SYNTHESIS OF MULTITERMINAL RC NETWORKS WITH THE AID OF A MATRIX TRANSFORMATION

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SYNTHESIS OF MULTITERMINAL RC NETWORKS WITH THE AID OF A MATRIX TRANSFORMATION

INTRODUCTION

In the synthesis of passive networks, one of the most important problems is to determine realizability conditions. That is, what are the necessary and sufficient conditions upon a set of network functions in order that there may be a physical network possessing these particular network functions? If the ordinary, lumped elements of all three kinds (R, L, and C) are allowed, including the existence of mutual inductance, and including the ideal transformers as elements; then the realizability conditions for the network transfer and driving point immittances are well known. These conditions are that $\sum X_r X_s F_{rs}$ be a positive real function for all arbitrary values of real variable x_r. Here F_{rs} designates the prescribed driving point or transfer immittance (3). In the general m-terminal pair networks, it is equivalent to say that the prescribed immittance matrix is positive real*(12).

Suppose more restrictions are introduced; such as the realization of RC networks without the use of ideal transformers, the situation is completely changed. The general case ($m > 1$) has not been completely solved though a lot of work has been done on this problem (2, 8, 11).

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* A real, symmetric matrix is defined as a "positive real" matrix if the quadratic form associated with this matrix is a positive real function for any real vector.

When all the terminals of a network share a common ground, then this network is referred to as a "m-terminal" network. Only this kind of network realization is discussed here. That is, the objective of this thesis is to make an investigation on the realization of RC multiterminal networks without the use of transformers.

The approach presented here is based on the principle of equivalent networks. With the aid of a matrix transformation, a group of node-admittance matrices (realizable and non-realizable) having the prescribed open-circuit impedance matrix can be determined. From this group of solutions a physical realizable network may be obtained if it does exist. On the other hand, a different attack has also been introduced in order to obtain a physical realizable network having the prescribed open-circuit impedance matrix. A particular structure, the ladder, has been assumed to have the prescribed open-circuit impedance matrix, and based on this assumption a synthesis procedure is described.

The cornerstone of the investigation is a general transformation theory for network synthesis. This transformation, involving the idea of equivalent networks, is very important to the whole development of this thesis, and will be introduced in the first section.

I. A GENERAL TRANSFORMATION THEORY FOR NETWORK SYNTHESIS

In network analysis one can describe any linear, bilateral network with lumped elements by either a system of loop equations or a system of node equations. In the following discussions, a set of n independent node equations has been chosen to describe an arbitrary linear, bilateral, lumped network; written in Laplace transform as:

$$
\begin{aligned}
I_1 &= Y_{11}E_1 + Y_{12}E_2 + \ldots + Y_{1n}E_n \\
I_2 &= Y_{21}E_1 + Y_{22}E_2 + \ldots + Y_{2n}E_n \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
I_m &= Y_{m1}E_1 + Y_{m2}E_2 + \ldots + Y_{mm}E_n \\
&\ldots\ldots\ldots\ldots\ldots\ldots \\
I_n &= Y_{n1}E_1 + Y_{n2}E_2 + \ldots + Y_{nn}E_n
\end{aligned}
\tag{1.1}
$$

Or, using matrix notation, (1.1) may be rewritten as,

$$(I)_1^n = (Y)_1^n \, (E)_1^n \tag{1.2}$$

where

$(I)_1^n$ = The column matrix of n rows for the current vector.

$(E)_1^n$ = The column matrix of n rows for the voltage vector.

$(Y)_n^n$ = The n x n node-admittance matrix; i.e.,

$$\begin{bmatrix} Y_{11} & Y_{12} & \cdots & Y_{1n} \\ Y_{21} & Y_{22} & \cdots & Y_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ Y_{m1} & Y_{m2} & \cdots & Y_{mn} \\ \cdot & \cdot & \cdot & \cdot \\ Y_{n1} & Y_{n2} & \cdots & Y_{nn} \end{bmatrix} \qquad (1.3)$$

Suppose the first m node-pairs are used as the external terminals of the network, then all the currents except those at these terminals will be identically equal to zero. That is,

$$I_{m+1} = 0, \quad I_{m+2} = 0, \; \cdots \; I_n = 0$$

The terminal characteristics of such a network with m-terminal pairs can be described by the "open-circuit impedance matrix." It will be seen that this matrix, written as $(Z)_m^m$, can be derived from the node-admittance matrix $(Y)_n^n$ described in (1.3) by means of a special matrix transformation.

Imagine that there are a group of networks, all of them are equivalent to the network specified in (1.3). Here "Equivalence" simply means that all of them have the same terminal characteristics; i.e., the same open-circuit impedance matrix $(Z)_m^m$. The simplest kind of networks will be one with exactly m node pairs. Thus, for this network, (1.2) becomes,

$$(I')_1^m = (Y')_m^m \, (E')_1^m$$

Since $(Y')_m^m$ is non-singular, this equation can be re-written as,

$$(E')_1^m = \left[(Y')_m^m\right]^{-1} (I')_1^m$$

According to the definition of equivalent networks, the inverse of $(Y)_m^m$ should be exactly equal to the $(Z)_m^m$, the open-circuit impedance matrix of the specified network. Moreover, the voltages and currents at the external terminals should be equal. That is,

$$I_1 = I_1' \quad I_2 = I_2' \; \ldots\ldots \; I_m = I_m'$$

$$I_{m+1} = 0 \; \ldots\ldots \; I_n = 0 \qquad (1.4)$$

and $E_1 = E_2' \quad E_2 = E_2' \; \ldots\ldots \; E_m = E_m' \qquad (1.5)$

In matrix notation, (1.4) can be rewritten as

$$(I)_1^n = (C)_m^n \, (I')_1^m \qquad (1.6)$$

Here $(C)_m^n$ is the transformation matrix. It is apparent that,

$$(C)_m^n = \begin{array}{c} \begin{bmatrix} 1,0 & \ldots\ldots & 0 \\ 0,1 & \ldots\ldots & 0 \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ 0,0 & \ldots\ldots & 1 \\ 0,0 & \ldots\ldots & 0 \\ 0,0 & \ldots\ldots & 0 \end{bmatrix} \end{array} \begin{array}{l} m \\ \\ \\ m \\ \\ n \end{array} \quad \text{or} \begin{bmatrix} U \\ \hdashline O \end{bmatrix} \quad \begin{array}{l} U = m \times m \text{ unit matrix} \\ O = (n-m) \times m \text{ null} \\ \quad\;\; \text{matrix} \quad (1.7) \end{array}$$

It is seen that the transformation matrix $(C)_m^n$ is a degenerative matrix with n rows and m columns. It is formed by a m x m unit matrix in the first m rows and zeroes in the remaining n-m rows.

Similarly, for the voltage vectors, (1.5) gives

$$(E')_1^m = (\widetilde{C})_n^m \, (E)_1^n \qquad (1.8)$$

where $(\widetilde{C})_n^m$ is the transpose of matrix $(C)_m^n$.

