INDEX AND ZEROS OF A FUNCTION OF A COMPLEX VARIABLE

by

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INTRODUCTION

STATEMENT OF THE PROBLEM. The purpose of this thesis is to consider various methods of determining the number and location of zeros of an analytic function of a complex variable.

METHOD OF INVESTIGATION. The methods for investigating these questions will involve mostly the geometric operations with complex numbers and certain principles which are based upon these operations.

The thesis begins with the definition of Cauchy Index. Theorems are derived with the aid of the index concerning the number and location of the zeros of an analytic function.

A lemma on index is derived to aid in determining the number and location of the zeros of a polynomial in the unit circle. In Chapter IV an interesting result is obtained concerning multiple zeros of a function.

Finally, a definition of the level curves of a complex variable is given with theorems and results obtained concerning the zeros and multiple points of the level curves. This is followed by an illustrated example of the level curves of a function.

INDEX AND ZEROS OF A FUNCTION OF A COMPLEX VARIABLE

CHAPTER I

CAUCHY INDEX

DEFINITION OF INDEX (CAUCHY). Let $g(t)$ and $h(t)$ be two continuous functions of the variable t which varies in the interval $a \leq t \leq b$. The functions $g(t)$ and $h(t)$ do not vanish simultaneously and $g(a) \neq 0$, $g(b) \neq 0$. As t varies from a to b the quotient $h(t)/g(t)$ may become infinite and change its sign. Let m denote the number of times it changes from $+\infty$ to $-\infty$, and n denote the number of times it changes from $-\infty$ to $+\infty$. The semi-difference $(m - n)/2$ will be called the <u>index</u> of $h(t)/g(t)$ in $[a,b]$ and will be denoted by

$$I_{[a,b]}(h/g) \quad \text{or simply} \quad I(h/g).$$

Theorem 1. There exists a unique continuous function ϕ of the variable t satisfying

1) $\tan \phi = h(t)/g(t)$

and for $t = a$ coinciding with the principal value of $\arctan [h(a)/g(a)]$.

Proof(7,p.183): Let T be an arbitrary number of the interval $[a,b]$ for which $g(T) \neq 0$. If in the interval $a \leq t \leq T$ the quotient $h(t)/g(t)$ changes from $+\infty$ to $-\infty$ $m(T)$ times, and from $-\infty$ to $+\infty$ $n(T)$ times, then

2) $\phi = \arctan[h(T)/g(T)] + \pi[m(T) - n(T)]$

is the unique continuous solution of the equation

3) $\tan\phi = h(t)/g(t)$

reducing to $\arctan[h(a)/g(a)]$ for $T = a$.

When t varies from a to b the total variation in ϕ is

$$\phi(b) - \phi(a) = \arctan[h(b)/g(b)] - \arctan[h(a)/g(a)] + \pi[m(b) - n(b)]$$

or simply

$$\phi(b) - \phi(a) = \arctan[h(b)/g(b)] - \arctan[h(a)/g(a)] + 2\pi I(h/g).$$

Q.E.D.

Corollary 1. Let C be a simple closed curve such that $h(b) = h(a) \neq 0$, and $g(b) = g(a) \neq 0$. Then the total variation in ϕ along C is

$$2\pi I(h/g).$$

Proof: From the results of the preceding theorem,

1) $\phi(b) - \phi(a) = \arctan[h(b)/g(b)] - \arctan[h(a)/g(a)] + 2\pi I(h/g).$

But since C is a simple closed curve

2) $\arctan[h(b)/g(b)] = \arctan[h(a)/g(a)]$

therefore

$$\phi(b) - \phi(a) = 2\pi I(h/g).$$

Q.E.D.

Let $f(z) = g(t) + ih(t)$ be an analytic function of the complex variable z within an open region D; continuous on its boundary C and not vanishing on C. The boundary can be represented parametrically by the continuous functions,

$x = x(t)$; $y = y(t)$

where in order to describe C in the positive sense, t varies from a to b.

At any point z_o of C, the complex number $f(z_o)$ has an infinite number of amplitudes, which differ from each other by multiples of 2π. If $a_o + ib_o$ is any one of the logarithms of $f(z_o)$, then b_o is an amplitude of $f(z_o)$. This is because

$$f(z_o) = e^{a_o + ib_o} = e^{a_o} e^{ib_o}$$

$$= e^{a_o}(\cos b_o + i \sin b_o).$$

The real and imaginary parts of f(z) on C are functions of t:

$g(t)$; $h(t)$

which satisfy the requirements of g(t) and h(t) for discussion of the index of the quotient

$h(t)/g(t)$.

This index does not depend on the way in which the closed curve C is represented and can be called the index of the closed curve C and denoted accordingly by

$I_C(h/g)$.

This index for a closed curve C is obviously always a positive or negative integer or zero. While most theorems concerning index are for closed curves, the next two theorems give interesting results for a straight line.

