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Title: MODES OF VIBRATION OF INEXTENSIBLE CABLES SUSPENDED FROM TWO POINTS

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The equations of motion of a suspended, inextensible cable in the absence of damping are studied. A method is developed and illustrated for determining the frequencies and mode shapes for small oscillations about the equilibrium configuration.

With the introduction of a new variable, the fourth order equation governing tangential displacement takes a form with polynomial coefficients. This form, previously undiscovered, permits application of the method of Frobenius for solution.

The system consisting of this differential equation and the boundary conditions determined by attachment points, contains a set of eigenvalues which correspond to possible frequencies of oscillation. A criterion for selection of the eigenvalues is developed from the boundary conditions.

A digital computer was used to perform the calculation of the fundamental frequency of several different cable configurations. The results agree favorably with previously published values.

Modes of Vibration of Inextensible Cables Suspended from Two Points

by

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MODES OF VIBRATION OF INEXTENSIBLE CABLES SUSPENDED FROM TWO POINTS

I. INTRODUCTION

The theory of vibration of stretched strings was presented by Daniel Bernoulli in the middle 1700's. The question of what happens when the string becomes slack was approached by J. H. Rohrs (4) approximately a hundred years later. The problem arose from concern over the failure of suspension bridges. Rohrs derived the equations of motion and used a Fourier series approach to obtain solutions for an almost horizontal cable. The problem became of interest again in the late 1800's. E. J. Routh (5) included a discussion of vibrating cables in some of his texts. His particular emphasis was on solving exactly the case of a cable with a cycloidal equilibrium configuration. A. G. Pugsley (3), D. S. Saxon and A. S. Cahn (6), and W. J. Goodey (1) have added to the literature over the last thirty years. They have provided empirical results and various approximate solutions. All of the above solutions are for cables with symmetry about a vertical line through the lowest point on the cable.

In this text the equations of motion are derived in a different form than used in any of the above publications. Normal mode motion is assumed and a change of variable is introduced which results in an equation which permits an exact solution. The unsymmetric cable can be handled using this method.

II. DERIVATION OF THE EQUATIONS OF MOTION

The equations of motion are developed by the direct application of Newton's second law of motion to an element of the cable. Two kinematic equations are developed by introducing the constraint that the cable be inextensible.

Consider the cable in the equilibrium configuration, Figure 1. The lowest point on the cable is taken as the origin of the coordinate system. The distance along the cable to an arbritrary point, s, on the cable, defines every point on the cable. As the cable is displaced from the equilibrium configuration, points on the cable are identified with the same value of s with which they were identified in the equilibrium configuration. At each point s a line tangent to the equilibrium configuration can be drawn. The angle between this tangent line and a horizontal line is called α.

Allow the cable to assume a displaced configuration, Figure 2. Point s has moved to a new point in the plane. The displacement can be described by the distance moved in the direction of the tangent line and the distance moved in the direction normal to the tangent line. The displacement in the direction of the tangent line is called v (positive in the direction of increasing s). The displacement normal to this is called u (positive in the direction toward the local center of curvature of the equilibrium configuration). The angle between the tangent line drawn through point s on the displaced curve and the tangent line

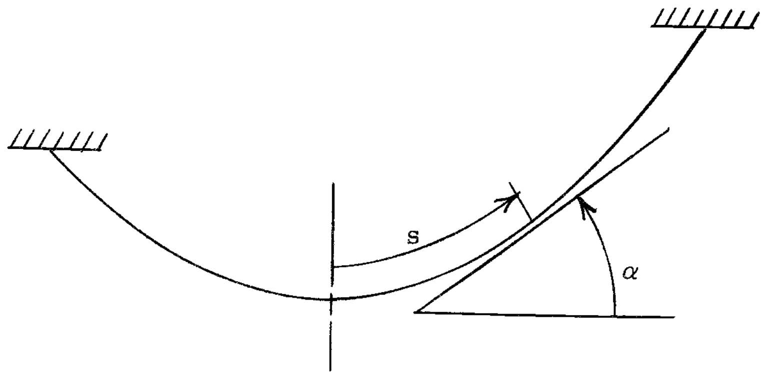

Figure 1. Equilibrium configuration of hanging cable.

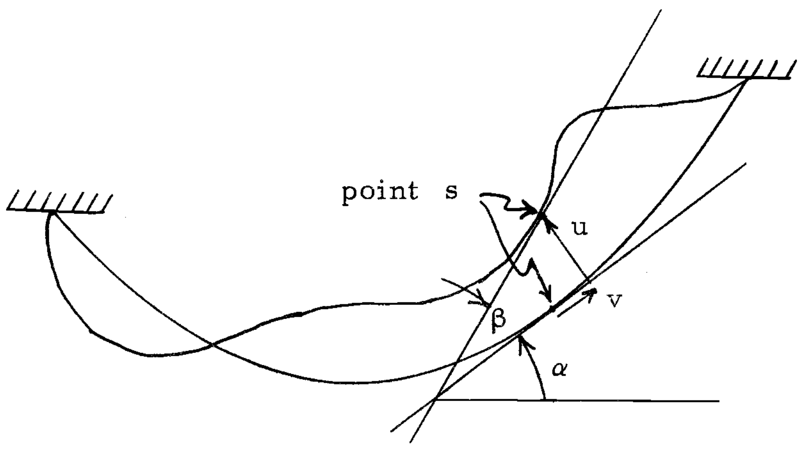

Figure 2. Displaced configuration of hanging cable showing displacement vectors and angles.

drawn through the point s on the equilibrium curve is called β.

A free-body diagram of an element of cable (length = Δs) is shown in Figure 3. Application of Newton's second law of motion in the tangential direction gives:

$$-mg(\Delta s)\sin(\alpha+\beta) + T(s+\Delta s, t)\cos(\Delta\alpha+\Delta\beta) - T(s,t)$$

$$= m(\Delta s)\left(\frac{\partial^2 v}{\partial t^2}\cos\beta + \frac{\partial^2 u}{\partial t^2}\sin\beta\right)$$

where m = mass per unit length of the cable

$T(s,t)$ = tension in the cable at point s and time t

g = acceleration due to gravity

Divide through by (Δs) and let $(\Delta s) \rightarrow 0$. The equation becomes:

$$m\left[\frac{\partial^2 v}{\partial t^2}\cos\beta + \frac{\partial^2 u}{\partial t^2}\sin\beta + g\sin(\alpha+\beta)\right] - \frac{\partial T}{\partial s} = 0 \qquad (1)$$

Similarly, in the normal direction:

$$T(s+\Delta s, t)\sin(\Delta\alpha+\Delta\beta) - mg(\Delta s)\cos(\alpha+\beta)$$

$$= m(\Delta s)\left[\frac{\partial^2 u}{\partial t^2}\cos\beta - \frac{\partial^2 v}{\partial t^2}\sin\beta\right]$$

which becomes:

$$m\left[\frac{\partial^2 u}{\partial t^2}\cos\beta - \frac{\partial^2 v}{\partial t^2}\sin\beta + g\cos(\alpha+\beta)\right] - T\left(\frac{\partial\beta}{\partial s} + \frac{d\alpha}{ds}\right) = 0 \qquad (2)$$

