EFFECTS OF SHEAR DEFORMATION IN THE CORE OF A FLAT RECTANGULAR SANDWICH PANEL

COMPRESSIVE BUCKLING OF SANDWICH PANELS HAVING DISSIMILAR FACINGS OF UNEQUAL THICKNESS

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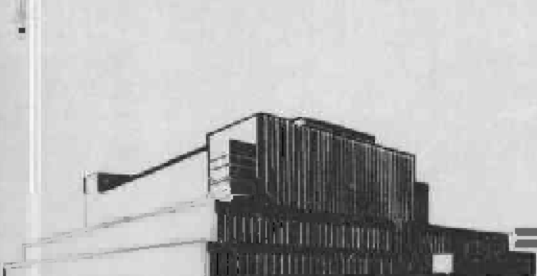

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In Cooperation with the University of Wisconsin

COMPRESSIVE BUCKLING OF SANDWICH PANELS HAVING DISSIMILAR FACINGS OF UNEQUAL THICKNESS[1, 2]

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Introduction

The buckling of a sandwich panel with facings of equal thickness under compressive end load was considered in U. S. Forest Products Laboratory Report No. 1583. In that report an approximate method for analyzing the effect of the transverse shear deformation in the core was used. This method of analysis led to relatively simple formulas for predicting the critical buckling load of sandwich panels with orthotropic core and facing materials and with various types of edge support, provided that the elastic properties of the sandwich in the two directions parallel to the edges were not too greatly different. Predictions by these formulas were found to be in satisfactory agreement with test results, as shown in U. S. Forest Products Laboratory Reports Nos. 1525-B, C, D, and E. For comparison of results with those obtained by different methods of analysis, reference is made to the analysis of the problem for simply supported panels by Reissner (9),[4] and by Seide and Stowell (11).

[1]This progress report is one of a series prepared and distributed by the Forest Products Laboratory under U. S. Navy, Bureau of Aeronautics No. NBA-PO-NAer 00619, Amendment No. 2, and U. S. Air Force No. USAF-PO-(33-038)49-4696E. Results here reported are preliminary and may be revised as additional data become available.

[2]Major revision by John J. Zahn, Engineer, of a report originally issued November 1950.

[3]Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[4]Underlined numbers in parentheses refer to Literature Cited at end of this report.

The problem for the case of simply supported loaded edges and clamped unloaded edges was considered by Seide (10). These publications, which were concerned with panels with isotropic facings and cores, indicate that for such constructions, the formulas developed in Report No. 1583 lead to substantially the same results as those derived on a more rigorous basis.

This report considers the compressive buckling of a rectangular sandwich panel having dissimilar facings of unequal thickness. As in Report No. 1583 the facing and core materials are taken to be orthotropic. Two of the orthotropic axes of the core and facings are assumed to be parallel to the edges of the panel and the third is then perpendicular to the facings.

The approximate method here employed is a generalization of that which was used in Report No. 1583 and which was previously applied in a number of British reports concerned with the behavior of sandwich panels with isotropic facings and core materials (12, 4, 3). In these prior publications the displacements in the core have been assumed to be such that plane sections initially perpendicular to the middle surface remain plane under deformation but rotate by an amount proportional to the slope of the normal displacement of the middle surface. A single proportionality factor has been used to specify these rotations about axes parallel to both the x and y axes, these axes being in coincidence with the edges of the panel.

In the present analysis, (Appendix A), two proportionality factors are used; one to specify rotations about axes parallel to the x-axis and the other to specify those about axes parallel to the y-axis.[5] By this means the results are made applicable to those sandwich constructions in which the elastic properties of the core material may be greatly different in the two directions. In order to analyze the displacements in a sandwich panel with facings of unequal thickness it is necessary to take surfaces other than the middle surface of the core as those which contain the fixed lines of the rotating planes.

For a panel with similar orthotropic facings it is found (Appendix C) that if the bending of the facings is neglected, the present method of analysis yields a formula for the buckling load of a simply supported panel which is the same as that obtained by the method of Libove and Batdorf (5), which consists in solving a set of differential equations subject to proper boundary conditions. For other types of support, however, the energy method applied here is expected to yield estimates of the buckling load that are somewhat greater than those which would be obtained by using the differential equations.

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[5] A quite similar type of analysis has been used by Dale and Smith (2), Appendix IV.

If the facings are considered as membranes it is possible to define a buckling load coefficient which, in the case of isotropic facing and core materials, depends mainly on two parameters, V and ρ. This dependence is shown in the form of curves for two of the types of support considered herein. Curves for the two remaining types of support which are discussed below are given in references (10) and (11).

Throughout the following analysis it is assumed that neither the core nor the facing materials are stressed beyond the elastic limit. A method by which the formulas may be applied in an approximate manner beyond the proportional limit of the facing material has been discussed in Report No. 1583. The problem of buckling in the plastic range is considered in reference (11) for simply supported panels.

Notation

a, b	dimensions of the panel with the sides b parallel to the line of action of the compressive load.
c	thickness of the core.
D	flexural stiffness of sandwich panel with dissimilar facings; defined by formula (2).
D_{f1}, D_{f2}	flexural stiffness of individual facings about their own middle planes; defined by formula (13).
E_f	Young's modulus of elasticity of similar isotropic facings.
E_{f1}, E_{f2}	Young's modulus of elasticity of dissimilar isotropic facings.
E_{x1}, E_{y1}, E_{x2}, E_{y2}	Young's moduli of elasticity of dissimilar orthotropic facings in the direction indicated by the subscripts.
f_1, f_2	thickness of the facings.
λ_f	$1 - \sigma^2$
λ_{f1}	$1 - \sigma_1^2$
λ_{f2}	$1 - \sigma_2^2$
λ_1	$1 - \sigma_{xy1}\sigma_{yx1}$
λ_2	$1 - \sigma_{xy2}\sigma_{yx2}$
μ'	shear modulus of isotropic core.
μ'_{zx}, μ'_{yz}	shear moduli of orthotropic core.
μ_{xy1}, μ_{xy2}	shear modulus of facings.
n	number of longitudinal half-waves in buckled surface.
P	compressive buckling load, pounds per inch.
p, p_f, p_M	buckling coefficients of sandwich panels with dissimilar facings; defined by formulas (1), (12), and (15).

ϕ	defined by formula (A36).
ρ	$\frac{a}{b}$ aspect ratio.
σ	Poisson's ratio of similar isotropic facings.
σ_1, σ_2	Poisson's ratio of dissimilar isotropic facings.
σ_{xy1}, σ_{yx1}	Poisson's ratios of dissimilar orthotropic facings.
σ_{xy2}, σ_{yx2}	Poisson's ratios of dissimilar orthotropic facings.
T_1, T_2	membrane stiffness of facings; defined by formula (2).
t	relative membrane stiffness parameter; defined by formula (6).
V, V_x, V_y	core shear parameters; defined by formulas (9), (7), and (8).
u, v, w	displacements in the x, y, and z directions.
x, y, z	coordinate and orthotropic axes (see figure 3).

Results and Discussion

The compressive buckling load per inch of panel edge applied at the neutral surface of a sandwich panel is defined in terms of a buckling load coefficient, p, by the relation

$$P = p \frac{\pi^2}{a^2} D \tag{1}$$

with

$$\left.\begin{aligned} D &= \frac{T_1 T_2}{T_1 + T_2}\left(c + \frac{f_1 + f_2}{2}\right)^2 \\ T_i &= f_i \frac{\sqrt{E_{xi} E_{yi}}}{\lambda_i}, \quad i = 1, 2 \end{aligned}\right\} \tag{2}$$

where i = 1, 2, denotes facing 1 or facing 2. (See figure 3 for orientation of axes.)

For a panel with dissimilar orthotropic facings and orthotropic core, the theoretical value of p depends upon nine physical constants. These are:

$$\left.\begin{aligned}
\alpha_i &= \sqrt{\frac{E_{xi}}{E_{yi}}} && (3)\\
\beta_i &= \frac{\lambda_i}{\sqrt{E_{xi}\,E_{yi}}}\left\{\frac{E_{xi}\,E_{yi}}{\lambda_i} + 2\mu_{xyi}\right\} && (4)\\
\gamma_i &= \frac{\mu_{xyi}\,\lambda_i}{\sqrt{E_{xi}\,E_{yi}}} && (5)
\end{aligned}\right\}\quad i = 1, 2$$

$$t = \frac{T_1}{T_1 + T_2} \tag{6}$$

$$V_x = \frac{cT_1\,T_2}{(T_1 + T_2)}\;\frac{\pi^2}{a^2\,\mu'_{zx}} \tag{7}$$

$$V_y = \frac{cT_1\,T_2}{(T_1 + T_2)}\;\frac{\pi^2}{a^2\,\mu'_{yz}} \tag{8}$$

where $i = 1, 2$, denotes facing 1 or 2, and T_1 and T_2 are given by the second of formulas (2).

