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Title: A MODEL OF PEANO'S AXIOMS IN EUCLIDEAN GEOMETRY

Abstract approved: Redacted for privacy
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The purpose of this paper is to show that arithmetic is consistent if Euclidean geometry is. Specifically, a model of Peano's axioms [2] is defined in the space of Euclidean geometry, where Hilberts axioms [3] are taken to be the axioms of Euclidean geometry.

A Model of Peano's Axioms in Euclidean Geometry

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A MODEL OF PEANO'S AXIOMS
IN EUCLIDEAN GEOMETRY

I. INTRODUCTION

An axiomatic system consists of a collection of undefined terms and unproved statements, called axioms, which concern these terms. An axiomatic system is called consistent if no contradiction can ever occur as a result of statements following logically from the axioms. The proof of the consistency of any axiomatic system is then a very complex problem. We may, however, prove that the consistency of an axiomatic system A implies the consistency of a system B by defining a model of B in A . That is, by interpreting the undefined terms of B to be objects in the system A and then showing that the axioms of B can be proved as theorems resulting from the axioms of system A . Then any inconsistency in B will imply, through the model, an inconsistency in A .

The following notation will be used throughout the thesis.

(1) Capital letters will be used to denote points.

(2) If A and B are distinct points then

(a) $\underline{AB}$ will denote the unique line incident upon A and B .

(b) AB will denote the set of points on $\underline{AB}$ which are between A and B .

(c) $\overline{AB} = AB \cup \{A,B\}$.

(d) $\overrightarrow{AB}$ will denote the open ray with origin A which contains the point B .

(3) $A \overset{*}{B} C$ will denote that B is between A and C .

(4) The triangle incident upon non collinear points A , B , and C will be denoted ΔABC .

(5) The phrase <u>such</u> <u>that</u> will be denoted by the symbol •)• .

(6) The usual symbols of set theory and for quantifiers will be used.

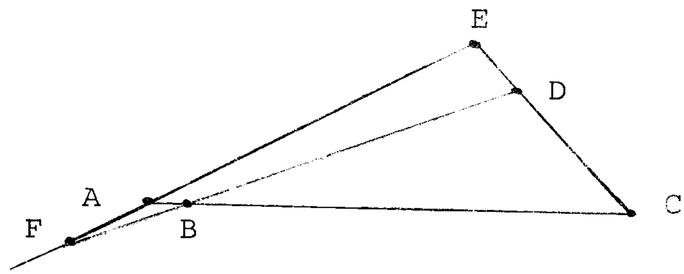
The theorems of Section I will be proved using the classical "steps-and-reasons" scheme. The reader will notice that the phrase <u>a</u> <u>lemma</u> will be used to substantiate some steps of proofs in this section. This phrase means that the step does not follow immediately from the pre-ceeding steps, but rather needs a trivial argument for substantiation. Most generally these lemmas involve a collinearity argument which is mechanical and uninterest-ing in nature.

II. SOME THEOREMS OF GEOMETRY

This section consists of a study of some of the problems of geometry [1] which are related to the betweeneness relation of Hilbert's axioms.

THEOREM 1.

If A and C are distinct points then there is a point $B \in AC$.



Proof:

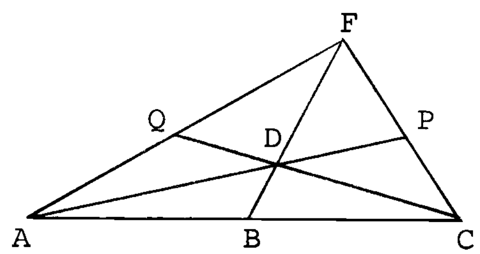
Steps	Reason
1. $\exists$ a point D not on $\underline{AC}$	1. Axiom I - 8.
2. $\exists$ a point ·)· $D \in CE$	2. Axiom II - 3.
3. $E \notin \overline{AC}$	3. a lemma
4. $\exists\ \Delta ACE$	4. def. of triangle
5. $\exists$ a point F ·)· $A \in EF$	5. Axiom II - 3.
6. $F \neq D$	6. a lemma
7. $\exists$ a line $\underline{DF}$	7. Axiom I - 1.
8. $\underline{DF}$ meets ΔACE at a point $B \neq D$	8. Step 2 and Paschs axiom
9. $B \notin \overline{CE}$	9. a lemma

10.	B $\notin \overline{AE}$	10.	a lemma.
11.	B $\in$ AC	11.	steps 8,9, and 10.

QED

THEOREM 2.

Given three distinct collinear points, denoted A, B, and C, exactly one of them is between the other two.



Proof:

	Steps		Reasons
1.	At most one of A, B, and C is between the other	1.	Axiom II - 2.
2.	Assume none of $\overset{*}{ABC}$, $\overset{*}{ACB}$ or $\overset{*}{BAC}$ is true	2.	A proof by contradiction.
3.	$\exists$ a point D not on $\underline{AB}$.	3.	Axiom I - 8.
4.	$\exists$ a point F on $\underline{BD}$ ·)· $\overset{*}{BDF}$	4.	Axiom II - 3.
5.	B, C, and F are noncollinear	5.	A lemma.