Being equipped with these equations, one can easily determine the relations between $(Y)_n^n$ and $(Z)_m^m$. (10).

Recalling $(E')_1^m = \left[(Y')_m^m\right]^{-1} (I')_1^m = (Z)_m^m (I')_1^m$, one has, with the aid of (1.6) and (1.8);

$$(Z)_m^m (I')_1^m = (E')_1^m = (\widetilde{C})_n^m (E)_1^m = (\widetilde{C})_n^m \left[(Y)_n^n\right]^{-1}$$

$$(C)_m^n (I')_1^m$$

Since this holds true for any vector $(I')_1^m$, thus $(I')_1^m$ can be cancelled,

$$(Z)_m^m = (\widetilde{C})_n^m \left[(Y)_n^n\right]^{-1} (C)_n^n \qquad (1.9)$$

(1.9) states that one can determine the open-circuit impedance matrix $(Z)_m^m$ by applying a so-called "m-affine, degenerative, congruence transformation" on the inverse of the node admittance matrix $(Y)_n^n$. That is, for any network with n-node pairs, having $(Z)_m^m$ as its open-circuit impedance matrix, it is necessary that its node-admittance matrix $(Y)_n^n$ has to satisfy the equation (1.9). In other words, one may interpret that the equation (1.9) determines a group of networks (described by their node-admittance matrices) all of them will have the prescribed $(Z)_m^m$ as their open-circuit impedance matrix. Hence, from the synthesis standpoint, in order to realize a network from its prescribed impedance matrix $(Z)_m^m$, a natural approach should be: solve the equation (1.9) for $(Y)_n^n$ subject to the realizability conditions!

The solution of equation (1.9); however, is clearly not unique. In order to solve this equation subject to certain realizability conditions, a kind of attack that is frequently used in applied mathematics has been chosen. First, a particular solution of equation (1.9) is found, then from this particular solution one can generate all the possible solutions by means of a matrix transformation. Finally the realizability conditions are applied and thus it is possible to pick out from the group of solutions those that can be realized as physical networks. The generating of all the possible solutions $(Y)_n^n$ from a particular solution, say $(Y_1)_n^n$ can be performed by the transformation (4,5):

$$(Y)_n^n = (\tilde{T})_n^n \ (Y_1)_n^n \ (T)_n^n \qquad (1.10)$$

where $(T)_n^n$ is a non-singular matrix in which the first m rows are rows from the unit matrix; i.e.,

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & 1 & 0 & 0 \\ t_{m+1,1} & t_{m+1,2} & \cdots & & t_{m+1,m+1} & t_{m+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ t_{n1} & t_{n,2} & \cdots & & t_{n,m+1} & t_{nn} \end{bmatrix} \qquad (1.11)$$

The proof of (1.10) comes directly from (1.9). If $(Y_1)_n^n$ is a solution of (1.9), then

$$(Z)_m^m = (\tilde{C})_n^m \left[(Y_1)_n^n\right]^{-1} (C)_m^n$$

Note that $(\tilde{C})_n^m \, (T)_n^n = (C)_n^m$, etc. one has, by using (1.10)

$$(Z)_m^m = (\tilde{C})_n^m \, (T)_n^n \left[(Y_n)_n^n\right]^{-1} (T)_n^n \, (C)_m^n$$

$$= (\tilde{C})_n^m \left[(Y)_n^n\right]^{-1} (C)_m^n$$

Therefore, if $(Y_1)_n^n$ is a solution of (1.9) then $(Y)_n^n$ given by (1.10) will also be a solution of (1.9).

Furthermore, the generating formula (1.10) is also complete, i.e. equation (1.10) is also necessary. If $(Y)_n^n$ is a solution of (1.9), then,

$$(\tilde{C})_n^m \left[(Y_1)_n^n\right]^{-1} (C)_m^n = (Z)_m^m = (\tilde{C})_n^m \left[(Y)_n^n\right]^{-1} (C)_m^n$$

put $(C)_n^m = (C)_n^m \left[(T)_n^n\right]^{-1}$ yields,

$$(C)_n^m \left[(\tilde{T})_n^n\right]^{-1} (Y_1)_n^n \left[(T)_n^n\right]^{-1} (C)_m^n = (Z)_m^m$$

One solution has to be:

$$(\tilde{T})_n^n \, (Y_1)_n^n \, (T)_n^n = (Y)_n^n$$

Thus, it is concluded that the equation (1.10) will generate all the possible solutions of the synthesis equation (1.9) from a particular solution $(Y_1)_n^n$. It will be seen in the following sections how a physical network might be found from this group of solutions.

II. SYNTHESIS OF MULTITERMINAL RC NETWORKS FROM A PRESCRIBED OPEN-CIRCUIT IMPEDANCE MATRIX

The general transformation theory derived in the previous section will be applied to the synthesis of RC multiterminal plus ground networks (i.e., a multiterminal network with one node serving as the common ground). Without the loss of generality, one may assume that there are at most one resistance and one capacitance connected in parallel between any two nodes of the network. The node-admittance matrix, in this case, becomes a linear function of s with matrix coefficients, i.e.,

$$Y_n = As + B^{*} \qquad (2.1)$$

where: $A = (a_{ij})_n^n$ is the capacitance matrix,

$B = (b_{ij})_n^n$ is the resistance matrix

The synthesis equation (1.9) becomes

$$Z_m = \tilde{C}(As + B)^{-1} C. \qquad (2.2)$$

Some relevant questions may arise before solving the equation (2.2):

a) What properties that the open-circuit impedance matrix Z_m must have in order that it can probably be realized as a multiterminal RC network? In other words, what are the necessary conditions

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* From now on upper-case letters will represent matrices. Their order can be understood from the context.

that the open-circuit impedance matrix Z_m has to satisfy?

b) What are the properties of the coefficient matrices A and B. Under what conditions (both necessary and sufficient) can they always be realized?

These two questions will be discussed before beginning the solution of the synthesis equation (2.2).

A. Open-Circuit Impedance Matrix Z_m.

Some necessary conditions of realizability that Z_m has to satisfy are well-known. They are cited here without proof (11, 15).

a) All poles are simple and restricted on the non-positive real axis.

b) The zeroes of the diagonal elements (the driving-point functions) must lie in the left-half plane, including the imaginary axis.

c) The matrix of residues at each pole is positive-semidefinite.*

The character of the matrix of residues plays an important role in the synthesis. In general, they are highly degenerate and of unit rank (6). When the matrix of

* A positive semidefinite matrix is defined as a real, symmetric matrix whose associated quadratic form is non-negative for all values of the real variables. Thus, according to this terminology, a positive-semidefinite matrix may be positive-definite.

residues at a certain pole is a singular matrix of unit rank, this pole will be referred to as a "compact" pole (15). If all the poles are compact, the network will be referred to as a "compact network".