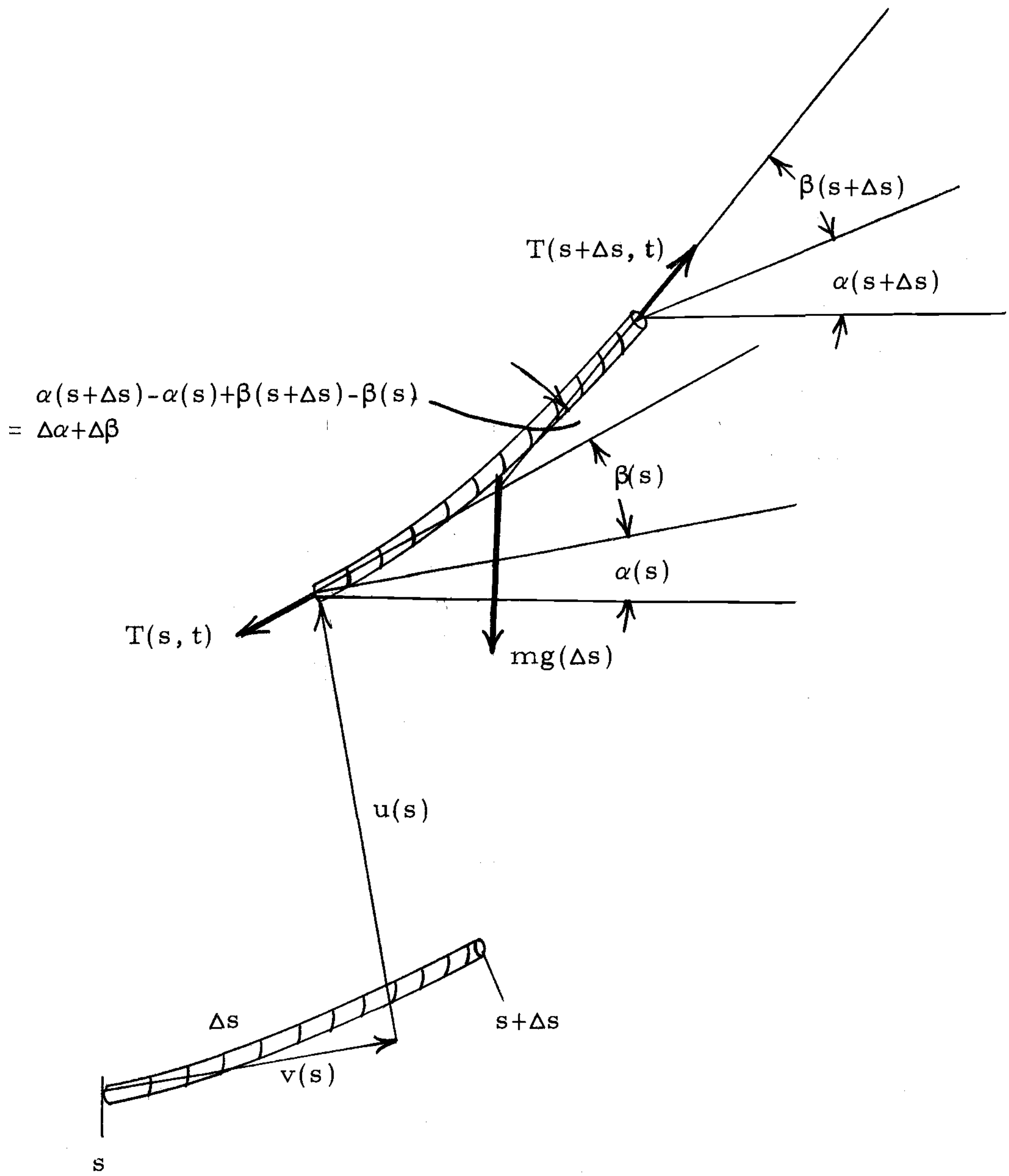

Figure 3. Free-body diagram of an element of cable in displaced configuration.

A sketch of an element of the cable in equilibrium and displaced configurations, showing the displacement components and angles, is shown in Figure 4. The cable is taken to be inextensible. Therefore the points s and s+Δs will remain a distance Δs , measured along the cable,. apart. As Δs becomes small the element can be approximated by a straight line segment. From this sketch two expressions for b can be written and equated:

$$b = (\Delta s)\cos\beta = (\Delta s)\cos\frac{\Delta\alpha}{2} + v(s+\Delta s)\cos(\Delta\alpha) - u(s+\Delta s)\sin(\Delta\alpha) - v(s)$$

Dividing through by Δs and letting $\Delta s \to 0$:

$$\cos\beta = 1 + \frac{\partial v}{\partial s} - u\frac{d\alpha}{ds} \qquad (3)$$

The same approach for h results in:

$$h = (\Delta s)\sin\beta = (\Delta s)\sin\frac{\Delta\alpha}{2} + v(s+\Delta s)\sin\Delta\alpha + u(s+\Delta s)\cos(\Delta\alpha) - u(s)$$

which becomes:

$$\sin\beta = \frac{\partial u}{\partial s} + v\frac{d\alpha}{ds} \qquad (4)$$

The tension may be written as the sum of two parts:

$$T(s, t) = T_0(s) + T_1(s, t), \qquad (5)$$

where $T_0(s)$ = the tension distribution in the cable in the equilibrium configuration

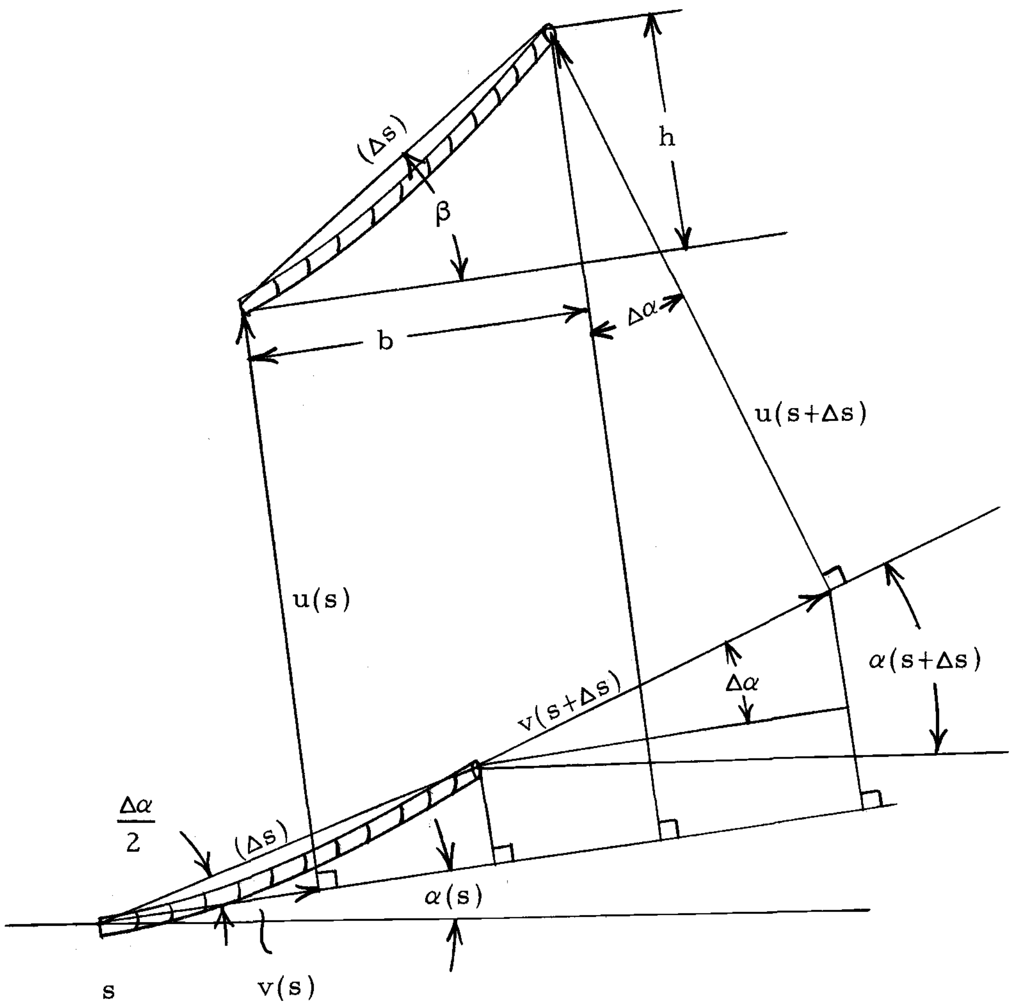

Figure 4. Element of the cable in equilibrium and displaced configurations.

