac{(n+1)n}{2!}\,3^n-\frac{(n+1)n(n-1)}{3!}\,2^n \\
&\qquad +\frac{(n+1)n(n-1)(n-2)}{4!}\,1^n.
\end{aligned} \tag{36}$$

From this an intelligent conjecture for $f_r(n)$ is evidently,

$$F_r(n) = \sum_{m=1}^{r} (-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n. \tag{37}$$

The following inductive proof that $F_r(n) = f_r(n)$ consists in showing,

$$F_r(n) = (n-r+1)F_{r-1}(n-1)+r\,F_r(n-1);\ 2 \leqq r \leqq n$$

and, (38)

$$F_1(n) = F_n(n) = 1$$

in this order.

Hence we first prove:

$$\sum_{m=1}^{r} (-1)^{m+1}\binom{n+1}{m-1}(r-m+1)^n = (n-r+1)\sum_{m=1}^{r-1}(-1)^{m+1}\cdot$$

$$\binom{n}{m-1}(r-m)^{n-1}+r\sum_{m=1}^{r}(-1)^{m+1}\binom{n}{m-1}(r-m+1)^{n-1}.$$

We proceed to reduce the expression on the right to that on the left. Changing the summand of the first term on the right and adjusting through the limits of summation, we obtain:

$$(n - r + 1)\sum_{m=2}^{r} (-1)^m \binom{n}{m-2}(r - m + 1)^{n-1}$$

$$+ r\sum_{m=1}^{r} (-1)^{m+1}\binom{n}{m-1}(r - m + 1)^{n-1}.$$

We have the two relations:

$$\binom{n}{m-2} = \frac{m - 1}{n + 1}\binom{n+1}{m-1},$$

$$\binom{n}{m-1} = \frac{n - m + 2}{n + 1}\binom{n+1}{m-1}, \tag{39}$$

which follow immediately from the expression of the binomial coefficients in their factorial form. Substitution from these equations in the above expression yields:

$$(n - r + 1)\sum_{m=2}^{r} (-1)^m \frac{m - 1}{n + 1}\binom{n+1}{m-1}(r - m + 1)^{n-1}$$

$$+ r\sum_{m=1}^{r} (-1)^{m+1} \frac{n - m + 2}{n + 1}\binom{n+1}{m-1}(r - m + 1)^{n-1}.$$

If the exponents of $\underline{-1}$ and $(r - m + 1)$ are adjusted, this becomes:

$$-(n - r + 1)\sum_{m=2}^{r} (-1)^{m+1} \frac{m - 1}{(n+1)(r-m+1)}\binom{n+1}{m-1}(r - m + 1)^n$$

$$+ r\sum_{m=1}^{r} (-1)^{m+1} \frac{n-m+2}{(n+1)(r-m+1)}\binom{n+1}{m-1}(r - m + 1)^n.$$

The limits of summation for the first sum may just as well be for m from 1 to r as the term corresponding to m = 0 is zero. Hence if we make this change, include the constant factors inside the summation, and add sums, we have:

$$\sum_{m=1}^{r} \left[ -\frac{(n-r+1)(m-1)}{(n+1)(r-m+1)} + \frac{r(n-m+2)}{(n+1)(r-m+1)} \right] (-1)^{m+1} \binom{n+1}{m-1}(r - m + 1)^n.$$

Upon algebraic reduction this becomes:

$$\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1}(r - m + 1)^n,$$

and the first equation of (38) is proved.

To continue, we evaluate $F_1(n)$ by substitution in (37) as follows:

$$F_1(n) = (-1)^2\binom{n+1}{0} 1^n = 1. \tag{40}$$

Finally we show $F_n(n) = 1$. To do this it will be convenient to consider an array which is our triangle augmented by the elements $F_n(n - 1)$, $n \geqq 2$, which satisfy the recursion relation of our triangle. Hence $F_n(n - 1)$ must be defined by (37) as this definition causes it to satisfy (38). Thus,

$$F_n(n - 1) = \sum_{m=1}^{n} (-1)^{m+1} \binom{n}{m-1}(n - m + 1)^{n-1}. \tag{41}$$