If the core material is isotropic, V_x and V_y both reduce to

$$V = \frac{cT_1\,T_2}{(T_1 + T_2)}\;\frac{\pi^2}{a^2\,\mu'} \tag{9}$$

The coefficient p is given by the formula

$$p = p_f + p_M \tag{10}$$

where p_f is the buckling load coefficient of the two facings acting independently, but with half-wave lengths so determined that the buckling load of the composite panel is a minimum, and p_M is the buckling load coefficient of the panel with the facings considered as membranes. In many cases p_f is so small that the approximation

$$p = p_M \tag{11}$$

may be used.

$$p_f = \frac{D_{f1}}{D} K_1 + \frac{D_{f2}}{D} K_2 \tag{12}$$

where

$$\left.\begin{aligned} D_{fi} &= \frac{f_i^{\,3}}{12} \; \frac{\sqrt{E_{xi}\; E_{yi}}}{\lambda_i} && (13) \\ K_i &= \alpha_i c_1 + 2\beta_i\, c_2 + \frac{c_3}{\alpha_i} && (14) \end{aligned}\right\} i = 1, 2$$

$$p_M = \frac{\Psi_1 K_2 + \left(\frac{V_x}{c_4} + V_y\right) F_2}{\Psi_2 + \Psi_3 L_2 + \frac{V_x \; V_y}{c_4} F_2} \tag{15}$$

where

$$\Psi_1 = t + (1 - t) \frac{K_1}{K_2} \frac{F_2}{F_1} \tag{16}$$

$$\Psi_2 = t^2 + 2t(1 - t) \frac{F_{12}}{F_1} + (1 - t)^2 \frac{F_2}{F_1} \tag{17}$$

$$\Psi_3 = t + (1 - t) \frac{L_1}{L_2} \frac{F_2}{F_1} \tag{18}$$

$$\left.\begin{aligned} L_i &= (\alpha_i c_1 + \gamma_i\, c_2) \frac{V_x}{c_4} + \left(\frac{c_3}{\alpha_i} + \gamma_i c_2\right) V_y && (19) \\ F_i &= c_1 c_3 - \beta_i^{\,2}\, c_2^{\,2} + \gamma_i c_2 K_i && (20) \end{aligned}\right\} i = 1, 2$$

$$F_{12} = \left(\frac{\alpha_1^{\,2} + \alpha_2^{\,2}}{2\alpha_1 \alpha_2}\right) c_1 c_3 - \beta_1 \beta_2 c_2^{\,2} + \frac{\gamma_1}{2} c_2 K_2 + \frac{\gamma_2}{2} c_2 K_1 \qquad (21)$$

The quantities c_j, $j = 1, 2, 3, 4$, are given in terms of $\rho = \frac{a}{b}$, the aspect ratio of the panel, and n, the integral number of longitudinal half waves into which the panel buckles. In each case the integer n must be chosen so that p is as small as possible. When formula (10) is used, the same value of n must be used for both p_f and p_M. The following formulas are taken from (A69), (A71), (A73), (A74), and (A76).[6]

Case I. All Edges Simply Supported

$$c_1 = \frac{1}{n^2 \rho^2}, \; c_2 = 1, \; c_3 = n^2\rho^2, \; c_4 = c_1 \qquad (22)$$

Case II. Loaded Edges Simply Supported and the Remaining Edges Clamped

$$c_1 = \frac{16}{3n^2 \rho^2}, \; c_2 = \frac{4}{3}, \; c_3 = n^2\rho^2, \; c_4 = \frac{c_1}{4} \qquad (23)$$

Case III. Loaded Edges Clamped and the Remaining Edges Simply Supported

$$c_1 = \begin{cases} \dfrac{3}{4\rho^2} & \text{if } n = 1 \\[2ex] \dfrac{1}{(n^2 + 1)\rho^2} & \text{if } n \geq 2 \end{cases}, \quad c_2 = 1,$$

$$c_3 = \frac{\rho^2(n^4 + 6n^2 + 1)}{n^2 + 1}, \quad c_4 = \begin{cases} \dfrac{3}{n\rho^2} & \text{if } n = 1 \\[2ex] \dfrac{1}{(n^2 + 1)\rho^2} & \text{if } n \geq 2 \end{cases} \qquad (24)$$

[6] The numbers of equations in the appendices are preceded by the appropriate letter.

Case IV. All Edges Clamped

$$c_1 = \begin{cases} \dfrac{4}{\rho^2} & \text{if } n = 1 \\[2ex] \dfrac{16}{3(n^2+1)\rho^2} & \text{if } n \geq 2 \end{cases}, \quad c_2 = \frac{4}{3},$$

$$c_3 = \frac{\rho^2(n^4 + 6n^2 + 1)}{n^2 + 1}, \quad c_4 = \begin{cases} \dfrac{1}{\rho^2} & \text{if } n = 1 \\[2ex] \dfrac{4}{3(n^2+1)\rho^2} & \text{if } n \geq 2 \end{cases} \qquad (25)$$

Dissimilar Isotropic Facings

For isotropic facings

$$\left.\begin{aligned} &\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1, \quad \gamma_1 = \frac{1 - \sigma_1}{2} \\ \text{and} \quad &\gamma_2 = \frac{1 - \sigma_2}{2} \end{aligned}\right\} \qquad (26)$$

It appears that the only essential difference in the properties of the two facings is that due to Poisson's ratio. The effect of Poisson's ratio is usually small, and this property does not vary much among structural materials which are used for facings; thus the assumption can be made that $\sigma_1 = \sigma_2$. Then

$$\left.\begin{aligned} &K_1 = K_2 = K \qquad L_1 = L_2 = L \qquad \text{and} \\ &F_1 = F_{12} = F_2 = F \end{aligned}\right\} \qquad (27)$$

and $\Psi_1 = \Psi_2 = \Psi_3 = 1.$

The expressions for p_f and p_M reduce to

$$p_f = \frac{D_{f1} + D_{f2}}{D} K \qquad (28)$$

and

$$p_M = \frac{K + \left(\frac{V_x}{c_4} + V_y\right) F}{1 + L + \frac{V_x V_y}{c_4} F} \tag{29}$$

where

$$K = c_1 + 2c_2 + c_3 \tag{30}$$

$$L = \left(c_1 + \frac{1 - \sigma}{2} c_2\right) \frac{V_x}{c_4} + \left(c_3 + \frac{1 - \sigma}{2} c_2\right) V_y \tag{31}$$

$$F = c_1 c_3 - c_2^2 + \frac{1 - \sigma}{2} c_2 K \tag{32}$$

Formula (29) for p_M is exactly the same as formula (6) of Forest Products Laboratory Report No. 1854, which presents design curves for panels with similar facings. This can be seen by substituting (30) and (31) into (29) and making the following change of notation:

Notation of Report No. 1854	Notation of Present Report
rV	V_x
V	V_y
A	F
K_M	p_M

With this interpretation of the variables appearing in Report No. 1854, the curves of that report can be used to find p_M for panels with dissimilar isotropic facings. Unless b/a is small or V_x and V_y large, p_f is negligible in comparison with p_M. However, p_f can be obtained from the curves of Report No. 1854 by noting that

$$p_f = \frac{D_{f1} + D_{f2}}{D} \; p_{M0} \tag{33}$$

where p_{M0} is the value of p_M at $V_x = V_y = 0$.

Similar Orthotropic Facings

A reduction very similar to that made for dissimilar isotropic facings can be made for similar orthotropic facings.

Again

$$K_1 = K_2 = K, \qquad L_1 = L_2 = L$$

$$F_1 = F_{12} = F_2 = F \quad \text{and} \quad \Psi_1 = \Psi_2 = \Psi_3 = 1 \tag{34}$$

Also

$$D = \frac{f_1 \; f_2}{f_1 + f_2} \; (c + \frac{f_1 + f_2}{2})^2 \; \frac{\sqrt{E_x E_y}}{\lambda} \tag{35}$$

$$V_x = \frac{c f_1 f_2}{f_1 + f_2} \; \frac{\pi^2}{a^2} \; \frac{\sqrt{E_x E_y}}{\lambda \mu'_{zx}} \tag{36}$$

$$V_y = \frac{c \; f_1 \; f_2}{f_1 + f_2} \; \frac{\pi^2}{a^2} \; \frac{\sqrt{E_x E_y}}{\lambda \mu'_{yz}} \tag{37}$$

$$p_f = \frac{D_{f1} + D_{f2}}{D} \; K \tag{38}$$

and

$$PM = \frac{K + (\frac{V_x}{c_4} + V_y) \; F}{1 + L + \frac{V_x V_y}{c_4} \; F} \tag{39}$$

where

$$K = \alpha c_1 + 2\beta c_2 + \frac{c_3}{\alpha} \tag{40}$$

$$F = c_1 c_3 - \beta^2 c_2^2 + \gamma c_2 K \tag{41}$$

$$L = (\alpha c_1 + \gamma c_2) \frac{V_x}{c_4} + (\frac{c_3}{\alpha} + \gamma c_2) \; V_y \tag{42}$$

Similar Isotropic Facings

The formulas given above for similar orthotropic facings apply to similar isotropic facings if:

$$\alpha = \beta = 1 \qquad \gamma = \frac{1 - \sigma}{2}$$

and

$$\frac{\sqrt{E_x E_y}}{\lambda} = \frac{E_f}{\lambda_f}$$

Column Formulas

The formula for the buckling load of a column can be obtained from formula (1) by taking the limit as the width, a, of the panel becomes infinite. For this process, formula (12) is taken for p_f, and formula (15) is taken for p_M. The number n in the formulas for c_j, $j = 1, 2, 3, 4$, is taken as 1, since the column is assumed to buckle into a single half wave.