Theorem 2. Let L be a line in the complex plane on which a given rth degree polynomial $f(z) = g(t) + ih(t)$ has no zeros. Let $\Delta_C \arg f(z)$ denote the net change in arg f(z) as point z traverses L in a specified direction and let p and q denote the number of zeros of f(z) to the left and to the right of this direction of L respectively. Then

$$p - q = 2I(h/g).$$

Proof: If $z_1, z_2, \ldots, z_p$ denote the zeros of f(z) to the left of L relative to a specified direction and $z_{p+1}, z_{p+2}, \ldots, z_q$ denote the zeros of f(z) to the right of L, then

$$1)\quad f(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_p)(z - z_{p+1})\ldots (z - z_q)$$

and

$$2)\quad \arg f(z) = \arg a_n + \arg(z - z_1) + \arg(z - z_2) + \ldots + \arg(z - z_p) + \ldots + \arg(z - z_q).$$

Now the net change in arg f(z) as point z traverses L is given by

3) $\Delta_C \arg f(z) = \sum_{i=1}^{p} \Delta_C \arg (z - z_i) + \sum_{j=p+1}^{q} \Delta_C \arg (z - z_j).$

It is obvious that as the point z traverses L in a specified direction the net change in arg $(z - z_m)$ is π or $-\pi$ according as z_m is to the left or to the right of L relative to the specified direction.

Then

4) $\sum_{i=1}^{p} \Delta_C \arg (z - z_i) = p\pi$

and

5) $\sum_{j=p+1}^{q} \Delta_C \arg (z - z_j) = -q\pi.$

Therefore

6) $\Delta_C \arg f(z) = p\pi - q\pi$

or

7) $p - q = (1/\pi)\Delta_C \arg f(z).$

But by Theorem 1,

8) $\Delta_C \arg f(z) = 2\pi I(h/g)$

then

$p - q = (1/\pi)[2\pi I(h/g)] = 2I(h/g).$

(Note: Above theorem cannot be extended, e.g., e^z)

Q.E.D.

Corollary 2. Let L be a line on which a given r-th degree polynomial $f(z) = g(t) + ih(t)$ has no zeros, and let the point z traverse L under the conditions of Theorem 2. Then

$$p = \tfrac{1}{2}[r + 2I(h/g)]$$

and

$$q = \tfrac{1}{2}[r - 2I(h/g)]$$

Proof: Now by the preceding Theorem 2,

1) $p - q = 2I(h/g)$

and since f(z) is an r-th degree polynomial

2) $p + q = r$

Then adding 1) and 2)

$$2p = r + 2I(h/g)$$

or

$$p = \tfrac{1}{2}[r + 2I(h/g)].$$

Subtracting 1) from 2)

$$2q = r - 2I(h/g)$$

or

$$q = \tfrac{1}{2}[r - 2I(h/g)].$$

Q.E.D.

Theorem 3. Let $f(z) = g(t) + ih(t)$ be analytic interior to a simple closed curve C and continuous and different from zero on C. Let K be the curve described in the W-plane by the point $W = f(z)$ and let $\Delta_C \arg f(z)$ denote the net change in arg f(z) as the point z traverses C once over in the counterclockwise direction.

Then the number p of zeros of f(z) interior to C, counted with their multiplicities, is

$$p = I_C(h/g).$$

That is, it is the number of times that K winds about the point $W = 0$.

Proof(1,p.240): If $z_1, z_2, \ldots, z_p$ denote the zeros of f(z) inside C and $z_{p+1}, z_{p+2}, \ldots, z_n$ denote those outside C, then

$$f(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_p)\ldots(z - z_n)F(z)$$

and

$$\arg f(z) = \arg a_n + \sum_{j=1}^{p} \arg(z - z_j) + \sum_{j=p+1}^{n} \arg(z - z_j)$$
$$+ \arg F(z).$$

Now as the point z describes C counterclockwise, (see Figure I) $\arg(z - z_j)$ changes by 2π when $1 \leq j \leq p$, and has a zero net change when $p < j \leq n$. Also there is a zero net change in $\arg F(z)$. Then

1) $\Delta_C \arg f(z) = 2\pi p.$

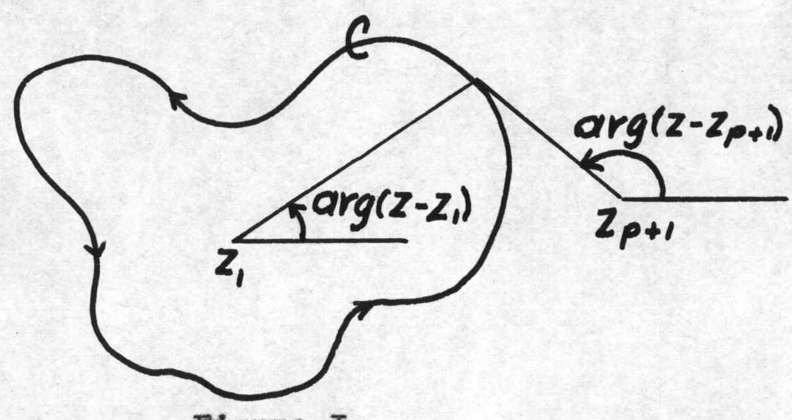

Figure I.

Now the total variation of the argument of $f(z)$ according to Corollary 1 of Theorem 1, is given by

2) $\Delta_C \arg f(z) = 2\pi I_C(h/g)$.

Substituting the value of $\Delta_C \arg f(z)$ from 2) into equation 1), the equation for the number of zeros of $f(z)$ interior to C is

$$2\pi p = 2\pi I_C(h/g)$$

or

$$p = I_C(h/g)$$

that is, the number of zeros of $f(z)$ interior to the closed curve C is simply the index of the quotient

$$h(t)/g(t)$$

Q.E.D.