Non-compact networks may also exist; however, their occurrence is comparatively rare. For simplicity, only compact networks are investigated. The synthesis procedure can be applied to non-compact networks with slight modifications.

B. Node-Admittance Matrices.

In this section the node-admittance matrices $Y_n = As + B$ will be discussed in detail. In particular, a theorm is given concerning the necessary and sufficient conditions that a matrix can be realized as a multiterminal RC network.

Definition (16): An n x n symmetric real matrix $M = (m_{ij})_n^n$ is defined as a "Dominant matrix", if each of its main-diagonal terms is not less than the sum of the absolute value of all the other elements in the same row, i.e.,

$$m_{ii} \geqq \sum_{j=1}^{n} |m_{ij}| \qquad (i = j, i = 1, 2, \ldots n)$$

Definition: An n x n symmetric real matrix $M = (m_{ij})_n^n$ is called "proper signed", if there exist n real numbers μ_i ($i = 1, 2, \ldots n$) such that every off-diagonal term of the matrix $(\mu_i \mu_j m_{ij})_n^n$ is nonpositive.

In terms of the definitions given above, the theorm concerning the realizability conditions on the node-admittance matrices $Y_n = As + B$ can be stated. In order to avoid an interruption in the context, their proof is given in Appendix A.

<u>Theorem</u>(3, 17). The node-admittance matrix

$$Y_n = As + B$$

can be realized as a physical network if, and only if,

i) A and B are "proper-signed dominant" matrices.

ii) The sum of A and B is positive-definite.

This theorem leads to some characteristics of these node-admittance matrices. They are discussed as follows:

a) The coefficient matrices A and B are positive-semidefinite. This is obvious since they are dominant, and every dominant matrix must be positive-semidefinite.

b) The natural modes of a RC network must be real and nonpositive. Since Y_n can be considered as the inverse of an open-circuit impedance matrix when every node is regarded as a terminal. It has already been shown in the previous section that every pole of a RC network is real and nonpositive. Therefore, the natural modes are all real and nonpositive. In other words, the zeroes of the determinant equation $|As + B| = 0$ are real and nonpositive.

c) The natural modes need not be all distinct. However, it will be seen that if they are all distinct; then the open-circuit impedance matrix corresponding to this network will have degenerative residues matrix of unit rank at every pole. In other words, every simple mode corresponds to a compact pole. For simplicity, it is assumed here all natural modes are distinct; i.e., a compact network.

d) It is well-known that a pair of symmetric matrices can be diagonalized simultaneously if one of them is positive-definite (13). This is actually a generalized eigenvalue problem. If all the eigenvalues are distinct, then it's always possible to find a set of independent eigenvectors to form a diagonalizing matrix. This matrix will simultaneously diagonalize the pair of symmetric matrices. The positive-definite matrix will be reduced into a unit matrix, while the other will become a diagonal matrix whose elements are the eigenvalues; i.e., the roots of the determinant equation.

Since the sum of the two coefficient matrices is positive-definite, there must exist a non-singular matrix p which can simultaneously diagonalize A + B and A; i.e.,

$$\tilde{P}\,(A + B)\;P = U. \quad \tilde{P}AP = R, \text{ where } R = \text{diag. } (\lambda_i) \quad (2.3)$$

where λ_i are roots of the determinant equation $|(A + B)\lambda - A| = 0$.

By a simple change of variables:

set $\lambda = \frac{1}{1-s}$

then $\lambda_i = \frac{1}{1+\sigma_i}$ $\quad (i = 1, 2, \ldots n)$

where $(-\sigma_i)$'s are the roots of determinant equation $|As + B| = 0$, i.e. the natural modes of the network. Hence, they must be real and nonpositive. In general, they can be assumed as ∞, 0, $-\sigma_1$, $-\sigma_2$, ... $-\sigma_p$ (p = n when all modes are finite and nonzero). The corresponding λ_i's in Eq. (2.3) will be 0, 1, λ_1, λ_2, ... λ_p.

i.e. $\quad R = \text{diag. } (0, 1, \lambda_1, \lambda_2, \ldots \lambda_p)$

Let $\quad Q = \text{diag. } (1, 1, \lambda_1^{-\frac{1}{2}}, \lambda_2^{-\frac{1}{2}}, \ldots \lambda_p^{-\frac{1}{2}})$, and

$\quad J = PQ$

Then $\quad \tilde{J}AJ = Q\,\tilde{P}APQ = QRQ = D \qquad (2.4)$

where $\quad D = \text{diag. } (0, 1, 1, \ldots 1)$

and $\tilde{J}(A+B)J = Q\tilde{P}\,(A+B)\,PQ \pm QUQ$

$\quad = \text{diag. } (1, 1, \lambda_1^{-1}, \lambda_2^{-1}, \ldots \lambda_p^{-1}) \quad (2.5)$

subtracting (2.4) from (2.5), yields

$$\tilde{J}BJ = E \quad \text{where } E = \text{diag. } (1, 0, \sigma_1, \ldots \sigma_p). \qquad (2.6)$$

Thus, the pair of coefficient matrices A and B can be simultaneously reduced to diagonal forms as shown in (2.4) and (2.6). Observe that in (2.4) the zero element represents an infinite mode, and the zero element in (2.6) represents a zero mode. All the modes are distinct, since the network is assumed to be compact. They will be confluent in the case of non-compact networks. That is, some diagonal elements in (2.4) and (2.6) will be repeated.

From (2.4) and (2.6), using the notation "$\dotplus$" for direct sums, one has,

$$\begin{aligned}
Y_n^{-1} &= (As + B)^{-1} = J\ (SD + E)^{-1}\ \tilde{J} \\
&= J\left[1 \dotplus s \dotplus (s+\sigma_1) \dotplus (s+\sigma_2) + \ldots \right. \\
&\qquad \left. \dotplus (s+\sigma_p)\ \right] \tilde{J} \\
&= J\left[1 \dotplus s^{-1} \dotplus (s+\sigma_1)^{-1} \dotplus (s+\sigma_2)^{-1} \dotplus \ldots \right. \\
&\qquad \left. \dotplus (s+\sigma_p)^{-1}\right]\ \tilde{J} \\
&= H_\infty + \frac{H_o}{s} + \sum_{i=1}^{p} \frac{H_i}{s+\sigma_i} \qquad (2.7)
\end{aligned}$$

where $H_\infty = J\ (1 \dotplus 0 \dotplus \ldots \dotplus 0)\ \tilde{J}$

$H_o = J\ (0 \dotplus 1 \dotplus 0 \dotplus \ldots \dotplus 0)\ \tilde{J}$

$H_i = J\ (0 \dotplus 0 \dotplus \ldots \dotplus 1 \dotplus \ldots)\ \tilde{J}$

Evidently these matrices of residues are degenerate and of unit rank. Thus, it is concluded that the inverse of node-admittance matrix can be expanded into partial fractions with unit rank matrix of residues at each pole.

