

INFINITELY MANY LATTICE SURFACES WITH SPECIAL PSEUDO-ANOSOV MAPS

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ABSTRACT. We give explicit pseudo-Anosov homeomorphisms with vanishing Sah–Arnoux–Fathi invariant. Any translation surface whose Veech group is commensurable to any of a large class of triangle groups is shown to have an affine pseudo-Anosov homeomorphism of this type. We also apply a reduction to finite triangle groups and thereby show the existence of nonparabolic elements in the periodic field of certain translation surfaces.

1. INTRODUCTION

By integration, a holomorphic 1-form ω induces the structure of a translation surface on a Riemann surface X . Veech [32] showed that the group $SL(X, \omega)$ of linear parts of the orientation-preserving affine automorphisms of the translation surface is a discrete subgroup of $SL(2, \mathbb{R})$; that is, it is a Fuchsian group. A *lattice surface*, also called a Veech surface, is a translation surface with $SL(X, \omega)$ a lattice in $SL(2, \mathbb{R})$.

The Sah–Arnoux–Fathi (SAF) invariant for interval exchanges can be extended to directional flows on translation surfaces by use of transversals. This invariant vanishes for any direction of a periodic flow. After a suitable normalization, the set of slopes of all directions on a lattice surface with vanishing Sah–Arnoux–Fathi invariant is the projective line over a field, called the periodic field [12]. When this field is of degree at most two over the rationals \mathbb{Q} (as is the case for any translation surface of genus at most two), the corresponding directions on the lattice surface are all periodic [27].

The hyperbolic elements of $SL(X, \omega)$ correspond to pseudo-Anosov maps. The above normalization results in $SL(X, \omega)$ being a subgroup of the special linear group over the periodic field. A hyperbolic element whose eigenvalues are in this field is said to be *special*, the corresponding pseudo-Anosov maps are also said to be special. We easily establish that any special affine pseudo-Anosov automorphism has vanishing Sah–Arnoux–Fathi invariant.

In fact, the periodic field of a lattice surface is also its *trace field* [12]. That is, it is the extension over \mathbb{Q} generated by the traces of the elements of $SL(X, \omega)$. One

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can also consider the *invariant trace field*, the field generated by the traces of the squares of elements. Hooper [21] noted that a result of Kenyon–Smillie [22] implies that a Fuchsian group with hyperbolic elements can only be realized as some $\mathrm{SL}(X, \omega)$ if its trace and invariant trace fields are equal.

1.1. Main Result. Denote by $\Delta(m, n, \infty)$ the triangle Fuchsian group formed by the orientation-preserving subgroup of the group generated by reflections in the sides of a hyperbolic triangle having one ideal vertex and the other two of angles π/m and π/n .

THEOREM 1. *Suppose that both m, n are even, and that the trace field and the invariant trace field for the triangle group $\Delta(m, n, \infty)$ coincide. If a translation surface has Veech group commensurable to $\Delta(m, n, \infty)$, then the surface admits a pseudo-Anosov affine automorphism whose stable flow has vanishing Sah–Arnoux–Fathi invariant.*

Note that Bouw–Möller [8], as confirmed by Hooper [21], have shown that up to commensurability every $\Delta(m, n, \infty)$ is realized as a Veech group. Hooper succeeded in giving an elementary construction of the surfaces found by Bouw and Möller (as Hooper [21] verified for all but the case of m, n both even, and Wright [34] for these remaining cases). Using these, we give fully explicit special affine pseudo-Anosov diffeomorphisms, see Proposition 8, Example 9 and Figure 1.

We also show that certain other triangle groups are such that any translation surface whose Veech group is commensurable to the triangle group must have infinitely many nonperiodic directions with vanishing Sah–Arnoux–Fathi invariant. Informed by this, we obtain a further new special affine pseudo-Anosov diffeomorphism, see Example 19 and Figure 2. This particular example is on a surface with cubic trace field, and hence the flow in the expanding direction of the pseudo-Anosov map has rank three in the sense of McMullen [28]; we thus can give a pictorial representation of this flow on a genus 15 surface of the type that McMullen [28] gives for each of the cubic example of Arnoux and Yoccoz and the recent cubic example of Lanneau, again see Figure 2.

We note here that we had previously [11] built continued fraction algorithms for the Veech groups of the translation surfaces studied by [33]—these groups are Fuchsian triangle groups $\Delta(3, n, \infty)$ —in anticipation of using them to find nonparabolic directions on the surfaces or even special pseudo-Anosov diffeomorphisms. The Ward examples have proven to be significantly more resistant to the search for such phenomena than were the original Veech examples, and of course than the translation surfaces that we treat here.

1.2. Motivation and History. A holomorphic 1-form ω , induces the structure of a *translation surface* on a Riemann surface X —integration gives an atlas off of the zeros of ω with transitions functions being translations (completing to the induced flat metric to the zeros results in cone singularities). An *affine diffeomorphism* of the translation surface (X, ω) to itself is a homeomorphism taking singularities to singularities that is a diffeomorphism with respect to the translation structure off of the singularities. In local coordinates, such a map

is affine, with linear part constant on the surface. Veech [32] showed that the group $SL(X, \omega)$ of all of linear parts of the orientation-preserving affine diffeomorphisms is a discrete subgroup of $SL(2, \mathbb{R})$. A *lattice surface*, also called a Veech surface, is a translation surface with $SL(X, \omega)$ a lattice in $SL(2, \mathbb{R})$; that is, $SL(X, \omega)$ is of finite covolume with respect to Haar measure.