6.	$\exists\ \Delta BCF$	6.	Definition of triangle.
7.	$D \neq A$	7.	D is not on $\underline{AB}$.
8.	$\underline{AD}$ meets ΔBCF in a point $P \neq D$.	8.	step 4 and Pasch's axiom.
9.	If $P \in \overline{BC}$ then $P = A$.	9.	A lemma.
10.	$P \notin \overline{BC}$	10.	Steps 2 and 9.
11.	If $P \in \overline{BF}$, then $\underline{BF} = \underline{AD}$	11.	A lemma.
12.	If $\underline{BF} = \underline{AD}$, then D is on $\underline{AB}$	12.	A lemma.
13.	$P \notin \overline{BF}$	13.	Steps 3, 11, and 12.
14.	$P \in FC$	14.	Steps 8, 10, and 13.
15.	$\exists\ Q \neq D \cdot)\cdot$ $Q \in FA$ and Q is on $\underline{CD}$	15.	Symmetry.
16.	$\{A, P, F\}$ is a non-colinear set.	16.	A lemma.
17.	$\exists\ \Delta APF$	17.	Definition of triangle.
18.	$\underline{QD}$ meets ΔAPF in a point $R \neq Q$.	18.	Step 15 and Pasch's axiom.
19.	If $R \in \overline{AF}$ then $\underline{AF} = \underline{CD}$, which implies D is on $\underline{AC} = \underline{AB}$.	19.	A lemma.

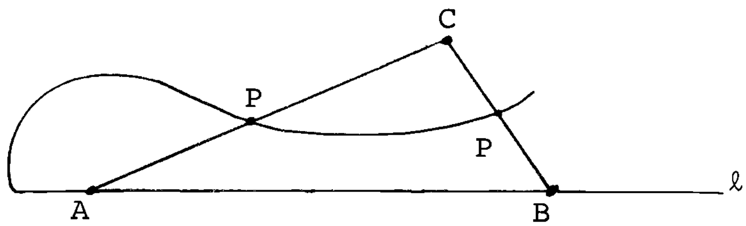
20.	$R \in \overline{AF}$	20.	Steps 3 and 19.
21.	If $R \in \overline{FP}$ then $\underline{CF} = \underline{CR} = \underline{CD}$ and then $\underline{CD} = \underline{DF} = \underline{BF}$, which implies D is on $\underline{BC} = \underline{AB}$.	21.	A lemma.
22.	$R \notin \overline{FP}$	22.	Steps 3 and 21.
23.	$R \in AP$	23.	Steps 18, 20, and 22.
24.	$R \neq D$ implies $\underline{AD} = \underline{AQ}$, which implies D is on $\underline{AB}$.	24.	A lemma.
25.	$R = D$	25.	Steps 3 and 24.
26.	$D \in AP$	26.	Steps 23 and 25.
27.	$\exists\ \triangle APC$	27.	A lemma.
28.	$\underline{DB}$ meets $\triangle APC$ in a point $T \neq D$.	28.	Step 26 and Pasch's axiom.
29.	$T \notin \overline{AP}$	29.	A lemma.
30.	If $T \in \overline{PC}$ then $T = F$	30.	A lemma.
31.	$T \in PC$	31.	Step 14, step 30, and axiom II - 2 .

32. $T \neq P$ and $T \neq C$	32. Step 14, step 30 and since between is a relation concerning points and pairs of *distinct other* points.
33. $T \in AC$	33. Steps 28, 29, 31, and 32.
34. If $T = B$ then $B \in AC$	34. Step 33.
35. $T \neq B$	35. Step 32 and step 2.
36. B and T are points common to $\underline{AB}$ and $\underline{DB}$	36. Steps 28 and 33 for T, and by definition for B.
37. $\underline{AB} = \underline{DB}$	37. Step 36 and a lemma.
38. D is on $\underline{AB}$	38. Step 37.
39. Assumption of step 2 is false.	39. Step 38 contradicts step 3.

QED

THEOREM 3.

A line incident upon two vertices of a triangle cannot have a point in common with each side of that triangle.



Proof:

Step	Reason
1. Let ΔABC be a triangle which has a line ℓ incident upon two of its vertices, say A and B	1. Hypothesis.
2. Assume there is a point P on ℓ and on AC	2. A proof by contradiction.
3. $\underline{AB} = \underline{AP} = \underline{AC}$	3. Axiom I - 2 (used twice).
4. A, B, and C are collinear	4. Def. of triangle.
6. There is no point P on both ℓ and on AC	6. logical consequence.

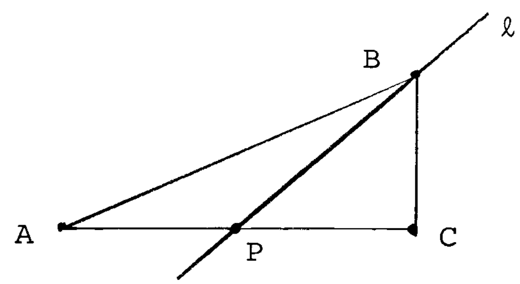
7. ℓ and BC have no point in common — 7. symmetry.

8. Steps 6 and 7 prove the lemma.

QED

THEOREM 4.

If a line ℓ contains one vertex of a triangle, then ℓ either contains another vertex or contains at most one other point of the triangle.



Proof:

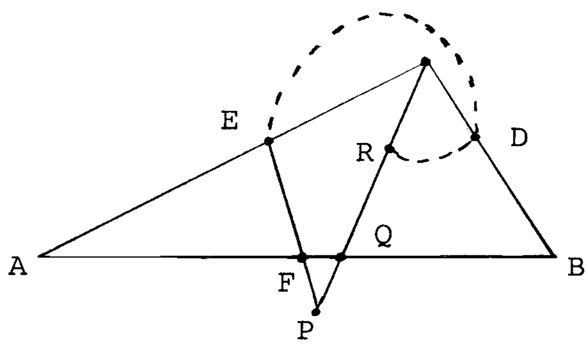
Step	Reason
1. let ℓ contain point B of ΔABC and no other vertex of ΔABC and further let $P \neq B$ be on both ℓ and ΔABC	1. assumptions.