Theorem 4. Let $P(z) = g(t) + ih(t)$ and $Q(z) = g_1(t) + ih_1(t)$ be analytic interior to a simple closed curve C, continuous and different from zero on C, and

$$|P(z)| < |Q(z)| \quad \text{on } C,$$

and let $F(z) = P(z) + Q(z) = g_2(t) + ih_2(t)$. Then $F(z)$ has the same number of zeros interior to C as does $Q(z)$. That is,

$$I_C(h_1/g_1) = I_C(h_2/g_2)$$

Proof: Let $F(z) = wQ(z)$, where

1) $w = 1 + P(z)/Q(z)$.

Now if q denotes the number of zeros of $Q(z)$ in C, then according to Theorem 3,

2) $(1/2\pi)\,\Delta_C \arg Q(z) = q = I_C(h_1/g_1)$.

Since $|P(z)/Q(z)| < 1$ on C, the point w defined in equations 1) describes (see Figure II) a closed curve Γ which lies interior to the circle with center at $w = 1$ and radius 1. Thus, point w remains always in the

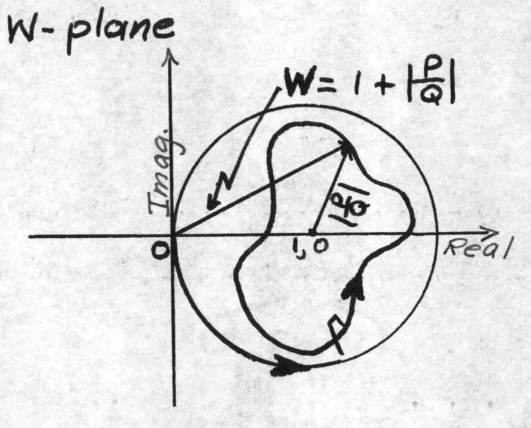

Figure II

right half-plane. The net change in arg w as w varies on Γ is therefore zero. This means according to equations 1) that

3) $\Delta_C \arg F(z) = \Delta_C \arg w + \Delta_C \arg Q(z) = \Delta_C \arg Q(z)$.

Then according to equations 2)

$$(1/2\pi)\,\Delta_C \arg F(z) = q = I_C(h_1/g_1)$$

that is, F(z) has the same number of zeros interior to C as does Q(z), and from the results of Theorem 3,

$$(1/2\pi)\,\Delta_C \arg F(z) = I_C(h_2/g_2)$$

therefore,

$$I_C(h_1/g_1) = I_C(h_2/g_2).$$

Q.E.D.

STATEMENTS OF THEOREMS CONCERNING ZEROS OF ANALYTIC FUNCTIONS. The proofs of the following stated theorems may be found in the literature cited following the statements of the theorems.

Theorem 5. If f(z) is analytic inside and on a closed contour C, and is not zero on the contour, then

$$(1/2\pi i)\int_C \frac{f'(z)}{f(z)}\,dz = N$$

where N is the number of zeros inside the contour (a zero of order m being counted m times). (8,p.115)

Theorem 6. Let C be a simple closed contour, inside and on which f(z) is analytic. Then if $R\{f(z)\}$ vanishes at 2k distinct points on C, f(z) has at most k zeros inside C. (8,p.123)

Theorem 7. If f(z) has n zeros inside C, then f'(z) has n-1 zeros inside C. (8,p.122)

Theorem 8. <u>Rouche's</u> <u>Theorem</u>: If f(z) and g(z) are analytic inside and on a closed contour C, and

$$|g(z)| < |f(z)| \quad \text{on } C,$$

then f(z) and f(z)+ g(z) have the same number of zeros inside C. (8,p.116)

CHAPTER II

A LEMMA ON INDEX

Lemma 1. If none of the four numbers $g(a)$, $g(b)$, $h(a)$, $h(b)$ is zero, then

$$I(h/g) + I(g/h) = \tfrac{1}{2}\delta$$

where

$$\delta = 0 \quad \text{if} \quad [h(a)/g(a)]\,[h(b)/g(b)] > 0,$$

$$\delta = 1 \quad \text{if} \quad h(a)/g(a) > 0,\ h(b)/g(b) < 0,$$

$$\delta = -1 \quad \text{if} \quad h(a)/g(a) < 0,\ h(b)/g(b) > 0.$$

Proof: Let t_1, t_2, ..., t_k be k points where either $g(t)$ or $h(t)$ vanishes and changes its sign. Let $\epsilon = \pm 1$ be a unit of the same sign as $h(a)/g(a)$. By the definition of index

$$I_{[a,t]}(h/g) + I_{[a,t]}(g/h) = 0 \quad \text{if } a < t < t_1$$

and

$$I_{[a,t]}(h/g) + I_{[a,t]}(g/h) = \tfrac{1}{2}\epsilon \quad \text{if } t_1 < t < t_2.$$

Next,

$$I_{[a,t]}(h/g) + I_{[a,t]}(g/h) = \tfrac{1}{2}\epsilon - \tfrac{1}{2}\epsilon \quad \text{if } t_2 < t < t_3$$

and

$$I_{[a,t]}(h/g) + I_{[a,t]}(g/h) = \tfrac{1}{2}\epsilon - \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon \quad \text{if } t_3 < t < t_4.$$

In general

$$I_{[a,b]}(h/g) + I_{[a,b]}(g/h) = \tfrac{1}{2}(\epsilon - \epsilon + \epsilon - \dots + (-1)^k\epsilon) = \begin{cases} \tfrac{1}{2}\epsilon & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

Since k is odd or even according as

$$[h(a)/g(a)]\,[h(b)/g(b)] < 0 \quad \text{or} > 0,$$

the desired result is obtained.

