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In 2014, W. Bogley identified a relation between the algebraic and geometric properties of cyclically presented groups $G_n(w)$ in the case where $w = x_0 x_k x_l$ is a positive word of length three. Specifically, it was shown that the dynamics of the shift θ_G on the group $G = G_n(x_0 x_k x_l)$ are connected to the finiteness of G and the combinatorial asphericity status of the corresponding cyclic presentation $\mathcal{P}_n(x_0 x_k x_l)$ that determines G. This dissertation provides an extension of those results to the case where $w = x_0 x_j x_k x_l$ is a positive word of length four. Additionally, this dissertation provides a complete classification of the cyclically presented groups $G_n(x_0 x_j x_k x_l)$ that are finite and, except for two families of presentations where the status is unresolved, a complete classification of the cyclic presentations $\mathcal{P}_n(x_0 x_j x_k x_l)$ that are combinatorially aspherical.

Shift Dynamics for Cyclically Presented Groups with Length Four Positive Relators

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I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes release of my dissertation to any reader upon request.

Forrest William Parker, Author

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Academic

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1 INTRODUCTION

Let n be a positive integer, and let $F = F(\mathbf{x})$ be the free group of rank n with basis $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$. The map $\theta_F : F \to F$ via $x_i \mapsto x_{i+1}$ (subscripts taken modulo n) defines an order n automorphism of F. Along with n, a word $w \in F$ determines the **cyclic group presentation**

$$\mathcal{P}_n(w) = \left\langle x_0, \ldots, x_{n-1} : w, \theta_F(w), \ldots, \theta_F^{n-1}(w) \right\rangle .$$

The presentation $\mathcal{P}_n(w)$, in turn, determines the **cyclically presented group**

$$G = G_n(w) = F(\mathbf{x}) \Big/ \langle\langle \mathbf{r} \rangle\rangle_F$$

where $\langle\langle \mathbf{r} \rangle\rangle_F$ is the normal closure of $\mathbf{r} = \left\{w, \theta_F(w), \ldots, \theta_F^{n-1}(w)\right\}$ in F. The automorphism $\theta_F \in \mathrm{Aut}(F)$ induces an automorphism $\theta_G \in \mathrm{Aut}(G)$ called the **shift** that satisfies $\theta_G^n = 1$. The order of the shift is therefore a divisor of n, and the shift may or may not be an inner automorphism of G.

1.1 Main Theorems

The following two classification problems have received attention in the literature. For example, see [7], [14], [25], [26].

Finiteness: For which n and w is the cyclically presented group $G_n(w)$ finite?

Combinatorial Asphericity: For which n and w is the cyclic presentation $\mathcal{P}_n(w)$ combinatorially aspherical?

Every group presentation $\mathcal{P}$ has a corresponding two-dimensional cellular model $K(\mathcal{P})$ whose fundamental group $\pi_1 K(\mathcal{P})$ is isomorphic to the group determined by $\mathcal{P}$. See e.g. [24] and Section 2.2.1 for more detail. The presentation $\mathcal{P}$ is **aspherical** if the two-complex $K(\mathcal{P})$ is aspherical (i.e. has a contractible universal covering complex). It follows that any group determined by an aspherical presentation therefore has geometric dimension at most two and, in particular, is torsion-free. Combinatorial asphericity is a more general concept of presentations that allows for the presence of freely redundant and proper power relators, in which case $\pi_2 K(\mathcal{P}) \neq 0$. See Section 2.2.2.

In [7], [14], the finiteness and combinatorial asphericity classification problems were completely solved for cyclic presentations $\mathcal{P}_n(w)$ where the defining word w is positive of length three; that is, when $w = x_0 x_k x_l$ for some integers k, l. This dissertation addresses the case where w is a positive word of length four. Under consideration are cyclic presentations of the form

$$\mathcal{P}_n(x_0 x_j x_k x_l) = \langle x_0, \ldots, x_{n-1} : x_i x_{i+j} x_{i+k} x_{i+l},\ 0 \leq i < n \rangle$$

where the integer parameters j, k, l are considered modulo n. (Note that the presentation $\mathcal{P}_n(w)$ is unchanged by replacing the word w with $\theta_F^i(w)$ for any integer i, so it suffices to consider only the case where $i = 0$.)

Modeling an approach developed in [14], the classifications are primarily framed in terms of three conditions (A), (B), and (C) on the parameters n, j, k, l outlined in Table 1.1. Next, the parameters n, j, k, l are used to calculate the **primary divisor**

$$d = \gcd(n, j, k, l)$$

and the **secondary divisor**

$$\gamma = \gcd(n, k - 2j, l - 2k + j, k - 2l, j + l)$$

which play a role in the finiteness classification. See Section 3.2.2. Finally, a number of presentation **types** are used to distinguish particular presentations $\mathcal{P}_n(x_0 x_j x_k x_l)$. Tables 1.2 and 1.3 list a number of exemplars upon which the isolated and unresolved types are defined.

Types (I) and (U): Section 3.2.1 introduces a group Γ_n of order $8n\phi(n)$ (where ϕ is the Euler totient function) such that Γ_n acts on the set of positive words of length four

(A) $2k \equiv 0$ or $2j \equiv 2l \mod n$

(B) $k \equiv 2j$ or $k \equiv 2l$ or $j + l \equiv 2k$ or $j + l \equiv 0 \mod n$

(C) $l \equiv j + k$ or $l \equiv j - k \mod n$

TABLE 1.1: Conditions (A), (B), (C) on the parameters n, j, k, l

Type	Exemplar
(I5)	$\mathcal{P}_5(x_0x_3x_1x_1)$
(I6′)	$\mathcal{P}_6(x_0x_4x_2x_3)$
(I6″)	$\mathcal{P}_6(x_0x_0x_1x_2)$
(I10)	$\mathcal{P}_{10}(x_0x_3x_6x_1)$
(I12)	$\mathcal{P}_{12}(x_0x_1x_2x_9)$
(I16)	$\mathcal{P}_{16}(x_0x_3x_6x_1)$
(I20)	$\mathcal{P}_{20}(x_0x_3x_6x_1)$
(I24)	$\mathcal{P}_{24}(x_0x_1x_2x_{15})$

TABLE 1.2: Exemplars for the Eight Isolated Types (I5)-(I24)

Type	Exemplar
(U24′)	$\mathcal{P}_{24}(x_0x_3x_6x_1)$
(U24″)	$\mathcal{P}_{24}(x_0x_1x_2x_{19})$

TABLE 1.3: Exemplars for the Two Unresolved Types (U24′) and (U24″)

in the generators $x_0, \ldots, x_{n-1}$. Under this action, the orbit of a word $w = x_ix_jx_kx_l$ under Γ_n consists of all shifts of cyclic permutations of words of the form $x_{ui}x_{uj}x_{uk}x_{ul}$ or $x_{ui}x_{ul}x_{uk}x_{uj}$ where u is an arbitrarily given unit $u \in \mathbb{Z}_n^*$. Applying this to the exemplar listed in Table 1.2 under (I5), the cyclic presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is of **type (I5)** if $n = 5$ and $x_0x_jx_kx_l$ is a shift of a cyclic permutation of a word of the form $x_0x_{3u}x_ux_u$ or $x_0x_ux_ux_{3u}$ where $u \in \mathbb{Z}_5^*$. Thus, the presentations of type (I5) are precisely those of the form $\mathcal{P}_5(x_0x_jx_kx_l)$ where $x_0x_jx_kx_l$ is a positive length four word appearing in the orbit of $x_0x_3x_1x_1$ under the action of the finite group Γ_5. In Section 3.2.1, it is shown that each such orbit gives rise to a single group up to isomorphism, combinatorial asphericity status, and shift dynamics. In the same way, the remaining exemplars determine types (I6′), (I6″), (I10), (I12), (I16), (I20), and (I24). Any cyclic presentation of one of these eight types is **isolated** and said to be of **type (I)**. Similarly, the exemplars listed in Table 1.3 determine the presentations of type (U24′) and (U24″) in terms of the action of the finite group Γ_{24}. Any presentation of type (U24′) or (U24″) is **unresolved** and of **type (U)**.

Types (I*) and (U*): For arbitrary parameters n, j, k, l, the presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is of **type (I*)** (respectively **type (U*)**) if the presentation $\mathcal{P}_{n/d}\left(x_0x_{j/d}x_{k/d}x_{l/d}\right)$ is of type (I) (respectively type (U)) where d is the primary divisor. Any presentation of type (I*) is not combinatorially aspherical (see Lemma 3.2.5 and Theorem 3.2.31), but the combinatorial asphericity status of any presentation of type (U*) is currently unresolved. Resolution of this ambiguity reduces to consideration of the exemplars $\mathcal{P}_{24}(x_0x_3x_6x_1)$ and $\mathcal{P}_{24}(x_0x_1x_2x_{19})$; see Lemma 3.2.5.

The main theorems regarding combinatorial asphericity and finiteness are below.

Theorem A (Combinatorial Asphericity Classification) *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l. If $\mathcal{P}$ is not of type* (U*)*, then it is combinatorially aspherical if and only if at least one of the following holds:*

(a) Conditions (A) *and* (C) *are both true;*

(b) Conditions (B) *and* (C) *are both false; or*

(c) Condition (B) *is true, conditions* (A) *and* (C) *are both false, and* $\mathcal{P}$ *is not of type* (I*)*.*

The presentation $\mathcal{P}$ *is aspherical if and only if it is combinatorially aspherical and there are no freely redundant or proper power relators. If* $\mathcal{P}$ *is combinatorially aspherical and either* $k \not\equiv 0$ *or* $j \not\equiv l \mod n$*, then* $\mathcal{P}$ *has no proper power relators, and so* G *is a torsion-free infinite group of geometric dimension at most two.*

Theorem B (Finiteness Classification) *Let* $G = G_n(x_0 x_j x_k x_l)$ *be a cyclically presented group with parameters* n, j, k, l*. Then* G *is finite if and only if at least one of the following holds:*

(a) $\gcd(n, 2k) = 1$ *and either*

(i) $l \equiv j + k \mod n$ *and* $\gcd(n, j) = 1$*; or*

(ii) $l \equiv j - k \mod n$ *and* $\gcd(n, l) = 1$*;*

(b) Conditions (A) *and* (B) *are true, condition* (C) *is false, and the secondary divisor* $\gamma = 1$*;*

(c) The cyclic group presentation $\mathcal{P}_n(x_0 x_j x_k x_l)$ *that determines* G *is of type* (I5), (I6′), *or* (I6″)*.*

In claim (a), G *is cyclic of order four. In claim (b),* G *is solvable. In type* (I5)*,* G *is metacyclic and non-nilpotent of order* 220*. The types* (I6′) *and* (I6″) *lead to nonisomorphic, nonsolvable groups* G *of order* $4088448 = 2^7 \cdot 3^3 \cdot 7 \cdot 13^2$*, each of which contains the simple group* $\mathrm{PSL}(3, 3)$*.*

These classifications are supported by simultaneous consideration of **shift dynamics**, which concern the action of the cyclic group $\mathbb{Z}_n$ by automorphisms on the nonidentity elements of $G = G_n(w)$. For arbitrary n and w, shift dynamics encompasses at least these two general problems.

Fixed Points: For which n and w does the shift on $G_n(w)$ have a nonidentity fixed point?

Freeness: For which n and w does the shift determine a free $\mathbb{Z}_n$-action on the nonidentity elements of $G_n(w)$?

Combinatorial asphericity of $\mathcal{P}_n(w)$ and freeness of the $\mathbb{Z}_n$-action are related by the following general result from [3]. A cyclic presentation $\mathcal{P}_n(w)$ is **orientable** if w is not a cyclic permutation of the inverse of any of its shifts. In the general case, a cyclic presentation $\mathcal{P}_n(w)$ fails to be orientable if and only if $n = 2m$ is even and $w = u\theta_F^m(u)^{-1}$ for some word u [3, Lemma 3.6] (in which case u is a fixed point of θ_G^m). Note that any cyclic presentation defined by a positive word w is orientable.

Theorem 1.1.1 ([3, Theorem A]) *If the cyclically presented group $\mathcal{P} = \mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then the cyclic group $\mathbb{Z}_n$ of order n acts freely via the shift on the nonidentity elements of the group $G_n(w)$ determined by $\mathcal{P}$.*

As a consequence, if a combinatorially aspherical cyclic presentation $\mathcal{P}_n(w)$ is orientable and determines a nontrivial group $G = G_n(w)$, then the shift automorphism θ_G has order n (compare [19]) and, in fact, determines an element of order n in the *outer* automorphism group $\mathrm{Out}(G)$. This is because every inner automorphism of a nontrivial group has a nonidentity fixed point. Of note, there are just a few known examples of orientable cyclic presentations that are not combinatorially aspherical and yet have free shift action. Examples include the Fibonacci groups $F(2,n) = G_n\left(x_0x_1x_2^{-1}\right)$ for $n = 5, 7$, which are finite cyclic having orders 11 and 29 respectively [25].

In [3, Theorems B, C], it was shown that if $w = x_0x_kx_l$ is a positive word of length three, then the relationships between finiteness, combinatorial asphericity, and shift dynamics are sharp:

- $\mathcal{P}_n(x_0x_kx_l)$ is combinatorially aspherical if and only if $\mathbb{Z}_n$ acts freely via the shift on the nonidentity elements of $G_n(x_0x_kx_l)$;

- $G = G_n(x_0x_kx_l)$ is finite if and only if the shift θ_G has a nonidentity fixed point.

The other principal aim of this dissertation is to extend the connection between finiteness and fixed points to the setting where $w = x_0x_jx_kx_l$ is a positive word of length four. The relation between combinatorial asphericity and freeness is not as tidy due to the unresolved cases in the asphericity classification.

Theorem C *If the cyclic group presentation $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ is not of type* (U*)*, then $\mathcal{P}$ is combinatorially aspherical if and only if the cyclic group $\mathbb{Z}_n$ of order n acts freely via the shift on the nonidentity elements of the group $G_n(x_0x_jx_kx_l)$ determined by $\mathcal{P}$.*

When combined with Theorem B, the next result explicitly determines those cyclically presented groups $G_n(x_0x_jx_kx_l)$ for which the shift has a nonidentity fixed point.

Theorem D *The shift on the cyclically presented group $G = G_n(x_0x_jx_kx_l)$ has a nonidentity fixed point if and only if either G is finite or both condition* (C) *is true and* $\gcd(n, 2k) = 1$.

Unlike for the cyclically presented groups $G_n(x_0x_kx_l)$, it can happen that $G = G_n(x_0x_jx_kx_l)$ is infinite and the fixed point subgroup $\mathrm{Fix}(\theta_G)$ is nontrivial. For example, it is observed in [3, Page 3] that the group $G_3(x_0^2x_1^2)$ is infinite, but the element $u_0 = x_0^2$ is an order two fixed point of the shift that generates a cyclically presented subgroup $G_2(u_0u_1) \cong \mathbb{Z}_2$. By analyzing similar infinite cases in detail, it becomes apparent that nonidentity fixed points for the shift always derive from finite cyclically presented subgroups.

Theorem E *The shift θ_G on the cyclically presented group $G = G_n(x_0x_jx_kx_l)$ has a nonidentity fixed point if and only if G possesses a finite, cyclically presented subgroup of the form $H = G_n(v)$ where $\theta_G|_H = \theta_H$ and* $\mathrm{Fix}(\theta_G) = \mathrm{Fix}(\theta_H) \neq 1$.

Combining with the results of [3], the following result is concluded.

Corollary F *Let $G = G_n(w)$ be a cyclically presented group with shift θ_G. Suppose w is a positive word of length L where $2 \leq L \leq 4$. Then the fixed point subgroup* $\mathrm{Fix}(\theta_G)$ *is finite. Moreover, if G is finite, then* $\mathrm{Fix}(\theta_G)$ *is nontrivial.*

1.2 Remarks on the Combinatorial Asphericity Classification

Recall that the shift θ_G of the cyclically presented group $G = G_n(w)$ satisfies $\theta_G^n = 1$. This implies that the cyclic group $\mathbb{Z}_n$ of order n acts via powers of the shift on G; this group action is called the **shift action**. The shift action induces the **shift extension**

$$E = E_n(w) = G_n(w) \rtimes_{\theta_G} \mathbb{Z}_n.$$

Working in E, the shift θ_G arises via conjugation by a chosen generator a of $\mathbb{Z}_n$; that is, $\theta_G(g) = aga^{-1}$ for $g \in G$. The shift extension is determined by a relative presentation

$$\mathcal{R}_n(w) = \langle \mathbb{Z}_n, x : W \rangle$$

where the word $W = W(a, x)$ is obtained from the word w via the substitutions $x_i = a^i x a^{-i}$ for $0 \leq i < n$. See Section 2.3.1. In the setting where $w = x_0 x_j x_k x_l$ is a positive word of length four, the word $W = xa^j xa^{k-j} xa^{l-k} xa^{-l}$. See Section 3.1.

Alternatively, one may begin with a group E that is determined by a relative presentation $\langle \mathbb{Z}_n, x : W \rangle$. If there exists a retraction $\nu : E \to \mathbb{Z}_n$, the kernel $\ker \nu \cong G_n(w)$ is cyclically presented, and E is the shift extension of $\ker \nu$. In this setting, the word $w = \rho^\nu(W)$ where ρ^ν is a Reidermeister-Schreier rewriting process that depends on the retraction ν. See [3, Theorem 2.3] and Section 2.3.2.

Now, consider a cyclically presented group $G_n(w)$ with shift extension $E = E_n(w)$ that is determined by the relative presentation $\mathcal{R}_n(w) = \langle \mathbb{Z}_n, x : W \rangle$. The word $w = \rho^{\nu^0}(W)$ is recovered via the rewriting process corresponding to the retraction $\nu^0 : E \to \mathbb{Z}_n$ defined by $\nu^0(a) = a$ and $\nu^0(x) = a^0 = 1$. Thus every cyclically presented group $G_n(w)$ is the kernel of such a retraction.

The relative presentation $\mathcal{R}_n(w)$ is **aspherical** if its **cellular model** M satisfies $\pi_2 M = 0$. See Section 2.2.3. A connection between the combinatorial asphericity status of the cyclic presentations $\mathcal{P}_n(x_0x_jx_kx_l)$ and the asphericity status of the associated relative presentations $\mathcal{R}_n(x_0x_jx_kx_l) = \langle \mathbb{Z}_n, x : xa^jxa^{k-j}xa^{l-k}xa^{-l} \rangle$ is given in [3] in the terms of retractions and is reproduced below. Recall that the cyclic presentation $\mathcal{P}_n(w)$ is orientable if w is a positive word.

Theorem 1.2.1 ([3, Theorem 4.1]) *Let n be a positive integer. Let M be the cellular model of a relative presentation $\langle \mathbb{Z}_n, x : W \rangle$ that determines a group E. Suppose that $\nu : E \to \mathbb{Z}_n$ is a retraction, and let $w = \rho^\nu(W) \in F(x_0, \ldots, x_{n-1})$.*

(a) If $\pi_2 M = 0$, then the cyclic presentation $\mathcal{P}_n(w)$ is combinatorially aspherical.

(b) If $\mathcal{P}_n(w)$ is orientable and combinatorially aspherical, then $\pi_2 M = 0$.

A classification of asphericity for relative presentations of the form $\langle H, x : xh_1xh_2xh_3xh_4 \rangle$ where the coefficients h_i are taken from an arbitrarily given group H is provided in [2] for all but a few cases in which the asphericity status remained unresolved at the time. The analysis first deals with the case where the coefficients h_i are pairwise distinct ([2, Theorem 2]), in which case the relative presentation is aspherical. This is essentially a C(4)-T(4) small cancellation situation as in [20] and corresponds to Theorem 3.2.18. The situation where $h_1 = h_3$ or $h_2 = h_4$ is dealt with in [2, Theorem 3] and corresponds to Theorem 3.2.18. Thus what remains is the case where two consecutive coefficients coincide in the relator viewed as a cyclic word. In the current context, where $H = \mathbb{Z}_n = \langle a \rangle$ and the relator has the form $xa^jxa^{k-j}xa^{l-k}xa^{-l}$, this simply means that condition (B) is true.

As in Lemma 3.2.13, a linear substitution $u = xa^p$ transforms the relative presentation $\mathcal{R}_n(x_0x_jx_kx_l) = \langle \mathbb{Z}_n, x : xa^jxa^{k-j}xa^{l-k}xa^{-l} \rangle$ that determines the shift extension $E_n(x_0x_jx_kx_l)$ into one of the form $\widehat{\mathcal{R}} = \langle \mathbb{Z}_n, u : u^3\alpha u\beta \rangle$ where $\alpha, \beta \in \mathbb{Z}_n = \langle a \rangle$. Because of the simple nature of this substitution, the cellular model of the relative presentation $\widehat{\mathcal{R}}$

is homotopy equivalent to that of $\mathcal{R}_n(x_0x_jx_kx_l)$, and so these relative presentations have the same asphericity status.

The statement of [2, Theorem 4] classifies the asphericity status of relative presentations

$$\widehat{\mathcal{R}} = \langle H, u : u^3\alpha u\beta \rangle$$

where $\alpha, \beta \in H$ under the assumption that $2 \leq \mathrm{o}(\alpha) \leq \mathrm{o}(\beta)$ and where $\mathrm{o}(-)$ refers to the order of an element in H. A symmetrized version of the result that does not include this restriction on the orders of α and β and which is updated in light of recent work is presented below.

The following circumstances were termed **exceptional** in [2].

(E1) $\alpha = \beta^2$ or $\beta = \alpha^2$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_6$

(E2) $\alpha = \beta^3$ or $\beta = \alpha^3$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_6$

(E3) $\alpha = \beta^4$ or $\beta = \alpha^4$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_6$

(E4) $\{\mathrm{o}(\alpha), \mathrm{o}(\beta)\} = \{2, 4\}$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$

(E5) $\{\mathrm{o}(\alpha), \mathrm{o}(\beta)\} = \{2, 5\}$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_5$

As in other asphericity studies [12] [17], these exceptional cases were unresolved for asphericity status in [2]. In recent work, Aldwaik and Edjvet [1] have shown that $\widehat{\mathcal{R}}$ is aspherical in cases (E4) and (E5). Now let L be the group determined by the presentation $\langle \mathbb{Z}_6, u : u^3t^3ut \rangle$ where t generates the coefficient group $\mathbb{Z}_6$. Williams [27] determined that L is finite of order $|L| = 24530688$. Any group L^* determined by a presentation of the form $\langle H, u : u^3t^3ut \rangle$ where $t \in H$ has order six decomposes as an amalgamated free product $L^* \cong H *_{\mathbb{Z}_6} L$, and it follows that L^* contains elements of finite order that are not conjugate to any element in H. By [4], this in turn implies that if $\mathrm{o}(\beta) = 6$, then the relative presentation $\langle H, u : u^3\beta^3u\beta \rangle$ in case (E2) is not aspherical. Similar remarks apply to the case $\langle H, u : u^3 \alpha u \alpha^3 \rangle$ upon replacing u by u^{-1}. Thus only the cases (E1) and (E3) remain unresolved for asphericity status. At this time, it is not known whether the groups determined by the presentations $\langle \mathbb{Z}_6, u : u^3 t^{\pm 2} u t \rangle$ (where t generates the coefficient group $\mathbb{Z}_6$) are finite.

Theorem 1.2.2 ([2, Theorem 4],[1],[27]) *Consider a relative presentation*

$$\widehat{\mathcal{R}} = \langle H, u : u^3 \alpha u \beta \rangle ,$$

where $1 \neq \alpha, \beta \in H$ *and suppose that if the group* $\langle \alpha, \beta \rangle \cong \mathbb{Z}_6$ *is cyclic of order six, then* $\alpha \neq \beta^{\pm 2}$ *and* $\beta \neq \alpha^{\pm 2}$. *Then* $\widehat{\mathcal{R}}$ *is aspherical if and only if none of the following conditions hold:*

(a) $\alpha = \beta^{\pm 1}$ *has finite order;*

(b) $\alpha = \beta^2$ *or* $\beta = \alpha^2$ *and* $\langle \alpha, \beta \rangle \cong \mathbb{Z}_4$ *or* $\mathbb{Z}_5$*;*

(c) $\alpha = \beta^3$ *or* $\beta = \alpha^3$ *and* $\langle \alpha, \beta \rangle \cong \mathbb{Z}_6$*;*

(d) $\{\mathrm{o}(\alpha), \mathrm{o}(\beta)\} = \{2, 3\}$ *and* $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3$*;*

(e) $\frac{1}{\mathrm{o}(\alpha)} + \frac{1}{\mathrm{o}(\beta)} + \frac{1}{\mathrm{o}(\alpha\beta^{-1})} > 1$ *where* $1/\infty = 0$.

2 PRELIMINARY MATERIAL

2.1 Group Presentations

2.1.1 Ordinary Group Presentations

Let $F(\mathbf{x})$ denote the free group with basis $\mathbf{x}$. An **ordinary group presentation** is an expression

$$\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$$

consisting of a set $\mathbf{x}$ and a set $\mathbf{r} \subset F(\mathbf{x})$. The elements contained in the sets $\mathbf{x}$ and $\mathbf{r}$ are called the **generators** and **relators** of the presentation $\mathcal{P}$, respectively. The presentation $\mathcal{P}$ is a **finite (ordinary group) presentation** if the sets $\mathbf{x}$ and $\mathbf{r}$ are both finite. The group presentation $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ determines the group

$$G = G(\mathcal{P}) = F(\mathbf{x}) \Big/ \langle\langle \mathbf{r} \rangle\rangle_{F(\mathbf{x})}$$

where $\langle\langle \mathbf{r} \rangle\rangle_{F(\mathbf{x})}$ is the normal closure of $\mathbf{r}$ in $F(\mathbf{x})$.

A relator $r \in \mathbf{r}$ is **freely redundant** if it is freely trivial or else is freely conjugate to another relator or its inverse. A relator $r \in \mathbf{r}$ is a **proper power** if it is freely equal to $r = \dot{r}^e$ where $\dot{r} \in F(\mathbf{x})$ and $e > 1$. When e is maximal, it is the **exponent** of r and $\dot{r}$ is the **root**.

Suppose a relator $r \in \mathbf{r}$ is written as $r = x_0^{\epsilon_0} x_1^{\epsilon_1} \cdots x_{L-1}^{\epsilon_{L-1}}$ where $x_i \in \mathbf{x}$ and $\epsilon_i = \pm 1$ for $i = 0, \ldots, L-1$ for some positive integer L. Then r is **cyclically reduced** if whenever $x_i = x_{i+1}$ then $\epsilon_i = \epsilon_{i+1}$ (all subscripts considered modulo L) for all $i = 0, \ldots, L-1$. If r is cyclically reduced, then the **length** of r is L. If $\epsilon_i = 1$ for all $i = 0, \ldots, L-1$, then r is **positive**. Note that a positive relator is cyclically reduced.

