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Skew Disperson and Continuity of Local Time

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Abstract

Results are provided that highlight the effect of interfacial discontinuities in the diffusion coefficient on the behavior of certain basic functionals of the diffusion, such as local times and occupation times, extending previous results in [2, 3] on the behavior of first passage times. The main goal is to obtain a characterization of large scale parameters and behavior by an analysis at the fine scale of stochastic particle motions. In particular, considering particle concentration modeled by a diffusion equation with piecewise constant diffusion coefficient, it is shown that the continuity of a natural modification of local time is the individual (stochastic) particle scale equivalent to continuity of flux at the scale of the (macroscopic) particle concentrations. Consequences of this involve the determination of a skewness transmission probability in the presence of an interface, as well as corollaries concerning interfacial effects on occupation time of the associated stochastic particles.

Keywords. Dispersion, discontinuous diffusion, skew Brownian motion, semimartingale local time, local time, occupation time, first passage time

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The statistical physics developed in the present article is motivated by applications of interfacial phenomena found in diverse areas of science and engineering. The recent survey [32] illustrates different examples in which dispersion in the presence of an interface of discontinuity in the dispersion coefficient is used in diverse problems from hydrology [6, 14, 15, 18], ecology [9, 24, 30, 37, 40], finance [27], astrophysics [12] and physical oceanography [25]. The present note completes some of the theory involved in such applications by adapting certain modifications of important quantities from the mathematical foundations of the theory of stochastic processes to the implied physical or biological situation. Specifically we analyze the effects of interfacial discontinuities in the diffusion coefficient on natural modifications of local time and occupation time in an effort to obtain a

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characterization of large scale parameters and behavior in terms of underlying stochastic particle motions.

Local time is a quantity with a long history in the mathematical theory of continuous semimartingale stochastic processes. The precise meaning of a "continuous semimartingale"stochastic process is explained in the next section, but it includes Brownian motion and all other diffusion processes. Although largely restricted to one spatial dimension in its fullest development, a certain notion of local time will be shown to be uniquely adaptable to the micro-scale physics from the perspective of 'continuity'. Local time concerns a measure of the "amount"of time a continuous semimartingale spends locally about a point, referred to as *semimartingale local time*. Complete and systematic expositions of this concept can be found in [39, 38]. There are other formulations of local time as well; for example that of Itô-McKean in [17]. The results to be presented here require a modification of the standard mathematical definitions of local time, to be referred to as *natural local time*, in order to properly capture the physics. In the case of *standard* Brownian motion, the formulae for semimartingale and natural local times, as well as the Itô-McKean local time, will be seen to match; the latter up to a factor of 1/2, see [39]. However, this match is an artifact of a (constant) unit diffusion coefficient. From the point of view of the physics, these are distinct formulations of local time.

Starting with a celebrated theorem of Trotter [42] establishing the continuity of local time for standard Brownian motion, the general problem of determining necessary and sufficient conditions for continuity of local time for stochastic processes is a well-studied problem at the foundations of the mathematical theory of stochastic processes. At the most fundamental level, the issue for physics naturally involves identifying the appropriate *units* of measurement. In particular, the units of semimartingale local time in the context of dispersion of particle concentrations are typically those of *spatial length*. The local time of Itô-McKean is dimensionless. Neither of these are unnatural in their mathematical context owing to the locally linear relationship between spatial variance and time when the dispersion coefficient is sufficiently smooth. However, as will be seen, this rationale breaks down in the presence of an interface of discontinuity in the diffusion coefficient. To properly capture the physics certain natural modifications of the usual mathematical concepts are required. From the point of view of physics, that such a modification is even possible in a way that at the volumetric scale of particle concentrations the *continuity of flux* can be viewed as *continuity of natural local time* at the stochastic particle scale is a main result of this paper. Specifically, the continuity of natural local time of the particles will be shown to precisely correspond to continuity of concentration flux. This will not be the case without the transformation to natural local time.