2.	$P \notin AB$	2.	otherwise A is on ℓ.
3.	$P \notin BC$	3.	otherwise C is on ℓ.
4.	$P \in AC$	4.	a process of elimination.
5.	Suppose there is a $P' \in \{P,B\}$ on both ℓ and ΔABC	5.	A proof by contradiction.
6.	$P' \in AC$	6.	by symmetry with P .
7.	ℓ and $\underline{AC}$ have P and P' in common	7.	steps 4, 6.
8.	$\ell = \underline{PP'} = \underline{AC}$	8.	Axiom I - 2 .
9.	C is on ℓ	9.	since $\ell = \underline{AC}$.
10.	Step 9 contradicts assumptions of step 1	10.	C and B are on ℓ and ΔABC .
11.	Step 5 and the assumptions of the theorem are contradictory and therefore the theorem is true.	11.	as proven by contradiction.

QED

THEOREM 5.

A line ℓ which is not incident upon a pair of vertices of a triangle ΔABC can have at most two points in common with that triangle.



Proof:

	Step		Reason
1.	$\ell \neq \overline{AB}$; $\ell \neq \overline{BC}$, $\ell \neq \overline{AC}$	1.	Axiom I - 2.
2.	suppose ℓ meets ΔABC in three distinct points D, E, and F .	2.	a supposition, for a proof by contradiction.
3.	ℓ has at most one point on any side of ΔABC in common with it.	3.	a lemma.
4.	A is not an point of ℓ	4.	theorem 4 and step 1 .
5.	B and C are not on ℓ	5.	symmetry
6.	ℓ has exactly one point in common with each side of ΔABC	6.	steps 2, 3, 4 and 5.

7.	let $D \in BC$, $E \in AC$; $F \in AB$	7.	an arbitrary selection
8.	$\exists$ a point P ·)· $F \in EP$	8.	Axiom II - 3.
9.	P is not on $\underline{CE}$	9.	a lemma.
10.	$\exists$ Δ PCE	10.	Def of triangle.
11.	AB has a point $Q \neq F$ on Δ PCE	11.	step 7; Pasch's axiom.
12.	$Q \notin \overline{PE}$	12.	a lemma.
13.	$Q \notin \overline{CE}$	13.	a lemma.
14.	$Q \in CP$	14.	steps 11, 12, and 13.
15.	$Q \notin BC$	15.	a lemma.
16.	Q is a point of $\underline{BC}$ iff $Q = B$	16.	a lemma.
17.	if $B \neq Q$ then ΔBQC	17.	step 16, Def. of a triangle.
18.	$\underline{DE}$ has a point $R \neq D$ which is also on ΔBQC	18.	step 7 and Pasch's axiom.
19.	$R \notin \overline{BC}$	19.	a lemma.
20.	$R \notin \overline{BQ}$	20.	a lemma.
21.	$R \in CQ$	21.	steps 18, 19 and 20.
22.	$\underline{PC} = \underline{CQ}$	22.	step 14, axiom I - 2.
23.	R is on $\underline{CQ}$	23.	steps 21 and 22.
24.	R is on $\underline{PD}$	24.	step 18, since DE = PD.

25.	$R \neq P$	25.	a lemma based upon axiom II - 2.
26.	$\underline{PC} = \underline{PD} = \underline{DE}$	26.	a lemma.
27.	C is a point of $\underline{DE}$	27.	since $\underline{PC} = \underline{DE}$.
28.	step 27 contradicts step 5	28.	since $\ell = \underline{DE}$.
29.	Supposition of steps one and two are impossible together.	29.	as proved, by contradiction.

QED

<u>Corollary 1</u>

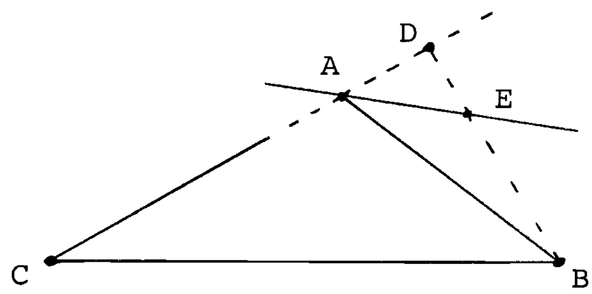
A line cannot have a point in common with each side of a triangle.

<u>Proof</u>: The corollary follows directly from theorems 3, 4, and 5 .

<u>Theorem 6</u>

Given a triangle ΔABC there is a line ℓ containing A and no other point of the triangle.

Proof:



Step	Reason
1. $\exists$ D $\cdot)\cdot$ A $\in$ CD	1. Axiom II′ - 3.
2. D $\notin$ $\overline{BC}$	2. a lemma.
3. $\exists$ ΔBCD	3. B, C, and D are noncollinear.
4. $\exists$ an E $\in$ BD	4. Theorem 1.
5. $\underline{AE} = \ell$ meets ΔBCD on two sides, at A on CD and at E on BD	5. steps 2, 4.
6. ℓ does not meet ΔBCD on BC	6. corollary 1.
7. neither B nor C is on ℓ	7. a lemma, based on step 2.
8. If ℓ contains a point of AC then it contains C	8. a lemma.

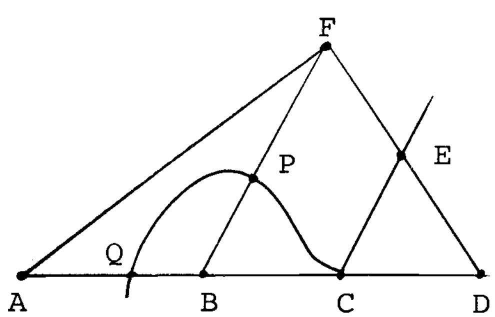
9. If ℓ contains a point of AB then it contains B	9. a lemma.
10. ℓ contains no points of AC or of AB	10. steps 6, 7, and 8
11. ℓ meets $\triangle ABC$ at A and no other point of $\triangle ABC$.	11. steps 4, 5, 6 and 9.