On occasion, a relator $r \in \mathbf{r}$ may be alternatively written as an equality $s = t$ where $s, t \in F(\mathbf{x})$ and $r = st^{-1}$. This is done particularly when it is desirable to focus on the equality of the elements s and t when considered as elements of the group $G(\mathcal{P})$ determined by $\mathcal{P}$.

It should be observed that multiple presentations may determine isomorphic groups. In many cases, the presentations may arise by disparate means. This is a well-studied problem known as the Group Isomorphism Problem.

Example 2.1.1 *The Fibonacci group $F(2,5)$ is the group determined by the presentation*

$$\langle x_0, \ldots, x_4 : x_i x_{i+1} = x_{i+2} \textit{ for } i = 0, \ldots, 4 \rangle$$

where subscripts are considered modulo 5. The group $F(2,5)$ is isomorphic to the cyclic group $\mathbb{Z}_{11}$ of order 11 [10] which is also determined by the superficially unrelated presentation $\langle a : a^{11} \rangle$.

2.1.2 Relative Group Presentations

A **relative group presentation** is an expression

$$\mathcal{R} = \langle H, \mathbf{x} : \mathbf{r} \rangle$$

consisting of a group H, a set $\mathbf{x}$ disjoint from H, and a set $\mathbf{r} \subset F(\mathbf{x}) * H$ where $F(\mathbf{x}) * H$ is the free product of the free group $F(\mathbf{x})$ with basis $\mathbf{x}$ and the group H. The elements contained in the group H and the sets $\mathbf{x}$ and $\mathbf{r}$ are respectively called the **coefficients**, **additional generators**, and **relators** of the presentation $\mathcal{R}$. The relative presentation $\mathcal{R}$ determines the group

$$G = G(\mathcal{R}) = (F(\mathbf{x}) * H) \Big/ \langle\langle \mathbf{r} \rangle\rangle_{F(\mathbf{x}) * H}.$$

As with ordinary group presentations, a relator $r \in \mathbf{r}$ of the relative presentation $\mathcal{R} = \langle H, \mathbf{x} : \mathbf{r} \rangle$ is a **proper power** if it is equal in the free product $F(\mathbf{x}) * H$ to $r = \dot{r}^e$ where $\dot{r} \in F(\mathbf{x}) * H$ and $e > 1$. When e is maximal, it is the **exponent** of r and $\dot{r}$ is the **root**. As an additional similarity, a relator $r \in \mathbf{r}$ may be alternatively written as an equality $s = t$ in the relative presentation $\mathcal{R} = \langle H, \mathbf{x} : \mathbf{r} \rangle$ where $s, t \in F(\mathbf{x}) * H$ and $r = st^{-1}$, particularly when it is desirable to focus on the equality of the elements s and t when considered as elements of the group $G(\mathcal{R})$ determined by $\mathcal{R}$.

Remark: Throughout this document, the word "presentation" by itself may refer either to an ordinary presentation or to a relative presentation, but the context will always be clear in such cases.

Of particular interest, consider the ordinary group presentation of the form

$$\mathcal{Q} = \langle a, x : a^n, W(a, x) \rangle$$

that determines the group $G(\mathcal{Q})$ where $W(a, x) \in F(a, x)$. By viewing $W(a, x)$ as an element W in the free product $Z_n * \langle x \rangle$ of the cyclic group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a with the infinite cyclic group $\langle x \rangle$ generated by x, there is a relative presentation

$$\mathcal{R} = \langle \mathbb{Z}_n, x : W \rangle$$

that determines the group $G(\mathcal{R})$ such that

$$G(\mathcal{R}) \cong (\mathbb{Z}_n * \langle x \rangle) \Big/ \langle\langle W \rangle\rangle_{\mathbb{Z}_n * \langle x \rangle} \cong F(a, x) \Big/ \langle\langle a^n, W(a, x) \rangle\rangle_{F(a,x)} \cong G(\mathcal{Q}),$$

and hence $\mathcal{Q}$ and $\mathcal{R}$ determine isomorphic groups. Note that this process may be reversed in a natural way by starting with a relative presentation $\mathcal{R} = \langle \mathbb{Z}_n, x : W \rangle$ and obtaining an ordinary presentation $\mathcal{Q} = \langle a, x : a^n, W(a, x) \rangle$ for which the groups determined by $\mathcal{Q}$ and $\mathcal{R}$ are isomorphic: $G(\mathcal{Q}) \cong G(\mathcal{R})$. Such an ordinary presentation $\mathcal{Q}$ is called a **lift** of the relative presentation $\mathcal{R}$.

2.2 Cellular Models and Types of Asphericity

The discussion of cellular models and the types of asphericity in this section follows those given in [3, Pages 6-8].

A topological space Y is **aspherical** if each spherical map $S^k \to Y$, $k \geq 2$, is homotopic to a constant map. A connected aspherical CW-complex with fundamental group G is a $\boldsymbol{K(G,1)}$**-complex** and its homotopy type is uniquely determined by G. The existence of a $K(G,1)$-complex for any group G is well-known.

2.2.1 Cellular Model of an Ordinary Group Presentation

Let $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ be an ordinary group presentation that determines the group $G = G(\mathcal{P})$. The **cellular model** of $\mathcal{P}$ is a connected two-dimensional CW-complex $K = K(\mathcal{P})$ whose one-skeleton $K^{(1)}$ consists of a single zero-cell c^0 together with a single one-cell c_x^1 for each generator $x \in \mathbf{x}$. Selection of an orientation for each one-cell c_x^1 determines an isomorphism $\pi_1 K^{(1)} \cong F$ between the fundamental group of the one-skeleton $K^{(1)}$ and the free group $F = F(\mathbf{x})$ with basis $\mathbf{x}$. The cellular model K has a two-cell c_r^2 for each relator $r \in \mathbf{r}$ whose boundary attaching map $S^1 \to K^{(1)}$ represents $r \in F \cong \pi_1 K^{(1)}$. The construction of K is such that there is an isomorphism $\pi_1 K \cong G$ between the fundamental group of K and the group G (see [24]).

2.2.2 Combinatorial Asphericity of an Ordinary Group Presentation

Let $\mathcal{P} = \langle \mathbf{x} : \mathbf{r} \rangle$ be an ordinary group presentation that determines the group $G = G(\mathcal{P})$. The free group $F = F(\mathbf{x})$ with basis $\mathbf{x}$ acts by automorphisms on the free group $\mathbb{F} = F(F \times \mathbf{r})$ (with basis $F \times \mathbf{r}$) via $v \cdot (u, r) = (vu, r)$. The homomorphism $\partial : \mathbb{F} \to F$ given by $\partial(u, r) = uru^{-1}$ is F-equivariant where F acts on itself by (left) conjugation. The image of ∂ is $\langle\langle \mathbf{r} \rangle\rangle_F$, so the cokernel of ∂ is isomorphic to G. Elements of the kernel $\mathbb{I} = \ker \partial$ are called **identities** (or **identity sequences**) for $\mathcal{P}$. This induces an exact sequence

$$1 \to \mathbb{I} \to \mathbb{F} \to F \to G \to 1 \tag{2.1}$$

for the ordinary presentation $\mathcal{P}$. Among the identities are **Peiffer identities** having the form $(u,r)^{\epsilon}(v,s)^{\delta}(u,r)^{-\epsilon}\left(ur^{\epsilon}u^{-1}v,s\right)^{-\delta}$ where $u,v \in F$, $r,s \in \mathbf{r}$, and $\delta,\epsilon = \pm 1$. If $\mathbb{P}$ denotes the normal closure of the Peiffer identities in $\mathbb{F}$, then the F-action on $\mathbb{F}$ descends to a G-action on the abelian quotient group $\mathbb{I}/\mathbb{P}$, and there the exact **fundamental sequence**

$$0 \to \mathbb{I}\big/\mathbb{P} \to \mathbb{F}\big/\mathbb{P} \overset{\partial}{\to} F \to G \to 1 \tag{2.2}$$

that provides a combinatorial description for the long exact homotopy sequence of the skeleton pair $\left(K, K^{(1)}\right)$ where $K = K(\mathcal{P})$ is the cellular model of the ordinary presentation $\mathcal{P}$. In particular, there exists an isomorphism $\pi_2 K \cong \mathbb{I}/\mathbb{P}$ between the second homotopy group of K and the group $\mathbb{I}/\mathbb{P}$ as $\mathbb{Z}G$-modules.

As in [8, Proposition 1.4], the ordinary presentation $\mathcal{P}$ is **combinatorially aspherical** if the homotopy module $\pi_2 K \cong \mathbb{I}/\mathbb{P}$ is generated as a $\mathbb{Z}G$-module by the classes of identities that have length two in the free group $\mathbb{F}$. As a sufficient condition, the presentation $\mathcal{P}$ is combinatorially aspherical if it satisfies any of the C(p)-T(q) small cancellation conditions and $1/p + 1/q \leq 1/2$ [8, 20]. In fact, such a presentation $\mathcal{P}$ satisfies even stronger conditions such as diagrammatic asphericity [9].

The cellular model $K(\mathcal{P})$ is aspherical if and only if the ordinary presentation $\mathcal{P}$ is combinatorially aspherical and each relator of $\mathcal{P}$ is neither freely redundant nor a proper power. See [8] or [3, Section 3]. Regardless of whether $K(\mathcal{P})$ is aspherical or not, there exists a $K(G,1)$-complex $\widehat{K}$ (recall $G = G(\mathcal{P})$) whose two-skeleton $\widehat{K}^{(2)} = K(\mathcal{P})$ is the cellular model of $\mathcal{P}$.

Example 2.2.1 *The presentation $\mathcal{P} = \langle a : a^n \rangle$ that determines the cyclic group $\mathbb{Z}_n$ of order n is combinatorially aspherical and possesses a $K(\mathbb{Z}_n, 1)$-complex that is constructed by attaching cells of dimension three and higher to the cellular model $K(\mathcal{P})$. See [3, Example 3.2].*

2.2.3 Asphericity of a Relative Group Presentation $\langle \mathbb{Z}_n, x : W \rangle$

Let $\mathcal{R} = \langle \mathbb{Z}_n, x : W \rangle$ be a relative group presentation where $W \in \mathbb{Z}_n * \langle x \rangle$ is a word in the free product of the cyclic group $\mathbb{Z}_n$ of order n and the infinite cyclic group $\langle x \rangle$. The **cellular model** $M = M(\mathcal{R})$ of $\mathcal{R}$ is a connected CW-complex that is described in [4, Section 4]. Working in the free product $\mathbb{Z}_n * \langle x \rangle$, first write $W = \dot{W}^e$ where $\dot{W} \in \mathbb{Z}_n * \langle x \rangle$ and e is maximal. Let K_n be a $K(\mathbb{Z}_n, 1)$-complex as in Example 2.2.1, and similarly let K_e be a $K(\mathbb{Z}_e, 1)$-complex unless $e = 1$, in which case let K_e be a disc; in either case, the one-skeleton $K_e^{(1)}$ of K_e is a circle S_e^1. Additionally, let S_x^1 be an oriented circle labeled with x. The CW-complex M is

$$M = \left(K_n \vee S_x^1\right) \cup K_e$$

where K_e is attached to the one-point wedge $K_n \vee S_x^1$ of K_n and S_x^1 via a map $S_e^1 \to \left(K_n \vee S_x^1\right)^{(1)}$ that represents the word $\dot{W} \in \mathbb{Z}_n * \langle x \rangle \cong \pi_1 K_n \vee S_x^1$. The homotopy type of M is determined by $\mathcal{R}$ (see [3, Section 3] or [4, Section 4] for details). The relative presentation $\mathcal{R}$ is **aspherical** if the relative homotopy group $\pi_2(M, K_n)$ is trivial. Equivalently, $\mathcal{R}$ is aspherical if and only if the inclusion $K_n \to M$ induces an injective homomorphism $\pi_1 K_n \to \pi_1 M$ on their respective fundamental groups and the absolute homotopy group $\pi_2 M$ is trivial. The point is that if $\mathcal{R} = \langle \mathbb{Z}_n, x : W \rangle$ is an aspherical relative presentation that determines the group $E = G(\mathcal{R})$, then $\mathbb{Z}_n$ embeds in E and M is a $K(E, 1)$-complex.

2.3 Cyclically Presented Groups

Let $G = G_n(w)$ be the cyclically presented group determined by the cyclic presentation $\mathcal{P}_n(w)$. Let θ_G be the shift on G.

2.3.1 The Shift Extension

Recall the shift extension $E = E_n(w) = G \rtimes_{\theta_G} \mathbb{Z}_n$ and that the shift on G corresponds to conjugation by an element a that generates $\mathbb{Z}_n$. This induces a natural presentation $\widehat{\mathcal{Q}}$ that determines the group E:

$$\widehat{\mathcal{Q}} = \left\langle a, x_0, \ldots, x_{n-1} : a^n, w, \theta_F(w), \ldots, \theta_F^{n-1}(w), ax_0a^{-1} = x_1, \ldots, ax_{n-1}a^{-1} = x_0 \right\rangle$$

It is well-known that the presentation $\widehat{\mathcal{Q}}$ transforms into a two-generator, two-relator ordinary presentation $\mathcal{Q}_n(w) = \langle a, x : a^n, W(a,x) \rangle$ that also determines the group E. By introducing $x = x_0$ and setting $x_i = a^i x a^{-i}\,(0 \le i < n)$, one can rewrite the word $w = w(x_0, \ldots, x_{n-1})$ as a word $W = W(a,x)$. Since $x_i = \theta_F^i(x_0)$, one has $\theta_F^i(w) = a^i W a^{-i}$, and so Tietze transformations are applied to $\widehat{\mathcal{Q}}$ as follows:

$$\begin{aligned}
\widehat{Q} \quad &= \quad \Big\langle a, x_0, \ldots, x_{n-1} \;:\; a^n, w, \theta_F(w), \ldots \theta_F^{n-1}(w), \\
&\qquad\qquad ax_0a^{-1} = x_1, \ldots, ax_{n-1}a^{-1} = x_0 \Big\rangle \\
&\overset{x=x_0}{\cong} \Big\langle a, x, x_0, \ldots, x_{n-1} \;:\; a^n, w, \theta_F(w), \ldots \theta_F^{n-1}(w), \\
&\qquad\qquad ax_0a^{-1} = x_1, \ldots, ax_{n-1}a^{-1} = x_0, x = x_0 \Big\rangle \\
&\overset{x_i=a^ixa^{-i}}{\cong} \Big\langle a, x, x_0, \ldots, x_{n-1} \;:\; a^n, w, \theta_F(w), \ldots \theta_F^{n-1}(w), \\
&\qquad\qquad axa^{-1} = x_1, \ldots, a^nxa^{-n} = x_0, x = x_0 \Big\rangle \\
&\overset{\theta_F^i(w)=a^iWa^{-i}}{\cong} \Big\langle a, x, x_0, \ldots, x_{n-1} \;:\; a^n, W, aWa^{-1}, \ldots, a^{n-1}Wa^{-(n-1)}, \\
&\qquad\qquad axa^{-1} = x_1, \ldots, a^nxa^{-n} = x_0, x = x_0 \Big\rangle \\
&\cong \quad \langle a, x \;:\; a^n, W \rangle
\end{aligned}$$

where the final congruence occurs by eliminating redundant relators and extraneous generators.

Finally, consider the word $W(a,x)$ as a word W in the free product $\mathbb{Z}_n * \langle x \rangle$ of the cyclic group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a and the infinite cyclic group $\langle x \rangle$ generated by x. Then, as in Section 2.1.2, the transformed presentation $\mathcal{Q}_n(w)$ is a lift of the relative presentation

$$\mathcal{R}_n(w) = \langle \mathbb{Z}_n, x : W \rangle$$

that also determines the group E.

2.3.2 Retractions

Let E be any group determined by a relative presentation of the form $\mathcal{R} = \langle \mathbb{Z}_n, x : W \rangle$ where the element a generates $\mathbb{Z}_n$ and $W = W(a, x)$ is a word in the free product $\mathbb{Z}_n * \langle x \rangle$. If $\nu^f : E \to \mathbb{Z}_n$ is a retraction satisfying $\nu^f(a) = a$ and $\nu^f(x) = a^f$, then the kernel $\ker \nu^f \cong G_n(w)$ is cyclically presented where the word $w = \rho^f(W)$ is obtained from following a Reidermeister-Schreier rewriting process [21, Theorem 2.9]. The following details of this process mirror those in [3, Page 5].

Let $W(a, x) = x^{\epsilon_1} a^{p_1} \cdots x^{\epsilon_L} a^{p_L}$ where $L \geq 1$, $\epsilon_i = \pm 1$, and the p_i are integers. For an integer f, the process involves two integer-valued functions. The first is defined by

$$v(1) = 0 \text{ and } v(i) = \sum_{j=1}^{i-1} \epsilon_j f + p_j$$

for $i = 2, \ldots, L$. The second is defined by

$$u(i) = v(i) + \frac{\epsilon_i - 1}{2} f = \begin{cases} v(i) & \text{if } \epsilon_i = 1 \\ v(i) - f & \text{if } \epsilon_i = -1. \end{cases}$$

Given the integer f and the word $W(a, x)$, the word $w = \rho^f(W(a, x))$ is defined to be

$$\rho^f(W(a, x)) = x_{u(1)}^{\epsilon_1} \cdots x_{u(L)}^{\epsilon_L}.$$

Example 2.3.1 *Let E be determined by the presentation $\langle \mathbb{Z}_n, x : W \rangle$ where the element a generates $\mathbb{Z}_n$ and $W = W(a, x) = xa^j xa^{k-j} xa^{l-k} xa^{-l}$. The map $\nu^0 : E \to \mathbb{Z}_n$ defined by $\nu^0(a) = a$ and $\nu^0(x) = a^0 = 1$ is a retraction with kernel $\ker \nu^0 = G_n(w)$ where*

$$w = \rho^0(W) = x_0 x_j x_{j+(k-j)} x_{j+(k-j)+(l-k)} = x_0 x_j x_k x_l.$$

A group E may have multiple retractions, and the cyclically presented kernels arising from different retractions need not be isomorphic. However, these retraction kernels are commensurable in E, and [3, Lemma 2.2] implies that for any retraction $\nu : E \to \mathbb{Z}_n$, there is an isomorphism $\ker\nu \cong E/\mathbb{Z}_n$ as left $\mathbb{Z}_n$-sets. In particular, the cyclically presented groups arising from all retractions of E onto $\mathbb{Z}_n$ have the same shift dynamics.

Example 2.3.2 *Consider the group L determined by the ordinary presentation $\mathcal{Q} = \langle t, u : t^6, u^3t^3ut \rangle$. Note that $\mathcal{Q}$ is a lift of the relative presentation $\mathcal{R} = \langle \mathbb{Z}_n, u : u^3t^3ut \rangle$. There exist retractions $\nu^f : L \to \mathbb{Z}_n$ (where $\mathbb{Z}_n = \langle t \rangle$ is the cyclic group of order n generated by t) defined by $\nu^f(t) = t$ and $\nu^f(u) = t^f$ precisely when $f = 2, 5$. By rewriting the relator u^3t^3ut of $\mathcal{R}$ in an essential way, the two retractions have cyclically presented kernels* $\ker\nu^2 = G_6\left(\rho^2(w)\right) = G_6(u_0u_2u_4u_3)$ *and* $\ker\nu^5 = G_6\left(\rho^5(w)\right) = G_6(u_0u_5u_4u_0)$*. As in Lemma 3.2.30, computations performed using GAP [15] show that $|L| = 24530688$ [27]. Since the kernels are of index six in L, it follows that they are both of order* 4088448*. However, further computations using the* `AbelianInvariants` *function reveal that $\left(\ker\nu^2\right)^{\mathrm{ab}} \cong \mathbb{Z}_7 \times \mathbb{Z}_8$ and $\left(\ker\nu^5\right)^{\mathrm{ab}} \cong \mathbb{Z}_8$. Therefore the two kernels are not isomorphic, but the dynamics of their respective shifts are the same. See Section 4.1 for more information regarding the usage of GAP.*

3 CYCLICALLY PRESENTED GROUPS WITH LENGTH FOUR POSITIVE RELATORS

3.1 Preliminary Remarks

Let n be a positive integer, and let $x_i x_j x_k x_l \in F$ be a length four positive word in the free group $F = F(\mathbf{x})$ of rank n with basis $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$. Consider the cyclic group presentation $\mathcal{P} = \mathcal{P}_n(x_i x_j x_k x_l)$ determined by n and $x_i x_j x_k x_l$. Let $G = G_n(x_i x_j x_k x_l)$ be the group determined by $\mathcal{P}$. Let θ_G be the shift on the group G. Note that the map $x_i \mapsto 1 \in \mathbb{Z}_4$ determines a surjective homomorphism $G \to \mathbb{Z}_4$, so G is nontrivial.

There are many isomorphisms amongst the groups under consideration. Section 3.2.1 provides an action of the group $\Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$ on the set Φ_n of length four positive words in F. The elements $c \in \Gamma_n$ induce isomorphisms between the cyclically presented groups that preserve the group structure, shift dynamics, and combinatorial asphericity status of the corresponding cyclic presentations. See Theorem 3.2.2 and Corollary 3.2.3. Note that it is sufficient to consider only the words $x_i x_j x_k x_l$ where $i = 0$. Unless otherwise stated, the discussions henceforth (and the statements of the main theorems) are provided in terms of the word $x_0 x_j x_k x_l$, and so the corresponding presentations $\mathcal{P}_n(x_0 x_j x_k x_l)$ and groups $G_n(x_0 x_j x_k x_l)$ are considered to be determined only by the four integer parameters n, j, k, l.

As in Section 2.3.1, the shift extension $E = E_n(x_0 x_j x_k x_l)$ of the cyclically presented group $G_n(x_0 x_j x_k x_l)$ is determined by the ordinary presentation

$$\mathcal{Q}_n(x_0 x_j x_k x_l) = \left\langle a, x : a^n, x a^j x a^{k-j} x a^{l-k} x a^{-l} \right\rangle$$

and by the relative presentation

$$\mathcal{R}_n(x_0 x_j x_k x_l) = \left\langle \mathbb{Z}_n, x : x a^j x a^{k-j} x a^{l-k} x a^{-l} \right\rangle.$$

One of the key uses of conditions (A), (B), and (C) outlined in Table 1.1 is to enable simplifications of the relator $W = xa^jxa^{k-j}xa^{l-k}xa^{-l}$ of $\mathcal{R}_n(x_0x_jx_kx_l)$ through the use of substitutions. For example, if $k \equiv 2j \mod n$ as in condition (B), then the substitution $u = xa^j$ transforms W into $u^3a^{l-3j}ua^{-l-j}$. Simplifications of this sort are developed in Section 3.2.3.

Finally, one of the main points of interest is in determining the structure of the fixed point subgroup of the shift and its nonidentity powers. To this end, the notation $\mathrm{Fix}\left(\theta_G^p\right)$ is used to refer to the fixed point subgroup of θ_G^p.

3.1.1 Organization of the Proof

The proof begins with Sections 3.2.1, 3.2.2, and 3.2.3 which greatly reduce the cyclic presentations that need to be considered. The proof is then organized around the eight combinations of truth values of conditions (A), (B), and (C). (See Table 1.1 for a statement of these conditions.) These combinations are arranged into four cases which are discussed in Sections 3.2.4 to 3.2.7. In each case, the treatment of combinatorial asphericity status, finiteness, and shift dynamics are all handled simultaneously. This is done to take advantage of the intrinsic connections between these concepts and to avoid needless repetition of common arguments. Finally, Section 3.2.8 connects the results of the prior sections to conclude the proof.

Table 3.1 provides and overview of the combinatorial asphericity and finiteness results arranged by the different combinations of truth values in the first column. In addition to the summaries combinatorial asphericity and finiteness results of columns two and three respectively, the fourth column provides information about the simplification that occurs when a specially chosen substitution of the form $u = xa^p$ is applied to the relator $xa^jxa^{k-j}xa^{l-k}xa^{-l}$ in relative presentation $\mathcal{R}_n(x_0x_jx_kx_l)$ that determines the shift extension $E_n(x_0x_jx_kx_l)$. These simplifications enable the application of [2, Theorems 2-4] (See Section 1.2 which contains an updated version of [2, Theorems 4] in light of recent

(A)	(B)	(C)	Comb. Aspherical	Group	W
T	T	T	Yes C(4)-T(4)	Finite $\iff n=1 \iff \mathbb{Z}_4$	z^4
T	F	T			$\left(z^2\alpha\right)^2$
T	F	F			$\Pi_{i=1}^4 x\alpha_i$
F	F	F			$\Pi_{i=1}^4 x\alpha_i$
F	T	T	No	Finite $\iff \mathbb{Z}_4$	$z^4\alpha$
F	F	T			$z^2\alpha z^2\beta$
T	T	F	No	Finite $\iff \gamma=1$ Finite $\Rightarrow$ Solvable	$u^3\alpha u\alpha^{\pm 1}$
F	T	F	(I*): No (U*): Unresolved Else: Yes	Finite $\iff$ (I5), (I6$'$), or (I6$''$)	$u^3\alpha u\beta$

TABLE 3.1: Summary of the Combinatorial Asphericity and Finiteness Classification Results

results [1] [27]). The bottom row of the table treats the most complex case which is discussed in Section 3.2.7. Notably, Lemma 3.2.24 shows that the isolated and unresolved presentations all occur within this case.

3.2 Proof of the Main Theorems

3.2.1 Isomorphisms

Classifications involving cyclic presentations are complicated by the fact that there are many isomorphisms that must be accounted for, not all of which are obvious or easily catalogued. See [26, Section 2] and [14, Lemma 2.1], for example.

For a fixed positive integer n, let Φ_n denote the set of all positive words of length four in the free group $F(x_0, \ldots, x_{n-1})$ of rank n with basis $x_0, \ldots, x_{n-1}$. Thus every element $w \in \Phi_n$ has the form $w = x_i x_j x_k x_l$ where the integer parameters i, j, k, l are defined modulo n. Consider the following bijective transformations on Φ_n:

$$\sigma(x_i x_j x_k x_l) = x_i x_l x_k x_j$$

$$\tau(x_i x_j x_k x_l) = x_l x_i x_j x_k$$

$$\theta_F(x_i x_j x_k x_l) = x_{i+1} x_{j+1} x_{k+1} x_{l+1}$$

$$u(x_i x_j x_k x_l) = x_{ui} x_{uj} x_{uk} x_{ul}$$

Here, the element u is taken from the multiplicative group $\mathbb{Z}_n^*$ of units modulo n. All subscripts are considered modulo n.