If the ends are simply supported, the limiting process yields

$$P = \frac{\pi^2}{b^2}\left\{\left[\frac{f_1{}^3}{12}\,\frac{E_{y1}}{\lambda_1} + \frac{f_2{}^3}{12}\,\frac{E_{y2}}{\lambda_2}\right] + \frac{\dfrac{f_1 E_{y1} f_2 E_{y2}}{f_1 E_{y1}\lambda_2 + f_2 E_{y2}\lambda_1}\left(c + \dfrac{f_1 + f_2}{2}\right)^2}{1 + \dfrac{c f_1 E_{y1} f_2 E_{y2}}{f_1 E_{y1}\lambda_2 + f_2 E_2\lambda_1}\,\dfrac{\pi^2}{b\,\mu'_{yz}}}\right\} \qquad (43)$$

If the ends are clamped, the limiting process yields

$$P = \frac{4\pi^2}{b^2}\left\{\left[\frac{f_1{}^3}{12}\,\frac{E_{y1}}{\lambda_1} + \frac{f_2{}^3}{12}\,\frac{E_{y2}}{\lambda_2}\right] + \frac{\dfrac{f_1 E_{y1} f_2 E_{y2}}{f_1 E_{y1}\lambda_2 + f_2 E_{y2}\lambda_1}\left(c + \dfrac{f_1 + f_2}{2}\right)^2}{1 + \dfrac{4 c f_1 E_{y1} f_2 E_{y2}}{f_1 E_{y1}\lambda_2 + f_2 E_{y2}\lambda_1}\,\dfrac{\pi^2}{b^2\mu'_{yz}}}\right\} \qquad (44)$$

The terms containing $f_1{}^3$ and $f_2{}^3$ represent the effect of the bending stiffness of the individual facings about their respective middle planes, and are often negligibly small as compared with the second term. The above formulas apply to a column with dissimilar facings of unequal thickness. If the facings are similar, formulas (43) and (44) reduce to

$$P = \frac{\pi^3}{b} \frac{E_y}{\lambda} \left\{ \frac{f_1^3 + f_2^3}{12} \quad \frac{\frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2} \right)^2}{1 + \frac{c f_1 f_2 \pi^2 E_y}{(f_1 + f_2)\, b^2 \mu'_{yz} \lambda}} \right\} \tag{45}$$

for simply supported ends, and

$$P = \frac{4\pi^2}{b^2} \frac{E_y}{\lambda} \left\{ \frac{f_1 + f_2}{12} + \frac{\frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2} \right)^2}{1 + \frac{4 c f_1 f_2 \pi^2 E_y}{(f_1 + f_2)\, b^2 \mu'_{yz} \lambda}} \right\} \tag{46}$$

for clamped ends.

Comparison of Results with Those of Other Investigators

The case of a sandwich panel with dissimilar isotropic facings and orthotropic core and with all edges simply supported has been solved by Chang and Ebcioglu (13) by a differential equation method. The assumption was made that the Poisson's ratios of the facings were the same or nearly equal. Equations (29) and (33) of the present report agree exactly with their results.

The case of a sandwich panel with equal isotropic facings and core material and with edges simply supported has been discussed by Seide and Stowell (11). These writers have assumed that the effect of the bending of the facings is negligible, and p is therefore determined by formula (11). In this case formula (39) with (22) for p_M is identical with that obtained by Seide and Stowell and that obtained from Report No. 1583. In their report Seide and Stowell have given a series of curves relating p to $1/\rho$ for various values of the parameter V.

Case II above has been discussed by Seide (10) for a panel with equal isotropic facings and core materials. In his analysis Seide has assumed that the effect of the bending of the facings is negligible. Under this assumption the present results are given by formula (11), with p_M determined by formulas (39) and

(23). The results of Seide, which are obtained by means of a transcendental relation, are given in a series of curves relating p to $1/\rho$ for various values of V. These results differ somewhat from those obtained from the present formulas. However, comparisons made at various check points indicate that the greatest discrepancy occurs when $V = 0$. When $V = 0$ there is no correction for transverse shear deformation in the core and p_M reduces to that obtained in ordinary plate theory when the form (A70) is assumed for w. On the other hand, when $V = 0$, the results of Seide are those obtained by the differential equation method for ordinary plates. The comparison with the results of Seide, therefore, indicate that the difference between results of Seide, therefore, indicate that the difference between results by the present energy method and those by the differential equations method are no greater than the difference between results by the two methods in ordinary plate theory.

Design Curves

The buckling load coefficient p_M has been computed on the basis of formula (39) for cases III and IV, for panels with isotropic facings and core materials. The results of these computations are given in figures 1 and 2 in the forms of plots of p_M as a function of $1/\rho$ for various values of V. In figure 1 the curves are given for case III, in which the loaded edges are clamped and the remaining edges are simply supported, and the curves for case IV, in which all edges are clamped, are given in figure 2.

APPENDIX A

Derivation of the Formula for the Buckling Load of a Panel

A1. Axes of Reference

The axes of reference, x, y, z, are so oriented that the x and y axes (fig. 3) are in coincidence with two edges of the panel and the z axis is perpendicular to underformed surfaces of the facings. The x, y plane is taken as the juncture of the core and the facings of thickness denoted by f_1, (fig. 4). The thickness of the core is denoted by c, the thicknesses of the two facings by f_1 and f_2, and the dimensions of the panel by a and b, with a measured along the x axis. It is convenient to identify a facing by the number 1 or 2, according to whether its thickness is f_1 or f_2. The displacements in the x, y, and z directions are denoted by u, v, and w. The displacement w is taken to be that of the neutral surface and it is assumed to be constant through the thickness of the panel.

A2. Expressions for Components of Displacement and Strain

In the present analysis the core is assumed to deform in such a way that each plane section originally parallel to the x, z or to the y, z plane remains plane but rotates about a line which it contains, this line being parallel to the x, y plane.[7] Specifically, the components of displacement in the core are taken as

$$\left.\begin{aligned} u_c &= -k(z-q)\frac{\partial w}{\partial x} \\ v_c &= -h(z-r)\frac{\partial w}{\partial y} \\ w_c &= w(x, y) \end{aligned}\right\} \qquad (A1)$$

[7] This method of analysis is an extension, to sandwich constructions with unequal facings, of that used by Williams, Leggett, and Hopkins (12) and other British investigators (4), (3), for isotropic sandwich constructions with equal facings and in Report No. 1583 for orthotropic constructions with equal facings. Its adoption leads to a uniform shear stress distribution in the core instead of the slightly variable actual shear stress distribution.

where the subscripts c identify the components as those of the core. It is seen from the formula for u_c that the amount of rotation of a section parallel to the y, z plane is prescribed by the parameter k and the line about which this plane section rotates is its intersection with the plane $z = q$. Similarly the amount of rotation and the fixed line of a section parallel to the x, z plane are prescribed by the parameters h and r, respectively. The parameters k, q, h, and r will be determined in the course of the analysis.

The continuity of displacements at the glue lines prescribes that the components (A1) evaluated at $z = 0$ and $z = c$ shall be those of the inner surfaces of the facings 1 and 2, respectively. Within each facing the components of displacement are assumed to be such that sections originally plane and perpendicular to the middle surface of the facing remain plane and perpendicular to the deformed middle surface. Accordingly the components which are identified by a subscript 1 or 2 according as the thickness of the facing is f_1 or f_2 are

$$\left.\begin{aligned} u_1 &= (kq - z)\frac{\partial w}{\partial x} \\ v_1 &= (hr - z)\frac{\partial w}{\partial y} \\ w_1 &= w(x, y) \end{aligned}\right\} \tag{A2}$$

$$\left.\begin{aligned} u_2 &= -\left\{k(c - q) + z - c\right\}\frac{\partial w}{\partial x} \\ v_2 &= -\left\{h(c - r) + z - c\right\}\frac{\partial w}{\partial y} \\ w_2 &= w(x, y) \end{aligned}\right\} \tag{A3}$$

Love's (6) notation will be used for the components of strain with the symbols c, 1 and 2 used as superscripts to denote components in the core, in facing 1 and in facing 2, respectively. From (A1) it follows that

$$e_{zx}^{(c)} = (1 - k)\frac{\partial w}{\partial x}, \quad e_{yz}^{(c)} = (1 - h)\frac{\partial w}{\partial y} \tag{A4}$$

For the present the remaining strains in the core associated with the displacements (A1) are assumed to have a negligible effect upon the results in the problem under consideration, and are neglected. This amounts to neglecting the bending energy of the core in the present energy method.

In the determination of the strain energy in the facings, it is convenient to consider the components of strain in the facings as the superposition of two states of strain. The first of these consists of the membrane strains or strains in their middle surfaces. Components of this type, as determined from formulas (A2) and (A3), are

$$\left.\begin{aligned} e_{xx}^{(1)} &= \left(kq + \frac{f_1}{2}\right) \frac{\partial^2 w}{\partial x^2} \\ e_{yy}^{(1)} &= \left(hr + \frac{f_1}{2}\right) \frac{\partial^2 w}{\partial y^2} \\ e_{xy}^{(1)} &= (kq + hr + f_1) \frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad \text{(A5)}$$

and

$$\left.\begin{aligned} e_{xx}^{(2)} &= -\left\{k(c - q) + \frac{f_2}{2}\right\} \frac{\partial^2 w}{\partial x^2} \\ e_{yy}^{(2)} &= -\left\{h(c - r) + \frac{f_2}{2}\right\} \frac{\partial^2 w}{\partial y^2} \\ e_{xy}^{(2)} &= -\left\{k(c - q) + h(c - r) + f_2\right\} \frac{\partial^2 w}{\partial x \partial y} \end{aligned}\right\} \quad \text{(A6)}$$

The second state of strain in the facings is that associated with the bending of the facings about their own middle surfaces. This state, in either facing, has the components

$$e'_{xx} = -z' \frac{\partial^2 w}{\partial x^2}, \quad e'_{yy} = -z' \frac{\partial^2 w}{\partial y^2}, \quad e'_{xy} = -2z' \frac{\partial^2 w}{\partial x \partial y} \quad \text{(A7)}$$

where z' is measured from the middle surface of the facing under consideration.

A3. Strain Energy in the Core and Facings

The strain energy in the core or facings is given by the expression (7), (8)

$$U = \frac{1}{2\lambda} \iiint_V \Big[E_x e_{xx}^2 + E_y e_{yy}^2 + 2 E_x \sigma_{yx} e_{xx} e_{yy}$$

$$+ \lambda \mu_{xy} e_{xy}^2 + \lambda \mu_{yz} e_{yz}^2 + \lambda \mu_{zx} e_{zx}^2 \Big] dz\, dy\, dx \qquad \text{(A8)}$$

where, for the material under consideration (core or facing), $\lambda = 1 - \sigma_{xy}\sigma_{yx}$, E_x and E_y are Young's moduli, μ_{xy}, μ_{yz}, and μ_{zx} are moduli of rigidity, and σ_{xy} and σ_{yx} are Poisson's ratios. Primed letters will denote the elastic constants of the core material and unprimed letters will denote those of the facing material. The integration indicated in formula (A8) is to be carried out over the entire volume of the core or facing.