QED

<u>Theorem 7</u>

Given four distinct collinear points A, B, C, and D such that $A\overset{*}{B}C$, then a necessary and sufficient condition for $B\overset{*}{C}D$ is $A\overset{*}{C}D$.

<u>Proof</u> <u>of</u> <u>sufficiency</u>:

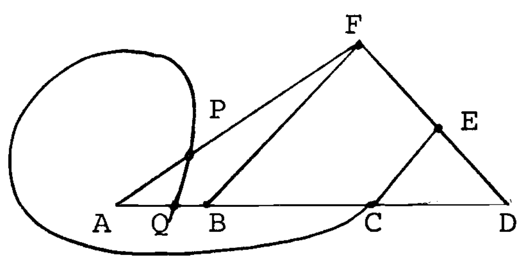


Step	Reason
1. suppose $\overset{*}{ABC}$ and $\overset{*}{ACD}$ hold for the given points	1. a supposition.
2. $\exists$ a point E not on $\underline{AB} = \underline{AC}$	2. Axiom I - 8.
3. $\exists$ a point F on $\underline{DE}$ $\cdot)\cdot$ $\overset{*}{DEF}$	3. Axiom II - 3.
4. F is not on $\underline{AB} = \underline{BD} = \underline{AD}$	4. a lemma.
5. $\exists$ ΔBFD, ΔABF, and ΔADF	5. step 4 and def. of triangle.
6. $\underline{CE}$ meets ΔBFD in a point $P \neq E$	6. step 3 and Pasch's axiom
7. $P \notin \overline{DF}$	7. a lemma.
8. If $P \in \overline{BD}$, then $P = C$, which implies $C \in BD$	8. a lemma.
9. suppose $P \in BF$	9. It will be shown that this is inconsistent with step 1 and hence $P \in \overline{BD}$.
10. $\underline{CE}$ meets ΔABF in a point $Q \neq P$	10. steps 6, 9 and Pasch's axiom.

11. $Q \notin \overline{BF}$	11. a lemma.
12. $Q \in \overline{AF}$	12. this would imply $\underline{CE}$ contains points of three sides of ΔADF, which contradicts corollary 1.
13. $Q \in AB$	13. steps 10, 11 and 12.
14. $Q \in AB$ implies $Q = C$	14. a lemma.
15. $C \in AB$	15. steps 13 and 14.
16. step 15 contradicts step 1	16. Axiom II - 2.
17. step 16 implies that $C \in BD$	17. step 8, reason 9.

QED

Proof of necessity:



Step	Reason
1. suppose $\overset{*}{ABC}$ and $\overset{*}{BCD}$	1. a supposition.
2. steps 2 - 5 of proof of sufficiency hold	2. as proven above.
3. $\underline{CE}$ meets ΔAFD in a point $P \neq E$	3. step 2 and Pasch's axiom.
4. $P \notin \overline{DF}$	4. a lemma.

5.	If $P \in \overline{AD}$ then $P = C$ and hence $C \in AD$	5.	a lemma.
6.	suppose $P \in AF$	6.	It will be shown that this supposition leads to a contradiction and hence $P \in \overline{AD}$.
7.	$\overline{CE}$ meets ΔABF in a point $Q \neq P$	7.	Pasch's axiom.
8.	$Q \in AB$	8.	symmetry from the above proof.
9.	$Q \in AB \Rightarrow Q = C$	9.	a lemma.
10.	$C \in AB$	10.	steps 8 and 9 .
11.	step 10 contradicts step 1.	11.	Axiom II - 2.
12.	$C \in AD$	12.	steps 5, 6, and 11; reason 6.

QED

<u>Corollary 1</u>

Given four distinct collinear points A, B, C, and D such that $A\overset{*}{B}C$ and $A\overset{*}{B}D$, then either $B\overset{*}{C}D$ or $B\overset{*}{D}C$, but not $D\overset{*}{B}D$.

<u>Proof</u>: The corollary is simply the contrapositive statement of Theorem 7, where the symbols A, B, C, and D of the theorem are renamed by the permutation (BDC).

Theorem 8

Given four distinct collinear points, denoted P, Q, R, and S, these points may be renamed A, B, C, and D in exactly two ways such that $A\overset{*}{B}C$ and $B\overset{*}{C}D$ (and hence $A\overset{*}{C}D$ and $A\overset{*}{B}D$).

Proof: assume without loss of generality that $P\overset{*}{Q}R$.

Step	Reason
1. Either $Q\overset{*}{R}S$, $Q\overset{*}{S}R$, or $R\overset{*}{Q}S$	1. Theorem 2.
2. If $Q\overset{*}{R}S$, rename as follows: $P \to A$, $Q \to B$, $R \to C$, $S \to D$	2. This gives $A\overset{*}{B}C$ and $B\overset{*}{C}D$.
3. another method of renaming P, Q, R, and S is found by the permutation (AD)(BC) of the above names	3. This also give $A\overset{*}{B}C$ and $B\overset{*}{C}D$, if $Q\overset{*}{R}S$.
4. In the case $Q\overset{*}{R}S$, naming the points other than as in steps 2 and 3 is not appropriate.	4. trial and error (may logically be reduced to three trials, by use of symmetry.)