In summary, the main results demonstrate (i) a special role for continuity of flux manifest in the continuity of (natural) local time, and conversely; (ii) a corresponding equivalent special role for continuity of (natural) local time for the determination of a transmission probability parameter at the interface; (iii) symmetric relationships (via martingales) between interfacial transmission probabilities, dispersion coefficients, and skewness parameters; and (iv) the effects of general point interface transmission probabilities on occupation times. These results follow on earlier results that depict the effects of interfacial discontinuities on first passage times; e.g. see [2, 3, 4, 32].

1 Natural Dispersion and Natural Occupation Time

The first theorem provides a useful summary in one-dimension of the interplay between diffusion coefficients and broader classes of possible interfacial conditions.

As warm-up consider the case of Brownian motion, i.e., diffusion with constant drift and diffusion coefficients. The definition of *semimartingale local time at a*, denoted $\ell_t^X(a)$, for a diffusion X_t

with constant drift v and constant diffusion coefficient $D > 0$ quantifies $d\ell_t^X(a)$ as an "amount of time" that X spends in the infinitesimal neighborhood $(a, a \pm da)$ prior to time t. More precisely,

$$\ell_t^X(a) = \lim_{\epsilon \downarrow 0} \frac{D}{2\epsilon} \int_0^t 1_{(a-\epsilon, a+\epsilon)}(X(s))ds. \tag{1.1}$$

Note the presence of the diffusion coefficient D. Thus, in the context of dispersion of particles in a fluid, for example, the diffusion coefficient has units of length^2 per unit time (L^2/T), and ϵ and X have units of length (L), so that semimartingale local time ℓ_t^X has units of spatial length (L).

The standard extension of the mathematical definition of local time for a continuous semimartingale X exploits the *quadratic variation* $\langle X \rangle_t, t \geq 0$, of the process. If X is a square-integrable martingale then $\langle X \rangle_t$ is defined by the property that the process $X^2(t) - \langle X \rangle_t, t \geq 0$, is a martingale; see [38]. So, in the case of a diffusion with constant diffusion coefficient D, one has that $\langle X \rangle_t = Dt, t \geq 0$. In the case of a continuous semimartingale, the quadratic variation is quadratic variation of its martingale component.

Semimartingale local time at a is more generally defined as an increasing continuous stochastic process $\ell_t^X(a), t \geq 0$ such that for $t > 0$,

$$|X(t) - a| = |X(0) - a| + \int_0^t \operatorname{sgn}(X(s) - a)dX(s) + \ell_t^X(a),$$

where $\operatorname{sgn}(x) = x/|x|, x \neq 0$, and $\operatorname{sgn}(0) = -1$ (by convention); see [39]. For purposes of calculation it is often convenient to consider (right and left) one-sided versions defined by

$$\ell_t^{X,+}(a) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1_{[a, a+\epsilon)}(X(s))d\langle X \rangle_s, \qquad t \geq 0, \tag{1.2}$$

$$\ell_t^{X,-}(a) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_0^t 1_{(a-\epsilon, a]}(X(s))d\langle X \rangle_s, \qquad t \geq 0. \tag{1.3}$$

Then

$$\ell_t^X(a) = \frac{\ell_t^{X,+}(a) + \ell_t^{X,-}(a)}{2}, \qquad t \geq 0. \tag{1.4}$$

The utility of these quantities rests in the following two well-known formulae connecting semimartingale local time to semimartingale occupation time:

Itô-Tanaka Formula: If X is a continuous semimartingale with semimartingale local time $\ell_t^X(a), a \in \mathbb{R}, t \geq 0$, and f is the difference of two convex functions then

$$f(X(t)) = f(X(0)) + \int_0^t f'_-(X(s))dX(s) + \frac{1}{2}\int_{\mathbb{R}} \ell_t^{X,+}(a)f''(da), \tag{1.5}$$

where $f''(da)$ is a positive measure corresponding to the second derivative of f in the sense of distributions, and $f'_-(x)$ denotes one sided left derivative.

