AN ABSTRACT OF THE DISSERTATION OF

Sarah A. Erickson for the degree of Doctor of Philosophy in Mathematics presented on July 29, 2020.

Title: Investigating Provers' Understandings of Combinatorial Proof.

Abstract approved: ____
Elise N. Lockwood

Enumerative combinatorics is an area of mathematics that is both highly accessible for students and widely applicable to other sciences and areas of mathematics (Kapur, 1970; Lockwood, Wasserman, & Tillema, 2020). One important class of problems in combinatorics is combinatorial proofs of binomial identities, which is a type of proof that argues for the veracity of an identity by arguing that each side enumerates a (finite) set of outcomes. The validity of a combinatorial proof lies in the fact that a set can have only one cardinality. Such proofs suggest an analytical proof scheme (Harel and Sowder, 1998) and have been considered to be examples of proofs that explain (in the sense of Hersh, 1993) with respect to an enumerative representation system (Lockwood, Caughman, & Weber, 2020). Combinatorial proofs also differ from other types of proofs students may encounter in several important ways. One feature of combinatorial proofs is that they are comprised exclusively of sentences and paragraphs; that is, a student producing a combinatorial proof must combinatorially interpret symbols appearing in the identity without algebraically manipulating those symbols. This feature has potential implications for students, since researchers have found that combinatorial reasoning can be a notoriously difficult for students (e.g., Batanero

et al., 1997, and Lockwood, 2014b) and that some students are less likely to accept an argument to be a rigorous mathematical proof if it does not contain symbolic manipulations (e.g., Martin & Harel, 1989). In addition, while there are a couple of prior studies that have looked at students' combinatorial proof activity (Engelke & CadwalladerOlsker, 2010; Lockwood, Reed, & Erickson, in press), much remains unknown regarding what students and mathematicians attend to as they produce combinatorial proofs. For instance, it has also never been verified with empirical evidence whether or not students or mathematicians do indeed consider combinatorial proofs to be proofs that explain, and even less is known regarding whether these populations consider combinatorial proofs to be proofs that convince.

In my dissertation study, I seek to answer the following research questions:

1. To what extent do experienced provers (including students and mathematicians) believe that combinatorial proofs of binomial identities are convincing and/or explanatory, and why?
2. What proof schemes do undergraduate students who are experienced provers use to discuss and characterize combinatorial proof?
3. What do the answers to these questions say about the nature of combinatorial proof (including how it may differ from other types of proof)?
4. What are some other insights about combinatorial proof that can be gained from interviewing experienced provers?

To answer these questions, I conducted clinical interviews with five upper-division mathematics students and eight mathematicians to investigate what they attended to as they produced and evaluated combinatorial proofs and how they viewed combinatorial proof as different from other types of proof. This dissertation begins with overall summaries of relevant literature, theory, and the methods involved in the overall study. Then, the results of the dissertation are presented in three manuscripts, where I describe the students' and mathematicians' perceptions of combinatorial proof using two theoretical frameworks: proofs that explain and/or convince (Hersh, 1993) and proof schemes (Harel & Sowder, 1998). I also use Lockwood's (2013) model and the construct of cognitive models to describe an important aspect of students' and mathematicians' combinatorial reasoning that had implications for their success producing combinatorial proofs: cognitive models of multiplication.

In the first manuscript chapter of my dissertation, I describe the results of my investigation into whether students and mathematicians viewed combinatorial proof as explanatory or convincing (Hersh, 1993), and why. I found that all 13 participants felt that combinatorial proofs are equally or more explanatory than other types of proofs, but participants demonstrated a variety of perspectives regarding the extent to which combinatorial proofs are convincing. These findings provide empirical evidence for Lockwood et al.'s (2020) claim that combinatorial proofs are usually proofs that explain within the enumerative representation system, as well as provide insights on the nature of combinatorial proof as a mathematics topic.

In the second manuscript chapter of my dissertation, I discuss the proof schemes (Harel & Sowder, 1998) that students used to discuss and characterize combinatorial proof compared with other types of proof. I found that students used authoritarian, ritual, perceptual empirical, transformational analytical, and contextual restrictive proof schemes, and that these proof schemes had implications for the students' perspectives regarding whether (and why) combinatorial proof constitutes rigorous mathematical proof. I also discuss whether and how other proof schemes may emerge for students engaging in combinatorial proof.

Finally, in my third manuscript chapter, I focus on a specific phenomenon that emerged during my interviews with mathematicians and students as they engaged in combinatorial proof production. In particular, participants used a wide variety of cognitive models to interpret multiplication by a constant when reasoning about binomial identities, some of which seemed to be more (or less) effective in helping produce a combinatorial proof. I present these cognitive models and describe episodes that illustrate implications of these cognitive models for my participants' work on proving binomial identities. My findings both inform research on combinatorial proof and highlight the importance of understanding subtleties of the familiar operation of multiplication.

Overall, in addition to the specific results and findings presented in each of the papers, these three manuscripts supported four main takeaways regarding students' and mathematicians' reasoning about and engagement with combinatorial proof: 1) students can successfully produce combinatorial proofs and recognize their activity constitutes proof; 2) combinatorial proof may be viewed by some students as intuitive arguments but not formal proofs; 3) the contexts used in combinatorial proofs are important; and 4) difficulties in solving counting problems can carry over to difficulties in combinatorial proof production. These findings have implications practitioners and researchers. For a start, both teachers and researchers should be aware that students may have a variety of conceptions about combinatorially proof as they teach and conduct proof-education research, respectively. In the classroom, instructors should understand that some students may believe combinatorial proof is less valid than algebraic, induction, or other types of proof for a variety of reasons, and so instructors should clarify for students why correct combinatorial proofs are indeed mathematically rigorous and logically valid. Instructors should also have discussions with their students about the element selection cognitive model of multiplication and highlight its relationship with the Multiplication Principle. Lastly, when researchers draw conclusions about student thinking about proof, they should be mindful that some of these conclusions may apply differently to student thinking about combinatorial proof.

Investigating Provers' Understandings of Combinatorial Proof

by
Sarah A. Erickson

A DISSERTATION

submitted to

Oregon State University

in partial fulfillment of
the requirements for the
degree of

Doctor of Philosophy

Presented July 29, 2020
Commencement June 2021

Doctor of Philosophy dissertation of Sarah A. Erickson presented on July 29, 2020

APPROVED:

Major Professor, representing Mathematics

Head of the Department of Mathematics

Dean of the Graduate School

I understand that my dissertation will become part of the permanent collection of Oregon State University libraries. My signature below authorizes the release of my dissertation to any reader upon request.

Sarah A. Erickson, Author

ACKNOWLEDGEMENTS

I'm filled with gratitude for the numerous people who have supported and assisted me throughout my tenure in graduate school and particularly during the last couple of years while I finished this dissertation.

First, I'd like to thank my awesome adviser, Dr. Elise Lockwood, who continually provided me invaluable support and guidance throughout my graduate career. My path to a PhD wasn't always exactly linear or smooth, and I appreciate Elise's patience and willingness to be flexible with me as I pursued a community-college teaching career concurrently with finishing my PhD. Elise is an incredible writer, presenter, and researcher in her field, and I feel especially fortunate that I could learn from her example as one of her graduate students. I am thankful that she helped me narrow down an interesting dissertation topic I could research successfully, and I'm so deeply grateful for the vast number of hours she spent advising me, answering my questions about study design and analysis, editing my writing, critiquing my presentations, and always pushing me to excel. Elise also brought me with her to four RUME conferences, which offered me indispensable learning opportunities and experiences to engage with others in the field of undergraduate mathematics education and develop as a researcher. Elise is truly a gem, and I'm so happy I was fortunate enough to be one of her Master's and PhD students for my six years as a student at OSU. I would also like to thank Jani Lockwood (Elise's mom) for helping to sew cloth face masks for me and my husband during the COVID-19 pandemic, and for always making sure I felt welcome and well-fed whenever I visited her or Elise's home.

I would also like to thank the other members of my PhD committee: Dr. Mary Beisiegel, Dr. Tom Dick, Dr. Wendy Aaron, and Dr. Bogdan Strimbu. Mary's Mathematical Knowledge for

Teachers course I took at the beginning of my graduate career was so inspiring and useful—I even drew on what I learned in her course during job interviews for community-college teaching jobs. I since took multiple additional education classes with Mary and always so enjoyed to have her as a professor. Tom has also been an invaluable member of my committee. I learned so much hearing his insights every time I attended Math Ed Seminar, and he's always willing to share fun, challenging math problems to solve. Wendy was in integral part of my graduate education by teaching my Learning Theories course, which was critical to my development as a mathematics education researcher. She also is a very kind, caring person, and I appreciate that she was always there to listen if I needed to rant about contract negotiations with OSU admin. Finally, I also want to thank Bogdan Strimbu for serving as the Graduate Council Representative on my committee.

I would next like to thank my academic siblings Branwen Purdy, Adaline De Chenne, Zack Reed, and Allison Dorko for being wonderful humans and for the fun times we've had together at math education conferences and workshops. I'm so grateful to know all of you and for the mutual support we've provided one another as we've navigated our graduate careers.

I additionally want to thank my wonderful comrades in the Coalition of Graduate Employees who I admire so much for their dedication to bettering the lives of graduate teaching and research assistants at Oregon State University. Because of CGE I was able to focus on my studies while here at OSU, and I didn't have to worry about food insecurity or whether I'd be able to afford rent or healthcare. I'd especially like to thank my fellow bargaining team members: Lzz Johnk, Nathan Waugh, Alex Avila, Kiran Ahmed, David Glennon, Emalydia Flenory, Erin Abernethy, Erin Kanzig, Fara Broughton, Gabrielle Miller, Hazel Daniels, Jon Ruff, Nick Fisher, Nicole Metildi, Patrick Carilli, and Zoe Juozapaitis. Y'all have incredible tenacity, and I'm proud to have struggled alongside you during contract negotiations this academic year.

I would also like to thank my family for their support and love throughout my life and especially during these past few years. I’m so grateful for my sister, Emily Wooldridge, for being there when I want to talk, making me laugh, and for being my lifelong friend. I also want to thank my parents, Brian and Pam Erickson, for supporting my education throughout my childhood and college career. They both worked incredibly hard homeschooling my sister and me growing up, and I’m grateful that my they always encouraged us to pursue mathematics and science. I have no doubt that I could not have been so successful in school if I had not had the privilege of a loving, supportive family. I also want to thank my dear friend, Miho Stanberry. I’m so grateful to have met them in a probability class our junior year of undergrad at Western Washington University, and they are always down to chat about social justice or play a round of Mario Kart with me. I’m also so glad they encouraged me to go vegan more than two years ago.

Finally, I want to express my deepest thanks to my husband and best friend in the whole world, Ricardo Reyes Grimaldo. Ricardo has always supported and encouraged me in my graduate studies, helping me to persevere during the most difficult of times when I considered dropping out. He comforted and helped re-focus me during times when I had episodes of depression and anxiety, and his work ethic, teaching skills, and mathematical modeling abilities are an inspiration to me. He is the most patient and caring person I know, and I never would have finished my PhD without him.

This dissertation is based upon work supported by the National Science Foundation under Grant No. 1650943.

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DEDICATION

This dissertation is dedicated to my parents. Words can't express how grateful I am for your love and encouragement, and for supporting my education in so many ways throughout my life.

CHAPTER 1 – Introduction

Enumerative combinatorics is an area of mathematics that is both highly accessible for students and widely applicable to other sciences and areas of mathematics (Kapur, 1970; Lockwood, Wasserman, & Tillema, 2020). One important topic in combinatorics education is combinatorial proof of binomial identities, which comes up in discrete mathematics, statistics, probability, number theory, and other contexts, and yet has received little attention in the mathematics education literature. Combinatorial proof is a proof method that establishes the veracity of an equation by arguing that the expressions on either side of the equation each enumerate a set (possibly the same set) of equal cardinality (Lockwood, Reed, & Erickson, in press; Rosen, 2012).