Any hyperbolic element of $SL(X, \omega)$ is the linear part of a pseudo-Anosov map (as defined by Thurston, see [31]); the stable directions of an affine pseudo-Anosov map correspond to the eigenvectors of its linear part. The first return map to any full transversal to the linear flow in a fixed direction of a translation surface defines an interval exchange transformation. The Sah–Arnoux–Fathi (SAF) invariant of the interval-exchange transformation vanishes for a periodic flow. McMullen [27] reformulates the invariant as his “flux” and shows in [28] that the vanishing of this flux for the flow in the expanding direction of an affine pseudo-Anosov diffeomorphism implies zero average drift for the leaves of the foliation; see also [3, 19, 20].

The first examples of an affine pseudo-Anosov with vanishing Sah–Arnoux–Fathi invariant were given by Arnoux and Yoccoz [5]; they used a construction involving suspension of interval-exchange transformations. Arnoux and Schmidt [4] found further examples on the lattice surfaces given by gluing two copies of the regular n -gon together along opposite edges for $n \in \{7, 9, 14, 18, 20, 24\}$. The discovery of these maps was especially surprising considering that the surfaces themselves are among the well-studied first examples of Veech of nonarithmetic lattice surfaces, [32]. Recently, McMullen [28] communicated an example found by Laneeau of a special pseudo-Anosov diffeomorphism with vanishing Sah–Arnoux–Fathi invariant on a genus three surface.

We give a large class of affine pseudo-Anosov diffeomorphisms with vanishing Sah–Arnoux–Fathi invariant. We do this by finding so-called *special pseudo-Anosov* diffeomorphisms. Long and Ried [24] say that a hyperbolic element of a Fuchsian group is *special* if its eigenvalues lie in the trace field of the group. Accordingly, an affine pseudo-Anosov diffeomorphism is said to be special if the eigenvalues of its linear part lie in the trace field of the Veech group of its translation surface. Calta and Smillie [12] show that under fairly mild hypotheses, see Lemma 2, one can normalize a translation surface so that the set of (cotangents of) directions for flows with vanishing Sah–Arnoux–Fathi invariant forms the trace field (and infinity). With this normalization, each special affine pseudo-Anosov diffeomorphism has vanishing Sah–Arnoux–Fathi invariant.

2. BACKGROUND

2.1. Translation surfaces and Veech groups. A *translation surface* is a 2-manifold with finitely many marked points and an atlas whose transition functions are translations. This is equivalent to the definition of a translation surface as a disjoint union of finitely many polygons P_1, \dots, P_n in \mathbb{R}^2 glued along parallel edges to form a closed surface. The marked points are cone points, which can arise at vertices of the P_i when too many polygons are glued around a single

vertex, resulting in a total angle at that vertex of $2k\pi$ where $k \geq 1$ is an integer. Equivalently, a translation surface can be seen as a pair (M, ω) where M is a Riemann surface and ω an abelian differential on M —away from the zeros of the abelian differential, integration of the abelian differential gives local coordinates with transition functions that are translations.

The group $\mathrm{SL}(2, \mathbb{R})$ acts on the moduli space of translation surfaces, preserving genus and the number and order of cone points. In the polygonal model, if $S = P_1, \dots, P_n$ is a surface and $g \in \mathrm{SL}(2, \mathbb{R})$ then gS is defined as $gP_1 \cup \dots \cup gP_n$. The stabilizer of a surface S under this action is called its *Veech group*, which is always a noncocompact Fuchsian group. Generically this group is empty but occasionally it is a *lattice* subgroup of $\mathrm{SL}(2, \mathbb{R})$, which is to say that it has finite covolume. In this case, we refer to the surface as a *lattice surface*. Veech [32] proved that a pair of regular n -gons glued together along parallel sides forms a lattice surface for $n \geq 4$. Whether or not a surface is a lattice surface has profound implications for the dynamics of the linear flow on the surface. Veech [32] proved that in any direction v on a lattice surface, the orbits of the linear flow in that direction are either closed or connect two cone points, or all orbits in that direction are uniformly distributed on the surface. In the literature, this dynamic dichotomy has come to be called “optimal dynamics”.

2.2. The Sah–Arnoux–Fathi Invariant. Suppose f is an interval-exchange transformation on a finite interval I , that is, a piecewise linear orientation-preserving isometry of I . Then by definition f exchanges n intervals I_i of lengths l_i for $i = 1, \dots, n$ by translating each I_i by the amount t_i . One can associate to f a certain invariant known as the Sah–Arnoux–Fathi (SAF) invariant that takes values in $\mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$ and is defined as $\sum_{i=1}^n l_i \wedge_{\mathbb{Q}} t_i$. The Sah–Arnoux–Fathi invariant is a central tool in the study of the dynamics of the linear flow on translation surfaces. Given a direction v on a surface, one can choose an interval transverse to the orbits of the flow in the direction v that meets every orbit. The first-return map to this interval is an interval-exchange transformation. If the flow in the direction v is periodic then the associated Sah–Arnoux–Fathi invariant is zero. The converse, however, is false. The dynamics of the flow in SAF-zero directions has been an object of recent interest and in this paper we show that there are surfaces for which the flow in a particular SAF-zero direction is the expanding direction of a pseudo-Anosov diffeomorphism.

There is another way to define the Sah–Arnoux–Fathi invariant of a direction on a translation surface using the J -invariant of Kenyon and Smillie [22]. The J -invariant of a polygon takes values in $\mathbb{R}^2 \wedge_{\mathbb{Q}} \mathbb{R}^2$. If the vertices of a polygon P are v_0, \dots, v_n , then $J(P) = \sum_{i=0}^n v_i \wedge v_{i+1}$, where $v_{n+1} = v_0$. Since a translation surface S can be realized as a disjoint union of polygons P_i for $i = 1, \dots, n$ glued along parallel sides, we define $J(S)$ to be $\sum_{i=1}^n J(P_i)$. Then the projection $J_v(S)$ of $J(S)$ in the direction v is the Sah–Arnoux–Fathi invariant of the interval-exchange transformation which is the first-return map on a full transversal to the linear flow in the direction v . It is not hard to see that the Sah–Arnoux–Fathi invariant

of a periodic interval-exchange transformation is zero. Thus the Sah–Arnoux–Fathi invariant in a parabolic direction on a surface is zero.