C. Synthesis Procedures.

The conclusion given at the end of the previous section is very important. It points out that the matrices of residues are degenerate and of unit rank. Hence, they may be factorized uniquely into the product of a column matrix by its transpose (7).

$H_i = h_i\widetilde{h_i}$, where h_i is a n x 1 column matrix.

Eq. (2.7) can be rewritten as

$$Y_n^{-1} = (As + B)^{-1} = h_\infty \widetilde{h_\infty} + \frac{h_o\widetilde{h_o}}{s} + \sum_{i=1}^{P} \frac{h_i\widetilde{h_i}}{s+\sigma_i} \quad (2.8)$$

Eq. (2.8) is an important result. It states that Y_n^{-1} can be expanded into partial fractions Every matrix of residues can be considered as the product of a column matrix by its transpose.

Similarly Z_m can also be expanded, as,

$$Z_m = K_\infty + \frac{K_o}{s} + \sum_{i=1}^{P} \frac{K_i}{s+\sigma_i}$$

$$= k_\infty \widetilde{k_\infty} + \frac{k_o\widetilde{k_o}}{s} + \sum_{i=1}^{P} \frac{k_i k_i}{s+\sigma_i} \quad (2.9)$$

Where K_∞, K_o, K_i are m x m residue matrix of unit rank; and k , k_o, and k_i are the corresponding m x i column matrices.

From (2.8) and (2.9) it is but one step to the solution of synthesis equation (2.2). However, it is felt pertinent to discuss some basic geometric idea involved in the transformation theory. The argument presented here is somewhat intuitive, but it might suggest a better point of view.

Recalling in the analysis of an arbitrary n node-pairs network, both the voltages and currents are written in the form of n x 1 column matrices. Actually they can be interpreted as two vectors in a n-dimensional space. These two n-dimensional vectors are related by an operator Y_n. That is, if the current vector is fixed, then the voltage vector is uniquely determined by the operator Y_n. In a selected reference frame, the geometric objects (V.I, and Y_n) may be expressed in the form of matrices. Then one can write $(V) = (Y_n)^{-1}$ (I) to express the relation among these quantities. In the synthesis problem, however, the operator Y_n is no longer kept unchanged. That is, for a fixed current vector, the voltage vector is allowed to vary in such a manner that will keep the open-circuit impedance invariant. In other words, for a prescribed open-circuit impedance of m-terminals network, the first m components of the voltage vector (i.e., the voltages at the external terminals) will not change, but the remaining n-m components may change in any way whatsoever. This is actually the case since a m-dimensional space is mapped "into" a n-dimensional space. Naturally a group of n-dimensional vectors will be generated by one m-dimensional vector by the transformation $V_m = C\ V_n$. Now if the n x 1 column matrix h_i in Eq. (2.8) is interpreted as a n-dimensional vector, and the corresponding m x 1 column matrix k_i in Eq. (2.9) as a m-dimensional vector, should they follow the same mapping relation?

That is, $k_i = \tilde{C}\, h_i$ (2.10)

The equation (2.10) is evidently true. Actually it can be easily derived by inserting (2.8) and (2.9) into the synthesis equation $Z_m = \tilde{C}\, Y_n{}^{-1}\, C$, and identifying the corresponding terms.

$$k_i \tilde{k}_i = \tilde{C}\, h_i\, \tilde{h}_i\, C = (\tilde{C} h_i)(\widetilde{\tilde{C} h_i})$$

Recalling the uniqueness of factorization, (2.10) is thus proved.

Referring back to the definition of matrices of residues H_i in (2.7), it is apparent that:

$$J = [h_\infty\,,\ h_o,\ h_i,\ \ldots\ h_p] \qquad (2.11)$$

using (2.4) and (2.6), one has,

$$\begin{aligned} Y_n &= As + B \\ &= \tilde{J}^{-1} (SD + E)\, J^{-1} \\ &= \tilde{J}^{-1} [\, 1 \dotplus S \dotplus (S + \sigma_1) \dotplus \ldots \dotplus (S + \sigma_p)]\, J^{-1} \qquad (2.12) \end{aligned}$$

The equations (2.12), (2.11) and (2.10) may be used to determine the general solution of synthesis equation (2.2). However, since the realizability conditions cited in the theorm (page 12) are difficult to apply, it seems much better to choose the attack described in the first section. First, a particular solution is determined, then Eq. (1.10) is used to generate all the possible solutions.

After such a process the realizability conditions might possibly be examined for these solutions.

The generating transformation in Eq. (1.10) can be considered as a series of elementary congruent transformations (9). The allowable transformations imply;

(a) Interchange of column (row) i and column (row) j, where $i, j > m$

(b) Multiply column (row) i by any non-zero constant, where $i > m$

(c) Adding to column (row) i by any multiple of column (row) j, where $j > k$.

If a physical realizable solution exists, it should be contained in the general solution. Hence, it would be possible to find it by operating on the particular solution the allowable transformations listed above. In order to illustrate the synthesis procedures, a simple example is given as follows:

Example:

$$Z_m = \frac{1}{16s(s+1)(s+2)} \begin{bmatrix} 81s^2+210s+128, & 24s^2+32s \\ 24s^2+32, & 64s^2+132s \end{bmatrix}$$

$$= \frac{1}{s}\begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+1}\begin{bmatrix} \frac{1}{16} & \frac{1}{2} \\ \frac{1}{2} & 4 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} \frac{1}{4} & 1 \\ 1 & \frac{1}{4} \end{bmatrix} \qquad (2.13)$$

Factorize the matrix of residues at each pole.

$$k_o = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad k_i = \begin{bmatrix} \frac{1}{4} \\ 2 \end{bmatrix} \quad k_2 = \begin{bmatrix} \frac{1}{2} \\ 2 \end{bmatrix}$$

The simplest solution of independent vectors h_i from (2.10) will be

$$h_o = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \quad h_1 = \begin{bmatrix} \frac{1}{4} \\ 2 \\ 0 \end{bmatrix} \quad h_2 = \begin{bmatrix} \frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$$

Inserting in (2.11), yields

$$J = \begin{bmatrix} 2 & \frac{1}{4} & \frac{1}{2} \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

A particular solution is, from (2.12)

$$Y_1 = \begin{bmatrix} 2 & 0 & 0 \\ \frac{1}{4} & 2 & 0 \\ \frac{1}{2} & 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} s & 0 & 0 \\ 0 & s+1 & 0 \\ 0 & 0 & s+2 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{4} & \frac{1}{2} \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 4s & -\frac{s}{2} & -s \\ -\frac{s}{2} & \frac{65}{16}s+4 & \frac{-31}{8}s-4 \\ -s & -\frac{31}{8}s-4 & \frac{49s}{4}+12 \end{bmatrix}$$

This is not realizable since it is not dominant. It can be made realizable by means of the allowable operations.