The energy in the core is obtained by substituting expressions (A4) into (A8), the remaining strains in the latter formula being neglected as previously stated. After integrating with respect to z over the thickness of the core, the expression for the energy, denoted by U_c, is

$$U_c = \frac{c}{2} \int_0^a \int_0^b \left[\mu'_{zx} (1 - k)^2 \left(\frac{\partial w}{\partial x}\right)^2 + \mu'_{yz} (1 - h)^2 \left(\frac{\partial w}{\partial y}\right)^2 \right] dy dx \qquad \text{(A9)}$$

The strain energy in the facings associated with the membrane strains is the sum of the energies obtained from (A5) and (A6). With the substitution of these expressions into (A8) one obtains, after integration with respect to z, the following expression which is denoted by U_M.

$$U_M = U_{M1} + U_{M2} \qquad \text{(A10)}$$

where

$$U_{M1} = \frac{1}{2\lambda_1} \int_o^a \int_o^b \left[E_{x1} f_1 \left(kq + \frac{f_1}{2}\right)^2 \left(\frac{d^2w}{dx^2}\right)^2 + E_{y1} f_1 \left(hr + \frac{f_1}{2}\right)^2 \left(\frac{d^2w}{dy^2}\right)^2 \right.$$

$$+ 2E_{x1}\, \sigma_{yx1} f_1 \left(kq + \frac{f_1}{2}\right)\left(hr + \frac{f_1}{2}\right) \frac{d^2w}{dx^2}\frac{d^2w}{dy^2}$$

$$\left. + \lambda_1\, \mu_{xy1}\, f_1 \left(kq + hr + f_1\right)^2 \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy \qquad \text{(A11)}$$

$$U_{M2} = \frac{1}{2\lambda_2} \int_o^a \int_o^b \left[E_{x2}\, f_2 \left(k\,(c - q) + \frac{f_2}{2}\right)^2 \left(\frac{d^2w}{dx^2}\right)^2 \right.$$

$$+ E_{y2} f_2 \left(h\,(c - r) + \frac{f_2}{2}\right)^2 \left(\frac{d^2w}{dy^2}\right)^2$$

$$+ 2E_{x2}\, \sigma_{yx2}\, f_2 \left(k\,(c - q) + \frac{f_2}{2}\right)\left(h\,(c - r) + \frac{f_2}{2}\right) \frac{d^2w}{dx^2}\frac{d^2w}{dy^2}$$

$$\left. + \lambda_2\, \mu_{xy2}\, f_2 \left(k\,(c - q) + h\,(c - r) + f_2\right)^2 \left(\frac{d^2w}{dx\,dy}\right)^2 \right] dx\, dy \qquad \text{(A12)}$$

The strain energy in the facings associated with the flexural strain, U_F, is obtained by substituting expressions (A7) into (A8) and integrating over the volume of each facing. After integrating with respect to z',

$$U_F = U_{F1} + U_{F2} \qquad \text{(A13)}$$

where

$$U_{F1} = \frac{f_1^{\,3}}{24\lambda_1} \int_0^a \int_0^b \left[E_{x1} \left(\frac{d^2w}{dx}\right)^2 + E_{y1} \left(\frac{d^2w}{dy^2}\right)^2 \right.$$

$$\left. + 2\, E_{x1}\sigma_{yx1} \frac{d^2w}{dx^2} \frac{d^2w}{dy^2} + \lambda_1 \mu_{xy1} \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy \qquad \text{(A14)}$$

$$U_{F2} = \frac{f_2^{\,3}}{24\lambda_2} \int_0^a \int_0^b \left[E_{x2} \left(\frac{d^2w}{dx^2}\right)^2 + E_{y2} \left(\frac{d^2w}{dy^2}\right)^2 \right.$$

$$\left. + 2\, E_{x2}\sigma_{yx2} \frac{d^2w}{dx^2} \frac{d^2w}{dy^2} + \lambda_2 \mu_{xy2} \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy \qquad \text{(A15)}$$

The total strain energy in the sandwich, U, is taken as the sum

$$U = U_c + U_M + U_F \qquad \text{(A16)}$$

This expression depends upon the parameters k, q, h, and r which will be determined in the following section so that the load is a minimum. For these determinations it is convenient to consider U as a function of kq, hr, k, and h, since it is a quadratic in these variables, and to introduce the notation

$$A_{1i} = \int_0^a \int_0^b \left[\frac{E_{xi}}{\lambda_i} \left(\frac{d^2w}{dx}\right)^2 + \mu_{xyi} \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy$$

$$A_{2i} = \int_0^a \int_0^b \left[\frac{E_{xi}\,\sigma_{yxi}}{\lambda_i} \frac{d^2w}{dx^2} \frac{d^2w}{dy^2} + \mu_{xyi} \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy \qquad i = 1, 2$$

$$A_{3i} = \int_0^a \int_0^b \left[\frac{E_{yi}}{\lambda_i} \left(\frac{d^2w}{dy^2}\right)^2 + \mu_{xyi} \left(\frac{d^2w}{dxdy}\right)^2 \right] dx\, dy \qquad \text{(A17)}$$

$$A_4 = \int_0^a \int_0^b \mu'_{zx} \left(\frac{\partial w}{\partial x}\right)^2 dy\, dx$$

$$A_5 = \int_0^a \int_0^b \mu'_{yz} \left(\frac{\partial w}{\partial y}\right)^2 dy\, dx$$

where $i = 1, 2$, denotes facing 1 or facing 2.

Then if U is expressed in the form,

$$\begin{aligned} 2U = {} & B_1 (k\,q)^2 + 2B_2 (k\,q)(h\,r) + B_3 (h\,r)^2 \\ & + 2B_4 (k\,q)\,k + 2B_5 (k\,q)\,h + 2B_5 (h\,r)\,k \\ & + 2B_6 (h\,r)\,h + B_7 k^2 + 2B_8\, kh + B_9 h^2 \\ & + 2B_{10} (k\,q) + 2B_{11} (h\,r) + 2B_{12} K + 2B_{13} h + B_{14} + B_{15} \end{aligned} \qquad \text{(A18)}$$

the coefficients B_i are given as follows:

$$
\left.
\begin{aligned}
&B_1 = f_1 A_{11} + f_2 A_{12}, \qquad && B_2 = f_1 A_{21} + f_2 A_{22} \\
&B_3 = f_1 A_{31} + f_2 A_{32}, && B_4 = -cf_2\, A_{12} \\
&B_5 = -cf_2\, A_{22}, && B_6 = -cf_2 A_{32} \\
&B_7 = cA_4 + c^2 f_2 A_{12}, && B_8 = c^2 f_2 A_{22} \\
&B_9 = cA_5 + c^2 f_2 A_{32}, && \\
&B_{10} = \frac{f_1^{\,2}}{2}(A_{11} + A_{21}) - \frac{f_2^{\,2}}{2}(A_{12} + A_{23}) && \\
&B_{11} = \frac{f_1^{\,2}}{2}(A_{21} + A_{31}) - \frac{f_2^{\,2}}{2}(A_{22} + A_{32}) && \\
&B_{12} = -cA_4 + \frac{cf_2^{\,2}}{2}(A_{12} + A_{22}) && \\
&B_{13} = -cA_5 + \frac{cf_2^{\,2}}{2}(A_{22} + A_{32}) && \\
&B_{14} = c(A_4 + A_5) + \frac{f_1^{\,3}}{4}(A_{11} + 2A_{21} + A_{31}) + \frac{f_2^{\,3}}{4}(A_{12} + 2A_{22} + A_{32}) && \\
&B_{15} = \frac{f_1^{\,3}}{12}(A_{11} + 2A_{21} + A_{31})\ t\ \frac{f_2^{\,3}}{4}(A_{12} + 2A_{22} + A_{32}) &&
\end{aligned}
\right\}
\quad (A19)
$$

The term B_{15} is twice the strain energy associated with flexural strains in the facings. This component of the strain energy has been written separately because it is often negligible, and when it is neglected the expression for the load is simplified considerably.

A4. The Buckling Load of a Panel Under Compressive End Load

If a panel is subjected to a uniform compressive end load of P pounds per inch of edge at the neutral surface of the panel in the direction of the y axis, the work done by the load during buckling is

$$W = \frac{P}{2} \int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\, dx \qquad \text{(A20)}$$

The condition for instability is expressed by the relation

$$W = U \qquad \text{(A21)}$$

where U is obtained from (A16). In order to write the expression for P obtained from this formula, it is convenient to introduce the notation

$$U' = \frac{U}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\, dx} \qquad \text{(A22)}$$

$$\left.\begin{aligned} A'_{j1} &= \frac{A_{j1}}{\int_0^a \int_0^b \left(\frac{dw}{dy}\right)^2 dx\, dy} \\ A'_{j2} &= \frac{A_{j2}}{\int_0^a \int_0^b \left(\frac{dw}{dy}\right)^2 dx\, dy} \end{aligned}\right\} \quad j = 1, \ldots, 5 \qquad \text{(A23)}$$

$$B'_i = \frac{B_i}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\, dx} \quad i = 1, \text{--}\ 15 \qquad \text{(A24)}$$

Then according to (A20), (A21), and (A22),

$$P = 2U' \tag{A25}$$

with U' obtained from (A 16) by means of (A22), (A23), and (A24).