Also note that Lemma 2.4 of the Appendix of Calta’s [10] directly implies that the Sah–Arnoux–Fathi invariant in the form of $J_\nu(S)$ is constant on $\mathrm{SL}_2(\mathbb{R})$ -orbits, in the sense that $J_\nu(S) = J_{A\nu}(A \circ S)$ for any $A \in \mathrm{SL}_2(\mathbb{R})$.

2.3. Fields and translation surfaces. If a translation surface has at least three directions of vanishing Sah–Arnoux–Fathi invariant, then Calta and Smillie [12] show that the surface can be normalized by way of the $\mathrm{SL}_2(\mathbb{R})$ -action so that the directions with slope 0, 1 and ∞ have vanishing Sah–Arnoux–Fathi invariant and prove that on the normalized surface the set of slopes of directions with vanishing invariant is a field union with infinity. A translation surface so normalized is said to be in *standard form*, and the field so described is called the *periodic direction field*. In this paper, we are primarily interested in directions on a surface that come from the periodic direction field.

On the other hand, Kenyon and Smillie [22] defined the *holonomy field* of a translation surface as the smallest field over which the set of holonomy vectors is contained in a two-dimensional vector space. A holonomy vector is associated via the developing map to a closed, nonsingular curve on the surface or to a closed curve that is a union of saddle connections.

Gutkin and Judge define the *trace field* of a surface to be the extension of \mathbb{Q} generated by the traces of the elements of its Veech group. Since the trace is a conjugacy invariant, the trace field of a given surface is the same for as that of any other surface in its $\mathrm{SL}(2, \mathbb{R})$ orbit. Calta and Smillie [12] show that if S is a lattice surface, then the holonomy, trace and periodic direction fields are all equal.

The following is a direct implication of the Calta–Smillie [12] result that the periodic field of a translation surface in standard form equals its trace field.

LEMMA 2. *On a translation surface in standard form, the stable directions of an affine pseudo-Anosov diffeomorphism have vanishing Sah–Arnoux–Fathi invariant if and only if the pseudo-Anosov diffeomorphism is special.*

2.4. Triangle groups and realizability. Of central importance to us is the fact that triangle groups are (up to finite index) realized as Veech groups.

THEOREM 3 (Bouw–Möller, Hooper). *Every hyperbolic triangle group with parabolic elements is commensurable to a group realized as the Veech group of a translation surface.*

However, not every full triangle group can be realized as a Veech group. Hubert and Schmidt [18] remarked that one can use the fundamental observation of Kenyon and Smillie [22] that the trace field of a translation surface is generated by the trace of any of its affine pseudo-Anosov diffeomorphisms to show that no triangle group of signature $(2, 2n, \infty)$ can be realized as a Veech group. Hooper [21] uses the observation in the form that a Fuchsian group can only be realized as a Veech group if its trace field equals the field generated over

the rationals by the traces of the squares of elements of the group; this latter field is called the *invariant trace field* of the group, as Margulis proved that it is an invariant of the (wide) commensurability class of the group, see [26] for a discussion.

3. SPECIAL AFFINE PSEUDO-ANOSOV DIFFEOMORPHISMS

In this section, we focus on the arithmetic of triangle groups in order to find the linear parts of special affine pseudo-Anosov diffeomorphisms. To do this, Lemma 5 is key. It allows us to prove a more precisely worded version of our main theorem, Theorem 7, thus giving fully explicit pseudo-Anosov maps, as shown by Example 9 and Figure 1. In the final subsection, we give some results about the groups for which the full triangle group is never a Veech group.

3.1. A special hyperbolic matrix. Our approach here is centered on properties of groups realized as Veech groups of translation surfaces. Since any parabolic direction on a translation surface has flow of vanishing Sah–Arnoux–Fathi invariant we use the following variation of a term used by Calta and Smillie [12].

DEFINITION 4. A Fuchsian group is in (group) *parabolic standard form* if its set of parabolic fixed points (for its action on the Poincaré upper half-plane) includes 0, 1 and ∞ .

Note that since the Sah–Arnoux–Fathi invariant of a periodic flow is zero, whenever the Veech group of a translation surface is in parabolic standard form, the surface itself is in the standard form defined by Calta and Smillie.

To expedite discussion, we take a specific representation for each of the triangle groups we consider. Let $G_{m,n}$ be generated by

$$(1) \quad A = \begin{pmatrix} 1 & 2\cos\pi/m + 2\cos\pi/n \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2\cos\pi/m & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -2\cos\pi/n & 1 \\ -1 & 0 \end{pmatrix},$$

and note that $C = AB$. The group is easily verified to be a Fuchsian triangle group of signature (m, n, ∞) . The trace field of $G_{m,n}$ is

$$K_{m,n} = \mathbb{Q}(\cos\pi/m, \cos\pi/n),$$

see [26, p. 159].

LEMMA 5. Suppose m, n are distinct and even. If α is a nonzero finite parabolic fixed point of $G_{m,n}$, then multiplication by α^{-1} defines a transformation that conjugates $G_{m,n}$ to a group in parabolic standard form, with special hyperbolic elements.

Proof. Being generated by A and B , the group $G_{m,n}$ clearly has infinity as a parabolic fixed point. Since B sends 0 to infinity, 0 is also a parabolic fixed point.