$$\begin{array}{cc}\begin{bmatrix} 4s & -\frac{s}{2} & -s \\ -\frac{s}{2} & \frac{65}{16}s+4 & -\frac{31}{16}s-4 \\ -s & -\frac{31}{8}s-4 & \frac{49}{4}s+12 \end{bmatrix} & \begin{matrix} \\ \\ \leftarrow -\frac{1}{2} \end{matrix} \\ \begin{matrix} \qquad\qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad\qquad\qquad -\frac{1}{2} \end{matrix} & \end{array} \longrightarrow \begin{bmatrix} 4s & -\frac{s}{2} & -\frac{s}{2} \\ -\frac{s}{2} & \frac{65}{16}s+4 & -\frac{31}{16}s-2 \\ -\frac{s}{2} & -\frac{31}{16}s-2 & \frac{49}{16}s+3 \end{bmatrix}$$

The realized network is shown in Fig. 1. Alternate forms are also possible, for instance,

$$\begin{array}{cc}\begin{bmatrix} 4s & -\frac{s}{2} & -\frac{s}{2} \\ -\frac{s}{2} & \frac{65}{16}s+4 & -\frac{31}{16}s-2 \\ -\frac{s}{2} & -\frac{31}{16}s-2 & \frac{49}{16}s+3 \end{bmatrix} & \begin{matrix} \\ \\ \leftarrow -1 \end{matrix} \\ \begin{matrix} \qquad\qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad\qquad\qquad -1 \end{matrix} & \end{array} \longrightarrow \begin{array}{cc}\begin{bmatrix} 4s & 0 & -\frac{s}{2} \\ 0 & 11s+11 & -5s-5 \\ -s & -5s-5 & \frac{49}{16}s+3 \end{bmatrix} & \begin{matrix} \\ \\ \leftarrow 2 \end{matrix} \\ \begin{matrix} \qquad\qquad\qquad\qquad\qquad \uparrow \\ \qquad\qquad\qquad\qquad\qquad 2 \end{matrix} & \end{array} \longrightarrow$$

$$\begin{bmatrix} 4s & 0 & -s \\ 0 & 11s+11 & -10s-10 \\ -s & -10s-10 & \frac{49}{4}s+12 \end{bmatrix}$$

This realization is shown in Fig. 2. It has less elements than the realization in Fig. 1.

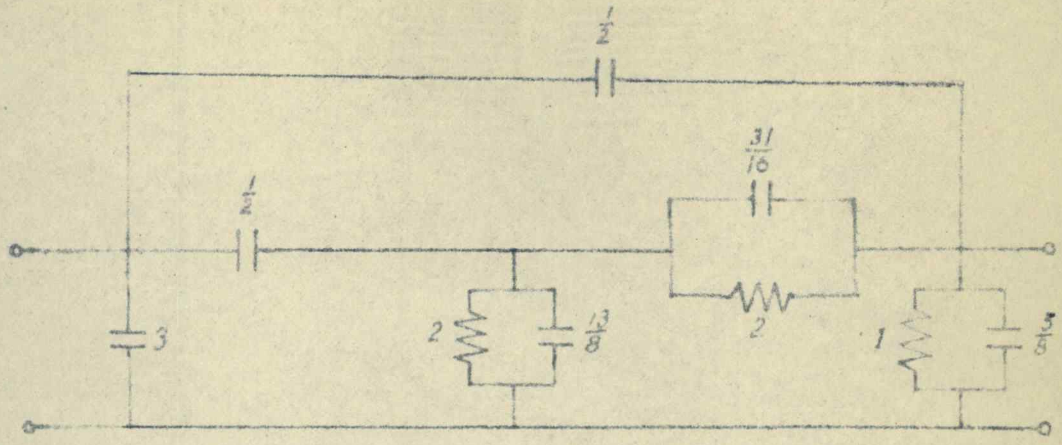

Fig. 1. Network Synthesizing $Z_m(s)$. Eq. (2.13)

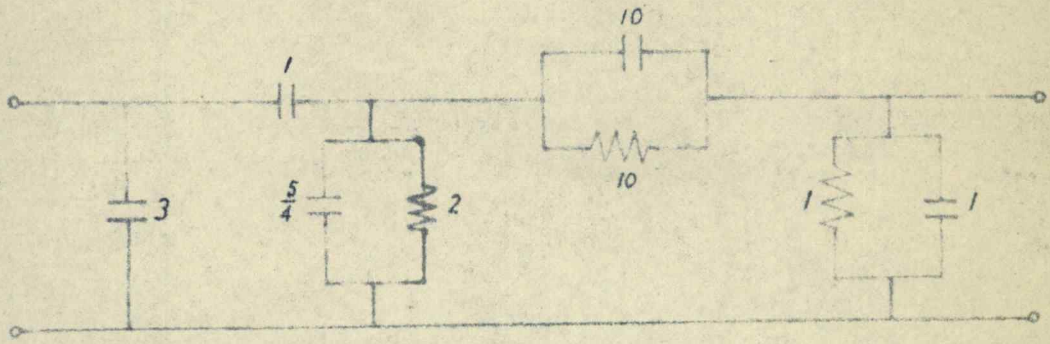

Fig. 2. Alternate Realization of $Z_m(s)$. Eq. (2.13)

III. RC LADDER SYNTHESIS

The synthesis method described in section 2 may lead to a physical realizable network having the prescribed open-circuit impedance matrix. However, since no sufficient condition has been obtained, there is no guarantee that the realizable solution exists.

In this section a different approach has been chosen. Instead of seeking for a physical realizable network from the general solution of the synthesis equation, a particular model is chosen to fit the solution. That is, a particular structure is forced to be identified, if possible, as a solution of the synthesis equation. The ladder network is a basic structure of primary importance; therefore, it is selected as the particular model to be identified as a solution of the synthesis equation. For simplicity, only the three terminal network is discussed. That is, the synthesis of a "RC Quadropole" from a prescribed 2 x 2 open-circuit impedance matrix, (i.e., the impedance function Z_{11}, Z_{12}, Z_{22}) is investigated.

A ladder, such as shown in Fig. 3 is represented by a special form of node-admittance matrix $Y_n = As + B$ as,*

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* For convenience, the matrix is labeled in a reverse order.

$$\begin{bmatrix} a_n s+b_n, & a_n' s+b_n', & 0 & \cdots & \cdots & \cdots \\ a_n' s+b_n', & a_{n-1}s+b_{n-1}, & a_{n-1}'s+b_{n-1}', & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & a_2 s+b_2, & a_2' s+b_2' \\ & & & & a_2' s+b_2' & a_1 s+b_1 \end{bmatrix} \qquad (3.1)$$

This matrix is called the "Tripel diagonal form". In this matrix all the elements except those on the main diagonal and those just above and below the main diagonal are equal to zero. It is also referred to as the "Jacobian matrix".