Consider, for example, Pascal's identity, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.[1] This identity can be proven by considering the set of committees of size *k* which can be formed from a group of *n* (distinct) people. The left side of the identity counts this, since $\binom{n}{k}$ counts the number of unordered selections of size *k* that can be formed from a set of *n* distinct things. For the right side, suppose without loss of generality that one of the *n* people is named Sofía. Then, $\binom{n-1}{k}$ counts the number of committees that can be formed excluding her (since there are $n - 1$ remaining people who can be on the committee), and $\binom{n-1}{k-1}$ enumerates the committees that include her (since there are $k - 1$ remaining spots on the committee and $n - 1$ remaining people). Since this case breakdown (those without and with Sofía, respectively) encompasses all possibilities, the left side also counts the same set of

[1] For this and all subsequent binomial identities in this dissertation, I consider the domain for the variables involved to be nonnegative integers, which was also made clear to all research participants. However, I also acknowledge that many of the identities in this paper may also hold for other real values.

committees[2]. The validity of a combinatorial proof lies in the fact that a set can have only one cardinality.

Because combinatorial proof of binomial identities does not involve algebraic manipulation but instead requires the prover to articulate combinatorial processes underlying binomial expressions, combinatorial proof can provide opportunities for students to encounter analytic proof schemes and proofs that explain (Harel & Sowder, 1998; Hersh, 1993). Despite the utility and pedagogical advantages of combinatorial proof, however, little is understood about student beliefs about combinatorial proof or what students attend to as they write combinatorial proofs. The few existing studies that target undergraduate students' activity with and beliefs about combinatorial proof rely only on artifact-based data (Engelke & CadwalladerOlsker, 2010; Engelke Infante & CadwalladerOlsker, 2011), or were conducted with novice provers who may have limited perspectives about combinatorial proof as a proof method (Lockwood et al., in press). Furthermore, even less is understood about how mathematicians may conceptualize or engage in these proofs.

In this dissertation, I report on a qualitative study investigating upper-division mathematics students' and mathematicians' conceptions of and engagement with combinatorial proof. To examine students' and mathematicians' conceptions, I applied the applied two well-studied frameworks in proof literature: *proofs that convince and proofs that explain* (Hersh, 1993) and *proof schemes* (Harel & Sowder, 1998). In addition, I used Lockwood's (2013) model of students' combinatorial thinking to characterize combinatorial proof for this study and as a theoretical lens to frame students' and mathematicians' engagement with combinatorial proof.

[2] This proof exemplifies an Approach 1 combinatorial proof (Lockwood et al., in press), and I acknowledge that there are other kinds of combinatorial proofs that exist (such as those that involve establishing a bijection). I do not discuss these other types of combinatorial proof in this dissertation, as they were not the focus of my study.

A broad goal of the study is to characterize how the nature of combinatorial proof differs from other types of proof, as well as the extent to which students may consider combinatorial proof to be a legitimate method of mathematical proof. Some of my own experience in research and in the classroom has suggested that some undergraduate students may be uncomfortable with the idea of combinatorial proof—feeling that it is not as convincing as an algebraic proof, or that a combinatorial proof resembles a fallacious "proof by example" since it establishes a general identity by placing it within a specific combinatorial context (e.g., counting committees of a particular size, counting binary strings, etc.). There have been two prior studies that have looked at what students may attend to as they produce combinatorial proofs (Engelke Infante & CadwalladerOlsker, 2011; Lockwood et al., in press), and there are many opportunities for investigation into a number of aspects of combinatorial proof. In particular, none of these studies have specifically targeted experienced provers' conceptions about combinatorial proof, and no prior studies have examined mathematicians' combinatorial proof production. I elaborate existing literature on combinatorial proof in more detail in subsequent sections of this dissertation.

1.1 Research Questions

To address these existing gaps in the literature, overall my qualitative study aims to answer the following research questions:

1. To what extent do experienced provers (including students and mathematicians) believe that combinatorial proofs of binomial identities are convincing and/or explanatory, and why?
2. What proof schemes do undergraduate students who are experienced provers use to discuss and characterize combinatorial proof?
3. What do the answers to these questions say about the nature of combinatorial proof (including how it may differ from other types of proof)?
4. What are some other insights about combinatorial proof that can be gained from interviewing experienced provers?

In this dissertation, these questions are motivated and contextualized in Chapters 2 and 3, where I expand on relevant literature and theory from combinatorics and proof education literature, respectively. Then, in Chapter 4, I describe the methods I used to conduct and analyze the interview data I collected for my study. The results of this study are presented in three manuscripts. The first, *Combinatorial Proofs as Proofs That Convince and Proofs That Explain* (Chapter 5), addresses Research Questions 1 and 3 by reporting on data regarding mathematicians' and students' perspectives about combinatorial proofs as more or less convincing and explanatory (Hersh, 1993) than other types of proof. The second manuscript of my dissertation, *Investigating Undergraduate Students' Proof Schemes and Perspectives about Combinatorial Proof as Rigorous Mathematical Proof* (Chapter 6), addresses Research Questions 2 and 3 by describing the results of my analysis looking at the proof schemes (Harel & Sowder, 1998) upper-division mathematics students used to characterize combinatorial proof as more or less rigorous than other types of proof (such as algebraic and induction), and why. The third manuscript, *Investigating Combinatorial Provers' Reasoning about Multiplication* (Chapter 7), addresses Research Question 4 by reporting on an unexpected interesting phenomenon that emerged in my thematic analysis (Braun & Clarke, 2006) of students' and mathematicians' combinatorial proof production, namely the emergence of different cognitive models for multiplication the students and mathematicians used. As I discuss, these cognitive models were not only surprisingly varied but also had implications for students' success at combinatorial proof tasks. I submitted a manuscript based on this chapter with Dr. Lockwood to the *International Journal of Research in Undergraduate Mathematics Education*, and it has been accepted with revision. Finally, in Chapter 8, I discuss overall conclusions from my study, as well as its limitations and potential avenues for future research. Overall, I describe some important ways that students and mathematicians characterize combinatorial proof as different from other types of proof using the lens of proofs that explain/convince (Hersh, 1993) and proof schemes (Harel & Sowder, 1998), and I argue for the importance of considering cognitive models when considering students' and mathematicians' combinatorial proof production.

Ultimately, this dissertation offers contributions both to combinatorics and proof education literature. My study sheds light on an important topic in combinatorics education—combinatorial proof—that has received little attention thus far, and it provides evidence for ways students and mathematicians may view combinatorial proof differently from other types of proof. In addition, my study offers a novel application of the widely used proof schemes framework (Harel & Sowder, 1998) and offers empirical contributions to conversations researchers have begun having about combinatorial proofs as convincing and explanatory (e.g., Lockwood et al., 2020).

CHAPTER 2 – Literature Review and Theory about Combinatorics

In this chapter, I review relevant literature in combinatorics education to provide context and background for my study. First, in Section 2.1, I summarize some of the work that has been done on students' solving of counting problems, and in Section 2.2 I review Lockwood's (2013) model of students' combinatorial thinking. In Section 2.3 I discuss all prior research that has been conducted on combinatorial proof. The purpose of this chapter is to provide a brief, general overview of the most relevant literature for my study; in Chapters 5-7 of this dissertation I provide more detailed discussions of existing literature that is most relevant to each manuscript.

2.1 Combinatorics Education

It has been said that the road to solving counting problems is strewn with pitfalls (Hadar & Hadass, 1981). Authors have described the difficulties associated with teaching students to count, because oftentimes there is no rigid formula or procedure that can be applied generally (Annin & Lai, 2010). Students can find it challenging to articulate a plan for approaching counting problems, or even articulate exactly what they are trying to count (Hadar & Hadass, 1981). Furthermore, there are many subtle errors one can easily commit while solving counting problems, even if at first glance the solution seems correct (Annin & Lai, 2010; Lockwood, 2014b). Indeed, even when a student's solution is correct, they can still lack sufficient ability to justify their solutions (Lockwood et al., 2015b). Because students can face so many difficulties, there a clear need for more investigations into ways to help students be more successful in solving combinatorial problems. Researchers have taken a variety of approaches to helping address these challenges students face solving counting problems, including categorizing common counting errors (Batanero et al., 1997), advocating for a set-oriented perspective (e.g., Lockwood, 2014a; Wasserman & Galarza, 2019), investigating aspects of students' reasoning about particular concepts such as multiplication (e.g., Lockwood & Purdy, 2019a; Tillema, 2013) or permutations and combinations (Lockwood, Wasserman, & McGuffey, 2018), and creating models to describe students' combinatorial thinking (Lockwood, 2013). In the next section, I expand on Lockwood's (2013) model, as it is both an important piece of the combinatorics education literature and informs the way in which I characterize combinatorial proof in my study.

2.2 A Model of Students' Combinatorial Thinking

Lockwood (2013) said there are three components that may be present in a student's reasoning about a counting problem: *sets of outcomes*, *counting processes*, and *formulas/expressions*. See Figure 2.1. Sets of outcomes represent collections of objects that are enumerated, which also encompasses different ways those objects may be represented or "encoded" (Lockwood et al., 2015a). Examples may include representing outcomes as binary strings or as sequences where order of the items does not matter. Counting processes describe the mental or physical operations a counter uses to generate or enumerate sets of outcomes. For instance, this could include use of the Multiplication Principle[3] or constructing a case breakdown. Finally, formulas/expressions include mathematical expressions whose numerical value(s) are the cardinality of the set of outcomes being enumerated. These are often considered the "answer" to the counting problem.

[3] Tucker (2002) offers my preferred statement of the Multiplication Principle: "Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes" (p. 170).

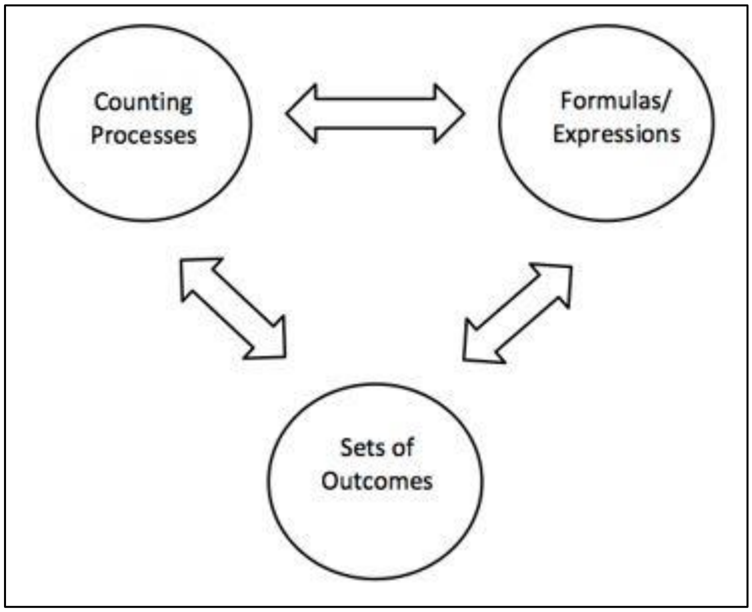

Figure 2.1. *Lockwood's (2013) model of students' combinatorial thinking (p. 253).*

Between each of these components there are also bidirectional relationships that Lockwood (2013) and Lockwood, Swinyard, and Caughman (2015b) describe. For example, a counting process which involves picking a committee (i.e., an unordered selection) of four people from a set of fifteen people and then picking one of those four people to be the chairperson of the committee would yield the expression $\binom{15}{4} \times \binom{4}{1}$. Similarly, and critically for combinatorial proof, a given expression may suggest a particular underlying counting process. The expression $\binom{n}{1} \times \binom{n-1}{k-1}$, for instance, may suggest a counting process in which 1 object is selected first from a group of n distinct objects, and then an unordered selection of $k - 1$ objects is then made from the remaining $n - 1$ objects. Other mathematical operations can suggest different underlying counting processes; for example, addition may indicate a counting process involving a case breakdown.

Several researchers have applied Lockwood's (2013) model to investigate various aspects of student thinking about counting problems (e.g. Halani, 2013; Hidayati et al., 2019; Lockwood, 2014; Lockwood et al., 2018; Lockwood & Erickson, 2017; Lockwood & Gibson, 2016; Lockwood & Purdy, 2019a). Only one previous study has applied her model to examine student thinking about combinatorial proof specifically (Lockwood et al., in press), outlining combinatorial proof activity as commonly involving starting with sets of outcomes, then moving to counting processes, and then sets of outcomes. Throughout my dissertation study I broadly used Lockwood's model in a similar manner as Lockwood et al. (in press), and in the third manuscript of this dissertation I applied Lockwood's model as a theoretical lens to study students' and mathematicians' combinatorial proof production. In the next subsection I review all pre-existing literature targeting combinatorial proof specifically, including the aforementioned study.