Recall that the product of any two distinct elliptic elements of order two in $\mathrm{PSL}(2, \mathbb{R})$ is a hyperbolic element. Here, certainly $B^{m/2}, C^{n/2}$ are elliptic of order

two, their product is thus hyperbolic. Now, for each integer k ,

$$(2) \quad B^k = \begin{pmatrix} b_{k+1} & b_k \\ -b_k & -b_{k-1} \end{pmatrix}, \quad C^k = (-1)^k \begin{pmatrix} c_{k+1} & -c_k \\ c_k & -c_{k-1} \end{pmatrix}$$

where $b_k = \sin \frac{k\pi}{m} / \sin \frac{\pi}{m}$, and $c_k = \sin \frac{k\pi}{n} / \sin \frac{\pi}{n}$, see, say [9]; thus, one finds that

$$B^{m/2} \cdot (\pm 1) = \mp 1, \quad C^{n/2} \cdot (\pm 1) = \mp 1.$$

Thus, their product $B^{m/2}C^{n/2}$ fixes both -1 and 1 .

Since $G_{m,n} \subset \mathrm{SL}_2(\mathcal{O}_K)$, all finite parabolic fixed points of $G_{m,n}$ lie in the field $K_{m,n}$. Let α be as in our hypotheses. By the triple transitivity of $\mathrm{SL}_2(\mathbb{R})$ acting on the real projective line, there is an element M that sends α to 1 while fixing both zero and infinity. But elementary considerations of 2×2 -matrices show that the action of M is simply multiplication by α^{-1} . The conjugation of $G_{m,n}$ by M is clearly in parabolic standard form and the hyperbolic fixed point 1 of $G_{m,n}$ corresponds to the point α^{-1} —this is an element of K fixed by some hyperbolic element of the conjugate group. That is, the conjugate group has special hyperbolic elements. \square

3.2. Special affine pseudo-Anosov diffeomorphisms. Recall that W. P. Hooper determined conditions under which the trace field and invariant trace field of a triangle group $\Delta(m, n, \infty)$ coincide; the conditions are given in terms of the indices m, n and their greatest common divisor. Inequality of the two fields is an obstruction to the group being a Veech group of any translation surface.

DEFINITION 6. Given a pair of integers m, n , let $\gamma = \gcd(m, n)$. We say that the pair m, n is *unobstructed* if neither of the following conditions hold:

1. $\gamma = 2$;
2. m/γ and n/γ are both odd.

Our main result can be more precisely stated as follows.

THEOREM 7. *Let m, n be an unobstructed pair of even integers, and suppose that \mathcal{S} is a translation surface whose Veech group is commensurable to $G_{m,n}$. Then some power of $B^{m/2}C^{n/2}$ is the linear part of a special affine pseudo-Anosov automorphism of \mathcal{S} .*

Proof. Suppose that \mathcal{S} is a translation surface whose Veech group is commensurable to $G_{m,n}$. Using the $\mathrm{SL}_2(\mathbb{R})$ -action, we may assume that \mathcal{S} has as its Veech group a finite-index subgroup of $G_{m,n}$. Lemma 5 now provides an element of $\mathrm{SL}_2(\mathbb{R})$ conjugating $G_{m,n}$ into parabolic standard form while conjugating $B^{m/2}C^{n/2}$ to a special hyperbolic matrix.

The set of parabolic fixed points is unaltered by passage to a finite-index subgroup, thus the image of \mathcal{S} by this conjugating element is in standard form. By the work of Calta and Smillie [12] the set of (nonvertical) directions for which the flow has vanishing Sah–Arnoux–Fathi invariant forms a field, the periodic field, and since the translation surface certainly has some affine pseudo-Anosov automorphism, this equals the trace field of the Veech group.

Now, there is some nonzero power of our (special) hyperbolic element of the larger group that belongs to the finite-index subgroup. Since the two fixed points are common to the cyclic subgroup generated by a hyperbolic element, this element of the Veech group fixes points in the periodic field. From this, the corresponding pseudo-Anosov map has vanishing Sah–Arnoux–Fathi invariant. Finally, by [10, Lemma 2.4 of the Appendix], any $M \in \mathrm{SL}_2(\mathbb{R})$ sends this direction to a direction on the M -image of this surface that also has vanishing Sah–Arnoux–Fathi invariant. In particular, we can return in this way to the original direction and surface we began with; the result thus holds. \square

Our construction easily leads to explicit examples. Hooper [21] explicitly realizes the Bouw–Möller translation surfaces in two different ways. (Wright [34] finishes the verification that Hooper’s surfaces are indeed those of Bouw–Möller.) First, by way of grid graphs presenting the combinatorics of the intersections of horizontal and vertical cylinders so as to apply the Thurston construction [31]; the resulting surfaces Hooper denotes by $(X_{m,n}, \omega_{m,n})$. Second, he denotes by $(Y_{m,n}, \eta_{m,n})$ the translation surfaces formed by appropriately identifying sides of semiregular polygons. In the case where both m and n are even, there is a natural involution on the surface. Hooper denotes the resulting respective quotients by $(X_{m,n}^e, \omega_{m,n}^e)$, $(Y_{m,n}^e, \eta_{m,n}^e)$. Hooper shows that the Veech group of $(X_{m,n}, \omega_{m,n})$ is an index-two subgroup of $G_{m,n}$ and that the transformation $z \mapsto D_\mu(z) = (\csc \pi/n)z - \cot \pi/n$ conjugates this to the Veech group of $(Y_{m,n}, \eta_{m,n})$.

PROPOSITION 8. *Suppose m, n is an obstructed pair of even integers. Then on each of Hooper’s translation surfaces $(Y_{m,n}, \eta_{m,n})$ and $(Y_{m,n}^e, \eta_{m,n}^e)$ the flow in the direction $(1 - \cos \pi/n)/(\sin \pi/n)$ is a stable direction of a special affine Anosov diffeomorphism. Furthermore, letting*

$$\lambda = \frac{\cos \frac{\pi}{m} \cos \frac{\pi}{n} + \cos \frac{\pi}{m} + \cos \frac{\pi}{n} + 1}{\sin \frac{\pi}{m} \sin \frac{\pi}{n}},$$

the dilatation of this Anosov diffeomorphism is λ if four divides $\gcd(m, n)$ and λ^2 otherwise.

