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BISPECTRUM ANALYSIS OF WIND AND CURRENTS OFF THE OREGON COAST: I DEVELOPMENT

by

Neng Chun G. Yao
and
S. Neshyba

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This is the first of two technical reports derived from the dissertation by Yao, Neng-Chun titled "Bispectral and Cross-bispectral Analysis of Winds and Currents off the Oregon Coast." In this first technical report we give the theoretical development of the rotary component approach to the bispectrum and cross-bispectrum among vector random processes, as well as the development of a set of energy transfer functions by which the nonlinear interaction among triplet frequency components of the processes can be quantized. Statistical

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In the subsequent technical report we give the results of application of the rotary component method of cross-bispectrum analyses to a set of winds and current data acquired from the research buoy TOTEM.

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INTRODUCTION

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In the subsequent technical report we give the results of application of the rotary component method of cross-bispectrum analyses to a set of winds and current data acquired from the research buoy TOTEM.

DEVELOPMENT

Most physical processes in the atmosphere and in the ocean are not strictly Gaussian in nature (Tukey, 1961) and the classical spectrum analysis does not reveal any correlation and any energy transform due to nonlinear interactions. Higher order spectrum analysis techniques are needed to address such a problem. Existing techniques of bispectrum and cross-bispectrum analysis are particularly useful for studying the quadratic interactions (lowest order of non-linear interactions). However, they have been developed along the assumption that all the processes are real. For studying vector random processes, such as wind and ocean current velocity, one is forced to resolve them into scalar quantities along certain rectangular coordinates. The results of the spectrum analysis from the resolved data not only depend highly on the choice of the coordinates but also are very difficult to interpret (Fofonoff, 1969; Mooers, 1970).

A new version of bispectrum and cross-bispectrum analysis, based on the rotary component concept given by Mooers (1970), is developed here to deal specifically with this particular problem.

An energy transfer equation in terms of rotary spectral parameters is given. It will show the certain amount of energy transformed due to particular quadratic interactions involved. A brief review of bispectrum and cross-bispectrum analysis as well as the rotary component spectrum analysis will be given before the detail of the development of the new technique is presented. The newly-developed technique will be used to study the quadratic interactions between the atmospheric wind and the near-surface ocean current.

A. Bispectrum and Cross-bispectrum Analysis

Since the suggestion made by Tukey (1961), several studies of the estimation of higher order spectra have been given (Rosenblatt and Van Ness, 1965; Van Ness, 1966; Brillinger and Rosenblatt, 1966b; Hinich and Clay, 1968; and Nagata, 1970). Bispectrum analysis has been applied to the study of non-Gaussian processes in geophysical problems (Hasselmann, Munk, McDonald, 1963; McDonald, 1963; Haubrich, 1965; Murty and Henry, 1972) as well as in the other fields (Godfrey, 1965; Hasselmann, 1966; Barnett, Johnson, Naitoh, Hicks and Nute, 1971; Huber , Kleiner, Gasser and Dumermuth, 1971). Application of cross-bispectrum analysis has been given by Hasselmann's (1966) study of nonlinear ship motion, and by Roden and Bendiner's (1973) study of oceanic parameters.

Higher order spectrum, such as the trispectrum, have not yet been used in practical problems because they require large amounts of computing time (Nagata, 1970). Even bispectrum and cross-bispectrum analysis are relatively new. However, they are useful in studying the low-order nonlinear interacting phenomenon associated with non-Gaussian processes.

A brief review of the definitions concerning their various functions and their physical meaning will be given in the following section. Comparisons with ordinary spectrum functions are made for better understanding. An effort is made to emphasize the difference between the spectral density function of a frequency and the part of spectrum contained in a resolved frequency band around that particular frequency.[1]

1. Definitions of Bispectrum Density, Biphase and Bicoherence

The spectral and bispectral density of a random process are

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[1] The physical meaning of spectrum density and the part of the spectrum contained in a resolved frequency band centered at the particular frequency are obviously different. However, in most literature, a distinction between these two quantities has not been established, and the definition of the spectrum density by Jenkins and Watts (1968) adds further confusion. They define the spectrum density function as the normalized power spectrum (i.e., the part of the spectrum in a resolved frequency band divided by the total variance of the random process). For the sake of consistency, their definition will not be used in this study.

defined as the frequency decomposition of the random processes second and third moment respectively (Hasselmann, et al., 1963). For example, if $\zeta(t)$ is a stationary random function of time (or distance) with zero mean, the spectrum density $F(\lambda)$ and bispectrum density $B(\lambda_1, \lambda_2)$ of $\zeta(t)$ are obtained from the Fourier transforms of $\zeta(t)$'s mean second- and third-order products:

$$F(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{-i2\pi\lambda\tau} d\tau \qquad (1)$$

where $R(\tau) = <\zeta(t)\zeta(t+\tau)>$, and

$$B(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} S(\tau_1, \tau_2) e^{-i2\pi(\lambda_1\tau_1 + \lambda_2\tau_2)} d\tau_1 d\tau_2 \qquad (2)$$

where $S(\tau_1, \tau_2) = <\zeta(t)\zeta(t+\tau_1)\zeta(t+\tau_2)>$.

$R(\tau)$ and $S(\tau_1, \tau_2)$ are called the covariance and bicovariance of $\zeta(t)$ respectively. τ is the time lag (or distance lag), and λ is the frequency (wave number in case that τ is the distance lag). The brackets $<>$ denote means. The inverse relations of (1), (2) are

$$R(\tau) = \int_{-\infty}^{\infty} F(\lambda) e^{i2\pi\lambda\tau} d\lambda \qquad (3)$$

$$S(\tau_1, \tau_2) = \int\int_{-\infty}^{\infty} B(\lambda_1, \lambda_2) e^{i2\pi(\lambda_1\tau_1 + \lambda_2\tau_2)} d\lambda_1 d\lambda_2 . \qquad (4)$$

For real $\zeta(t)$,

$$F(\lambda) = F^{*}(-\lambda) \tag{5}$$

$$B(\lambda_1, \lambda_2) = B^{*}(-\lambda_1, -\lambda_2) . \tag{6}$$

Here * denotes the conjugate of the quantity. As $\zeta(t)$ is assumed to be stationary with real-valued components, the known symmetry relations

$$R(\tau) = R(-\tau) \tag{7}$$

$$S(\tau_1, \tau_2) = S(\tau_2, \tau_1) = S(-\tau_2, \tau_1 - \tau_2)$$

$$= S(\tau_1 - \tau_2, -\tau_2) = S(-\tau_1, \tau_2 - \tau_1) = S(\tau_2 - \tau_1, \tau_1) \tag{8}$$

follow (Hasselmann et al., 1963; Brillinger and Rosenblatt, 1966a, b). In terms of the spectrum and bispectrum densities, (7) and (8) become

$$F(\lambda) = F(-\lambda) \tag{9}$$

$$B(\lambda_1, \lambda_2) = B(\lambda_2, \lambda_1) = B(\lambda_1, -\lambda_1 - \lambda_2)$$

$$= B(-\lambda_1 - \lambda_2, \lambda_1) = B(\lambda_2, \lambda_1 - \lambda_2) = B(-\lambda_1 - \lambda_2, \lambda_2) . \tag{10}$$

From (5), (6), (9) and (10), it follows that the spectrum density is real and is determined by its value on a half line, whereas the bispectrum density is complex (see below) and is determined by

its values in an octant; for example

$$0 \leq \lambda_2 \leq \infty \ , \quad 0 \leq \lambda_1 \leq \lambda_2 \ .$$

The spectrum density and bispectrum density of a random process can also be expressed in terms of the components $dZ(\lambda)$ of the Fourier-Stieltjes representation of the random process $\zeta(t)$ as follows (Hasselmann et al., 1963; MacDonald, 1963; Huber et al., 1971):

$$\zeta(t) = \int_{-\infty}^{\infty} dZ(\lambda) e^{i2\pi\lambda t}$$

$$\begin{aligned} \langle dZ(\lambda_1) dZ(\lambda_2) \rangle &= F(\lambda_1) d\lambda_1, \quad \text{if } \lambda_1 + \lambda_2 = 0 \\ &= 0 \ , \qquad \text{if } \lambda_1 + \lambda_2 \neq 0 \ , \end{aligned} \tag{11}$$

i.e., $\langle dZ(\lambda_1) dZ^*(\lambda_1) \rangle = F(\lambda_1) d\lambda_1$, and

$$\begin{aligned} \langle dZ(\lambda_1) dZ(\lambda_2) dZ(\lambda_3) \rangle &= B(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2, \quad \text{if } \lambda_1 + \lambda_2 + \lambda_3 = 0 \\ &= 0 \ , \qquad \text{if } \lambda_1 + \lambda_2 + \lambda_3 \neq 0 \ , \end{aligned} \tag{12}$$

i.e., $\langle dZ(\lambda_1) dZ(\lambda_2) dZ^*(\lambda_1 + \lambda_2) \rangle = B(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2$.

These expressions not only provide an alternative computational procedure for estimating the spectrum and bispectrum densities but also suggest some physical interpretations of them. From equation (11), one can see that $F(\lambda_1) d\lambda_1$, the part of spectrum in the resolved

frequency band centered at λ_1, represents the contribution to the mean square of the random process, $\overline{\zeta^2(t)}$, from the products of its Fourier components of frequencies λ_1 and $-\lambda_1$. From equation (12) one finds that $B(\lambda_1, \lambda_2)d\lambda_1 d\lambda_2$ the part of bispectrum in the two-dimensional frequency domain bounded by the resolved frequency bands centered at λ_1 and λ_2 respectively, represents the contribution to the mean cube of the random process, $\overline{\zeta^3(t)}$, from the product of its three Fourier components of frequency λ_1, λ_2 and $-(\lambda_1+\lambda_2)$ respectively. It is interesting to note that bispectrum density is a function of two independent frequencies only. Since the mean cube, or third moment, of a random process generally results from a second order interaction (Nagata, 1970), $B(\lambda_1, \lambda_2)$ is a measure of the amount of such interaction involving the random processes components whose frequencies are λ_1, λ_2 and $-(\lambda_1+\lambda_2)$ respectively. Since $B(\lambda_1, \lambda_2)$ is a complex function, (as shown by equation (12), the interpretation of the interactions among the components would be clearer if $B(\lambda_1, \lambda_2)$ is expressed in polar form.

$$B(\lambda_1, \lambda_2) = \left| B(\lambda_1, \lambda_2) \right| e^{i\phi(\lambda_1, \lambda_2)}$$

$$\phi(\lambda_1, \lambda_2) = \tan^{-1} \frac{\mathrm{Im}[B(\lambda_1, \lambda_2)]}{\mathrm{Re}[B(\lambda_1, \lambda_2)]} \qquad (13)$$

$$-\pi \leq \phi(\lambda_1, \lambda_2) \leq \pi \quad .$$

$|B(\lambda_1, \lambda_2)|$ is the amplitude of the bispectrum density. If it is not significantly larger than zero, there are no interactions among the components of the three frequencies. $\phi(\lambda_1, \lambda_2)$ is the phase of the bispectrum density. It is determined by the ratio of the imaginary part of the bispectrum density to the real part of it, and was formally called biphase by Barnett et al. (1971).