5.	If $Q\overset{*}{S}R$, then $P\overset{*}{Q}R$ implies that $P\overset{*}{Q}S$	5.	Theorem 7.
6.	If $Q\overset{*}{S}R$, then rename $P \to A$, $Q \to B$, $S \to C$, $R \to D$ or permute these names by (AD)(BC)	6.	these are the only appropriate ways of renaming the points in this case, by symmetry from the above cases.
7.	If $R\overset{*}{Q}S$, then either $P\overset{*}{S}Q$ or $S\overset{*}{P}Q$	7.	$P\overset{*}{Q}S$ is impossible by Corollary 2.
8.	If $P\overset{*}{Q}S$ and $P\overset{*}{S}Q$, then rename $P \to A$, $S \to B$, $Q \to C$, $R \to D$, or permute these names by (AD)(BC)	8.	same reason as step 6.
9.	If $R\overset{*}{Q}S$ and $S\overset{*}{P}Q$, then rename $S \to A$, $P \to B$, $Q \to C$, $R \to D$, or permute these names by (AD)(BC)	9.	same as 8.

10. all cases have been exhausted and in each case the theorem was shown to be true.

QED

Theorem 9

If $P_1, P_2, \ldots, P_n$ are n distinct collinear points, where $n \geq 4$, then there are exactly two ways in which these points may be renamed $A_1, A_2, \ldots, A_n$ so that $A_j \in A_i A_k$ iff $i < j < k$.

Proof:

Step	Reason
1. suppose the theorem is true for $n = k - 1$, $k \geq 5$	1. a supposition for a proof by induction.
2. Given k distinct collinear points, $P_1, P_2, \ldots, P_k$, we may rename $P_1, P_2, \ldots, P_{k-1}$ by $A_1, \ldots, A_{k-1}$ in exactly two ways •)• $A_j \in A_i A_k$ iff $i < j < k$	2. step 1
3. either $A_1 \overset{*}{A}_{k-1} P_k$, $A_1 \overset{*}{P}_k A_{k-1}$, or $P_k \overset{*}{A}_1 A_{k-1}$	3. Theorem 2.

4. If $A_1 \overset{*}{A}_{k-1} P_k$, then $A_i \overset{*}{A}_j P_k$ for every pair (i,j) iff $i < j$	4. Theorem 7.
5. If $A_1 \overset{*}{A}_{k-1} P_k$, naming $P_k \rightarrow A_k$ satisfies the conditions needed in the renaming of the points	5. step 2 and 4.
6. Since there is no choice in naming P_k as A_k , there are two rênamings of $P_1, \ldots, P_k$ as $A_1, \ldots, A_k$.)· the theorem conclusion holds	6. step 2 and 5.
7. If $P_k \overset{*}{A}_1 A_{k-1}$, then rename $A_i \rightarrow A_{i+1}$ $i = 1, 2, \ldots k - 1$ and name $P_k \rightarrow A_1$	7. a renaming.
8. Step 7 results in exactly two satisfactory renamings of $P_1, \ldots, P_k$, depending upon the original two renamings by $A_1, \ldots, A_k$ of step 2	8. There is no choice for the new name of P_k .

9.	If $A_1 \overset{*}{P}_k A_{k-1}$, then suppose A_{k-1} for some j ·)· $1 \leq j \leq k$ and rename the points of $\{P_{1}, \ldots, P_k\} - \{P_j\}$ by $A'_1, \ldots, A'_{k-1}$ so that the conditions of the theorem hold	9.	renaming by step 1.
10.	The we have $A'_1 \overset{*}{A}'_{k-1} P_j$	10.	follows from step 9.
11.	Rename P_j by A'_k	11.	a renaming.
12.	step 9 and 11 constitute a satisfactory renaming of $P_1, \ldots, P_k$	12.	same reason as step 8.
13.	In each of the three cases stated in step 3, the theorem is true	13.	step 6, 8, 12.
14.	The truth of the theorem for $n = k - 1$ implies the truth for $n = k$ $k \geq 5$	14.	step 13.
15.	Since the theorem is true for $k = 4$, step 14 implies that the theorem is true	15.	axiom of induction.

QED

Theorem 10

Every line has an infinite number of points upon it.

Proof:

	Step		Reason
1.	Every line has at least two distinct points, A and C	1.	Axiom I - 5.
2.	$\exists\ D \cdot)\cdot\ C \in AD$	2.	Axiom II - 3.
3.	$\exists\ B \cdot)\cdot\ B \in AC$	3.	Theorem 1.
4.	$D \neq B$	4.	Axiom II - 2, steps 2,3.
5.	A, B, C, and D are distinct	5.	steps 2, 3, 4.
6.	every line contains at least 4 distinct points	6.	step 5.
7.	assume that there is a line ℓ containing n points $P_1, \ldots, P_n$, where $4 \leq n$. With n a finite number	7.	an assumption for a proof by contradiction.
8.	The n points may be renamed $A_1, A_2, \ldots, A_n$ so that $A_j \in A_iA_k$ iff $i < j < k$	8.	Theorem 9.

9. $\exists$ P in ℓ $\cdot)\cdot$ $A_{n-1}\overset{*}{A}_nP$	9. Axiom II - 3.
10. $P \neq A_j$ $j = 1,2, \ldots, n$	10. since $j \leq n$, this follows from step 8.
11. step 10 contradicts assumption of step 7	11. ℓ contains at least $n + 1$ points.
12. Theorem is true	12. follows from the contradiction.

QED

Theorem 11

A point P of a line ℓ separates the set of points on ℓ into three disjoint nonempty classes $\cdot)\cdot$ distinct points Q and R are in the same class iff $P \notin \overline{QR}$.

Proof.

Step	Reason
1. $\exists$ a point $A \neq P$ on ℓ	1. Axiom I - 5.