Proof. Since any power of $B^{m/2}C^{n/2}$ fixes the direction $z = 1$, we find that the direction $D_\mu(1) = (1 - \cos \pi/n)/(\sin \pi/n)$ determines a flow on each of $(Y_{m,n}, \eta_{m,n})$ and $(Y_{m,n}^e, \eta_{m,n}^e)$ with vanishing Sah–Arnoux–Fathi invariant. Hooper shows that B^2, C^2 (in our notation) are in the Veech group of $(X_{m,n}, \omega_{m,n})$, whereas neither B nor C is. It follows that $B^{m/2}C^{n/2}$ itself is in this group if and only if four divides $\gcd(m, n)$. Otherwise it is the square of this special hyperbolic element that belongs to the group.

The dilatation of an affine pseudo-Anosov automorphism is the larger of the two eigenvalues of its linear part, and here this is the same as that of $B^{m/2}C^{n/2}$ or of its square. Thus, the dilatation is as claimed. \square

EXAMPLE 9. In Figure 1, we show the result when $(m, n) = (8, 4)$; the translation surface $(Y_{8,4}^e, \eta_{8,4}^e)$ is a suspension surface over an interval-exchange transformation on eleven intervals with permutation in the usual redundant notation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 7 & 5 & 2 & 10 & 3 & 1 & 11 & 8 & 4 & 6 & 9 \end{pmatrix}.$$

The dilatation of this pseudo-Anosov map is the quartic number $3 + 2\sqrt{2} + \sqrt{20 + 14\sqrt{2}}$.

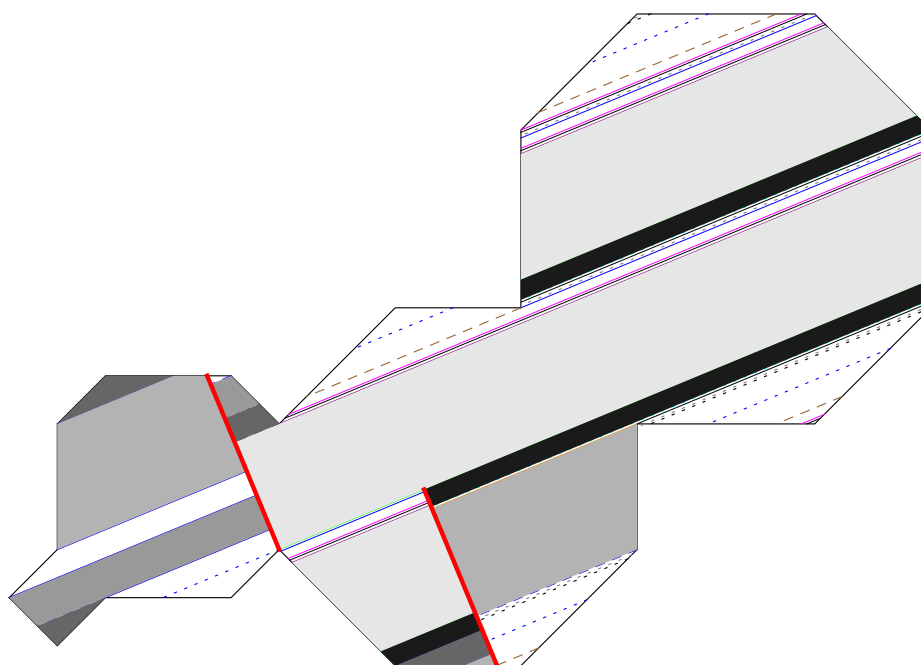


FIGURE 1. A pseudo-Anosov map with vanishing Sah–Arnoux–Fathi invariant, indicated as zippered rectangles on Hooper’s translation surface $(Y_{8,4}^e, \eta_{8,4}^e)$. Thick (red) intervals comprise the transversal to the flow; rectangles 1,2,3,9,11 are shaded.

3.3. Remarks on the “obstructed” setting. We now show that Lemma 5 cannot provide information about the existence of special pseudo-Anosov maps when the trace field of the group $G_{m,n}$ differs from its invariant trace field.

PROPOSITION 10. *If m, n are such that the trace field of $G_{m,n}$ does not equal its invariant trace field, then no nonzero parabolic fixed point of $G_{m,n}$ lies in its invariant trace field.*

Proof. Recall that the trace field $K_{m,n}$ of $G_{m,n}$ is generated over \mathbb{Q} by the pair $\{\cos \pi/m, \cos \pi/n\}$ and Hooper shows that the triple

$$\{\cos 2\pi/m, \cos 2\pi/n, \cos \pi/m \cos \pi/n\}$$

generates the invariant trace field, $k_{m,n}$, see also p. 159 of [26]. Define $\delta = 2(\cos 2\pi/n + 1)$; note that $\delta > 0$ since $n > 2$. One easily shows that $K_{m,n} = k_{m,n}(\sqrt{\delta})$ since first a standard double angle formula implies that $2\cos \pi/n = \sqrt{\delta}$, and then of course $\cos \pi/m = (\cos \pi/m \cos \pi/n) / \cos \pi/n$.