For convenience, (3.1) is rewritten as,

$$Y_n = \begin{bmatrix} P_n & -q_n & 0 & \cdots & \cdots & \cdots \\ -q_n & P_{n-1} & -q_{n-1} & \cdots & \cdots & \cdots \\ 0 & -q_{n-1} & P_{n-2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & -q_3 & 0 \\ \cdots & \cdots & \cdots & -q_3 & P_2 & -q_2 \\ \cdots & \cdots & \cdots & 0 & -q_2 & P_1 \end{bmatrix} \qquad (3.2)$$

where $P_i = a_i s+b_i \qquad i = 1, 2, \ldots n$ *

$-q_i = a_i' s+b_i \qquad k = 2, 3, \ldots n$

The special feature of a Jacobian matrix lies in the fact that any minors of this matrix can be represented as

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* It is obvious that P_i and q_i need not be restricted as the parallel combination of a resistance and a capacitance as described in section 2.

the product of its elements with its principal minors. A detailed discussion is given in Appendix B.

Now, suppose a ladder is chosen to be a physical realization of a certain prescribed open-circuit impedance matrix, then necessarily the solution of the corresponding synthesis equation (1.9) must be in the form of a Jacobian matrix. In other words, the prescribed open-circuit impedance matrix Z_m has to be restricted in some special manner. As usual, before the investigation for a synthesis procedure, an attempt is made to establish some necessary conditions.

The necessary conditions that a prescribed open-circuit impedance matrix Z_m has to satisfy in order that it may possibly be realized as an RC ladder network have been well established (9). Besides those conditions cited on page 10 there are some special characteristics of ladder networks, namely,

(a) The poles of the transfer impedance Z_{12} must also be the poles of the driving point impedance Z_{11} and Z_{22}; hence, they must be real and negative. Furthermore, they have to be finite and different from zero.

(b) The zeroes of the transfer impedance Z_{12} must be restricted to the negative real axis.

(c) The poles may be repeated. However, in the present discussions it will be assumed that they are distinct. That is, only the "compact" network is discussed. The matrix of residue at each pole is highly degenerative and of unit rank. This matrix of residues, in case of a three-terminal network, can be written as,

$$K_i = \begin{bmatrix} k_{11}^{(i)} & k_{12}^{(i)} \\ k_{12}^{(i)} & k_{22}^{(i)} \end{bmatrix} \qquad (i = 1, 2, \ldots n)$$

If the matrix should be degenerative and of unit rank, then necessarily its determinant must vanish, i.e.,

$$k_{11}^{(i)} \; k_{22}^{(i)} - \left[k_{12}^{(i)}\right]^2 = 0 \qquad i = 1, 2, \ldots n.$$

Thus, if $k_{12}^{(i)}$ and the residue of one of the driving point impedance at a certain pole, say $k_{22}^{(i)}$, are fixed, then the other one, $k_{11}^{(i)}$ will be uniquely determined. In other words, if a "compact network" has been found to possess the prescribed quantity Z_{12} as its transfer impedance, and Z_{12} as its driving point impedance at one terminal, then necessarily it will have Z_{11} as the driving-point impedance at the other terminal. Hence, only the realization of Z_{12} and Z_{22} (or Z_{11}) need be considered in a synthesis procedure. The third quantity Z_{11} (or Z_{22}) is realized as a consequence.

Returning to Eq. (3.2), it is possible to determine the open-circuit impedance matrix Z_m from the node-admittance matrix Y_n. However, it is noted that the transformation matrix introduced in the first section needs a slight justification. Since in the present case the first and last, instead of the first m node are considered as the access terminals. If a new symbol "L" is used to denote this modified transformation matrix, evidently,

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ \cdots & \cdots \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad (3.3)$$

And the corresponding synthesis equation is simply

$$Z_m = \tilde{L}\, Y_n^{-1}\, L \qquad (3.4)$$

where the node-admittance matrix Y_n is in the form of a "Jacobian matrix" in the present case of ladder realization.

From Eq. (3.2), (3.3), (3.4) it is possible to relate the elements in the Jacobian matrix with the prescribed quantities Z_{11}, Z_{12}, and Z_{22}. It is proved in Appendix B that

$$Z_{11} = \frac{1|}{|p_n} - \frac{q_n^2|}{|p_{n-1}} - \frac{q_{n-1}^2|}{|p_{n-2}} - \cdots - \frac{q_3^2|}{|p_2} - \frac{q_2|}{|p_1} \qquad (3.5)^*$$

$$Z_{22} = \frac{1|}{|p_1} - \frac{q_2^2|}{|p_2} - \frac{q_3^2|}{|p_3} - \cdots - \frac{q_{n-1}^2|}{|p_{n-1}} - \frac{q_n^2|}{|p_n} \qquad (3.6)$$

and

---

* The conventional notation for continued fraction, (i.e., stieltjes fraction) is adapted.

$$Z_{12} = \frac{\prod_{i=2}^{n} (-q_i)}{|Y_n|} \tag{3.7}$$

where $|Y_n|$ represents the determinant of the Jacobian matrix Y_n.

Eq. (3.7) states that any zero of the transfer impedance Z_{12} must be a root of the equation ${}_{i=2}^{\;n} q_i = 0$. Hence, the zeroes must be restricted to the negative real axis. Furthermore, it also suggests a possible synthesis procedure. It is observed that the series admittance q_i is a linear factor of the numerator of the transfer impedance Z_{12}, thus by the factorization of the numerator of the prescribed transfer impedance Z_{12}, it is possible to choose one of its linear factors as q_2; and thus, in turn, determine P_1 and the first section of the ladder. This process may be continued if the remaining impedance is still a RC impedance function.* Thus, a complete realization is possible and a three-terminal RC ladder having the prescribed quantities Z_{12} Z_{22} can be thus constructed. This network will also possess Z_{11} as one of its driving point impedance

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* A RC impedance function is defined as a rational function of s which can be realized as a driving point impedance using resistance and capacitance only.

in case of a "compact" network. The synthesis procedure is best explained by the following example:

$$Z_m = \frac{1}{27(s+1)(s+2)(s+3)} \begin{bmatrix} 9s^2+39s+38 & 6s+10 \\ 6s+10 & 18s^2+57s+41 \end{bmatrix}$$

$$= \frac{1}{27(s+1)} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} + \frac{1}{27(s+2)} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

$$+ \frac{1}{27(s+3)} \begin{bmatrix} 1 & -4 \\ -4 & 16 \end{bmatrix} \qquad (3.8)$$

It is easily seen that the prescribed open-circuit impedance matrix Z_m satisfies the necessary conditions cited on page 10. And, from the partial fraction expansion, evidently Z_m is compact at each pole.