Now since $\Delta = E - 1$,

$$\Delta^n x^{n-1} = (E - 1)^n x^{n-1}$$

which becomes by the binomial formula,

$$\sum_{m=1}^{n+1} (-1)^{m+1} \binom{n}{m-1} E^{n-m+1} x^{n-1}.$$

If we apply the E operator we have,

$$\sum_{m=1}^{n+1} (-1)^{m+1} \binom{n}{m-1}(x + n - m + 1)^{n-1}.$$

Hence we have,

$$\Delta^n x^{n-1} \Big|_{x=0} = \sum_{m=1}^{n+1} (-1)^{m+1} \binom{n}{m-1} (n - m + 1)^{n-1},$$

or as the term for m = n + 1 equals zero,

$$\Delta^n x^{n-1} \Big|_{x=0} = \sum_{m=1}^{n} (-1)^{m+1} \binom{n}{m-1} (n - m + 1)^{n-1}. \qquad (42)$$

(1, p.19-20)

If we compare (41) with (42) we have:

$$F_n(n - 1) = \Delta^n x^{n-1} \Big|_{x=0} .$$

Now $\Delta^{n-1} x^{n-1} = (n - 1)!$ means

$$\Delta^n x^{n-1} = 0.$$

Hence we have:

$$F_n( n - 1) = 0.$$

Now from the proved recursion relationship of (38), when r = n,

$$F_n(n) = F_{n-1}(n - 1) + n\, F_n(n - 1),$$

or as $F_n(n - 1) = 0$,

$$F_n(n) = F_{n-1}(n - 1).$$

Since n is arbitrary,

$$F_n(n) = F_{n-1}(n - 1) = F_{n-2}(n - 2) = \dots = F_1(1).$$

Now since $F_1(1) = 1$,

$$F_n(n) = 1. \qquad (43)$$

This completes the proof that,

$$f_r(n) = F_r(n),$$

or that,

$$f_r(n) = \sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1}(r - m + 1)^n. \tag{44}$$

Now with $f_r(n)$ we have from (30) and (33):

$$\sum_{k=1}^{\infty} k^n x^k = \frac{x}{(1-x)^{n+1}} \sum_{r=1}^{n} \left[\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1}(r - m + 1)^n\right] x^{r-1}, \tag{45}$$

or in a slightly different form,

$$\sum_{k=1}^{\infty} k^n x^k = \frac{\sum_{r=1}^{n} \left[\sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1}(r-m+1)^n\right] x^r}{(1 - x)^{n+1}}. \tag{46}$$

The application of all this information about (26) to (25) is immediate.

A summary of the results of this investigation and a discussion of their application is now in order. The methods and relationships mentioned at the first of this chapter have now been developed. We have the finite expression of the sum (25) by applying (45) or (46). We have the recursion relationships (29), (31), and (34). The best method of evaluation of the sum is by use of the summation triangle with (30) and (33).

In the summation of (25) the value of our methods depends on the following: The magnitude of n, the proximity of $/x/$ to unity, and the required degree of accuracy of the evaluation. Individual considerations must be given each series. For example, if n is small but $/x/$ is near unity our methods are of value, but if n is large and $/x/$ is small the convergence of the series is not seriously slow and our results are of less value practically speaking.

The evaluation of (25) for some particular value of x may be accomplished in various ways such as direct substitution in a finite number of terms or by expression of the polynomial coefficient of $x^k$ in factorials followed by summation by parts. However, the methods presented here are of greatest value for this evaluation.

For clarification we give typical examples where these methods are particularly suited.