A third function which is derived from the normalization of the bispectrum has been given various names in literature. Hasselmann et al. (1963) relates the normalized bispectrum to the skewness with the following dimensionless equation

$$\frac{<\zeta_1 \zeta_2 \zeta_3>}{[<\zeta_1^2><\zeta_2^2><\zeta_3^2>]^{\frac{1}{2}}} = \frac{B(\lambda_1, \lambda_2)d\lambda^2}{[F(\lambda_1)F(\lambda_2)F(\lambda_3)d\lambda^3]^{\frac{1}{2}}} \qquad (14)$$

where ζ_1, ζ_2, ζ_3 are the frequency components of $\zeta(t)$ obtained by passing it through three filters centered on λ_1, λ_2, $\lambda_3 = -\lambda_1 - \lambda_2$ and of bandwidth $d\lambda$. Godfrey (1965) calls the result of the same equation the bispectral coefficient. Since it is a measure of skewness of the random process, it is also called the bispectral skewness (Hinich and Clay, 1968). The name bicoherence also appears in the published papers (Haubrich, 1965; Barnett et al., 1971; Huber et al., 1971), but it has been defined rather inconsistently by various authors. Haubrich (1965) and Barnett et al. (1971) show

$$bic^2(\lambda_1, \lambda_2) = \frac{\left| B(\lambda_1, \lambda_2) \right|^2}{< \left| X(\lambda_1) \right|^2 \left| X(\lambda_2) \right|^2 \left| X(\lambda_1 + \lambda_2)^2 \right| >} \qquad (15)$$

where $bic(\lambda_1, \lambda_2)$ is the bicoherence. Here $(B(\lambda_1, \lambda_2)$ is defined as bispectrum (rather than bispectrum density) and $B(\lambda_1, \lambda_2) = < X(\lambda_1) \cdot X(\lambda_2) X^*(\lambda_1 + \lambda_2) >$ and $X(\lambda_j)$ is the Fourier coefficient of the random process obtained from discrete Fourier transform.

Equation (15) is essentially the same one with which Hinich and Clay (1968) used to define the bispectral skewness, except they explicitly express it as the spectrum over the resolved band centered at frequency λ. Equation (15) appears differently from equation (14) given by Hasselmann et al. (1963) because different scales of Fourier transform parameters are used. (See Hinich and Clay, 1968, for details.) Physically, however, they are the same. Thus by definition, the bicoherence or bispectral skewness is a dimensionless real value with a range from zero to one.

A definition given by Huber et al. (1971) seems different. He also calls the bicoherence the normalized bispectrum, but he gives a mathematical relationship as

$$bic(\lambda_1, \lambda_2) = \frac{B(\lambda_1, \lambda_2)}{(F(\lambda_1) F(\lambda_2) F(\lambda_3))^{\frac{1}{2}}} \qquad (16)$$

where $B(\lambda_1, \lambda_2)$ is the bispectrum density and $F(j)$ is the spectrum

density as defined by equations (10) and (11). Bicoherence so defined has a dimension of $(d\lambda)^{-\frac{1}{2}}$ and is not a real value. The amplitude varies from zero to $(d\lambda)^{-\frac{1}{2}}$. For the sake of consistency, this definition of bicoherence will not be used. In this study, bicoherence will be considered as the dimensionless absolute value of the normalized bispectrum or bispectral skewness defined in equation (14).

Hinich and Clay (1968) show that if the component at $\lambda_1 + \lambda_2$ is basically due to a multiplicative interaction between the components of the random process at λ_1 and λ_2, then the value of bicoherence is equal to one and the biphase is the phase lag of the nonlinear process,

$$\phi(\lambda_1, \lambda_2) = \phi(\lambda_1) + \phi(\lambda_2) - \phi(\lambda_3) \qquad (17)$$

where $\phi(\lambda_j)$ are the phases of the interacting components of the random process. The case in which $\phi(\lambda_1, \lambda_2) = 0$ or π is called phase locking by Barnett et al. (1971).

2. Definitions of Cross-bispectrum Density, Cross-biphase, and Cross-bicoherence

The definitions concerning cross-bispectrum have been derived following the same line of the bispectrum analysis by Hasselmann (1966) and Roden and Bendiner (1973). For stationary random functions of time (or distance) with zero mean, say x(t), y(t), z(t), the cross-

spectrum, $F_{xy}(\lambda)$, and cross-bispectrum densities, $B_{xyz}(\lambda_1, \lambda_2)$, are defined respectively as the Fourier transforms of $R_{xy}(\tau)$, covariance of x(t) and y(t); and $S_{xyz}(\tau_1, \tau_2)$, cross-bicovariance of x(t), y(t) and z(t):

$$F_{xy}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-i2\pi\lambda\tau} d\tau \tag{18}$$

where

$$R_{xy}(\tau) = < x(t+\tau) y(t) > , \tag{19}$$

$$B_{xyz}(\lambda_1, \lambda_2) = \frac{1}{(2\pi)^2} \int\int_{-\infty}^{\infty} S_{xyz}(\tau_1, \tau_2) e^{-2\pi(\lambda_1\tau_1 + \lambda_2\tau_2)} d\tau_1 d\tau_2 \tag{20}$$

where

$$S_{xyz}(\tau_1, \tau_2) = < x(t+\tau_1) y(t+\tau_2) z(t) > . \tag{21}$$

In terms of the components of the Fourier Stietjes representation of

$$x(t) = \int_{-\infty}^{\infty} dZ_x(\lambda) e^{i2\pi\lambda t}$$

$$y(t) = \int_{-\infty}^{\infty} dZ_y(\lambda) e^{i2\pi\lambda t}$$

$$z(t) = \int_{-\infty}^{\infty} dZ_z(\lambda) e^{i2\pi\lambda t}$$

then

$$< dZ_x(\lambda_1) dZ_y(\lambda_2) > = F_{xy}(\lambda_1) d\lambda_1, \text{ if } \lambda_1 + \lambda_2 = 0$$
$$= 0 \quad \text{if } \lambda_1 + \lambda_2 \neq 0 \tag{22}$$

$$\text{i.e. } \langle dZ_x(\lambda_1) dZ_y^*(\lambda_1) \rangle = F_{xy}(Y_1) d\lambda_1$$

$$\langle dZ_x(\lambda_1) dZ_y(\lambda_2) dZ(\lambda_3) \rangle = B_{xyz}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \text{ if } \lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$= 0 \quad \text{if } \lambda_1 + \lambda_2 + \lambda_3 \neq 0 \tag{23}$$

$$\langle dZ_x(\lambda_1) dZ_y(\lambda_2) dZ_z^*(\lambda_1 + \lambda_2) \rangle = B_{xyz}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 .$$

The symmetry relations are

$$R_{xy}(\tau) = R_{yx}(-\tau) \tag{24}$$

$$S_{xyz}(\tau_1, \tau_2) = S_{yxz}(\tau_2, \tau_1) = S_{yzx}(\tau_2 - \tau_1, -\tau_1)$$

$$= S_{zyx}(-\tau_2, \tau_1 - \tau_2) \tag{25}$$

and hence

$$F_{xy}(\lambda) = F_{yx}(-\lambda) \tag{26}$$

$$B_{xyz}(\lambda_1, \lambda_2) = B_{yxz}(\lambda_2, \lambda_1) = B_{yzx}(\lambda_2, -\lambda_1 - \lambda_2)$$

$$= B_{zyx}(-\lambda_1 - \lambda_2, \lambda_1) . \tag{27}$$

The physical meaning of cross-spectrum density is well known. It is a complex quantity. Its amplitude shows whether a frequency component in one random process (say x(t)) is associated with larger or smaller amplitudes at the same frequency in other random process (say y(t)). The phase shows whether the frequency component in one

random process lags or leads the component at the same frequency in the other random process. As pointed out by Roden and Bendiner (1973), the cross-bicorrelation measures the joint interaction between three sets of variables displaced relative to each other in the time domain and gives an indication of the persistence of the interaction. The cross-bispectrum density is a measure of such interaction in frequency domain. If any two of the three random processes are the same, then the cross-bispectrum density gives the measure of the quadratic interaction between two frequency components, say λ_1, λ_2 in one random process, and the frequency component of $-(\lambda_1+\lambda_2)$ in the other random process. If all three are the same process, the cross-bispectrum density is just the bispectrum density of the process; subsequently, all the subscripts in the formula can be dropped.

Cross-bispectrum density is also a complex function. Its phase can be defined the same way as the biphase:

$$\phi_{xyz}(\lambda_1, \lambda_2) = \tan^{-1} \frac{\mathrm{Im}[\, B_{xyz}(\lambda_1, \lambda_2)\,]}{\mathrm{Re}[\, B_{xyz}(\lambda_1, \lambda_2)]} \tag{28}$$

where $\phi_{xyz}(\lambda_1, \lambda_2)$ is called cross-biphase by Roden and Bendiner (1973).

By the same analogy, the cross-bicoherence can be defined as

$$\mathrm{bic}^2_{xyz}(\lambda_1, \lambda_2) = \frac{\left| B_{xyz}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \right|^2}{(F_x(\lambda_1) F_y(\lambda_2) F_z(\lambda_3) d\lambda^3)} \tag{29}$$

where $bic_{xyz}(\lambda_1, \lambda_2)$ is the cross-bicoherence, $B_{xyz}(\lambda_1, \lambda_2)$ is the cross-bispectrumdensity and $F_x(\lambda_1), F_y(\lambda_2), F_z(\lambda_3)$ are the spectrum densities of random processes x(t), y(t) and z(t) at frequencies λ_1, λ_2, λ_3 respectively. The cross-bicoherence is, by definition, a real quantity equal to or larger than zero. In cases where the nonlinear process is due to quadratic interaction between the independent frequencies λ_1 and λ_2 such that the component at $\lambda_1 + \lambda_2$ is equal to or proportional to the product of the components at λ_1 and λ_2, the cross-bicoherence (and also the bicoherence) is unity. It follows, therefore, that if the cross-bicoherence differs from unity significantly, the nonlinear processes cannot be of the simple type (Roden and Bendiner, 1973).

The interpretation of cross-biphase can also be obtained by the analogy to that of biphase. In general, one can consider that bispectrum analysis is the special case of cross-bispectrum analysis.

B. Derivation of Rotary Bispectrum and Rotary Cross-bispectrum Functions

1. Fourier Representation of Rotary Components of a Two-dimensional Vector Random Process

The idea of rotary components was introduced by Mooers (1970). His method is based on a decomposition of the complex-valued series into polarized components. For a two-dimensional velocity vector

expressed as a complex valued quantity, oscillatory elements execute periodic, elliptical orbits in the hodograph plane. The trajectory of the velocity vector in this plane then is composed of a superposition of those ellipses of all oscillatory elements. The velocity vector series can be decomposed, for each frequency, into two counter-rotating circular components. Each component has its own amplitude and phase. The counter-clockwise rotating components correspond to oscillatory motions at positive frequencies; the clockwise rotating components correspond to those at negative frequencies. In other words, the oscillatory element at each frequency is composed of two counter-rotating components which have angular velocities equal to the positive and negative angular frequencies respectively (Gonella, 1972).