Referring to Eq. (3.7, 3.8),

$$Z_{12} = \frac{q_2 q_3}{|Y_n|} = \frac{6s+10}{27(s+1)(s+2)(s+3)} \qquad (3.9)$$

$$Z_{12} = \frac{1|}{|p_1} - \frac{q_2^2|}{|p_2} - \frac{q_3^2|}{|p_3} = \frac{18s^2+57s+41}{27(s+1)(s+2)(s+3)} \qquad (3.10)$$

choose $q_2 = \alpha(s+\frac{5}{3})$, where α is a constant.

Subs. the above eq. into (3.10).

$$\frac{1}{Z_{22}(s)} = \frac{27(s+1)(s+2)(s+3)}{18s^2+57s+41} = p_1 - \frac{\alpha^2(s+\frac{5}{3})^2}{\left(\frac{1|}{|p_2} - \frac{q_3^2|}{|p_3}\right)} \qquad (3.11)$$

putting $s = -\frac{5}{3}$ yields,

$$p_1(-\tfrac{5}{3}) = 2$$

Differentiate $\frac{1}{Z_{22}(s)}$ and then set $s = -\frac{5}{3}$

$$p_1'(-\tfrac{5}{3}) = 3$$

$$p_1 = 3s+7.$$

$$\alpha^2 \left[\frac{1}{p_2} - \frac{q_3{}^2}{p_3}\right] = \frac{27s+45}{18s^2+57s+41}$$

choose $\alpha = 3$. i.e., $q_2 = 3s+5$, $q_3 = 2$, $p_1 = 35+7$,

and

$$\frac{1}{p_2} - \frac{4}{p_3} = \frac{3s+5}{18s^2+57s+41} \qquad (3.12)$$

Thus, the first section of the RC ladder network can be constructed. The whole network can be found by continuing the procedures.

From Eq. (12), it is seen,

$$p_2 = \frac{4}{p_3} = 6s+9 - \frac{4}{3s+5}$$

Therefore, $p_2 = 6s+9$

$p_3 = 3s+5$

This ladder network is shown in Fig. 4.

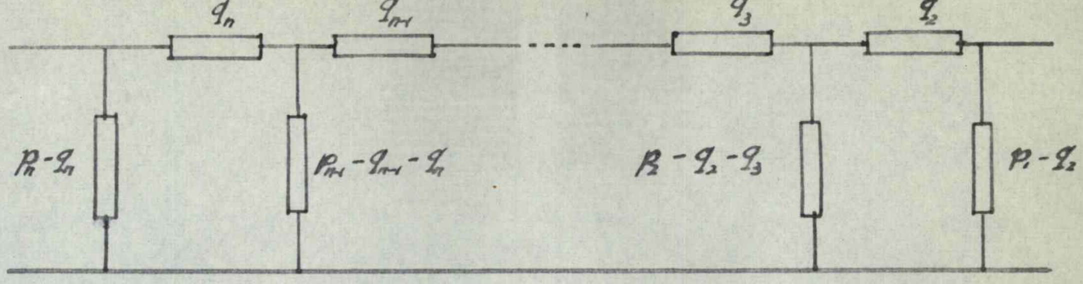

Fig. 3. A Ladder Network

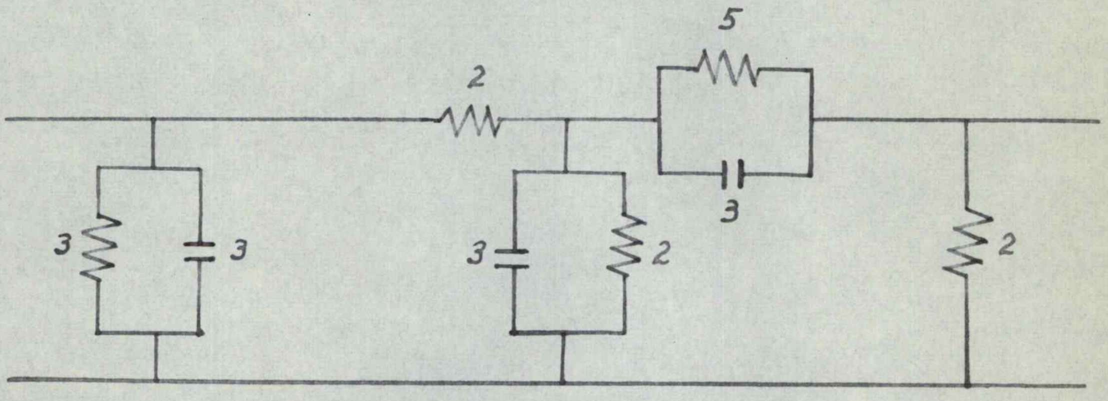

Fig. 4. RC Ladder Realization
of $Z_m(s)$. Eq. (3.8)

IV. CONCLUSION

The synthesis method described in the previous sections can also be applied to the case of RL and LC networks by a simple change of variables. It is well known that the substitutions $s = \frac{1}{t}$ and $s = t^2$ will change the RL and LC networks to an identical form of RC networks respectively. The technique is treated in most books on network synthesis and thus its discussion here is unnecessary.

The network realization described here can be modified to meet the synthesis problems involving active elements. Transfer functions with poles and zeroes anywhere in the complex plane can always be provided even with only two kinds of elements if active elements are utilized. The situation becomes somewhat simpler with the use of active elements since the dominant requirement on the node admittance matrix no longer has to be satisfied. And, this is just the cumbersome condition that usually renders the problem unsolvable. Actually this transformation approach is more natural and has been recognized as a powerful tool in the area of active networks.

It would be appropriate to give a comment on this synthesis method based on the equivalence of networks and some possible directions for further investigations.

This synthesis method, though apparently a powerful tool, actually does not lead to a fruitful result. This fact may be seen from the examples given in the previous sections. This approach is based on the idea of equivalence of networks. And this seems to be a natural and general attack of the synthesis problem. Unfortunately, it is scarcely possible to solve a problem based on a general approach. The reason for the failure of this method is that there is no one-to-one correspondence between the parameter matrix, and the associated network unless the network is restricted to a particular reference frame. And even if a rather convenient reference frame is selected resulting in relatively simple realizability conditions, the simplicity will be lost immediately as a result of subsequent manipulations. Synthesis based on network equivalence is, so far, one of the most subtle and least well-understood aspects in the area of network synthesis. However, it is felt that a further investigation is still needed since the method provides a fundamental concept to the problem of network synthesis. Moreover, it would yield the complete solution of network synthesis if its inherent defects could be overcome.

It is felt that in order to overcome the inherent defects of matrix method, the transformation should be interpreted in terms of linear graphs. Matrices can only represent a "cross section view", so to speak, of the real geometric object. How could one be so fortunate to pick out a particular reference frame, among a great number of choices, in which there is a well-posed physical interpretation?

Thus, it seems more practical to select certain network structures such as the ladder realization described in section 3 and attempt to identify these as a solution of the synthesis equation. The disadvantage of this "constructed-solution" method is that it must be restricted to the special form of selected structure. As a consequence more restrictions are imposed on the prescribed open-circuit impedance matrix Z_m. From this point of view one will naturally prefer to take the more general approach. It is believed that the general principle of equivalent networks will be proved useful and deserve more investigation.