Occupation Time Formula: For a non-negative Borel function g and $t \geq 0$, one has a.e. that

$$\int_0^t g(X(s))d\langle X \rangle_s = \int_{\mathbb{R}} g(a)\ell_t^X(a)da. \tag{1.6}$$

To proceed we require a class of special continuous semimartingales referred to as *α-skew Browniam motions*, where $0 < \alpha < 1$, introduced in [16] as, for given $0 < \alpha < 1$, the diffusion $B^{(\alpha)}$ on $\mathbb{R}$ whose infinitesimal generator is $\frac{1}{2}\frac{d^2}{dx^2}$ on the domain

$$\mathcal{D}_\alpha := \{u \in C(\mathbb{R}) : \alpha u_x(0^+) = (1-\alpha)u_x(0^-), u_{xx}(0^-) = u_{xx}(0^+)\} \cap C^2(\infty, 0] \cap C^2[0, \infty). \tag{1.7}$$

Notice that in the case $\alpha = 1/2$, this is the generator of standard Brownian motion B. The parameter α is referred to as a *transmission probability.* The diffusion constructed by Itô and McKean in [16] is obtained by first enumerating the excursions of the paths of a standard Brownian motion reflected at zero, namely excursions of $|B|$, and then using independent coin tosses to determine if an excursion should be flipped (sign-changed) to $-|B|$, or not. The sign changes occur with probability $1 - \alpha$. This creates the desired flux of particles across zero required in the specification of the infinitesimal generator; see [22] for a comprehensive overview.

John Walsh [43] astutely observed that α-skew Brownian motion, with $\alpha \neq 1/2$, provides a first example of a continuous semimartingale on all of $\mathbb{R}$ having discontinuous semimartingale local time. In the case of standard Brownian motion ($\alpha = 1/2$), the continuity of local time is the aforementioned theorem of Trotter [42]. We shall return to this point in the last section.

For the following theorem let us write $D(x) = D^+, x \geq 0$, and $D(x) = D^-, x < 0$. Also $\mathcal{D}_\lambda$ is defined by (1.7) with λ in place of α.

Theorem 1.1. *Let D^+, D^- be arbitrary positive numbers and let $0 < \alpha, \lambda < 1$. Define $Y^{(\alpha)}(t) = \sigma(B_t^{(\alpha)}), t \geq 0$, where $B^{(\alpha)}$ is skew Brownian motion with transmission parameter α and $\sigma(x) = \sqrt{D^+}x1_{[0,\infty)}(x) + \sqrt{D^-}x1_{(-\infty,0]}(x),\ x \in \mathbb{R}$. Then*

$$M(t) = f(Y^{(\alpha)}(t)) - \frac{1}{2}\int_0^t D(Y^{(\alpha)}(u))f''(Y^{(\alpha)}(u))du, \quad t \geq 0,$$

is a martingale for all $f \in \mathcal{D}_\lambda$ if and only if

$$\alpha = \alpha(\lambda) = \frac{\lambda\sqrt{D^-}}{\lambda\sqrt{D^-} + (1-\lambda)\sqrt{D^+}}.$$

Equivalently,

$$\lambda = \lambda(\alpha) = \frac{\alpha\sqrt{D^+}}{(1-\alpha)\sqrt{D^-} + \alpha\sqrt{D^+}}.$$

Proof: In the case of skew Brownian motion one has the following relationships between semimartingale local time at zero and its one-sided variants, see [29]. For $t \geq 0$

$$\ell_t^{B^{(\alpha)},+}(0) = 2\alpha\ell_t^{B^{(\alpha)}}(0), \tag{1.8}$$

$$\ell_t^{B^{(\alpha)},-}(0) = 2(1-\alpha)\ell_t^{B^{(\alpha)}}(0). \tag{1.9}$$

Moreover, $B^{(\alpha)}$ is the unique strong solution to the stochastic differential equation [13, 19],

$$dB^{(\alpha)}(t) = \frac{2\alpha - 1}{2\alpha}d\ell_t^{B^{(\alpha)},+}(0) + dB(t). \tag{1.10}$$

In particular, considering the integrated version, B is the martingale component and $\frac{2\alpha-1}{2\alpha}\ell_t^{B^{(\alpha)}}$ is the finite variation component in the view of $B^{(\alpha),+}$ as a semimartingale.