The oscillatory elements can be represented as the Fourier components of the velocity vector series (Perkins, 1970). Gonella (1972) gives the Fourier transform, for the angular velocity ω of the complex time series $\vec{u}(t)$ as

$$U(\omega) = \frac{1}{T}\int_0^T \vec{u}(t)e^{-i\omega t}dt$$

and

$$U(\omega) = \left| U(\omega)\right| e^{i\Phi\omega} \qquad (30)$$

where T is the duration of the complex time series $\vec{u}(t)$, and

$$\vec{u}(t) = u_1(t) + iu_2(t) .$$

u_1 and u_2 are the scalar components of $\vec{u}$ along horizontal rectangular axes $0x_1$, $0x_2$. The complex coefficient $U(\omega)$ gives the amplitude $|U(\omega)|$ and phase Φ_ω at the initial time, of the rotary components with an angular velocity ω. The angular velocity ω will be an integral multiple, positive or negative, of $2\pi/T$. $U(\omega)$ can be readily obtained in terms of the sine and cosine Fourier coefficients of $u_1(t)$ and $u_2(t)$ corresponding to the angular frequency $\sigma = |\omega|$,

$$U(\omega) = \frac{1}{T}\int_0^T [u_1(t) + iu_2(t)][\cos\omega t - i \sin \omega t]\, dt$$

$$= [A_1(\omega) + B_2(\omega)] + i[A_2(\omega) - B_1(\omega)] \qquad (31)$$

where

$$A_j(\omega) = \frac{1}{T}\int_0^T u_j(t)\cos\omega t dt$$

$$B_j(\omega) = \frac{1}{T}\int_0^T u_j(t)\sin\omega t dt$$

A_j and B_j are the sine and cosine Fourier coefficients of $u_j(t)$.

Let $$U_+(\sigma) = U(\omega) \quad \text{if } \omega = \sigma$$

$$U_-(\sigma) = U(\omega) \quad \text{if } \omega = -\sigma$$

Then
$$U_+(\sigma) = [A_1(\sigma) + B_2(\sigma)] + i[A_2(\sigma) - B_1(\sigma)]$$

$$U_-(\sigma) = [A_1(\sigma) - B_2(\sigma)] + i[A_2(\sigma) + B_1(\sigma)] \qquad (32)$$

$U_+(\sigma)$, $U_-(\sigma)$ are complex quantities. They represent the amplitude and the phase of the rotary components of angular frequencies $+\sigma$ and $-\sigma$ respectively. They are called rotary Fourier coefficients in this study. It is obvious that the rotary Fourier coefficient of $(+\sigma)$ does not necessarily equal the conjugate of the rotary Fourier coefficient of $(-\sigma)$.

The inverse of Fourier transform will be

$$\vec{u}(t) = u_1(t) + iu_2(t) = \sum_{n=-\infty}^{+\infty} U(\omega)e^{i\omega t} \tag{33}$$

where $$\omega = \frac{2\pi n}{T} \quad .$$

In terms of rotary Fourier coefficient

$$u_1(t) + iu_2(t) = \sum_{\sigma=0}^{\infty} U_+(\sigma)e^{i\sigma t} + U_-(\sigma)e^{-i\sigma t} \quad . \tag{34}$$

If $u_{1\sigma}(t)$, $u_{2\sigma}(t)$ represent the scalar components of the oscillatory element with angular frequency σ, then the ellipse equation is

$$u_{1\sigma}(t) + iu_{2\sigma}(t) = U_+(\sigma)e^{+i\sigma t} + U_-(\sigma)e^{-i\sigma t} \tag{35}$$

By equation (32), one can see[2]

---

[2]The amplitude of rotary component derived in this study is twice of that given by Gonella (1972).

$$u_{1\sigma}(t) = 2[A_1(\sigma)\cos\sigma t + B_1(\sigma)\sin\sigma t]$$
$$u_{2\sigma}(t) = 2[A_2(\sigma)\cos\sigma t + B_2(\sigma)\sin\sigma t] \qquad (36)$$

The rotary Fourier coefficients derived here are the linear functions of the sine and cosine coefficients of the scalar components of the vector time series. Since the scalar components are resolved along an arbitrary chosen rectanglar axes, their Fourier coefficients are not independent of the coordinates. In fact, if one rotates the coordinate, the Fourier coefficients will obey the same coordinate transform as the scalar components (Fofonoff, 1969), i.e.:

$$A_1' = A_1 \cos\theta + A_2 \sin\theta$$
$$A_2' = -A_1 \sin\theta + A_2 \cos\theta$$
$$B_1' = B_1 \cos\theta + B_2 \sin\theta \qquad (37)$$
$$B_2' = -B_1 \sin\theta + B_2 \cos\theta$$

where θ is the angle (measured counterclockwise) between the primed and unprimed axes. Substituting equation (37) into equation (32) yields

$$U_+'(\sigma) = U_+(\sigma)e^{-i\theta}$$
$$U_-'(\sigma) = U_-(\sigma)e^{-i\theta} \qquad (38)$$

From equation (38), it is obvious that the amplitude of the rotary

Fourier coefficient is invariant under the coordinate rotation but the phase does change by an amount equal to the angle of the rotation; i.e.

$$\left| U_{\pm}'(\sigma) \right| = \left| U_{\pm}(\sigma) \right|$$

$$\Phi'_{\pm\sigma} = \Phi_{\pm\sigma} - \theta \quad . \tag{39}$$

In the terminology of an ellipse, $\left| \left| U_{+}(\sigma) \right| + \left| U_{-}(\sigma) \right| \right|$ is the length of semimajor axis of the ellipse, and $\left| \left| U_{+}(\sigma) \right| - \left| U_{-}(\sigma) \right| \right|$ is the length of its semiminor axis. The orientation of the major axes, from the data coordinate system, is

$$\frac{\Phi_{\sigma_+} - \Phi_{\sigma_-}}{2}$$

and that of the minor axes is

$$\frac{\Phi_{\sigma_+} + \Phi_{\sigma_-}}{2}$$

(Mooers, 1970). Therefore, if $\left| U_{+}(\sigma) \right|$ equals $\left| U_{-}(\sigma) \right|$ the oscillatory element is in rectilinear oscillation; if either $\left| U_{+}(\sigma) \right|$ or $\left| U_{-}(\sigma) \right|$ is zero, the oscillatory element describes pure circular motion rotating in the direction of the non-zero component. For a scalar random process, the rotary Fourier coefficients will be the same as the ordinary Fourier coefficients (see equation 32). In this special

$$U_{+}(\sigma) = U_{-}^{*}(\sigma) , \tag{40}$$

case, and the oscillatory element is also a rectilinear oscillation.

2. Advantages of Using Rotary Component Method for Spectrum Analysis of Two-dimensional Vector Random Processes

For a two-dimensional vector random process, spectrum analysis can be conducted on the scalar component series resolved along rectangular coordinates. The spectral functions generated from the two scalar series are dependent upon the orientation of the coordinates along which the observations were made or analyzed. Also, the two component series are not independent. These problems were noted by Fofonoff (1969). He proposes that, for each frequency, a normal set of coordinates can be introduced such that the non-invariant cospectrum between the scalar component series vanishes. This is a tedious method for analyzing one set of such vector random process alone, to say nothing of analyzing a pair of them. The rotary component method can overcome these problems as shown by Perkins (1970), Mooers (1970, 1973) and Gonella (1972).

Since certain oscillatory components in the ocean current are expected to have regular motion with one sense of rotation, for instance inertial oscillation and tidal oscillations, this method appears more adequate to analyze the oceanic current velocity data than the resolved scalar component series method. The rotary spectral functions and rotary cross-spectral functions have been well defined and discussed by Gonella (1972) and Mooers (1973). Some of these functions will be reviewed briefly where the rotary

component method is extended to third order spectral analysis.

3. Rotary Spectrum- and Rotary Bispectrum-Functions

In earlier sections, the spectrum and bispectrum densities have been defined as the frequency decomposition of a real random process' second and third moments respectively. For a vector random process, the same operations yield general second and third-order spectrum densities (Brillinger and Rosenblatt, 1966b). If the decomposition is carried out in terms of angular velocities on the covariance and bicovariance functions of a two-dimensional random process, the results are defined as the rotary spectrum density and the rotary bispectrum density respectively. Let $\vec{u}(t)$ be the stationary two-dimensional vector random process. Its covariance is defined as

$$\begin{aligned} \vec{R}(\tau) &= < \vec{u}^*(t)\vec{u}(t+\tau) > \\ &= < \vec{u}(t)\vec{u}^*(t-\tau) > \end{aligned} \tag{41}$$

(Bendat and Piersol, 1966). By the same token, its bicovariance is defined as

$$\begin{aligned} \vec{S}(\tau_1, \tau_2) &= < \vec{u}^*(t)\vec{u}(t+\tau_1)\vec{u}(t+\tau_2) > \\ &= < \vec{u}^*(t)\vec{u}(t+\tau_2)\vec{u}(t+\tau_1) > . \end{aligned} \tag{42}$$

The second and third-order spectrum densities of $\vec{u}(t)$ are then

$$P(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{R}(\tau)e^{-i\omega\tau}d\tau \qquad (43)$$

and

$$RB(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \vec{S}(\tau_1, \tau_2)e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \qquad (44)$$

where ω is the angular velocity of an oscillatory component.

Expressed in terms of the Fourier-Stieltjes representations,

$$\vec{u}(t) = \sum_{n=-\infty}^{\infty} U(\omega)e^{i\omega t}, \text{ with}^3\ \omega = \frac{2\pi n}{T}$$

$$\langle U(\omega_1)\, U^*(\omega_2)\rangle = P(\omega)d\omega \quad \text{if } \omega_1 = \omega_2$$

$$= 0 \qquad \text{if } \omega_1 \neq \omega_2 \qquad (45)$$

and

$$\langle U(\omega_1)\, U(\omega_2)\, U^*(\omega_3)\rangle = RB(\omega_1, \omega_2)d\omega^2 \quad \text{if } \omega_1+\omega_2 = \omega_3$$

$$= 0 \qquad \text{if } \omega_1+\omega_2 \neq \omega_3 . \qquad (46)$$

For $\omega = \pm\sigma$, equation (45) is rewritten as

$$P_{u\pm u\pm}(\sigma)d\sigma = \langle U_\pm(\sigma)U_\pm^*(\sigma)\rangle = \langle |U_\pm(\sigma)|^2\rangle \qquad (47)$$

where $P_{u+u+}(\sigma)$ and $P_{u-u-}(\sigma)$ are the rotary spectrum densities of

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[3] In practice, w is limited by the Nyquist frequency, Ny, i.e.,

$$\frac{-1}{2\Delta t} \leq \frac{n}{T} \leq \frac{1}{2\Delta t}, \quad -\frac{\pi}{\Delta t} \leq \omega \leq \frac{\pi}{\Delta t}$$

where Δt is the data interval.

$\vec{u}(t)$ for positive and negative angular velocities respectively. In view of the characteristics of the rotary Fourier coefficients given in the previous section, it is obvious that:

1) $P_{u+u+}(\sigma)$, $P_{u-u-}(\sigma)$ are independent of the coordinates along which the scaler components of $\vec{u}(t)$ are measured; and

2) $P_{u+u\pm}(\sigma)$ is not equal to $P_{u-u-}(\sigma)$ in general, except when the angular frequency w is zero or the amplitudes of the rotating components are equal (rectilinear oscillation); and

3) the sum of $P_{u+u+}(\sigma)d\sigma$ and $P_{u-u-}(\sigma)d\sigma$ is the contribution to the total variance of $\vec{u}(t)$ from the oscillatory element of the frequency[4] $|\sigma|$.