Any pair of the elements in (1) generate $G_{m,n}$, thus this group is generated by

$$C = \begin{pmatrix} -\sqrt{\delta} & 1 \\ -1 & 0 \end{pmatrix}, A = \begin{pmatrix} 1 & b\sqrt{\delta} \\ 0 & 1 \end{pmatrix},$$

with

$$b = \frac{\delta + 4\cos \frac{\pi}{m} \cos \frac{\pi}{n}}{\delta}.$$

Of course, $b \in k_{m,n}$ and thus $b\sqrt{\delta} \in K_{m,n} \setminus k_{m,n}$. We thus define two types of elements of $G_{m,n}$ —an *even* element is of the form

$$\begin{pmatrix} a & b\sqrt{\delta} \\ c\sqrt{\delta} & d \end{pmatrix},$$

with $a, b, c, d \in K_{m,n}$; similarly, an *odd* element is of the form

$$\begin{pmatrix} a\sqrt{\delta} & b \\ c & d\sqrt{\delta} \end{pmatrix};$$

in particular, A is of even type while C is of odd type. Multiplication thus is similar to addition of integers in that the product of two elements of the same type is an even element, whereas the product of two elements of distinct types is an odd element. Now, any element of $G_{m,n}$ is a product in powers of A and C , and therefore is of one of the two types. But, the image of ∞ under such a group element is then of the form $a/c\sqrt{\delta}$ or $a\sqrt{\delta}/c$; any nonzero $a/c \in k_{m,n}$ of course has a multiplicative inverse in this field, and hence were the image of infinity to be in $k_{m,n}$, the contradiction of $\sqrt{\delta}$ also belonging to this field would follow. \square

The notion of even and odd subgroups of $G_{m,n}$ defined in the proof of Proposition 10 generalizes a notion for the Hecke triangle groups, of signature $(2, q, \infty)$ with q even; see, say Rankin's review MR0529968 in Mathematical Reviews.

LEMMA 11. *Suppose that m, n are such that the trace field of $G_{m,n}$ does not equal its invariant trace field. If H is a subgroup of $G_{m,n}$ that contains a parabolic element and H is realized as a Veech group of some translation surface, then H is contained in the even subgroup of $G_{m,n}$.*

Proof. The Kenyon–Smillie result shows that if a hyperbolic element of $G_{m,n}$ lies in some Veech group, then this element is contained in the even subgroup. Suppose now that some subgroup H is *not* contained in the even subgroup of $G_{m,n}$. Since parabolic elements of $G_{m,n}$ are conjugates of powers of A they are even elements; thus H either contains a hyperbolic element and we are done, or it contains an odd elliptic element. Denote this odd elliptic element by E .

Choose a parabolic element $P \in H$, so $P = MA^r M^{-1}$ for some $M \in G_{m,n}$ and some integer r . Let $F = M^{-1}EM$, so that

$$PE = MA^r M^{-1} MFM^{-1} = MA^r FM^{-1}$$

has the trace of $A^r F$. Represent F as a 2×2 real matrix with $(2, 1)$ -entry c ; this entry is nonzero as F is elliptic. Thus, the trace of PE equals $\text{tr} F + r(2 \cos \pi/m + 2 \cos \pi/n)c$. However, we can replace $P \in H$ by any power of P and hence ensure that the trace of this *odd* element of H is greater than 2 in absolute value. That is, we have found an odd hyperbolic element in H , obstructing this subgroup from being the Veech group of any translation surface. \square

4. NONPARABOLIC DIRECTIONS IN THE PERIODIC FIELD

We show that certain infinite families of $G_{m,n}$ are such that any translation surface with Veech group commensurable to $G_{m,n}$ has infinite classes of non-parabolic directions in its periodic field. A search through these classes can reveal special affine pseudo-Anosov automorphisms, as we found for the $(m, n) = (7, 7)$ case; see Example 19. (This type of informed search was used in [4].)

The technique we employ is number-theoretic. Since all entries of the generators given in (1) are algebraic integers it easily follows that $G_{m,n}$ is a subgroup of $\text{SL}_2(\mathcal{O}_K)$, where \mathcal{O}_K is the ring of integers of the field $K = K_{m,n}$. The quotient of \mathcal{O}_K by any of its prime ideals is a finite field \mathbb{F} , and there is induced homomorphism from $G_{m,n}$ to a subgroup of $\text{SL}_2(\mathbb{F})$, as well as from the projectivisation of $G_{m,n}$ to $\text{PSL}_2(\mathbb{F})$; these *reduction homomorphisms* are defined by entrywise reduction of the matrices in our group, that is each entry is sent to its equivalence class modulo the ideal. Now, $\text{SL}_2(\mathbb{F})$ acts on the finite projective line $\mathbb{P}^1(\mathbb{F})$, and if the image of $G_{m,n}$ fails to act transitively there, then $G_{m,n}$ itself must fail to act transitively on $\mathbb{P}^1(K)$. In this case, the preimages of elements not in the orbit of infinity are then elements of K that are not parabolic fixed points.

This method goes back to Borho [6] and Borho–Rosenberger [7], in the setting of the Hecke triangle groups, where it was further pursued by a school about Leutbecher, see [16] for a recent usage in that setting. Underlying the method is the classification of the subgroups of the various $\text{PSL}_2(\mathbb{F})$ by Dickson [14] and Macbeath's [25] application of this to study finite triangle groups. We call the kernel of the reduction homomorphism a *congruence subgroup* of $G_{m,n}$. Congruence subgroups of the Hecke triangle groups have been studied for various reasons, see for example [30, 15]. The action of the full Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the algebraic curves uniformized by those congruence subgroups of Hecke groups corresponding to surjective reduction maps is studied in [29]. Indeed, as Macbeath showed, the reduction of a triangle group (with parabolics) is almost always the full finite matrix group, see the recent work of Clark and Voight [13] for further discussion.

REMARK 12. There is a more elementary manner to prove the existence of non-parabolic points in special cases. Indeed, the trace field $K = K_{m,n}$ of $G_{m,n}$ is totally real, thus in the setting where $G_{m,n}$ is a subgroup of $\text{SL}_2(\mathcal{O}_K)$, the fact that

the cusps of the Hilbert modular group $\mathrm{SL}_2(\mathcal{O}_K)$ are in 1-to-1 correspondence with the elements of the class group of K shows that whenever the class number of K is greater than one there must be elements of K that are nonparabolic fixed points. See [4] for further discussion.