$T_1(s, t)$ = the difference between the actual tension and $T_0(s)$.

If the cable is in the equilibrium configuration, $\beta = u = v = T_1 = 0$, and $T(s, t) = T_0(s)$. Substitute these into the dynamic equations, (1) and (2).

$$mg \sin\alpha - \frac{dT_0}{ds} = 0 \tag{6}$$

$$mg \cos\alpha - T_0 \frac{d\alpha}{ds} = 0 \tag{7}$$

Dividing the first by the second gives:

$$\tan\alpha d\alpha - \frac{dT_0}{T_0} = 0$$

Integration yields:

$$T_0 = \frac{\mathcal{T}_0}{\cos\alpha} \tag{8}$$

where log $\mathcal{T}_0$ is the constant of integration. Put Equation (8) into Equation (7) and integrate to get:

$$s = a \tan\alpha \text{ , where } a = \frac{\mathcal{T}_0}{mg}$$

Thus:

$$\sin\alpha = \frac{s}{\sqrt{s^2+a^2}}, \quad \cos\alpha = \frac{a}{\sqrt{s^2+a^2}}, \quad \frac{d\alpha}{ds} = \frac{a}{a^2+s^2}, \quad \text{and} \quad T_0 = mg\sqrt{s^2+a^2} .$$

Substitute (5) into the dynamic Equations (1) and (2), and simplify using (6) and (7).

$$m\left[\frac{\partial^2 v}{\partial t^2}\cos\beta+\frac{\partial^2 u}{\partial t^2}\sin\beta\right]+mg(\cos\beta-1)\sin\alpha+mg\sin\beta\cos\alpha-\frac{\partial T_1}{\partial s}=0 \tag{9}$$

$$m\left[\frac{\partial^2 u}{\partial t^2}\cos\beta-\frac{\partial^2 v}{\partial t^2}\sin\beta\right]+mg(\cos\beta-1)\cos\alpha$$

$$-mg\sin\beta\sin\alpha-T_0\frac{\partial\beta}{\partial s}-T_1\left(\frac{\partial\beta}{\partial s}+\frac{d\alpha}{ds}\right)=0 \tag{10}$$

Define dimensionless variables by:

$$s=a\sigma,\quad u=a\xi\ ,\quad v=a\eta$$

$$t=\sqrt{\frac{a}{g}}\,\theta\ ,\quad T_1=mg\,\tau$$

$$(\ \ )'=\frac{\partial(\ \ )}{\partial\sigma}\ ,\quad (\dot{\ \ })=\frac{\partial(\ \ )}{\partial\theta}=\sqrt{\frac{a}{g}}\,\frac{\partial(\ \ )}{\partial t}$$

Substitution of these into Equations (3), (4), (9), and (10) gives:

$$\eta'-\frac{1}{1+\sigma^2}\,\xi+1-\cos\beta=0$$

$$\xi'+\frac{1}{1+\sigma^2}\,\eta-\sin\beta=0$$

$$\ddot{\eta}\cos\beta+\ddot{\xi}\sin\beta+\frac{\sigma}{\sqrt{1+\sigma^2}}(\cos\beta-1)+\frac{\sin\beta}{\sqrt{1+\sigma^2}}-\tau'=0$$

$$\ddot{\xi}\cos\beta-\ddot{\eta}\sin\beta+\frac{(\cos\beta-1)}{\sqrt{1+\sigma^2}}-\frac{\sigma\sin\beta}{\sqrt{1+\sigma^2}}-(\sqrt{1+\sigma^2}-\tau)\beta'-\frac{\tau}{1+\sigma^2}=0$$

Linearize these to obtain:

$$\eta'-\frac{1}{1+\sigma^2}\,\xi=0 \tag{11}$$

$$\xi' + \frac{1}{1+\sigma^2}\eta - \beta = 0 \tag{12}$$

$$\ddot{\eta} + \frac{\beta}{\sqrt{1+\sigma^2}} - \tau' = 0 \tag{13}$$

$$\ddot{\xi} - (\sqrt{1+\sigma^2}\beta)' - \frac{\tau}{1+\sigma^2} = 0 \tag{14}$$

III. NORMAL MODE MOTIONS

We can expect this linear, conservative system to admit solutions which represent normal mode oscillations, i.e.,

$$
\begin{aligned}
\tau(\sigma, \theta) &= A(\sigma)\sin\lambda\,\theta \\
\beta(\sigma, \theta) &= B(\sigma)\sin\lambda\,\theta \\
\xi(\sigma, \theta) &= Z(\sigma)\sin\lambda\,\theta \\
\eta(\sigma, \theta) &= H(\sigma)\sin\lambda\,\theta
\end{aligned}
\tag{15}
$$

where λ is the dimensionless frequency of oscillation.

Substitution of (15) into Equations (11) - (14) yields:

$$A' - \frac{B}{\sqrt{1+\sigma^2}} + \lambda^2 H = 0 \tag{16}$$

$$(\sqrt{1+\sigma^2}\, B)' + \frac{A}{1+\sigma^2} + \lambda^2 Z = 0 \tag{17}$$

$$Z' + \frac{H}{1+\sigma^2} - B = 0 \tag{18}$$

$$H' - \frac{1}{1+\sigma^2} Z = 0 \tag{19}$$

The variables A, B, and Z may be eliminated from this system to yield the single fourth-order equation for $H(\sigma)$:

$$
\begin{aligned}
&(1+\sigma^2)^{5/2}\frac{d^4 H}{d\sigma^4} + 10\sigma(1+\sigma^2)^{3/2}\frac{d^3 H}{d\sigma^3} + [3(3+8\sigma^2)\sqrt{1+\sigma^2} \\
&+ \lambda^2(1+\sigma^2)^2]\frac{d^2 H}{d\sigma^2} + 4\sigma(1+\sigma^2)[\frac{3}{\sqrt{1+\sigma^2}} + \lambda^2]\frac{dH}{d\sigma} - \lambda^2 H = 0
\end{aligned}
\tag{20}
$$

IV. CHANGE OF INDEPENDENT VARIABLE

The variable coefficients, and especially the appearance of the form $\sqrt{1+\sigma^2}$, make solution of Equation (20) difficult. A suitable change of variable which transforms the coefficients to polynomials is:

$$\mu = \frac{\sigma}{|\sigma|}\left(\sqrt{1+\sigma^2} - 1\right) \tag{21}$$

The inverse transformation is:

$$\sigma = \frac{\mu}{|\mu|}\sqrt{(1+|\mu|)^2 - 1} \tag{22}$$

Now, in order to transform Equation (20), the following derivatives are needed:

$$\frac{d\mu}{d\sigma} = \frac{|\sigma|}{\sqrt{1+\sigma^2}}$$

$$\frac{d^2\mu}{d\sigma^2} = \frac{\sigma}{|\sigma|\,(1+\sigma^2)^{3/2}}$$

$$\frac{d^3\mu}{d\sigma^3} = \frac{-3|\sigma|}{(1+\sigma^2)^{5/2}} + 2\delta(\sigma)$$

$$\frac{d^4\mu}{d\sigma^4} = \frac{-3\sigma(1-4\sigma^2)}{|\sigma|\,(1+\sigma^2)^{7/2}} + 2\frac{d\delta(\sigma)}{d\sigma}$$

where $\delta(\sigma)$ is the familiar Dirac delta function.