In case of the third-order spectrum density, all three angular velocities can have both positive and negative values. The constaint relationships of ω_j, j = 1, 2, 3, have eight permutations:

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[4]Gonella (1972) gives a complete set of functions of the rotary component method for spectrum and cross-spectrum analysis. $P_{u\pm u\pm}(\sigma)d\sigma$ defined here is twice the value of his mean kinetic energy spectrum $S\pm(\sigma)$:

$$S_{\pm}(\sigma) = 1/2 < U_{\pm}(\sigma)U_{\pm}^{*}(\sigma) >.$$

He also shows that all the rotary spectral- and rotary cross spectral functions can be related linearly to the ordinary spectral- and cross spectral functions of the resolved scalar component series.

It should also be noted that common usage in spectrum analysis is to refer to the second-order spectrum estimates as "energy" even though the described quantity does not have "energy" dimensions. This conventional terminology is also adopted in this study e.g., the energy spectrum of wind stress.

$$
\begin{array}{ll}
|\sigma_2| > |\sigma_1| & |\sigma_2| < |\sigma_1| \\
1)\ \sigma_1 + \sigma_2 = \sigma_3 & 2)\ \sigma_2 + \sigma_1 = \sigma_3 \\
3)\ -\sigma_1 + \sigma_2 = \sigma_3 & 4)\ \sigma_1 - \sigma_2 = \sigma_3 \\
5)\ \sigma_1 - \sigma_2 = -\sigma_3 & 6)\ -\sigma_1 + \sigma_2 = -\sigma_3 \\
7)\ -\sigma_1 - \sigma_2 = -\sigma_3 & 8)\ -\sigma_1 - \sigma_2 = -\sigma_3
\end{array}
\tag{48}
$$

The sum of these permutations can be seen clearer in a frequency domain diagram of Euclidean representation (Fig. 1).

Due to the symmetric relation of ω_1 and ω_2 (Equation 46) there are only four independent permutations, i.e., the third spectrum density of a two-dimensional vector random process can be defined in the half-plane of the frequency domain.[5] Correspondingly, the rotary bispectrum densities covering the four independent sections in the frequency domain are

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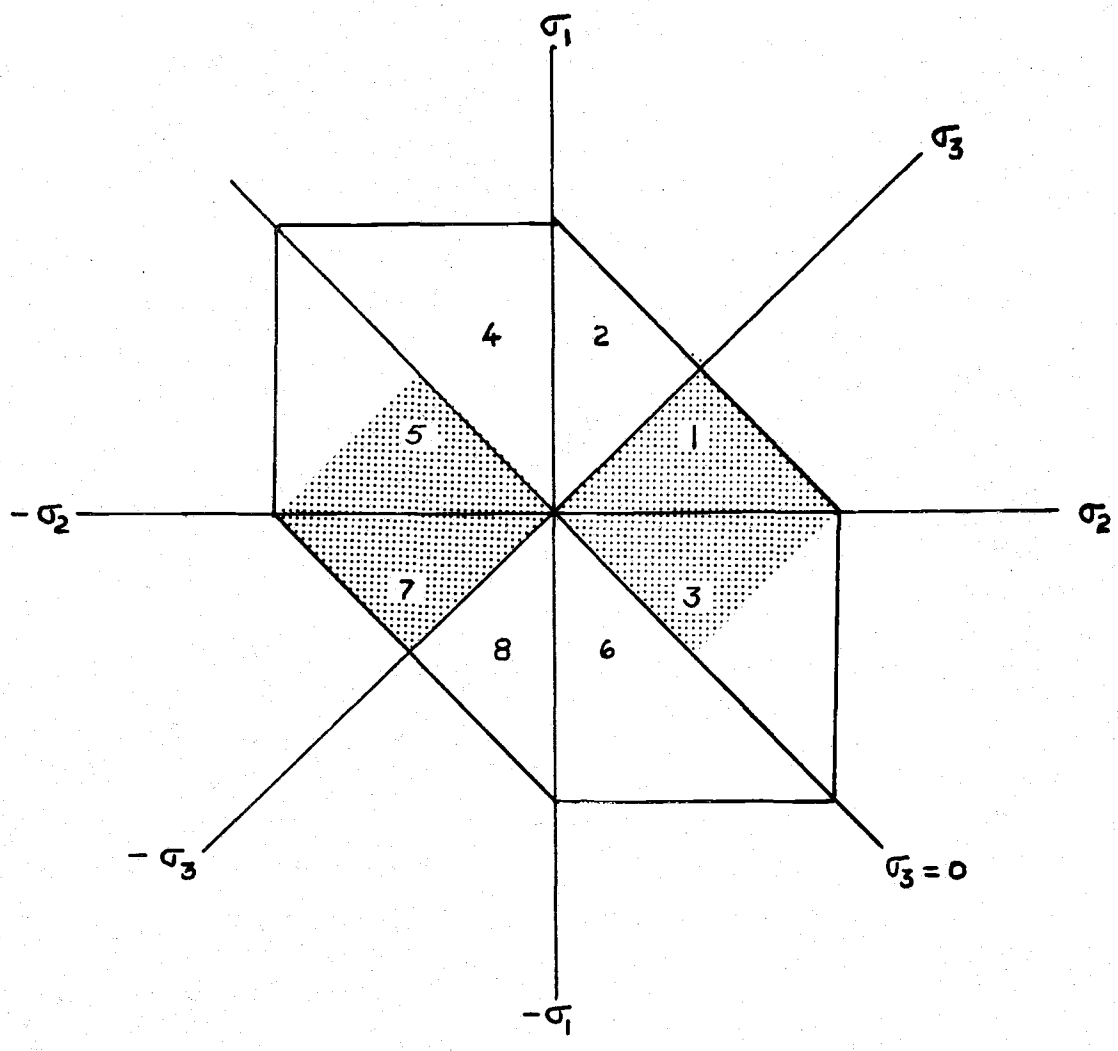
[5] The third order spectrum density of the two dimensional vector random processes can also be represented by the linear combination of ordinary bispectrum density and cross-bispectrum density of the vector's resolved scalar component series. There are eight independent bispectrum and cross-bispectrum densities for each such combination. It is not only very tedious to compute them but also very difficult to interpret them.



$$0 \leq \sigma_j \leq Ny \ , \ j = 1, \ 2, \ 3$$

$$\text{and } [\pm \sigma_1] + [\pm \sigma_2] = [\pm \sigma_3]$$

Fig. 1. The tri-frequency plane upon which the amplitudes of the rotary bispectrum and cross-bispectrum may be plotted:

i) case of Rotary bispectrum of a single random vector process- sections 1, 3, 5, 7 are the independent permutations.

ii) case of Rotary cross-bispectrum of two random vector processes- sections 1, 3, 5, 7 are the independent permutations.

iii) case of Rotary cross-bispectrum of three random vector processes- all eight sections are required.

$$RB(\sigma_1,\sigma_2)d\sigma^2 = \langle U_+(\sigma_1)U_+(\sigma_2)U_+^*(\sigma_3)\rangle,\ \sigma_1+\sigma_2=\sigma_3$$

$$RB(-\sigma_1,\sigma_2)d\sigma^2 = \langle U_-(\sigma_1)U_+(\sigma_2)U_+^*(\sigma_3)\rangle,\ -\sigma_1+\sigma_2=\sigma_3 \qquad |\sigma_2|>|\sigma_1| \qquad (49)$$

$$RB(\sigma_1,-\sigma_2)d\sigma^2 = \langle U_+(\sigma_1)U_-(\sigma_2)U_-^*(\sigma_3)\rangle,\ \sigma_1-\sigma_2=\sigma_3 \qquad |\sigma_2|>|\sigma_1|$$

$$RB(-\sigma_1,-\sigma_2)d\sigma^2 = \langle U_-(\sigma_1)U_-(\sigma_2)U_-^*(\sigma_3)\rangle,\ -\sigma_1-\sigma_2=-\sigma_3 \ .$$

Since the rotary bispectral density is a complex quantity, it can be expressed in polar form, for instance:

$$RB(\sigma_1,\sigma_2) = \left| RB(\sigma_1,\sigma_2)\right| e^{iR\phi(\sigma_1,\sigma_2)} \qquad (50)$$

where $R\phi(\sigma_1,\sigma_2)$ is defined as rotary biphase:

$$R\phi(\sigma_1,\sigma_2) = \tan^{-1}\frac{Im[RB(\sigma_1,\sigma_2)]}{Re[RB(\sigma_1,\sigma_2)]}\ . \qquad (51)$$

The corresponding bicoherence is defined as the rotary bicoherence, $Rbic(\sigma_1,\sigma_2)$:

$$Rbic^2(\sigma_1,\sigma_2) = \frac{\left| RB(\sigma_1,\sigma_2)d\sigma^2\right|^2}{(P_{u+u+}(\sigma_1)P_{u+u+}(\sigma_2)P_{u+u+}(\sigma_3)d\sigma^3)} \ . \qquad (52)$$

It is easy to see that the amplitude of the rotary bispectrum density and the rotary bicoherence are independent of the rectangular

coordinates along which the scalar components of the vector random process are measured. However, the rotary biphase is not an independent quantity. If the coordinate system is rotated by some angle, the value of the rotary biphase will change the same amount in the direction opposite to that of the coordinate rotation. This can be verified by using equations (38), (49), (50) and (51).

The rotary bispectrum is then the measure of low order nonlinear interactions among the rotary components of the oscillatory elements.

4. Rotary Cross-spectrum Functions and Rotary Cross-bispectrum Functions

The rotary cross-spectrum and rotary cross-bispectrum functions are obtained here following the same line of reasoning as in the previous section. Let $\vec{x}(t)$, $\vec{y}(t)$ and $\vec{z}(t)$ be the stationary two-dimensional random processes. The cross-covariance is defined as (Bendat and Piersol, 1966).

$$\vec{R}_{xy}(\tau) = < \vec{x}^{*}(t)\vec{y}(t+\tau) > = < \vec{x}(t)\vec{y}^{*}(t-\tau) > \tag{53}$$

and the cross-bicovariance is then

$$\begin{aligned} \vec{S}_{xyz}(\tau_1, \tau_2) &= < \vec{x}^{*}(t)\vec{y}(t+\tau_1)\vec{z}(t+\tau_2) > \\ &= < \vec{x}^{*}(t)\vec{z}(t+\tau_2)\vec{y}(t+\tau_1) > \\ &= \vec{S}_{xzy}(\tau_2, \tau_1) \end{aligned} \tag{54}$$

$$P_{xy}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \vec{R}_{xy}(\tau)e^{-i\omega\tau}d\tau \tag{55}$$

$$RB_{xyz}(\omega_1, \omega_2) = \frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} \vec{S}_{xyz}(\tau_1, \tau_2)e^{-i(\omega_1\tau_1+\omega_2\tau_2)}d\tau_1 d\tau_2 \tag{56}$$

where $P_{xy}(\omega)$ is the cross-spectrum density between $\vec{x}(t)$ and $\vec{y}(t)$, and $RB_{xyz}(\omega_1, \omega_2)$ is the cross-bispectrum density among $\vec{x}(t)$, $\vec{y}(t)$ and $\vec{z}(t)$.