The conditions

$$\frac{\partial P}{\partial (kq)} = 0, \quad \frac{\partial P}{\partial (hr)} = 0, \quad \frac{\partial P}{\partial k} = 0, \quad \frac{\partial P}{\partial h} = 0$$

are now imposed for the determination of (kq), (hr), k, and h. From equations (A18), (A22), (A24), and (A25) these yield respectively

$$\left.\begin{aligned} B'_1\,(kq) + B'_2\,(hr) + B'_4\,k + B'_5\,h + B'_{10} &= 0 \\ B'_2\,(kq) + B'_3\,(hr) + B'_5\,k + B'_6\,h + B'_{11} &= 0 \\ B'_4\,(kq) + B'_5\,(hr) + B'_7\,k + B'_8\,h + B'_{12} &= 0 \\ B'_5\,kq + B'_6\,hr + B'_8\,k + B'_9\,h + B'_{13} &= 0 \end{aligned}\right\} \tag{A26}$$

If the first of these is multiplied by (kq), the second by (hr), the third by k, and the fourth by h, and the resulting expressions are substituted into (A25) using (A18), (A22), and (A24), then

$$P = B'_{10}\,(kq) + B'_{11}\,(hr) + B'_{12}\,k + B'_{13}\,h + B'_{14} + B'_{15} \tag{A27}$$

If (kq), (hr), k, and h obtained from the solution of the system (A26) are substituted into this equation it is found that

$$p = \frac{\begin{vmatrix} B'_1 & B'_2 & B'_4 & B'_5 & B'_{10} \\ B'_2 & B'_3 & B'_5 & B'_6 & B'_{11} \\ B'_4 & B'_5 & B'_7 & B'_8 & B'_{12} \\ B'_5 & B'_6 & B'_8 & B'_9 & B'_{13} \\ B'_{10} & B'_{11} & B'_{12} & B'_{13} & B'_{14} \end{vmatrix}}{\begin{vmatrix} B'_1 & B'_2 & B'_4 & B'_5 \\ B'_2 & B'_3 & B'_5 & B'_6 \\ B'_4 & B'_5 & B'_7 & B'_8 \\ B'_5 & B'_6 & B'_8 & B'_9 \end{vmatrix}} + B'_{15} \tag{A28}$$

When the expressions (A19), using (A23), and (A24), are substituted for B_i', $i = 1, \ldots 14$, the determinants in this formula may be evaluated. This evaluation is most easily accomplished by eliminating elements by the process of adding multiples of one row or column to another. A detailed account of this process is given in Appendix C.

With the evaluation of these determinants the expression for P may be written

$$P = P_f + P_M \tag{A29}$$

with

$$P_f = \frac{f_1^{\,3}}{12} (A'_{11} + 2A'_{21} + A'_{31}) + \frac{f_2^{\,3}}{12}(A'_{12} + 2A'_{22} + A'_{32}) \tag{A30}$$

and

$$P_M = \frac{n}{d} \tag{A31}$$

where

$$n = I \left[\frac{f_1(A'_{11}A'_{31} - A'^{\,2}_{21})(A'_{12} + 2A'_{22} + A'_{32}) + f_2(A'_{12}A'_{32} - A'^{\,2}_{22})(A'_{11} + 2A'_{21} + A'_{31})}{f_1 + f_2} + (A'_{11}A'_{31} - A'^{\,2}_{21})(A'_{12}A'_{32} - A'^{\,2}_{22})\left(\frac{\phi}{A'_4} + \frac{\phi}{A'_5}\right) \right] \tag{A32}$$

$$d = A_{f1}A_{f3} - A_{f2}^{\,2} + \frac{f_1(A'_{11}A'_{31} - A'^{\,2}_{21})}{f_1 + f_2}\left(\frac{A'_{12}\phi}{A'_4} + \frac{A'_{32}\phi}{A'_5}\right) + \frac{f\,(A'_{12}A'_{32} - A'^{\,2}_{22})}{f_1 + f_2}\left(\frac{A'_{11}\phi}{A'_4} + \frac{A'_{31}\phi}{A'_5}\right) + \frac{\phi^2(A'_{11}A'_{31} - A'^{\,2}_{21})(A'_{12}A'_{32} - A'^{\,2}_{22})}{A'_4\,A'_5} \tag{A33}$$

$$A_{fj} = \frac{f_1 A'_{j1} + f_2 A'_{j2}}{f_1 + f_2}, \quad j = 1, 2, 3 \tag{A34}$$

$$I = \frac{f_1 f_2}{f_1 + f_2} \left[c + \frac{f_1 + f_2}{2} \right]^2 \tag{A35}$$

and

$$\phi = \frac{c f_1 f_2}{f_1 + f_2} \tag{A36}$$

The term P_f in the formula for P, comes from B'_{15} and is interpreted as the load required to buckle the two facings, each acting independently of the other, into the form assumed for w. This term is often negligible. The second term, P_M, in formula (A29) is the buckling load of the panel, with correction for shear deformation, with the facings considered as membranes. The quantity I, formula (A35), is the moment of inertia of a section of the panel with the facings considered as membranes, taken about an axis which gives a minimum moment of inertia.

A5. The Buckling Load Coefficient

It is often more convenient to deal with a dimensionless buckling load coefficient rather than the buckling load itself. Such a coefficient may be defined as

$$p = \frac{P}{\frac{\pi^2}{a^2} D} \tag{A37}$$

$$D = \frac{f_1\sqrt{E_{x1}E_{y1}} \quad f_2\sqrt{E_{x2}E_{y2}}}{f_1\sqrt{E_{x1}E_{y1}}\,\lambda_2 + f_2\sqrt{E_{x2}E_{y2}}\,\lambda_1} \left(c + \frac{f_1 + f_2}{2}\right)^2 \tag{A38}$$

Also define

$$p_f = \frac{P_f}{\frac{\pi^2}{a^2} D} \tag{A39}$$

and

$$p_M = \frac{P_M}{\frac{\pi^2}{a^2} D} \tag{A40}$$

so that

$$p = p_f + p_M \tag{A41}$$

For the reduction of formula (A31) to dimensionless form it is convenient to introduce the notation

$$\left.\begin{aligned}
\frac{\pi^2}{a^2} c_1 &= \frac{\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x^2}\right)^2 dy\,dx}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\,dx} \\
\frac{\pi^2}{a^2} c_2 &= \frac{\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dy\,dx}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\,dx} \\
\frac{\pi^2}{a^2} c_3 &= \frac{\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial y^2}\right)^2 dy\,dx}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\,dx} \\
c_4 &= \frac{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\,dx}{\int_0^a \int_0^b \left(\frac{\partial w}{\partial y}\right)^2 dy\,dx}
\end{aligned}\right\} \tag{A42}$$

Also define

$$\left.\begin{aligned}
&\left.\begin{aligned}
\alpha_i &= \sqrt{\frac{E_{xi}}{E_{yi}}} \\
\beta_i &= \frac{\lambda_i}{\sqrt{E_{xi} E_{yi}}} \left\{ \frac{E_{xi}\, \sigma_{yxi}}{\lambda_i} + 2\mu_{xyi} \right\} \\
\gamma_i &= \frac{\mu_{xyi}\, \gamma_i}{\sqrt{E_{xi}\, E_{yi}}} \\
T_i &= f_i \frac{\sqrt{E_{xi} E_{yi}}}{\lambda_i}
\end{aligned}\right\} i = 1,\ 2 \\
&V_x = \frac{c\ T_1\, T_2}{T_1 + T_2} \frac{\pi^2}{a^2 \mu'_{zx}} \\
&V_y = \frac{c\ T_1\, T_2}{T_1 + T_2} \frac{\pi^2}{a^2 \mu'_{yz}}
\end{aligned}\right\} \qquad \text{(A43)}$$

where i = 1, 2, denotes facing 1 or facing 2.

If the core is isotropic, V_x and V_y both reduce to

$$V = \frac{c\, T_1\ T_2}{T_1 + T_2} \frac{\pi^2}{a^2 \mu'} \qquad \text{(A44)}$$

In this notation formulas (A30) and (A39) yield

$$p_f = \frac{D_{f1}}{D} K_1 + \frac{D_{f2}}{D} K_2 \qquad \text{(A45)}$$

where

$$\left.\begin{aligned} D_{fi} &= \frac{f_i^3}{12} \frac{\sqrt{E_{xi} E_{yi}}}{\lambda_i} && \text{(A46)} \\ K_i &= \alpha_i c_1 + 2\beta_i c_2 + \frac{c_3}{\alpha_i} \quad \underline{8} && \text{(A47)} \end{aligned}\right\} i = 1, 2$$

and by formulas (A31) through (A36) and (A40),

$$p_M = \frac{\Psi_1 K_2 + (\frac{V_x}{c_4} + V_y) F_2}{\Psi_2 + \Psi_3 L_2 + \frac{V_x V_y}{c_4} F_2} \qquad \text{(A48)}$$

---

8 In the reduction of A'_{21} and A'_{22} it has been assumed that

$$\int_0^a \int_0^b \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dy\, dx = \int_0^a \int_0^b \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} dy\, dx$$

This relation holds for each form used for w.

where

$$
\left.
\begin{aligned}
\Psi_1 &= t + (1 - t)\,\frac{K_1}{K_2}\,\frac{F_2}{F_1} \\
\Psi_2 &= t^2 + 2t(1 - t)\,\frac{F_{12}}{F_1} + (1 - t)^2\,\frac{F_2}{F_1} \\
\Psi_3 &= t + (1 - t)\,\frac{L_1}{L_2}\,\frac{F_2}{F_1} \\
t &= \frac{T_1}{T_1 + T_2} \\
&\left.
\begin{aligned}
L_i &= (\alpha_i c_1 + \gamma_i c_2)\,\frac{V_x}{c_4} + \left(\frac{c_3}{\alpha_i} + \gamma_i c_2\right) V_y \\
F_i &= c_1 c_3 - \beta_i^{\,2} c_2^{\,2} + \gamma_i c_2 K_i
\end{aligned}
\right\} i = 1, 2 \\
F_{12} &= \left(\frac{\alpha_1^{\,2} + \alpha_2^{\,2}}{2\alpha_1\alpha_2}\right) c_1 c_3 - \beta_1 \beta_2 c_2^{\,2} + \frac{\gamma_1}{2} c_2 K_2 + \frac{\gamma_2}{2} c_2 K_1
\end{aligned}
\right\}
\qquad (A49)
$$

where, as usual, $i = 1, 2$, denotes facing 1 or facing 2.