Lemma 3.2.1 *Within the group* $\mathrm{Sym}(\Phi_n)$ *of permutations of* Φ_n*, the transformations* σ*,* τ*,* θ_F*, and* $u \in \mathbb{Z}_n^*$ *generate a subgroup that is a homomorphic image of* $\Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$*, where* D_4 *is the dihedral group of order eight generated by* σ *and* τ *and the semi-direct product involves the natural action of* $\mathbb{Z}_n^* \cong \mathrm{Aut}(\mathbb{Z}_n)$ *via multiplication on the additive cyclic group* $\mathbb{Z}_n$ *of order* n*.*

Proof Working in $\mathrm{Sym}(\Phi_n)$, one checks that $\sigma^2 = \tau^4 = (\sigma\tau)^2 = \theta_F^n = 1$, $\sigma\theta_F = \theta_F\sigma$, $\sigma u = u\sigma$, $\tau\theta_F = \theta_F\tau$, $\tau u = u\tau$, and $u\theta_F = \theta_F^u u$ where $u \in \mathbb{Z}_n^*$ is arbitrary. •

The orbit of $w = x_i x_j x_k x_l \in \Phi_n$ under the action of Γ_n consists of all words that arise as cyclic permutations of shifts of words of the form $u(w) = x_{ui} x_{uj} x_{uk} x_{ul}$ or $\sigma(u(w)) = x_{ui} x_{ul} x_{uk} x_{uj}$ where $u \in \mathbb{Z}_n^*$.

Theorem 3.2.2 *For each element* $c \in \Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$ *and word* $w = x_i x_j x_k x_l \in \Phi_n$*, there exists an isomorphism* $c_w : G_n(w) \to G_n(c(w))$ *between the cyclically presented groups* $G_n(w)$ *and* $G_n(c(w))$ *such that*

$$c_w \circ \theta_{G_n(w)} = \theta_{G_n(c(w))}^{\psi(c)} \circ c_w \tag{3.1}$$

where $\psi : \Gamma_n \to \mathbb{Z}_n^*$ *is the natural projection. Moreover, the cyclic presentation* $\mathcal{P}_n(w)$ *is combinatorially aspherical if and only if the cyclic presentation* $\mathcal{P}_n(c(w))$ *is combinatorially aspherical.*

Proof Let $F = F(\mathbf{x})$ be the free group of rank n with basis $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$. Given $c \in \Gamma_n$ and $w = x_i x_j x_k x_l \in \Phi_n$, denote the relator sets for the presentations $\mathcal{P}_n(w)$ and $\mathcal{P}_n(c(w))$ by

$$\mathbf{r}_w = \left\{\theta_F^q(w) : 0 \le q < n\right\} = \{x_{i+q}x_{j+q}x_{k+q}x_{l+q} : 0 \le q < n\}$$

and

$$c(\mathbf{r}_w) = \left\{\theta_F^q(c(w)) : 0 \le q < n\right\}$$

respectively. Consider the homomorphisms $c_w : F \to F$ and $\widehat{c}_w : \mathbb{F}_w \to \mathbb{F}_{c(w)}$ where $\mathbb{F}_w = F(F \times \mathbf{r}_w)$ and $\mathbb{F}_{c(w)} = F(F \times c(\mathbf{r}_w))$ are the free groups with bases $F \times \mathbf{r}_w$ and $F \times c(\mathbf{r}_w)$ respectively. For $c = \sigma$, τ, θ_F, or $u \in \mathbb{Z}_n^*$ and $w \in \Phi_n$, the assignments

$$\begin{aligned} \sigma_w(x_p) = x_p^{-1}; &\quad \widehat{\sigma}_w\left(v, \theta_F^q(w)\right) = \left(\sigma_w(v)\,\theta_F^q(x_i)^{-1}, \theta_F^q\sigma(w)\right)^{-1} \\ \tau_w = 1_F; &\quad \widehat{\tau}_w\left(v, \theta_F^q(w)\right) = \left(v\theta_F^q(x_l)^{-1}, \theta_F^q\tau(w)\right) \\ (\theta_F)_w = \theta_F; &\quad \widehat{(\theta_F)}_w\left(v, \theta_F^q(w)\right) = \left(\theta_F(v), \theta_F^{q+1}(w)\right) \\ u_w(x_p) = x_{up}; &\quad \widehat{u}_w\left(v, \theta_F^q(w)\right) = \left(u_w(v), \theta_F^{uq}(u(w))\right) \end{aligned}$$

define length-preserving homomorphisms $c_w : F \to F$ and $\widehat{c}_w : \mathbb{F}_w \to \mathbb{F}_{c(w)}$. These homomorphisms are constructed to satisfy $\partial \circ \widehat{c}_w = c_w \circ \partial$ for $c = \sigma, \tau, \theta_F, u \in \Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$. Viewed in $\mathrm{Aut}(F)$, the isomorphisms $\sigma_w, \tau_w, \theta_F, u_w$ are compatible with the relations of $\Gamma_n = D_4 \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$. Specifically, one notes that

$$\sigma_w^2 = \tau_w^4 = (\sigma_w \circ \tau_w)^2 = \theta_F^n = 1, t_w \circ u_w = (tu)_w \quad (t, u \in \mathbb{Z}_n^*),$$

the homomorphisms σ_w, τ_w commute pairwise with θ_F and u_w, and that $u_w \circ \theta_F = \theta_F^u \circ u_w$ (for $u \in \mathbb{Z}_n^*$). Since Γ_n acts on Φ_n, this shows that Γ_n acts on F by automorphisms and, in fact, this action is independent of w. (The notation c_w is retained in order to distinguish between $c \in \Gamma_n$ and $c_w \in \operatorname{Aut}(F)$.) For each $c \in \Gamma_n$, there is an automorphism $c_w \in \operatorname{Aut}(F)$ that is length-preserving and which is compatible with the boundary maps ∂ from the identity sequences (see Equation 2.1) for the presentations $\mathcal{P}_n(w)$ and $\mathcal{P}_n(c(w))$.

$$\begin{array}{ccc} \mathbb{F}_w = F(F \times \mathbf{r}_w) & \xrightarrow{\partial} & F \\ {\scriptstyle \hat{c}_w}\downarrow & & \downarrow{\scriptstyle c_w} \\ \mathbb{F}_{c(w)} = F(F \times c(\mathbf{r}_w)) & \xrightarrow{\partial} & F \end{array}$$

It follows that for any $c \in \Gamma_n$, the automorphism $c_w \in \operatorname{Aut}(F)$ induces an isomorphism $G_n(w) \to G_n(c(w))$, which will also be denoted by c_w. As for shift-equivariance, observe that

$$\sigma_w \circ \theta_{G_n(w)} = \theta_{G_n(\sigma(w))} \circ \sigma_w$$

$$\tau_w \circ \theta_{G_n(w)} = \theta_{G_n(\tau(w))} \circ \tau_w$$

$$u_w \circ \theta_{G_n(w)} = \theta^u_{G_n(u(w))} \circ u_w$$

Moreover, $G_n(\theta_F(w)) = G_n(w)$ and $(\theta_F)_w = \theta_{G_n(w)}$. This verifies equivariance claim as in Equation 3.1.

It remains to verify that combinatorial asphericity is preserved under the action of Γ_n on the set of presentations of the form $\mathcal{P}_n(w), w \in \Phi_n$. Note that the homomorphisms $\widehat{\sigma}_w, \widehat{\tau}_w$ depend on w. It turns out that the homomorphisms $\widehat{\sigma}_w, \widehat{\tau}_w, \widehat{(\theta_F)}_w, \widehat{u}_w$ are compatible with some of the relations of Γ_n, but not all. (As a matter of notational convenience, let $\widehat{c}^m_w = \widehat{c}_{c^{m-1}_w(w)} \circ \widehat{c}^{m-1}_w$ for $m > 1$ and $c \in \Gamma_n$.) To illustrate that $\widehat{\sigma}^2_w = 1_{\mathbb{F}_w}$, use the fact that $\sigma^2 = 1_{\Phi_n}$, that σ_w commutes with θ_F in $\operatorname{Aut}(F)$, that $\sigma_w = \sigma_{\sigma_w(w)}$, that $\sigma^2_w = 1_F$, and that the first letter of $w = x_i x_j x_k x_l$ is the same as for $\sigma(w) = x_i x_l x_k x_j$, and so

$$\begin{aligned} \widehat{\sigma}^2_w\left(v, \theta^q_F(w)\right) &= \widehat{\sigma}_{\sigma_w(w)}\left(\sigma_w(v)\,\theta^q_F(x_i)^{-1}, \theta^q_F(\sigma(w))\right)^{-1} \\ &= \left(\sigma_{\sigma_w(w)}\left(\sigma_w(v)\,\theta^q_F(x_i)^{-1}\right) \cdot \theta^q_F(x_i)^{-1}, \theta^q_F\left(\sigma^2(w)\right)\right) \\ &= \left(\sigma^2_w(v)\,\theta^q_F\left(\sigma_w(x_i)^{-1}\right) \cdot \theta^q_F(x_i)^{-1}, \theta^q_F(w)\right) \end{aligned}$$

$$= \left(v\theta_F^q(x_i)\cdot\theta_F^q(x_i)^{-1}, \theta_F^q(w)\right)$$
$$= \left(v, \theta_F^q(w)\right).$$

However, $\widehat{\tau}_w^4 : \mathbb{F}_w \to \mathbb{F}_w$ is not the identity. For $w = x_ix_jx_kx_l$, observe that

$$\begin{aligned}
\widehat{\tau}_w^4\left(v, \theta_F^q(w)\right) &= \widehat{\tau}_{\tau_w(w)}^3\left(v\theta_F^q(x_l)^{-1}, \theta_F^q(x_lx_ix_jx_k)\right)^{-1} \\
&= \widehat{\tau}_{\tau_w^2(w)}^2\left(v\theta_F^q(x_l)^{-1}\theta_F^q(x_k)^{-1}, \theta_F^q(x_kx_lx_ix_j)\right) \\
&= \widehat{\tau}_{\tau_w^3(w)}\left(v\theta_F^q(x_l)^{-1}\theta_F^q(x_k)^{-1}\theta_F^q(x_j)^{-1}, \theta_F^q(x_jx_kx_lx_i)\right)^{-1} \\
&= \left(v\theta_F^q(x_l)^{-1}\theta_F^q(x_k)^{-1}\theta_F^q(x_j)^{-1}\theta_F^q(x_i)^{-1}, \theta_F^q(x_ix_jx_kx_l)\right) \\
&= \left(v\theta_F^q(w)^{-1}, \theta_F^q(w)\right)
\end{aligned}$$

Nevertheless, the fact that $\theta_F^q(w)^{-1} = 1$ in the group $G_n(w) = G_n(x_ix_jx_kx_l)$ implies that $\left(v, \theta_F^q(w)\right) = \left(v\theta_F^q(w)^{-1}, \theta_F^q(w)\right)$ modulo the group $\mathbb{P}_w$ of Peiffer identities for the presentation $\mathcal{P}_n(w)$. See [24, Lemma 2.4]. Along with the previous two calculations, the following verify that by working modulo the Peiffer identities, the homomorphisms $\widehat{\sigma}_w, \widehat{\tau}_w, \widehat{(\theta_F)}_w, \widehat{u}_w$ are compatible with all of the relations of Γ_n in the following sense:

- $\widehat{\sigma}_w^2 = \widehat{\tau}_w^4 = \left(\widehat{\sigma}_{\tau_w(w)} \circ \widehat{\tau}_w\right)^2 = \widehat{(\theta_F)}_w^n = 1_{\mathbb{F}_w}$,
- $\widehat{c}_{d_w(w)} \circ \widehat{d}_w = \widehat{d}_{c_w(w)} \circ \widehat{c}_w$ for $c = \sigma, \tau$ and $d = \theta_F, u$ where $u \in \mathbb{Z}_n^*$,
- $\widehat{(\theta_F)}_{u_w(w)}^u \circ \widehat{u}_w = \widehat{u}_{(\theta_F)_w(w)} \circ \widehat{(\theta_F)}_w$ where $u \in \mathbb{Z}_n^*$, and
- $\widehat{t}_{u_w(w)} \circ \widehat{u}_w = \widehat{(tu)}_w$ where $t, u \in \mathbb{Z}_n^*$.

To avoid the excessive use of subscripts on subscripts in the following calculations, every homomorphism $c_{d_w(w)} : F \to F$ will be written as c_w since these homomorphisms do not depend on the words $w, d_w(w)$ where $c, d \in \Gamma_n$.

$$\left(\widehat{\sigma}_{\tau_w(w)} \circ \widehat{\tau}_w\right)^2\left(v, \theta_F^q(w)\right) = \widehat{\sigma}_{\tau_w\circ\sigma_w\circ\tau_w(w)} \circ \widehat{\tau}_{\sigma_w\circ\tau_w(w)} \circ \widehat{\sigma}_{\tau_w(w)}$$

$$
\begin{aligned}
&\left(v\theta_F^q\left(x_l\right)^{-1}, \theta_F^q\left(x_l x_i x_j x_k\right)\right)^{-1} \\
&= \widehat{\sigma}_{\tau_w \circ \sigma_w \circ \tau_w(w)} \circ \widehat{\tau}_{\sigma_w \circ \tau_w(w)} \\
&\quad \left(\sigma_w\left(v\theta_F^q\left(x_l\right)^{-1}\right) \cdot \theta_F^q\left(x_l\right)^{-1}, \theta_F^q\left(x_l x_k x_j x_i\right)\right)^{-1} \\
&= \widehat{\sigma}_{\tau_w \circ \sigma_w \circ \tau_w(w)} \circ \widehat{\tau}_{\sigma_w \circ \tau_w(w)} \\
&\quad \left(\sigma_w\left(v\right)\theta_F^q\left(x_l\right)\theta_F^q\left(x_l\right)^{-1}, \theta_F^q\left(x_l x_k x_j x_i\right)\right)^{-1} \\
&= \widehat{\sigma}_{\tau_w \circ \sigma_w \circ \tau_w(w)} \circ \widehat{\tau}_{\sigma_w \circ \tau_w(w)} \left(\sigma_w\left(v\right), \theta_F^q\left(x_l x_k x_j x_i\right)\right)^{-1} \\
&= \widehat{\sigma}_{\tau_w \circ \sigma_w \circ \tau_w(w)} \left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}, \theta_F^q\left(x_i x_l x_k x_j\right)\right) \\
&= \left(\sigma_w\left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}\right) \cdot \theta_F^q\left(x_i\right)^{-1}, \theta_F^q\left(x_i x_j x_k x_l\right)\right) \\
&= \left(\sigma_w^2\left(v\right)\theta_F^q\left(x_i\right)\theta_F^q\left(x_i\right)^{-1}, \theta_F^q\left(w\right)\right) \\
&= \left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\left(\theta_F\right)}_w^n\left(v, \theta_F^q\left(w\right)\right) &= \left(\theta_F^n\left(v\right), \theta_F^{q+n}\left(w\right)\right) \\
&= \left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\sigma}_{\theta_F(w)} \circ \widehat{\left(\theta_F\right)}_w\left(v, \theta_F^q\left(w\right)\right) &= \widehat{\sigma}_{\theta_F(w)}\left(\theta_F\left(v\right), \theta_F^{q+1}\left(w\right)\right) \\
&= \left(\sigma_w\left(\theta_F\left(v\right)\right) \cdot \theta_F^{q+1}\left(x_i\right)^{-1}, \theta_F^{q+1}\left(\sigma\left(w\right)\right)\right) \\
&= \left(\theta_F\left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}\right), \theta_F^{q+1}\left(\sigma\left(w\right)\right)\right) \\
&= \widehat{\left(\theta_F\right)}_{\sigma_w(w)}\left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}, \theta_F^q\left(\sigma\left(w\right)\right)\right) \\
&= \widehat{\left(\theta_F\right)}_{\sigma_w(w)} \circ \widehat{\sigma}_w\left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\sigma}_{u_w(w)} \circ \widehat{u}_w\left(v, \theta_F^q\left(w\right)\right) &= \widehat{\sigma}_{u_w(w)}\left(u_w\left(v\right), \theta_F^{uq}\left(u\left(w\right)\right)\right) \\
&= \left(\sigma_w\left(u_w\left(v\right)\right) \cdot \theta_F^{uq}\left(u\left(x_i\right)\right)^{-1}, \theta_F^{uq}\left(\sigma\left(u\left(w\right)\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \left(u_w\left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}\right), \theta_F^{uq}\left(u\left(\sigma\left(w\right)\right)\right)\right) \\
&= \widehat{u}_{\sigma_w(w)}\left(\sigma_w\left(v\right)\theta_F^q\left(x_i\right)^{-1}, \theta_F^q\left(\sigma\left(w\right)\right)\right) \\
&= \widehat{u}_{\sigma_w(w)} \circ \widehat{\sigma}_w\left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\tau}_{\theta_F(w)} \circ \widehat{(\theta_F)}_w\left(v, \theta_F^q\left(w\right)\right) &= \widehat{\tau}_{\theta_F(w)}\left(\theta_F\left(v\right), \theta_F^{q+1}\left(w\right)\right) \\
&= \left(\theta_F\left(v\right)\theta_F^{q+1}\left(x_l\right)^{-1}, \theta_F^{q+1}\left(\tau\left(w\right)\right)\right)^{-1} \\
&= \widehat{(\theta_F)}_{\tau_w(w)}\left(v\theta_F^q\left(x_l\right)^{-1}, \theta_F^q\left(\tau\left(w\right)\right)\right)^{-1} \\
&= \widehat{(\theta_F)}_{\tau_w(w)} \circ \widehat{\tau}_w\left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\tau}_{u_w(w)} \circ \widehat{u}_w\left(v, \theta_F^q\left(w\right)\right) &= \widehat{\tau}_{u_w(w)}\left(u_w\left(v\right), \theta_F^{uq}\left(u\left(w\right)\right)\right) \\
&= \left(u_w\left(v\right)\theta_F^{uq}\left(u\left(x_l\right)\right)^{-1}, \theta_F^{uq}\left(\tau\left(u\left(w\right)\right)\right)\right)^{-1} \\
&= \left(u_w\left(v\theta_F^q\left(x_l\right)^{-1}\right), \theta_F^{uq}\left(u\left(\tau\left(w\right)\right)\right)\right)^{-1} \\
&= \widehat{u}_{\tau_w(w)}\left(v\theta_F^q\left(x_l\right)^{-1}, \theta_F^q\left(\tau\left(w\right)\right)\right)^{-1} \\
&= \widehat{u}_{\tau_w(w)} \circ \widehat{\tau}_w\left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{u}_{\theta_F(w)} \circ \widehat{(\theta_F)}_w\left(v, \theta_F^q\left(w\right)\right) &= \widehat{u}_{\theta_F(w)}\left(\theta_F\left(v\right), \theta_F^{q+1}\left(w\right)\right) \\
&= \left(u_w\left(\theta_F\left(v\right)\right), \theta_F^{uq+u}\left(u\left(w\right)\right)\right) \\
&= \left(\theta_F^u\left(u_w\left(v\right)\right), \theta_F^{uq+u}\left(u\left(w\right)\right)\right) \\
&= \widehat{(\theta_F)}^u_{u_w(w)}\left(u_w\left(v\right), \theta_F^{uq}\left(u\left(w\right)\right)\right) \\
&= \widehat{(\theta_F)}^u_{u_w(w)} \circ \widehat{u}_w\left(v, \theta_F^q\left(w\right)\right)
\end{aligned}
$$

$$
\widehat{t}_{u_w(w)} \circ \widehat{u}_w\left(v, \theta_F^q\left(w\right)\right) = \widehat{t}_{u_w(w)}\left(u_w\left(v\right), \theta_F^{uq}\left(u\left(w\right)\right)\right)
$$

$$= \left(t_w\left(u_w\left(v\right)\right), \theta_F^{tuq}\left(t\left(u\left(w\right)\right)\right)\right)$$
$$= \left(\left(tu\right)_w\left(v\right), \theta_F^{tuq}\left(\left(tu\right)\left(w\right)\right)\right)$$
$$= \widehat{\left(tu\right)}_w\left(v, \theta_F^{q}\left(w\right)\right)$$

For each $c \in \Gamma_n$ and $w \in \Phi_n$, one obtains a commutative diagram that compares the fundamental sequences (see Equation 2.2) for the presentations $\mathcal{P}_n\left(w\right)$ and $\mathcal{P}_n\left(c\left(w\right)\right)$. The vertical arrows all represent isomorphisms.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & \mathbb{I}_w/\mathbb{P}_w & \longrightarrow & \mathbb{F}_w/\mathbb{P}_w & \xrightarrow{\partial} & F & \longrightarrow & G_n(w) & \longrightarrow & 1 \\
 & & \big\downarrow & & \hat{c}_w \big\downarrow & & \big\downarrow c_w & & \big\downarrow & & \\
0 & \longrightarrow & \mathbb{I}_{c(w)}/\mathbb{P}_{c(w)} & \longrightarrow & \mathbb{F}_{c(w)}/\mathbb{P}_{c(w)} & \xrightarrow{\partial} & F & \longrightarrow & G_n(c(w)) & \longrightarrow & 1
\end{array}$$

It remains to note that the isomorphism $\mathbb{I}_w/\mathbb{P}_w \to \mathbb{I}_{c(w)}/\mathbb{P}_{c(w)}$ is length-preserving, and so it follows that $\mathbb{I}_w/\mathbb{P}_w$ is generated by classes of length two identity sequences if and only if the same is true for $\mathbb{I}_{c(w)}/\mathbb{P}_{c(w)}$. •

Corollary 3.2.3 *If $w = x_i x_j x_k x_l \in \Phi_n$ and $c \in \Gamma_n = D_4 \times \left(\mathbb{Z}_n \rtimes \mathbb{Z}_n^*\right)$, then the cyclically presented groups $G_n\left(w\right)$ and $G_n\left(c\left(w\right)\right)$ are isomorphic via an isomorphism that preserves shift dynamics. Thus:*

(a) The shift on $G_n\left(w\right)$ has a nonidentity fixed point if and only if the shift on $G_n\left(c\left(w\right)\right)$ has a nonidentity fixed point; and

(b) The cyclic group $\mathbb{Z}_n$ of order n acts freely via the shift on the nonidentity elements of $G_n\left(w\right)$ if and only if $\mathbb{Z}_n$ acts freely via the shift on the nonidentity elements of $G_n\left(c\left(w\right)\right)$.

Proof The group isomorphisms of Theorem 3.2.2 are shift-equivariant and so define isomorphisms of left $\mathbb{Z}_n$-sets, where the shift action by $\mathbb{Z}_n$ is twisted as necessary via an automorphism $u \in \mathbb{Z}_n^* \cong \mathrm{Aut}\left(\mathbb{Z}_n\right)$. •

Example 3.2.4 *The isomorphisms* $(\theta_F)_w$ *enable the obvious fact that while studying the cyclic presentations* $\mathcal{P}_n(x_i x_j x_k x_l)$*, there is no loss of generality in assuming that* $i = 0$*. Only slightly less obvious are the isomorphisms*

$$G_n(x_0 x_j x_k x_l) \cong G_n(x_0 x_l x_k x_j) \quad (\textit{via } \sigma) \ \textit{ and}$$

$$G_n(x_0 x_j x_k x_l) \cong G_n(x_0 x_{uj} x_{uk} x_{ul}) \quad (\textit{via } u \in \mathbb{Z}_n^*).$$

Combining the isomorphisms arising from σ *and* τ *yields*

$$G_n(x_0 x_j x_k x_l) \cong G_n(x_0 x_l x_k x_j) \cong G_n(x_l x_k x_j x_0).$$

That is, the word $x_0 x_j x_k x_l$ *may be read backwards without changing the group structure, combinatorial asphericity status, or dynamics of the shift.*

Remark: A similar discussion can be carried out for positive words of any given length, where positive words of length L in the free group of rank n are acted upon by the group $D_L \times (\mathbb{Z}_n \rtimes \mathbb{Z}_n^*)$. The actions give rise to group isomorphisms that preserve combinatorial asphericity status and shift dynamics as in Corollary 3.2.3. In particular, the isomorphisms of [14, Lemma 2.1] for the case $L = 3$ all arise in this way, and so they too preserve the dynamics of the shift.

3.2.2 Divisor Criteria

To analyze cyclic presentations of the form $\mathcal{P}_n(x_0 x_j x_k x_l)$, it is well known [13] that one can generally restrict to the case where the **primary divisor** $d = \gcd(n, j, k, l)$ of the parameters n, j, k, l is equal to one. This includes considerations of combinatorial asphericity [8, Theorem 4.2], finiteness [7, Lemma 2.4], and shift dynamics [3, Lemma 5.1]. The reason for this is that if $d = \gcd(n, j, k, l)$, then the cyclic presentation $\mathcal{P} = \mathcal{P}_n(x_0 x_j x_k x_l)$ is a disjoint union of cyclic subpresentations and the group $G = G_n(x_0 x_j x_k x_l)$ determined by $\mathcal{P}$ decomposes as a free product

$$G \cong *_{i=1}^{d} G_{n/d}\left(x_0 x_{j/d} x_{k/d} x_{l/d}\right)$$

where the shift on G transitively permutes the free factors of the product. In particular, if $d \neq 1$, then G is infinite and its shift has no nonidentity fixed points. Lemma 3.2.5 formalizes the claim regarding considerations of combinatorial asphericity, and Corollary 3.2.8 formalizes the claims regarding considerations of finiteness and shift dynamics.

Lemma 3.2.5 *The cyclic group presentation $\mathcal{P}_{nd}(x_0 x_{jd} x_{kd} x_{ld})$ is combinatorially aspherical if and only if the cyclic presentation $\mathcal{P}_n(x_0 x_j x_k x_l)$ is combinatorially aspherical.*

Proof As in [13], $\mathcal{P}_{nd}(x_0 x_{jd} x_{kd} x_{ld})$ decomposes as a disjoint union of d subpresentations that can each be identified with $\mathcal{P}_n(x_0 x_j x_k x_l)$, so this follows from [8, Theorem 4.2]. •

A more general but equally checkable criterion involves the **secondary divisor**

$$\gamma = \gcd(n, k-2j, l-2k+j, k-2l, j+l).$$

The terms $k-2j, l-2k+j, -2l+k, -j-l$ are the differences of consecutive exponents on the coefficient a in the relator $W = xa^j xa^{k-j} xa^{l-k} xa^{-l}$ (considered cyclically) of the relative presentation $\mathcal{R}_n(x_0 x_j x_k x_l) = \langle \mathbb{Z}_n, x : W \rangle$ for the shift extension $E = E_n(x_0 x_j x_k x_l)$ where $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a. The dependence relation

$$(k-2j) + (l-2k+j) + (k-2l) + (j+l) = 0$$

implies that any one of the four bracketed terms can be deleted when calculating γ.