	Statement		Reason
2.	Define two classes of point of the set of points on ℓ, denoted S_1 and S_2 as follows: a point B is in S_1 if $P \in AB$ and B is in S_2 if $B \neq P$ and $P \notin AB$ and define a third class as $\{P\}$	2.	definitions.
3.	$S_1 \cap S_2 = \emptyset$	3.	Axiom II - 2 .
4.	$P \notin S_1$	4.	$P \in AP$.
5.	$P \notin S_2$	5.	$P = P$.
6.	S_1 , S_2 and $\{P\}$ are disjoint (Pairwise)	6.	steps 3, 4, 5 .
7.	$\{P\} \neq \emptyset$	7.	$P \in \{P\}$.
8.	$\exists$ a point B $\ni$ $P \in AB$	8.	Axiom II - 3 .
9.	$S_1 \neq \emptyset$	9.	step 8 .
10.	$S_2 \neq \emptyset$	10.	$A \neq P$, $P \notin AA = \emptyset \Rightarrow A \in S_2$
11.	Let Q be a point of ℓ	11.	Show that $Q \in S_1$ or $Q \in S_2$ or $Q \in \{P\}$.
12.	If $Q = P$, then $Q \in \{P\}$ and if $Q = A$ then $Q \in S_2$	12.	Reason 10.

13. If $Q \neq P$, then either $P \in AQ$ or $P \in AQ$, which implies $Q \in S_1$ or $Q \in S_2$ respectively	13. Theorem 2.
14. $\{P\}$, S_1, and S_2 are nonempty, disjoint sets whose union consists of all the points on ℓ	14. steps 6, 7, 8, 9, 10, and 13.
15. Suppose $P \notin \overline{QR}$, where Q and R are distinct points on ℓ	15. show this implies that Q and R are in the same class.
16. $P \notin QR$ implies $P\overset{*}{Q}R$ or $P\overset{*}{R}Q$	16. Theorem 2.
17. If $P = Q$ then $P \in \overline{PR} = \overline{QR}$	17. $\overline{PR} = \{P,R\} \cup PR$.
18. $P \neq Q$	18. steps 15 and 17.
19. $Q \in S_1$ implies $A\overset{*}{P}Q$	19. step 2.
20. $Q \in S_1$ and $P\overset{*}{Q}R$ implies $A\overset{*}{P}R$, which implies $R \in S_1$	20. step 19 and Theorem 7.
21. $Q \in S_1$ and $P\overset{*}{R}Q$ implies $A\overset{*}{P}R$, which implies $R \in S_1$	21. step 19 and Theorem 7.

22.	$P \notin QR$ and $Q \in S$, implies $R \in S_1$ and $R \in S_1$ implies $Q \in S_1$ (by symmetry)	22. Steps 16, 20, and 21.
23.	$Q \in S_2$ implies $P \notin AQ$ and $Q \neq P$	23. step 2.
24.	$Q \in S_2$ and $P\overset{*}{R}Q$ or $P\overset{*}{Q}R$ implies $R \in S_2$	24. a lemma.
25.	Distinct points Q and R are in the same set, among S_1 and S_2 , if $P \notin QR$	25. steps 16, 20, 21, 24.
26.	Now change the assumption of step 15 to read $P \in \overline{QR}$, for distinct points Q and R on ℓ	26. show that this implies that Q and R are not in the same set among S_1 , S_2 and $\{P\}$.
27.	If $Q = P$ then $P \neq R$ and hence $R \notin \{P\}$, but $Q \in \{P\}$.	27. Since R and Q are distinct.
28.	If $Q \in S_1$ then $Q\overset{*}{P}R$ and $A\overset{*}{P}Q$ implies that $P \notin AR$.	28. Corollary 2.
29.	$Q \in S_1$ implies $R \in S_2$ or $R \in \{P\}$	29. step 14 and step 28.

30.	If $Q \in S_2$ then $Q\overset{*}{P}R$ and $P\overset{*}{A}Q$ or $P\overset{*}{Q}A$ implies that $P \in \overline{AR}$	30.	Theorem 7.
31.	$Q \in S_2$ implies $P \in AR$ or $P = R$ and hence $R \notin S_2$	31.	step 30, step 2.
32.	The theorem has been proven except there is left to show that the construction is independent of the arbitrary point A of step 1.	32.	remains to be shown.
33.	suppose we choose $A' \neq A$ of step 1 .	33.	show that replacing A by A' in step 2 will separate the points of ℓ into the same three sets, except possibly for a permutation of names of sets S_1 and S_2 .

34. A' separates the points of ℓ into the same three disjoint sets as does the point A	34. A lemma shows that the conditions of step 2, with A replaced by A', imply the result of this step since Q and R (distinct) are in the same set iff $P \notin \overline{QR}$.

QED

The sets S_1 and S_2 of theorem 11 are called sides of ℓ with respect to P , or rays of ℓ with origin P . Since the sets $\{P\}$, S_1 , and S_2 form a partition of the points on ℓ , any one of these sets may be designated by designating a point of that set. Hence we may use the notation $\overrightarrow{PA}$ to designate the ray of ℓ , with origin P , which contains the point A . Special note should be made of the fact that $P \notin \overrightarrow{PA}$.

It should be noted that mathematical induction is used as a method of proof in theorems nine and ten. These two theorems are introduced simply for intuitive security and are not used in the development of the model of Piano's axioms found in the next section. Noting this, we see that no circularity of argument results from the use of induction found in these proofs.

The purpose of theorem ten is to assure us that the cardinality of the set of points on a given line is sufficiently large to enable us to find a one-to-one correspondence between the set of natural numbers and some subset of this set. Theorem 9 convinces us that, given such a subset, we may "line up" the points of this subset in a manner analogous to the way the natural numbers are "lined up".