In the event that the facings are isotropic (but dissimilar),

$$
\alpha_i = \beta_i = 1, \; \gamma_i = \frac{1 - \sigma_i}{2}, \; i = 1, 2 \qquad (A50)
$$

If only one facing is isotropic, reduction (A50) can be made for only that facing which is isotropic.

If both facings are isotropic, an important simplification can be achieved by assuming that $\sigma_1 = \sigma_2$; i.e. that $\gamma_1 = \gamma_2$. Then

$$K_1 = K_2, \; L_1 = L_2, \; F_1 = F_{12} = F_2$$

so that subscripts on K, L, and F can be dropped, and

$$\Psi_1 = \Psi_2 = \Psi_3 = 1$$

Finally, (A45) and (A48) reduce to

$$p_f = \frac{D_{f1} + D_{f2}}{D} K \tag{A51}$$

and

$$p_M = \frac{K + (\frac{V_x}{c_4} + V_y) F}{1 + L + \frac{V_x V_y}{c_4} F} \tag{A52}$$

where

$$\left.\begin{aligned} K &= c_1 + 2c_2 + c_3 \\ L &= (c_1 + \frac{1-\sigma}{2} c_2) \frac{V_x}{c_4} + (c_3 + \frac{1-\sigma}{2} c_2) V_y \\ F &= c_1 c_3 - c_2^2 + \frac{1-\sigma}{2} c_2 K \end{aligned}\right\} \tag{A53}$$

If the facings are similar, considerable simplification is possible. Define A'_j, j = 1, 2, 3, as in formulas (A23) but without a second subscript to denote facing. Then formula (A30) reduces

$$P_f = I_f (A'_1 + 2A'_2 + A'_3) \tag{A54}$$

where

$$I_f = \frac{f_1^3 + f_2^3}{12} \tag{A55}$$

and formulas (A31), (A32), and (A33) reduce to

$$P_M = \frac{I\left\{A_1' + 2A_2' + A_3' + (A_1' A_3' - A_2'^2)\left(\frac{\phi}{A_4'} + \frac{\phi}{A_5'}\right)\right\}}{1 + \frac{A_1'\phi}{A_4'} + \frac{A_3'\phi}{A_5'} + (A_1'A_3' - A_2'^2)\frac{\phi^2}{A_4'A_5'}} \tag{A56}$$

In order to make this last reduction note that, for similar facings, A_{f2} reduces to A_j', $j = 1, 2, 3$, and a factor $(A_1'A_3' - A_2'^2)$ cancels from numerator and denominator.

The definition of the buckling load coefficient, (A37), is replaced, in the case of similar facings, by

$$k = \frac{P}{\frac{\pi^2}{a^2} I \frac{\sqrt{E_x E_y}}{\lambda}} \tag{A57}$$

and, similarly,

$$k_f = \frac{P_f}{\frac{\pi^2}{a^2} I \frac{\sqrt{E_x E_y}}{\lambda}} \tag{A58}$$

$$k_M = \frac{P_M}{\frac{\pi^2}{a^2} I \frac{\sqrt{E_x E_y}}{\lambda}} \tag{A59}$$

so that $k = k_f + k_M$, as before.

Introduce the c_j, $j = 1, \ldots, 4$, according to (A41) and define

$$
\left.
\begin{aligned}
\alpha &= \sqrt{\frac{E_x}{E_y}} \\
\beta &= \frac{\lambda}{\sqrt{E_x E_y}} \left\{ \frac{E_x \sigma_{yx}}{\lambda} + 2\mu_{xy} \right\} \\
\gamma &= \frac{\mu_{xy}\lambda}{\sqrt{E_x E_y}} \\
S_x &= \frac{\phi \pi^2 \sqrt{E_x E_y}}{a \lambda \mu'_{zx}} \\
S_y &= \frac{\phi \pi^2 \sqrt{E_x E_y}}{a^2 \lambda \mu'_{yz}}
\end{aligned}
\right\} \tag{A60}
$$

Note that if the core is isotropic, S_x and S_y both reduce to

$$
S = \frac{\phi\pi^2}{a^2\mu'} \frac{\sqrt{E_x E_y}}{\lambda} \tag{A61}
$$

In this notation formulas (A54), (A55), (A58), and (A60) yield

$$
k_f = I_f \left(\alpha c_1 + 2\beta c_2 + \frac{c_3}{\alpha}\right) \tag{A62}
$$

and by formulas (A56), (A59), and (A60),

$$
k_M = \frac{\alpha c_1 + 2\beta c_2 + \frac{c_3}{\alpha} + \left(\frac{S_x}{c_4} + S_y\right) F}{1 + (\alpha c_1 + \gamma c_2)\frac{S_x}{c_4} + \left(\frac{c_3}{\alpha} + \gamma c_2\right) S_x + \frac{S_x S_y F}{c_4}} \tag{A63}
$$

where

$$F = c_1c_3-\beta^2c_2{}^2 + \gamma c_2 (\alpha c_1 + 2\beta c_2 + \frac{c_3}{\alpha}) \tag{A64}$$

In the event that the facings are similar and isotropic,

$$\alpha = \beta = 1, \ \gamma = \frac{1 - \sigma}{2} \tag{A65}$$

and $\frac{E_f}{\lambda_f}$ replaces $\frac{E_xE_y}{\lambda}$ in the formulas for S_x and S_y. Formulas (A62) and (A63) reduces to

$$k_f = I_f K \tag{A66}$$

and

$$k_M = \frac{K + (\frac{S_x}{c_4} + S_y) F}{1 + (c_1 + \frac{1 - \sigma}{2} c_2)\frac{S_x}{c_4} + (c_3 + \frac{1 - \sigma}{2} c_2) S_y + \frac{S_xS_y}{c_4} F} \tag{A67}$$

where K and F are given by (A58).

Note that (A52) and (A67) have exactly the same functional form, thus making it possible to plot one set of curves for both similar isotropic facings and dis-similar isotropic facings, provided the "dissimilar" facings have the same, or nearly equal, Poisson's ratios.

Boundary Conditions

The quantities c_i, $i = 1, 2, 3, 4$ defined by formulas (A42), may be evaluated when the function w (x, y), which represents the deflection is specified. A suitable form for w will now be chosen and these quantities will be determined for each of the boundary conditions considered. The form assumed for w is in each case a trigonometric expression, and the quantities c_i are functions of $\rho = \frac{a}{b}$, and n, the number of longitudinal half waves into which the panel buckles. In each case, the integer n must be chosen so that the coefficient p is as small as possible.

Case I. All Edges Simply Supported

A suitable form for w is[9]

$$w = c \sin \frac{\pi x}{a} \sin \frac{n\pi y}{b} \tag{A68}$$

Then by formulas (A42)

$$c_1 = \frac{1}{n^2\rho^2}, \quad c_2 = 1, \quad c_3 = n^2\rho^2, \quad c_4 = c_1 \tag{A69}$$

Case II. Loaded Edges Simply Supported and the Remaining Edges Clamped

The deflection in this case is taken as

$$w = c \sin^2 \frac{\pi x}{a} \sin \frac{n\pi y}{b} \tag{A70}$$

From (A42),

$$c_1 = \frac{16}{3n^2\rho^2}, \quad c_2 = \frac{4}{3}, \quad c_3 = n^2\rho^2, \quad c_4 = \frac{c_1}{4} \tag{A71}$$

Case III. Loaded Edges Clamped and the Remaining Edges Simply Supported

The form for the deflection surface is taken to be

$$w = c \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} \tag{A72}$$

---

[9] In this case, the form assumed for w is "exact" in the sense that no other form can yield a lower theoretical buckling load. In the following three cases, the form assumed for w is an approximate expression chosen for its mathematical simplicity. Such an approximation will always yield a theoretical buckling load which is too great. However, the error is small, since geometric boundary conditions are satisfied.

From (A42),

$$c_1 = \frac{1 + \frac{\delta_{1n}}{2}}{\rho^2 (n^2 + 1)}, \quad c_2 = 1, \quad c_3 = \frac{\rho^2 (n^4 + 6n^2 + 1)}{n^2 + 1}, \quad c_4 = c_1 \tag{A73}$$

where

$$\begin{aligned} \delta_{1n} &= 1 \text{ if } n = 1 \\ &= 0 \text{ if } n \geq 1 \end{aligned} \tag{A74}$$

Case IV. All Edges Clamped

A form for w in this case is

$$w = c \sin^2 \frac{\pi x}{a} \sin \frac{\pi y}{b} \sin \frac{n \pi y}{b} \tag{A75}$$