2.3 Literature on Combinatorial Proof

There have been some studies which have looked at children's combinatorial understandings of binomial identities (for example, Maher, Powell, & Uptegrove, 2011), but these studies do not look at combinatorial proof as I have defined it. There have been few studies that specifically addressed combinatorial proof at the undergraduate level. The first of these was conducted by Engelke Infante and CadwalladerOlsker (described both in Engelke and CadwalladerOlsker, 2010, and Engelke Infante and CadwalladerOlsker, 2011), who looked at upper-division undergraduate and graduate students' written solutions to combinatorial proof problems on exams. In their study, they rated the students' proofs on a scale from 1-4 based on how successful the proofs were, and they categorized difficulties that they observed students seemed to encounter with combinatorial proof. They additionally found some evidence that having students ask a specific "How many…?" question may help students be more successful at completing a correct combinatorial proof, and they also posited that some students may engage in "pseudo-semantic proof production" (2011, p. 96), a construct based on the distinction between semantic and syntactic proof production articulated by Weber and Alcock (2004).

More recently, Lockwood, Caughman, and Weber (2020) wrote a theoretical piece that focused on giving researchers tools and insights to more effectively understand and use the constructs of convincing and explanatory proofs (in the sense of Hersh, 1993), and they illustrated their theory by applying it to combinatorial proof. They argued that depending on the reader, combinatorial proofs are generally considered explanatory proofs within the enumerative representation system, because they can explain why a binomial identity holds combinatorially (but they do not explain algebraically, for instance, why a binomial identity holds). I expound upon this paper in the first manuscript chapter (Chapter 5), which involves discussing proofs that convince and/or explain.

Finally, the most recent study I identified in the literature targeting combinatorial proof was conducted by Lockwood et al. (in press). They found that the students in their study benefitted from two particular instantiations while trying to construct combinatorial proofs: *contextual instantiation* and *numerical instantiation*. By contextual instantiation, the authors referred to having students focus on one particular context in which to situate their combinatorial thinking, such as committees, and they used numerical instantiation to mean having students substitute specific values in for the variables appearing in a binomial identity. Lockwood et al. additionally found that combinatorial proof required the students to reconsider previous concepts they had internalized about algebraic expressions being "different." In particular, when the numerical equivalence of two expressions was apparent, the students occasionally struggled to distinguish between each side of the identity as counting a set of outcomes in two different ways.

Lockwood et al. (in press) used Lockwood's (2013) model to frame their investigation into students' thinking about combinatorial proof, which was a novel application of the model that was originally intended to frame student thinking about counting problems. I now review how they applied Lockwood's model, because I characterize combinatorial proof in a similar manner in this dissertation (this thus elaborates my own understanding of combinatorial proof as situated within existing literature). When a student is engaging in combinatorial proof activity, they can be considered as moving counterclockwise around Lockwood's model. See Figure 2.2. First, when a student is given a binomial identity to prove combinatorially, they must begin by picking one side of the identity to consider. That side of the identity is a formula/expression which the student must interpret as having an underlying counting process. That counting process enumerates or generates a particular set of outcomes, which includes the context (say, committees) that the students chooses to use. Then the student must start back again in the formulas/expressions component with the other side of the binomial identity, and they must interpret it as having some other underlying counting process which enumerates the same set of outcomes[4]. This manner of applying Lockwood's model as a theoretical lens to study to combinatorial proof worked effectively for Lockwood et al. (in press), and in the third manuscript of this dissertation (Chapter 7) I use Lockwood's model as a theoretical lens in the same way. Additionally, throughout this dissertation I broadly characterize combinatorial proof using the language of formulas/expressions, counting processes, and sets of outcomes for the purposes of my study.

[4] This process describes one type of combinatorial proof, specifically those that utilize "Approach 1" (Lockwood et al., in press). While this type of combinatorial proof is the focus of this paper, I again acknowledge that other types of combinatorial proof, such as bijective proofs, do also exist.

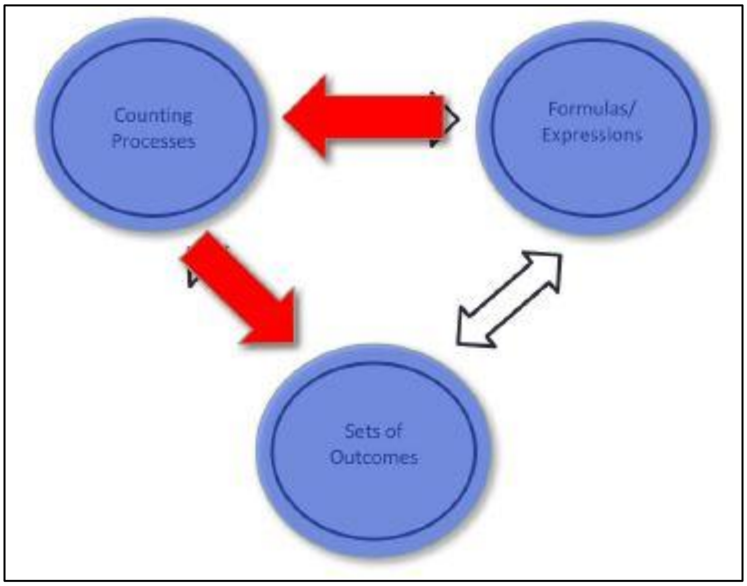

Figure 2.2. Lockwood's (2013) model as a lens for combinatorial proof (Lockwood et al., in press).

In conclusion, combinatorial proof is a topic that has received little attention in the mathematics education community, and much remains unanswered about how students engage with combinatorial proof tasks, or how they may think about combinatorial proof as different or similar to other types of proof. Furthermore, there have been no prior studies that have looked at mathematicians' reasoning about and engagement with combinatorial proof. In the following section, I describe two additional theoretical perspectives I used from the proof education literature to address some of these questions: proofs that convince and proofs that explain (Hersh, 1993) and Harel and Sowder's (1998) proof schemes.

CHAPTER 3 – Literature Review and Theory about Proof

In this chapter, I now turn to reviewing relevant literature from the proof education literature. I begin in Section 3.1 by clarifying what I took to constitute proof in my study, and in Sections 3.2 and 3.3 I discuss two theoretical lenses I am bringing to answer Research Questions 1, 2, and 3. These two theoretical lenses are proofs that explain and proofs that convince (Hersh, 1993) and Harel and Sowder’s (1998) proof schemes. As with Chapter 2, the purpose of this chapter is to provide a brief, overarching review of the literature I drew from for my study. I provide more details on the proof literature most relevant to the respective manuscripts in Chapters 5 and 6.

3.1 How I Am Taking Proof

Since this study centers combinatorial proof, I specify what I take to constitute proof. Currently in the mathematics education community, there is a wide array of perspectives on what should be taken as a mathematical proof. There is even debate around issues as structurally basic as whether a proof without words, such as a proof consisting only of a picture, really constitutes mathematical proof (e.g., Gierdien, 2007). Some researchers have articulated a dichotomy of formal proofs and acceptable proofs. For instance, Hanna (1990) explained that formal proofs are theoretical and exist as a string of sentences such that the first sentence is an axiom, and each consequent sentence either follows from those previous or is an axiom. However, Hanna (1990) also recognized that this is not how mathematics is usually done in the real world, and consequently Hanna (1990) said that acceptable proof can be thought of as what mathematicians actually do: produce proofs that are considered acceptable and valid within a qualified community. Other characterizations of proof by mathematics education researchers have distinguished between an argument that may be found personally convincing versus a proof that could persuade a broader community. Harel and Sowder (2007) used the terms *ascertaining* and *persuading* to describe these different types of proofs, respectively, and Raman (2003) discussed a similar distinction between private and public arguments.

The above represents a small sample of the ways that proof and proof production have been defined and characterized by mathematics education researchers. For the purposes of my dissertation study, I sought a definition of proof that is student-centered and that attends to the way students choose to represent particular mathematical objects (such as $\binom{n}{k}$). Thus, I adapted the definition given below by Stylianides (2007, p. 291; emphasis in original):

> *Proof* is a *mathematical argument*, a connected sequence of assertions against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.

Stylianides (2007) used observations from a third-grade classroom to elaborate on elements of this definition and to illustrate its applicability, hence his use of the term “classroom community.” In my dissertation study, I utilize a broader meaning of the term “community” to also encompass other communities (such as a group of peers, the community in a prior class, or the larger mathematics community) that may be important to the perspectives of students and mathematicians.

3.2 Proofs That Convince and Proofs That Explain

As mentioned previously in Section 2.3, Lockwood et al. (2020) said that combinatorial proofs can be considered proofs that explain *why* two quantities are equal, rather than merely convincing a reader that they are equal. To fully understand what they mean, in this section I discuss the ongoing conversation among mathematics-education researchers regarding the distinction between proofs that explain and proofs that convince (Hersh, 1993).

Similarly to Hanna's (1990) distinction between formal and acceptable proof, Hersh (1993) observed that in real-world mathematical practice, proofs are not often presented in an absolute sense, as if they exist purely as a sequence of statements manipulated using formal rules of logic apart from human activity. Instead, the term “proof” often has less to do with formal logic and is used more broadly to mean, “convincing argument, as judged by qualified judges” (p. 389). Based on this, Hersh (1993) articulated that proof can be divided into two categories, depending on the context and purpose of the proof. He said that in mathematical research, the purpose of proof is to convince, and to do so it must reach some standard of rigor and honesty as defined by the mathematical community. In the classroom however, he said that the purpose of proof is to explain, that is, proofs should be enlightening and stimulate students’ mathematical understanding. The value of proofs that explain is not limited to the classroom though. Hersh (1993) stated, “More than whether a conjecture is correct, mathematicians want to know why it is correct” (p. 390), and he used the historical example of Paul Halmos' complaints regarding the Appel-Haken theorem, which used computation to aid in proving the Four-Color Theorem.

Other researchers have also expanded on the categories of proofs that convince and proofs that explain. Hanna (2000) stated similarly that the two fundamental functions of proof are verification and explanation, and Weber (2010) touched on this distinction in a study investigating the different ways that mathematicians view proof. Weber (2010) found that often mathematicians find great value in proofs that explain (not just proofs that convince). For example, he noted that proofs are read by mathematicians to help them gain new insights and proof techniques within their field, as well as provide new ways of thinking about mathematical objects. Weber (2010) also expanded on Hersh's (1993) conception of proofs that explain by saying that explanatory proof should be thought of as an interaction between the proof and its reader, rather than considering 'being explanatory" to be a factor inherent to a proof, separate from any human interaction. He stated,

> "I conceptualize a proof that explains as a proof that enables the reader of the proof to reverse the connection—that is, this proof allows the reader to translate the formal argument that [they are] reading to a less formal argument in a separate semantic representation system" (p. 34).

As mentioned previously, Lockwood et al. (2020) have since adopted the similar term *representation system*, which they define as "consist[ing] of configurations that are used to represent mathematical objects and inferential schemes that can be used to deduce new facts about these objects" (p. 3). This reframing allows the discussion regarding proofs that convince and proofs that explain to be more reader-centered, opening the possibility that—depending on the representation system used—a proof could be considered explanatory or not depending on the reader. With this more flexible and student-centered framing in mind, in my study I draw upon the definitions of proofs that convince and proofs that explain from Weber (2002):

- "A proof that convinces begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown.... The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true" (p. 14).
- "A proof that explains also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous.... The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main ideas" (p. 14).

From these definitions, proofs that are carried out solely by manipulating symbols or employing an "algebraic trick" are usually considered proofs that convince (and not proofs that explain). Additionally, other researchers have argued that proofs by induction or by contradiction are usually only proofs that convince (Hanna, 2000; Lange, 2009), though it certainly is not a settled issue (for instance, Stylianides, Sandefur, & Watson, 2016, outlined some criteria in which a proof by induction could be considered explanatory). Here, again, the use of representation systems can be useful, because an algebraic or induction proof could be considered explanatory to some readers if they are situated in an algebraic or inductive representation system (Lockwood et al., 2020).