APPLICATION TO RATIONAL BOUNDARY CURVES. If the boundary C is a rational curve, so that

$$x = x(t) \text{ and } y = y(t)$$

are rational functions of t, or composed of a finite number of rational arcs and f(z) is a polynomial, then the index of C can be found by regular operations.

Let S be an arc of a rational curve C. On this arc

$$h(t)/g(t) = P_1/P$$

where P_1 and P are polynomials in t not vanishing simultaneously. Now divide P by P_1 and let Q_1 be the quotient and $-P_2$ the remainder, so that

$$P = P_1Q_1 - P_2$$

Repeat the above using P_1 and P_2 obtaining

$$P_1 = P_2Q_2 - P_3$$

Continuing this process, we obtain a finite number of equations

$$P = P_1Q_1 - P_2$$

$$P_1 = P_2Q_2 - P_3$$

$$\cdots\cdots\cdots$$

$$P_{r-2} = P_{r-1}Q_{r-1} - P_r,$$

where P_r is a constant different from 0.

Now we have

$$I_S(P/P_1) = I_S(Q_1 - \frac{P2}{P_1}) = -I_S(P_2/P_1)$$

Next

$$I_S(P_2/P_1) + I_S(P_1/P_2) = \tfrac{1}{2}\delta_1,$$

where δ_1 is determined by the preceding Lemma 1.

Then

$$I_S(P/P_1) = -\tfrac{1}{2}\delta_1 + I_S(P_1/P_2)$$

Similarly

$$I_S(P_1/P_2) = -\tfrac{1}{2}\delta_2 + I_S(P_2/P_3),$$

where

$$\tfrac{1}{2}\delta_2 = I_S(P_2/P_3) + I_S(P_3/P_2).$$

Continue until

$$I_S(P_{r-2}/P_{r-1}) = -\tfrac{1}{2}\delta_{r-1} + I_S(P_{r-1}/P_r).$$

However, since P_r is a constant,

$$I_S(P_{r-1}/P_r) = 0.$$

Hence

$$I_S(P/P_1) = -\tfrac{1}{2}(\delta_1 + \delta_2 + \ldots + \delta_{r-1})$$

From

$$I_S(P_1/P) + I_S(P/P_1) = \tfrac{1}{2}\delta,$$

it follows that

$$I_S(P_1/P) = \tfrac{1}{2}(\delta + \delta_1 + \ldots + \delta_{r-1}).$$

This sum $\delta + \delta_1 + \ldots + \delta_{r-1}$ has an explanation in terms of the number of variations of sign in the sequence of polynomials.

Let $V(t)$ be the number of variations of sign in the sequence $P(t), P_1(t), \ldots, P_r(t)$. From the definition of $\delta, \delta_1, \ldots, \delta_{r-1}$, it follows that

$$\delta + \delta_1 + \ldots + \delta_{r-1} = V(b) - V(a).$$

Then

$$I_S(P_1/P) = \frac{V(b) - V(a)}{2}.$$

This last result was derived assuming that neither of the sequences

$$P(a), P_1(a), \ldots , P_r(a)$$

$$P(b), P_1(b), \ldots , P_r(b)$$

contains zeros. The result is correct, however, even if this should be the case, provided

$$P(a) \neq 0, P(b) \neq 0.$$

CHAPTER III

THE NUMBER OF ZEROS OF A POLYNOMIAL IN THE UNIT CIRCLE $|z| = 1$

It is profitable to transform this circle $|z| = 1$ into a semiplane by using the transformation

1) $z = \frac{\varsigma - i}{\varsigma + i}$.

To show that this transformation transforms $|z| = 1$ into the upper half ς-plane it is sufficient to show that the circumference of the circle in the z-plane transforms into the real axis in the ς-plane, that is, for any point on the unit circle in the z-plane, ς has a real value; and that any point interior to the circle in the z-plane transforms into a point in the upper half ς-plane.

Solving 1) for ς,

$$\varsigma = -i \frac{z+1}{z-1}$$

Now on the unit circle in the z-plane

$$z = re^{i\theta} = \cos\theta + i\sin\theta, \text{ since } r = 1.$$

Then

$$\varsigma = -i \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$\varsigma = -i \frac{\cos\theta + 1 + i\sin\theta}{\cos\theta - 1 + i\sin\theta} \cdot \frac{\cos\theta - 1 - i\sin\theta}{\cos\theta - 1 - i\sin\theta}$$

After a slight simplification this reduces to

$$\varsigma = -\frac{\sin\theta}{1-\cos\theta}$$

Therefore ς is real for any point on the unit circle $|z| = 1$.

It is obvious that any point interior to the circle $|z| = 1$ transforms into the upper half ς-plane. Just take any interior point of the unit circle and substitute in the transformation for z. For $z = 0$, the origin in the z-plane, $\varsigma = i$. That is, the origin in the z-plane transforms into the point i in the ς-plane.

In the light of the preceding discussion it is evident that if z/r be written for z in the transformation 1) the result is a general transformation that transforms the circle $|z| = r$ in the z-plane into the upper half ς-plane.