From (A42)

$$c_1 = \frac{16 \left(1 + \frac{\delta_{1n}}{2}\right)}{3 \rho^2 (n^2 + 1)}, \quad c_2 = \frac{4}{3}, \quad c_3 = \frac{\rho^2 (n^4 + 6n^2 + 1)}{n^2 + 1} \tag{A76}$$

$$c_4 = \frac{c_1}{4}$$

APPENDIX B

Solution by Libove and Batdorf Method When

Panel is Simply Supported

In the case of simple support, an expression for the buckling load can be derived quite simply from the equations of Libove and Batdorf (11). For this type of support the functions

$$
\begin{aligned}
w &= A \sin \frac{\pi x}{a} \sin \frac{n\pi y}{b} \\
Q_x &= B \cos \frac{\pi x}{a} \sin \frac{n\pi y}{b} \qquad \text{(B1)} \\
Q_y &= C \sin \frac{\pi x}{a} \cos \frac{n\pi y}{b}
\end{aligned}
$$

may be used in solving the system of differential equations (11, pp. 13-14). These functions satisfy the boundary condition (11, conditions 10) and lead to an expression for

$$k = \frac{P}{\dfrac{\pi^2 \sqrt{D_x D_y}}{a^2 (1 - \mu_x \mu_y)}} \qquad \text{(B2)}$$

which is identical with p_M given by formula (A60) above, provided the following interpretations are given to the physical constants appearing in that formula

$$
\left.\begin{aligned}
\alpha &= \sqrt{\frac{D_x}{D_y}} \\
\beta &= \frac{1 - \mu_x\mu_y}{\sqrt{D_x D_y}}\left\{D_{xy} + \frac{D_x\mu_y + D_y\mu_x}{2(1 - \mu_x\mu_y)}\right\} \\
\gamma &= \frac{D_{xy}\ (1 - \mu_x\mu_y)}{2\sqrt{D_x D_y}} \\
S_x &= \frac{\pi^2}{a^2}\,\frac{\sqrt{D_x D_y}}{D_{Qx}(1 - \mu_x\mu_y)} \\
S_y &= \frac{\pi^2}{a^2}\,\frac{\sqrt{D_x D_y}}{D_{Qy}(1 - \mu_x\mu_y)}
\end{aligned}\right\} \qquad \text{(B3)}
$$

The symbols appearing in the right-hand member of each of these expressions are those which have been used by Libove and Batdorf. When expressions (B2) and (B3) are compared with (A57) and (A60), it is found that the solutions by the two methods are identical provided the Libove and Batdorf constants, which are given on the left below, are interpreted in the present notation as follows

$$
\left.\begin{aligned}
\mu_x &= \sigma_{xy} \\
\mu_y &= \sigma_{yx} \\
D_x &= IE_x = \frac{f_1 f_2}{f_1 + f_2} \left(c + \frac{f_1 + f_2}{2}\right)^2 E_x \\
D_y &= IE_y \\
D_{xy} &= 2I\,\mu_{xy} \\
D_{Qx} &= \frac{\left(c + \frac{f_1 + f_2}{2}\right)^2}{c} \mu'_{zx} \\
D_{Qy} &= \frac{\left(c + \frac{f_1 + f_2}{2}\right)^2}{c} \mu'_{yz}
\end{aligned}\right\} \quad \text{(B4)}
$$

Among these the first five reduce to those which have been used in the NACA publications (11), (10), when the facings are isotropic and of equal thickness. The last two reduce to the form suggested by Bijlaard (1) when the facings are of equal thickness and the core isotropic.

For the case under consideration, the identity of the present solution for p_M with that obtained by the differential equations method could be anticipated on the basis that the same form for w was assumed for both solutions; and the expressions for Q_x and Q_y derived from formulas (A4) are of the same forms as the second and third expressions of equations (B1) respectively. Under these circumstances the processes of Appendix A can be expected to yield the same results as that obtained by the differential equations method.

APPENDIX C

Reduction of Determinants in (A28)

Rows will be numbered from top to bottom and will be referred to as R1, R2, . . . , R-5. Columns will be numbered from left to right and referred to as C1, C2, . . . , C5. Substitutions will be indicated as follows

$$R1 + c \cdot R2 \rightarrow R1$$

which reads "substitute (termwise) c times row 2 plus row 1 for row 1."

Consider the numerator of the first term of (A28). The following sequence of substitutions

$$\left.\begin{aligned}
& R3 + c \cdot R1 \rightarrow R3 \\
& R4 + c \cdot R2 \rightarrow R4 \\
& C5 - \frac{f_2}{2} \cdot C1 - \frac{f_2}{2} \cdot C2 \rightarrow C5 \\
& R5 + \frac{f_2}{2} R1 + \frac{f_2}{2} R2 + R3 + R4 \rightarrow R5 \\
& C3 + C4 + C5 \rightarrow C5 \\
& R1 \div cf_2 \rightarrow R1 \\
& R2 \div cf_2 \rightarrow R2 \\
& C1 \div cf_1 \rightarrow C1 \\
& C2 \div cf_1 \rightarrow C2 \\
& R2 - \left(\frac{A'_{22} + A'_{32}}{A'_{12} + A'_{22}}\right) \cdot R1 \rightarrow R2 \\
& C2 - \left(\frac{A'_{21} + A'_{31}}{A'_{11} + A'_{21}}\right) \cdot C1 \rightarrow C2
\end{aligned}\right\} \qquad (C1)$$

results in a third order determinant. At this point the numerator is

$$n' = -c^2 f_1 f_2 \left(\frac{h+c}{2c}\right)^2 (A'_{11} + A'_{21})(A'_{12} + A'_{22}) \begin{vmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{vmatrix} \quad \text{(C2)}$$

where

$$\left.\begin{aligned}
J_{11} &= -\frac{1}{c^2}\left[\left(\frac{A'_{21}}{f_2} + \frac{A'_{22}}{f_1}\right)\right. \\
&\quad \left. - \left(\frac{A'_{11}}{f_2} + \frac{A'_{12}}{f_1}\right)\left(\frac{A'_{22} + A'_{32}}{A'_{12} + A'_{22}}\right)\right]\left(\frac{A'_{21} + A'_{31}}{A'_{11} + A'_{21}}\right) \\
J_{12} &= -A'_{22} + A'_{12}\left(\frac{A'_{22} + A'_{32}}{A'_{12} + A'_{22}}\right) \\
J_{13} &= -A'_{32} + A'_{22}\left(\frac{A'_{22} + A'_{32}}{A'_{12} + A'_{22}}\right) \\
J_{21} &= A'_{21} - A'_{11}\left(\frac{A'_{21} + A'_{31}}{A'_{11} + A'_{21}}\right) \\
J_{22} &= cA'_4 \\
J_{23} &= 0 \\
J_{31} &= A'_{31} - A'_{21}\left(\frac{A'_{21} + A'_{31}}{A'_{11} + A'_{21}}\right) \\
J_{32} &= 0 \\
J_{33} &= cA'_5
\end{aligned}\right\} \quad \text{(C3)}$$

This determinant can be expanded literally. Thus n' finally becomes

$$n' = c^4 f_1^2 f_2^2 \left(\frac{h+c}{2}\right)^2 \left\{ \frac{A'_4 A'_5}{c^2} \left[\frac{1}{f_2} (A'_{11} A'_{31} - A'^2_{21})(A'_{12} + 2A'_{22} + A'_{32}) + \frac{1}{f_1} (A'_{12} A'_{32} - A'^2_{22})(A'_{11} + 2A'_{21} + A'_{31}) \right] + \frac{1}{c}(A'_4 + A'_5)(A'_{11} A'_{31} - A'^2_{21})(A'_{12} A'_{32} - A'^2_{22}) \right\} \qquad \text{(C4)}$$

Consider next the denominator of the first term of (A28). The following sequence of substitutions

$$\left.\begin{array}{l}
R3 + c \cdot R1 \longrightarrow R3 \\
R4 + c \cdot R2 \longrightarrow R4 \\
R1 \div cf_2 \longrightarrow R1 \\
R2 \div cf_2 \longrightarrow R2 \\
C1 \div cf_1 \longrightarrow C1 \\
C2 \div cf_1 \longrightarrow C2 \\
R1 + \dfrac{A'_{22}}{cA'_5} \cdot R4 \longrightarrow R1 \\
R2 + \dfrac{A'_{32}}{cA'_5} \cdot R4 \longrightarrow R2 \\
R1 + \dfrac{A'_{12}}{cA'_4} \cdot R3 \longrightarrow R1 \\
R2 + \dfrac{A'_{22}}{cA'_4} \cdot R3 \longrightarrow R2 \\
R1 \cdot cA'_4 \longrightarrow R1 \\
R2 \cdot cA'_5 \longrightarrow R2
\end{array}\right\} \qquad \text{(C5)}$$

results in a second order determinant. Introduce the notation

$$\tau_i = \frac{A_{i1}}{f_2} + \frac{A_{i2}}{f_1}, \quad i = 1, 2 \tag{C6}$$

and the denominator, d' , becomes

$$
\begin{aligned}
d' = c^4 f_1{}^2 f_2{}^2 \Bigg\{ & \frac{A'_4 A'_5}{c^2} (\tau_1 \tau_3 - \tau_2{}^2) \\
& + \frac{1}{f_2} (A'_{11} A'_{31} - A'_{21}{}^2) \left(\frac{A'_{32} A'_4}{c} + \frac{A'_{12} A'_5}{c}\right) \\
& + \frac{1}{f_1} (A'_{12} A'_{32} - A'_{22}{}^2) \left(\frac{A'_{31} A'_4}{c} + \frac{A'_{11} A'_5}{c}\right) \\
& + (A'_{11} A'_{31} - A'_{21}{}^2) (A'_{12} A'_{32} - A'_{22}{}^2) \Bigg\}
\end{aligned} \tag{C7}
$$

Upon dividing n' and d' by $c^2 (f_1 + f_2)^2 A'_4 A'_5$ and introducing A_{fj}, j = 1, 2, 3, I, and ϕ as defined by (A34), (A35), and (A36), P_M can be written as $\frac{n}{d}$ according to (A31) through (A33).