In my study, I am motivated to learn more about combinatorial proof and its status among other types of proof, particularly among experienced provers, and so I view the distinction between proofs that explain and proofs that convince as a way to understand more about how people view relationships among types of proof. In particular, a broad goal was to empirically investigate Lockwood et al.'s (2020) theoretical assertions about combinatorial proof and its status as explanatory and/or convincing.

3.3 Proof Schemes

The other theoretical lens I am using from the proof literature is Harel and Sowder's (1998) proof schemes. This framework has been used by researchers to study proof comprehension and/or proof production in children, undergraduate students, and pre- and in-service teachers in several mathematical areas (e.g., Blanton & Stylianou, 2014; Çontay & Duatepe Paksu, 2018; Ellis, 2007; Fonseca, 2018; Healy & Hoyles, 2000; Housman & Porter, 2003; Jankvist & Niss, 2018; Kanellos, 2014; Koichu, 2010; Liu & Manouchehri, 2013; Gülcin Oflaz et al., 2016; Ören, 2007; Pence, 1999; Sen & Guler, 2015; Şengül, 2013). The fact that many researchers have used this framework in a variety of content areas and with different populations speaks to its broad applicability and utility in characterizing proof in mathematics education. However, perhaps in part because combinatorial proof has not been studied extensively to date, no researcher has previously applied

Harel and Sowder's framework to combinatorial proof. For my study, I chose to use proof schemes to analyze students' thinking about combinatorial proof, believing that it would be productive to interpret combinatorial proof through the well-known lens of proof schemes. Further, because it is a widely used and accepted perspective, I suggest that it can help to better inform how combinatorial proof compares to other kinds of mathematical proof, which is related to Research Question 3. Finally, I posit that this research can contribute to the large existing body of work that uses proof schemes as a lens to understand proof in mathematics education. My examination of combinatorial proof can provide insight into how the proof schemes framework might be applied to a type of proof to which it has not previously been applied.

To elaborate the proof schemes framework, Harel and Sowder (1998) contended that generally there are three non-mutually exclusive categories of proof schemes that a prover can use (each of which have subcategories): *external conviction*, *empirical*, and *analytical* (see Figure 3.1). External proof schemes describe situations where students' doubts are removed by the presence (or absence) of certain ritualistic characteristics of an argument, the word of an authority, or the symbolic form of an argument. For instance, if a student rejects a given combinatorial proof because it does not contain symbolic manipulation, the student would be using an external conviction proof scheme. On the other hand, a student can be said to be using an empirical proof scheme when, "conjectures are validated, impugned, or subverted by appeals to physical facts or sensory experiences" (p. 252). Harel and Sowder further distinguished between inductive and perceptual empirical proof schemes. Finally, Harel and Sowder stated that analytical proof schemes involve validation conjectures by means of logical deductions (p. 258). In total, these three categories of proof schemes represent hierarchical cognitive stages in a student's mathematical development, with external conviction being the least sophisticated and analytical being the most sophisticated. Further details about these three categories of proof schemes and each of their subcategories can be found in Chapter 6 of this dissertation.

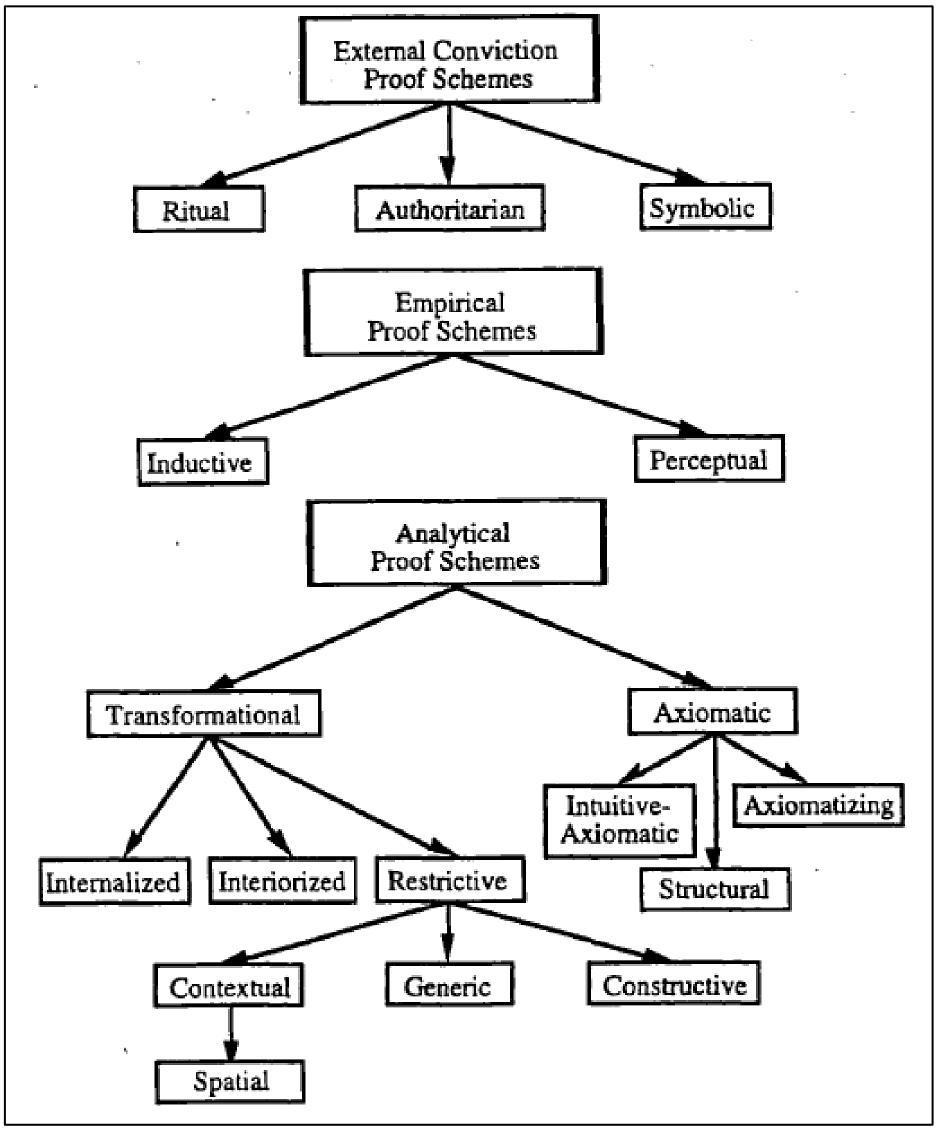

Figure 3.1. Harel and Sowder's (1998) proof schemes framework (p. 245).

This is just a brief overview of the relevant proof literature for my investigation into experienced provers' perceptions of combinatorial proof. I will draw on the lens of proofs that convince and proofs that explain (Hersh, 1993) to answer Research Questions 1 and 3, and I will use Harel and Sowder's (1998) proof schemes to address Research Questions 2 and 3. Further details about these two proof schemes can be found in Chapters 5 and 6 of this dissertation.

CHAPTER 4 – Methods

In this section I describe the methodology I used for data collection and subsequent analysis for my study. In the following subsections I discuss the participants involved in data collection, the recruitment of these participants, procedures used for data collection, and techniques for data analysis.

4.1 Participants

I collected data from two populations for this study, on which I elaborate below.

4.1.1 Mathematicians. I recruited eight mathematicians from three different universities for this study. The mathematicians were a convenience sample of professors to whom I had access, and they were recruited via email. I sought mathematicians who had a range of experiences teaching and/or conducting research involving binomial identities and combinatorial proofs. Specifically, I recruited between 2-3 mathematicians from each of the following groups: i) mathematicians who conduct research in combinatorics and who teach combinatorics and discrete mathematics (3), ii) mathematicians whose research area is not in combinatorics yet who still teach combinatorics and discrete mathematics at least occasionally (2); and iii) mathematicians whose research area is not in combinatorics and who do not typically teach combinatorics or discrete mathematics (3). I did not conduct selection interviews for the mathematicians, and they were compensated monetarily for their time.

4.1.2 Students. I recruited five suitable students for the study. To recruit the students, I handed out fliers at upper-division mathematics courses at a large university in the western United States. The reason for recruiting in upper-division courses was that I sought students with some experience with college-level proof-based mathematics. I conducted selection interviews with the seven students who responded to the fliers, and five of them who fit the selection criteria participated in the study. Selection criteria included demonstrating: i) knowledge of counting problems and binomial coefficients, ii) competency with basic proof techniques, and iii) an ability and willingness to clearly explain their reasoning aloud. The students were compensated monetarily for their time.

4.2 Data Collection

The design of my study was to conduct semi-structured, individual, task-based clinical interviews with each participant (Hunting, 1997). Each interview was audio- and video-recorded. Most of the interviews were conducted with me as the sole interviewer, although for two mathematician interviews and nine student interviews, another researcher was also present and helped run the camera. This other researcher's involvement in data collection was limited by scheduling constraints.

4.2.1 Mathematician data collection. The 90-minute interviews with each mathematician were conducted in person. I first asked the mathematicians the following series of questions aimed at gaining an understanding of their prior experience with combinatorial proof:

- What is your research area? How long have you been conducting mathematics research?
- How would you define a mathematical proof?
- How would you define a combinatorial proof?
- Do you ever use combinatorial proof in your research? How important is combinatorial proof in your field?
- Do you ever teach classes that cover combinatorial proof of binomial identities? How frequently? When did you teach combinatorial proof most recently?

Then, I asked the mathematicians to prove a series of binomial identities using combinatorial proof. This portion of the interviews ensured that later when I asked these mathematicians questions about combinatorial proof, they would have recent prior combinatorial proof activity upon which they could reflect. I also intended to study how these mathematicians approached combinatorial proof and intended to compare this with how undergraduate students in my study approached combinatorial proof. The binomial identities I gave the mathematicians are discussed in further detail in Section 4.3.1. This portion of the interviews allowed me to address Research Question 4 from Section 1.1.

I next gave the mathematicians noncombinatorial and combinatorial proofs of three binomial identities, and then I provided them with the following definitions of a proof that convinces and a proof the explains from Weber (2002) to ensure we had a shared understanding of these terms:

- "A *proof that convinces* begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown…. The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true" (p. 14).
- "A *proof that explains* also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous…. The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main idea" (p. 14).

After giving the mathematicians time to examine these noncombinatorial and combinatorial proofs and read the definitions, I asked the mathematicians reflection questions aimed at gauging their perspectives on how convincing and/or explanatory (in the sense of Hersh, 1993) combinatorial proofs are compared to other types of proof and why. This part of the interviews directly targeted Research Questions 1, 2, and 3 from Section 1.1.

4.2.2 Student data collection. The 60-minute selection interviews with students were conducted in person. The tasks in the selection interviews are elaborated in Section 4.3.2. In the selection interviews, I first asked the students to solve a series of counting problems aimed at probing their knowledge of binomial coefficients and proficiency with counting. Next, I asked the students to write proofs for three straightforward theorems about integers to test their proof-writing competency. Without this base skill set (being able to correctly apply combinations to counting problems and structure a basic mathematical proof), it was unlikely that a student in my study would be able to provide meaningful data aimed at understanding how students with prior counting and proof experience engage with and think about combinatorial proof of binomial identities. Students in these selection interviews who solved many of the counting problems incorrectly or who could not navigate a basic proof were not selected for my study. I also asked the students reflection questions targeting their understanding of what a mathematical proof is and what the purpose of proof is in mathematics.

The students who showed competency using combinations in counting problems and proving straightforward mathematical theorems were selected to participate in my study; I subsequently interviewed these students individually in four 60-minute in-person sessions (for approximately four total hours per person after the selection interviews). In these interviews, I first asked the students to solve more counting problems involving combinations, as well as tasks aimed at helping them to think about what counting processes (Lockwood, 2013) may underlie given binomial expressions. As the students solved these tasks, I asked them to articulate their thinking and to justify their combinatorial reasoning, such as how they knew to apply multiplication rather than, say, addition when solving a counting problem. The purpose of these tasks was to reinforce the skills I anticipated they would need to be successful at producing combinatorial proofs: solving counting problems correctly, making connections between the components of Lockwood's (2013) model (sets of outcomes, counting processes, and formulas/expressions), articulating how mathematical operations structure and organize the outcomes of a counting problem, and creating bijections. See Section 4.3.3 for a more detailed discussion of these combinatorial tasks.