REMARK 13. In answer to a question from the referee, we point out that there is to date no known example of a point in the periodic field of a lattice surface that is neither a parabolic fixed point nor a hyperbolic fixed point. There is, however at least one candidate for such a point, given by considering the double 11-gon translation surface of the initial Veech [32] examples, in conjunction with Example 3 of [16].

When considering quotients by prime ideals of rings of integers of fields generated by cosine values, the following lemma of Leutbecher is of great utility.

LEMMA 14 (Leutbecher [23]). *Given an integer $m \geq 3$, let $\lambda = \lambda_m = 2\cos\pi/m$. If m is not twice a power of a prime, then λ is a unit in the ring of integers \mathcal{O}_K of $K = \mathbb{Q}(\lambda)$. Otherwise, if $m = 2p^k$ for some prime p , $\lambda^{\phi(m)}$ is an associate in this ring of p ; here as usual $\phi(\cdot)$ denotes Euler's totient function.*

Similarly, we need the following.

LEMMA 15. *If at least one of m, n is odd, then $G_{m,n}$ is in parabolic standard form.*

Proof. Recall that both 0 and ∞ are parabolic fixed points of any $G_{m,n}$. From (2), we find that if $m = 2\ell + 1$ or $n = 2j + 1$ is odd, then

$$B^\ell \cdot (-1) = 0, \quad C^j \cdot (1) = 0,$$

respectively. This as, $\sin \frac{(\ell+1)\pi}{2\ell+1} = \sin \frac{\ell\pi}{2\ell+1}$, and similarly in the other case. Thus, if at least one of m, n is odd, then all three of 0, 1, ∞ are parabolic fixed points of our group. \square

4.1. Nonparabolic directions: when $m = 2^d$ and $n \neq 2^f + 1$ is odd.

PROPOSITION 16. *Suppose $m = 2^d$ with $d > 1$ and that n is odd with $n \neq 2^f + 1$ for any f . Then $G_{m,n}$ is in parabolic standard form and is integrally normalized; furthermore, any finite-index subgroup of $G_{m,n}$ that is realized as a Veech group is such that the corresponding translation surface has nonparabolic directions with vanishing Sah–Arnoux–Fathi invariant.*

Proof. By Lemma 15 $G_{m,n}$ is in parabolic standard form. By Lemma 14 with $m = 2^d$, we find that the rational integral ideal (2) factors as $(2\cos\pi/m)^{\phi(m)}$. Now, with $K = K_{m,n}$ choose any prime ideal of \mathcal{O}_K lying above $(2\cos\pi/m)$, say \mathfrak{p} . We have $\mathcal{O}_K/\mathfrak{p} \cong \mathbb{F}_{2^f}$, where f is the residue degree of \mathfrak{p} . This induces a group homomorphism $\mathrm{PSL}_2(\mathcal{O}_K) \rightarrow \mathrm{PSL}_2(\mathbb{F}_{2^f})$ that sends B to an element of order two, and hence the image of our group is a dihedral group of order $2n$. Arguing as in [7], this dihedral group is transitive on $\mathbb{P}^1(\mathbb{F}_{2^f})$ only if $n = 2^f + 1$. (Since (2) is totally ramified to $\mathbb{Q}(2\cos\pi/m)$, the residue degree of \mathfrak{p} is the residue degree of the ideal of $\mathbb{Q}(2\cos\pi/n)$ that \mathfrak{p} lies above.)

Thus, when n is not of the form $2^f + 1$ the orbit of infinity under $G_{m,n}$ does not equal all of $\mathbb{P}^1(K)$. That is, there are elements of $\mathbb{P}^1(K)$ that are not parabolic fixed points. Since $K = K_{m,n}$ also equals the invariant trace field $k_{m,n}$ by [26, p. 159], and as verified in detail by Hooper, K is the trace field of any finite-index subgroup of $G_{m,n}$. But, the union of the parabolic fixed points of any such subgroup is simply the set of parabolic fixed points of $G_{m,n}$. This is hence a proper subset of the trace field of the subgroup. Since $G_{m,n}$ is in parabolic standard form, so is any finite-index subgroup; thus, by Calta–Smillie, $\mathbb{P}^1(K)$ is the set of directions with vanishing Sah–Arnoux–Fathi invariant for the corresponding surface. \square

EXAMPLE 17. Let $m = 4$ and $n = 7$. Recall that $\mathbb{Z}[2\cos\pi/7]$ is the full ring of integers of $\mathbb{Q}(2\cos\pi/7)$. The minimal polynomial of $2\cos\pi/7$ over \mathbb{Q} (and hence over \mathbb{Z} , as this is an algebraic integer) is $p(x) = x^3 - x^2 - 2x + 1$. The reduction of $p(x)$ modulo two is irreducible; from this, the ideal (2) is inert to $\mathbb{Q}(2\cos\pi/7)$, and the quotient field $\mathbb{Z}[2\cos\pi/7]$ modulo this ideal is thus a finite field of order 2^3 .

Indeed, the orbit of ∞ modulo 2 is (by calculations, based on the fact that the orbit of 0 is given by its orbit under just the reduction of B , and using the arithmetic of $\mathbb{Q}(2\cos\pi/7)$ to simplify expressions)

$$0, \infty, \lambda, \lambda^2, 1, \lambda + 1, \lambda^2 + \lambda$$

where $\lambda = 2\cos\pi/7$. Thus, any element of K in the $G_{4,7}$ orbit of any element of \mathcal{O}_K equivalent to $\lambda^2 + \lambda + 1$ is *not* a parabolic fixed point.