It is also straightforward to relate semimartingale local times of $Y^{(\alpha)}$ and $B^{(\alpha)}$ through the one-sided formulae as:

$$\ell_t^{Y^{(\alpha)},+}(0) = \sqrt{D^+}\ell_t^{B^{(\alpha)},+}(0), \qquad \ell_t^{Y^{(\alpha)},-}(0) = \sqrt{D^-}\ell_t^{B^{(\alpha)},-}(0) \tag{1.11}$$

Applying the Itô-Tanaka formula to the positive and negative parts of $Y^{(\alpha)}$ together with (1.10), one has

$$d(Y^{(\alpha)}(t))^+ = \sqrt{D^+}\{1_{[B^{(\alpha)}(t)>0]}dB(t) + \frac{1}{2}d\ell_t^{B^{(\alpha)},+}(0)\},$$

and

$$\begin{aligned} d(Y^{(\alpha)}(t))^- &= \sqrt{D^-}\{-1_{[B^{(\alpha)}(t)\le 0]}dB(t) \\ &\quad -\frac{(2\alpha-1)}{2\alpha}d\ell_t^{B^{(\alpha)},+}(0) + \frac{1}{2}d\ell_t^{B^{(\alpha)},+}(0)\}. \end{aligned}$$

Thus, since $Y^{(\alpha)}$ is the difference of its positive and negative parts, one has

$$\begin{aligned} dY^{(\alpha)}(t) &= \sqrt{D(B^{(\alpha)}(t))}dB(t) \\ &\quad + \left(\frac{\sqrt{D^+}-\sqrt{D^-}}{2} + \sqrt{D^-}\frac{2\alpha-1}{2\alpha}\right)d\ell_t^{B^{(\alpha)},+}(0). \end{aligned} \tag{1.12}$$

For a difference of convex functions f, one has

$$f''(da) = f''(a)da + (f'_+(0) - f'_-(0))\delta_0(da).$$

In particular, for $f \in \mathcal{D}_\lambda$ one has

$$f''(da) = f''(a)da + f'_-(0)(\frac{1-\lambda}{\lambda} - 1)\delta_0(da).$$

Using Itô-Tanaka formula together with (1.8) and (1.12), one has

$$\begin{aligned} & f(Y^{(\alpha)}(t)) - f(Y^{(\alpha)}(0)) \\ =& \int_0^t f'_-(Y^{(\alpha)}(s))dY^{(\alpha)}(s) + \frac{1}{2}\int_{\mathbb{R}} \ell_t^{Y^{(\alpha)},+}(a)f''(da) \\ =& \int_0^t f'_-(Y^{(\alpha)}(s))\sqrt{D(Y^{(\alpha)}(s))}dB(s) + \frac{\alpha\sqrt{D^+}-(1-\alpha)\sqrt{D^-}}{2\alpha}f'_-(0)\ell_t^{B^{(\alpha)},+}(0) \\ +& \frac{1}{2}\int_{\mathbb{R}} \ell_t^{Y^{(\alpha)},+}(a)f''(a)da + \frac{1}{2}(\frac{1-\lambda}{\lambda} - 1)f'_-(0)\ell_t^{Y^{(\alpha)},+}(0). \end{aligned} \tag{1.13}$$

By the occupation time formula relating semimartingale occupation time and semimartingale local time, and noting that $d\langle Y^{(\alpha)}\rangle_s = D(Y^{(\alpha)}(s))ds$, it also follows that

$$\frac{1}{2}\int_{\mathbb{R}} f''(a)\ell_t^{Y^{(\alpha)},+}(a)da = \frac{1}{2}\int_0^t D(Y^{(\alpha)}(s))f''(Y^{(\alpha)}(s))ds.$$