Next, I asked the students to justify some binomial identities (see Section 4.3.3) by arguing that each side of the identity enumerates some set of outcomes. As they worked, I continued in asking them to articulate their thinking out loud and to explain their reasoning. During this section of the interviews, I asked the students some of the following questions if they were "stuck" trying to come up with a combinatorial justification for why a binomial identity held:

- What could this be counting?
- What if you tried plugging in a specific value for *n*?
- Could you think of this side (of the identity) as counting something related to [committees, binary strings, or some other specific context]?

These questions (particularly the last two) are in alignment with Lockwood et al. (in press), who found that contextual and numerical instantiations were useful in helping students successfully produce combinatorial proofs. Although I was careful to avoid referring to the students' activity as "proof" (since this may influence their conceptions regarding the extent to which combinatorial proofs are rigorous mathematical proofs), these tasks were aimed at addressing Research Question 4 from Section 1.1.

The next portion of the interviews was centered around investigating the extent to which the students characterized combinatorial proofs as convincing and explanatory (Hersh, 1993) compared with other types of proof. To do this, I began by showing them noncombinatorial and combinatorial proofs of three binomial identities (see Section 4.3.3). Some of the combinatorial proofs I showed the students used abstract contexts like sets and subsets, and some of the combinatorial proofs used more concrete contexts such as binary strings and committees. After getting the students' initial impressions of these six total proofs, I then asked the students what they thought it meant for a proof to be convincing and what it meant for a proof to be explanatory, and then I read to them Weber's (2002) definitions of a proof that convinces and a proof that explains out loud. I then proceeded to ask the students a line of questioning aimed at determining whether the students characterized combinatorial proofs as convincing and/or explanatory compared with other types of proof (such as algebraic and induction), and I also asked the students whether they thought combinatorial proofs constituted rigorous mathematical proof (and why). This task and line of questioning were aimed at addressing Research Questions 1, 2, and 3.

After this, I asked the students to write combinatorial proofs of more challenging binomial identities, continuing to ask them to articulate their thinking and from time to time asking them again whether they thought their combinatorial proving activity constituted rigorous mathematical proof. By "more challenging," I mean that these latter identities may require an Approach 2 combinatorial proof (Lockwood et al., in press), which I conjecture to be more difficult for students to produce, or the identities are less readily transferable to an intuitively conceived context. For example, to prove that

$$\binom{n+1}{k+1} = \sum_{i=k}^{n} \binom{i}{k}$$

it is useful to consider the RHS of the identity as counting subsets of $\{1, 2, \ldots, n, n+1\}$ of size $k + 1$ where the largest element is $i + 1$. The purpose of giving students these more challenging binomial identities to prove was to further address Research Question 4 (stated in the Section 1.1). A list of these identities can be found in Section 4.3.3. As the students proved these more challenging binomial identities, I again encouraged the students if they got stuck to try using specific values in the place of variables, or I asked them what a particular expression may be counting within a specific context (following Lockwood et al., in press). These tasks allowed me to further observe their combinatorial proving activity and address Research Questions 2, 3, and 4.

4.3 Interview Tasks

In this subsection, I discuss in more detail the specific interview tasks I gave to the research participants and provide justification for why these tasks were chosen.

4.3.1 Interview tasks for the mathematicians. In this section, I provide the binomial identities that I gave to the mathematicians to prove in order to study their approach to combinatorial proving. Each mathematician was asked to prove a subset of the identities given in Table 4.1.

Table 4.1. Identities given to the mathematicians to provide a combinatorial argument.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$
$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i\geq 0} \binom{n}{2i+1}$	$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$

These binomial identities were chosen because I wanted these mathematicians to have to work with a variety of basic operations in binomial identities to see how they would approach proving them combinatorially. This list also represents a wide array of difficulty levels for combinatorial proofs, allowing me to see what these mathematicians' "go-to" combinatorial contexts were (such as committees, block-walking, etc.) as well as what reasoning they used and what contexts they explored for more difficult combinatorial proofs. Full details of my protocol for the interviews with the mathematicians can be found in Appendix A.

4.3.2 Interview tasks for the selection interviews. In this subsection I describe the counting problems I asked the students to solve and the theorems they were asked to prove during the selection interviews. First, the following are the four counting problems I gave to the students, one at a time.

> **Domino Problem.** *A domino is a small, thin rectangular tile that has dots on one of its broad faces. That face is split into two halves, and there can be zero through six dots on each of those halves. Suppose you want to make a set of dominos (i.e., include every possible domino). How many distinguishable dominos would you make for a complete set?*[5]

I selected this problem because it cannot readily be solved using any of the four fundamental counting formulas (n^r, $n!$, nPr, and $\binom{n}{r}$). However, there are only 28 outcomes, and so it was possible for the students to list all the outcomes. Thus, this problem was useful to see whether the students were attuned to the sets of outcomes component of Lockwood's (2013) model, or if they were more prone to applying counting formulas without justification.

> **Committees Problem.** *A university department has 30 faculty members.*
> - a) *How many ways could a 5-member hiring committee be formed?*
> - b) *How many ways could a 5-member hiring committee be formed if one of the committee members must be the chairperson?*
> - c) *In the university department, 17 faculty members are professors and 13 are instructors. How many ways could a 5-member hiring committee be formed if the committee must consist of 3 professors and 2 instructors? (The committee won't have a chairperson.)*

I selected this problem to see how comfortable and proficient the students would be with using binomial coefficients to solve counting problems, and to see if they knew how to apply the Multiplication Principle. This counting problem also represented the context for a combinatorial proof of the binomial identity ($\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$), which students would see later in the selection interview and, if asked to participate in subsequent interviews, be asked to justify combinatorially.

> **Power Set Problem.** *Let S be a set containing 5 (distinct) elements. How many subsets are there of the set S? (That is, what is the cardinality of $P(S)$, the power set of S?)*

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[5] This problem is used with permission from (Lockwood, Swinyard, Caughman, 2015).

This problem offered another opportunity to see if the students could either correctly apply the Multiplication Principle, or else use addition to solve the problem via a case breakdown. In particular, students who were familiar with the strategy of enumerating subsets by considering whether each individual element is contained in the subset or not might find that the solution is 2^5 using the Multiplication Principle. Otherwise, if they had never seen this strategy or did not think to use it, they could also find that the solution is $\binom{5}{0}+\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}+\binom{5}{5}$ by counting all subsets of a fixed size.

Binary Strings Problem. *A binary string is a finite sequence containing only 1s and 0s.*

a) How many binary strings of length 8 contain exactly 5 0s?

b) *How many binary strings of length n contain exactly k 0s?*

I included this problem because Lockwood, Swinyard, and Caughman (2015a) found that students who are otherwise very successful counters can struggle to encode the outcomes of these kind of combination problems in a way that lets them leverage binomial identities. They found that students may be more likely to know that $\binom{n}{k}$ counts the number of groups of a certain size where order does not matter, but they may not realize that they can count these binary strings by enumerating groups of size 5 (or k) from a set of 8 (or n) positions. To successfully prove binomial identities combinatorially, it is essential to have the ability to flexibly apply combinations to a variety of contexts, so this problem was included to help me determine the flexibility of the students' understanding and use of combinations.

Next, I present the three theorems I asked the students to prove after solving the four counting problems described above.

Theorem 1. *The sum of two even integers is an even integer.*

This was the first theorem I included in my selection interviews since it should have been straightforward for any student in an upper-level mathematics course. When the students proved this theorem, I was checking to ensure they could navigate a basic proof, including defining the relevant variables and writing using coherent, logically correct, complete sentences.

Theorem 2. *Let* n *be a nonnegative integer. Then,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

This theorem was included in the selection interviews so that I could see if the students were able to navigate a straightforward proof by induction (if that was the proof method they chose), or otherwise ensure they could interpret and write a correct, coherent proof of a statement involving a summation.

Theorem 3. *Let n and k be nonnegative integers such that $n \geq k$. Then,*

$$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$$

This theorem was given to the students as written, although I did tell them the factorials formula for $\binom{n}{k}$ if they asked for it (i.e. $\binom{n}{k} = \frac{n!}{(n-k)!\cdot k!}$). This was also one theorem where it might be natural for some students to try a combinatorial proof, and so another reason I included it was to see if they would approach the proof combinatorially.

In summary, the selection interviews included tasks aimed at evaluating various aspects of students' skills at solving counting problems and writing proofs of basic mathematical theorems. While the students needed not be expert counters or provers, the tasks helped me to ensure that the students who were selected for future interviews had the pre-requisite skills necessary to provide meaningful data when asked to prove binomial identities in subsequent interviews/situations. Full details of my selection interview protocol can be found in Appendix B.

4.3.3 Subsequent interview tasks for those students who met selection criteria. In this section, I will review the tasks I gave to the students who I selected to participate in my study, and I will justify why I chose these tasks. During the four follow-up clinical interviews with each student, I asked them to complete a sequence of combinatorial tasks intended to help build scaffolding for these students to successfully produce combinatorial proofs. See Table 4.2 for a list of these tasks. Several of these tasks were intended to ensure the students had a robust, flexible understanding of combinations, that is, problems where the solution can be readily expressed using one or more binomial coefficients. Lockwood et al. (2018) found that students distinguish between two different types of problems that can be solved with binomial coefficients, so I felt it was important to ensure they could solve tasks involving either type of problem. Additionally, the fifth task I asked the students to solve, the Reverse Counting Problem, asked students to interpret an expression as the solution to a counting problem. The ways of thinking the students had to engage in to solve this task is very similar to that needed to prove a binomial identity combinatorially, so this task provided the most direct scaffolding for subsequent combinatorial proof tasks.

Before continuing, I want to make two points about the combinatorial tasks I gave the students. First, during the interviews I felt I did not have to give them more than these five tasks, because in the selection interviews these students had already shown that they were familiar with and could solve counting problems, so I did not feel it was necessary to have them solve too many more counting problems before asking them to engage in combinatorial argumentation. Second, several of the tasks in this table were intended to lay the necessary groundwork for students to make

Table 4.2. Combinatorial tasks for students to scaffold combinatorial proof.

Task	**Intended Purpose**
1. Spoonbill Problem. The scientific name of the roseate spoonbill (a species of large, wading bird) is *Platalea ajaja*. How many	Ensure students are familiar (or to familiarize them) with combination problems involving ordered sequences of two indistinguishable

arrangements are there of the letters in the word AJAJA? Can you list all of the outcomes?	objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
2. Subsets Problem. How many 3-element subsets are there of the set {1, 2, 3, 4, 5}? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving unordered selections of distinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
3. Find-a-Bijection Problem. Describe a bijection between the outcomes in the Spoonbill Problem and the Subsets Problem.	Facilitate a robust, flexible understanding of combinations. Lay groundwork for students to solve bijective combinatorial-proof problems.
4. Even- and Odd-Sized Sets Problem. Let S={1, 2, 3, 4, 5, 6}. (a) List all of the even-sized subsets of S. How many should there be? (b) List all of the odd-sized subsets of S. How many should there be? (c) Find a bijection between the subsets in parts (a) and (b) by considering whether the subsets contain the item 1.	Continue to facilitate a solid understanding of combinations. Provide scaffolding for students to eventually prove the identity $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ using a bijective combinatorial proof.
5. Reverse Counting Problem. (a) Write down a counting problem whose answer is 2^5. (b) Write down a counting problem whose answer is $15 \times \binom{14}{3}$.	Provide scaffolding for the concept of a combinatorial proof by asking students to interpret expressions in a combinatorial context.

bijective combinatorial proofs of binomial identities[6]. I included these tasks in this section for completeness, and because these tasks helped to ensure the students could work with binomial coefficients effectively. However, bijective combinatorial proofs are not a focus of this paper, and so I do not include details of their work on these types of proofs here.

After the students completed these tasks, I gave them a sequence of binomial identities (where all variables involved were nonnegative integers), and with these identities I gave the prompt: "Argue that the identity holds by arguing that each side counts something." I was careful to avoid

[6] That is, proofs which involve arguing that each side of the binomial identity counts a different set, and then making a bijection between the sets to establish their cardinalities are the same—and therefore the identity holds.

using the word "proof" to describe the tasks, because I planned to ask the students later in the interviews whether they felt their combinatorial arguments constituted proofs that explain or proofs that convince (or even whether combinatorial arguments can be proofs at all), and I did not want to influence the students' opinions. Each student was given a subset of the identities in Table 4.3 (as time in the interviews permitted).