III. A GEOMETRIC MODEL OF PEANO'S AXIOMS

In order to develop a model of Peano's axioms in Euclidean geometry the undefined terms of Peano's axioms must be defined as objects of Euclidean geometry. This implies that we must define a set N of Euclidean objects, the elements of which will be called natural numbers, and we must define a map $\mathscr{S}$ from N into N, which will be known as the successor map. These definitions must be made such that Peano's axioms, applied to N and $\mathscr{S}$, may be proven, using only Hilbert's axioms and Aristotelian logic, as theorems of geometry. It should be noted that neither a recursive definition nor any other use of induction may be used to define N or $\mathscr{S}$, since this would lead to a circular argument when an attempt is made to prove Peano's axiom of induction. Keeping this in mind we proceed as follows:

Peano's axioms for the natural numbers may be stated as follows: The undefined terms of Peano's axioms are a set N, called the set of natural numbers and whose elements are called natural numbers, and a mapping $\mathscr{S}$, of N into N, called the successor map. The image of a natural number n under the successor map is called the successor of n. Concerning these undefined terms we have the following axioms.

I. $N \neq \emptyset$.

II. $\mathscr{S}$ is one-to-one.

III. $\mathscr{S}$ is not onto N (The range of $\mathscr{S}$ is not N .)

IV. If M is a subset of N such that (1) M contains an element of N which is not in the range of $\mathscr{S}$ and (2) if n is in M then the successor of n is in M , then $M = N$.

Giving the undefined terms of Peano's axioms concrete definitions as objects of Euclidean geometry is the next step in developing a model. Intuitively, the natural numbers will be defined as "equally spaced" points on a given half-line with origin P_o . Then the successor of a natural number P will be defined as the point Q among these "equally spaced" points that is "directly after" P . Figure 1 shows that for a given "spacing"

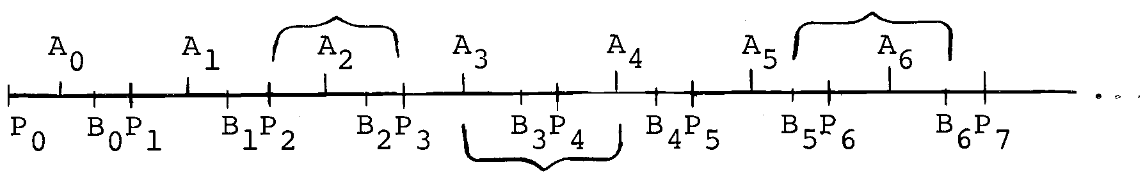


Figure 1. Three sets of "equally spaced" points.

there are many different sets of "equally spaced" points on a given half-line. The definition of the set of natural numbers must then select one of these sets of equally spaced points.

The "spacing" mentioned above will be defined as follows. First, given a set M of points, two points A and B in M are said to be <u>adjacent</u> provided no member of M is between A and B. Next, a pair of points A and B are called <u>CD-spaced</u>, where C and D are arbitrary points, provided $\overline{AB} \cong \overline{CD}$. Finally, a set of points are said to be <u>CD-spaced</u> iff the set is a collinear set and adjacent points of the set are CD-spaced. Hence a set of points that are CD-spaced will be equally spaced points along a line.

Choose now a line l and a point P_0 on l. P_0 divides the set of points on l into three disjoints sets, the set $\{P\}$ and the two rays along l with origin P_0. Choose one of these two rays by choosing arbitrarily a point $P_1 \neq P_0$ on l, and then considering the ray containing P_1. This ray, denoted by $\overrightarrow{P_0P_1}$, or by $\vec{r}$ will be used as the source of points from which the points destined to be defined as the natural numbers of our model will be chosen.

Preliminary to selecting points from the ray $\vec{r}$ the following definition will be made.

<u>Definition 1</u>. Define a function $T\colon \overline{P_0P_1} \longmapsto \overline{P_0P_1}$ by

$T(P) = Q$, where $P_0 \overset{*}{P} Q$ and $\overline{P_0P_1} \cong \overline{PQ}$.

To prove that T is a well defined function we must show that $T(P)$ is uniquely defined and that the range of T is in the ray $\overline{P_0P_1}$. Knowing that P is in the open line segment $P_0T(P)$ is enough to designate the ray of l with origin P_0 to which $T(P)$ belongs. Then axiom III - 1 of Hilbert's axioms assures us that there is a unique point Q on this designated ray such that $\overline{P_0P_1} \cong \overline{PQ}$ and hence $T(P) = Q$ is unique. That Q is on the ray $\overline{P_0P_1}$ follows from the fact that $P \in P_0Q$ and $P \in P_0P_1$.[1]

Now, using the function T and set intersection, the set N of natural numbers will be defined. Let A be any subset of the ray $\overline{P_0P_1}$. Then let A be called a <u>T-subset</u> of $\overline{P_0P_1}$ if and only if (1) $P_1 \in A$ and (2) $P \in A$ implies $T(P) \in A$. Designate the set of all T-subsets of $\overline{P_0P_1}$ by $\mathcal{J}$.

<u>Definition 2</u>. Define the set N of natural numbers by the

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[1] To show Q is on the given ray note that $P \in \overline{P_0P_1}$ implies that $P_0 \overset{*}{P_1} P$ or $P_0 \overset{*}{P} P_1$. Then in the first case $P \in P_0Q$ and $P_0 \overset{*}{P_1} P$ implies by lemma that $P_1 \in P_0Q$ and hence Q is on the given ray. In the second case the conditions $P_0 \overset{*}{P} P_1$ and $P \in P_0Q$ imply by lemma that P_0 is not in QP_1 and hence Q is on the given ray.

identity $N = \bigcap_{A \in \mathcal{J}} A$.