Note first of all that since $\frac{d\mu}{d\sigma}$ vanishes at the origin, it is quite likely that $\frac{dH}{d\mu}$ will exhibit singular behavior as $\mu \to 0$. Also, the factor $\frac{\sigma}{|\sigma|}$, introduced in the definition of μ so that the correspondence between σ and μ would be one-to-one, gives rise to impulse and doublet singularities in the third and fourth derivatives. The equation governing the tangential displacement, $V(\mu) \equiv H[\sigma(\mu)]$, in terms of the variable μ, is:

$$(|\mu|+1)[(|\mu|+1)^2-1]\frac{d^4V}{d\mu^4} + \frac{\mu}{|\mu|}[(|\mu|+1)^2-1][10(|\mu|+1)^2-4]\frac{d^3V}{d\mu^3}$$

$$+[24(|\mu|+1)^3-21(|\mu|+1)+((|\mu|+1)^2-1)(|\mu|+1)^2\lambda^2$$

$$+8\frac{\mu}{|\mu|}(|\mu|+1)^4((|\mu|+1)^2-1)^{1/2}\delta(\mu)]\frac{d^2V}{d\mu^2} + \frac{\mu}{|\mu|}[12(|\mu|+1)^2-6$$

$$+\lambda^2(|\mu|+1)(4(|\mu|+1)^2-3) + 20\frac{\mu}{|\mu|}((|\mu|+1)^2-1)^{1/2}(|\mu|+1)^3\delta(\mu)$$

$$+2\frac{\mu}{|\mu|}(|\mu|+1)^5\frac{d\delta(\mu)}{d\mu}]\frac{dV}{d\mu} - \lambda^2 V = 0$$

If the portion of the cable in which $\mu > 0$ is considered, $|\mu| = \mu$ and the Dirac delta function and its derivative may be left out of the equation. Then expanding the coefficients into polynomials results in:

$$[(\mu^5+5\mu^4+8\mu^3+4\mu^2)]\frac{d^4V}{d\mu^4} + [(10\mu^4+40\mu^3+46\mu^2+12\mu)]\frac{d^3V}{d\mu^3}$$

$$+ [(24\mu^3+72\mu^2+51\mu+3)+\lambda^2(\mu^4+4\mu^3+5\mu^2+2\mu)]\frac{d^2V}{d\mu^2} \qquad (23)$$

$$+ [(12\mu^2+24\mu+6)+\lambda^2(4\mu^3+12\mu^2+9\mu+1)]\frac{dV}{d\mu} - \lambda^2 V = 0$$

In obtaining solutions this simpler equation will be dealt with, recognizing that it does <u>not</u> represent the physical system under consideration for $\mu \leq 0$.

V. SOLUTION OF THE DIFFERENTIAL EQUATION

The method of Frobenius is used to obtain series solutions to Equation (23). The method is applied in the form presented in Chapter 16 of Ordinary Differential Equations by E. L. Ince (2).

Solutions of the form

$$V(\mu) = \sum_{k=0}^{\infty} a_k \mu^{k+r}, \quad a_0 \neq 0 \tag{24}$$

are desired. The singular points of the differential equation are given by the roots of the coefficient of $\frac{dV^4}{d\mu^4}$ set equal to zero. These points are $\mu=-1, 0, -2$. A series solution expanded about $\mu=0$ will converge uniformly with a radius of convergence equal to the distance to the nearest singular point, that is equal to 1. The values of r which allow a series solution of this form are called the characteristic exponents and are given by the roots of the indicial equation, that is, the result of equating to zero the coefficient of the lowest power of μ in the equation which results when (24) is substituted into (23). The characteristic exponents obtained are $0, 1, \frac{1}{2}$, and $\frac{3}{2}$. When two of these differ by an integer, it may happen that a solution is of a form different from (24). In this case, an analysis is necessary to determine the form of the corresponding solutions (see Ince, Section 16.3). This analysis shows that the solutions all have the form of (24). Four linearly independent solutions may in this case be obtained by direct substitution of (24) into

(23) with each of the four values of r. Setting the coefficient of μ^k equal to zero results in recurrence relationships for the constants a_k.

The four solutions are:

$$y_1(\mu) = \sum_{k=0}^{\infty} a_k \mu^k$$

$$a_0 = 1, \quad a_1 = -1-\lambda^2$$

$$\begin{aligned} a_k = & -\{[\lambda^2(k-4)(k-5)+4\lambda^2(k-4)]a_{k-4}+[(k-3)(k-4)(k-5)(k-6) \\ & +10(k-3)(k-4)(k-5)+24(k-3)(k-4)+4\lambda^2(k-3)(k-4)+12(k-3) \\ & +12\lambda^2(k-3)]a_{k-3}+[5(k-2)(k-3)(k-4)(k-5)+40(k-2)(k-3)(k-4) \\ & +72(k-2)(k-3)+5\lambda^2(k-2)(k-3)+24(k-2)+9\lambda^2(k-2)-\lambda^2]a_{k-2} \\ & +[8(k-1)(k-2)(k-3)(k-4)+46(k-1)(k-2)(k-3)+51(k-1)(k-2) \\ & +\lambda^2(k-1)(k-2)+6(k-1)+\lambda^2(k-1)]a_{k-1}\}/[4(k)(k-1)(k-2)(k-3) \\ & +12(k)(k-1)(k-2)+3(k)(k-1)] \qquad k > 1 \end{aligned}$$

$$y_2(\mu) = \mu \sum_{k=0}^{\infty} b_k \mu^k$$

$$b_0 = 1, \quad b_1 = \frac{-(6+\lambda^2)}{6}$$

$$\begin{aligned} b_k = & -\{[\lambda^2(k-3)(k)]b_{k-4}+[(k-2)(k-3)(k-4)(k-5)+10(k-2)(k-3)(k-4) \\ & +24(k-2)(k-3)+4\lambda^2(k-2)(k-3)+12(k-2)+12\lambda^2(k-2)]b_{k-3} \\ & +[5(k-1)(k-2)(k-3)(k-4)+40(k-1)(k-2)(k-3)+72(k-1)(k-2) \\ & +5\lambda^2(k-1)(k-2)+24(k-1)+9\lambda^2(k-1)-\lambda^2]b_{k-2}+[8(k)(k-1)(k-2)(k-3) \end{aligned}$$

$$+46(k)(k-1)(k-2) \ +51(k)(k-1)+2\lambda^2(k)(k-1)+6(k)+\lambda^2(k)]\, b_{k-1}\Big\}\Big/$$

$$[4(k+1)(k)(k-1)(k-2)+12(k+1)(k)(k-1)+3(k+1)(k)] \qquad k > 1$$

$$y_3(\mu) \ = \ \mu^{3/2} \sum_{k=0}^{\infty} c_k \mu^k$$

$$c_0 = 1, \ \ c_1 = -\frac{(69+6\lambda^2)}{60}$$

$$c_k \ = \ -\{[\lambda^2(k-\tfrac{5}{2})(k-\tfrac{7}{2})+4\lambda^2(k-\tfrac{5}{2})]\, c_{k-4}+[(k-\tfrac{3}{2})(k-\tfrac{5}{2})(k-\tfrac{7}{2})(k-\tfrac{9}{2})$$

$$+10(k-\tfrac{3}{2})(k-\tfrac{5}{2})(k-\tfrac{7}{2})+24(k-\tfrac{3}{2})(k-\tfrac{5}{2})+4\lambda^2(k-\tfrac{3}{2})(k-\tfrac{5}{2})+12(k-\tfrac{3}{2})$$

$$+12\lambda^2(k-\tfrac{3}{2})]\, c_{k-3}+[5(k-\tfrac{1}{2})(k-\tfrac{3}{2})(k-\tfrac{5}{2})(k-\tfrac{7}{2})+40(k-\tfrac{1}{2})(k-\tfrac{3}{2})(k-\tfrac{5}{2})$$

$$+72(k-\tfrac{1}{2})(k-\tfrac{3}{2})+5\lambda^2(k-\tfrac{1}{2})(k-\tfrac{3}{2})+24(k-\tfrac{1}{2})+9\lambda^2(k-\tfrac{1}{2})+\lambda^2]\, c_{k-2}$$

$$+[8(k+\tfrac{1}{2})(k-\tfrac{1}{2})(k-\tfrac{3}{2})(k-\tfrac{5}{2})+46(k+\tfrac{1}{2})(k-\tfrac{1}{2})(k-\tfrac{3}{2})+51(k+\tfrac{1}{2})(k-\tfrac{1}{2})$$