First, let us evaluate the infinite power series,

$$A = 1 + x + 8x^2 + 27x^3 + 64x^4 + \dots + k^3x^k + \dots,$$

for $x = 0.90000$ to five decimal accuracy. The exact value may be found by formulas(30)and(33)and the summation triangle and substitution for x. Thus we have,

$$S = 1 + \sum_{k=1}^{\infty} k^3x^k$$

$$= 1 + \frac{(-1)^4 x}{(x - 1)^4}(1 + 4x + x^2)$$

$$= 48{,}691.$$

Second, evaluate the convergent infinite series,

$$S = \frac{2 + 3 + 1}{2} + \frac{2.2 + 3.2^2 + 2^4}{2^2} + \frac{2.3 + 3.3^2 + 3^4}{2^3} + \ldots$$

$$\ldots + \frac{2k + 3k^2 + k^4}{2^k} + \ldots$$

Again using formulas(30)and(33)and the summation triangle, we have:

$$S = \sum_{k=1}^{\infty} \frac{2k + 3k^2 + k^4}{2^k}$$

$$= 2\sum_{k=1}^{\infty} k(\tfrac{1}{2})^k + 3\sum_{k=1}^{\infty} k^2(\tfrac{1}{2})^k + \sum_{k=1}^{\infty} k^4(\tfrac{1}{2})^k$$

$$= 172.$$

Finally, let us find a finite expression for the sum of an infinite series whose k'th term is the product of the k'th term of an arithmetic progression, $b$, $b + d$, $b + 2d$, ..., $b + (k - 1)d$, and the k'th term of a geometric progression, $a$, $ar$, $ar^2$, ..., $ar^{k-1}$. The series is convergent for $r < 1$.

$$S = ab + \sum_{k=1}^{\infty} (b + kd)ar^k$$

$$= ab + ab \sum_{k=1}^{\infty} r^k + ad \sum_{k=1}^{\infty} kr^k$$

$$= a \frac{b - br + dr}{(1 - r)^2} .$$

A study of the summation triangle.

Some interesting properties of the summation triangle follow immediately from its law of formation.

First, the elements of each row are symmetrical about the element or elements at the middle of the row. That is,

$$f_r(n) = f_{n-r+1}(n); \ 1 \leqq r \leqq n. \tag{47}$$

The proof is inductive with respect to n. It is evidently true for $n = 1$. If it is true for n, we have the two relations:

$$(n - r + 2)f_{r-1}(n) = (n - r + 2)f_{n-r+2}(n), \ 2 \leqq r \leqq n + 1.$$

$$r\, f_r(n) = r\, f_{n-r+1}(n), \ 1 \leqq r \leqq n.$$

If we add equals to equals in the above relationships, we have:

$$(n - r + 2)f_{r-1}(n) + r\, f_r(n) = r\, f_{n-r+1}(n) + (n - r + 2)f_{n-r+2}(n),$$

$$2 \leqq r \leqq n.$$

Now by (34) we have:

$$f_r(n + 1) = f_{n-r+2}(n + 1), \ 2 \leqq r \leqq n.$$

Now since $f_1(n + 1) = f_{n+1}(n + 1) = 1$, this relation holds for $1 \leqq r \leqq n + 1$. Thus the symmetry holds for $n + 1$ and the proof of (47) is complete.

Second, for each line the elements increase from the ends to the middle, that is,

$$\begin{matrix} f_r(n) < f_{r+1}(n) \\ f_{n-r+1}(n) > f_{n-r}(n) \end{matrix} \qquad r \leqq \begin{cases} \frac{n}{2} - 1 & n \text{ even} \\ \frac{n-1}{2} & n \text{ odd} \end{cases} \tag{48}$$

It is sufficient to prove the first inequality as the second inequality follows from the first by the symmetry just proved.

The proof is inductive with respect to n. Evidently the inequality is true for $n = 3$. Now we will assume this inequality holds for n odd and prove it holds for $n + 1$. Repeating for n even completes the proof.

Hence $f_r(n) < f_{r+1}(n)$ for $r \leqq \frac{n-1}{2}$ and n odd gives us the inequalities:

$$f_r(n) < f_{r+1}(n); \; f_{r-1}(n) < f_r(n); \; f_{r-1}(n) < f_{r+1}(n).$$

If we multiply the inequalities by $r$, $n - r + 1$, and $1$ respectively, add, and use (34) we obtain:

$$f_r(n + 1) < f_{r+1}(n + 1), \; r \leqq \frac{n+1}{2} - 1.$$

Now for n even we use the relations $f_r(n) < f_{r+1}(n)$ for $r \leqq \frac{n}{2} - 1$ and $f_r(n) = f_{r+1}(n)$ for $r = \frac{n}{2}$. This latter equation is from the symmetry. Proceeding as above we obtain:

$$f_r(n + 1) < f_{r+1}(n + 1), \; r \leqq \frac{(n+1) - 1}{2}.$$

Thus the induction is completed and (48) is proved.