Since the shift θ_G arises through conjugation by a in E, the fixed point set for any power θ_G^d of the shift is expressible in terms of a centralizer in E:

$$\mathrm{Fix}\left(\theta_G^d\right) = G_n(w) \cap \mathrm{Cent}_E\left(a^d\right).$$

In the proof of Theorem 3.2.7, it is shown that if $\gamma \neq 1$, then the presentation $\mathcal{Q}_n(x_0 x_j x_k x_l)$ that determines E transforms into a presentation that reveal an amalgamated free product decomposition $E = \mathbb{Z}_n *_{\mathbb{Z}_{n/\gamma}} D$ with the cyclic group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a as a vertex group. The following general result is then applied, which itself is derived from [21, Theorem 4.5]. A proof using Bass-Serre theory is included.

Lemma 3.2.6 (Centralizer Lemma) *Suppose that* $\Pi = A *_C B$ *is an amalgamated free product of groups* A *and* B *with amalgamated subgroup* C*. If* $a \in A$ *and* $g \in \mathrm{Cent}_{\Pi}(a)$*, then either* $g \in A$ *or there exists* $\alpha \in A$ *such that* $a \in \alpha C \alpha^{-1}$*.*

Proof The group Π acts on the standard graph T of $\Pi = A *_C B$ with orbit graph consisting of a single edge joining two distinct vertices. The vertices of T are the cosets wA and wB where $w \in \Pi$. The edges of T are the cosets wC where $w \in \Pi$. The endpoints of the edge wC are wA and wB, and the action of Π is the natural left action on cosets. Since $ga = ag$, the element $a \in A$ fixes both $1A$ and gA. If $g \notin A$, then the fact that T is a tree (see [11, Theorem I.7.6]) implies that a fixes the nontrivial geodesic joining the distinct vertices $1A$ and gA in T, and hence a fixes an edge adjacent to $1A$. This means that there exists $\alpha \in A$ such that $a\alpha C = \alpha C$, and so $a \in \alpha C \alpha^{-1}$. •

Theorem 3.2.7 (Secondary Divisor Criterion) *Let* $G = G_n(x_0 x_j x_k x_l)$ *be a cyclically presented group with parameters* n, j, k, l *and shift* θ_G*. Let* γ *be the secondary divisor.*

(a) If $\mathrm{Fix}\left(\theta_G^p\right) \neq 1$ *for some integer* p*, then* $\gamma \mid p$*.*

(b) If $\gamma > 1$*, then* G *is infinite and* $\mathrm{Fix}(\theta_G) = 1$*.*

Proof By applying the substitution $u = xa^j$ to the presentation

$$\mathcal{Q}_n(x_0 x_j x_k x_l) = \left\langle a, x : a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \right\rangle,$$

for the shift extension $E = E_n(x_0 x_j x_k x_l)$, an amalgamated free product decomposition emerges:

$$\begin{aligned} \mathcal{Q}_n(x_0 x_j x_k x_l) &= \left\langle a, x : a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \right\rangle \\ &\cong \left\langle a, u, x : a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l}, u = xa^j \right\rangle \\ &\cong \left\langle a, u, x : a^n, u^2 a^{k-2j} ua^{l-k-j} ua^{-l-j}, u = xa^j \right\rangle \end{aligned}$$

$$\cong \left\langle a, b, u : a^n, u^2 a^{k-2j} u a^{-(j+l)-(k-2j)} u a^{-(j+l)}, b = a^\gamma \right\rangle$$
$$\cong \left\langle a, b, u : a^n, b^{n/\gamma}, u^2 b^{(k-2j)/\gamma} u b^{(l-k-j)/\gamma} u b^{-(j+l)/\gamma}, b = a^\gamma \right\rangle$$

which shows $E \cong \mathbb{Z}_n *_{\mathbb{Z}_{n/\gamma}} D$, where the group $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a, the group D is determined by the presentation

$$\mathcal{D} = \left\langle b, u : b^{n/\gamma}, u^2 b^{(k-2j)/\gamma} u b^{(l-k-j)/\gamma} u b^{-(j+l)/\gamma} \right\rangle,$$

and the amalgamated subgroup $\mathbb{Z}_{n/\gamma} = \langle b \rangle = \langle a^\gamma \rangle$ is the cyclic group of order n/γ generated by $b = a^\gamma$. The presentation $\mathcal{D}$ yields the natural presentation $\left\langle u : u^4 \right\rangle$ that determines the quotient group $D/\langle\langle b \rangle\rangle_D$, which shows $D/\langle\langle b \rangle\rangle_D \cong \mathbb{Z}_4$ is cyclic of order four, and hence the amalgamated subgroup $\mathbb{Z}_{n/\gamma} = \langle b \rangle$ is a proper subgroup of D.

Now, suppose that $\text{Fix}\left(\theta_G^p\right) \neq 1$. Then $G \cap \text{Cent}_E\left(a^p\right) \neq 1$, so using the fact that $G \cap \langle a \rangle = 1$, the Centralizer Lemma (Lemma 3.2.6) implies that $a^p \in \langle a^\gamma \rangle$. Since $\gamma \mid n$, this implies that $\gamma \mid p$. Furthermore, suppose $\gamma \neq 1$. Then $\text{Fix}\left(\theta_G\right) = 1$ as above, and the amalgamated subgroup $\mathbb{Z}_{n/\gamma} = \langle a^\gamma \rangle$ is a proper subgroup of $\mathbb{Z}_n = \langle a \rangle$. This implies that E is infinite, and hence its index n subgroup G is also infinite. •

Corollary 3.2.8 (Primary Divisor Criterion) *Let $G = G_n\left(x_0 x_j x_k x_l\right)$ be a cyclically presented group with parameters n, j, k, l and shift θ_G. Let d be the primary divisor.*

(a) If $\text{Fix}\left(\theta_G^p\right) \neq 1$ for some integer p, then $d \mid p$.

(b) If $d > 1$, then G is infinite and $\text{Fix}\left(\theta_G\right) = 1$.

Proof By noting that any common divisor of n, j, k, l (in particular, d) is also a divisor of the secondary divisor γ, the result follows immediately. •

Example 3.2.9 *For the following tuples (n, j, k, l), the primary divisor $d = 1$ gives no information about the cyclically presented group $G_n\left(x_0 x_j x_k x_l\right)$, but the secondary divisor*

γ is greater than one, so the conclusions of Theorem 3.2.7 apply. Compare Tables 1.2 and 1.3.

$$(10,3,6,1) \quad (12,1,2,9) \quad (16,3,6,1) \quad (20,3,6,1)$$
$$(24,1,2,15) \quad (24,3,6,1) \quad (24,1,2,19)$$

3.2.3 Results on Conditions (A), (B), (C)

This section contains a number of implications involving the truth values of conditions (A), (B), and (C) that are intended to supplement the proofs of the Lemmas, Theorems, and Corollaries in Sections 3.2.4 through 3.2.7.

Lemma 3.2.10 *Given the parameters n, j, k, l, if conditions* (A) *and* (C) *are true, then both equivalence statements in* (A) *are true and both equivalence statements in* (C) *are true.*

Proof Denote the statements in (A) and (C) as follows:

$$\begin{array}{llll} \text{(A1)} & 2k \equiv 0 & \text{(C1)} & l \equiv j + k \\ \text{(A2)} & 2j \equiv 2l & \text{(C2)} & l \equiv j - k \end{array}$$

The following implications will be shown:

(A1) and (C1) $\Rightarrow$ (A2) and (C1) $\Rightarrow$ (A2) and (C2) $\Rightarrow$ (A2) and (C1) $\Rightarrow$ (A1) and (C1).

If statements (A1) and (C1) are true, then

$$2l \equiv 2\,(j + k) \equiv 2j + 2k \equiv 2j,$$

and hence statement (A2) is also true.

If statements (A2) and (C1) are true, then

$$l \equiv j + k \equiv 3j + k - 2j \equiv 3j + k - 2l \equiv 3j + k - 2\,(j + k) \equiv j - k,$$

and hence statement (C2) is also true.

If statements (A2) and (C2) are true, then

$$2k \equiv 2j + 2k - 2j \equiv 2l + 2k - 2j \equiv 2l - 2(j-k) \equiv 2l - 2l \equiv 0,$$

and hence statement (A1) is also true.

Finally, if statements (A1) and (C2) are true, then

$$l \equiv j - k \equiv j - k + 2k \equiv j + k,$$

and hence statement (C1) is also true. •

Lemma 3.2.11 *Let $G = G_n(x_0 x_j x_k x_l)$ be a cyclically presented group with parameters n, j, k, l. Suppose that condition* (C) *is true. Then there exists an integer p such that the shift extension $E = E_n(x_0 x_j x_k x_l)$ is determined by a presentation of the form*

$$\left\langle a, z : a^n, z^2 a^{k-2p} z^2 a^{-k-2p} \right\rangle \cong \left\langle a, u, z : a^n, ua^k ua^{-k}, ua^{2p} = z^2 \right\rangle$$

and $G = \ker \nu$ for a retraction $\nu : E \to \mathbb{Z}_n$ defined by $\nu(a) = a$, $\nu(u) = 1$, and $\nu(z) = a^p$. Specifically, if $l \equiv j + k \mod n$ as in condition (C)*, then take $p = j$. Otherwise, if $l \equiv j - k \mod n$ as in condition* (C)*, then take $p = -l$.*

Proof If $l \equiv j + k \mod n$, then

$$\begin{aligned}
\mathcal{Q}_n(x_0 x_j x_k x_l) &= \left\langle a, x : a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \right\rangle \\
&\cong \left\langle a, x : a^n, xa^j xa^{k-j} xa^j xa^{-k-j} \right\rangle \\
&\cong \left\langle a, x, z : a^n, z = xa^j, z^2 a^{k-2j} z^2 xa^{-k-2j} \right\rangle \\
&\cong \left\langle a, u, z : a^n, u = z^2 a^{-2j}, ua^k ua^{-k} \right\rangle
\end{aligned}$$

and so $p = j$ in this case.

In a similar fashion, if $l \equiv j - k \mod n$, then note that $j \equiv l + k \mod n$, and so

$$\mathcal{Q}_n(x_0 x_j x_k x_l) = \left\langle a, x : a^n, xa^j xa^{k-j} xa^{l-k} xa^{-l} \right\rangle$$

$$\cong \left\langle a, x : a^n, xa^{l+k}xa^{-l}xa^{l-k}xa^{-l} \right\rangle$$
$$\cong \left\langle a, x : a^n, xa^{-l}xa^{l+k}xa^{-l}xa^{l-k} \right\rangle$$

where cyclic permutation the second relator of the presentation establishes the latter congruence. Continuing,

$$\begin{aligned} \mathcal{Q}_n(x_0x_jx_kx_l) &\cong \left\langle a, x : a^n, xa^{-l}xa^{l+k}xa^{-l}xa^{l-k} \right\rangle \\ &\cong \left\langle a, x, z : a^n, z = xa^{-l}, z^2a^{k+2l}z^2xa^{-k+2lj} \right\rangle \\ &\cong \left\langle a, u, z : a^n, u = z^2a^{2l}, ua^kua^{-k} \right\rangle \end{aligned}$$

and so $p = -l$ in this case.

In either case, note that $\nu(z) = a^p$ and $\nu(u) = 1$. •

Lemma 3.2.12 *If the parameters n, j, k, l satisfy condition* (B)*, then there exists an element $c \in \Gamma_n$ such that $c(x_0x_jx_kx_l) = x_0x_{j'}x_{k'}x_{l'}$ where*

(a) $k' \equiv 2j' \mod n$ *as in condition* (B)*;*

(b) The primary divisors d and d' of the respective parameters n, j, k, l and n, j', k', l' are equal: $d = d'$

(c) The secondary divisors γ and γ' of the respective parameters n, j, k, l and n, j', k', l' are equal: $\gamma = \gamma'$;

(d) The parameters n, j, k, l satisfy condition (A) *if and only if the parameters n, j', k', l' satisfy condition* (A)*; and*

(e) The parameters n, j, k, l satisfy condition (C) *if and only if n, j', k', l' satisfy condition* (C)*.*

Proof All equivalences are considered modulo n. If the parameters n, j, k, l satisfy condition (B), then either $k \equiv 2j$, $k \equiv 2l$, $j + l \equiv 0$, or $j + l \equiv 2k$. If $k \equiv 2j$, then take

$c = 1 \in \Gamma_n$ and the remaining conclusions follow trivially. The remaining possibilities are handled in cases.

For the first case, suppose $k \equiv 2l$. By taking $c = \sigma \in \Gamma_n$, note that

$$x_0 x_{j'} x_{k'} x_{l'} = \sigma \left(x_0 x_j x_k x_l \right) = x_0 x_l x_k x_j.$$

By setting the tuple $(n, j', k', l') = (n, l, k, j)$, first note that $k' = k \equiv 2l = 2j'$. Next, observe that the primary and secondary divisors are equal:

$$\gcd \left(n, j', k', l' \right) = \gcd \left(n, l, k, j \right) = \gcd \left(n, j, k, l \right)$$

and

$$\begin{aligned} & \gcd \left(n, k' - 2j', l' - 2k' + j', k' - 2l', j' + l' \right) \\ = \; & \gcd \left(n, k - 2l, j - 2k + l, k - 2j, l + j \right) \\ = \; & \gcd \left(n, k - 2j, l - 2k + j, k - 2l, j + l \right). \end{aligned}$$

Finally, note each of the following:

$$2k \equiv 0 \Leftrightarrow 2k' \equiv 0$$

$$2j \equiv 2l \Leftrightarrow 2l' \equiv 2j' \Leftrightarrow 2j' \equiv 2l'$$

$$l \equiv j + k \Leftrightarrow j' \equiv l' + k' \Leftrightarrow l' \equiv j' - k'$$

$$l \equiv j - k \Leftrightarrow j' \equiv l' - k' \Leftrightarrow l' \equiv j' + k'.$$

Thus the parameters n, j, k, l satisfy condition (A) (respectively (C)) if and only if the parameters n, j', k', l' satisfy condition (A) (respectively (C)).

For the next case, suppose $j + l \equiv 0$, which implies $l \equiv -j$. By taking $c = \tau \circ \theta_F^j \in \Gamma_n$, note that

$$x_0 x_j x_k x_l = x_0 x_j x_k x_{-j}$$

$$\xrightarrow{\theta_F^j} x_j x_{2j} x_{j+k} x_0$$

$$\xrightarrow{\tau} x_0 x_j x_{2j} x_{j+k}.$$

By setting the tuple $(n, j', k', l') = (n, j, 2j, j+k)$, first note that $k' = 2j = 2j'$. Next, observe that the primary and secondary divisors are equal:

$$\begin{aligned} & \gcd\left(n, j', k', l'\right) \\ = \; & \gcd\left(n, j, 2j, j+k\right) \\ = \; & \gcd\left(n, j, k\right) \\ = \; & \gcd\left(n, j, k, -j\right) \\ = \; & \gcd\left(n, j, k, l\right) \end{aligned}$$

and

$$\begin{aligned} & \gcd\left(n, k' - 2j', l' - 2k' + j', k' - 2l', j' + l'\right) \\ = \; & \gcd\left(n, 2j - 2j, j + k - 2\left(2l\right) + j, 2j - 2\left(j+k\right), j + j + k\right) \\ = \; & \gcd\left(n, k - 2j, -2k, k + 2j\right) \\ = \; & \gcd\left(n, k - 2j, -2k + (j + l), k - 2l, j + l\right) \\ = \; & \gcd\left(n, k - 2j, l - 2k + j, k - 2l, j + l\right). \end{aligned}$$

Finally, note each of the following:

$$2k \equiv 0 \Leftrightarrow 2j \equiv 2j + 2k \Leftrightarrow 2j \equiv 2\left(j+k\right) \Leftrightarrow 2j' \equiv 2l'$$

$$2j \equiv 2l \Leftrightarrow 2j - 2l \equiv 0 \Leftrightarrow 4j \equiv 0 \Leftrightarrow 2k' \equiv 0$$

$$l \equiv j + k \Leftrightarrow j + k \equiv -j \Leftrightarrow j + k \equiv j - 2j \Leftrightarrow l' \equiv j' - k'$$

$$l \equiv j - k \Leftrightarrow k \equiv 2j \Leftrightarrow j + k \equiv j + 2j \Leftrightarrow l' \equiv j' + k'.$$

Thus the parameters n, j, k, l satisfy condition (A) (respectively (C)) if and only if the parameters n, j', k', l' satisfy condition (A) (respectively (C)).

For the final case, suppose $j + l \equiv 2k$. This case will reduce to the previous one. Note that

$$\begin{aligned} & x_0 x_j x_k x_l \\ \xrightarrow{\tau^2} \; & x_k x_l x_0 x_j \\ \xrightarrow{\sigma} \; & x_k x_j x_0 x_l \\ \xrightarrow{\theta_F^{-k}} \; & x_0 x_{j-k} x_{-k} x_{l-k}. \end{aligned}$$

By setting the tuple $(n, j', k', l') = (n, j-k, -k, l-k)$, first note that $j'+l' = j-k+l-k \equiv j+l-2k \equiv 0$. Next, observe that the primary and secondary divisors are equal:

$$\gcd\left(n, j', k', l'\right) = \gcd\left(n, j-k, -k, l-k\right) = \gcd\left(n, j, k, l\right)$$

and

$$\begin{aligned} & \gcd\left(n, k'-2j', l'-2k'+j', k'-2l', j'+l'\right) \\ = \; & \gcd\left(n, -k-2\left(j-k\right), l-k-2\left(-k\right)+j-k, -k-2\left(l-k\right), j-k+l-k\right) \\ = \; & \gcd\left(n, k-2j, j+l, k-2l, l-2k+j\right) \\ = \; & \gcd\left(n, k-2j, l-2k+j, k-2l, j+l\right). \end{aligned}$$

Finally, note each of the following:

$$2k \equiv 0 \Leftrightarrow -2k \equiv 0 \Leftrightarrow 2k' \equiv 0$$

$$2j \equiv 2l \Leftrightarrow 2j - 2k \equiv 2l - 2k \Leftrightarrow 2\left(j-k\right) \equiv 2\left(l-k\right) \Leftrightarrow 2j' \equiv 2l'$$

$$l \equiv j + k \Leftrightarrow l - k \equiv j - k + k \Leftrightarrow l' \equiv j' - k'$$

$$l \equiv j - k \Leftrightarrow l - k \equiv j - k - k \Leftrightarrow l' \equiv j' + k'.$$

Thus the parameters n, j, k, l satisfy condition (A) (respectively (C)) if and only if the parameters n, j', k', l' satisfy condition (A) (respectively (C)). Hence this case reduces to the case $j + l \equiv 0$. •

Lemma 3.2.13 *Let* $G = G_n(x_0x_jx_kx_l)$ *be a cyclically presented group with parameters* n, j, k, l*. Suppose that* $k \equiv 2j \mod n$ *as in condition* (B)*. Let* γ *be the secondary divisor of* n, j, k, l*. With the substitution* $u = xa^j$*, the presentation* $\mathcal{Q}_n(x_0x_jx_kx_l) = \langle a, x : a^n, xa^jxa^{k-j}xa^{l-k}xa^{-l}\rangle$ *that determines the shift extension* $E = E_n(x_0x_jx_kx_l)$ *transforms into a presentation of the form* $\langle a, u : a^n, u^3\alpha u\beta\rangle$ *where* $\alpha = a^{l-3j}$ *and* $\beta = a^{-l-j}$ *in the cyclic group* $\langle a\rangle \cong \mathbb{Z}_n$ *of order* n *generated by* a*. Moreover, with consideration of the parameters* n, j, k, l*:*

(a) Condition (A) *is true if and only if* $\alpha = \beta^{\pm 1}$*;*

(b) If condition (A) *is true, then the element* $\alpha \in \langle a\rangle \cong \mathbb{Z}_n$ *has order* n/γ*; and*

(c) Condition (C) *is true if and only if* $\alpha = 1$ *or* $\beta = 1$*.*

Proof All equivalences are considered modulo n. Using the substitution $u = xa^j$, the presentation $\mathcal{Q}_n(x_0x_jx_kx_l)$ is transformed as follows:

$$\begin{aligned}\mathcal{Q}_n(x_0x_jx_kx_l) &= \langle a, x : a^n, xa^jxa^{k-j}xa^{l-k}xa^{-l}\rangle \\ &\overset{k\equiv 2j}{\cong} \langle a, x : a^n, xa^jxa^jxa^{l-2j}xa^{-l}\rangle \\ &\overset{u=xa^j}{\cong} \langle a, u : a^n, u^3a^{l-3j}ua^{-l-j}\rangle .\end{aligned}$$

Thus $\alpha = a^{l-3j}$ and $\beta = a^{-l-j}$, and claim (a) is verified as follows:

$$\alpha = \beta \Leftrightarrow l - 3j \equiv -l - j \Leftrightarrow 2l \equiv 2j$$

$$\alpha = \beta^{-1} \Leftrightarrow l - 3j \equiv l + j \Leftrightarrow 4j \equiv 0 \overset{k\equiv 2j}{\Leftrightarrow} 2k \equiv 0.$$

For claim (b), it will be shown that if either $2k \equiv 0$ or $2j \equiv 2l$, then the secondary divisor γ has the simplified description

$$\gamma = \gcd(n, k - 2j, l - 2k + j, k - 2l, j + l) = \gcd(n, j + l).$$

Proceeding in cases, suppose $2k \equiv 0$. Using $k \equiv 2j$, and working modulo any common divisor of n and $j + l$, observe that

$$\gamma = \gcd(n, k - 2j, l - 2k + j, k - 2l, j + l)$$

$$= \gcd(n, 0, 0, 2j - 2l, j + l)$$

$$= \gcd(n, 2j - 2l + 2(j + l), j + l)$$

$$= \gcd(n, 4j, j + l)$$

$$= \gcd(n, j + l).$$

Instead, suppose $2j \equiv 2l$. Using $k \equiv 2j$, and working modulo any common divisor of n and $j + l$, observe that

$$\gamma = \gcd(n, k - 2j, l - 2k + j, k - 2l, j + l)$$

$$= \gcd(n, 0, -2k, 2j - 2l, j + l)$$

$$= \gcd(n, -4j, 0, j + l)$$

$$= \gcd(n, -2j - 2l, j + l)$$

$$= \gcd(n, -2(j + l), j + l)$$

$$= \gcd(n, j + l).$$

In both cases, it follows that the order of the element $\alpha = \beta^{\pm 1} \in \langle a \rangle \cong \mathbb{Z}_n$ is

$$\mathrm{o}(\alpha) = \mathrm{o}\left(\beta^{\pm 1}\right) = n/\gcd(n, j + l) = n/\gamma.$$

For claim (c), note that

$$\alpha = 1 \Leftrightarrow l - 3j \equiv 0 \Leftrightarrow l \equiv 3j \equiv j + 2j \equiv j + k$$

and

$$\beta = 1 \Leftrightarrow -l - j \equiv 0 \Leftrightarrow l \equiv -j \equiv j - 2j \equiv j - k$$

as in condition (C). •

3.2.4 C(4)-T(4) Cases

This section addresses the situation in which the cyclic presentation $\mathcal{P}_n(x_0 x_j x_k x_l)$ satisfies the C(4)-T(4) small cancellation condition. One result of note is that in all but one case, finiteness and combinatorial asphericity are mutually exclusive.

Lemma 3.2.14 *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l, and let $G = G_n(x_0x_jx_kx_l)$ be the group determined by $\mathcal{P}$. Suppose that $\mathcal{P}$ is combinatorially aspherical.*

(a) If G has nontrivial torsion, then $k \equiv 0 \mod n$ and $j \equiv l \mod n$, so conditions (A) *and* (C) *are true.*

(b) If G is finite, then $n = 1$, so conditions (A), (B), *and* (C) *are true, $\mathcal{P} = \langle x_0 : x_0^4 \rangle$, and $G \cong \mathbb{Z}_4$ is cyclic of order four.*

Proof Since $\mathcal{P}$ is combinatorially aspherical and G has nontrivial torsion, [18, Theorem 3] implies that the relator $x_0x_jx_kx_l$ or one of its cyclic shifts must be a proper power in the free group $F(\mathbf{x})$ of rank n with basis $\mathbf{x} = \{x_0, \ldots, x_{n-1}\}$. It follows that $k \equiv 0 \mod n$ and $j \equiv l \mod n$, so $G \cong G_n\left((x_0x_j)^2\right)$, proving claim (a). If G is finite, it is nontrivial, and so $k \equiv 0 \mod n$ and condition (C) is true. By Lemma 3.2.11, the shift extension $E = E_n(x_0x_jx_kx_l)$ is determined by the presentation $\langle a, u, z : a^n, u^2, ua^{2p} = z^2 \rangle$, and so E contains the free product $\mathbb{Z}_n * \mathbb{Z}_2 \cong \langle a, u \rangle$ as a subgroup. The free product is finite if and only if $n = 1$. Since G is a finite group of finite index in E, then E and all of its subgroups must be finite, hence $n = 1$, proving claim (b). •

A word $v \in F$ is a **piece** for the presentation $\mathcal{P}_n(w)$ if v occurs as a common initial subword of two distinct cyclic permutations of the relators or their inverses.

Example 3.2.15 *The word x_1x_2 is a piece of the cyclic presentation*

$$\mathcal{P}_5(x_0x_1x_2x_3)$$

because it is the common initial subword of $x_1x_2x_3x_0$ and $x_1x_2x_3x_4$ which are cyclic permutations of the relators $x_0x_1x_2x_3$ and $\theta_F(x_0x_1x_2x_3) = x_1x_2x_3x_4$ respectively.

When $w = x_0x_jx_kx_l$ is a positive word of length four, there is a connection between pieces of length two and the C(4)-T(4)small cancellation condition for $\mathcal{P}_n(w)$.

Lemma 3.2.16 *The cyclic group presentation $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ satisfies the* C(4)-T(4) *small cancellation condition if and only if no length two cyclic subword of $w = x_0x_jx_kx_l$ is a piece.*

Proof Since the relators of $\mathcal{P}$ have length four, the C(4) condition is satisfied if and only if each piece has length one. On the other hand, if all pieces have length one, then the T(4) condition is also satisfied because the relators are all positive words. (See [16].) The cyclic symmetry of $\mathcal{P}$ implies that it suffices to verify that no length two cyclic subword of $w = x_0x_jx_kx_l$ is a piece. •

Lemma 3.2.17 *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l. If conditions* (B) *and* (C) *are false, then $\mathcal{P}$ satisfies the* C(4)-T(4) *small cancellation condition, hence $\mathcal{P}$ is combinatorially aspherical.*

Proof If conditions (B) and (C) are false, then j, $k-j$, $l-k$, and $-l$ are pairwise distinct modulo n. By Lemma 3.2.16, it suffices to show that none of x_0x_j, x_jx_k, x_kx_l, x_lx_0 is a piece. If any of these is an initial subword of a cyclic permutation of $\theta_F^i(x_0x_jx_kx_l) = x_ix_jx_kx_l$, then $i = 0$ because the subscript differences $(i+j)-i$, $(i+k)-(i+j)$, $(i+l)-(i+k)$, $i-(i-l)$ are likewise pairwise distinct. In the same way, since the subwords x_0x_j, x_jx_k, x_kx_l, and x_lx_0 are pairwise distinct, none occurs as an initial subword of any nontrivial cyclic permutation of $x_0x_jx_kx_l = \theta_F^0(x_0x_jx_kx_l)$, and so none is a piece.