Table 4.3 Identities given to the students to provide a combinatorial argument.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i\geq 0} \binom{n}{2i+1}$
$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$	$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$
$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$	

Finally, in the last or second-to-last interview with each student (depending on how far they had progressed), I gave them the same six proofs (three combinatorial, and three non-combinatorial) to read as the mathematicians, except that on the students' handout each proof was labeled an "argument." Again, this was because I did not want to influence their opinions regarding whether they thought combinatorial proofs constituted convincing or explanatory mathematical proofs (or not). The details of these six proofs are given in Table 4. As with the mathematicians, I asked the students to read each handout and first give me their overall impression of the arguments they read. Once the students had finished giving me their impressions, I asked them what they thought it might mean for a proof to be convincing, and what they thought it might mean for a proof to be explanatory. Once they gave their answers, I then read them the same definitions of proofs that convince and proofs that explain from Weber (2002) that I had given the mathematicians. I next asked them a series of reflection questions aimed at probing their beliefs about combinatorial arguments as proofs, as well as combinatorial and non-combinatorial arguments (e.g., induction or algebraic) as proofs that explain and proofs that convince. Full details of my protocol for the interviews with the students can be found in Appendix C.

4.4 Data Analysis

Directly after data collection, the interview data were de-identified by assigning a code (a number, then a pseudonym) for each participant, and I referred to participants exclusively by these codes in all subsequent analysis. After conducting and recording interviews, I created enhanced transcripts with relevant screenshots from the video-recorded data capturing the participants work on the interview tasks. Broadly, I analyzed the data by drawing on multiple frameworks/perspectives, including Lockwood's model (2013), proofs that convince and proofs that explain (Hersh 1993), and Harel and Sowder's (1998) proof schemes. I utilized thematic analysis (Boyatzis, 1998; Braun & Clarke, 2006) since it is a well-established methodology in social science research and could provide me with a detailed and nuanced account of qualitative data. In this section, I first discuss the thematic analysis qualitative research methodology, and then I give more specific details describing how I used thematic analysis to analyze my interview data with the mathematicians and students. Additional details about data analysis are included in each of the manuscript chapters, Chapter 5, 6, and 7 of this dissertation.

4.4.1 Thematic analysis. Braun and Clarke (2006, p. 87) broadly outlined six phases of thematic analysis:

1. Familiarize yourself with the data;

2. Generate initial codes;
3. Search for themes (i.e., gather codes into potential themes);
4. Review themes (including checking to ensure they accurately represent the data);
5. Define and name themes (as part of ongoing analysis to refine themes);
6. Produce the report (including selecting compelling episodes from the data to illustrate the theme).

Familiarizing oneself with the data may involve tasks such as re-watching interview videos, transcribing data, and noting initial ideas. *Generating initial codes* involves systematically organizing interesting occurrences or ideas throughout the data set to produce an initial coding scheme. The phases *searching for themes* and *reviewing themes* include gathering the initial codes with relevant data into themes, ensuring the themes work across the entire data set, and generating a "thematic 'map'" of analysis (Braun & Clarke, p. 87). Themes are further refined in the *defining and naming themes* phase, and finally *producing the report* includes the production of a scholarly report of the results of data analysis complete with compelling, illustrative examples from the data. I carried out these six phases for my analysis of the interview data with the mathematicians and with the students; more details are provided in Sections 4.4.2 and 4.4.3, as well as in the three result manuscript chapters of this dissertation.

Braun and Clarke (2006) also articulated that the thematic analysis can be *inductive* or *deductive*, that themes can be *semantic* or *latent*, and that the researcher can adopt an *essentialist* or *constructionist* epistemology. For my study, I assumed an essentialist epistemology, since it was sufficient for me to characterize meanings related to combinatorial proof as essential to each individual participant, rather than necessarily to seek sociocultural explanations for the utterances and activity related to combinatorial proof exhibited by the mathematician and student participants. Also, the themes I searched for were semantic, because in order to answer my research questions they only needed to be descriptive of the data without additionally requiring interpretation of underlying ideas beyond what the participants said or wrote. Finally, some portions of my thematic analysis were inductive, and some were deductive depending on the research question I was focusing on in that moment. In particular, to answer Research Questions 1, 2, and 3, I had specific theoretical perspectives I was bringing to my coding of the data—namely Hersh's (1993) notion of proofs that convince/explain and Harel and Sowder's (1998) proof schemes. Therefore, that portion of the thematic analysis was deductive. However, to answer Research Question 4 (which I view as being broader than the other three), I wanted to be more open to whatever salient themes may emerge related to mathematicians' and students' perceptions of combinatorial proof and combinatorial proving activity. Generally, I characterized their combinatorial proving activity using the components of Lockwood's (2013) model, but I also engaged in thematic analysis to be more data-driven without trying to apply a prescriptive theoretical perspective.

While the previous broad description summarizes my overarching approach to analyzing my interview data via thematic analysis, in the following subsections I offer more detail into how I specifically analyzed the mathematicians' and students' utterances and activity related to combinatorial proof. Throughout the next two sections, I connect my analysis back to the six phases of analysis as characterized by Braun and Clarke (2006). I organize my discussion by first describing my analysis of the data related to how the mathematicians and students perceived combinatorial proof (which directly addresses Research Questions 1, 2, and 3) and, second, by describing my analysis of the mathematicians' and students' combinatorial proving activity (which addresses Research Question 4). Further details can also be found in Chapters 5, 6, and 7 of this dissertation.

4.4.2 Analysis of mathematicians' and students' perceptions of combinatorial proof. For my interviews with the mathematicians, I began with Phases 1 and 2 of thematic analysis (Braun

& Clarke, 2006) by looking for instances in the data where they described their impressions of combinatorial proof (either a specific proof or combinatorial proof in general) and coded these impressions. I also recorded the mathematicians' responses to Weber's (2002) definitions of proofs that convince and explain. Following Phase 3 of thematic analysis, I organized my codes into potential themes, including to what extent the mathematicians felt that combinatorial proofs are proofs that explain or proofs that convince (or neither or both), how combinatorial proof compares to other types of proof (e.g., induction or algebraic), as well as other interesting themes as they emerged from the data analysis.

For my interviews with the students, I also looked for instances in the data where they described their impressions of combinatorial argumentation and coded these impressions following Phases 1 and 2 of thematic analysis (Braun & Clarke, 2006). I coded to what extent the students felt that combinatorial arguments are proofs that explain or proofs that convince compared with other types of proof (such as algebraic or induction arguments) and why. In addition, I flagged episodes in the data when the students discussed whether they considered combinatorial proof to be rigorous mathematical proof in comparison to other types of proof, and I coded their reasoning in these cases using Harel and Sowder's (1998) proof schemes. For instance, if a student made utterances about the correctness of the logical structure of a combinatorial argument, I took that as evidence that the student was using an analytical proof scheme. If the student alluded to an authority (e.g., claiming they did not think their instructor would accept a combinatorial proof) or appealed to ritualistic features of a combinatorial proof (e.g., claiming a combinatorial argument did not constitute proof because it did not involve symbolic manipulation), then I took that to signify that the student was using an authoritarian or ritual proof scheme, respectively. I also recorded each students' concept definitions (Tall & Vinner, 1981) of a proof, as well as what they thought it might mean for a proof to be convincing and/or explanatory. These codes were organized into initial themes following Phase 3 of thematic analysis.

After completing Phases 1-3 of thematic analysis (Braun & Clarke, 2006) with my interview data with the students and mathematicians, I carried on to Phase 4 by reviewing the interview data, verifying that my themes were appropriate, and checking to see if there were any other episodes that may warrant further analysis. Any ambiguous episodes were discussed thoroughly with another experienced researcher until both that researcher and I were confident that the themes being applied were appropriate. Continuing to Phase 5 of thematic analysis, I discussed the themes with the other researcher to refine my overall themes to ensure they were clear and well-defined. Eventually I completed the final stage of thematic analysis (Phase 6) by drafting the results and discussion sections for the first and second manuscripts of this dissertation, including choosing compelling and representative excerpts from the data for each theme. I read these drafts and revised them, in some cases adjusting my organization and structure to make sure I was accurately characterizing the themes that had emerged from the data.

In summary, I followed deductive thematic analysis to code relevant episodes in the interview data to determine whether the mathematicians and students considered combinatorial proofs to be convincing or explanatory (and why), and I further coded relevant episodes in my interviews with the students to determine the proof schemes they used to characterize combinatorial proof. This enabled me to address Research Questions 1, 2, and 3. Further details can be found in Chapters 5 and 6 of this dissertation.

4.4.3 Analysis of mathematicians' and students' combinatorial proof production. As I mentioned above in Section 4.4.1, I used an inductive approach to analyze the data that were relevant to Research Question 4. This enabled me to proceed with coding the interview data without being constrained by a prescriptive coding frame—though I did broadly interpret mathematicians' and students' combinatorial proving activity using the components of Lockwood's (2013) model. First, following Phase 1 of thematic analysis (Braun & Clarke, 2006), I familiarized with the data by re-watching the interview videos and making note of episodes that seemed relevant to Research Question 4. I then re-examined these episodes in Phases 2-3 of thematic analysis and noticed patterns occurring in the ways that the students and mathematicians engaged with the combinatorial proof interview tasks. These patterns enabled me to conceive of initial coding schemes, which were then refined into more robust themes as I made multiple passes through the data with my evolving codes. I also went back to the literature and looked for relevant prior research that could help me think of how to appropriately categorize the phenomena I was observing. I continued through Phases 4 and 5 of thematic analysis until I had clear, distinct themes that saturated the data, all while discussing key episodes and findings that were emerging with another experienced researcher. We discussed the themes that were being used to ensure that they faithfully represented the data, and any episodes that were challenging to assign to a theme were discussed thoroughly until both my research colleague and I were confident that the themes being applied were appropriate. Finally, in Phase 6 of thematic analysis I produced a report by drafting the results and discussion sections for the third manuscript of this dissertation, including choosing compelling, representative excerpts from the data for each theme. Further details can be found in Chapter 7 of this dissertation.

Chapter 5 (Paper 1) – Combinatorial Proofs as Proofs That Convince and Proofs That Explain

Abstract. Combinatorial proof, an important topic in enumerative combinatorics, has received relatively little attention from the mathematics education community. No prior studies have examined whether students and mathematicians view combinatorial proofs as explanatory or convincing (in the sense of Hersh, 1993). I interviewed 13 experienced provers (five upper-division mathematics students and eight mathematicians) to investigate whether they considered combinatorial proofs to be proofs that explain and/or convince compared to other types of proof, and why. All 13 participants felt that combinatorial proofs are equally or more explanatory than other types of proofs, but participants demonstrated a variety of perspectives regarding the extent to which combinatorial proofs are convincing. These findings further ongoing discussions in proof education literature on proofs that explain and/or convince, as well as help address gaps in combinatorics education literature on students thinking about combinatorial proof.

Keywords: Combinatorics, Combinatorial proof, Proofs that convince, Proofs that explain

1. Introduction

Enumerative combinatorics is an area of mathematics that is both highly accessible for students and widely applicable to other sciences and areas of mathematics (Kapur, 1970; Lockwood, Wasserman, & Tillema, 2020). One important topic in combinatorics education that comes up in discrete mathematics, statistics, probability, number theory, and other contexts is combinatorial proof of binomial identities. Combinatorial proof is a proof method that establishes the veracity of an equation by arguing that the expressions on either side of the equation each enumerate a set (possibly the same set) of equal cardinality (Lockwood, Reed, & Erickson, in press; Rosen, 2012).