From this, and using the relation (1.11) between the natural one sided local times of the processes $Y^{(\alpha)}$ and $B^{(\alpha)}$ one obtains

$$\begin{aligned} f(Y^{(\alpha)}(t)) - f(Y^{(\alpha)}(0)) - \frac{1}{2}\int_0^t D(Y^{(\alpha)}(s))f''(Y^{(\alpha)}(s))ds &= \int_0^t f'_-(Y^{(\alpha)}(s))\sqrt{D(Y^{(\alpha)}(s))}dB(s) \\ + \left(\frac{\alpha\sqrt{D^+}-(1-\alpha)\sqrt{D^-}}{2\alpha} + \frac{\sqrt{D^+}}{2}(\frac{1-\lambda}{\lambda} - 1)\right)&f'_-(0)\ell_t^{B^{(\alpha)},+}(0) \end{aligned}$$

The stated martingale property follows noting that the coefficient of the local time terms vanishes iff $\alpha = \alpha(\lambda)$, i.e., if and only if $\frac{1-\alpha}{\alpha}\frac{\sqrt{D^-}}{\sqrt{D^+}} = \frac{1-\lambda}{\lambda}$. That this is sufficient now follows from more standard theory, e.g., Theorem 2.4 in [38]. □

Some implications for the examples noted at the outset will be indicated in the next section, however one may also note that one has the following consequence at the scale of Kolmogorov's backward equation. Recall $D(x) = D^+, x \geq 0$, and $D(x) = D^-, x < 0$.

Corollary 1.2. *Let D^+, D^- be arbitrary positive numbers and let $0 < \lambda < 1$. Then for $c_0 \in \mathcal{D}_\lambda$, the unique solution to*

$$\frac{\partial c}{\partial t} = \frac{1}{2} D(x) \frac{\partial^2 c}{\partial x^2}, \quad \lim_{t \to 0^+} c(t,x) = c_0(x),$$

$$\lambda \frac{\partial c}{\partial x}(t, 0^+) = (1-\lambda)\frac{\partial c}{\partial x}(t, 0^-), t > 0,$$

is given by

$$c(t,x) = \mathbb{E}_x c_0(Y^{(\alpha(\lambda))}(t)), \quad t \geq 0,$$

where $\alpha(\lambda), \lambda$ and $Y^{(\alpha(\lambda))}$ are as given in Theorem 1.1.

Notation. As a matter of notation it is sometimes convenient to expess the (spatial) jump condition of the form $\lambda^+ \frac{\partial c}{\partial x}(0^+) - \lambda^- \frac{\partial c}{\partial x}(0^-) = 0$ in the bracket notation $[\lambda \frac{\partial c}{\partial x}] = 0$.

2 Examples

It is illuminating to consider Theorem 1.1 and Corollary 1.2 in the context of a few of the indicated applications.

Theorem 1.1 provides a generalization of the results obtained in [34] and [2] for the case of dispersion problems across an interface arising in porous media. One may check that

$$\lambda = \frac{D^+}{D^+ + D^-}, \qquad \alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}}, \tag{2.1}$$

follow from Theorem 1.1 for this application. This coincides with the results of [34] and [2] obtained by other methods.

The following definition is made with reference to both the diffusion coefficient and the interface parameter in the context of this and the other examples.

Definition 2.1. *With the choice of $\alpha \equiv \alpha(\lambda)$ given by Theorem 1.1, we refer to the process $Y^{(\alpha(\lambda))}$ as the natural diffusion corresponding to the dispersion coefficients D^+, D^- and interface parameter λ.*

Remark: We sometimes refer to the natural diffusion corresponding to $\lambda = \frac{D^+}{D^+ + D^-}, \alpha^* = \frac{\sqrt{D^+}}{\sqrt{D^+}+\sqrt{D^-}}$ as the *physical diffusion*. We refer to the diffusion in the case $\lambda = 1/2, \alpha^\sharp = \frac{\sqrt{D^-}}{\sqrt{D^+}+\sqrt{D^-}} = 1 - \alpha^*$ as the *Stroock-Varadhan diffusion* since it is the solution to the particular martingale problem originating with these authors [41].