Alternatively, [2, Theorem 2] provides that the relative presentation $\mathcal{R}_n(x_0x_jx_kx_l) = \langle \mathbb{Z}_n, x : xa^jxa^{k-j}xa^{l-k}xa^{-l} \rangle$ that determines the shift extension $E_n(x_0x_jx_kx_l)$ is aspherical, and so $\mathcal{P}_n(x_0x_jx_kx_l)$ is combinatorially aspherical by Theorem 1.2.1. •

Theorem 3.2.18 *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l, and let $G = G_n(x_0x_jx_kx_l)$ be the group determined by $\mathcal{P}$. If condition* (C) *is true, then the following are equivalent:*

(a) Condition (A) *is true;*

(b) $\mathcal{P}$ satisfies the C(4)-T(4) *small cancellation condition;*

(c) $\mathcal{P}$ is combinatorially aspherical;

(d) The cyclic group $\mathbb{Z}_n$ of order n acts freely via the shift on the nonidentity elements of G.

Proof All equivalences are considered modulo n. Since condition (C) is true, the word $w = x_0 x_j x_k x_l$ takes the form $w = x_0 x_j x_k x_{j \pm k}$. The following implications will be shown to complete the proof:

$$\text{(a)} \Rightarrow \text{(b)} \Rightarrow \text{(c)} \Rightarrow \text{(d)} \Rightarrow \text{(a)}.$$

As noted in Section 2.2.2, ordinary presentations that satisfy the C(4)-T(4) small cancellation condition are combinatorially aspherical, hence (b) $\Rightarrow$ (c). Next, Theorem 1.1.1 implies (c) $\Rightarrow$ (d).

To show claim (a) implies claim (b), suppose that condition (A) is true. Then Lemma 3.2.10 implies $2k \equiv 0$ and $w = x_0 x_j x_k x_{j+k}$. Up to cyclic shifts, the only length two cyclic subwords of w are $x_0 x_j$ and $x_j x_k$, so by Lemma 3.2.16, it suffices to show that neither of these words is a piece.

Suppose that the initial subword $x_0 x_j$ of $w = x_0 x_j x_k x_{j+k}$ is equal to an initial subword of a cyclic permutation v of a shift $\theta_F^i(w) = x_i x_{i+j} x_{i+k} x_{i+j+k}$ of w. It will be shown that $w = v$ in this situation by considering four possible cases.

If $v = \theta_F^i(w) = x_i x_{i+j} x_{i+k} x_{i+j+k}$, then $i \equiv 0$ and thus $w = v$.

If $v = x_{i+j} x_{i+k} x_{i+j+k} x_i$, then $x_{i+j} x_{i+k} = x_0 x_j$. This implies $i + j \equiv 0$ and $i + k \equiv j$, which further implies that $i + j + k \equiv k$ and, using the fact that $2k \equiv 0$, that $i \equiv i + 2k \equiv (i + k) + k \equiv j + k$. Thus

$$w = x_0 x_j x_k x_{j+k} = x_{i+j} x_{i+k} x_{i+j+k} x_i = v.$$

If $v = x_{i+k}x_{i+j+k}x_0x_{i+j}$, then $x_{i+k}x_{i+j+k} = x_0x_j$, so $i + k \equiv 0$ and $i + j + k \equiv j$. Using $2k \equiv 0$, this implies that $i \equiv -k \equiv k$ and $i + j \equiv j - k \equiv j + k$, and thus

$$w = x_0x_jx_kx_{j+k} = x_{i+k}x_{i+j+k}x_ix_{i+j} = v.$$

Finally, if $v = x_{i+j+k}x_ix_{i+j}x_{i+k}$, then $x_{i+j+k}x_i = x_0x_j$, so $i + j + k \equiv 0$ and $i \equiv j$. Using $2k \equiv 0$, this implies that $i + j \equiv -k \equiv k$ and $i + k \equiv j + k$, and thus

$$w = x_0x_jx_kx_{j+k} = x_{i+j+k}x_ix_{i+j}x_{i+k} = v.$$

On the other hand, let $w' = x_jx_kx_{j+k}x_0$ be the cyclic permutation of w with the initial subword x_jx_k and suppose that x_jx_k is equal to an initial subword of a cyclic permutation v of a shift $\theta_F^i(w) = x_ix_{i+j}x_{i+k}x_{i+j+k}$ of w. It will be shown that $w' = v$ in this situation by considering four possible cases.

If $v = \theta_F^i(w) = x_ix_{i+j}x_{i+k}x_{i+j+k}$, then $x_ix_{i+j} = x_jx_k$, so $i \equiv j$ and $i + j \equiv k$. Using $2k \equiv 0$, this implies that $i + k \equiv j + k$ and $i + j + k \equiv i + j - k \equiv 0$, and thus

$$w' = x_jx_kx_{j+k}x_0 = x_ix_{i+j}x_{i+k}x_{i+j+k} = v.$$

If $v = x_{i+j}x_{i+k}x_{i+j+k}x_i$, then $x_{i+j}x_{i+k} = x_jx_k$. This implies $i \equiv 0$, and thus

$$w' = x_jx_kx_{j+k}x_0 = x_{i+j}x_{i+k}x_{i+j+k}x_i = v.$$

If $v = x_{i+k}x_{i+j+k}x_0x_{i+j}$, then $x_{i+k}x_{i+j+k} = x_jx_k$, so $i + k \equiv j$ and $i + j + k \equiv k$. Using $2k \equiv 0$, this implies that $i \equiv j - k \equiv j + k$ and $i + j \equiv 0$, and thus

$$w' = x_jx_kx_{j+k}x_0 = x_{i+k}x_{i+j+k}x_ix_{i+j} = v.$$

Finally, if $v = x_{i+j+k}x_ix_{i+j}x_{i+k}$, then $x_{i+j+k}x_i = x_jx_k$, so $i + j + k \equiv j$ and $i \equiv k$. Using $2k \equiv 0$, this implies that $i + j \equiv j + k$ and $i + k \equiv 2k \equiv 0$, and thus

$$w' = x_jx_kx_{j+k}x_0 = x_{i+j+k}x_ix_{i+j}x_{i+k} = v.$$

All cases having been considered, no length two cyclic subword of w is a piece.

Finally, to show claim (d) implies claim (a), it will be proven that if condition (A) is false and condition (C) is true, then some nontrivial power (i.e. not divisible by n) of the shift θ_G on G has a nonidentity fixed point. From Lemma 3.2.11, the shift extension $E = E_n(x_0x_jx_kx_l)$ is determined by the ordinary presentation $\langle a, u, z : a^n, ua^kua^{-k}, ua^{2p} = z^2 \rangle$. This presentation has elements $a, u \in E$ satisfying the relation $ua^kua^{-k} = 1$. Observe that the quotient group $E/\langle\langle a \rangle\rangle_E$ is determined by the presentation

$$\langle u, z : u^2, u = z^2 \rangle \cong \langle u, z : z^4, u = z^2 \rangle \cong \langle z : z^4 \rangle .$$

This shows $E/\langle\langle a \rangle\rangle_E \cong \mathbb{Z}_4$ is cyclic of order four. Moreover, it also implies $u \notin \langle\langle a \rangle\rangle_E$, and so the element $g = u\nu(u)^{-1}$ is a nonidentity element of $G = \ker \nu$ where $\nu : E \to \mathbb{Z}_n$ is a retraction from E to the cyclic group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a satisfying $\nu(a) = a$. Moreover, since $ua^kua^{-k} = 1$, then $a^kua^{-k} = u^{-1}$, and so

$$a^{2k}ua^{-2k} = a^ku^{-1}a^{-k} = u.$$

This implies $u \in \mathrm{Cent}_E(a^{2k})$, hence $1 \neq g \in G \cap \mathrm{Cent}_E(a^{2k}) = \mathrm{Fix}(\theta_G^{2k})$. Since condition (A) is false, then $n \nmid 2k$, and hence $\mathbb{Z}_n$ does not act freely via the shift on the nonidentity elements of G. •

3.2.5 Cases FTT and FFT

This section addresses the situation in which condition (C) is true and condition (A) is false. Before proceeding further in this setting, observe that Theorem 3.2.18 concludes that the cyclic presentation $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ is not combinatorially aspherical and that the action of $\mathbb{Z}_n$ via the shift on the nonidentity elements of the group $G_n(x_0x_jx_kx_l)$ determined by $\mathcal{P}$ is not free.

Lemma 3.2.19 *Let n be a positive integer, and let $G = G_n(u_0u_1)$ be a cyclically presented group. If n is odd, then $G \cong \mathbb{Z}_2$ is cyclic of order two and the shift is the identity automorphism. If n is even, then $G \cong \mathbb{Z}$ is infinite cyclic and the shift maps each element to its inverse, and thus has no nonidentity fixed points.*

Proof First note that G is determined by the cyclic presentation

$$\mathcal{P} = \mathcal{P}_n(u_0u_1) = \langle u_0, u_1, \ldots, u_{n-1} : u_0u_1, u_1u_2, \ldots, u_{n-1}u_0 \rangle .$$

By rewriting the relators of $\mathcal{P}$, observe that $u_k = u_{k+1}^{-1}$ (subscripts modulo n), implying that $G = \langle u_0 \rangle$ is a cyclic group with generator u_0. In particular, by further utilizing these equalities, observe that $u_0 = u_p^{-1}$ if and only if $p \equiv 2q+1 \mod n$ for some integer q. The proof proceeds in cases on n.

If n is odd, then $0 \equiv n \equiv 2q+1 \mod n$ for some q, and hence $u_0 = u_0^{-1}$. This implies that the order of u_0 divides two, and hence G is of order 1 or 2. Now, note that the map $\{u_0, u_1, \ldots, u_{n-1}\} \to \mathbb{Z}_2$ via $u_k \mapsto 1$ (where $1 \in \mathbb{Z}_2$ is the generator of the additive cyclic group $\mathbb{Z}_2$ of order 2) extends to a surjective homomorphism $G \to \mathbb{Z}_2$ since each relator u_ku_{k+1} (subscripts modulo n) of $\mathcal{P}$ maps to the identity element $0 \in \mathbb{Z}_2$. This implies G is nontrivial, and therefore $G \cong \mathbb{Z}_2$ is cyclic of order two. Moreover, since the automorphism group $\mathrm{Aut}(G) \cong \mathrm{Aut}(\mathbb{Z}_2) = 1$, the shift is the identity automorphism.

Instead, if n is even, then the map $\{u_0, u_1, \ldots, u_{n-1}\} \to \mathbb{Z}$ via $u_k \mapsto (-1)^k$ extends to a surjective homomorphism $G \to \mathbb{Z}$ since each relator u_ku_{k+1} (subscripts modulo n) of $\mathcal{P}$ maps to the identity element $0 \in \mathbb{Z}$, the identity element of $\mathbb{Z}$. This implies G is infinite, and therefore $G \cong \mathbb{Z}$ is infinite cyclic. In particular, observe that this homomorphism has $u_0 \mapsto 1$ and $u_1 \mapsto -1$, so $u_0 \neq u_1$ in G. Moreover, since $\theta_G(u_0) = u_1 = u_0^{-1}$ and u_0 generates G, the shift θ_G takes every element $g \in G$ to its inverse g^{-1}. Since infinite cyclic groups contain no nonidentity elements of finite order (in particular, none of order two), $\mathrm{Fix}(\theta_G) = 1$. •

Theorem 3.2.20 *Let $G = G_n(x_0x_jx_kx_l)$ be a cyclically presented group with parameters n, j, k, l. If condition* (C) *is true and condition* (A) *is false, then the following are equivalent:*

(a) The shift θ_G has a nonidentity fixed point;

(b) *The group* D *determined by the presentation* $\left\langle a, u : a^n, ua^k ua^{-k} \right\rangle$ *is finite;*

(c) *The group* D *determined by the presentation* $\left\langle a, u : a^n, ua^k ua^{-k} \right\rangle$ *is cyclic of order* $2n$*;*

(d) $\gcd(n, 2k) = 1$.

Proof From Lemma 3.2.11, the shift extension $E = E_n(x_0 x_j x_k x_l)$ is determined by the presentation

$$\left\langle a, u, z : a^n, ua^k ua^{-k}, z^2 = ua^{2p} \right\rangle.$$

Letting D be the group determined by the presentation $\left\langle a, u : a^n, ua^k ua^{-k} \right\rangle$, by [22], the group E is obtained from D by adjoining a square root of $ua^{2p} \in D$; so $E \cong D *_Y Z$ where the group $Z = \langle z \rangle$ is the (possibly infinite) cyclic group generated by z and the group $Y = \left\langle ua^{2p} \right\rangle = \left\langle z^2 \right\rangle = D \cap Z$ is the amalgamated subgroup. In particular, D embeds as a subgroup of E.

Now, the map $\nu : E \to \mathbb{Z}_n$ (where $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a) defined by $\nu(a) = a$, $\nu(u) = 1$, and $\nu(z) = a^p$ is a retraction, so the kernel $G = \ker \nu$. Moreover, the restriction $\nu|_D : D \to \mathbb{Z}_n$ of ν to D defines a retraction on D, and thus it also has a cyclically presented kernel $H = \ker \nu|_D = G \cap D \cong G_n(u_0 u_k)$. (See [3, Theorem 2.3].) Letting $\kappa = \gcd(n, k)$, Lemma 3.2.19 implies that

$$H \cong *_{i=1}^{\kappa} G_n(u_0 u_1) \cong \begin{cases} *_{i=1}^{\kappa} \mathbb{Z} & \text{if } n \text{ is even} \\ *_{i=1}^{\kappa} \mathbb{Z}_2 & \text{if } n \text{ is odd.} \end{cases}$$

Since H is of finite index in D, then D is finite if and only if H is finite. This occurs if and only if $\kappa = 1$ and n is odd. Thus claim (b) is true if and only if claim (d) is true. In addition, if claim (c) is true, then $D \cap G = \ker \nu|_D$ is cyclic of order two and is fixed by θ_G since D is abelian. Thus claim (c) implies claim (a).

Next, it will be shown that claim (a) implies claim (d). Suppose θ_G has a nonidentity fixed point as in claim (a), so there exists $1 \neq g \in G \cap \mathrm{Cent}_E(a)$. Recalling that $E \cong D *_Y Z$

and that $a \in D$, the Centralizer Lemma (Lemma 3.2.6) implies that either $g \in D$ or that there exists $\alpha \in D$ such that $a \in \alpha Y \alpha^{-1}$. It will be shown that in either case, n is odd and $\gcd(n,k) = 1$.

If $g \in D$, then $1 \neq g \in \mathrm{Fix}(\theta_H)$ where θ_H is the shift on the group H. This implies $\gcd(n,k) = 1$ by the corresponding version of the Primary Divisor Criterion (Corollary 3.2.8) involving positive relators of length two. Since $\mathrm{Fix}(\theta_H) \neq 1$, Lemma 3.2.19 implies that n must be odd.

Instead, if $a \in \alpha Y \alpha^{-1}$ for some $\alpha \in D$, then a is in the normal closure $\langle\langle Y \rangle\rangle_D$ of Y in D. But the quotient group $D/\langle\langle Y \rangle\rangle_D$ is determined by the ordinary presentation $\left\langle a, u : a^n, ua^kua^{-k}, u = a^{-2p} \right\rangle$ whose relator $u = a^{-2p}$ implies $u \in \langle\langle Y \rangle\rangle_D$, hence $D = \langle\langle Y \rangle\rangle_D$. Observe that

$$\left\langle a, u : a^n, ua^kua^{-k}, u = a^{-2p} \right\rangle \cong \left\langle a : a^n, a^{-4p} \right\rangle \cong \left\langle a : a^{\gcd(n,4p)} \right\rangle$$

which determines the cyclic group $D/\langle\langle Y \rangle\rangle_D \cong \mathbb{Z}_{\gcd(n,4p)}$ of order $\gcd(n,4p)$. Since $D/\langle\langle Y \rangle\rangle_D = 1$, then $\gcd(n,4p) = 1$ which implies that n is odd.

Now, working in D, the element a is a power of $\alpha ua^{2p}\alpha^{-1}$ where $\alpha \in D$, so that $\alpha ua^{2p}\alpha^{-1} \in \mathrm{Cent}_D(a)$. Observe then that the presentation $\left\langle a, u : a^n, ua^kua^{-k} \right\rangle$ that determines D transforms as

$$\begin{aligned} &\left\langle a, u : a^n, ua^kua^{-k} \right\rangle \\ \cong\ &\left\langle a, b, u : a^n, ua^kua^{-k}, b = a^k \right\rangle \\ \cong\ &\left\langle a, b, u : b^{n/\gcd(n,k)}, ubub^{-1}, b = a^k \right\rangle. \end{aligned}$$

The transformed presentation reveals the amalgamated free product $D \cong \mathbb{Z}_n *_B C$ where the group $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a, the group C is determined by the presentation $\left\langle b, u : b^{n/\gcd(n,k)}, ubub^{-1} \right\rangle$, and the group $B = \langle b \rangle = \left\langle a^k \right\rangle \cong \mathbb{Z}_{n/\gcd(n,k)}$ is the amalgamated subgroup. By the Centralizer Lemma (Lemma 3.2.6), it follows that either $\alpha ua^{2p}\alpha^{-1} \in \langle a \rangle$ or $a \in \beta B \beta^{-1} = B = \left\langle a^k \right\rangle$ for some $\beta \in \langle a \rangle$. However, note that the quotient group $D/\langle\langle a\rangle\rangle_D$ is determined by the presentation $\langle u : u^2\rangle$ which determines the (nontrivial) cyclic group $D/\langle\langle a\rangle\rangle_D \cong \mathbb{Z}_2$ of order two. This shows that u maps naturally to the nontrivial element of the quotient group $D/\langle\langle a\rangle\rangle_D$, implying $u \notin \langle\langle a\rangle\rangle_D$. So $ua^{2p} \notin \langle\langle a\rangle\rangle_D$, hence $\alpha ua^{2p}\alpha^{-1} \notin \langle a\rangle$. Thus $a \in \langle a^k\rangle$, and so $\gcd(n,k) = 1$. Since n is odd, then $\gcd(n,2k) = 1$. Thus claim (a) implies claim (d).

Next it will be shown that claim (d) implies claim (c). Suppose that $\gcd(n,k) = 1$ and that n is odd. Working in D as determined by the presentation $\langle a, u : a^n, ua^k ua^{-k}\rangle$, the relators imply that $a^k ua^{-k} = u^{-1}$ and $a^{nk} = 1$, so $u = u^{(-1)^n} = u^{-1}$ since n is odd. This implies $u^2 = 1$, and so D is determined by

$$\left\langle a, u : a^n, ua^k ua^{-k}\right\rangle \cong \left\langle a, u : a^n, u^2, ua^k u^{-1}a^{-k}\right\rangle.$$

Furthermore, since $\gcd(n,k) = 1$, then $a \in \langle a^k\rangle$, hence $uau^{-1}a^{-1} = 1$. Thus D is abelian, so $D \cong \mathbb{Z}_n \times \mathbb{Z}_2$. Since n is odd, this implies $D \cong \mathbb{Z}_{2n}$, as in claim (c).

Finally, note that claim (c) implies claim (b) trivially, thus completing the proof. •

Theorem 3.2.21 *Let $G = G_n(x_0x_jx_kx_l)$ be a cyclically presented group with parameters n, j, k, l. If condition* (C) *is true and condition* (A) *is false, then the following are equivalent:*

(a) The shift extension $E = E_n(x_0x_jx_kx_l) \cong \mathbb{Z}_{4n}$ is cyclic of order $4n$;

(b) $G \cong \mathbb{Z}_4$ is cyclic of order 4;

(c) G is finite;

(d) $\gcd(n,2k) = 1$ *and either*

- *(i)* $l \equiv j + k \mod n$ *and* $\gcd(n,j) = 1$*; or*
- *(ii)* $l \equiv j - k \mod n$ *and* $\gcd(n,l) = 1$.

Proof The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are obvious since G is an index n subgroup of E. Now, by Lemma 3.2.11, E is determined by the ordinary presentation

$$\left\langle a, u, z : a^n, ua^k ua^{-k}, z^2 = ua^{2p} \right\rangle .$$

Observe that this presentation yields an amalgamated free product decomposition $E \cong D *_Y Z$ where the group D is determined by the presentation $\langle a, u : a^n, ua^k ua^{-k} \rangle$, the group $Z \cong \langle z \rangle$ is the (possibly infinite) cyclic group generated by z, and the group $Y = \langle ua^{2p} \rangle = \langle z^2 \rangle = D \cap Z$ is the amalgamated subgroup. Note that the group G is finite if and only if E is finite, which in turn implies that D must also be finite. By Theorem 3.2.20, the group D is finite if and only if $\gcd(n, 2k) = 1$ as in claim (d), and hence is a necessary condition for G to be finite as in claim (c). Furthermore, this means that any of claims (a), (b), or (c) imply $\gcd(n, 2k) = 1$.

A couple observations will now be made to simplify the remaining implications. Note that the presentation $\langle a, u, z : a^n, ua^k ua^{-k}, z^2 = ua^{2p} \rangle$ that determines E transforms as

$$\begin{aligned}
&\left\langle a, u, z : a^n, ua^k ua^{-k}, z^2 = ua^{2p} \right\rangle \\
\cong\ &\left\langle a, u, z : a^n, z^2 a^{k-2p} z^2 a^{-k-2p}, u = z^2 a^{-2p} \right\rangle \\
\cong\ &\left\langle a, z : a^n, z^2 a^{k-2p} z^2 a^{-k-2p} \right\rangle \\
\cong\ &\left\langle a, v, z : a^n, z^2 a^{k-2p} z^2 a^{-k-2p}, v = z^2 a^{k-2p} \right\rangle \\
\cong\ &\left\langle a, v, z : a^n, v^2 a^{-2k}, z^2 = va^{2p-k} \right\rangle ,
\end{aligned}$$

and that the transformed presentation yields a second amalgamated free product decomposition $E \cong D' *_Y Z$ where the group D' is determined by the presentation

$$\left\langle a, v : a^n, v^2 a^{-2k} \right\rangle \cong \left\langle a, v : a^n, v^2 = a^{2k} \right\rangle$$

with the group $Y = \langle z^2 \rangle = \langle va^{2p-k} \rangle$ as the amalgamated subgroup. Note that if

$\gcd(n,2k) = 1$, the presentation $\langle a, v : a^n, v^2 = a^{2k} \rangle$ implies that $\langle v^2 \rangle = \langle a^{2k} \rangle = \langle a \rangle$, which in turn implies $D' = \langle v \rangle \cong \mathbb{Z}_{2n}$ is cyclic of order $2n$ generated by v.

Next, it will be shown that claim (c) implies claim (a), so suppose that G is finite. This implies that E is finite, which in turn occurs if and only if either $Y = \langle va^{2p-k} \rangle = \langle v \rangle = D'$ or $Y = \langle z^2 \rangle = \langle z \rangle = Z$. In both cases, $\gcd(n,2k) = 1$ since G is finite, and hence $D' \cong \mathbb{Z}_{2n}$ is cyclic of order $2n$. However, the latter case implies $E = D' \cong \mathbb{Z}_{2n}$, which in turn implies $G \cong \mathbb{Z}_2$ has order two. Yet G maps surjectively onto the cyclic group $\mathbb{Z}_4$ of order four, and therefore it must instead be the case that $Y = \langle va^{2p-k} \rangle = \langle v \rangle = D' \cong \mathbb{Z}_{2n}$. This implies that the element $va^{2p-k} \in D' \cong \mathbb{Z}_{2n}$ has order $2n$, hence $E = Z = \langle z \rangle \cong \mathbb{Z}_{4n}$ is cyclic of order $4n$ as in claim (a). Thus claim (c) implies claim (a), and therefore (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

Finally, it will be shown that claim (c) is true if and only if claim (d) is true. Recall that the equality $\gcd(n,2k) = 1$ is necessary for claim (c) to hold, so suppose that this is the case. One also recalls that this implies $D' \cong \mathbb{Z}_{2n}$. Let k' be the multiplicative inverse of $2k$ modulo n so that $2kk' \equiv 1 \mod n$. Working in D', this means $v^{2k'} = a^{2kk'} = a$, hence the element $va^{2p-k} = v^{1+2k' \cdot 2p - 2kk'} \in D' = \langle v \rangle \cong \mathbb{Z}_{2n}$ is of order $2n/q$ where

$$\begin{aligned} q &= \gcd\left(2n, 1 + 4pk' - 2kk'\right) \\ &= \gcd\left(n, 1 + 4pk' - 2kk'\right) \\ &= \gcd\left(n, 1 + 4pk' - 1\right) \\ &= \gcd\left(n, 4pk'\right) \\ &= \gcd(n, p). \end{aligned}$$

Thus G is finite (as in claim (c)) if and only if both $\gcd(n,2k) = 1$ and $\gcd(n,p) = 1$. Since condition (C) is true, then $l \equiv j \pm k \mod n$. By Lemma 3.2.11, if $l \equiv j + k$, then $p = j$, whereas if $l \equiv j - k$, then $p = -l$, as desired in claim (d). •

Example 3.2.22 *If condition* (C) *is true and condition* (A) *is false, then* $G_n(x_0 x_j x_k x_l) \cong$

$\mathbb{Z}_4$ can occur regardless of whether condition (B) *is true or false, as the following examples demonstrate.*

- *If $G = G_n(x_0x_1x_2x_3)$ and $n > 4$, then condition* (A) *is false, conditions* (B) *and* (C) *are true, and G is finite $\Leftrightarrow G \cong \mathbb{Z}_4 \Leftrightarrow n$ is odd.*
- *If $G = G_n(x_0x_1x_3x_4)$ and $n > 6$, then conditions* (A) *and* (B) *are false, condition* (C) *is true, and G is finite $\Leftrightarrow G \cong \mathbb{Z}_4 \Leftrightarrow \gcd(n, 6) = 1$.*

3.2.6 Case TTF

This section addresses the situation where conditions (A) and (B) are true and condition (C) is false. In this setting, the shift extension $E_n(x_0x_jx_kx_l)$ is determined by a presentation of the form $\langle a, u : a^n, u^3\alpha u\beta\rangle$ where $\alpha = \beta^{\pm 1}$ (see Lemma 3.2.13).