The problem of finding the number of zeros of the polynomial $f(z)$ of degree n which are situated in the domain $|z| < 1$ is equivalent to the problem of finding the number of roots of the equation

$$(\varsigma + i)^n f\left(\frac{\varsigma - i}{\varsigma + i}\right) = F(\varsigma) = P + iP_1 = 0.$$

in the semiplane $\mathcal{Q}(\varsigma) > 0$.

Denoting this number by N we have

$$N = I(P_1/P)$$

the index of P_1/P corresponding to the boundary of a semicircle of sufficiently large radius R to contain all the roots of $F(\zeta) = 0$ with $\mathcal{I}(\zeta) > 0$. If we denote the semicircular arc by C, we have

2) $N = I_{[-R,R]}(P_1/P) + I_C(P_1/P)$

Part A: The index $I_C(P_1/P) = n/2$ if C is a semicircle of sufficiently large radius as mentioned above.

Let $F(\zeta) = \zeta^n + c_1\zeta^{n-1} + \ldots + c_n$,

where $c_j = r_j e^{i\theta_j}$

Now for $\zeta = Re^{i\varphi}$ we have

$$P = R^n \cos n\varphi + r_1 R^{n-1} \cos[\theta_1 + (n-1)\varphi] + \ldots + r_n \cos\theta_n$$

$$P_1 = R^n \sin n\varphi + r_1 R^{n-1} \sin[\theta_1 + (n-1)\varphi] + \ldots + r_n \sin\theta_n.$$

Then

$$P_1/P = \frac{\sin n\varphi + (r_1/R)\sin[\theta_1 + (n-1)\varphi] + \ldots + (r_n/R^n)\sin\theta_n}{\cos n\varphi + (r_1/R)\cos[\theta_1 + (n-1)\varphi] + \ldots + (r_n/R^n)\cos\theta_n}$$

or

3) $P_1/P = \dfrac{\sin n\varphi + \alpha(R,\varphi)}{\cos n\varphi + \beta(R,\varphi)}$ with obvious abbreviations.

Now for an $\epsilon > 0$, arbitrarily small, we can take R_0 so large that

$$|\beta(R,\varphi)| < \epsilon, \qquad \left|\frac{d\beta(R,\varphi)}{d\varphi}\right| < \epsilon,$$

$$|\alpha(R,\varphi)| < \epsilon, \qquad \text{for } R \geq R_0$$

Let $\Delta = \pi/4n$ and divide the interval $(0,\pi)$ into the $2n+1$ intervals

(o,Δ), $(\Delta,3\Delta)$, $(3\Delta,5\Delta)$, ..., $[(4n-3)\Delta,(4n-1)\Delta]$,

$[(4n-1)\Delta,\pi]$.

In the intervals $[(4k-1)\Delta,(4k+1)\Delta]$, $k=0, 1, \ldots, n$,

$$|\cos n\varphi| \leq 1/\sqrt{2}$$

and

$$|\cos n\varphi + \beta(R,\varphi)| > (1/\sqrt{2}) - \epsilon > 0;$$

hence these intervals do not contribute anything to the index of P_1/P. In the intervals $[(4k+1)\Delta,(4k+3)\Delta]$, $k=0, 1, \ldots, n-1$, the numerator of 3) has the sign of $(-1)^k$ and does not vanish, since

$$(-1)^k \sin n\varphi \geq \sin(\pi/4) = 1/\sqrt{2}$$

and

$$(-1)^k \sin n\varphi + (-1)^k\alpha(R,\varphi) > (1/\sqrt{2}) - \epsilon > 0.$$

Similarly the denominator is a monotone function for the derivative of the denominator does not change sign. For

$\varphi=(4k+1)\Delta$

$$(-1)^k [\cos n\varphi + \beta(R,\varphi)] > (1/\sqrt{2}) - \epsilon > 0,$$

and for $\varphi = (4k+3)\Delta$

$$(-1)^{k-1} [\cos n\varphi + \beta(R,\varphi)] > (1/\sqrt{2}) - \epsilon > 0.$$

Therefore, in each of the n intervals, the quotient P_1/P passes through infinity and changes sign from + to - , and

$$I_C(P_1/P) = n/2.$$

Part B: Returning to 2), we now have

$$N = \frac{N_\infty - N_{-\infty} + n}{2}.$$

This is based on the assumption that the equation $f(z) = 0$ has no roots on the unit circle. In order to determine whether this is the case and to see how to proceed in such an eventuality, observe first that the possible root $z = 1$ can easily be discovered and removed, so that we can suppose to begin with that $f(1) \neq 0$. The real roots of $F(\varphi) = 0$ correspond to possible roots on the unit circle. If there are such roots the algorithm described in obtaining the formula

$$I_S(P_1/P) = \frac{V(b) - V(a)}{2}$$

will show this; namely, the first remainder which divides the preceding one will be not a constant but a polynomial P_r containing all the common roots of $P = 0$ and $P_1 = 0$. The number of roots on the unit circle will be exactly given by the degree of P_r. To find the number of roots of $F(\varphi) = 0$ with positive imaginary parts, consider the equation

$$F(\varphi)/P_r = (P/P_r) + i(P_1/P_r) = P' + iP_1'$$

having no real zeros. By division obtain the following

$$P' = P_1'Q_1 - P_2'$$

$$P_1' = P_2'Q_2 - P_3'$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P'_{r-2} = P'_{r-1}Q_{r-1} - P'_r$$