Expressed in terms of the Fourier-Stieltjes representations

$$\vec{x}(t) = \sum_{n=-\infty}^{\infty} U_x(\omega)e^{i\omega t}$$

$$\vec{y}(t) = \sum_{n=-\infty}^{\infty} U_y(\omega)e^{i\omega t} \tag{57}$$

$$\vec{z}(t) = \sum_{n=-\infty}^{\infty} U_z(\omega)e^{i\omega t}$$

$$< U_x(\omega_1)U_y^*(\omega_2) >= P_{xy}(\omega_1)d\omega \quad \text{if } \omega_1 = \omega_2$$

$$= 0 \qquad \text{if } \omega_1 \neq \omega_2 \tag{58}$$

$$< U_x(\omega_1)U_y(\omega_2)U_z^*(\omega_3) > = RB_{xyz}(\omega_1, \omega_2)d\omega^2 \quad \text{if } \omega_1+\omega_2 = \omega_3$$

$$= 0 \qquad \text{if } \omega_1+\omega_2 \neq \omega_3 \tag{59}$$

it is obvious that,

$$P_{xy}(\omega) = P^*_{yx}(\omega)$$

and

$$RB_{xyz}(\omega_1, \omega_2) = RB_{xzy}(\omega_2, \omega_1) \quad \bullet \tag{60}$$

For $\omega = \pm\sigma$, equation (55) is rewritten as

$$P_{x\pm y\pm}(\sigma)d\sigma = < U_{x\pm}(\sigma)U_{y\pm}^{*}(\sigma) > \tag{61}$$

where $P_{x+y+}(\sigma)$ and $P_{x-y-}(\sigma)$ are the rotary cross spectrum densities between x(t) and y(t) for the positive and negative angular frequencies respectively.

In the case of rotary cross-bispectrum density, the permutation and symmetric relations are the same as those for rotary bispectrum density. There,

$$\begin{aligned}
RB_{xyz}(\sigma_1,\sigma_2)d\sigma^2 &= < U_{x+}(\sigma_1)U_{y+}(\sigma_2)U_{z+}^{*}(\sigma_3) > \\
&\qquad \sigma_1 + \sigma_2 = \sigma_3 \\
RB_{xyz}(-\sigma_1,\sigma_2)d\sigma^2 &= < U_{x-}(\sigma_1)U_{y+}(\sigma_2)U_{z+}^{*}(\sigma_3) > \\
&\qquad -\sigma_1 + \sigma_2 = \sigma_3 \quad |\sigma_2| > |\sigma_1| \\
RB_{xyz}(\sigma_1,-\sigma_2)d\sigma^2 &= < U_{x+}(\sigma_1)U_{y-}(\sigma_2)U_{z-}^{*}(\sigma_3) > \\
&\qquad \sigma_1 - \sigma_2 = -\sigma_3 \quad |\sigma_2| > |\sigma_1| \\
RB_{xyz}(-\sigma_1,-\sigma_2)d\sigma^2 &= < U_{x-}(\sigma_1)U_{y-}(\sigma_2)U_{z-}^{*}(\sigma_3) > \\
&\qquad -\sigma_1 - \sigma_2 = -\sigma_3
\end{aligned} \tag{62}$$

and

$$RB_{xyz}(\sigma_1,\sigma_2) = \left|RB_{xyz}(\sigma_1,\sigma_2)\right| e^{iR\phi_{xyz}(\sigma_1,\sigma_2)} \tag{63}$$

where $R\phi_{xyz}(\sigma_1, \sigma_2)$ is defined as rotary cross-biphase

$$R\phi_{xyz}(\sigma_1, \sigma_2) = \tan^{-1} \frac{Im[RB_{xyz}(\sigma_1, \sigma_2)]}{Re[RB_{xyz}(\sigma_1, \sigma_2)]} . \tag{64}$$

Correspondingly, the rotary cross-bicoherence is

$$Rbic^2_{xyz}(\sigma_1, \sigma_2) = \frac{\left| RB_{xyz}(\sigma_1, \sigma_2) d\sigma^2 \right|^2}{(P_{x+x+}(\sigma_1) P_{y+y+}(\sigma_2) P_{z+z+}(\sigma_3) d\sigma^3)} . \tag{65}$$

Again, one can see that the rotary bispectrum functions are the special cases of the rotary cross-bispectrum functions. If all three vector random processes are identical, all the subscripts, x, y, z, can be dropped, and one has a set of rotary bispectrum functions. But the rotary cross-biphase has a higher dependence on the coordinates. It is easy to see

$$R'\phi_{xyz} = R\phi_{xyz} - \theta_x - \theta_y + \theta_z \tag{65}$$

where θ_x, θ_y, θ_z are the amount of rotation of which the coordinates of $\vec{x}(t)$, $\vec{y}(t)$, $\vec{z}(t)$ undertake respectively.

The rotary cross-bispectrum is the measure of low order non-linear interactions among the rotary components of the different vector random processes. If the first two processes are identical, then the rotary cross bispectrum is the measure of quadratic interactions between the two different processes.

5. Estimation Procedure

The rotary bispectrum and rotary cross-bispectrum functions can be easily estimated with the rotary Fourier coefficients of the respective two-dimensional vector random processes. The algorithm is readily supplied by the related equations given in the previous sections. The proposed procedures are outlined and given in the Appendix.

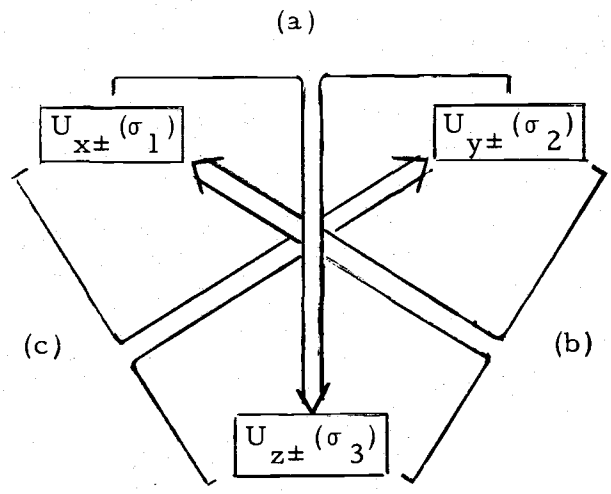
6. Applicability

The rotary bispectrum is a reliable tool for examining the non-Gaussian nature of a vector random process. Being independent of the rectangular coordinates along which the scalar components of the process are described, the amplitude of the rotary bispectral density gives the unique measure of the low-order nonlinear interactions (the quadratic interactions) internal to the process itself. However, the amplitude does not show the cause and effect of the quadratic interaction as it is only a measure of the contribution to third moment of the process by combinations of three oscillatory interacting components whose frequencies are related in the manner described by equation (48), i.e., one frequency is the algebraic sum of the other two frequencies. It does not show explicitly the amount of internal energy transferred among the oscillatory components (e.g., the third moment of a current series has dimension of L^3/T^3, which is

not an energy dimension). The rotary biphase shows the phase lag of the quadratic interaction. Since the rotary biphase is not a unique quantity, i.e., not independent of the coordinates, it does not yield meaningful information if considered alone.

The amplitude of the rotary cross-bispectrum density a quantity also independent of coordinate description, gives a unique measure of the quadratic interactions among three different vector processes at the triple frequencies of which one is the algebraic sum of the other two. If a pair of the three processes are the same process, then the measure is of quadratic interaction between the two independent processes (in this case, two of the three oscillatory elements belong to one process). Since the rotary cross-bispectrum analysis involves two or three different vector process, the role of the cause and effect of the quadratic interactions may be assigned to each process by examining the physical nature of the interacting process. One assigns the "cause" to the two oscillatory components and the "effect" to that component whose frequency is the algebraic sum of the frequencies of the "cause" components. The possible combinations of such an assignment are illustrated in Fig. 2.

In this study, the quadratic interactions to be examined are the effect of two interacting oscillating wind stress components upon the energy contained in an ocean current at the third frequency. Thus, one can postulate the relation



a) $[\pm\sigma_1]+[\pm\sigma_2]=[\pm\sigma_3]$ b) $[\pm\sigma_2]+[\pm\sigma_3]=[\pm\sigma_1]$ c) $[\pm\sigma_1]+[\pm\sigma_3]=[\pm\sigma_2]$

$$0 \leq \sigma_1 \leq N_y \qquad 0 \leq \sigma_2 \leq N_y \qquad 0 \leq \sigma_3 \leq N_y$$

Figure 2. Possible combinations of cause and effect for quadratic interactions shown by the rotary cross-bispectrum.

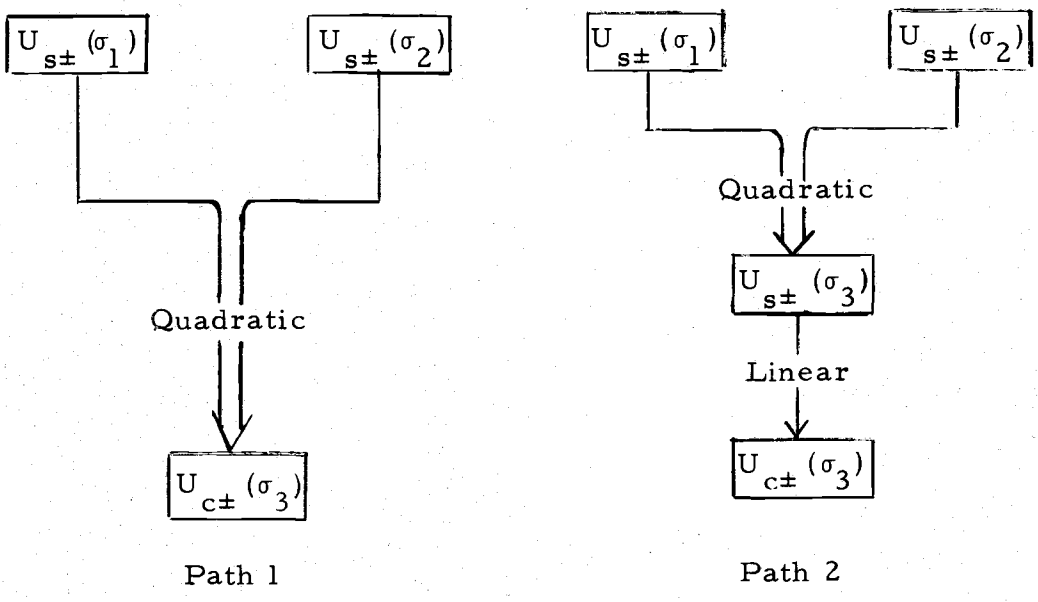


Figure 3. Quadratic energy transfer paths from process $\vec{s}$ to process $\vec{c}$.