In an application to the coastal upwelling model given by [25], for example, one obtains the relationship

$$\lambda = 1/2, \qquad \alpha(1/2) = \alpha^\sharp = \frac{\sqrt{D^-}}{\sqrt{D^+} + \sqrt{D^-}}. \tag{2.2}$$

This particular interface parameter provides continuity of the derivative at the interface. Mathematically, the natural diffusion for this example may be checked to coincide with the Stroock-Varadhan martingale in this case; cf. [41]. However, we will see that this is more properly interpreted as a continuity of flux from the point of view of the physics.

The following modification of the usual notion of semimartingale occupation time, where the integration is with respect to quadratic variation and in units of squared-length, provides a quantity in units of time that we refer to as natural occupation time.[1]

Definition 2.2. *Let X be a continuous semimartingale. The natural occupation time of G by time t, is defined by $\tilde{\Gamma}(G,t) = \int_0^t 1(X(s) \in G)ds, t \geq 0$, for an arbitrary Borel subset G of $\mathbb{R}$.*

The following result illustrates another way in which the specification of interfacial conditions is indeed a sensitive matter in consideration of occupation times. The proof exploits the basic property of skew Brownian motion that for any $t > 0$,

$$P(B^{(\alpha)} > 0) = \alpha. \tag{2.3}$$

This is easily checked from definition and, intuitively, reflects the property that the excursion interval $J_{n(t)}$ of $|B|$ containing t must result in a $[A_{n(t)} = +1]$ coin flip, an event with probability α.

Theorem 2.3. *Let $Y^{(\alpha(\lambda))}$ denote the natural diffusion for the dispersion coefficients D^+, D^- and interface parameter λ. Denote natural occupation time processes by*

$$\tilde{\Gamma}^+_\lambda(t) = \int_0^t 1[Y^{(\alpha(\lambda))}(s) > 0]ds, \quad t \geq 0.$$

Similarly let $\tilde{\Gamma}^-_\lambda(t) = t - \tilde{\Gamma}^+_\lambda(t), t \geq 0$. Then,

$$\mathbb{E}\tilde{\Gamma}^+_\lambda(t) > \mathbb{E}\tilde{\Gamma}^-_\lambda(t) \ \forall t > 0 \quad \Longleftrightarrow \lambda > \frac{\sqrt{D^+}}{\sqrt{D^+} + \sqrt{D^-}},$$

with equality when $\lambda = \frac{\sqrt{D^+}}{\sqrt{D^+}+\sqrt{D^-}}$. In fact,

$$\lim_{t\to\infty} \frac{\tilde{\Gamma}^+_\lambda(t)}{\tilde{\Gamma}^-_\lambda(t)} = \frac{\lambda\sqrt{D^-}}{(1-\lambda)\sqrt{D^+}} \quad \textit{almost surely.}$$

Proof: Using the definition of natural diffusion $Y^{(\alpha(\lambda))}$ for the parameters $D^\pm, \lambda$ and the above noted property (2.3) of skew Brownian motion, one has

$$\begin{aligned} \mathbb{E}\tilde{\Gamma}^+_\lambda(t) &:= \mathbb{E}\int_0^t 1[Y^{(\alpha(\lambda))}(s) > 0]ds \\ &= \int_0^t P(B^{(\alpha(\lambda))}(s) > 0)ds = t\alpha(\lambda). \end{aligned}$$

[1]While we emphasize "natural"choices from the point of view of modeling (and units), there are very sound and important reasons for the standard mathematical definitions. In particular, no suggestion to change the mathematical definition is intended. Indeed, as the proof of Theorem 1.1 demonstrates, the notion of semimartingale local time and occupation time and their relationship is extremely powerful in singling out the special value of $\alpha(\lambda)$ for given interface parameter λ and dispersion coefficients $D^\pm$.