$$+2\lambda^2(k+\tfrac{1}{2})(k-\tfrac{1}{2})+6(k+\tfrac{1}{2})+\lambda^2(k+\tfrac{1}{2})]\, c_{k-1}\Big\}\Big/[4(k+\tfrac{3}{2})(k+\tfrac{1}{2})(k)(k+1)]$$

$$k > 1$$

$$y_4(\mu) \ = \ \mu^{\frac{1}{2}} \sum_{k=0}^{\infty} d_k \mu^k$$

$$d_0 = 1, \ \ d_1 = \frac{-(5+2\lambda^2)}{6}$$

$$d_k \ = \ -\{[\lambda^2(k-\tfrac{7}{2})(k-\tfrac{9}{2})+4\lambda^2(k-\tfrac{7}{2})]\, d_{k-4}+[(k-\tfrac{5}{2})(k-\tfrac{7}{2})(k-\tfrac{9}{2})(k-\tfrac{11}{2})$$

$$+10(k-\tfrac{5}{2})(k-\tfrac{7}{2})(k-\tfrac{9}{2})+24(k-\tfrac{5}{2})(k-\tfrac{7}{2})+4\lambda^2(k-\tfrac{5}{2})(k-\tfrac{7}{2})+12(k-\tfrac{5}{2})$$

$$+12\lambda^2(k-\tfrac{5}{2})]\, d_{k-3}+[5(k-\tfrac{3}{2})(k-\tfrac{5}{2})(k-\tfrac{7}{2})(k-\tfrac{9}{2})+40(k-\tfrac{3}{2})(k-\tfrac{5}{2})(k-\tfrac{7}{2})$$

$$+72(k-\tfrac{3}{2})(k-\tfrac{5}{2})+5\lambda^2(k-\tfrac{3}{2})(k-\tfrac{5}{2})+24(k-\tfrac{3}{2})+9\lambda^2(k-\tfrac{3}{2})-\lambda^2]\, d_{k-2}$$

$$+\left[8\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\left(k-\frac{5}{2}\right)\left(k-\frac{7}{2}\right)+46\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\left(k-\frac{5}{2}\right)+51\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)\right.$$

$$\left.+6\left(k-\frac{1}{2}\right)+2\lambda^2\left(k-\frac{1}{2}\right)\left(k-\frac{3}{2}\right)+\lambda^2\left(k-\frac{1}{2}\right)\right] d_{k-1}\Big\} \Big/ \left[4\left(k+\frac{1}{2}\right)\left(k-\frac{1}{2}\right)(k)(k-1)\right]$$

$$k > 1$$

In the above recurrence formulas, all variables with negative subscripts are equal to zero.

Any solution of the differential equation may be written as a linear combination of these four solutions.

$$V(\mu) = C_1 y_1(\mu) + C_2 y_2(\mu) + C_3 y_3(\mu) + C_4 y_4(\mu)$$

Recall that this is valid for $\mu > 0$, only.

For a cable with no discontinuity in any of its properties, a solution of the equations of motion will require that η, ξ, β, and T be continuous functions of σ. This means that A, B, Z, and H must be continuous functions of σ. A study of the term-by-term continuity of Equations (17)-(19) shows that H, H', H'', and H''' must be continuous functions. The argument is as follows. Given Z continuous, the second term in Equation (19) is continuous and thus the first term, H', is continuous. The continuity of H and B provides for the continuity of the second and third terms of Equation (18), and therefore Z' is continuous. Differentiation of Equation (19) along with the continuity of Z and Z' shows that H'' is continuous. The continuity of A, B, and Z in Equation (17) shows that B' is continuous. B, H, B' and H' continuous in the derivative of Equation (18) results in the continuity of Z''.

The second derivative of Equation (19) with the continuity of Z, Z', and Z'' gives H''' is continuous. Thus H, H', H'', and H''' are continuous functions.

For the portion of the cable in which $\mu > 0$, let $\mu_1 = \mu$, $\sigma_1 = \sigma$, $V_1 = V$, and $H_1 = H$. For the portion in which $\mu < 0$, let $\mu_2 = -\mu$, $\sigma_2 = -\sigma$, $V_2 = -V$, and $H_2 = -H$ (see Figure 5). With these substitutions, solutions for the positive and negative portions of the cable will be:

$$V_1(\mu_1) = A_1 y_1(\mu_1) + A_2 y_2(\mu_1) + A_3 y_3(\mu_1) + A_4 y_4(\mu_1)$$

$$V_2(\mu_2) = B_1 y_1(\mu_2) + B_2 y_2(\mu_2) + B_3 y_3(\mu_2) + B_4 y_4(\mu_2)$$

For small μ, or σ, μ can be approximated by $\mu = \frac{1}{2}\sigma^2$. So, for small μ, approximating each series by its dominant term:

$$H_1(\sigma_1) = V_1[\mu_1(\sigma_1)]$$

$$= [A_1(1) + A_2(\tfrac{1}{2}\sigma_1^2) + A_3(\tfrac{1}{2\sqrt{2}}\sigma_1^3) + A_4(\tfrac{1}{\sqrt{2}}\sigma_1)]$$

$$H_2(\sigma_2) = V_2[\mu_2(\sigma_2)]$$

$$= [B_1(1) + B_2(\tfrac{1}{2}\sigma_2^2) + B_3(\tfrac{1}{2\sqrt{2}}\sigma_2^3) + B_4(\tfrac{1}{\sqrt{2}}\sigma_2)]$$

For H, H', H'', and H''' to be continuous functions of σ the following conditions are implied,

$$H_1(\sigma_1)\big|_{\sigma_1 = 0} = -H_2(\sigma_2)\big|_{\sigma_2 = 0}$$

$$H_1'(\sigma_1)\big|_{\sigma_1 = 0} = H_2'(\sigma_2)\big|_{\sigma_2 = 0}$$

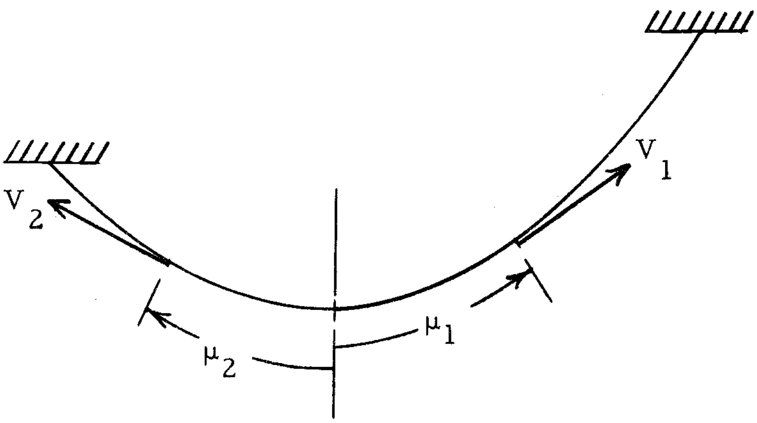

Figure 5. New definition of displacement components for $\mu < 0$.

$$H_1''(\sigma_1)\Big|_{\sigma_1=0} = -H_2''(\sigma_2)\Big|_{\sigma_2=0}$$

$$H_1'''(\sigma_1)\Big|_{\sigma_1=0} = H_2'''(\sigma_2)\Big|_{\sigma_2=0}$$

Application of the above to Equations (25), results in the requirements:

$$A_1 = -B_1,\ A_2 = -B_2,\ A_3 = B_3,\ A_4 = B_4$$

The solution takes the form:

$$\begin{aligned} V_1(\mu_1) &= A_1 y_1(\mu_1) + A_2 y_2(\mu_1) + A_3 y_3(\mu_1) + A_4 y_4(\mu_1) \\ V_2(\mu_2) &= -A_1 y_1(\mu_2) - A_2 y_2(\mu_2) + A_3 y_3(\mu_2) + A_4 y_4(\mu_2) \end{aligned} \qquad (26)$$

VI. BOUNDARY CONDITIONS

The boundary conditions for this problem are obtained from the constraint that the ends of the cable are firmly mounted. That is, the ends of the cable are not permitted to undergo any displacement. In terms of the dimensionless variables:

$$V_1(\mu_{10}) = Z[\sigma(\mu_{10})] = V_2(\mu_{20}) = Z[\sigma(\mu_{20})] = 0$$

where $\mu_1 = \mu_{10}$ is the value of μ defining the point of attachment in the region $\mu > 0$,

$\mu_2 = \mu_{20}$ is the value of μ defining the point of attachment in the region $\mu < 0$. .