$$\vec{c}(t) = a\vec{s}(t) + b\vec{s}^{2}(t) \qquad (67)$$

where $\vec{s}(t)$ and $\vec{c}(t)$ are the wind stress and current fields respectively, a is the coefficient of the linear dependence term and b is the coefficient of the quadratic, nonlinear dependence of current on the wind stress. Two cases are examined in a subsequent technical report: 1) wind stress vs. current at 14 m, and 2) wind stress vs. current at 34 m. In a similar manner, quadratic interactions between the ocean currents at 14 m and 34 m levels are examined (in this case, the left hand side of the equation (67) will denote current field at 34 m and the right hand side will denote current field at 14 m). Thus, the rotary cross-bispectrum analysis involves only two processes of which, in case of wind stress vs. current, the wind stress field is assigned the role of "cause", and, in the case of current vs. current, the current field at 14 m is assigned that of "cause" process.[6]

The amount of energy transferred still cannot be shown quantitatively as the rotary cross-bispectrum density only shows the amount of common cube shared by the three interacting components, and its dimension is the product of the energy of the "cause" process and the

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[6]Different method for examining the nonlinear relationship is given by Dewan (1969). However, his method only deals with the interaction among the two components with same frequency in one process and third component in another process, whose frequency is either zero or twice of that of the other two components. Thus, it only covers the axes of $\sigma_3 = 0$, $\sigma_1 = \sigma_2$ in the frequency domain shown by Fig. 1.

half power of the "effect" process. Besides, the energy transformed quadratically from one process to another can take two paths: 1) direct quadratic interaction among the two components in the "cause" process and the interacted component in the "effect" process, 2) linear interaction between an "effect" component and a "cause" component which is itself produced by quadratic interaction internal to the "cause" process. These two paths are shown in Fig. 3. They may both exist in nature but the cross-bispectrum cannot show the latter even qualitatively, in order to assess the amount of energy transformed quadratically from one process to another, an additional scheme is developed in the following section.

C. Energy Transfer Functions between a Pair of Two-dimensional Vector Random Processes

1. Rotary Fourier Representation of Quadratic Interaction between Two Different Vector Random Processes

Hasselmann (1966) developed a Fourier representation of the quadratic interactions between two random processes. A modification makes it useful for the rotary component representation of two different vector random processes. The rotary Fourier coefficients previously defined are used in place of the Fourier Stieltjes representations. Also, where Hasselmann used the ordinary conjugate to denote a function of negative frequency, the rotary component method

requires use of the function of negative frequency itself. With this modification, Hasselmann's equation (7) is applied to the rotary component analysis, as

$$\begin{aligned}
\vec{u}_c(t) &= \sum_{\sigma} \left[U_{c+}(\sigma) e^{i\sigma t} + U_{c-}(\sigma) e^{-i\sigma t} \right] \\
&= \sum_{\sigma} \left[T_{+}(\sigma) U_{s+}(\sigma) e^{i\sigma t} + T_{-}(\sigma) U_{s-}(\sigma) e^{-i\sigma t} \right] \\
&\quad + \sum_{\sigma_1} \sum_{\sigma_2} \left[K_{+}(\sigma_1 + \sigma_2) U_{s+}(\sigma_1) U_{s+}(\sigma_2) e^{i(\sigma_1 + \sigma_2)t} \right. \\
&\qquad + K_{-}(\sigma_1 + \sigma_2) U_{s-}(\sigma_1) U_{s-}(\sigma_2) e^{-i(\sigma_1 + \sigma_2)t} \\
&\qquad + V(\sigma_1 - \sigma_2) U_{s+}(\sigma_1) U_{s-}(\sigma_2) e^{i(\sigma_1 - \sigma_2)t} \\
&\qquad \left. + V(-\sigma_2 + \sigma_1) U_{s-}(\sigma_1) U_{s+}(\sigma_2) e^{i(-\sigma_1 + \sigma_2)t} \right]
\end{aligned} \tag{68}$$

where

1) $U_{s\pm}(\sigma)$, $U_{c\pm}(\sigma)$ are the rotary Fourier coefficients of random processes $\vec{s}(t)$, $\vec{u}(t)$ respectively, e.g. wind stress and current velocity.

2) σ is the frequency with a range of $0 < \sigma < \infty$. For the discrete time series, the upper limit of σ is the Nyquist frequency.

3) $T_{\pm}(\sigma)$ are the time-independent linear transfer functions, and $T_{\pm}(\sigma) \neq T_{\pm}^{*}(\sigma)$ in general.

4) $K_{\pm}(\sigma_1+\sigma_2)$ are the time-independent quadratic sum transfer functions and

$$K_{\pm}(\sigma_1+\sigma_2) = K_{\pm}(\sigma_2+\sigma_1)$$

but $K_{\pm}(\sigma_1+\sigma_2) \neq K_{\pm}^{*}(\sigma_1+\sigma_2)$ in general.

5) $V(\sigma_1-\sigma_2)$ and $V(-\sigma_1+\sigma_2)$ are the time independent quadratic difference transfer functions.

$$V(\sigma_1-\sigma_2) = V_+(\sigma_1-\sigma_2) \quad \text{if } \sigma_1-\sigma_2 > 0$$

$$= V_-(\sigma_1-\sigma_2) \quad \text{if } \sigma_1-\sigma_2 < 0$$

$$V(-\sigma_1+\sigma_2) = V_+(-\sigma_1-\sigma_2) \quad \text{if } -\sigma_1+\sigma_2 > 0$$

$$= V_-(-\sigma_1+\sigma_2) \quad \text{if } -\sigma_1+\sigma_2 < 0 \ .$$

Thus it is obvious

$$V_{\pm}(\sigma_1-\sigma_2) = V_{\pm}(-\sigma_2+\sigma_1)$$

and $V_+(\sigma_1-\sigma_2) \neq V_-^{*}(\sigma_1-\sigma_2)$ in general.

For any particular frequency σ_3 equation (58) can be separated into two different equations concerning positive angular velocity $\omega_3 = \sigma_3$ and negative angular velocity $\omega_3 = -\sigma_3$ components. Thus if $|\sigma_2| \geq |\sigma_1|$

$$U_{c+}(\sigma_3)e^{i\sigma_3 t} = T_+(\sigma_3)U_{s+}(\sigma_3)e^{i\sigma_3 t} + 2\sum_{\sigma_1}\sum_{\sigma_2} K_+(\sigma_1+\sigma_2)U_{s+}(\sigma_1)U_{s+}(\sigma_2)e^{i(\sigma_1+\sigma_2)t} \times \delta[(-\sigma_1+\sigma_2)-\sigma_3]$$

$$+ 2\sum_{\sigma_1}\sum_{\sigma_2} V_+(-\sigma_1+\sigma_2)\, U_{s-}(\sigma_1)\, U_{s+}(\sigma_2) e^{i(-\sigma_1+\sigma_2)t} \times \delta[(-\sigma_1+\sigma_2) - \sigma_3] \tag{69}$$

and

$$U_{c-}(\sigma_3)e^{-\sigma 3t} = T_-(\sigma_3)\, U_{s-}(\sigma_3)e^{-i\sigma 3t} + 2\sum_{\sigma_1}\sum_{\sigma_2} K_-(\sigma_1+\sigma_2)\, U_{s-}(\sigma_1)\, U_{s-}(\sigma_2) \times e^{-i(\sigma_1+\sigma_2)t}\, \delta[-(\sigma_1+\sigma_2)+\sigma_3] \tag{70}$$

$$+ 2\sum_{\sigma}\sum_{\sigma} V_-(\sigma_1-\sigma)\; U_{s+}(\sigma_1)\, U_{s-}(\sigma_2) e^{-i(\sigma_1-\sigma_2)t} \times \delta[-(\sigma_1-\sigma_2)+\sigma_3]$$

where $\delta[\;\;]$ is the Dirac's Delta function serving as a constraint device for the double summation e.g., $\delta[(\sigma_1+\sigma_2) - \sigma_3]$ implies the summations only cover the range $\sigma_1+\sigma_2 = \sigma_3$. Because of this constraint, these two equations can be simplified as

$$U_{c\pm}(\sigma_3) = T_\pm(\sigma_3)\;\; {}_{s\pm}(\sigma_3) + 2\sum_{\sigma_1}\sum_{\sigma_2} K_\pm(\sigma_1+\sigma_2) U_{s\pm}(\sigma_1)\, U_{s\pm}(\sigma_2) \times \delta[\pm(\sigma_1+\sigma_2) \pm \sigma_3] \tag{71}$$

$$+ 2\sum_{\sigma_1}\sum_{\sigma_2} V_\pm(\pm\sigma_1 \pm \sigma_2) U_{s\pm}(\sigma_1)\; U_{s\pm}(\sigma_2) \times \delta[\pm(\sigma_1-\sigma_2)\pm\sigma_3]$$

where[7]

$$|\sigma_1| \leq |\sigma_2| \quad .$$

One has to notice also that each $U_{s\pm}(\sigma_3)$ may be expressed in equation (71) form if there are any quadratic interactions existing in $\vec{u}_s(t)$. However, for simplicity, it will be assumed that $\vec{u}_s(t)$ is a Gaussian process. But this assumption shall be corrected later when the quadratic transform functions K and V are defined (see Hasselmann, 1966).

2. The Transfer Functions

The transfer functions T, K, V, can be defined in terms of the Rotary spectrum parameters.

a. Linear Transfer Functions $T_{\pm}(\sigma)$

The linear transfer function is given by Hasselmann (1966) as the ratio of the cross-spectrum density between the driving process and the responding process, to the spectrum density of the driving process. For rotary spectrum expression, one has

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[7] Dimensional wise. $[T_{\pm}(\sigma)] = [U_{c\pm}(\sigma) U_{s\pm}(\sigma)^{-1}]$

$[K_{\pm}(\sigma_1 + \sigma_2)] = [V_{\pm}(\pm\sigma_1 \pm \sigma_2)] = [U_{c\pm}(\sigma_3) U_{s\pm}(\sigma_1)^{-1} U_{s\pm}(\sigma_2)^{-1}]$

$$T_{+}(\sigma) = \frac{P^{*}_{s+c+}(\sigma)}{P_{s+s+}(\sigma)} \tag{72}$$

and

$$T_{-}(\sigma) = \frac{P^{*}_{s-c-}(\sigma)}{P_{s-s-}(\sigma)} \tag{73}$$

b. Quadratic Sum Transfer Functions, $K_{\pm}(\sigma_1 + \sigma_2)$

Multiplying the positive frequency equation of equation (71) by $U^{*}_{s+}(\sigma_1)$ and $U^{*}_{s-}(\sigma_2)$, with $\sigma_1 + \sigma_2 = \sigma_3$, and taking the ensemble average, one has[8]

$$RB^{*}_{ssc}(\sigma_1, \sigma_2)d\sigma^2 = T_{+}(\sigma_3)RB^{*}_{sss}(\sigma_1 + \sigma_2)d\sigma^2 + 2K_{+}(\sigma_1 + \sigma_2)P_{s+s+}(\sigma_1)P_{s+s+}(\sigma_2)d\sigma^2 \tag{74}$$

If σ is zero frequency, $\sigma_3 = \sigma_0 = 0$. Then

$$RB^{*}_{ssc}(\sigma_0, \sigma_2)d\sigma^2 = T_{+}(\sigma_3)RB^{*}_{sss}(\sigma_0, \sigma_2)d\sigma^2 + 4K_{+}(\sigma_0 + \sigma_2)P_{s+s+}(\sigma_0)P_{s+s+}(\sigma_2)d\sigma^2 \tag{75}$$