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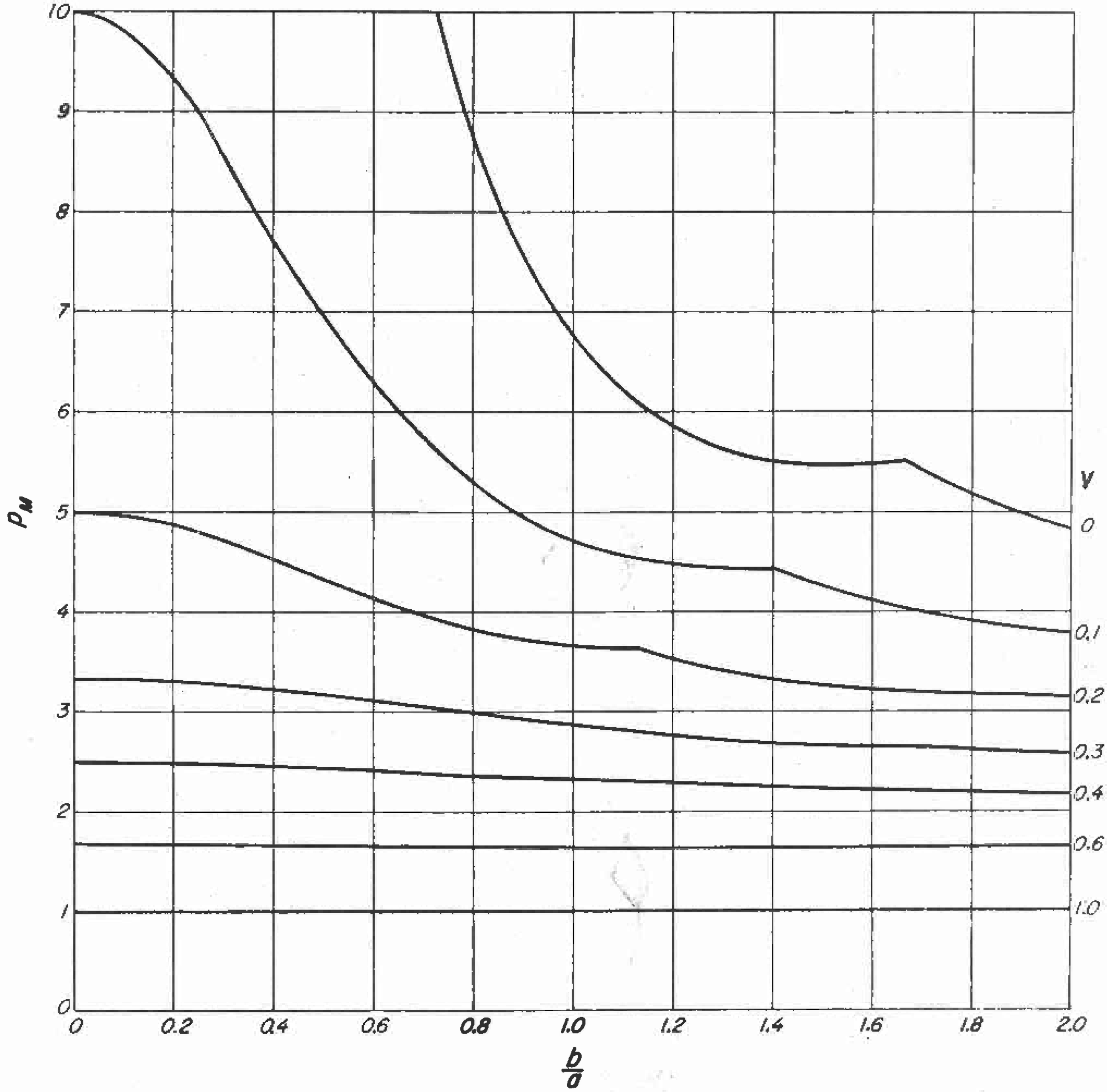

Figure 1. --The compressive buckling load coefficient p_M for a rectangular panel with clamped loaded edges and the remaining edges simply supported. Isotropic facing and core material. $\sigma = \frac{1}{3}$

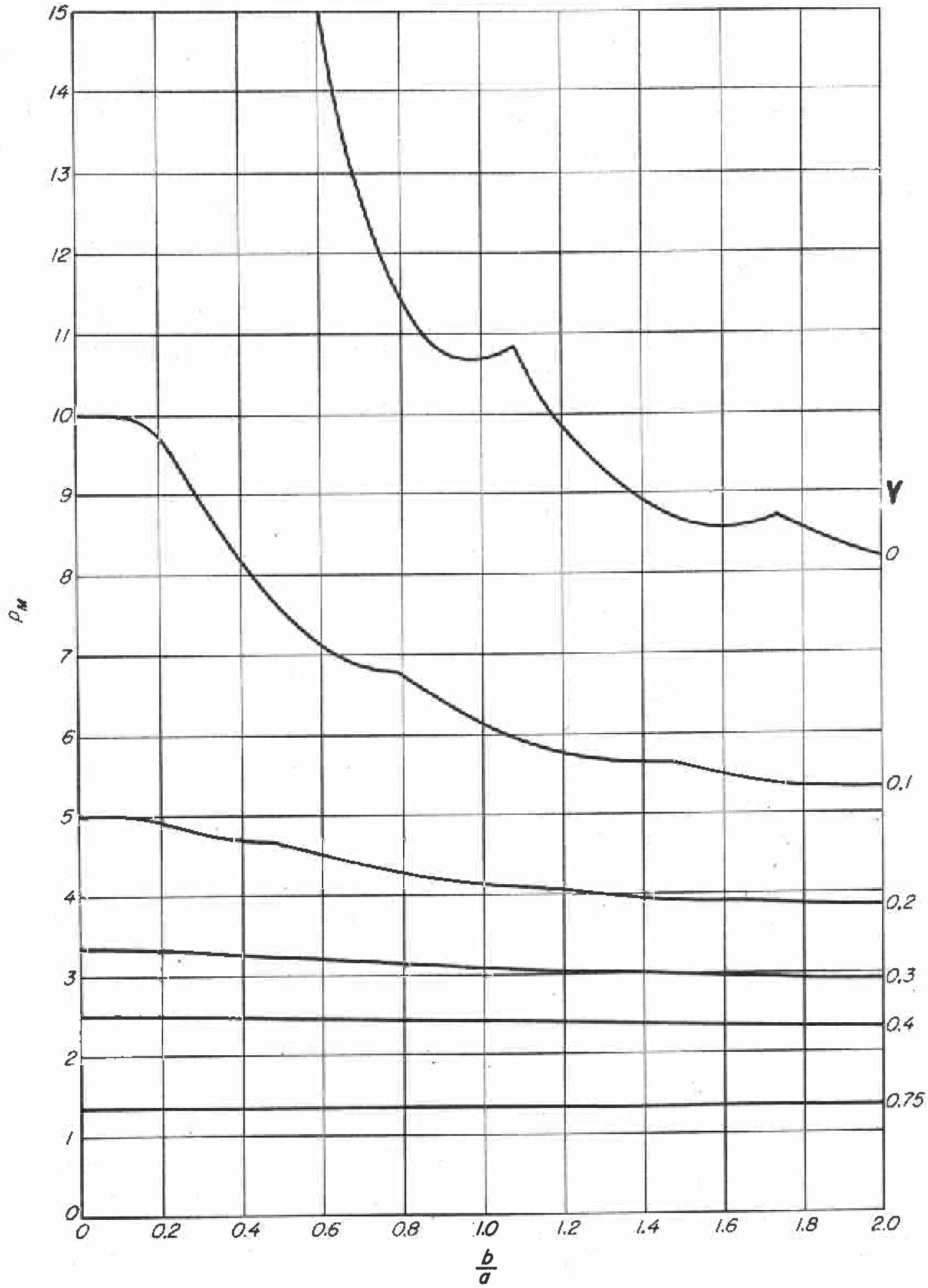

Figure 2.--The compressive buckling load coefficient p_M for a rectangular panel with all edges clamped. Isotropic facing and core material. $\sigma = \frac{1}{3}$

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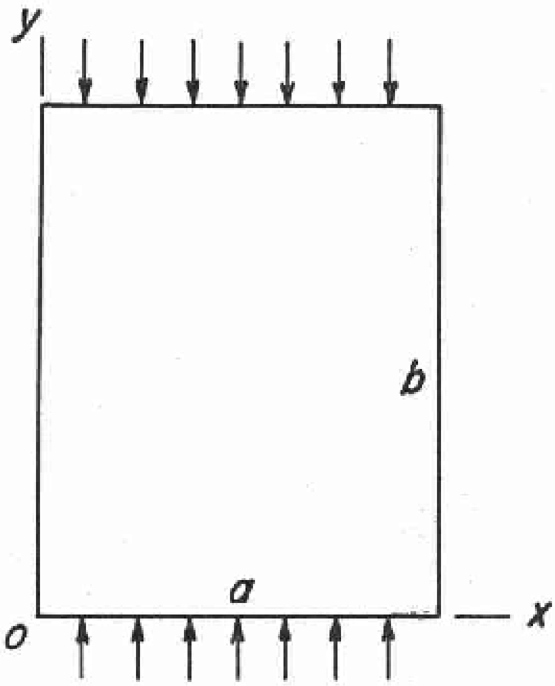

Figure 3. --Flat sandwich panel in compression.

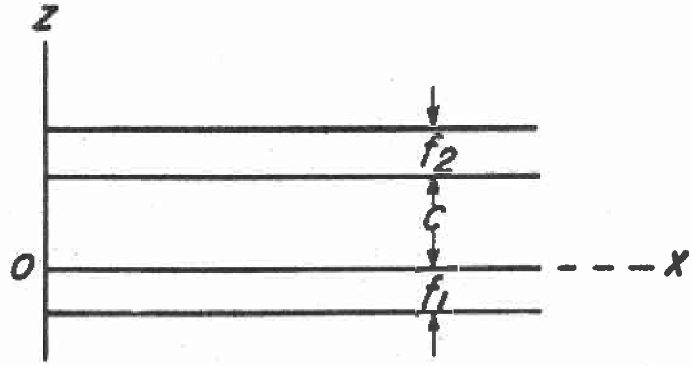

Figure 4. --Cross section of sandwich panel.