From Equation (19), $Z[\sigma(\mu)] = 0$ implies that $\frac{dV}{d\mu}(\mu) = 0$. The boundary conditions may be rewritten as:

$$V_1(\mu_{10}) = \frac{dV_1}{d\mu_1}(\mu_{10}) = V_2(\mu_{20}) = \frac{dV_2}{d\mu_2}(\mu_{20}) = 0$$

Using these in conjunction with Equations (26), the following are obtained.

$$A_1 y_1(\mu_{10}) + A_2 y_2(\mu_{10}) + A_3 y_3(\mu_{10}) + A_4 y_4(\mu_{10}) = 0$$

$$A_1 \frac{dy_1}{d\mu_1}(\mu_{10}) + A_2 \frac{dy_2}{d\mu_1}(\mu_{10}) + A_3 \frac{dy_3}{d\mu_1}(\mu_{10}) + A_4 \frac{dy_4}{d\mu_1}(\mu_{10}) = 0 \qquad (27)$$

$$-A_1 y_1(\mu_{20}) - A_2 y_2(\mu_{20}) + A_3 y_3(\mu_{20}) + A_4 y_4(\mu_{20}) = 0$$

$$-A_1 \frac{dy_1}{d\mu_2}(\mu_{20}) - A_2 \frac{dy_2}{d\mu_2}(\mu_{20}) + A_3 \frac{dy_3}{d\mu_2}(\mu_{20}) + A_4 \frac{dy_4}{d\mu_2}(\mu_{20}) = 0$$

(27) cont.

From a well known theorem of Linear Algebra a non-trivial set of A_1, A_2, A_3, and A_4 are obtained if and only if the determinant of coefficients is equal to zero. Thus:

$$\begin{vmatrix} y_1(\mu_{10}) & y_2(\mu_{10}) & y_3(\mu_{10}) & y_4(\mu_{10}) \\ \frac{dy_1}{d\mu}(\mu_{10}) & \frac{dy_2}{d\mu}(\mu_{10}) & \frac{dy_3}{dn}(\mu_{10}) & \frac{dy_4}{d\mu}(\mu_{10}) \\ -y_1(\mu_{20}) & -y_2(\mu_{20}) & y_3(\mu_{20}) & y_4(\mu_{20}) \\ -\frac{dy_1}{d\mu}(\mu_{10}) & -\frac{dy_2}{d\mu}(\mu_{10}) & \frac{dy_3}{d\mu}(\mu_{20}) & \frac{dy_4}{d\mu}(\mu_{20}) \end{vmatrix} = 0$$

(28)

Recall that the coefficients in the series solutions depend upon λ . The problem is to find combinations of μ_{10}, μ_{20}, and λ that make this determinent equal to zero.

VII. COMPUTER RESULTS

A computer routine was developed to evaluate the determinant of the preceding section for given combinations of μ_{10}, μ_{20}, and λ (see Appendix). Each infinite series was approximated by the first 48 terms of that series. Some combinations of μ_{10}, μ_{20}, and λ which make the determinant equal to zero are presented in Table 1. In each case the value of λ is the lowest value, corresponding to the fundamental frequency of oscillation, for the given μ_{10} and μ_{20} to make the determinant equal zero. The program is capable of determining the higher order modes. A check on the methods developed in this text was the main purpose of the computer program, so higher order values were not obtained. The results are plotted in Figure 6 with some values from Saxon and Cahn (6) for cables with endpoints at equal heights .

As a secondary output from the computer program, a listing of the first seventy-three coefficients of each series and the sum of each series after eight, eighteen, forty-eight, and seventy-three terms for the case $\lambda=2.09$, $\mu_{10}=\mu_{20}=0.70$ was obtained. This was the largest value of μ_{10} used and it would be expected that the slowest convergence would correspond. The coefficients in the first three series approach, after ten terms, a constant value to five significant figures with alternating signs. The fourth series has coefficients that

Table 1. Fundamental frequencies for some cable configurations.

μ_{10}	μ_{20}	$\frac{\mu_{20}}{\mu_{10}}$	λ_1	α_{10}	α_{20}
.70	.70	1	2.09	54°	54°
.60	.60	1	2.31	51.3°	51.3°
.50	.50	1	2.59	48.2°	48.2°
.40	.40	1	2.98	44.4°	44.4°
.30	.30	1	3.55	39.8°	39.8°
.20	.20	1	4.52	33.6°	33.6°
.10	.10	1	6.67	24.7°	24.7°
.70	.525	.75	2.32	54°	49°
.70	.35	.50	2.81	54°	42.2°
.40	.30	.75	3.29	44.4°	39.8°
.40	.20	.50	3.95	44.4°	33.6°
.20	.15	.75	4.96	33.6°	29.2°
.20	.10	.50	5.87	33.6°	24.7°