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[8] In a manner analogous to that given in Hasselmann (1966),

$$< U_s(\omega_1)\, U_s(\omega_2) U_s^{*}(\omega_3) U_s^{*}(\omega_4) > = 0$$

unless $\omega_1 = \omega_3$, $\omega_2 = \omega_4$. If, for the rotary component method $\omega_2 = \omega_4 = -\sigma_2$, $\omega_1 = \omega_3 = \sigma_1$, then

$$< U_{s+}(\sigma_1)\, U_{s-}(\sigma_2)\, U_{s+}^{*}(\sigma_1)\, U_{s-}^{*}(\sigma_2) > = P_{s+s+}(\sigma_1)P_{s-s-}(\sigma_2)d\sigma^2$$

as $K_+(\sigma_0+\sigma_2) = V_+(-\sigma_0+\sigma_2)$ and $P_{s+s+}(\sigma_0) = P_{s-s-}(\sigma_0)$. Here the rotary bispectral density $RB^*_{sss}(\sigma_1,\sigma_2)$ is included to account for the non-Gaussian nature of the $\vec{u}(t)$ process itself. Solving for the quadratic sum transfer function, one obtains

$$K_+(\sigma_1+\sigma_2) = \frac{RB^*_{ssc}(\sigma_1,\sigma_2)}{2\,P_{s+s+}(\sigma_1)P_{s+s+}(\sigma_2)} - T_+(\sigma_3)\frac{RB^*_{sss}(\sigma_1,\sigma_2)}{2\,P_{s+s+}(\sigma_1)P_{s+s+}(\sigma_2)},$$

$$\sigma_1+\sigma_2=\sigma_3 . \tag{76}$$

The first and second terms on the right hand side of equation (76) account for the path 1 and path 2 illustrated in Fig. 3. If $\vec{u}(t)$ is indeed a Gaussian process, then the second term does not exist because $RB^*_{sss}(\sigma_1,\sigma_2)$ will not be significantly greater than zero. On the other hand, if there are no quadratic interactions between the two random processes, but the driving process has its own internal quadratic interaction, then the σ_3 component of the responding process still will be affected by σ_1, σ_2 components in the driving process provided that the linear transfer function $T_+(\sigma_3)$ is significant. The quadratic sum transfer function is then

$$K_+(\sigma_1+\sigma_2) = -T_+(\sigma_3)\frac{RB^*_{sss}(\sigma_1,\sigma_2)}{2P_{s+s+}(\sigma_1)P_{s+s+}(\sigma_2)} .$$

This interaction is not explicit. The same relation also holds for negative components.

$$K_{-}(\sigma_1+\sigma_2) = \frac{RB^{*}_{ssc}(-\sigma_1, -\sigma_2)}{2P_{s-s-}(\sigma_1)P_{s-s-}(\sigma_2)}$$

$$- T_{-}(\sigma_3)\frac{RB^{*}_{sss}(-\sigma_1, -\sigma_2)}{2P_{s-s-}(\sigma_1)P_{s-s-}(\sigma_2)} \qquad (77)$$

c. Quadratic Difference Transfer Functions, $V_{\pm}(\mp\sigma_1 \pm \sigma_2)$

The quadratic difference transfer functions can be derived the same way as the quadratic sum transfer functions. Thus

$$V_{\pm}(\mp\sigma_1 \pm \sigma_2) = \frac{RB^{*}_{ssc}(\mp\sigma_1, \pm\sigma_2)}{2P_{s\mp s\mp}(\sigma_1)P_{s\pm s\pm}(\sigma_2)}$$

$$- T_{\pm}(\sigma_3)\frac{RB^{*}_{sss}(\mp\sigma_1, \pm\sigma_2)}{2P_{s\mp s\mp}(\sigma_1)P_{s\pm s\pm}(\sigma_2)} . \qquad (78)$$

3. The Energy Transfer Equation

The energy transfer equation between a pair of two-dimensional vector random processes can be readily obtained from equation (71). As $P_{c\pm c\pm}(\sigma_3)d\sigma = \langle U_{c\pm}(\sigma_3)\, U_{c\pm}^{*}(\sigma_3)\rangle$, one has[9]

$$P_{c\pm c\pm}(\sigma_3)d\sigma = \left|T_{\pm}(\sigma_3)\right|^2 P_{s\pm s\pm}(\sigma_3)d\sigma +$$

[9] Mathematically, there should be a factor of four in the second and third terms on the right hand side of equation (78) to account for the permutations of each pair of the interacting frequency components. The factor of four is not included in this equation as only one of the permutations represents the true energy process.

$$+\sum_{\sigma_1}\sum_{\sigma_2}\left|K_{\pm}(\sigma_1+\sigma_2)\right|^2 P_{s\pm s\pm}(\sigma_1)P_{s\pm s\pm}(\sigma_2)d\sigma^2\cdot\delta[(\pm\sigma_1\pm\sigma_2)\mp\sigma_3]$$

$$+\sum_{\sigma_1}\sum_{\sigma_2}\left|V_{\pm}(\mp\sigma_1\pm\sigma_2)\right|^2 P_{s\mp s\mp}(\sigma_1)P_{s\pm s\pm}(\sigma_2)d\sigma^2\cdot\delta[(\mp\sigma_1\pm\sigma_2)\mp\sigma_3] \quad (79)$$

where $|\sigma_2|\geq|\sigma_1|$.

The first term on the right hand side of the equation denotes the amount of energy being transformed from the driving process to the responding process through linear action. The second and third terms denote the amount of energy being transformed through quadratic interactions.

D. Statistical Considerations

The development of this proposed technique is based on two hypotheses; first that the random processes are not normally distributed (non-Gaussian), and second that the random processes are stationary. Therefore, any set of data used for this analysis should first pass a statistical test of these hypotheses. Since this analysis is carried out on a limited sample, significance test should also be applied to function estimates such as rotary cross-spectrum density, rotary bispectrum density and rotary cross-bispectrum density. The estimates should be significantly greater than zero to yield any useful information. These tests are discussed in the following sections. For detailed testing procedures the reader is referred to referenced literature.

1. Normality Test

The well known χ^2 test for goodness of fit of normality is chosen as the main testing scheme. It has been well documented (Crow, Davis and Maxfield, 1960), and the proper computer programs needed for the test are also available (Bevington, 1969). A complimentary testing scheme using Cornu ratio and skewness (Crew and Bodvarsson, 1971) will also be used for comparison. The tests are applied to the scalar component series for the two dimensional vector random process. If both of the scalar component series fail all of the tests then this vector random process is not normally distributed (Jenkins and Watts, 1968).

2. Stationarity Test

For practical reasons, the rigorous testing of stationarity of a random process is not feasible (Bendat and Piersol, 1966). However, in this proposed analysis, all of the data series used must satisfy the condition of weak stationarity, to second order, as a minimum condition. This is due to the fact that all the functions defined here are derived directly or indirectly from the Fourier coefficients of the scalar component series. It is rather important to ensure that the Fourier representation of the series is valid.

Haubrich (1965) gives such a test in detail, based on the idea that the coherence squared between the Fourier coefficients at

different frequencies will be significantly greater than zero if the random process is not stationary. This idea has been further proven by Lumley (1970). He states that, for a stationary process, Fourier components at two different frequencies are uncorrelated, though not statistically independent.

Therefore, the stationarity test by Haubrich (1965) is utilized for this analysis for the sake of compatibility and practicality. It will be conducted on each resolved scalar data series to ensure that its Fourier representation is valid.

3. Significance Test of the Rotary Cross-spectrum Estimates

The rotary cross-spectrum density is meaningful only when its corresponding coherence estimate is significantly greater than zero. The rotary coherence $\gamma_{12}(\pm\sigma)$ is defined as (Mooers, 1973)

$$\gamma_{12}^{2}(\pm\sigma) = \frac{P_{1\pm 2\pm}(\sigma)^{2}}{P_{1\pm 1\pm}(\sigma)\, P_{2\pm 2\pm}(\sigma)} \tag{80}$$

where

$$0 \leq \gamma_{12}^{2} \leq 1 ,$$

and $P_{1\pm 1\pm}(\sigma)$ is the rotary cross-spectrum density between processes 1 and 2, and $P_{1\pm 1\pm}(\sigma)$, $P_{2\pm 2\pm}(\sigma)$ are rotary spectrum densities of processes 1 and 2 at $\pm\sigma$. It provides a non-dimensional measure of the correlation between two time series as a function of frequency.

Jenkins and Watts (1968) give a simple significance test scheme for such correlation based on least squares analysis in the frequency domain. They show that the random variable

$$\frac{(EDF-2)\,\hat{\gamma}_{12}^{2}(\sigma)}{2(1-\hat{\gamma}_{12}^{2}(\sigma))} \tag{81}$$

is distributed approximately as the $F_{2,(EDF-2)}$ where EDF is the equivalent degree of freedom of the spectrum estimates $F_{2,(EDF-2)}$ is the Fisher F distribution with 2 and (EDF-2) degrees of freedom, and $\hat{\gamma}_{12}$ is the estimated value of the coherence squared. Thus,

$$\frac{(EDF-2)\,\hat{\gamma}_{12}^{2}(\sigma)}{2(1-\hat{\gamma}_{12}^{2}(\sigma))} \leq f_{2,(EDF-2)}(1-\alpha) \tag{82}$$

then γ_{12}^{2} is not greater than zero at the α significance level.

This test is derived from ordinary spectrum and cross-spectrum functions. It can also be applied to the rotary component spectrum functions by following the same analogy.

4. Significance Test of the Rotary Bispectrum and Rotary Cross-bispectrum Estimates

The third order spectrum estimates of a random process (vector-valued or real-valued) are asymptotically unbiased and asymptotically

normal under mild conditions (Brillinger and Rosenblatt, 1966a). Thus the estimates of the coherence squared asymptotically has a χ^2 distribution with two degrees of freedom since it is sum of squares of two terms each of which is asymptotically normally distributed. Haubrich (1965) gives that the expected values of the estimate of the bicoherence squared, obtained from truncated data, is approximately 2/EDF if its corresponding bispectrum density estimates have true vanishing value.

Therefore, the bispectrum density estimates will be significantly greater than zero at the 5% level if its correspinding estimates of the bicoherence squared is greater than or equal to 6/EDF. This testing criterion can also be applied to the cross-bispectrum estimates which have been shown as the general case of the bispectrum estimates in earlier section. By the same analogy, it can also be applied to the rotary bispectrum and cross-bispectrum estimates. Thus, if the estimate of the rotary bicoherence squared is less than 6/EDF, the corresponding rotary bispectral density is not significantly greater than zero at the 5% level. The same rule holds for rotary cross-bispectum density.

Due to the fact that the bicoherence can only be estimated when the three interacting components all have a reasonable amount of spectrum density (Hinich and Clay, 1968), a significance test should be applied to the spectrum density estimates of the rotary components

before their bispectrum density are tested for significance. One can see that this requirement is also true for the significance test of the rotary cross-bispectrum density estimates.