Consider for example Pascal's identity, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. This identity can be proven by considering the set of committees of size k which can be formed from a group of n (distinct) people. The left side of the identity counts this, since $\binom{n}{k}$ counts the number of unordered selections of size k that can be formed from a set of n distinct things. For the right side, suppose without loss of generality that one of the n people is named Sofía. Then, $\binom{n-1}{k}$ counts the number of committees that can be formed excluding her (since there are n – 1 remaining people who can be on the committee), and $\binom{n-1}{k-1}$ enumerates the committees that include her (since there are k – 1 remaining spots on the committee and n – 1 remaining people). Since this case breakdown (those without and with Sofía, respectively) encompasses all possibilities, the left side also counts the same set of committees. Since both the left and right sides of the identity enumerate the same set, we can conclude they must be equal in value. The validity of a combinatorial proof lies in the fact that a set can have only one cardinality.

Because combinatorial proof of binomial identities does not involve algebraic manipulation but instead requires the prover to articulate combinatorial processes underlying binomial expressions, combinatorial proof can provide opportunities for students to engage in semantic proof production (Weber & Alcock, 2004), use analytical proof schemes (Harel & Sowder, 1998), and encounter proofs that explain (Hersh, 1993). Despite its utility and these potential pedagogical advantages of combinatorial proof, however, little is known about students' perceptions regarding the nature of combinatorial proof. The few existing studies that target undergraduate students' thinking about combinatorial proof rely on artifact-based data (Engelke Infante & CadwalladerOlsker, 2011; Engelke & CadwalladerOlsker, 2010) or were conducted with novice provers who may have limited experience with mathematical proof (Lockwood et al., in press). Additionally, even less is known about how mathematicians may conceptualize these proofs. In this paper, I attempt to address these gaps in the literature by presenting findings from a study in which I conducted clinical interviews with upper-division mathematics students and mathematicians, aimed at answering the following research questions:

1. To what extent do experienced provers consider combinatorial proofs of binomial identities to be convincing or explanatory compared with other types of proofs, and why?

2. What do experienced provers' perceptions of combinatorial proof as convincing or explanatory tell us about the nature of combinatorial proof (including its similarities and differences to other types of proof)?

Addressing these questions can contribute both to answering open questions in the relevant combinatorics education literature, as well as add to ongoing discussions in proof education literature about proofs that convince and/or explain. To these ends, I gave upper-division mathematics students and mathematicians tasks aimed at eliciting combinatorial proof activity and asked them reflection questions on their conceptions of combinatorial proof. Specifically, I sought to determine if the students and mathematicians found combinatorial proof to be convincing or explanatory (in the sense of Hersh, 1993).

In the proceeding section, I situate this research by first discussing relevant studies in the combinatorics education literature, including Lockwood's (2013) model of students' combinatorial thinking and the few existing studies that have focused specifically on combinatorial proof. Later, I also discuss the relevant literature from proof education research I drew from to inform my investigations.

2. Literature Review

Enumerative combinatorics is widely acknowledged as an important area of mathematics (Kapur, 1970), and combinatorial proof is a highly useful topic within combinatorics and has applications ranging from statistics and probability to computer science. Combinatorics and combinatorial proof are also ideal settings for students to grapple with difficult, important mathematical ideas (such as isomorphism, relation, and equivalence), since counting is highly accessible and does not require a lot of prior mathematical background such as calculus (Kapur, 1970). In the classroom, counting and combinatorial proof provide ample opportunities for students not only to problem-solve, but also to justify why and how the solution to a mathematics problem works (Hurdle, Warshauer, & White, 2016). Unfortunately, it is also widely acknowledged that students of all ages struggle to solve counting problems correctly (e.g., Annin & Lai, 2010; Batanero, Navarro Pelayo, & Godino, 1997; Lockwood & Gibson, 2016). While some research has been conducted to address such difficulties with counting generally, much less has been formally studied about how students understand combinatorial proof or what they think about as they go about writing them.

Combinatorial proof is also an area that is ripe for advancing the mathematics education research area of proof. While proof at secondary and tertiary levels has been studied for decades (e.g., Hanna, 2000; Harel & Sowder, 1998; Mejía-Ramos et al., 2015; Mingus & Grassl, 1999; Raman et al., 2009; J. Selden & Selden, 1995; Stylianou et al., 2015), the majority of these studies have focused on proof in domains such as analysis, number theory, and algebra. There have been very few studies conducted that focus on proof in the combinatorial domain, and researchers such as Lockwood et al. (in press) point out that studying student beliefs about and activity in combinatorial proof could provide new insights for the proof literature.

I now proceed with a discussion of the existing literature relevant to combinatorial proof as situated in combinatorics education. Later, in Section 3, I expand on combinatorial proof as situated within discussions of proof literature

2.1 Relevant Literature on Combinatorics Education

In this section, I present an overview of the work researchers have done toward understanding student thinking about counting generally and combinatorial proof. First, I describe work that has documented difficulties students encounter solving counting problems, which will help to situate my subsequent presentation of results in this paper. Then, in Section 2.1.2 I expand on a useful model of students' combinatorial thinking developed by Lockwood (2013). In Section 2.2 I then discuss some of the work that has been done on combinatorial proof which will be relevant for subsequent discussions in the paper.

2.1.1 Student difficulties solving counting problems. It has been said that the road to solving counting problems is strewn with pitfalls (Hadar & Hadass, 1981). Authors have described the difficulties associated with teaching students to count, because oftentimes there is no rigid formula or procedure that can be applied generally (Annin & Lai, 2010). Students can find it challenging to articulate a plan for approaching counting problems, or even articulate exactly what they are trying to count (Hadar & Hadass, 1981). It can be tricky to know how to encode outcomes in a useful way (Lockwood et al., 2015a; Spira, 2018), and even when solutions to counting problems are found they can be notoriously difficult to verify (Eizenberg & Zaslavsky, 2004). There are many subtle errors one can easily commit while solving counting problems even if at first glance the solution seems correct (Annin & Lai, 2010; Lockwood, 2014b). Finally, even when a student's solution is correct, they can still lack sufficient ability to justify their solutions (Lockwood et al., 2015b). I mention these documented difficulties not to paint a negative picture of students' abilities in these areas, but to emphasize the need for more investigations into ways to help students be more successful in solving combinatorial problems. Researchers have taken a variety of approaches to helping address these challenges students face solving counting problems, including categorizing common counting errors (Batanero et al., 1997), advocating a set-oriented perspective (Lockwood, 2014a), and creating a model to describe students' combinatorial thinking (Lockwood, 2013). In the next section, I expand on this model, as it is both an important piece of the combinatorics education literature and informs the way in which I characterize combinatorial proof in my study.

2.1.2 Lockwood’s (2013) model of students’ combinatorial thinking. One important contribution to the research on combinatorics education literature is Lockwood's (2013) model of students combinatorial thinking. See Figure 5.1 for a diagram illustrating the model. Lockwood (2013) developed her model using in Thompson’s (2008) theoretical notion of a mathematical conceptual analysis, as well as empirical observation of student activity solving counting problems. One component of Lockwood’s model is *sets of outcomes*, which are the collection(s)

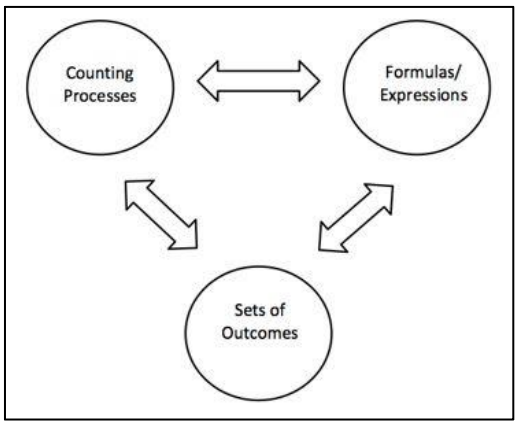

Figure 5.1. Lockwood’s (2013) model of students’ combinatorial thinking.

of objects being counted. This component of the model includes the combinatorial activity of *encoding*, which Lockwood, Swinyard, and Caughman (2015) defined as determining the nature of what is being counted. Another component of Lockwood’s model is *counting processes*, which describes the processes by which a counter enumerates the set(s) of outcomes. Counting processes can be thought of as leveraging or imposing a particular organization of the set of outcomes. The third component of Lockwood's is *formulas/expressions*, which are mathematical expressions that are often thought of as the answer to a counting problem. Lockwood noted that the way a formula/expression is written may suggest an underlying counting process, a key realization for a student who is coming up with a combinatorial proof. For instance, a student may arrive at the expression $\binom{26}{5} \times 5$ as the solution for a counting problem, which suggests an underlying counting process of first carrying out a task in which there are $\binom{26}{5}$ ways to complete the task, and then carrying out a second task in which there are 5 ways to complete the task, using the Multiplication Principle[7]. Another important point Lockwood (2013) made is that even if two expressions have the same numerical value, the different forms could suggest different underlying counting processes.

Since Lockwood's (2013) model was developed, it has been used as an analytical lens through which to examine data in many studies investigating students' thinking and activity solving counting problems (e.g. Halani, 2013; Hidayati et al., 2019; Lockwood, 2014; Lockwood et al., 2018; Lockwood & Erickson, 2017; Lockwood & Gibson, 2016; Lockwood & Purdy, 2019a). In the next section, I transition to discussing prior work done on student thinking about and engagement in combinatorial proof, including a prior study that used Lockwood's model for the first time as a lens to study combinatorial proof (rather than just a lens to study students' solving of counting problems).

2.2 Relevant Literature on Combinatorial Proof

There is agreement among much of the mathematics education community that proof is a critical mathematical topic for students to learn and should be introduced earlier and utilized more frequently in secondary and tertiary mathematics curriculum (Hanna, 2000; Harel & Sowder, 1998; Mingus & Grassl, 1999; G. J. Stylianides et al., 2017). Since combinatorics is an accessible domain in mathematics (Kapur, 1970), combinatorial proof of binomial identities could provide

[7] Tucker (2002) offers my preferred statement of the Multiplication Principle: "Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes" (p. 170).

an excellent setting for students to gain experience with proof and help address the myriad of difficulties students face when attempting to write or even comprehend mathematical proofs (Raman et al., 2009; A. Selden & Selden, 2008; J. Selden & Selden, 1995; Stylianou et al., 2015). Below, I review the limited existing literature related to combinatorial proof of binomial identities, and I will discuss proof more generally later in Section 3 of this paper. While there have been dozens of studies spanning several decades that have been conducted on proof in the mathematics education literature, I focus on proof literature that is specifically relevant to combinatorial proof.

2.2.1 Research on secondary students' justifications of binomial identities. Research conducted on grade-school children has provided evidence that even young children can demonstrate a combinatorial understanding of binomial identities (Maher, Powell, & Uptegrove, 2015). For instance, Maher, Muter, and Kiczek (2007) described an episode where a 10$^{\text{th}}$-grade student showed that there are 32 total pizzas that can have up to 5 different toppings by constructing a bijection between these pizzas and binary strings, and in Maher et al. (2015) 11$^{\text{th}}$-grade students found interesting connections between numbers in Pascal's triangle and outcomes for counting problems involving pizzas and block towers of two colors. Another student used combinatorial reasoning and numerical patterning to justify Fermat's formula,

$$\binom{n}{r+1} = \binom{n}{r} \cdot \frac{n-r}{r+1}$$

in the context of block towers. Finally, Maher and Speiser (1997) also chronicled the progress of a 14-year-old participant investigating binomial coefficients and combinations. This student made meaningful mathematical connections to Pascal's triangle and was able to justify combinatorially why Pascal's addition identity,

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

holds.

While this work does not explicitly address combinatorial proof, the combinatorial insight that these secondary students demonstrated in justifying binomial identities is a promising sign that undergraduate students can engage in rigorous combinatorial proof activity.

2.2.2 Studies on undergraduate students' combinatorial proof activity and thinking. Although there have been few studies that address combinatorial proof at the undergraduate level, I now review what work has been done in this specific area. Engelke Infante and CadwalladerOlsker (2011) used student work on midterm and final exams in a class they were taking to study how successful these students were at combinatorial proof. Engelke Infante and CadwalladerOlsker rated the students' proofs on a scale from 1 to 4 based on how successful the proofs were, and they found that students most often seemed to struggle with the following difficulties (these are definitions I articulated from p. 95-96):

1. Language mimicking—attempting to copy the syntax of a combinatorial argument they had previously encountered, but not paying enough attention to potentially important details of the binomial identity at hand.
2. Inflexibility of context—attempting to apply the same context (e.g. selecting jobs for people to perform) to all combinatorial proofs, even when not inappropriate.
3. Misunderstanding of combinatorial functions—for example, not comprehending what the choose function (i.e. combinations or binomial coefficients) represents.
4. Failure to count the same set—when the student argued that each side of a binomial identity counted different sets of outcomes.