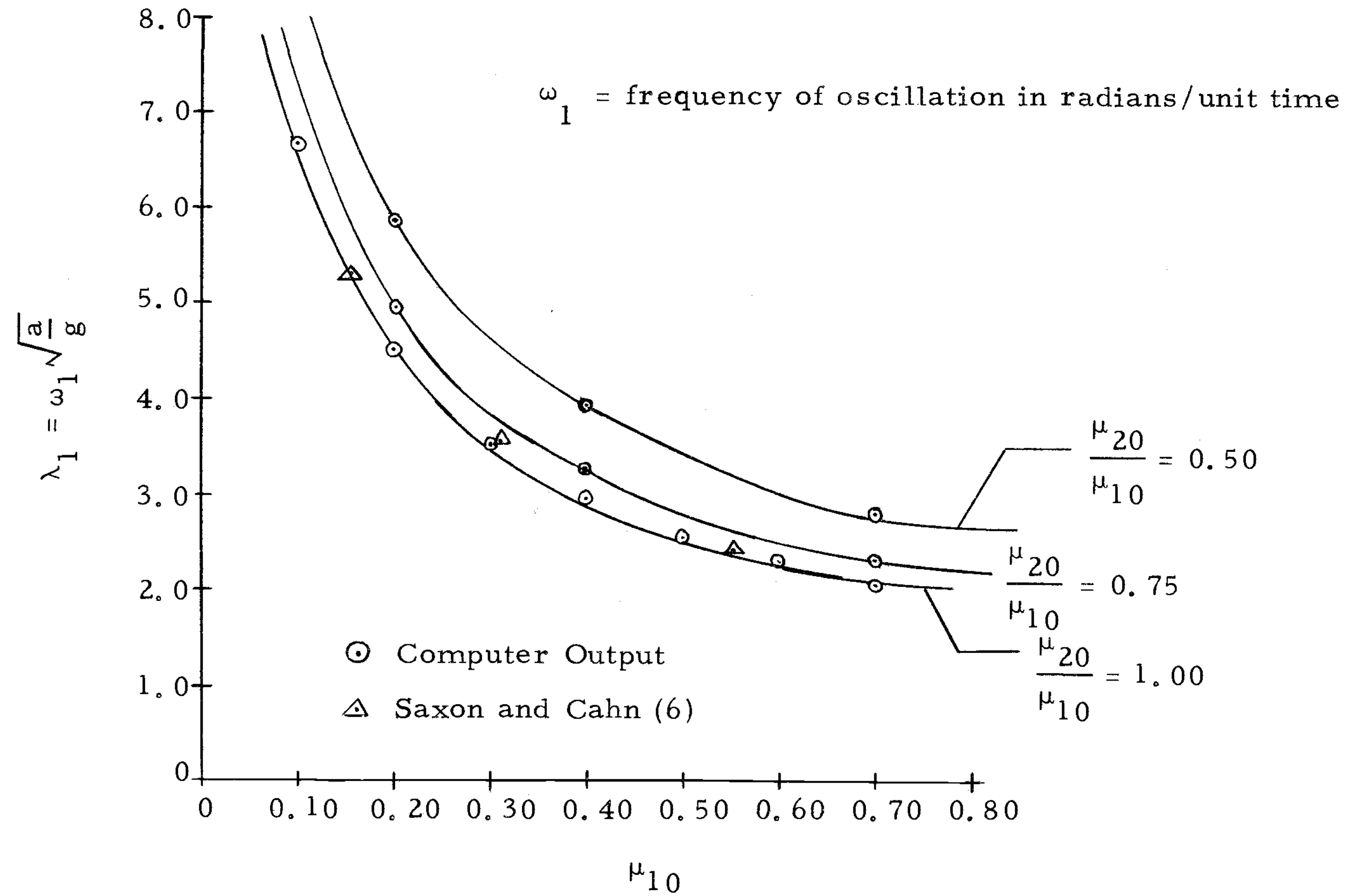

Figure 6. Frequency as a function of μ_{10} and μ_{20}.

continue to decrease with a change in sign every one or two terms. A comparison of the value of the sum of each series for different number of terms indicated that after forty-eight terms the sum was reasonably correct to four significant figures.

Figure 7 has the same information as Figure 6, but with the angles between the ends of the cable and the horizontal as the independent variables. The angle between the cable and the horizontal is given by:

$$\alpha = \arc\tan \sigma = \arc\tan [(1+\mu)^2 - 1]^{1/2} \quad .$$

At the ends of the cable:

$$\alpha_{10} = \arc\tan [(1+\mu_{10})^2 - 1]^{1/2}$$

$$\alpha_{20} = \arc\tan [(1+\mu_{20})^2 - 1]^{1/2}$$

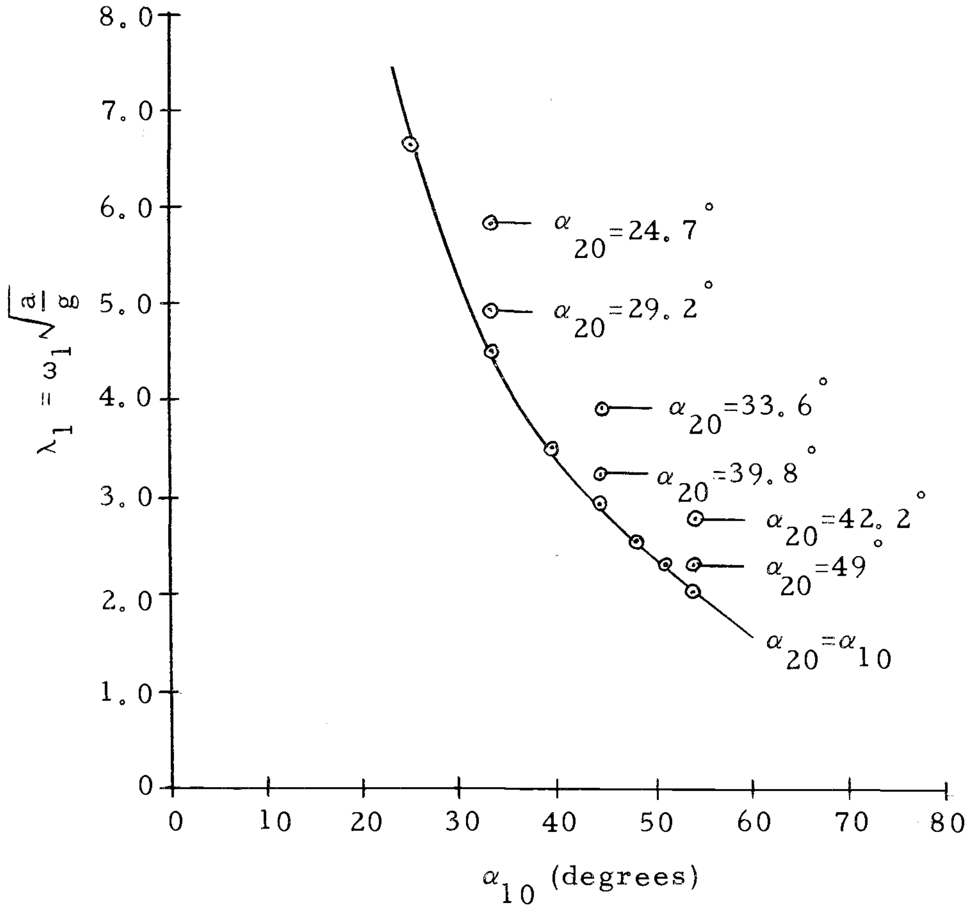

Figure 7. Frequency as a function of α_{10} and α_{20}.

VIII. DISCUSSION

The four infinite series developed for solution of the differential equation converge for $|\mu| < 1$. This limits the solution to cables having equilibrium configurations with $\alpha_{10}, \alpha_{20} \leq \tan^{-1}\sqrt{3} = 60^\circ$. Another set of series can be developed by expansion about another point, say $\mu=1$, to extend the solution to a larger range of lengths.

The modes shapes corresponding to a particular eigenvalue can be constructed by solving Equations (27) for A_1, A_2, A_3, and A_4, and forming the appropriate sum of the solutions. These are of interest if an initial condition problem is to be solved.

Some extensions of this work would be the solution of the problem including damping, elasticity in the cables, or driving forces such as prescribed motion at one end of the cable.