5. Significance Test of the Rotary Spectrum Density Estimates

For a discrete time series, the spectrum density estimates should decrease to zero before the Nyquist frequency. If the spectrum density estimate at the Nyquist frequency appears greater than zero, this estimated value is the noise introduced by aliasing of the frequency components above the Nyquist frequency. This noise cannot be completely eliminated in practice. Heuristically, the noise is regarded as the systematical error for the true zero spectrum density estimate, and it can be used as a threshold value for testing other non-zero spectrum estimates. The confidence limits for the spectrum density estimate at a certain significance level can be easily obtained with the estimate, its equivalent degrees of freedom, and the variable associated with the significance level (Jenkins and Watts, 1968). A simple test is then proposed in this study to compare the lower limit of any rotary spectrum density estimate against the upper limit of that at the Nyquist frequency. If the former value is less than the latter, the tested rotary spectrum density is considered not to be significantly greater than zero at that certain level.

It is obvious that all of the significance tests are essential to

the reliability of this proposed analyzing technique. Thus, in order to have valid results, all the interpretations and calculations should only be based on the rotary spectrum parameters which have passed these tests.

This process has been applied to a set of wind and current data gathered from the TOTEM buoy in 1970. The data, the analyses, and the results are given in Bispectrum and Cross-Bispectrum Analysis of Wind and Currents off the Oregon Coast: II Data and Examples.

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APPENDIX

Estimation Procedures of the Rotary Spectral, Rotary Cross-spectral, Rotary Bispectral and Rotary Cross-bispectral Functions

A. Editing the Data

For practical reasons, the existing data of two-dimensional vector random process are often recorded in the form of a finite length discrete time series of their resolved scalar components or of their magnitude and directions. Since rotary Fourier coefficients are obtained from the linear combinations of the cosine and sine Fourier coefficients of the scalar components and since their amplitudes are independent of the coordinates along which the scalar components are resolved, data should be in the form of resolved scalar component series along E-W (ox_1) and N-S (ox_2) directions with E, N designated as the positive axes. The choice of coordinates is arbitrary; to be consistent, conventional notation is used. For the sake of saving computation effort, the cosine and sine Fourier coefficients are computed with the help of the relatively new, fast Fourier transform algorithm (Welch, 1967). Also, in order to increase the number of degrees of freedom of the estimates of various functions, (i.e., to ensure the proper statistical stability), each data record is broken into several truncations, from which the averaged

estimated functions are obtained (Haubrich, 1965; Hinich and Clay, 1968; Huber et al., 1971). Therefore, the data records have to be edited to achieve the proper estimation with less computational expense.

1. Rearranging the Data Interval and Removal of the Mean

The data sampling interval should be of proper length so that the Nyquist frequency will not be excessively higher than the upper limit of the frequency range of interest. This can be done with either a block average of several closely spaced data values or with a weighted moving average technique. The latter method is a combination of low pass filtering and averaging. With properly chosen weighting coefficients, the aliases can be avoided. However, certain data points on both ends of the record will be sacrificed. Then the mean of the data should be removed in order to be consistent with the assumption that the random process has zero mean.

2. Adjusting the Data Record

The data record may have to be adjusted to provide the proper number of truncations so that the averaged function estimates can have proper equivalent degrees of freedom. Because of the requirements of the Fast Fourier transform, the number of data points in each truncation must be a power of two. In order to have the proper

number of truncations, it may be necessary to adjust the data length by either adding zeros at the end of each truncation (Henry and Graefe, 1971) or by overlapping a part of data in each truncation (Haubrich, 1965; Welch, 1967; Huber et al., 1971). The latter method is proposed in this study because the equivalent degrees of freedom EDF of the spectral estimates from the overlapping truncated data can be calculated explicitly (Welch, 1967). Let $x(t)$, $t = 0, \ldots N-1$ be the finite discrete time series of a random process $x(t)$. If $x(t)$ has a record length of T and a data interval of Δt, the total number of data points N is $T/\Delta t$. Overlapping can be done by taking truncations of length $m\Delta t$ (m data points in each truncation) with the starting point D data points apart. Then the truncated data is represented as

$$x(k, j) = x(j + (K-1)D) \quad j = 0, \ldots N-1$$
$$k = 1, \ldots K \qquad (A-1)$$

where k is the total number of the truncations.

$$K = \left(\frac{N}{M} - 1\right)\frac{M}{D} + 1 \qquad (A-2)$$

There are M-D data points overlapped in each truncation. If no overlapping is allowed, one simply lets D equal M. Then equation (A • 2) becomes $K = N/M$ (Welch, 1967).

3. Applying weighting function to the truncated data

This modification can reduce the leakage of spectral power from a spectral peak to frequencies far away (Haubrich, 1965; Welch, 1967; Huber et al., 1971). However, weighting the data not only will degrade the frequency resolution but also will change the expectations and the variances of the spectral estimates (Huber et al., 1971). Since the data available for this study are already limited in length this technique will not be used for the sake of keeping adequate frequency resolution.

B. Approximating the Rotary Fourier Coefficients of the Truncated Data

For each truncation of M data values, (1 + M/2) number of the rotary Fourier coefficients, with the frequencies ranging from zero up to Nyquist, can be obtained from the linear combinations of the sine and cosine Fourier coefficients of the resolved scalar components in that truncation.

Let $u_1(k, j)$ and $u_2(k, j)$ be the scalar components of the kth truncation. Then the discrete fast Fourier transform is

$$Z_p(k, \lambda) = \frac{1}{M} \sum_{j=0}^{M-1} u_p(k, j) W(j) e^{-i2\pi jq/M}$$

$$p = 1, 2 \quad ; \qquad q = 0, \ldots, \frac{M}{2} \qquad \text{(A -3)}$$

(Welch, 1967)

where, $Z_p(k, \lambda)$ is the complex Fourier coefficient; λ is the frequency of cycle per data interval CPH and $\lambda = q/M$; $W(j)$ is the weighting function or data window. If no weighting function is used, then $W(j) = 1$, for $j = 0, \ldots, M-1$. Physically $Z_p(k, \lambda)$ represents the amplitude and the phase of the qth frequency component of the scalar components; the resolved frequency band of the component is $1/M\Delta t$ CPD if no data window is applied[10]. The cosine and sine coefficients of $u_p(k, j)$ are the real and imaginary parts of $Z_p(k, \lambda)$ respectively by definition. Thus

$$A_p(k, \lambda) = \mathrm{Re}[Z_p(k, \lambda)]$$

$$B_p(k, \lambda) = \mathrm{Im}[Z_p(k, \lambda)] \qquad (A-4)$$

By equation (32), the rotary Fourier coefficients of the truncated data $u(k, j)$ are

$$U_+(k, \lambda) = [A_1(k, \lambda) + B_2(k, \lambda)] + i[A_2(k, \lambda) - B_1(k, \lambda)]$$

$$U_-(k, \lambda) = [A_1(k, \lambda) - B_2(k, \lambda)] + i[A_2(k, \lambda) + B_1(k, \lambda)] \qquad (A-5)$$

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[10] The definition of the discrete Fourier transform is not uniform in the literature. Some authors use a quantity equivalent to $Z(\lambda)M\Delta t$ or $Z(\lambda)(M\Delta t)^{1/2}$, others use a positive exponent instead of a negative one (Jenkins and Watts, 1968; Hinich and Clay, 1968). These variants appear not only in magnitude but also in physical meaning. For example, the quantity $Z_p(\lambda)(M\Delta t)$ represents the distribution of $u_p(t)$ per resolved frequency band, and $Z_p(\lambda)(M\Delta t)^{1/2}$ represents the distribution of $u_p(t)$ per square root of the resolved frequency band. For the purpose of consistency with the derivation, the coefficients of DFT defined as given in equation (A-3).

C. Computing the Spectrum Estimates

Once the rotary Fourier coefficients are calculated, computing the spectrum estimates is fairly straightforward. Using the definitions given before, the formulae for computing various spectrum estimates are given in the following equations.

For a single vector random process, $x(t)$

$$\hat{P}_{x\pm x\pm}(\lambda)d\lambda = \frac{1}{K}\sum_{k=1}^{K}\left| U(k,\lambda)\right|^2 \qquad (A-6)$$

$$\hat{R}B(\pm\lambda_1, \pm\lambda_2)d\lambda^2 = \frac{1}{K}\sum_{k=1}^{K} U_{x\pm}(k,\lambda_1)\, U_{x\pm}(k,\lambda_2)\, U_{x\pm}{}^{*}(k,\pm\lambda_3)$$

$$\pm\lambda_1 \pm\lambda_2 = \pm\lambda_3 \qquad (A-7)$$

$$\hat{R}B(\pm\lambda_1, \pm\lambda_2) = \left|\hat{R}B(\pm\lambda_1, \pm\lambda_2)\right| e^{iR\phi(\lambda_1, \lambda_2)} \qquad (A-8)$$

$$\hat{R}bic^2(\pm\lambda_1, \pm\lambda_2) = \frac{K\left|\sum_{k=1}^{K} U_{x\pm}(k,\lambda_1)\, U_{x\pm}(k,\lambda_2)\, U_{x\pm}{}^{*}(k,\lambda_3)\right|^2}{\sum_{k=1}^{K}\left|U_{x\pm}(k,\lambda_1)\right|^2\sum_{k=1}^{K}\left|U_{x\pm}(k,\lambda_2)\right|^2\sum_{k=1}^{K}\left|U_{x\pm}(k,\lambda_3)\right|^2} \qquad (A-9)$$

where ^ denotes estimating value.[11]

[11] More sophisticated average procedures are suggested by Huber et al. (1971) for their study of ordinary bispectrum analysis. Bust these procedures need too lengthy a data record and too much computing time to be used in this study.

In the case of multiple random processes, let $U_{x\pm}(k, \lambda)$, $U_{y\pm}(k, \lambda)$, $U_{z\pm}(k, \lambda)$ represent the rotary Fourier coefficients of their truncated data. Then

$$\hat{P}_{x\pm y\pm}(\lambda)d\lambda = \frac{1}{K}\sum_{k=1}^{K} U_{x\pm}(k, \lambda)\, U_{y\pm}{}^{*}(k, \lambda) \qquad (A\text{-}10)$$

$$\hat{R}B_{xyz}(\pm\lambda_1, \pm\lambda_2)d\lambda^2$$

$$= \frac{1}{K}\sum_{k=1}^{K} U_{x\pm}(\lambda_1)\, U_{y\pm}(\lambda_2)\, U_{z\pm}{}^{*}(\lambda_3)$$

$$\pm\lambda_1 \pm\lambda_2 = \pm\lambda_3 \qquad (A\text{-}11)$$

$$\hat{R}bic^2{}_{xyz}\pm\lambda_1, \pm\lambda_2)$$

$$= \frac{K\left|\sum_{k=1}^{K} U_{x\pm}(k, \lambda_1)\, U_{y\pm}(k, \lambda_2)\, U_{z\pm}{}^{*}(k, \lambda_3)\right|^2}{\sum_{k=1}^{K}\left|U_{x\pm}(k, \lambda_1)\right|^2 \sum_{k=1}^{K}\left|U_{y\pm}(k, \lambda_2)\right|^2 \sum_{k=1}^{K}\left|U_{z\pm}(k, \lambda_3)\right|^2}$$

$$(A\text{-}12)$$

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