Similarly $\mathbb{E}\tilde{\Gamma}_\lambda^-(t) = t(1-\alpha(\lambda))$. Thus

$$\frac{\mathbb{E}\tilde{\Gamma}_\lambda^+(t)}{\mathbb{E}\tilde{\Gamma}_\lambda^-(t)} = \frac{\alpha(\lambda)}{1-\alpha(\lambda)} = \frac{\lambda\sqrt{D^-}}{(1-\lambda)\sqrt{D^+}}.$$

The first assertion now follows. The ratio limit theorem of the second assertion is a direct consequence of the Chacon-Ornstein ergodic theorem and (2.3), see [20], since skew Brownian motion is null-recurrent with an absolutely continuous invariant measure; see [36]. □

Remark 2.4. *Theorem 2.3 shows that in fact the conservative interface condition (defined by this choice of* λ*) would not be appropriate for models of animal movement for which the faster dispersion occurs in more hostile environments [24, 9, 30]. The ratio limit explicitly provides the effect of relative parameter values in terms of the relative times spent on the positive and negative half-lines, respectively, i.e,* $\lambda\sqrt{D^-} > (1-\lambda)\sqrt{D^+}$ *if and only if* $\lambda > \sqrt{D^+}/(\sqrt{D^+}+\sqrt{D^-})$*. There is indeed something to be learned from data as to what exactly might apply, but such theoretical insights can provide a useful guide, and help to prevent mistaken assumptions when transferring more well-developed physical principles to biological/ecological phenomena.*

3 Continuity of Natural Local Time

We now discuss another issue pertaining to the definition of semimartingale local time and the basis for the suggested notion of natural local time. To resolve the physical issues for modeling at the stochastic particle scale, it is convenient to consider the (physically) more comprehensive diffusion equation of the form, recalling the bracket notation to express jumps,

$$\eta(x)\frac{\partial c}{\partial t} = \frac{1}{2}\frac{\partial}{\partial x}(\kappa(x)\frac{\partial c}{\partial x}), \quad [\kappa\frac{\partial c}{\partial x}] = 0, \tag{3.1}$$

where both coefficients $\eta, \kappa > 0$ are piecewise constant about the interface at zero. While mathematically this may be viewed as an equivalent problem to that in Corollary 1.2 by multiplying the latter equation by $\eta = \lambda/D$, from the point of view of the physics the roles (and units) of the two coefficients η and κ are quite distinct. The approach taken here is designed to both respect and exploit the difference between the spatial flux on the right side and the time rate of change on the left side of the equation (3.1).

Let us first examine this situation in the context of natural diffusions with $\eta \equiv 1$, and κ an arbitrary positive constant. Since the quadratic variation of skew Brownian motion coincides with that of standard Brownian motion $B = B^{(\frac{1}{2})}$, one has $\langle\sqrt{\kappa}B^{(\alpha)}\rangle \equiv \langle\sqrt{\kappa}B\rangle$. Within this class of natural diffusions, one may then ask what distinguishes the particular diffusion $X \equiv \sqrt{\kappa}B$? Of course, the answer is that X is determined by $\alpha = \lambda = 1/2$. If one views this choice of diffusion in the context of the flux in particle concentration, then it provides continuity of flux. On the other hand, in view of the respective theorems of Trotter [42] and Walsh [43], it is also the unique choice of α from among all skew Brownian motions to make local time continuous. The latter may be viewed as a stochastic particle determination of the physical diffusion model, among natural diffusions, for constant diffusion coefficient D.

The next theorem, a version of which was originally conceived in [2, 3], extends this to the more general framework of the present paper, in particular to include the case $\eta \equiv 1$ constant, but $\kappa^+ \neq \kappa^-$. However, it requires the following modification of the definition of semimartingale local time, already alluded to above, and referred to here as *natural local time.*

Definition 3.1. *Let X be a continuous semimartingale. The natural local time at a $\tilde{\ell}_t^X = \frac{\tilde{\ell}_t^{X,+}+\tilde{\ell}_t^{X,-}}{2}$ of X is defined by*

$$\tilde{\ell}_t^{X,+}(a) = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\int_0^t 1_{[a,a+\epsilon)}(X(s))ds,$$

and

$$\tilde{\ell}_t^{X,-}(a) = \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\int_0^t 1_{(a-\epsilon,a]}(X(s))ds,$$

provided that the indicated limits exist almost surely.