Theorem 3.2.23 *Let $G = G_n(x_0x_jx_kx_l)$ be a cyclically presented group with parameters n, j, k, l. Suppose that conditions* (A) *and* (B) *are true and that condition* (C) *is false.*

(a) The action via the shift by the cyclic group $\mathbb{Z}_n$ of order n on the nonidentity elements of G is not free, hence the cyclic group presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ that determines G is not combinatorially aspherical by Theorem 1.1.1.

(b) The following are equivalent:

(i) G is finite;

(ii) The secondary divisor $\gamma = 1$;

(iii) The shift θ_G has a nonidentity fixed point.

(c) If G is finite, then G is solvable.

Proof By Lemma 3.2.12, there exists an element $c \in \Gamma_n$ such that $c(x_0x_jx_kx_l) = x_0x_{j'}x_{k'}x_{l'}$ where $k' \equiv 2j' \mod n$, the respective primary and secondary divisors of the parameters

n, j, k, l and n, j', k', l' are equal, the truth value of condition (A) for the respective parameters are identical, and the truth value of (C) for the respective parameters are identical. Moreover, Theorem 3.2.2 and Corollary 3.2.3 imply that the asphericity status, group structure, and shift dynamics are identical for the cyclic presentations and cyclically presented groups determined by the respective parameters n, j, k, l and n, j', k', l'. Thus, it may be assumed that the parameters n, j, k, l are such that $k \equiv 2j \mod n$, condition (A) is true, and condition (C) is false.

By Lemma 3.2.13, the shift extension $E = E_n(x_0 x_j x_k x_l)$ is determined by the ordinary presentation $\langle a, u : a^n, u^3 \alpha u \beta \rangle$ where $1 \neq \alpha = \beta^{\pm 1}$ has order n/γ in the cyclic group $\langle a \rangle \cong \mathbb{Z}_n$ of order n generated by a. The proof of claims (a) through (c) will be handled in the cases $\alpha = \beta^{-1}$ and $\alpha = \beta$ separately. For both cases, recall that G is the kernel of a retraction $\nu : E \to \mathbb{Z}_n$ where $\mathbb{Z}_n = \langle a \rangle$ with $\nu(a) = a$.

First, suppose that $1 \neq \alpha = \beta^{-1}$, so $\beta = \alpha^{-1}$. Formally, this means that E is determined by the presentation

$$\langle a, u : a^n, u^3 \alpha u \alpha^{-1} \rangle \cong \left\langle a, u, \alpha : a^n, \alpha^{n/\gamma}, u^3 \alpha u \alpha^{-1}, \alpha = a^{m\gamma} \right\rangle$$

where $\gcd(n, m) = 1$. This presentation yields an amalgamated free product decomposition

$$E \cong \mathbb{Z}_n *_{\mathbb{Z}_{n/\gamma}} M$$

where the group $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a, the group M is determined by presentation $\langle \alpha, u : \alpha^{n/\gamma}, u^3 \alpha u \alpha^{-1} \rangle$, and the cyclic group $\mathbb{Z}_{n/\gamma} = \langle a^{m\gamma} \rangle = \langle \alpha \rangle$ of order n/γ is the amalgamated subgroup.

By [6, Lemma 2.2], the group M is the split metacyclic group $M \cong \langle u \rangle \rtimes \langle \alpha \rangle$, where u has order $\mu = 3^{n/\gamma} - (-1)^{n/\gamma}$ and $\langle u \rangle \cap \langle \alpha \rangle = 1$ in E. The order μ is divisible by four, and since condition (C) is false, claim (c) of Lemma 3.2.13 implies that $n/\gamma > 1$, so $\mu > 4$. This means that the element $v = u^{\mu/4} \in M$ is a nontrivial element of E that lies outside of $\langle \alpha \rangle = \langle a \rangle \cap M$. Observe that

$$\alpha v \alpha^{-1} = \alpha u^{\mu/4} \alpha^{-1} = u^{-3\mu/4} = u^{(1-4)\mu/4} = v u^{-\mu} = v$$

and so $v \in \mathrm{Cent}_E(\alpha)$. Since $v \notin \langle a \rangle$, it follows that $g = v\nu(v)^{-1} \in G \cap \mathrm{Cent}_E(\alpha)$ is nontrivial. Since $\alpha \neq 1$ in $\langle a \rangle \cong \mathbb{Z}_n$, then $\alpha = a^{m\gamma}$ is a nonidentity power of a that centralizes a nonidentity element $g \in G$ in the shift extension E. Thus $\mathbb{Z}_n$ does not act freely via the shift on the nonidentity elements of G as in claim (a).

Now, since the group $\langle \alpha \rangle$ is a proper subgroup of M, it must be the case that

$$G \text{ is finite} \Leftrightarrow E \text{ is finite} \Leftrightarrow E = M \Leftrightarrow \langle a \rangle = \langle \alpha \rangle \Leftrightarrow \gamma = 1,$$

in which case G is a subgroup of the metacyclic group M. This implies that G is also metacyclic, hence G is solvable as in claim (c). If $\gamma = 1$, then $1 \neq g \in G \cap \mathrm{Cent}_E(\alpha) = G \cap \mathrm{Cent}_E(a) = \mathrm{Fix}(\theta_G)$, whereas if $\gamma \neq 1$, then $\mathrm{Fix}(\theta_G) = 1$ by Theorem 3.2.7, thereby showing claim (b).

Next, suppose that $1 \neq \alpha = \beta$. Formally, this means that E is determined by the presentation

$$\left\langle a, u : a^n, u^3 \alpha u \alpha \right\rangle \cong \left\langle a, u, \alpha : a^n, \alpha^{n/\gamma}, u^3 \alpha u \alpha, \alpha = a^{m\gamma} \right\rangle$$

where $\gcd(n, m) = 1$. This presentation yields an amalgamated free product decomposition

$$E \cong \mathbb{Z}_n *_{\mathbb{Z}_{n/\gamma}} \Delta$$

where the group $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a, the group Δ is determined by presentation $\left\langle \alpha, u : \alpha^{n/\gamma}, u^3 \alpha u \alpha \right\rangle$, and the cyclic group $\mathbb{Z}_{n/\gamma} = \langle a^{m\gamma} \rangle = \langle \alpha \rangle$ of order n/γ is the amalgamated subgroup.

Working in Δ, observe that

$$u^2 \alpha \cdot u \alpha u = 1 \quad \text{and} \quad \alpha u^2 \cdot u \alpha u = 1,$$

which implies that $u^2 \in \mathrm{Cent}_E(\alpha)$, and so $g = u^2\nu\left(u^2\right)^{-1} \in G \cap \mathrm{Cent}_E(\alpha)$. Furthermore, the quotient group $E/\langle\langle a\rangle\rangle_E$ is determined by the ordinary presentation $\langle u : u^4\rangle$. This shows $E/\langle\langle a\rangle\rangle_E \cong \mathbb{Z}_4$ is cyclic of order four. Moreover, this implies $u^2 \notin \langle\langle a\rangle\rangle_E$, and hence $1 \neq g \in G$. The fact that condition (C) is false means that $1 \neq \alpha \in \langle a\rangle \cong \mathbb{Z}_n$, and so a nonidentity power of a centralizes a nonidentity element of $g \in G$ in the shift extension E. Thus $\mathbb{Z}_n$ does not act freely via the shift on the nonidentity element of G as in claim (a)

Now, since $u^2 \in \mathrm{Cent}_E(\alpha)$, then u^2 is central in Δ. The central quotient group $\Delta/\langle u^2\rangle$ is determined by the ordinary presentation

$$\left\langle \alpha, u : \alpha^{n/\gamma}, u^2, u^3\alpha u\alpha\right\rangle \cong \left\langle \alpha, u : \alpha^{n/\gamma}, u^2, (u\alpha)^2\right\rangle$$

which determines the dihedral group $\Delta \cong D_{n/\gamma}$ of order $2n/\gamma$. Consider the five-term homology sequence

$$H_2\Delta \to H_2D_{n/\gamma} \to \langle u^2\rangle \to H_1\Delta \to H_1D_{n/\gamma} \to 0.$$

Utilizing the presentation latter presentation, the group Δ has the 2×2 relation matrix

$$\begin{bmatrix} n/\gamma & 2 \\ 0 & 4 \end{bmatrix}$$

which has determinant $4n/\gamma \neq 0$, so $H_2\Delta = 0$ and $|H_1\Delta| = 4n/\gamma$. For the dihedral group, observe that $\left|H_1D_{n/\gamma}\right| - \left|H_2D_{n/\gamma}\right| = 2$, and so the five-term sequence implies

$$0 = |H_2\Delta| - \left|H_2D_{n/\gamma}\right| + \left|\langle u^2\rangle\right| - |H_1\Delta| + \left|H_1D_{n/\gamma}\right| = \left|\langle u^2\rangle\right| + 2 - 4n/\gamma.$$

Thus Δ is a central extension of the dihedral group of order $2n/\gamma$ with cyclic kernel $\langle u^2\rangle$ of order $-2 + 4n/\gamma$. In particular, Δ is solvable. As in the previous case, the group $\langle\alpha\rangle$ is a proper subgroup of Δ, and so

$$G \text{ is finite} \Leftrightarrow E \text{ is finite} \Leftrightarrow E = \Delta \Leftrightarrow \langle a\rangle = \langle\alpha\rangle \Leftrightarrow \gamma = 1,$$

in which case G is a subgroup of the solvable group Δ, hence G is solvable as in claim (c). If $\gamma = 1$, then $1 \neq g \in G \cap \mathrm{Cent}_E(\alpha) = G \cap \mathrm{Cent}_E(a) = \mathrm{Fix}(\theta_G)$, whereas if $\gamma \neq 1$, then $\mathrm{Fix}(\theta_G) = 1$ by Theorem 3.2.7, thereby showing claim (b). •

3.2.7 Case FTF

Thus far, seven of the eight possible combinations of truth values for conditions (A), (B), and (C) have been considered. The remaining possibility, when condition (B) is true and conditions (A) and (C) are false, is the most complex and the most interesting. It is within this setting that the presentations of type (I*) or (U*) occur (Lemma 3.2.24), and moreover, it is shown that a cyclic group presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ in this setting is combinatorially aspherical unless it is of type (I*) or (possibly) type (U*) (Theorem 3.2.26).

Lemma 3.2.24 *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l. If $\mathcal{P}$ is of type* (I) *or* (U), *then condition* (B) *is true, conditions* (A) *and* (C) *are false, and the primary divisor $d = 1$.*

Proof For each exemplar presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ of type (I) or type (U), one calculates the primary divisor d and checks that the parameters n, j, k, l satisfy condition (B) but neither condition (A) nor condition (C). Lemma 3.2.12 implies that this is sufficient. •

Recall that the shift extension $E_n(x_0x_jx_kx_l)$ for a cyclically presented group $G = G_n(x_0x_jx_kx_l)$ is determined by the relative presentation

$$\mathcal{R}_n(x_0x_jx_kx_l) = \left\langle \mathbb{Z}_n, x : xa^jxa^{k-j}xa^{l-k}xa^{-l} \right\rangle$$

with the finite cyclic coefficient group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a.

Theorems 1.2.2 and 1.2.1 will be applied to the relative presentations obtained in Lemma 3.2.13 and thence to the cyclic presentations $\mathcal{P}_n(x_0x_jx_kx_l)$. However, it is beneficial to prove a small result first.

Lemma 3.2.25 *Let n, p, q be integers. Then*

$$\Sigma = \gcd(n,p) + \gcd(n,q) + \gcd(n,p-q) > n$$

if and only if at least one of the following holds:

(a) $p \equiv 0 \mod n$;

(b) $q \equiv 0 \mod n$; *or*

(c) $p \equiv q \mod n$.

Proof All equivalences are considered modulo n. Suppose $\Sigma > n$. Let

$$(\alpha,\beta,\gamma) \in \{(p,q,p-q),(p,p-q,q),(q,p,p-q),(q,p-q,p),(p-q,p,q),(p-q,q,p)\}$$

be the tuple such that

$$\gcd(n,\alpha) \geq \gcd(n,\beta) \geq \gcd(n,\gamma).$$

Observe as in Table 3.2 that $\gamma = \pm\alpha \pm \beta$ in each of the six possible tuples (α,β,γ).

Tuple (α,β,γ)	γ	Calculation
$(p,q,p-q)$	$\alpha-\beta$	$\gamma = p-q = \alpha-\beta$
$(p,p-q,q)$	$\alpha-\beta$	$\gamma = q = p-(p-q) = \alpha-\beta$
$(q,p,p-q)$	$-\alpha+\beta$	$\gamma = p-q = -q+p = -\alpha+\beta$
$(q,p-q,p)$	$\alpha+\beta$	$\gamma = p = q+(p-q) = \alpha+\beta$
$(p-q,p,q)$	$-\alpha+\beta$	$\gamma = q = -(p-q)+p = -\alpha+\beta$
$(p-q,q,p)$	$\alpha+\beta$	$\gamma = p = (p-q)+q = \alpha+\beta$

TABLE 3.2: Calculations to show $\gamma = \pm\alpha \pm \beta$

Now, it follows that if $\gcd(n,\alpha) \leq n/3$, then $\Sigma \leq n/3 + n/3 + n/3 = n$, hence $\gcd(n,\alpha) \geq n/2$. If $\gcd(n,\alpha) = n$, then

$$p \equiv 0 \quad \text{or} \quad q \equiv 0 \quad \text{or} \quad p-q \equiv 0$$

where the final possibility implies $p \equiv q$, as in claims (a)-(c). Otherwise, suppose that n is even and $\gcd(n,\alpha) = n/2$. It follows that if $\gcd(n,\beta) \leq n/4$, then $\Sigma \leq n/2 + n/4 + n/4 = n$, hence

$$n/2 = \gcd(n,\alpha) \geq \gcd(n,\beta) \geq n/3.$$

Suppose $\gcd(n,\beta) = n/2$. Since

$$\gcd(n,\alpha) = \gcd(n,-\alpha) = \gcd(n,\beta) = \gcd(-\beta) = \frac{n}{2},$$

it follows that $\pm\alpha \pm \beta \equiv 0$, and so $\gamma \equiv 0$ as in Table 3.2. But this means

$$\gcd(n,\gamma) = n > n/2 = \gcd(n,\alpha),$$

contrary to the assumption that $\gcd(n,\alpha) \geq \gcd(n,\gamma)$.

Instead, suppose that $3 \mid n$ and $\gcd(n,\beta) = n/3$, implying $6 \mid n$. Moreover, since $n/6$ is a common divisor of $n/2$ and $n/3$ which themselves divide α and β respectively, it follows that $\frac{n}{6} \mid \pm\alpha \pm \beta$, hence Table 3.2 implies $\frac{n}{6} \mid \gamma$. Now, if $\frac{n}{k} \mid \gamma$ for some integer k, then $\operatorname{lcm}(n/6, n/k) \mid \gamma$. Thus, in the cases where $k = 4$ and $k = 5$, it follows that $\frac{n}{2} \mid \gamma$ and $n \mid \gamma$ respectively. This means $\gcd(n,\gamma) \geq n/2$, contradicting the assumption that $\gcd(n,\gamma) \leq \gcd(n,\beta) = n/3$. However, if $\gcd(n,\gamma) \leq n/6$, then $\Sigma \leq n/2 + n/3 + n/6 = n$, and so it must be the case that $\gcd(n,\gamma) = n/3$. This implies $\frac{n}{3} \mid \pm\beta \pm \gamma$, hence $\frac{n}{3} \mid \alpha$ follows from Table 3.2. However, since $\operatorname{lcm}(n/3, n/2) = n$, this means $n \mid \alpha$, contradicting the assumption that $\gcd(n,\alpha) = n/2$. This completes the proof of the forward implication.

The reverse implication is obvious as each of claims (a)-(c) imply that one of the summands is equal to n, and thus $\Sigma \geq n + 2$. •

Theorem 3.2.26 *Let $\mathcal{P} = \mathcal{P}_n(x_0 x_j x_k x_l)$ be a cyclic group presentation with parameters n, j, k, l. Suppose that condition* (B) *is true, conditions* (A) *and* (C) *are false, and that the primary divisor $d = 1$. If $\mathcal{P}$ is not combinatorially aspherical, then $\mathcal{P}$ is either of type* (I) *or of type* (U).

Proof By Lemma 3.2.12 and Theorem 3.2.2, it suffices to prove the claims in the case where $k \equiv 2j \mod n$, the parameters n, j, k, l have primary divisor $d = 1$, and both conditions (A) and (C) are false. Let $G = G_n(x_0x_jx_kx_l)$ be the group determined by $\mathcal{P}$. As a consequence of Lemma 3.2.13, the shift extension $E = E_n(x_0x_jx_kx_l)$ is determined by a relative presentation of the form $\langle \mathbb{Z}_n, u : u^3\alpha u\beta \rangle$ where

$$\alpha = a^{l-3j} \quad \text{and} \quad \beta = a^{-l-j}$$

are such that $\alpha, \beta \neq 1$ and $\alpha \neq \beta^{\pm 1}$ in the cyclic group $\langle a \rangle \cong \mathbb{Z}_n$ of order n generated by a. By noting that the map $\nu^j : E \to \mathbb{Z}_n$ (where $\mathbb{Z}_n = \langle a \rangle$) defined by $\nu(a) = a$ and $\nu(u) = a^j$ is a retraction with

$$\rho^j\left(u^3\alpha u\beta\right) = u_0u_ju_{2j}u_l \overset{k\equiv 2j}{=} u_0u_ju_ku_l \quad \text{(subscripts modulo } n\text{)},$$

Theorem 1.2.1 implies that $\mathcal{P}$ is combinatorially aspherical if and only if the relative presentation $\langle \mathbb{Z}_n, u : u^3\alpha u\beta \rangle$ is aspherical.

It follows from Theorem 1.2.2 that the relative presentation $\langle \mathbb{Z}_n, u : u^3\alpha u\beta \rangle$ is aspherical unless one of the following conditions is satisfied:

(a) $\alpha = \beta^2$ and $4 \leq \mathrm{o}(\beta) \leq 6$

(a') $\beta = \alpha^2$ and $4 \leq \mathrm{o}(\alpha) \leq 6$

(b) $\alpha = \beta^{-2}$ and $\mathrm{o}(\beta) = 6$

(b') $\beta = \alpha^{-2}$ and $\mathrm{o}(\alpha) = 6$

(c) $\alpha = \beta^3$ and $\mathrm{o}(\beta) = 6$

(c') $\beta = \alpha^3$ and $\mathrm{o}(\alpha) = 6$

(d) $\{\mathrm{o}(\alpha), \mathrm{o}(\beta)\} = \{2, 3\}$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3$

(e) $\frac{1}{\mathrm{o}(\alpha)} + \frac{1}{\mathrm{o}(\beta)} + \frac{1}{\mathrm{o}(\alpha\beta^{-1})} > 1$

Using the fact that $d = 1$, it will be shown that if any of these conditions hold, then $\mathcal{P}$ is either of type (I) or type (U).

First, consider the element $c = -1 \cdot \theta_F^{-k} \tau^2 \sigma \in \Gamma_n$. Observe that

$$\begin{aligned} x_0 x_j x_k x_l &\overset{\sigma}{\to} x_0 x_l x_k x_j \\ &\overset{\tau^2}{\to} x_k x_j x_0 x_l \\ &\overset{\theta_F^{-k}}{\to} x_0 x_{j-k} x_{-k} x_{l-k} \\ &\overset{\times -1}{\to} x_0 x_{k-j} x_k x_{k-l} = c\left(x_0 x_j x_k x_l\right) = x_0 x_{j'} x_{k'} x_{l'} \end{aligned}$$

has, as above,

$$\alpha' = a^{l'-3j'} = a^{k-l-3(k-j)} = a^{3j-2k-l} \overset{k\equiv 2j}{\equiv} a^{-l-j} = \beta$$

and

$$\beta' = a^{-l'-j'} = a^{l-k-(k-j)} = a^{l-2k+j} \overset{k\equiv 2j}{\equiv} a^{l-3j} = \alpha$$

while preserving the desired assumption that

$$k' \equiv k \equiv 2k - k \overset{k\equiv 2j}{\equiv} 2k - 2j \equiv 2\left(k-j\right) \equiv 2j' \mod n.$$

Thus it suffices to consider just one from each of the paired conditions (a)-(a'), (b)-(b'), and (c)-(c') as the roles of α and β may be switched by utilizing the Γ_n-action on the set Φ_n.

Next, observe that because the group $\langle \alpha, \beta \rangle$ generated by α and β is finite cyclic, Lemmas 3.2.12 and 3.2.25 show that condition (e) implies that either condition (A) or condition (C) is true, contradicting the hypothesis. (Apply Lemma 3.2.25 by dividing both sides of the inequality by n and setting $p = l - 3j$ and $q = -l - j$.) It remains to consider the conditions (a), (b), (c), and (d) individually. Unless stated otherwise, all equivalences are considered modulo n.

(a) $\alpha = \beta^2$ and $4 \leq \mathrm{o}(\beta) \leq 6$: In this case, $l - 3j \equiv -2l - 2j$ and $4 \leq \mathrm{o}(\beta) = n/\gcd(n, j+l) \leq 6$. That is,

$$j \equiv 3l \quad \text{and} \quad n = \mathrm{o}(\beta)\gcd(n, j+l).$$

Since $k \equiv 2j$, then $k \equiv 6l$. Now the fact that $1 = d = \gcd(n, j, k, l) = \gcd(n, 3l, 6l, l)$ implies that $\gcd(n, l) = 1$, and so $x_0x_jx_kx_l = x_0x_{3l}x_{6l}x_l$ where $l \in \mathbb{Z}_n^*$. Now $j + l \equiv 3l + l \equiv 4l$, so $n = \mathrm{o}(\beta)\gcd(n, 4l) = \mathrm{o}(\beta)\gcd(n, 4)$.

If $\mathrm{o}(\beta) = 4$, then $n = 4\gcd(n, 4)$, and so it follows that $n = 16$. Then $\mathcal{P}$ is of isolated type (I16).

If $\mathrm{o}(\beta) = 5$, then $n = 5\gcd(n, 4)$, and it follows that $n = 5, 10$ or 20. If $n = 5$, then $x_0x_{3l}x_{6l}x_l = x_0x_{3l}x_lx_l$, and hence $\mathcal{P}$ is of isolated type (I5). Otherwise, if $n = 10, 20$, then $\mathcal{P}$ of isolated type (I10), (I20) respectively.

If $\mathrm{o}(\beta) = 6$, then $n = 6\gcd(n, 4)$, and it follows that $n = 24$, and so $\mathcal{P}$ is of unresolved type (U24′).

(b) $\alpha = \beta^{-2}$ and $\mathrm{o}(\beta) = 6$: In this case, $l - 3j \equiv 2j + 2l$ and $n = 6\gcd(n, j + l)$. Thus, $l \equiv -5j$ and $k \equiv 2j$, so

$$1 = d = \gcd(n, j, k, l) = \gcd(n, j, 2j, -5j) = \gcd(n, j).$$

Thus $j \in \mathbb{Z}_n^*$ and $x_0x_jx_kx_l = x_0x_jx_{2j}x_{-5j}$. Next, $n = 6\gcd(n, j + l) = 6\gcd(n, -4j) = 6\gcd(n, 4)$, and so it follows that $n = 24$. Thus $x_0x_jx_kx_l = x_0x_jx_{2j}x_{-5j} = x_0x_jx_{2j}x_{19j}$, so $\mathcal{P}$ is of unresolved type (U24″).

(c) $\alpha = \beta^3$ and $o(\beta) = 6$: In this case, $n = 6\gcd(n, j + l)$ and $l - 3j \equiv 3(-l - j)$, implying $4l \equiv 0$. The fact that $d = \gcd(n, j, k, l) = 1$ and $k \equiv 2j$ provides that

$$\begin{aligned}
1 &= \gcd(n, j, k, l) \\
&= \gcd(n, j, 2j, l) \\
&= \gcd(n, j, l) \\
&= \gcd(n, j + l, l) \\
&= \gcd(l, \gcd(n, j + l)) \\
&= \gcd(l, n/6).
\end{aligned}$$

Since $4l \equiv 0$, it follows that $\frac{n}{2} \mid 2l$, and so $\frac{n}{6} \mid 2l$. The fact that $\gcd(l, n/6) = 1$ implies that $n/6 = 1$ or 2, hence $n = 6$ or 12. Now, the element $\alpha = a^{l-3j}$ has order two in the group $\langle a \rangle \cong \mathbb{Z}_n$, so $n = 2\gcd(n, l-3j)$. Since $6 \mid n$, this implies $3 \mid l - 3j$, and thus $3 \mid l$.

Suppose that $n = 6$, so that $l \equiv 0$ or 3. If $l \equiv 0$, then $1 = \gcd(n, j, l) = \gcd(n, j)$, so $j \in \mathbb{Z}_n^*$ and $x_0x_jx_kx_l = x_0x_jx_{2j}x_0 = \sigma^{-1}(x_0x_0x_jx_{2j})$. Thus $\mathcal{P}$ is of isolated type (I6″). If $l \equiv 3$, then $n = 6\gcd(n, j+l) = 6\gcd(n, j+3) = 6$, so $\gcd(n, j+3) = 1$. This implies that $j \equiv 2$ or 4. Thus $x_0x_jx_kx_l = x_0x_2x_4x_3 = x_0x_{-4}x_{-2}x_3$ or $x_0x_jx_kx_l = x_0x_4x_2x_3$, and so $\mathcal{P}$ is of isolated type (I6′).

Instead, suppose that $n = 12$, so that $l \equiv 0$, 3, 6, or 9. Then $12 = 6\gcd(n, j+l)$, so $j + l$ is even. Moreover, $1 = d = \gcd n, j, k, l = \gcd(n, j, l)$ implies that either j or l is odd, and hence both j and l are odd. Hence $l \equiv 3$ or 9. Neither 3 or 4 is a divisor of $j+l$, so $j + l \equiv 2$ or 10. The only arrangements meeting these requirements are $(j, l) \equiv (7, 3)$, $(11, 3)$, $(1, 9)$, or $(5, 9)$. Observe that in each case, $l \equiv 9j$, and so $x_0x_jx_kx_l = x_0x_jx_{2j}x_{9j}$ where $j \in \mathbb{Z}_n^*$. Thus $\mathcal{P}$ is of isolated type (I12).