However, one might have a better chance of success if the approach based on equivalent networks is studied in the language of linear graphs instead of matrix transformations. Hence, it is felt that "topology" may assume a fruitful role in a further investigation of the network synthesis problem.

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APPENDICES

APPENDIX A

Necessary and Sufficient Conditions of Realizability of the Node-Admittance Matrix $Y_n = As + B$

The theorem cited in section 2 (page 12) concerning the realizability of node admittance matrix $Y_n = As + B$ will be proved in this Appendix.

Lemma: A n x n real matrix $M = (m_{ij})_n^n$ can be realized as the node-admittance matrix of a n node-pair resistive network if, and only if, the matrix is a "proper-signed" dominant matrix.

The lomma is just a modified form of a theorm given in many relevant papers (17). The usual condition requires that each off-diagonal term of the matrix be non-positive. The proof for such a situation is really routine.

Necessity comes directly from writing the node-equations of a common ground network. Sufficiency can be proved by a simple synthesis procedure, corresponding to each non-zero element m_{ij}, a conductance equals to m_{ij} is connected between node i and node j. And between each node j and the common ground a conductance is connected with the value $m_j = \sum_{i=1}^{n} m_{ij}$, since the matrix is dominant, hence $m_j - 0 \quad j = 1, 2, \ldots n$.

It is attempted to argue here that requiring all off-diagonal terms non-negative is not essential. Any dominant matrix even with some positive off-diagonal terms is realizable if it can be converted into one with all off-diagonal elements non-positive when both voltages and currents are reversed in direction at some terminals. Since a reversal of both voltage and current at a certain node, say the j-th node, will just cause a change in sign for all the off-diagonal terms in the j-th row and j-ith column. Therefore, all the possible changes in sign distribution may be performed as the matrix $(\mu_i \mu_j m_{ij})$. Thus, if there exist n non-zero numbers μ_i $(i=1, 2, \ldots n)$ such that all the off-diagonal terms of the matrix $(\mu_i \mu_j m_{ij})$ be non-positive, then a physical realization is always possible for the dominant matrix (m_{ij}). In other words, the matrix must be a "proper-signed" dominant matrix. Hence, the lamma is proved.

<u>Theorem</u>. The node-admittance matrix

$$Y_n = As + B$$

can be realized as a multiterminal RC plus ground network if, and only if,

a) A and B are "proper-signed" dominant matrices.

b) The sum of A and B is positive-definite.

"Proof."

<u>Necessity</u>: (a) is obviously true according to the lemma. And since $Y_n = As + B$ must be a positive real

matrix when every node is considered as an external terminal. Thus, As + B should be positive-definite for s real and positive. Letting s = 1, it is seen that A + B has to be positive definite.

Sufficiency: The network can be constructed in the manner described in the lemma. The condition (a) implies that each element in the network is real and non-negative. One needs only prove that (b) implies the network contains a tree. This may be proved by applying the Maxwell Topological rules as follows (14):

$Y_n = As + B = \sum$ tree admittance products of the network. Taking s = 1, Y_n will be positive-definite, this it must be non-singular. This implies that the summation of tree admittance product not be equal to zero. Thus, at least one tree product is different from zero. Hence, the network contains a tree, and the proof is completed.

APPENDIX B

Evaluation of Eqs. (3.5), (3.6), and (3.7)

The Jacobian matrix is a "Triple diagonal form", that is, all elements are zero, except those on the main-diagonal and those just above and below the main diagonal, i.e.

$$Y_n = \begin{vmatrix} p_n, & q_n & 0 & \cdots & \cdots & \cdots \\ q_n & p_{n-1} & q_{n-1} & \cdots & \cdots & \cdots \\ 0 & q_{n-1} & p_{n-2} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & p_3 & q_3 & 0 \\ \cdots & \cdots & \cdots & q_3 & p_2 & q_2 \\ \cdots & \cdots & \cdots & 0 & q_2 & p_1 \end{vmatrix} \qquad (3.2)$$

A set of submatrices can be obtained by deleting the first row and column, the first two rows and columns,.... ...etc., and the corresponding determinants of these submatrices are designated by D_{n-1}, D_{n-2}, ... D_2, D_1 respectively. Besides, another set of submatrices can be also constructed by deleting the last row and column, the last two rows and columns, ... etc., and the corresponding determinants of these submatrices are designated by D_{n-1}', D_{n-2}', ... D_2', D_1'. From Eq. (3.2), it is observed that,

$$D_i = p_i \, D_{i-1} - q_i^2 \, D_{i-2}$$

and $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i=2, 3, \ldots n)$

$$D_i' = p_{n-i+1} \, D_{i-1}' - q_{n-i+2}^2 \, D_{i-2}'$$

also $D_1 = p_1$, $D_1' = p_n$.

and $D_n = D_n' = Y_n$.

With the aid of these relations, the Eqs. (3.5, 3.6, 3.7) can be easily derived.

Inserting Eq. (3.2) into the synthesis Eq. $Z_m = L\, Y_n\, L$ yields,

$$Z_m = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{pmatrix} = \frac{1}{|Y_n|} \begin{bmatrix} D_{n-1} & \prod_{i=2}^{n}(-q_i) \\ \prod_{i=2}^{n}(-q_i) & D'_{n-1} \end{bmatrix}$$

Hence $$Z_{12} = \frac{\prod_{i=2}^{n}(-q_i)}{|Y_n|}$$

$$Z_{11} = \frac{D_{n-1}}{D_n} = \frac{D_{n-1}}{p_n D_{n-1} - q_n^2 D_{n-1}} = \frac{1}{p_n - q_n^2 \frac{D_{n-2}}{D_{n-1}}}$$

$$= \cfrac{1}{p_n - \cfrac{q_n^2}{p_{n-1} - \cfrac{q_{n-1}^2 D_{n-3}}{D_{n-2}}}} = \ldots\ldots\ldots$$

$$= \frac{1|}{|p_n} - \frac{q_n^2|}{|p_{n-1}} - \frac{q_{n-1}^2|}{|p_{n-2}} - \ldots - \frac{q_3^2|}{|p_2} - \frac{q_2^2|}{|p_1} \qquad (3.5)$$

Similarly

$$Z_{22} = \frac{D'_{n-1}}{D_n} = \frac{D'_{n-1}}{p_1 D'_{n-1} - q_2^2 D'_{n-2}} = \cfrac{1}{p_1 - \cfrac{q_2^2}{p_2 - \cfrac{q_3^2 D'_{n-3}}{D'_{n-2}}}}$$

$$= \ldots = \frac{1|}{|p_1} - \frac{q_2^2|}{|p_2} - \ldots \frac{q_{n-1}^2|}{|p_{n-1}} - \frac{q_n^2|}{|p_n} \qquad (3.6)$$