where

$$P'_k = P_k/P_r , \quad P'_r = 1.$$

The numbers of variations in the sequence

$$P', P'_1, P'_2, \ldots, P'_r$$

for $\varrho = -\infty$ and $\varrho = +\infty$, however, are the same as for the original series

$$P, P_1, P_2, \ldots, P_r.$$

However in place of n, the degree of f(z), we must substitute n - k, where k is the degree of P_r. The general formula for the number of roots of f(z) = 0 inside the unit circle will be

$$N = \frac{N_{\infty} - N_{-\infty} + n - k}{2}$$

ILLUSTRATED EXAMPLE. The discussion above will be illustrated for the polynomial

1) $f(z) = 2z^5 - 4z^4 + 5z^3 - 5z^2 + 3z - 1$

Since z = 1 is a root, divide f(z) by z-1. Then

2) $f(z) = (z-1)(2z^4 - 2z^3 + 3z^2 - 2z + 1)$

Using the transformation

$z = (\varrho - i)/(\varrho + i)$ in the last factor of 2) we have

3) $f\left(\frac{(\varrho-i)}{(\varrho+i)}\right) = 2\frac{(\varrho-i)^4}{(\varrho+i)^4} - 2\frac{(\varrho-i)^3}{(\varrho+i)^3} + 3\frac{(\varrho-i)^2}{(\varrho+i)^2} - 2\frac{(\varrho-i)}{(\varrho+i)} + 1$

and writing 3) in the form

$$F(\varrho) = (\varrho + i)^4 f\left(\frac{\varrho - i}{\varrho + i}\right)$$

then

4) $$F(\rho) = 2(\rho-i)^4 -2(\rho-i)^3(\rho+i) + 3(\rho-i)^2(\rho+i)^2 -2(\rho-i)(\rho+i)^3 + (\rho+i)^4$$

Expanding the terms in 4) and collecting like terms

$$F(\rho) = P + iP_1 = (2\rho^4 -12\rho^2 +10) + i(-4\rho^3 + 4\rho)$$

so that

$$P = 2\rho^4 -12\rho^3 +10, \quad P_1 = -4\rho^3 + 4\rho$$

Now dividing P by P_1

$$P = P_1Q_1 - P_2 = (-4\rho^3 + 4\rho)(-\tfrac{1}{2}\rho) - (-10\rho^2 + 10)$$

so that

$$P_2 = 10\rho^2 - 10$$

Then dividing P_1 by P_2 to get

$$P_1 = P_2Q_2 - P_3 = (10\rho^2 - 10)(-\tfrac{4}{10}\rho) - (0)$$

therefore P_3 is zero. The four polynomials are then

5) $$P = 2\rho^4 -12\rho^2 +10$$
$$P_1 = -4\rho^3 + 4\rho$$
$$P_2 = 10\rho^2 - 10$$
$$P_3 = 0$$

Now since the order of P_2 is 2, there are two more roots on the unit circle making a total of 3. To determine N_∞ and $N_{-\infty}$ substitute $\rho = \infty$ and $\rho = -\infty$ in the polynomials 5) and count the variations in sign.

	P	P_1	P_2	P_3
For $\varrho = +\infty$	+	−	+	
For $\varrho = -\infty$	+	+	+	

Evidently $N_{\infty} = 2$ and $N_{-\infty} = 0$. Then substituting these values in the general formula for the number of roots of $f(z) = 0$ inside the unit circle

$$N = \frac{N_{\infty} - N_{-\infty} + n - k}{2} = \frac{2 - 0 + 4 - 2}{2} = 2$$

Therefore there are two roots inside the unit circle.

The original function under consideration can be factored into

$$f(z) = (z - 1)(z^2 + 1)(2z^2 - 2z + 1)$$

from which we can see that the roots are located as determined above.

As another example of slightly different nature, take for the polynomial

$$f(z) = 2z^5 + 4z^4 + 11z^3 + 17z^2 + 12z + 1$$

and proceeding in exactly the same manner as before, we find for the sequence of polynomials:

$$P = 25\varrho^4 - 18\varrho^2 + 5$$

$$P_1 = 20\varrho^3 + 4\varrho$$

$$P_2 = 23\varrho^2 - 5$$

$$P_3 = -192/23\,\varrho$$

$$P_4 = 5$$

Now since the order of P_4 is zero, there are no more roots on the unit circle other than $z=-1$. This root, as before, was removed from the polynomial before the method was undertaken.

Again determine N_{∞} and $N_{-\infty}$ by substituting $\varrho = +\infty$ and $\varrho = -\infty$ in the sequence of polynomials and counting the variations of sign.

	P	P_1	P_2	P_3	P_4
For $\varrho = +\infty$	+	+	+	−	+
For $\varrho = -\infty$	+	−	+	+	+

Therefore $N_{\infty} = 2$ and $N_{-\infty} = 2$. Then substituting these values in the general formula for the number of roots of $f(z) = 0$ inside the unit circle

$$N = \frac{N_{\infty} - N_{-\infty} + n - k}{2} = \frac{2 - 2 + 4 - 0}{2} = 2$$

Then there are two roots inside the unit circle. Evidently there are two other roots outside the unit circle, and this is seen to be the case by observing the location of the roots of $f(z)$ in factored form.

$$f(z) = (z+1)(z^2+4)(2z^2+2z+1)$$

From the factored form it is obvious that there are two roots outside the unit circle, two roots inside, and one root on the circle. This is in agreement with the results found above.