4.2. Nonparabolic directions: $m = n$ odd and not divisible by any $2^f + 1$; another special pseudo-Anosov map.

PROPOSITION 18. *Suppose that $m = n$ is odd and not divisible by any integer $2^f + 1$ for positive f . Any finite-index subgroup of $G_{m,n}$ that is realized as a Veech group is such that the corresponding translation surface has nonparabolic directions with vanishing Sah–Arnoux–Fathi invariant.*

Proof. Again, $G_{m,n}$ is in parabolic standard form. The matrix A is now clearly congruent to the identity modulo $2\mathcal{O}_K$. We thus choose a prime \mathcal{O}_K -ideal \mathfrak{p} dividing the ideal $2\mathcal{O}_K$. By Lemma 14, neither B nor C is trivial when entries are reduced modulo \mathfrak{p} . Thus, $G_{m,m}$ projects to a nontrivial cyclic subgroup of $\mathrm{SL}(\mathbb{F}_{2^f})$ where f is the residue degree of \mathfrak{p} . The order of this homomorphic image must divide the orders of B and C , that is must divide m . Since $\mathbb{P}^1(\mathbb{F}_{2^f})$ has $2^f + 1$ elements, we conclude that this homomorphic image is too small to act transitively. But then $G_{m,m}$ fails to act transitively on $\mathbb{P}^1(K)$. \square

EXAMPLE 19. One again finds that the class of $\lambda^2 + \lambda + 1$ is not in the orbit of infinity. In fact, this element itself is fixed by a hyperbolic element for $G_{7,7}$; for simplicity, we take a conjugate element to get a simpler appearing matrix. Let

$$M = AC^5 = \begin{pmatrix} -1 - 2\lambda^2 & -2 + 3\lambda + 2\lambda^2 \\ -\lambda & -1 + \lambda^2 \end{pmatrix}$$

(we reduced entries modulo $p(x)$, the minimal polynomial of λ), so M fixes

$$\frac{3\lambda + \sqrt{7\lambda^2 + \lambda - 1}}{2\lambda}.$$

One calculates that $\beta = (\alpha + 13)/(\alpha - 16)$ is a square root of $\alpha = 7\lambda^2 + \lambda - 1$. Thus M does have fixed points in the trace field. There is correspondingly a special affine pseudo-Anosov diffeomorphism of totally real cubic dilatation $-9\lambda^2 + 10\lambda + 16$ on Hooper's $(Y_{7,7}, \eta_{7,7})$.

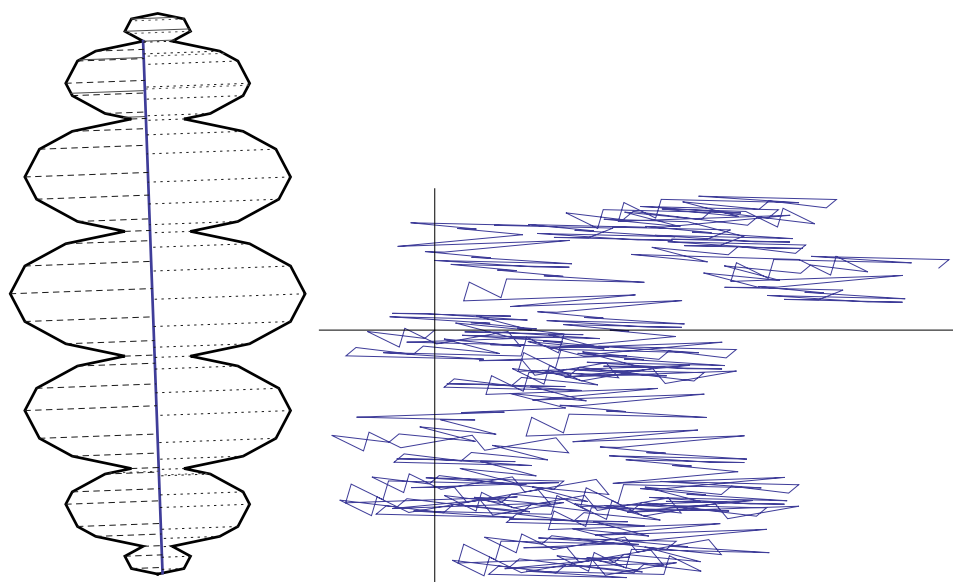


FIGURE 2. A pseudo-Anosov map with vanishing Sah-Arnoux-Fathi invariant, indicated as zippered rectangles on the normalization of Hooper's translation surface $(Y_{7,7}, \eta_{7,7})$ such that all vertices have coordinates in the periodic field. A deterministic walk approximating à la McMullen [28] a zero flux leaf of this rank three flow, as represented by using Galois automorphisms.

The periodic field here is totally real and cubic (over the rationals), but as $(Y_{7,7}, \eta_{7,7})$ is of genus 15, this is certainly not the example of a totally real cubic case of pseudo-Anosov map found by Lanneau and discussed by McMullen in [28]. We now pursue McMullen's idea of focussing on the *rank* of the flow. To normalize the surface so that all of the coordinates lie in the periodic field, we divide all x -coordinates by $\sin \pi/7$ (including adjusting the flow direction, of course), see the left-hand side of Figure 2. We choose a transversal (again in a direction perpendicular to that of our pseudo-Anosov with vanishing Sah-Arnoux-Fathi invariant) and explicitly find the interval-exchange transformation given by first return to this transversal; both the set of widths and of translations for this transformation are contained in the periodic field. We can now consider

an initial piece of the orbit under the interval-exchange map of any point x on the interval, $(x_n)_{n \leq N}$ and map this to $((x_n - x)', (x_n - x)'')_{n \leq N}$, the ordered pairs of the conjugates of the difference of the n^{th} image from x , see the right-hand side of Figure 2. (There, after scaling the transversal interval to have length one, we have taken $x = \lambda^3/8$.) As McMullen discusses, this deterministic walk approximates the continuous leaf when identifying the first homology of the period torus of the real part of (the normalization of) $\eta_{7,7}$ with the periodic field itself.

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