(d) $\{\mathrm{o}(\alpha), \mathrm{o}(\beta)\} = \{2, 3\}$ and $\langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_3$: It suffices to consider the case where $\mathrm{o}(\alpha) = \mathrm{o}\left(a^{l-3j}\right) = 2$ and $\mathrm{o}(\beta) = \mathrm{o}\left(a^{-l-j}\right) = 3$. Here, $2l \equiv 6j$ and $3j \equiv -3j$, so $l \equiv 3l - 2l \equiv -3j - 6j \equiv -9j$. Coupled with the fact that $k \equiv 2j$, it must be that $1 = d = \gcd(n, j, k, l) = \gcd(n, j, 2j, -9j) = \gcd(n, j)$, hence $j \in \mathbb{Z}_n^*$ with $x_0x_jx_kx_l = x_0x_jx_{2j}x_{-9j}$. Now, from $\mathrm{o}(\alpha) = \mathrm{o}\left(a^{l-3j}\right) = 2$, it follows that

$$n = 2\gcd(n, l-3j) = 2\gcd(n, -12j) = 2\gcd(n, 12),$$

which implies that 8 is a divisor of n. With $\mathrm{o}(\beta) = \mathrm{o}\left(a^{-l-j}\right) = 3$, it follows that

$$n = 3\gcd(n, -l-j) = 3\gcd(n, 8j) = 3\gcd(n, 8) = 24.$$

Thus $x_0x_jx_kx_l = x_0x_jx_{2j}x_{15j}$, and so $\mathcal{P}$ is of isolated type (I24). •

The following four computational results help to address the isolated presentations; that is, the cyclic presentations of type (I*).

Lemma 3.2.27 *The group J_4 determined by the presentation $\langle t, u : t^4, u^3t^2ut \rangle$ is finite metacyclic of order 272. The centralizer $\mathrm{Cent}_{J_4}(t)$ is cyclic of order 16.*

Proof Using the functions `Size`, `DerivedSubgroup`, `AbelianInvariants`, `Centralizer`, and `StructureDescription`, GAP indicates that $|J_4| = 272$, $J_4/J_4' \cong \mathbb{Z}_{16}$, $J_4' \cong \mathbb{Z}_{17}$, and $\mathrm{Cent}_{J_4}(t) \cong \mathbb{Z}_{16}$. Note that J_4 is the group $J_4(4,1)$ of [5], where it is shown that J_4 is a semidirect product $\mathbb{Z}_{17} \rtimes \mathbb{Z}_{16}$. •

Lemma 3.2.28 *The group K determined by the presentation $\langle t, u : t^5, u^3t^2ut \rangle$ is finite metacyclic of order 1100. The centralizer $\mathrm{Cent}_K(t)$ is an elementary abelian group of order 25.*

Proof Computing as in Lemma 3.2.27, GAP indicates that $|K| = 1100$, $K/K' \cong \mathbb{Z}_{20}$, $K' \cong \mathbb{Z}_{55}$, and $\mathrm{Cent}_K(t) \cong \mathbb{Z}_5^2$. •

Lemma 3.2.29 *The group J_6 determined by the presentation $\langle t, u : t^6, u^3t^3ut^2 \rangle$ is finite metacyclic of order 4632. The centralizer $\mathrm{Cent}_{J_6}(t)$ is cyclic of order 24.*

Proof Computing as in Lemma 3.2.27, GAP indicates that $|J_6| = 4632$, $J_6/J_6' \cong \mathbb{Z}_{24}$, $J_6' \cong \mathbb{Z}_{193}$, and $\mathrm{Cent}_{J_6}(t) \cong \mathbb{Z}_{24}$. Note that J_6 is the group $J_6(4,1)$ of [5], where it is shown that J_6 is a semidirect product $\mathbb{Z}_{193} \rtimes \mathbb{Z}_{24}$. •

Lemma 3.2.30 *The group L determined by the presentation $\langle t, u : t^6, u^3t^3ut \rangle$ is a nonsolvable group of order $24530688 = 2^8 \cdot 3^4 \cdot 7 \cdot 13^2$ with second derived subgroup L'' isomorphic to the simple group $\mathrm{PSL}(3,3)$ of order 5616. The centralizer $\mathrm{Cent}_L(t)$ contains a noncyclic abelian subgroup of order 72.*

Proof Although the computations are somewhat lengthy, the order and nonsolvability claims for L can be verified directly in GAP using the `Size`, `DerivedSubgroup`, and `PefectIdentification` functions [27]. The centralizer claim is more difficult and is resistant to brute force application of the `Centralizer` function. See Section 4.2.

A coset enumeration using `Size` indicates that $|L| = 24530688$. Now, consider the word

$$v = ut^4u^2t^2ut^4ut^2ut^5ut^4ut^4u^2. \tag{3.2}$$

A second coset enumeration reveals that the quotient group Q determined by the presentation

$$\left\langle t, u : t^6, u^3t^3ut, tvt^{-1}v^{-1} \right\rangle$$

has identical order $|Q| = 24530688$, and so $v \in \mathrm{Cent}_L(t)$. Two additional coset enumerations show that the groups P_4 and P_6 determined by the respective presentations

$$\left\langle t, u : t^6, u^3t^3ut, v^4 \right\rangle \text{ and } \left\langle t, u : t^6, u^3t^3ut, v^6 \right\rangle$$

have orders $|P_4| = 4368$ and $|P_6| = 12265344$. Since $|P_6| = |L|/2$ and 2 is prime, it must be the case that $v \in L$ has order $\mathrm{o}(v) \equiv 6 \equiv m/2 \mod m$ for some integer m. This implies $\mathrm{o}(v) = 4$ or 12, but since $|P_4| \neq |L|$, v must have order 12. Hence, the centralizer $\mathrm{Cent}_L(t)$ contains the group $\langle t, v\rangle \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}$ as a subgroup of order 72. •

Remark: Instructions for reproducing the results of Lemmas 3.2.27, 3.2.28, 3.2.29, and 3.2.30 in GAP [15] are found in Chapter 4. This includes details about how the word v in Equation (3.2) was discovered utilizing new GAP functions found in [23]. Moreover, it is shown using these new functions that $\mathrm{Cent}_L(t) = \langle t, v\rangle \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}$, but to verify this using only functions included in the standard GAP distribution would take a substantial amount of time to complete without a flash of mathematical inspiration to reduce the computations necessary.

Theorem 3.2.31 *If the cyclic group presentation $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ is of type* (I*)*, then the cyclic group $\mathbb{Z}_n$ of order n does not act freely via the shift on the nonidentity elements of the group $G = G_n(x_0x_jx_kx_l)$ determined by $\mathcal{P}$, and thus $\mathcal{P}$ is not combinatorially aspherical by Theorem 1.1.1. If $\mathcal{P}$ is of type* (I5), (I6′), *or* (I6″)*, then G is finite and the shift θ_G has a nonidentity fixed point.*

Proof Let d be the primary divisor of the parameters n, j, k, l. If $\mathcal{P}$ is of type (I*), then the cyclic presentation $\mathcal{P}_{n/d}\left(x_0 x_{j/d} x_{k/d} x_{l/d}\right)$ is of type (I) and G is isomorphic to the free product $G \cong *_{i=1}^{c} H$ where $H = G_{n/d}\left(x_0 x_{j/d} x_{k/d} x_{l/d}\right)$. Further, as described in Section 3.2.2 and [3, Lemma 5.1] in the case of length three relators, the shift θ_H on H arises as a restriction of a power of the shift on G: $\theta_H = \theta_G^d\big|_H$. It follows that if the cyclic group $\mathbb{Z}_{n/d}$ of order n/d does not act freely via the shift on the nonidentity elements of H, then the cyclic group $\mathbb{Z}_n$ of order n does not act freely via the shift on the nonidentity elements of G. Thus it suffices to prove the claims under the assumption that $\mathcal{P}$ is of type (I). Moreover, Theorem 3.2.24 and Lemma 3.2.12 together show that it is sufficient to consider only the exemplars for each type (I$-$). The proof proceeds in cases.

For $n = 5$, 10, 16, or 20 where $\mathcal{P}$ is of type (In), consider that the cyclically presented group $G = G_n\left(x_0 x_3 x_6 x_1\right)$ has the shift extension $E = E_n\left(x_0 x_3 x_6 x_1\right)$ that is determined by the presentation

$$\left\langle a, x : a^n, xa^3xa^3xa^{-5}xa^{-1}\right\rangle \overset{u=xa^3}{\cong} \left\langle a, u : a^n, u^3a^{-8}ua^{-4}\right\rangle .$$

Moreover, the group G is the kernel of a retraction $\nu : E \to \mathbb{Z}_n$ onto the cyclic group $\mathbb{Z}_n = \langle a \rangle$ of order n generated by a.

If $n = 5$, then E is determined by the presentation $\left\langle a, u : a^5, u^3a^2ua\right\rangle$, showing that $E \cong K$ as in Lemma 3.2.28. Then the centralizer $\mathrm{Cent}_E\left(a\right)$ has order 25, and since $a \in \mathrm{Cent}_E\left(a\right)$, the retraction ν restricts to a surjection of $\mathrm{Cent}_E\left(a\right)$ onto $\mathbb{Z}_5 = \langle a \rangle$. This implies $\mathrm{Fix}\left(\theta_G\right) = G \cap \mathrm{Cent}_E\left(a\right)$ is cyclic of order 5.

If $n = 10$ or 20, then E is determined by the presentation

$$\begin{aligned}\left\langle a, u : a^n, u^3a^{-8}ua^{-4}\right\rangle &\cong \left\langle a, t, u : a^n, u^3a^{-8}ua^{-4}, t = a^{-4}\right\rangle \\ &\cong \left\langle a, t, u : a^n, t^5, u^3t^2ut, t = a^{-4}\right\rangle .\end{aligned}$$

This presentation exposes an amalgamated free product decomposition $E \cong \mathbb{Z}_n *_Y K$ for E where the group $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a, the group K

is determined by the presentation $\langle t, u : t^5, u^3t^2ut \rangle$ as in Lemma 3.2.28, and the group $Y = \langle a^{-4} \rangle = \langle t \rangle \cong \mathbb{Z}_5$ is the amalgamated subgroup. Since the centralizer $\mathrm{Cent}_K(t)$ has order 25 and $t = a^{-4} \in \mathrm{Cent}_K(t)$, the retraction ν maps $\mathrm{Cent}_K(t)$ onto the the cyclic subgroup $\langle a^4 \rangle \cong \mathbb{Z}_5$ of order 5 in $\mathbb{Z}_n = \langle a \rangle$, and hence $\mathrm{Fix}(\theta_G^4) = G \cap \mathrm{Cent}_E(a^4)$ contains a cyclic subgroup of order 5.

If $n = 16$, then E is determined by the presentation

$$\begin{aligned}\langle a, u : a^{16}, u^3a^{-8}ua^{-4} \rangle &\cong \langle a, t, u : a^{16}, u^3a^{-8}ua^{-4}, t = a^{-4} \rangle \\ &\cong \langle a, t, u : a^{16}, t^4, u^3t^2ut, t = a^{-4} \rangle .\end{aligned}$$

This presentation exposes an amalgamated free product decomposition $E \cong \mathbb{Z}_{16} *_Y J_4$ for E where the group $\mathbb{Z}_{16} = \langle a \rangle$ is the cyclic group of order 16 generated by a, the group J_4 is determined by the presentation $\langle t, u : t^4, u^3t^2ut \rangle$ as in Lemma 3.2.27, and the group $Y = \langle a^{-4} \rangle = \langle t \rangle \cong \mathbb{Z}_4$ is the amalgamated subgroup. Since the centralizer $\mathrm{Cent}_{J_4}(t)$ has order 16 and $t = a^{-4} \in \mathrm{Cent}_{J_4}(t)$, the retraction ν maps $\mathrm{Cent}_{J_4}(t)$ onto the cyclic subgroup $\langle a^4 \rangle \cong \mathbb{Z}_4$ of order 4 in $\mathbb{Z}_{16} = \langle a \rangle$, and hence $\mathrm{Fix}(\theta_G^4) = G \cap \mathrm{Cent}_E(a^4)$ contains a cyclic subgroup of order 4.

For $\mathcal{P}$ of types (I6′) or (I6″), consider the group L determined by the presentation $\langle a, u : a^6, u^3a^3ua \rangle$ as in Lemma 3.2.30 and choose an element $v \in \mathrm{Cent}_L(a)$ that is not a power of a. For $f = 2, 5$, there are retractions $\nu^f : L \to \mathbb{Z}_6$ (where $\mathbb{Z}_6 = \langle a \rangle$ is the cyclic group of order six generated by a) satisfying $\nu^f(a) = a$ and $\nu^f(u) = a^f$. Using the rewriting scheme described in [3, Theorem 2.3] together with isomorphisms from Theorem 3.2.2, note that

$$\ker \nu^2 \cong G_6(u_0u_2u_4u_3) \overset{\times -1}{\cong} G_6(u_0u_4u_2u_3)$$

and

$$\ker \nu^5 \cong G_6(u_0u_5u_4u_0) \overset{\tau}{\cong} G_5(u_0^2u_5u_4) \overset{\times -1}{\cong} G_6(u_0^2u_1u_2)$$

which are the groups determined by the exemplars of types (I6′) and (I6″) respectively.

For $f = 2, 5$, the element $g_f = v\nu^f(v)^{-1} \in \ker \nu^f \cap \mathrm{Cent}_L(a)$ is nontrivial, hence the respective shifts on the groups $\ker \nu^2 \cong G_6(u_0u_4u_2u_3)$ and $\ker \nu^5 \cong G_6(u_0^2u_1u_2)$ each contain a nonidentity fixed point.

For $\mathcal{P}$ of type (I12), the cyclically presented group $G = G_{12}(x_0x_1x_2x_9)$ is the kernel of a retraction $\nu : E \to \mathbb{Z}_{12}$ (where $\mathbb{Z}_{12} = \langle a \rangle$ is the cyclic group of order 12 generated by a) on the shift extension $E = E_{12}(x_0x_1x_2x_9)$. The group E is determined by the presentation

$$\begin{aligned}\left\langle a, x : a^{12}, xaxaxa^7xa^{-9}\right\rangle &\overset{u=xa}{\cong} \left\langle a, u : a^{12}, u^3a^6ua^2\right\rangle \\ &\cong \left\langle a, t, u : a^{12}, u^3a^6ua^2, t = a^2\right\rangle \\ &\cong \left\langle a, t, u : a^{12}, t^6, u^3t^3ut, t = a^2\right\rangle.\end{aligned}$$

This presentation exposes an amalgamated free product decomposition $E \cong \mathbb{Z}_{12} *_Y L$ for E where the group $\mathbb{Z}_{12} = \langle a \rangle$ is the cyclic group of order 12 generated by a, the group L is determined by the presentation $\langle t, u : t^6, u^3t^3ut\rangle$ as in Lemma 3.2.30, and the group $Y = \langle a^2 \rangle = \langle t \rangle \cong \mathbb{Z}_6$ is the amalgamated subgroup. There exists an element $v \in \mathrm{Cent}_L(t)$ that is not a power of $t = a^2$, which in turn determines a nonidentity element $g = v\nu(v)^{-1} \in G \cap \mathrm{Cent}_E(a^2) = \mathrm{Fix}(\theta_G^2)$.

For $\mathcal{P}$ of type (I24), the cyclically presented group $G = G_{24}(x_0x_1x_2x_{15})$ is the kernel of a retraction $\nu : E \to \mathbb{Z}_{24}$ (where $\mathbb{Z}_{24} = \langle a \rangle$ is the cyclic group of order 24 generated by a) on the shift extension $E = E_{24}(x_0x_1x_2x_{15})$. The group E is determined by the presentation

$$\begin{aligned}\left\langle a, x : a^{24}, xaxaxa^{13}xa^{-15}\right\rangle &\overset{u=xa}{\cong} \left\langle a, u : a^{24}, u^3a^{12}ua^8\right\rangle \\ &\cong \left\langle a, t, u : a^{24}, u^3a^{12}ua^8, t = a^4\right\rangle \\ &\cong \left\langle a, t, u : a^{24}, t^6, u^3t^3ut^2, t = a^4\right\rangle.\end{aligned}$$

This presentation exposes an amalgamated free product decomposition $E \cong \mathbb{Z}_{12} *_Y J_6$ for E where the group $\mathbb{Z}_{24} = \langle a \rangle$ is the cyclic group of order 24 generated by a, the group J_6

is determined by the presentation $\langle t, u : t^6, u^3t^3ut^2 \rangle$ as in Lemma 3.2.29, and the group $Y = \langle a^4 \rangle = \langle t \rangle \cong \mathbb{Z}_6$ is the amalgamated subgroup. Since the centralizer $\mathrm{Cent}_{J_6}(t)$ has order 24 and $t = a^4 \in \mathrm{Cent}_{J_6}(t)$, the retraction ν maps $\mathrm{Cent}_{J_6}(t)$ onto the cyclic subgroup $\langle a^4 \rangle \cong \mathbb{Z}_6$ of order 6 in $\mathbb{Z}_{24} = \langle a \rangle$, and hence $\mathrm{Fix}\left(\theta_G^4\right) = G \cap \mathrm{Cent}_E\left(a^4\right)$ contains a cyclic group of order 4. •

Theorem 3.2.32 *Let $\mathcal{P} = \mathcal{P}_n(x_0x_jx_kx_l)$ be a cyclic group presentation with parameters n, j, k, l. Let $G = G_n(x_0x_jx_kx_l)$ be the group determined by $\mathcal{P}$. Suppose that condition* (B) *is true and that both conditions* (A) *and* (C) *are false.*

(a) If $\mathcal{P}$ is not of type (U*)*, then the following are equivalent:*

(i) $\mathcal{P}$ is combinatorially aspherical;

(ii) The cyclic group $\mathbb{Z}_n$ of order n acts freely via the shift on the nonidentity elements of G;

(iii) $\mathcal{P}$ is not of type (I*)*.*

(b) The following are equivalent:

(i) G is finite;

(ii) $\mathcal{P}$ is of type (I) *or* (U) *and the secondary divisor $\gamma = 1$;*

(iii) $\mathcal{P}$ is of type (I5), (I6′)*, or* (I6″)*;*

(iv) The shift θ_G has a nonidentity fixed point.

Proof First suppose that $\mathcal{P}$ is not of type (U*). The implications (i) ⇒ (ii) ⇒ (iii) in claim (a) follow from Theorems 1.1.1 and 3.2.31 respectively. Let d be the primary divisor of the parameters n, j, k, l. Now, if $\mathcal{P}$ is not of type (I*) or (U*), then the cyclic group presentation $\mathcal{P}_{n/d}\left(x_0x_{j/d}x_{k/d}x_{l/d}\right)$ is combinatorially aspherical by Theorem 3.2.26, and hence $\mathcal{P}$ is combinatorially aspherical by Lemma 3.2.5. This shows that (iii) ⇒ (i), hence proving claim (a).

Now, suppose that G is finite. Then the primary divisor $d = 1$ and secondary divisor $\gamma = 1$ by Corollary 3.2.8 and Theorem 3.2.7 respectively. Since condition (A) is false, it follows that $n \neq 1$, and so $\mathcal{P}$ is not combinatorially aspherical by Lemma 3.2.14. It follows that $\mathcal{P}$ is either of type (I) or type (U) from Theorem 3.2.26. This shows that (i) $\Rightarrow$ (ii). The secondary divisor calculations in Example 3.2.9 show that (ii) $\Rightarrow$ (iii). The implications (iii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (i) follow directly from Theorem 3.2.31. Finally, if the shift θ_G has a nonidentity fixed point, then the primary and secondary divisors are both equal to one by Corollary 3.2.8 and Theorem 3.2.7 respectively, and $\mathcal{P}$ is not combinatorially aspherical by Theorem 1.1.1, whence $\mathcal{P}$ is either of type (I) or type (U) by Theorem 3.2.26. Thus (iv) $\Rightarrow$ (ii), hence proving of claim (b). •

3.2.8 Finishing the Proofs of the Main Theorems

Avoiding presentations of the unresolved type (U*), Theorem A is a consequence of Lemma 3.2.17 and Theorems 3.2.18, 3.2.23(a), and 3.2.32(a). Each result is considered in turn, and the relevant lines of Table 3.1 are provided. Lemma 3.2.17 handles the situation where conditions (B) and (C) are false, corresponding to lines three and four of the table. Theorem 3.2.18 handles the situation where condition (C) is true, corresponding to lines one, two, four, and five of the table. Theorem 3.2.23(a) handles the situation where conditions (A) and (B) are true and condition (C) is false, corresponding to line seven of the table. Finally, Theorem 3.2.32(a) handles the final case where conditions (A) and (C) are false and condition (B) is true, corresponding to line eight of the table. This completes the combinatorial asphericity classification. Now, If $k \not\equiv 0 \mod n$ or $j \not\equiv l \mod n$, then the cyclic presentation $\mathcal{P} = \mathcal{P}_n(x_0 x_j x_k x_l)$ has no proper power relators, in which case combinatorial asphericity implies that after removing freely redundant relators from the two-dimensional model of $\mathcal{P}$, the group $G_n(x_0 x_j x_k x_l)$ determined by $\mathcal{P}$ is the fundamental group of an aspherical two-complex, and so is torsion-free with geometric dimension at most two. This completes the proof of the theorem.

Theorem B follows from Lemma 3.2.14(b) and Theorems 3.2.21, 3.2.23(b), and 3.2.32(b). These results also detail the structural claims for the finite groups that occur. Again, each result is considered in turn, and relevant lines of Table 3.1 are provided. First, Lemma 3.2.14(b) handles the situation that the presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is combinatorially aspherical, corresponding particularly to lines one through four of the table. Note that in this situation, $n = 1$ which implies that the gcd and equivalence statements of claim (a) of Theorem B all hold. The cases where $\mathcal{P}_n(x_0x_jx_kx_l)$ is not combinatorially aspherical follow from the remaining theorems. Theorem 3.2.21 handles the situation where condition (A) is false and condition (C) is true, corresponding to lines five and six of the table. Theorem 3.2.23(b) handles the situation where conditions (A) and (B) are true and condition (C) is false, corresponding to line seven of the table. Finally, Theorem 3.2.32(b) handles the situation where conditions (A) and (C) are false and condition (B) is true, corresponding to line eight of the table. Lemma 3.2.14(b) and Theorem 3.2.21 together handle the situation in claim (a) to show that $G_n(x_0x_jx_kx_l) \cong \mathbb{Z}_4$. Theorem 3.2.23(c) handles the situation in claim (b) to show that $G_n(x_0x_jx_kx_l)$ is solvable. The group $G_5(x_0x_3x_1x_1)$ determined by the exemplar for type (I5) is a normal subgroup of index five in its shift extension, which is the group K of Lemma 3.2.28. Thus $G_5(x_0x_3x_1x_1)$ is metacyclic of order 220 and in fact is a semidirect product $\mathbb{Z}_5 \rtimes \mathbb{Z}_{44}$. As a nonabelian group with cyclic abelianization, it is not nilpotent. As in the proof of Theorem 3.2.31, the groups

$$G_1 = G_6(x_0x_4x_2x_3) \quad \text{and} \quad G_2 = G_6(x_0x_0x_1x_2)$$

determined by presentations of types (I6$'$) and (I6$''$) respectively have shift extensions isomorphic to the group L as in Lemma 3.2.30. The group L has order 24530688 with second derived subgroup L'' isomorphic to the simple group $\mathrm{PSL}(3,3)$ of order 5616. Both G_1 and G_2 are thus nonsolvable of order 4088448 and contain $L'' \cong \mathrm{PSL}(3,3)$. Note that the elements $-1 \in \mathbb{Z}_6^*$ and $\tau^{-1} \in \Gamma_6$ act as in Lemma 3.2.1 on the words $x_0x_4x_2x_3$ and

$x_0x_0x_1x_2$:

$$x_0x_4x_2x_3 \overset{\times -1}{\to} x_0x_{-4}x_{-2}x_{-3} = x_0x_2x_4x_3$$

and

$$x_0x_0x_1x_2 \overset{\tau^{-1}}{\to} x_0x_1x_2x_0 \overset{\times -1}{\to} x_0x_{-1}x_{-2}x_0 = x_0x_5x_4x_0.$$

This implies $G_1 \cong G_6\left(x_0x_2x_4x_3\right)$ and $G_2 \cong G_6\left(x_0x_5x_4x_0\right)$ by Theorem 3.2.2. As in Example 2.3.2, the groups $G_1^{\mathrm{ab}} \cong \mathbb{Z}_8$ and $G_2^{\mathrm{ab}} \cong \mathbb{Z}_7 \times \mathbb{Z}_8$ are not isomorphic, and thus $G_1 \ncong G_2$. Finally, the claims regarding the exemplars of presentation types (I5), (I6′), and (I6″) extend to any group determined by a cyclic presentation of any of these respective types via the Γ_n-action in Theorem 3.2.2. This completes the proof of the theorem.

For the proof of Theorem C, since cyclic presentations with positive relators are orientable, Theorem 1.1.1 indicates that combinatorial asphericity always implies freeness of the shift action on the nonidentity elements. Following the proof of Theorem A above, the situations where the presentation $\mathcal{P}_n\left(x_0x_jx_kx_l\right)$ is not combinatorially aspherical are handled in Theorems 3.2.18, 3.2.23(a), and 3.2.32(a). In each of these theorems it is shown that if the presentation $\mathcal{P}_n\left(x_0x_jx_kx_l\right)$ is not combinatorially aspherical, then the $\mathbb{Z}_n$-action via the shift on the nonidentity elements of the group $G_n\left(x_0x_jx_kx_l\right)$ is not free. This completes the proof of the theorem.

For the proof of Theorem D, by following the proof of Theorem B above, the treatment of finite groups is handled in Lemma 3.2.14(b) and Theorems 3.2.21, 3.2.23(b), and 3.2.32(b). In each of these results it is shown that either the shift action is trivial (as when $n = 1$ in Lemma 3.2.14(b) or as in Theorem 3.2.21), or else it has been shown that the shift automorphism has a nonidentity fixed point (as in Theorems 3.2.23(b) and 3.2.32(b)). Now suppose that condition (C) is true and $\gcd(n, 2k) = 1$. This implies that n is odd and $\gcd(n, k) = 1$ which, in turn, implies that either $n = 1$ or condition (A) is false since

$$n = 1 \iff 2k \equiv 0 \iff k \equiv 0 \iff j \equiv l \iff 2j \equiv 2l$$

where all equivalences are considered modulo n. In the situation that $n \neq 1$ and condition (A) is false, then the shift on $G_n(x_0x_jx_kx_l)$ has a nonidentity fixed point by Theorem 3.2.20. Conversely, if the shift automorphism on the group $G_n(x_0x_jx_kx_l)$ has a nonidentity fixed point, then Theorems A and 1.1.1 imply that at least one of the following must be true:

- $n = 1$;
- Condition (A) is false and condition (C) is true;
- Conditions (A) and (B) are true and condition (C) is false; or
- The presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is of type (I*) or type (U*).