CHAPTER IV

A RESULT CONCERNING MULTIPLE ZEROS

The zeros of $f(z) = u + iv$ are the intersections of the curves $u = 0$, $v = 0$.

Theorem 9. At an n-tuple zero of the analytic function $f(z)$, each of the curves $u = 0$, $v = 0$ has an n-tuple point; and the two curves intersect at an angle of $(1/n)(\pi/2)$ at the n-tuple zero.

Proof: No generality is lost by taking the zero of $f(z)$ to be at $z = 0$, and writing

$$f(z) = ae^{i\alpha} z^{n} + O(|z|^{n+1})$$

so that

1) $\quad u = ar^{n} \cos(\alpha + n\theta_1) + O(|r|^{n+1})$

and

2) $\quad v = ar^{n} \sin(\alpha + n\theta_2) + O(|r|^{n+1})$.

Now by the fundamental theorem of algebra each of the curves $u = 0$, $v = 0$ has an n-tuple point at $r = 0$.

For the curves in 1) and 2) the directions of the tangents to $u = 0$, $v = 0$, are given by

3) $\quad \alpha + n\theta_1 = \left(\frac{2n+1}{2}\right)\pi, \qquad n = 0, 1, 2, \ldots$

and

4) $\quad \alpha + n\theta_2 = n\pi, \qquad n = 0, 1, 2, \ldots$

or

$$n\theta_1 = \tfrac{1}{2}(2n+1)\pi - \alpha$$

$$n\theta_2 = n\pi - \alpha .$$

Then

$$\theta_1 = (1/2n)(2n+1)\pi - \alpha/n$$

$$\theta_2 = (1/n)(n\pi - \alpha).$$

Now the angle between the two curves $u=0$, $v=0$, will be the difference between the angles θ_1 and θ_2. That is,

$$\theta_1 - \theta_2 = (1/n)(n\pi + \pi/2 - \alpha - n\pi + \alpha)$$

$$= (1/n)(\pi/2).$$

Q.E.D.

AN EXAMPLE FOR $n = 3$. Consider the function

$$f(z) = (z-2)^3 = [(x+iy) - 2]^3$$

$$= [(x-2)^3 - 3y^2(x-2)] + i\,[(-y^3) + y(3x^2-12x+12)]$$

Then

1) $u = (x-2)^3 - 3y^2(x-2) = 0$

2) $v = (-y^3) + y(3x^2 - 12x + 12) = 0$

Solving for x in terms of y in $u=0$, and for y in terms of x in $v=0$, obtain from 1)

3) $x-2=0$; $x-2 = \sqrt{3}y$; $x-2 = -\sqrt{3}y$

and from 2)

4) $y=0$; $y = \sqrt{3}x - \sqrt{12}$; $y = -(\sqrt{3}x - \sqrt{12})$.

These curves are plotted in Figure III, and an inspection of the slopes shows clearly that the curves intersect at an angle of 30°. This agrees with the results of the last theorem for $(1/3)(\pi/2)$ is 30°.

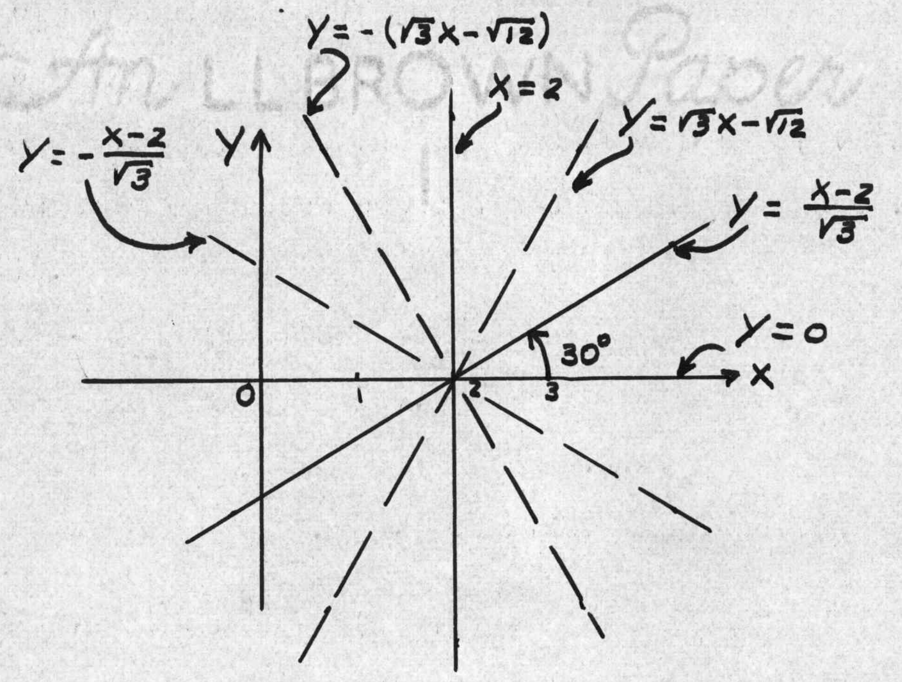

Figure III.

CHAPTER V

ZEROS OF LEVEL CURVES

DEFINITION OF A LEVEL CURVE. The locus of a point z which moves in the plane of the complex variable z so that the modulus of a function of $f(z)$ remains constant is defined as a level curve of $f(z)$. The equation of such a curve may be written in the form $|f(z)| = M$ where M is the constant modulus. By giving M all values from zero to $+\infty$, we obtain an infinite number of curves. Clearly, one and only one of these passes through any given point in the plane.