<u>An Abel sum</u>.

If a series diverges, it may be possible to use some other combination than the first $n$ terms as an approximation to the "sum" of the divergent series, (9, p.261-265). These sums are set up by definition and not by analogy with sums of convergent series. Obviously these sums have a different meaning than in the case of convergent series, but they are often of practical value. This is true especially when this sum is regular, that is, the methods used to obtain the sum will sum a convergent series to the ordinary sum.

Now we have an example of Abels sum for the oscillating series obtained by setting $x = -1$ in the expression for $K_n(x)$. We have from the definition of Abels sum,

$$A = \lim_{x \to 1-} \sum_{k=0}^{\infty} u_k x^k,$$

and (46),

$$\sum_{k=1}^{\infty} (-1)^k k^n = \lim_{x \to -1} \frac{1}{(1-x)^{n+1}} \sum_{r=1}^{n} \left[ \sum_{m=1}^{r} (-1)^{m+1} \cdot \binom{n+1}{m-1} (r-m+1)^n \right] x^r$$

$$= \frac{1}{2^{n+1}} \sum_{r=1}^{n} \left[ \sum_{m=1}^{r} (-1)^{m+1} \binom{n+1}{m-1} (r-m+1)^n \right] (-1)^r, \qquad (49)$$

by the method of Abel.

$K_n(x)$ for negative integral n and 0.

To gain perspective on our previous results and to obtain some new results we will develop a finite expression for $K_n(x)$ for $n = -1$, and develop a method whereby $K_n(x)$ may be approximated for some particular value of x where $n \leqq -2$. We will use recursion relation (29), evidently true for any complex number n, in the form:

$$\sum_{k=1}^{\infty} k^n x^k = (xD)^{-1} \sum_{k=1}^{\infty} k^{n+1} x^k, \tag{50}$$

with the initial condition, $K_o(x) = \frac{x}{1 - x}$, for this extension.

Hence we have,

$$\sum_{k=1}^{\infty} k^{-1} x^k = (xD)^{-1} \sum_{k=1}^{\infty} x^k$$

$$= (xD)^{-1} \frac{x}{1 - x}$$

$$= \int \frac{dx}{(1 - x)} + C$$

$$= -\ln(1 - x) + C.$$

Since $\sum_{k=1}^{\infty} k^{-1} x^k = 0$ for $x = 0$, $C = 0$. This gives,

$$\sum_{k=1}^{\infty} k^{-1} x^k = -\ln(1 - x), \tag{51}$$

The known expression for the sum of this common series.

For $n = -2$ we obtain in the same way a non-finite expression in the form of an integral, an expected result for the sum of such a series. It is convenient to consider the integral in its definite form with limits 0 and x. The constant drops out and we obtain:

$$\sum_{k=1}^{\infty} k^{-2}x^k = -\int_0^x \frac{\ln(1-x)}{x}\,dx$$

$$= \int_0^x \frac{\ln\frac{1}{1-x}}{x}\,dx. \qquad (52)$$

Repeating this process for successive negative integers, we obtain a multiple integral of multiplicity (n - 1) for $K_n(x)$. We have:

$$\sum_{k=1}^{\infty} k^n x^k = \int_0^x \frac{1}{x}\int_0^x \frac{1}{x}\ldots\int_0^x \frac{1}{x}\ln\frac{1}{1-x}\,(dx)^{n-1}. \qquad (53)$$

From a practical standpoint approximate values of these infinite series may be found by numerical integration of (53). The utility of (53) decreases as /n/ increases for two reasons, the integration becomes more tedius and evaluation by other means becomes easier due to the increasing convergency of the series. Note that these series converge for x = -1 when $n \leq -1$.

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