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APPENDIX

COMPUTER PROGRAM

```
      PROGRAM ROGER CABLE
      DIMENSION A(100),B(100),C(100),D(100),X(5),S(5),T(5),U(5),
V(5)
      REAL LAM
C     FREQ0=INITIAL VALUE OF LAMBDA.  DFREQ=INCREMENTAL CHANGE I
N LAMBDA.
C     ITER=NUMBER OF ITERATIONS ON LAMBDA.  NUM=NUMBER OF TERMS
IN SERIES.
    5 READ(60,10)X(1),X(2),FREQ0,DFREQ,ITER,NUM
      IF(X(1).EQ.0) GO TO 100
   10 FORMAT (4F8.0,2I4)
      ITER0=0
      A(1)=0
      A(2)=0
      A(3)=1.
      B(1)=0
      B(2)=0
      B(3)=1.
      C(1)=0
      C(2)=0
      C(3)=1.
      D(1)=0
      D(2)=0
      D(3)=1.
      WRITE(61,11)
   11 FORMAT('0',18X,'X(1)',6X,'X(2)',7X,'NUM',6X,'LAM',12X,'DET
')
   15 LAM=FREQ0*FREQ0
      A(4)=-1.-LAM
      B(4)=(-6.-LAM)/6.
      C(4)=(-69.-6.*LAM)/60.
      D(4)=(-5.-2.*LAM)/6.
      DO 20 J=5,NUM
      G=J
      K=J-4
      L=J-3
      M=J-2
      N=J-1
      A(J)=-((LAM*(G-7.)*(G-8.)+4.*LAM*(G-7.))*A(K)+((G-6.)*(G-7
.)*(G-8.
     1)*(G-9.)+10.*(G-6.)*(G-7.)*(G-8.)+24.*(G-6.)*(G-7.)+4.*LAM
*(G-6.)*
     2(G-7.)+12.*(G-6.)+12.*LAM*(G-6.))*A(L)+(5.*(G-5.)*(G-6.)*(
G-7.)*(G
     3-8.)+40.*(G-5.)*(G-6.)*(G-7.)+72.*(G-5.)*(G-6.)+5.*LAM*(G-
5.)*(G-6
     4.)+24.*(G-5.)+9.*LAM*(G-5.)-LAM)*A(M)+(8.*(G-4.)*(G-5.)*(G
-6.)*(G-
     57.)+46.*(G-4.)*(G-5.)*(G-6.)+51.*(G-4.)*(G-5.)+2.*LAM*(G-4
.)*(G-5.
     6)+6.*(G-4.)+LAM*(G-4.))*A(N))/(4.*(G-3.)*(G-4.)*(G-5.)*(G-
6.)+12.*
     7(G-3.)*(G-4.)*(G-5.)+3.*(G-3.)*(G-4.))
      B(J)=-((LAM*(G-6.)*(G-3.))*B(K)+((G-5.)*(G-6.)*(G-7.)*(G-8
.)+10.*(
     1G-5.)*(G-6.)*(G-7.)+24.*(G-5.)*(G-6.)+4.*LAM*(G-5.)*(G-6.)
+12.*(G-
     25.)+12.*LAM*(G-5.))*B(L)+(5.*(G-4.)*(G-5.)*(G-6.)*(G-7.)+4
0.*(G-4.
```

```
     3)*(G-5.)*(G-6.)+72.*(G-4.)*(G-5.)+5.*LAM*(G-4.)*(G-5.)+24.*(G-4.)+
     49.*LAM*(G-4.)-LAM)*B(M)+(8.*(G-3.)*(G-4.)*(G-5.)*(G-6.)+46.*(G-3.)
     5*(G-4.)*(G-5.)+51.*(G-3.)*(G-4.)+2.*LAM*(G-3.)*(G-4.)+6.*(G-3.)+LA
     6M*(G-3.))*B(N))/(4.*(G-2.)*(G-3.)*(G-4.)*(G-5.)+12.*(G-2.)*(G-3.)*
     7(G-4.)+3.*(G-2.)*(G-3.))
      C(J)=-((LAM*(G-5.5)*(G-6.5)+4.*LAM*(G-5.5))*C(K)+((G-4.5)*(G-5.5)*
     1(G-6.5)*(G-7.5)+10.*(G-4.5)*(G-5.5)*(G-6.5)+24.*(G-4.5)*(G-5.5)+4.
     2*LAM*(G-4.5)*(G-5.5)+12.*(G-4.5)+12.*LAM*(G-4.5))*C(L)+(5.*(G-3.5)
     3*(G-4.5)*(G-5.5)*(G-6.5)+40.*(G-3.5)*(G-4.5)*(G-5.5)+72.*(G-3.5)*(
     4G-4.5)+5.*LAM*(G-3.5)*(G-4.5)+24.*(G-3.5)+9.*LAM*(G-3.5)-LAM)*C(M)
     5+(8.*(G-2.5)*(G-3.5)*(G-4.5)*(G-5.5)+46.*(G-2.5)*(G-3.5)*(G-4.5)+5
     61.*(G-2.5)*(G-3.5)+2.*LAM*(G-2.5)*(G-3.5)+6.*(G-2.5)+LAM*(G-2.5))*
     7C(N))/(4.*(G-1.5)*(G-2.5)*(G-3.)*(G-2.))
 20   D(J)=-((LAM*(G-6.5)*(G-7.5)+4.*LAM*(G-6.5))*D(K)+((G-5.5)*(G-6.5)*
     1(G-7.5)*(G-8.5)+10.*(G-5.5)*(G-6.5)*(G-7.5)+24.*(G-5.5)*(G-6.5)+4.
     2*LAM*(G-5.5)*(G-6.5)+12.*(G-5.5)+12.*LAM*(G-5.5))*D(L)+(5.*(G-4.5)
     3*(G-5.5)*(G-6.5)*(G-7.5)+40.*(G-4.5)*(G-5.5)*(G-6.5)+72.*(G-4.5)*(
     4G-5.5)+5.*LAM*(G-4.5)*(G-5.5)+24.*(G-4.5)+9.*LAM*(G-4.5)-LAM)*D(M)
     5+(8.*(G-3.5)*(G-4.5)*(G-5.5)*(G-6.5)+46.*(G-3.5)*(G-4.5)*(G-5.5)+5
     61.*(G-3.5)*(G-4.5)+6.*(G-3.5)+2.*LAM*(G-3.5)*(G-4.5)+LAM*(G-3.5))*
     7D(N))/(4.*(G-1.5)*(G-2.5)*(G-3.)*(G-2.))
      S(1)=0
      S(2)=0
      S(3)=0
      S(4)=0
      T(1)=0
      T(2)=0
      T(3)=0
      T(4)=0
      U(1)=0
      U(2)=0
      U(3)=0
      U(4)=0
      V(1)=0
      V(2)=0
      V(3)=0
      V(4)=0
      DO 30 J=1,NUM
      G=J
      S(1)=A(J)*(X(1)**J)/(X(1)**3)+S(1)
      S(2)=B(J)*(X(1)**J)/(X(1)**2)+S(2)
```

```
      S(3)=C(J)*(X(1)**J)/(X(1)**1.5)+S(3)
      S(4)=D(J)*(X(1)**J)/(X(1)**2.5)+S(4)
      T(1)=A(J)*(G-3.)*(X(1)**J)/(X(1)**4)+T(1)
      T(2)=B(J)*(G-2.)*(X(1)**J)/(X(1)**3)+T(2)
      T(3)=C(J)*(G-3.5)*(X(1)**J)/(X(1)**2.5)+T(3)
      T(4)=D(J)*(G-2.5)*(X(1)**J)/(X(1)**3.5)+T(4)
      U(1)=A(J)*(X(2)**J)/(X(2)**3)+U(1)
      U(2)=B(J)*(X(2)**J)/(X(2)**2)+U(2)
      U(3)=C(J)*(X(2)**J)/(X(2)**1.5)+U(3)
      U(4)=D(J)*(X(2)**J)/(X(2)**2.5)+U(4)
      V(1)=A(J)*(G-3.)*(X(2)**J)/(X(2)**4)+V(1)
      V(2)=B(J)*(G-2.)*(X(2)**J)/(X(2)**3)+V(2)
      V(3)=C(J)*(G-3.5)*(X(2)**J)/(X(2)**2.5)+V(3)
   30 V(4)=D(J)*(G-2.5)*(X(2)**J)/(X(2)**3.5)+V(4)
      DET=S(1)*(T(2)*U(3)*V(4)-U(2)*V(3)*T(4)-V(2)*T(3)*U(4)+V(2
)*U(3)*T
     1(4)-V(3)*U(4)*T(2)+V(4)*U(2)*T(3))-S(2)*(T(1)*U(3)*V(4)-U(
1)*V(3)*
     2T(4)-V(1)*T(3)*U(4)+V(1)*U(3)*T(4)-V(3)*U(4)*T(1)+V(4)*U(1
)*T(3))+
     3S(3)*(-T(1)*U(2)*V(4)+U(1)*V(2)*T(4)-V(1)*T(2)*U(4)-V(1)*U
(2)*T(4)
     4+V(2)*U(4)*T(2)+V(4)*U(1)*T(2))-S(4)*(-T(1)*U(2)*V(3)+U(1)
*V(2)*T(
     53)-V(1)*T(2)*U(3)-V(1)*U(2)*T(3)+V(2)*U(3)*T(1)+V(3)*U(1)*
T(2))
      WRITE(61,12)X(1),X(2),NUM,FREQ0,DET
   12 FORMAT('0',17X,F5.2,5X,F5.2,7X,I3,4X,F7.4,2X,E15.7)
      FREQ0=FREQ0+DFREQ
      ITERO=ITERO+1
      IF(ITERO-ITER)15,15,99
   99 GO TO 5
  100 STOP
      END
```