Theorem 10. For an analytic function $f(z)$ a level curve has a double point, if, and only if, it passes through a zero of $f'(z)$.

Proof(8,p.121): The equation of a level curve is

$$u^2 + v^2 = c^2$$

and this has a double point if, and only if,

1) $uu_x + vv_x = 0$

2) $uu_y + vv_y = 0$

Both of these conditions are satisfied if $f'(z) \quad 0$.

Conversely, equation 2) may be written

3) $-uv_x + vu_x = 0$.

Then squaring and adding equations 1) and 3)

$$(u_x^2 + v_x^2)(u^2 + v^2) = 0.$$

Hence, $u_x = 0$ and $v_x = 0$, that is,

$$f'(z) = 0.$$

Q.E.D.

Theorem 11. If C is a simple closed level curve, and f(z) is analytic inside and on C, then f(z) has at least one zero inside C.

Proof(8,p.121): Let

$f(z) = u + iv = ce^{i\theta}$ on C, so that c is a constant. Then

$$c = \sqrt{u^2 + v^2}, \quad \theta = \arctan (v/u).$$

Let S be the length of C measured from some fixed point on it. Then

1) $0 = \frac{dc}{ds} = (u\frac{du}{ds} + v\frac{dv}{ds})(1/c)$

2) $\frac{d\theta}{ds} = (u\frac{dv}{ds} - v\frac{du}{ds})(1/c^2)$

Now $d\theta/ds$ cannot vanish on C. For if it did on squaring and adding equations 1) and 2)

$$(u^2 + v^2)\left[(du/ds)^2 + (dv/ds)^2\right] = 0, \text{ that is,}$$

$$du/ds = 0, \; dv/ds = 0.$$

Now

3) $du/ds = u_x(dx/ds) + u_y(dy/ds)$,

4) $dv/ds = v_x(dx/ds) + v_y(dy/ds) = -u_y(dx/ds) + u_x(dy/ds)$,

so that squaring and adding equations 3) and 4)

5) $(u_x^2 + u_y^2)\left[(dx/ds)^2 + (dy/ds)^2\right] = 0.$

The last factor is 1, so that $u_x = 0$, $u_y = 0$, that is,

$f'(z) = 0.$

This is impossible on a level curve without double points by the previous theorem.

It follows that $d\theta/ds$ has the same sign at all points of C, i.e., that θ increases or decreases steadily round the contour. Hence its variation round the contour is not zero. Then by Theorem 3 there is at least one zero inside C.

Q.E.D.

AN EXAMPLE OF A LEVEL CURVE. As an example of the level curves of a function, suppose that

$f(z) = \sin z.$

Then(3,p.40)

$$|f(z)|^2 = |\sin(x+iy)|^2 = [\sin(x+iy)][\sin(x-iy)]$$
$$= \tfrac{1}{2}(\cosh 2y - \cos 2x)$$

and the level curves are given by

$\cosh 2y - \cos 2x = 2M^2$

where M ranges from 0 to $+\infty$.

Since $\cosh 2y$ and $\cos 2x$ are both even functions, the curves are symmetrical about both axes of coordinates. Also since $\cos 2x$ is periodic, it is sufficient to trace the curves which lie in the strip bounded by the lines $x = \pm(\pi/2)$. If M does not exceed unity, the curves meet the x-axis where $\sin x = \pm M$; otherwise the curves do not meet OX at all. When x vanishes

$$M = |\sin iy| = \pm \sinh y,$$

according as y is positive or negative. Thus for all values of M the curves meet OY in two points equidistant from the origin.

Consider the curve for which $M = 1$. Its equation may be reduced to the form

$$\sinh y = \pm \cos x$$

The curve passes through the points

$$(\pm \pi/2, 0) \quad \text{and} \quad [0, \pm \log(1 + \sqrt{2})].$$

At each of the first two points it has a node, the tangents at which make angles of $\pi/4$ with OX.

The form of the curves is indicated in the figure below.

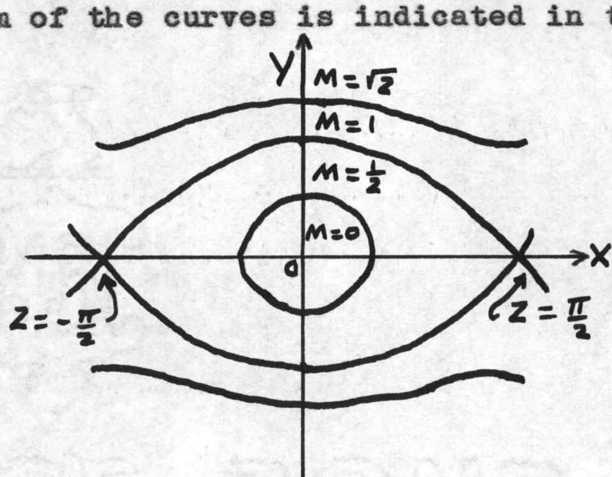

FIGURE IV

When M is less than unity we have a series of ovals with their centers at the points $(n\pi, 0)$, where $n = 0, \pm 1, \pm 2, \ldots$. When M is equal to unity, we obtain a curve which cuts OX at the points where x is equal to an odd multiple of $\pi/2$. For values of M greater than unity the curve is in two distinct branches above and below the x-axis.

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