The units of natural local time are time per unit length ($\frac{T}{L}$), appropriate to a measurement of (occupation) time in the vicinity of a spatial location a. While the purpose here is not to explore the generality for which natural local time exists among all continuous semimartingales, according to the following theorem it does exist for natural skew diffusion. Moreover, continuity of natural local time has a special significance for the determination of parameters at the particle scale as follows.

Theorem 3.2. *Let $Y^{(\alpha(\lambda))}$ be the natural skew diffusion with parameters $D^{\pm}, \lambda$. Then the natural local time of $Y^{(\alpha(\lambda))}$ at 0 is continuous if and only if $\lambda = \frac{D^+}{D^+ + D^-}$, i.e., if and only if $\alpha(\lambda) = \alpha^*$ and thus $Y^{(\alpha^*)}$ is the physical diffusion.*

Proof In view of (1.8) observe that

$$\frac{\ell_t^{B^{(\alpha)},+}(0)}{\ell_t^{B^{(\alpha)},-}(0)} = \frac{\alpha}{1-\alpha}, \qquad \forall t \geq 0.$$

Moreover,

$$\tilde{\ell}_t^{Y^{(\alpha)},+}(0) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_0^t 1[0 \leq Y^{(\alpha)}(s) < \epsilon]ds = \frac{1}{\sqrt{D^+}}\ell_t^{B^{(\alpha)},+}(0).$$

Similarly, $\tilde{\ell}_t^{Y^{(\alpha)},-}(0) = \frac{1}{\sqrt{D^-}}\ell_t^{B^{(\alpha)},-}(0)$. Thus,

$$\frac{\tilde{\ell}_t^{Y^{\alpha(\lambda)},+}(0)}{\tilde{\ell}_t^{Y^{\alpha(\lambda)},-}(0)} = \frac{\alpha(\lambda)}{1-\alpha(\lambda)}\frac{\sqrt{D^-}}{\sqrt{D^+}}.$$

Now simply observe that this ratio is one if and only if $\lambda D^- = (1-\lambda)D^+$, which establishes the assertion. □

Finally, the role of piecewise constant coefficient η in (3.1) with respect to the interface at zero can be identified by rescaling local time accordingly. Namely, one has the following generalization of Corollary 1.2 appropriate to the problem (3.1).

Corollary 3.3. *Consider the problem (3.1) and let $Y^{(\alpha(\lambda))}$ be the natural skew diffusion with parameters $D = \frac{\kappa}{\eta}, \lambda = \kappa$. Then $[\frac{\tilde{\ell}}{\eta}] = 0$ if and only if $\lambda = \frac{\kappa^+}{\kappa^+ + \kappa^-}$, i.e., the interface condition is continuity of flux.*

This last corollary fully resolves the modeling issues inherent in the physical and biological examples discussed above at the scale of both (i) the dispersion equation in Corollary 1.2, and (ii) the stochastic particle motion. Namely, in the case of dispersion in porous media, the continuity of flux is determined by the dispersion coefficient as $[D\frac{\partial c}{\partial x}] = 0$ with $\eta \equiv 1$ or, equivalently, continuity of natural local time of the rescaled skew Brownian motion with $\alpha = \alpha^*$. For the case of the upwelling problem, the continuity of flux is specified by $[\frac{\partial c}{\partial x}] = 0$, and by $\eta = \frac{1}{D}$, and $\kappa = 1$. In particular, α is determined by continuity of $\frac{\tilde{\ell}}{\eta} = D\tilde{\ell}$. Finally, in the case of animal movement, the jump condition $[\lambda\frac{\partial c}{\partial x}] = 0$ prescribes continuity of flux for $\kappa = \lambda$ and $\eta = \frac{\lambda}{D}$, with α determined by continuity of $\frac{\tilde{\ell}}{\eta}$ at the stochastic particle scale.

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