If $n = 1$, then the group $G_n(x_0x_jx_kx_l) = G_1\left(x_0^4\right)$ is finite. If condition (A) is false and condition (C) is true, then Theorem 3.2.20 implies $\gcd(n, 2k) = 1$. If conditions (A) and (B) are true and condition (C) is false, then Theorem 3.2.23(b) implies that the group $G_n(x_0x_jx_kx_l)$ is finite. For the final case, suppose that the presentation $\mathcal{P}_n(x_0x_jx_kx_l)$ is of type (I*) or type (U*). Corollary 3.2.8(b) implies that the primary divisor $d = 1$, thus $\mathcal{P}_n(x_0x_jx_kx_l)$ is of type (I) or type (U). Lemma 3.2.24 then implies that conditions (A) and (C) are false and condition (B) is true. Finally, Theorem 3.2.32(b) shows that the group $G_n(x_0x_jx_kx_l)$ is finite. This completes the proof of the theorem.

To prove Theorem E, let $G = G_n(x_0x_jx_kx_l)$ and suppose that $\mathrm{Fix}(\theta_G) \neq 1$ where G is infinite (otherwise, take $G = H$). By Theorem D, condition (C) is true and $\gcd(n, 2k) = 1$. By Lemma 3.2.11, there exists an integer p such that the shift extension $E = G \rtimes_{\theta_G} \mathbb{Z}_n$ is determined by a presentation of the form

$$\left\langle a, u, z : a^n, ua^kua^{-k}, ua^{2p} = z^2 \right\rangle$$

where $G = \ker \nu$ for a retraction $\nu : E \to \mathbb{Z}_n$ (where $\mathbb{Z}_n = \langle a \rangle$ is the cyclic group of order n generated by a) satisfying $\nu(a) = a$, $\nu(u) = 1$, and $\nu(z) = a^p$. Finally, since

G is infinite, Theorem 3.2.21 implies that $\gcd(n, p) \neq 1$. Now, $E \cong D *_Y Z$ where the group $D \cong \mathbb{Z}_{2n}$ is determined by the presentation $\langle a, u : a^n, ua^k ua^{-k} \rangle$, the group $Z = \langle z \rangle$ is the cyclic group generated by z, and the group $Y = \langle ua^{2p} \rangle = \langle z^2 \rangle$ is the amalgamated subgroup. As in [3, Theorem 2.3], it follows that $G = \ker \nu$ contains the finite group $H = \ker \nu|_D \cong G_n(u_0 u_k) \cong \mathbb{Z}_2$, and so $\mathrm{Fix}(\theta_G)$ contains $\mathrm{Fix}(\theta_H) = \mathbb{Z}_2 \neq 1$. It remains to show that $\mathrm{Fix}(\theta_G) = \mathrm{Fix}(\theta_H)$, so assume that $g \in \mathrm{Fix}(\theta_G) = \mathrm{Cent}_E(a) \cap G$. By the Centralizer Lemma (Lemma 3.2.6), either $g \in D \cap G = H$ or else there exists an element $d \in D$ such that $a \in dYd^{-1}$. The former case implies $g \in \mathrm{Fix}(\theta_H)$ as desired. However, note that $\gcd(n, p) \neq 1$, so $\nu(ua^{2p})$ generates a proper subgroup of $\nu(E) = \mathbb{Z}_n = \langle \nu(a) \rangle$, and so $a \notin dYd^{-1}$ for any element $d \in D$. Thus $\mathrm{Fix}(\theta_G) = \mathrm{Fix}(\theta_H)$, completing the proof of the theorem.

4 GAP

Much of the information about specific GAP functions comes from the GAP manual which contains far more information than one will find here. This chapter is not intended to be a replacement in any sense.

4.1 Proofs for Claims Relying on GAP

Several claims found in Example 2.3.2, Lemmas 3.2.27, 3.2.28, 3.2.29, and 3.2.30, and Chapter 5 rely on computations performed using GAP [15]. Each claim is backed by the results of functions that come as part of the standard distribution of GAP. A log showing the specific inputs and outputs of the relevant functions is provided below for those who wish to verify the claims for themselves. Some additional commentary is included for those who may be unfamiliar with syntax of the GAP language.

Remark: The discovery of the word v in Equation 3.2 comes from new GAP functions [23] originally written to overcome the computational complexity of the `Centralizer` function that was encountered while analyzing the group L in Lemma 3.2.30. See Section 4.2. However, the computations to verify the claims that the word v lies in the centralizer $\mathrm{Cent}_L(t)$ of t and that v has order 12 in L do not rely on any of the new functions, and thus are included below.

First, a free group `free` of rank two is defined. Due to the way GAP handles groups, the generators of `free` are assigned to the variables `free.1` and `free.2` by default. The second line below simply assigns the variables `t` and `u` to the same respective generators for convenience. It is standard syntax for each GAP command to end in either a single or double semicolon, the difference being that the output of a command (if any) is suppressed when followed by a double semicolon.

```
gap> free := FreeGroup("t","u");;
gap> t := free.1;; u := free.2;;
```

For each of the groups $\widehat{E} = J_4$, K, and J_6, the same sequence of functions is used to define the group, its derived subgroup, and the centralizer $\mathrm{Cent}_{\widehat{E}}(t)$. Like with the group assigned to the variable `free`, the variables `J4.1`, `K.1`, and `J6.1` refer to the image of `t` in each respective finitely presented group assigned to the variables `J4`, `K`, and `J6`.

The `Size` function outputs the order, whether finite or infinite, of a given input group. The `StructureDescription` function gives a (non-unique) description of the group given as its input. The description it provides is intended to make the group more easily understood. For the outputs encountered here, the character `C` followed by a number n indicates a cyclic group of order n, and in the case of the group $\mathrm{Cent}_K(t)$ assigned to the variable `Cent_K_t`, the character `x` indicates a Cartesian product.

```
gap> J4 := free/[t^4, u^3*t^2*u*t];;
gap> J4prime := DerivedSubgroup(J4);;
gap> Cent_J4_t := Centralizer(J4,J4.1);;
gap> Size(J4);
272
gap> StructureDescription(J4/J4prime);
"C16"
gap> StructureDescription(J4prime);
"C17"
gap> StructureDescription(Cent_J4_t);
"C16"

gap> K := free/[t^5, u^3*t^2*u*t];;
gap> Kprime := DerivedSubgroup(K);;
gap> Cent_K_t := Centralizer(K,K.1);;
gap> Size(K);
```

```
1100
gap> StructureDescription(K/Kprime);
"C20"
gap> StructureDescription(Kprime);
"C55"
gap> StructureDescription(Cent_K_t);
"C5 x C5"
```

```
gap> J6 := free/[t^6, u^3*t^3*u*t^2];;
gap> J6prime := DerivedSubgroup(J6);;
gap> Cent_J6_t := Centralizer(J6,J6.1);;
gap> Size(J6);
4632
gap> StructureDescription(J6/J6prime);
"C24"
gap> StructureDescription(J6prime);
"C193"
gap> StructureDescription(Cent_J6_t);
"C24"
```

Due to the large size of the group L, the coset enumeration of the `Size` function on the variable `L` takes much longer to complete than in the previous cases, and even longer for the groups assigned to the variables `Q`, `P_4`, and `P_6`; on the same computer, `Size(L)` took an average of 64308 ms to complete, `Size(Q)` took an average of 268361 ms, `Size(P_4)` took an average of 134706 ms, and `Size(P_6)` took an average of 330092 ms, with each computation taken over three separate trials. See Section 4.3. For comparison, the prior `Size` computations for the groups above each took only a couple milliseconds to complete.

```
gap> L := free/[t^6, u^3*t^3*u*t];;
gap> v := u*t^4*u^2*t^2*u*t^4*u*t^2*u*t^5*u*t^4*u*t^4*u^2;;
```

```
gap> Q := free/[t^6, u^3*t^3*u*t, t*v*t^-1*v^-1];;
gap> P_4 := free/[t^6, u^3*t^3*u*t, v^4];;
gap> P_6 := free/[t^6, u^3*t^3*u*t, v^6];;
gap> Size(L);
24530688
gap> Size(Q);
24530688
gap> Size(P_4);
4368
gap> Size(P_6);
12265344
```

A number of libraries of predefined groups are part of the standard distribution of GAP. One such library contains a large number of perfect groups. Associated with this library is the function `PerfectIdentification` that is used to output the library's internal index (a list of two integers) of a given input of a perfect group. In particular, the first entry in the internal index is the order of the input group.

In addition to the libraries, GAP includes a number of functions that can be used to determine whether a given group satisfies a host of different properties; for example, the functions `IsPerfect` and `IsSimple` are used to determine whether a given group is perfect or simple respectively. Although it is well-known that the group $\mathrm{PSL}(3,3)$ is perfect and simple, GAP can also verify these facts.

```
gap> Lprime := DerivedSubgroup(L);;
gap> Lprimeprime := DerivedSubgroup(Lprime);;
gap> IsPerfect(Lprimeprime);
true
gap> PerfectIdentification(Lprimeprime);
[ 5616, 1 ]
gap> IsPerfect(PSL(3,3));
```

```
true
gap> IsSimple(PSL(3,3));
true
gap> PerfectIdentification(PSL(3,3));
[ 5616, 1 ]
```

The group L has two retractions $\nu^f : L \to \mathbb{Z}_n$ with $f = 2, 5$ (as in Example 2.3.2) with corresponding cyclically presented kernels $\ker \nu^2 = G_6(u_0u_2u_4u_3)$ and $\ker \nu^5 = G_6(u_0u_5u_4u_0)$. Each kernel $\ker \nu^f$ is a subgroup of L with generating set $\{t^i u t^{-i-f} : i = 0, \ldots, 5\}$ by [3, Lemma 2.2]. The `Subgroup` function is used output the subgroup of a given input group generated by a set of elements from the group as a second input.

The `AbelianInvariants` function produces a list of non-negative integers that give the order of each cyclic factor in the abelianization of the given input group and where a value of `0` indicates an infinite cyclic factor. As before, the variables `L.1` and `L.2` refer to the images of `t` and `u` respectively in the quotient group assigned to the variable `L`.

```
gap> G6_2gens := [];;
gap> G6_5gens := [];;
gap> for i in [0..5] do
> Add(G6_2gens, L.1^i*L.2*L.1^(-2-i));;
> Add(G6_5gens, L.1^i*L.2*L.1^(-5-i));;
> od;;
gap> G6_2 := Subgroup(L, G6_2gens);;
gap> G6_5 := Subgroup(L, G6_5gens);;
gap> AbelianInvariants(G6_2);
[ 8 ]
gap> AbelianInvariants(G6_5);
[ 7, 8 ]
```

Finally, Chapter 5 contains a discussion regarding the cyclic presentations of type

(U*) wherein several claims are made regarding groups $G_1 = G_{24}(x_0x_3x_6x_1)$ and $G_2 = G_{24}(x_0x_1x_2x_{19})$ (whose cyclic presentations are the exemplars of types (U24′) and (U24″) respectively) and their corresponding shift extensions $E_1 = E_{24}(x_0x_3x_6x_1)$ and $E_2 = E_{24}(x_0x_1x_2x_{19})$. For convenience, the claims involving the shift extensions are addressed here first and involve GAP functions used previously. The generators a and x will also be replaced with `t` and `u` respectively.

```
gap> E1 := free/[t^24, u*t^3*u*t^3*u*t^-5*u*t^-1];;
gap> E1prime := DerivedSubgroup(E1);;
gap> E2 := free/[t^24, u*t*u*t*u*t^17*u*t^-19];;
gap> E2prime := DerivedSubgroup(E2);;
gap> AbelianInvariants(E1prime);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 5, 5, 5, 5, 5, 5, 5, 5, 7, 7, 7, 7 ]
gap> AbelianInvariants(E2prime);
[ 0, 0, 0, 0, 0, 0, 0, 0, 0, 3, 3, 3, 3, 73, 73, 73, 73 ]
```

The group G_1 (respectively G_2) embeds into E_1 (respectively E_2) as the subgroup generated by $\{x, axa^{-1}, \ldots, a^{23}xa^{-23}\}$. The computations continue with the substitutions `t` and `u` for a and x respectively.

```
gap> G1gens := [];;
gap> G2gens := [];;
gap> for i in [0..23] do
> Add(G1gens, E1.1^i*E1.2*E1.1^-i);;
> Add(G2gens, E2.1^i*E2.2*E2.1^-i);;
> od;;
gap> G1 := Subgroup(E1, G1gens);;
gap> G2 := Subgroup(E2, G2gens);;
gap> AbelianInvariants(G1);
[ 0, 0, 0, 5, 5, 7 ]
gap> AbelianInvariants(G2);
```

```
[ 0, 0, 0, 3, 73 ]
```

4.2 New GAP Functions

4.2.1 Motivation and Approach Taken

The large order of the group L determined by the presentation $\langle t, u : t^6, u^3t^3ut \rangle$ (as in Lemma 3.2.30) makes identifying elements in the centralizer $\mathrm{Cent}_L(t)$ difficult. Using the eponymous `Centralizer` function to perform this computation necessitates, in the case of L, allocating a large quantity of memory (greater than six gigabytes) to GAP for the computation to complete.

Remark: It is currently unknown what amount of memory GAP requires in order for the `Centralizer` computation to complete when computing $\mathrm{Cent}_L(t)$. When the computation was attempted, GAP produceed an error that provided an option either to increase the amount of memory allocated to GAP or otherwise to terminate the computation. The computer on which the computation was taking place has a limited amount of memory, and it was decided that allocating more than six gigabytes of memory to GAP was not worthwhile considering the following alternative was more efficient in terms of both runtime and memory usage.

The other approach for identifying the centralizer $\mathrm{Cent}_L(t)$ relies on performing a Todd-Coxeter coset enumeration of the (right) cosets of $C = \langle t \rangle$ in L. As a matter of fact, many of the functions included in the standard GAP distribution cause such a coset enumeration to take occur while working with a finitely presented group, including, for example, the `Size` function when the number of relators of the presentation is greater than or equal to the number of generators. The following result can then be applied.

Lemma 4.2.1 *Let E, H, and K be groups with $H \leq E$ and K abelian. Suppose there exists a homomorphism $\varphi : E \to K$ such that its restriction $\varphi|_H$ to H is injective. Given* *elements $h \in H$ and $e \in E$, then $He = Heh$, if and only if $e \in \mathrm{Cent}_E(h)$.*

Proof Suppose $He = Heh$. Then $eh \in He$, so there exists an element $h' \in H$ such that $eh = h'e$. Observe that

$$\varphi(e)\varphi(h) = \varphi(eh) = \varphi(h'e) = \varphi(h')\varphi(e) = \varphi(e)\varphi(h').$$

Since $\varphi|_H$ is injective, then $h = h'$, and hence $eh = h'e = he$. Therefore $e \in \mathrm{Cent}_E(h)$. The converse is obvious. •

Consider a finite group E determined by an ordinary presentation of the form $\langle t, u : t^n, W(t,u)\rangle$. If there exists a map $E \to \mathbb{Z}_{kn}$ with $t \mapsto k$ for some positive integer k, then an analysis of the (right) cosets of $C = \langle t\rangle$ is sufficient to determine the centralizer $\mathrm{Cent}_E(t)$. In particular, note that the shift extension $E_n(w)$ of the cyclically presented group $G_n(w)$ naturally meets these criteria.

Remark: It is also important to note that the coset enumeration algorithm completes in a finite number of steps only if the number of cosets is finite. In the setting given in the previous paragraph, since the subgroup $C = \langle t\rangle$ is finite, this implies the coset enumeration of $C\backslash E$ will complete in GAP only if the shift extension $E = E_n(w)$ is finite.

4.2.2 Identifying $\mathrm{Cent}_L(t)$ Using New GAP Functions

Consider a finitely presented group E with subgroup C of finite index m. Underlying GAP's enumeration of the (right) coset space $C\backslash E$ is an assignment of each coset Ce (where $e \in E$) to an index value $1, \ldots, m$ via a bijection $C\backslash E \to \{1, \ldots, m\}$ chosen by GAP. In every such coset enumeration, the index value 1 is assigned to the subgroup $C = C1_E$. The functions in GAP that utilize a coset enumeration use these index values both internally and externally (as in an output) to refer to the cosets of C in E. One of the most useful of these is the `CosetTable` function which produces a $2p \times m$ matrix $M_{E,C}$ (where p is the number of generators in the presentation for E) where each row is a permutation of the indexing set $\{1, \ldots, m\}$. The matrix $M_{E,C}$ gives a representation of the (right) action of E on the quotient $C\backslash E$. To understand the representation given by $M_{E,C}$, first enumerate the generators of E as $\{t_1, \ldots, t_p\}$ in the same order as entered into GAP. The entries in column j give, in order by row index, the index values of the cosets Ce_jt_1, $Ce_jt_1^{-1}$, ..., Ce_jt_p, $Ce_jt_p^{-1}$ where $e_j \in E$ is an element contained in the coset of C associated with the index value j.

In the case of the group L, it is relatively simple to use the matrix created by the `CosetTable` function to determine the number of elements in the centralizer $\mathrm{Cent}_L(t)$; Lemma 4.2.1 implies that one need only count the number of columns of $M_{L,\langle t\rangle} = (M_{ij})$ in which the entry $M_{i_tj} = j$ where the row i_t corresponds to the generator t, then multiply that result by six since $t \in L$ has order six. However, this count does not explicitly identify the elements within the cosets themselves nor does it identify the structure of the centralizer $\mathrm{Cent}_L(t)$.

The standard GAP distribution includes the `TracedCosetFpGroup` function which computes the product Cev given a coset Ce and a word $v = v(t_1, \ldots, t_p) \in E$ (where GAP works with the two cosets in terms of their index values), but it lacks an "inverse" function to produce a word $e_j = e_j(t_1, \ldots, t_p) \in E$ from a given integer $1 \leq j \leq m$ such that the coset Ce_j is assigned the index value j. Such functionality is provided by combining the `MakeWordlist` and `MakeWord` functions in [23]. In the case of the group L, this function can be used to explicitly identify the words in the centralizer $\mathrm{Cent}_L(t)$ and allows for the creation of the group in GAP using the `Subgroup` function. The following log shows how the new GAP functions identify $\mathrm{Cent}_L(t)$ along with some relevant commentary regarding the functions used from [23].

The `Read` function causes GAP to read a file as an input stream, executing each command in the file as though it were typed into the GAP prompt.

```
gap> Read("cpgtools.gap");;
gap> free := FreeGroup("t","u");;
```

```
gap> t := free.1;; u := free.2;;
gap> L := free/[t^6, u^3*t^3*u*t];;
gap> t := L.1;; u := L.2;;
gap> Z6 := Subgroup(L,[t]);;
```

GAP includes a mechanism to prevent coset enumerations from "running away" when such an enumeration will require a large number of steps to complete. This is set by the `CosetTableDefaultMaxLimit` variable which controls the maximum number of cosets allowed in an enumeration. It has a default value of 4096000. Even though $\mathbb{Z}_6 = \langle t \rangle$ has 4088448 cosets in L, the default value is not sufficient for the enumeration to complete. To compensate in this case, the default value is being multiplied by 1000.

```
gap> CosetTableDefaultMaxLimit := 4096000000;;
```

The remaining commands found in this log all contain functions from [23]. The `MakeUnified` function creates a list that contains the matrix from `CosetTable(L,Z6)` together with additional data to be used in choosing elements $e_j \in L$ such that the coset $\langle t \rangle e_j$ has index value j in GAP.

```
gap> U := MakeUnified(CosetTable(L,Z6),6);;
```

The `CentralizingIndices` function takes as an input the output of `MakeUnified` and an integer p to output a list of index values for the cosets containing elements that commute with $t^p \in L$. Observe that `centralizingindices` below contains 12 entries which implies that $\mathrm{Cent}_L(t)$ contains 72 elements.

```
gap> centralizingindices := CentralizingIndices(U,1);
[ 1, 1072887, 1750306, 2337191, 2841805, 3316281, 3369220, 3400320,
  3905641, 4084176, 4086124, 4088353 ]
gap> Size(centralizingindices);
12
```

The `MakeWordlist` and `MakeWord` functions work together to create a word $e_j \in L$ (in terms of specified input characters; in this case, `t` and `u`) such that the coset $\langle t \rangle e_j$ has index value j in GAP.

```
gap> wordlist := MakeWordlist(U,centralizingindices[4]);
[ 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1,
    1, 0, 1, 1, 1, 1, 0, 1, 1, 1, 1, 0, 0 ]
gap> v := MakeWord(wordlist,"t","u");
"u*t^4*u^2*t^2*u*t^4*u*t^2*u*t^5*u*t^4*u*t^4*u^2"
```

Let $e_j \in L$ be a word in L such that the coset $\langle t \rangle e_j$ has index value j in GAP. The `MakePowers` function creates the list of index values of the cosets $\langle t \rangle e_j, \langle t \rangle e_j^2, \ldots$ which terminates when $e_j^p \in \langle t \rangle$, hence the final entry is always `1`. Observe that `powers` below contains 12 entries which implies that 12 divides the order of the word `v` $= v \in L$.

```
gap> powers := MakePowers(U,wordlist);
[ 2337191, 3369220, 1750306, 4086124, 3905641, 4088353, 3400320,
   4084176, 1072887, 3316281, 2841805, 1 ]
gap> Size(powers);
12
```

Since the group $\langle v \rangle$ has order at least 12, the group $\langle t \rangle$ has order six, and $\langle t, v \rangle$ has order 72 and is a subset of the centralizer $\mathrm{Cent}_L(t)$ which also has order 72, it must be the case that $\mathrm{Cent}_L(t) = \langle t, v \rangle \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}$.

4.3 Timing of GAP Functions

Table 4.1 shows the time taken for GAP to complete each of the selected functions found in Chapter 4. Each function was executed three distinct times. Since GAP contains functionality to save the results of particular functions, each execution was performed with a new instance of each input. The times were obtained using the `time` command.

GAP command	Runtime (in ms)
`Size(L)`	64512
	64012
	64402
`Size(Q)`	98779
	99274
	607032
`Size(P_4)`	72743
	72602
	258773
`Size(P_6)`	443729
	102889
	443658
`CosetTable(L,Z6)`	474982
	424447
	422080

TABLE 4.1: Runtimes for select GAP functions

It is interesting to note that the `Size` computations for the groups Q, P_4, and P_6 each had one execution that had a highly varying runtime from the other two. The cause of this is unknown to the author, but there is little reason to believe it was due to other operations taking place on the same computer. The computer these computations were completed on was given a clean installation of its operating system, disconnected from all networks, had no additional software installed on it (other than GAP), and only GAP was running at the time the commands were executed. Further understanding of the way in which the `Size` function chooses a subgroup with which to perform the coset enumeration would be warranted, but this is beyond the scope of this dissertation.

4.4 Closing Remarks

Recall that the GAP functions contained in [23] are not necessary for using GAP to determine the existence of elements of the centralizer $\mathrm{Cent}_L(t)$ which are not powers of t, but they do enable GAP to identify the specific elements of L that commute with t. This, in turn, has the effect of lowering the memory requirements and runtime of the GAP commands necessary to verify the existence of elements in $\mathrm{Cent}_L(t)$ that are not themselves a power of t should one wish to test the validity of these claims for oneself. In particular, the `CosetTable` function took an average of 440503 ms to complete over three trials, whereas the `Size` computation of the group Q took an average of 268361 ms, with one outlying runtime causing this value to be high. See Section 4.3. The latter computation is clearly both significantly more efficient in terms of runtime and - personally - far more mathematically satisfying. The claim that v is not a power of t is made by noting that u has exponent sum $10 \equiv 2 \mod 4$ in v, and hence v maps naturally to a nonidentity element of the quotient $L/\langle\langle t\rangle\rangle_L \cong \mathbb{Z}_4$ which is determined by the presentation $\langle u : u^4 \rangle$, therefore not requiring any additional computation with GAP.

In addition to the functions presented in Section 4.2.2, there are a number of functions included in [23] that, when applicable, utilize the group structure (corresponding to a right group action) in [3, Lemma 2.2] by implicitly applying it to the data in the matrix M from the `CosetTable` function. This allows these functions to perform computations on the cyclically presented kernel without the restrictions of having to work within the data structure within GAP that contains the shift extension nor of having to create a separate data structure within GAP that contains the kernel as a finitely presented group.

5 CONCLUDING REMARKS

It would be of interest to see a resolution of Theorems A and C for the presentations $\mathcal{P}_n(x_0x_jx_kx_l)$ of type (U*). This comes down to consideration of the two exemplars from Table 1.3:

$$\mathcal{P}_{24}(x_0x_3x_6x_1) \text{ and } \mathcal{P}_{24}(x_0x_1x_2x_{19}).$$

The secondary divisor $d = 4$ for the parameters of both presentations, so by the Secondary Divisor Criterion (Theorem 3.2.7), the groups defined by these presentations are infinite and their shifts are fixed point free. It is unknown at this time whether these presentations are combinatorially aspherical or whether either shift action is free on the nonidentity elements of the corresponding groups. However, the groups $G_1 = G_{24}(x_0x_3x_6x_1)$ and $G_2 = G_{24}(x_0x_1x_2x_{19})$ are not isomorphic as computations in GAP [15] indicate that they abelianize to

$$G_1^{\mathrm{ab}} \cong \mathbb{Z}^3 \times \mathbb{Z}_5^2 \times \mathbb{Z}_7 \text{ and } G_2^{\mathrm{ab}} \cong \mathbb{Z}^3 \times \mathbb{Z}_3 \times \mathbb{Z}_{73}.$$

The associated shift extensions $E_1 = E_{24}(x_0x_3x_6x_1)$ and $E_2 = E_{24}(x_0x_1x_2x_{19})$ determined by the presentations

$$\left\langle a, x : a^{24}, xa^3xa^3xa^{-5}xa^{-1} \right\rangle \text{ and } \left\langle a, x : a^{24}, xaxaxa^{17}xa^{-19} \right\rangle$$

are also not isomorphic, having second derived quotients

$$E_1'/E_2'' \cong \mathbb{Z}^9 \times \mathbb{Z}_5^8 \times \mathbb{Z}_7^4 \text{ and } E_2'/E_2'' \cong \mathbb{Z}^9 \times \mathbb{Z}_3^4 \times \mathbb{Z}_{73}^4.$$

See Section 4.1.

Recall that when a cyclic presentation $\mathcal{P}_n(w)$ is combinatorially aspherical, the shift θ_G is an outer automorphism of the group $G = G_n(w)$. It would be of interest to explore the extent to which that is true for those cyclic presentations which are not combinatorially aspherical. Some other areas that may be worth studying include the relation between the order of the shift and the size of the orbits of the shift action and the structure of the fixed point subgroups $\mathrm{Fix}\left(\theta_G^p\right)$, in particular when $p = 1$.

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