Definition 3. Define the successor map $\mathcal{S} : N \longmapsto N$ to be the restriction of the map T to the subset N of $\overline{P_0P_1}$.

Intuitively it can be seen that a T-subset of $\overline{P_0P_1}$ is a collection of "infinite" $\overline{P_0P_1}$ - spaced sets, one of which is the subset N . Then we might guess that N itself is a T-subset, and hence the smallest T-subset, and also N could be conjectured to be the largest P_0P_1 - spaced subset of $\overline{P_0P_1}$ which contains the point P_1 . Both of these conjectures will prove to be true.

Before proceeding any further we should ask if the set of natural numbers and the successor map have been well defined. First, considering the definition of T-subset, it can be seen that a condition on the points of $\overline{P_0P_1}$ has been made and this condition specifies certain subsets of $\overline{P_0P_1}$ to be T-subsets. Then we might ask if $\mathcal{J}$ is a well defined set. That $\mathcal{J}$ is well defined follows from the fact that it is the subset of the power set of $\overline{P_0P_1}$ whose elements are specified to be T-subsets. Hence the definition of N as an intersection of T-subsets is well defined. Next we see that $\mathcal{S}$ is single valued, since T is, and hence the successor map is well defined. We need also to show that the range of $\mathcal{S}$ is actually contained in N , as indicated above. This leads to the following theorem.

<u>THEOREM 1</u>. $\mathcal{S}$ maps N into N .

Proof. Let P be a point in N . Then the definition of N implies that T(P) is in N . But since $P \in N$, $T(P) = \mathcal{S}(P)$ and, by definition of $\mathcal{S}$, $\mathcal{S}(P)$ is in N. Hence for every $P \in N$, $\mathcal{S}(P) \in N$.

QED

The set N and the map $\mathcal{S}: N \longmapsto N$ have been defined from geometric objects and terms without the use of induction. The proof of Peano's axioms as theorems of geometry follows. The theorems derived from Peano's axioms in our model will be called propositions.

<u>Proposition I</u>. $N \neq \emptyset$.

Proof. $P_1 \in N$, since $N = \bigcap_{A \in \mathcal{J}} A$, where $P_1 \in A$ for each $A \in \mathcal{J}$.

QED

<u>Proposition II</u>. $\mathcal{S}$ is one-to-one.

Proof. Suppose $\mathcal{S}(R) = Q$ and $\mathcal{S}(P) = Q$. This implies that R and P are in the open line segment P_0Q and hence both on the ray $\overline{QP_0}$. However, axiom III - 1 assures us that there is a unique point B in $\overline{QP_0}$ such that $\overline{P_0P_1} \cong \overline{BQ}$. But since the definition of $\mathcal{S}$ implies that $PQ \cong P_0P_1 \cong RQ$, it must be the case that $P = R$.

QED

Proposition III. $\mathscr{S}$ is not onto.

Proof. Consider the point P_1 , which is in N . The existence of a point $P \in N$ such that $\mathscr{S}(P) = P_1$ implies that $P \in P_0P_1$ and $\overline{PP_1} = \overline{P_0P_1}$. Now $P \in P_0P_1$ implies that $P \in \overline{P_1P_0}$. Then axiom III - 1 assures us that there is a unique point B in $\overline{P_1P_0}$ such that $\overline{BP_1} \cong \overline{P_1P_0}$. However $P_0 \in \overline{P_1P_0}$ and $\overline{P_0P_1} \cong \overline{P_0P_1}$. Hence it must be the case that $P_0 = P$. However P_0 is not in $\overline{P_0P_1}$ and, since $N \in \overline{P_0P_1}$, $P_0 \in N$. Hence there is no point in N whose image under $\mathscr{S}$ is P_1 .

QED

Before proving Peano's axiom of induction the following lemma will be proved.

Lemma. P_1 is the only element of N not in the range of $\mathscr{S}$.

Proof. Suppose Q is a point of N distinct from P_1 which is not in the range of $\mathscr{S}$. We need to show that there is a set $A \in \mathscr{J}$ which does not contain Q , thereby showing that Q is not in the intersection of all T-subsets of $\overline{P_0P_1}$, and hence not in N .

Case 1. Suppose Q is not in the range of T , which is defined in definition I. Then let A be any T-subset of $\overline{P_0P_1}$ which contains Q . Then the set $A - \{Q\}$ will be a T-subset also, since $P_1 \in A$ implies $P_1 \in A - \{Q\}$ and for every $P \in A$, $T(P) \in A$, since Q is not in the range of T . Hence Q is not in the T-subset $A - \{Q\}$. Hence $Q \notin N$.

Case II. Suppose Q is in the range of T . Then by the hypothesis, Q is not in the range of $\mathcal{S}$, and it must be true that $T(R) = Q$ implies that R is not in N. Hence, for each such R there is a T-subset which does not contain R . Then the intersection of all such T-subsets is obviously a T-subset which does not contain any element R whose image under T is Q . Denote this intersection by A_R . Then if A is any T-subset containing Q , then the intersection A with A_R will be a T-subset and also $A \cap A_R - \{Q\}$ will be a T-subset. But $A \cap A_R - Q$ does not contain Q and hence Q is not in N .

Proposition IV. Let P be an element of N which is not in the range of $\mathcal{S}$, and let $M \subseteq N$ such that (1) P is in M and (2) Q in M implies that $\mathcal{S}(Q)$ is in M . Then $M = N$.

Proof. By the proceeding lemma, $P = P_1$. Then P_1 is in M and for every Q in M , $\mathcal{S}(Q)$ is in M . Hence M is a T-subset. Thus we have that $N = \bigcap_{A \in \mathcal{J}} A \subseteq M$ and by the hypothesis, $M \subseteq N$. Ergo $M = N$.

QED

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