Engelke Infante and CadwalladerOlsker (2011) also noted that many student-written proofs in the study contained errors of logic as well, for instance claiming that, "Since the LHS=RHS, they count the same thing" (p. 96). While it is difficult to determine conclusively from artifact-based (rather than interview-based) data, Engelke Infante and CadwalladerOlsker also found that some students they studied seem to engage what they called, "pseudo-semantic proof production," which is based on the semantic/syntactic proof production distinction introduced by Weber and Alcock (2004).

As far as I could determine, the only other empirical study existing in the mathematics education literature that specifically targeted combinatorial proof at the undergraduate level was conducted recently by Lockwood et al. (in press). They carried out a paired teaching experiment (Steffe & Thompson, 2000) with two undergraduate students who were taking vector calculus at the time of the study and who had little previous combinatorial experience. The researchers met with the students for a total of 15 hour-long sessions, and in the final three sessions the pair of students were able to articulate successfully combinatorial justifications of binomial identities with the knowledge and insights they had learned throughout the teaching experiment. Lockwood et al.'s work provided some evidence that having students build up to combinatorial proof by first solving counting problems two ways and generalizing from specific cases may help facilitate reasoning that lends itself nicely to productive combinatorial proof activity. However, since the students in the study had never taken upper-division mathematics courses, it is unclear whether the students themselves felt that they were engaging in proof.

Finally, Lockwood, Caughman, and Weber (2020) provided a theoretical contribution to the proof literature by examining proofs that convince and proofs that explain using combinatorial proof as an example. Lockwood et al. argued that proof researchers should adopt the perspective of the reader when discussing constructs such as a proofs that explain and/or convince (Hersh, 1993), and they argued that researchers should refer to the appropriate *representation system* when discussing the constructs of a convincing and explanatory proof. (For example, combinatorial proof as exemplified in the Introduction of this paper would be considered as situated within the *enumerative* representation system.) They also discussed that while the labels *explanatory* and *convincing* are reader-dependent, there is some regularity regarding what types of proofs readers may find convincing or explanatory, and they stated that many would consider combinatorial proofs to be proofs that explain.

In summary, while some work has been done that explores combinatorial proof within the mathematics education literature, there is a need for more investigations to help the field better understand how provers (including students and mathematicians) conceptualize combinatorial proof. In particular, there is great opportunity to study combinatorial proof among more experienced provers, which would shed light on their perceptions of combinatorial proof as convincing or explanatory. In the proceeding section, I describe the theoretical perspective I adopted for this paper as well as situate my characterization of combinatorial proof within relevant literature.

3. Theoretical Perspectives and Characterizing Combinatorial Proof

In this section, I discuss the theoretical framing for this paper. In Section 3.1, I discuss how I am conceptualizing proof for this study. Next, in Section 3.2, I expand on the theoretical lens I used to design and analyze the data for this paper: proofs that convince and proofs that explain (Hersh, 1993). Finally, in Section 3.3, I revisit Lockwood's (2013) model to describe how I applied it to characterize combinatorial proof throughout my study.

3.1 Characterizing Proof in This Study

Since this study centers on combinatorial proof, I first specify what I take to constitute proof. Currently in the mathematics education community, there is a wide array of perspectives on what should be taken as a mathematical proof. There is even debate around issues as structurally basic as whether a proof without words, such as a proof consisting only of a picture, really constitutes mathematical proof (e.g. Gierdien, 2007). Some researchers have articulated a dichotomy of formal proofs and acceptable proofs. For instance, Hanna (1990) explained that formal proofs are theoretical and exist as a string of sentences such that the first sentence is an axiom, and each consequent sentence either follows from those previous or is an axiom. However, Hanna (1990) also recognized that this is not how mathematics is usually done in the real world, and consequently Hanna (1990) said that acceptable proof can be thought of as what mathematicians actually do: produce proofs that are considered acceptable and valid within a qualified community. Other characterizations of proof by mathematics education researchers have distinguished between an argument that may be found personally convincing versus a proof that could persuade a broader community. Harel and Sowder (2007) used the terms *ascertaining* and *persuading* to describe these different types of proofs, respectively, and Raman (2003) discussed a similar distinction between private and public arguments.

The above represents a small sample of the ways that proof and proof production have been characterized by mathematics education researchers. For the purposes of my study, I sought a definition of proof that is student-centered and that attends to the way students choose to represent particular mathematical objects (such as $\binom{n}{k}$). Thus, I adapted the definition given below by Stylianides (2007, p. 291; emphasis in original):

> *Proof* is a *mathematical argument*, a connected sequence of assertions against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.

Stylianides (2007) used observations from a third-grade classroom to elaborate on elements of this definition and to illustrate its applicability, hence his use of the term “classroom community.” In my investigation, I utilized a broader meaning of the term “community” to also encompass other communities (such as a group of peers, the community in a prior class, or the larger mathematics community) that may be important to the perspective of a student.

3.2 Proofs that Explain Versus Proofs that Convince

As mentioned previously in the Literature Review, Lockwood et al. (2020) said that combinatorial proofs are considered by many to be proofs that explain *why* two quantities are equal, rather than merely convincing a reader that they are equal. To fully understand what they mean, in this section I discuss the ongoing conversation among mathematics education researchers regarding the distinction between proofs that explain and proofs that only convince (Hersh, 1993).

Similarly to Hanna's (1990) distinction between formal and acceptable proof, Hersh (1993) observed that in real-world mathematical practice, proofs are not often presented in an absolute sense, as if they exist purely as a sequence of statements manipulated using formal rules of logic apart from human activity. Instead, the term "proof" often has less to do with formal logic and is used more broadly to mean, "convincing argument, as judged by qualified judges" (p. 389). Based on this, Hersh (1993) articulated that proof can be divided into two categories, depending on the context and purpose of the proof. He said that in mathematical research, the purpose of proof is to convince, and to do so it must reach some standard of rigor and honesty as defined by the mathematical community. In the classroom however, he said that the purpose of proof is to explain that is, proofs should be enlightening and stimulate students' mathematical understanding. The value of proofs that explain is not limited to the classroom though. Hersh (1993) stated, "More than whether a conjecture is correct, mathematicians want to know why it is correct" (p. 390), and he used the historical example of Paul Halmos' complaints regarding the Appel-Haken theorem, which used computation to aid in proving the Four-Color Theorem.

Other researchers have also expanded on the convincing/explanatory distinction in the research field of proof. Hanna (2000) stated similarly that the two fundamental functions of proof are verification and explanation, and Weber (2010) touched on this distinction in a study investigating the different ways that mathematicians view proof. Weber (2010) found that often mathematicians find great value in proofs that explain (not just proofs that convince). For example, he noted that proofs are read by mathematicians to help them gain new insights and proof techniques within their field, as well as provide new ways of thinking about mathematical objects. Weber (2010) also argued that explanatory proof should be thought of as an interaction between the proof and its reader, rather than considering "being explanatory" to be a factor inherent to a proof, separate from any human interaction. He stated,

> I conceptualize a proof that explains as a proof that enables the reader of the proof to reverse the connection—that is, this proof allows the reader to translate the formal argument that [they are] reading to a less formal argument in a separate semantic representation system (p. 34).

As mentioned previously, Lockwood et al. (2020) have since adopted the similar term *representation system*, which they define as "consist[ing] of configurations that are used to represent mathematical objects and inferential schemes that can be used to deduce new facts about these objects" (p. 3). This reframing allows the discussion regarding proofs that convince and proofs that explain to be more reader-centered, opening the possibility that—depending on the semantic representation system used—a proof could be considered explanatory or not depending on the reader. With this more flexible and student-centered framing in mind, in my study I drew upon the definitions of proofs that convince and proofs that explain from Weber (2002):

- "A proof that convinces begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is unknown....The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true" (p. 14).

- "A proof that explains also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity is not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous....The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main ideas" (p. 14).

From these definitions, proofs that are carried out solely by manipulating symbols or employing an "algebraic trick" are usually considered proofs that convince (and not proofs that explain). Additionally, other researchers have argued that proofs by induction or by contradiction are usually only proofs that convince (Hanna, 2000; Lange, 2009), though it certainly is not a settled issue (for instance, Stylianides, Sandefur, & Watson, 2016, outlined some criteria in which a proof by induction could be considered explanatory). Here, again, the use of representation systems can be useful, because an algebraic or induction proof could be considered explanatory to some readers if they are situated in an algebraic or inductive representation system (Lockwood et al., 2020).

While I believe that the distinction between proofs that explain versus proofs that only convince can be useful, I also acknowledge the criticisms that some researchers have had of the distinction. For instance, Stylianides et al. (2017) expressed concern that the explanatory and convincing distinction in proof literature is inadequately defined, with the precise characterization of what constitutes a proof that explains remaining especially unclear. In addition, Mingus and Grassl (1999) interviewed preservice teachers and found that some proofs fell into neither category for some of the teachers, and Weber (2002) offered an expansion of the binary categorization and acknowledged that they are overlapping categories. Other researchers have argued the dichotomy of proofs that convince versus proofs that explain is too restrictive and does not capture other ways that students or mathematicians may assess a proof (e.g., Inglis & Aberdein, 2016).

I appreciate and acknowledge these criticisms, and I argue that one practice that can help address some of the issues raised above is to specify the reader of the proof that is being categorized, and the representation system in which they are situated. For instance, mathematicians might consider combinatorial proofs to be explanatory, but upper-division mathematics students may not find combinatorial proof to be explanatory or even convincing if they are not used to working in the enumerative representation system. It similarly may be the case that combinatorial proofs might be explanatory in the enumerative domain (depending on the reader), but not in the algebraic domain. This also means that if a reader wants to understand algebraically why a binomial identity might hold, a combinatorial proof would not explain this even if it could provide a combinatorial explanation. Lockwood et al. (2020) therefore emphasized the importance of a nuanced perspective when labeling a proof as convincing or explanatory, including considering the representation system in use and the perspective of the reader. This also means that it may be misleading to label all algebraic or inductive proofs as non-explanatory (see also Stylianides et al., 2016), since students may be comfortable in the algebraic or inductive representation system and find these proofs to be more personally explanatory or convincing than a combinatorial proof. Unpacking and utilizing this nuance is a goal of the research conducted in this study. Finally, I point out that while I hold that a proof should be considered explanatory (or not) depending on the reader and representation system, I nevertheless acknowledge there may be some uniformity that exists about which proofs individuals consider explanatory, such as proofs that include some kind of visualization (Lockwood et al., 2020).

Additionally, while labeling a proof as "explanatory" requires nuance, so too does the label of "convincing." Multiple proof researchers have distinguished among the differing level of conviction that a reader gets from reading or writing a proof. For instance, Harel and Sowder (2007) distinguish between a prover ascertaining or persuading in their proof activity, where ascertaining is defined as, "the process an individual (or a community) employs to remove [their] own doubts about the truth of an assertion," (p. 6) and persuading is defined as, "the process an individual or a community employs to remove others' doubts about the truth of an assertion" (p. 6). This is similar to the distinction between a private argument and public argument articulated by Raman (2003), and she explained that, for example, a private argument might consist of informal reasoning that convinces an individual of the truth of a statement and that could, if the individual were pressed, become a public argument with the addition of sufficient rigor. It is additionally important to remember that while a proof should not be considered "explanatory" without reference to an individual and representation system, whether or not a proof is "convincing" also depends on the individual and the representation system. Stylianides et al. (2016) also suggested following a subjective perspective on proof that is more focused on the prover or reader rather than objective qualities inherent to the proof.

In conclusion, while I drew from and utilize Weber's (2002) definitions of proofs that explain and proofs that convince, I acknowledge there has been a rich discussion on this issue in the mathematics community and intended to take care when applying this as a theoretical framework. In particular, I followed Lockwood et al. (2020) by considering both the reader and their representation system when considering a student or mathematician's proof as convincing and/or explanatory. Ultimately, despite some reservation in the community about the usefulness of this distinction, I still think it is valuable to better understand how combinatorial proof is viewed as convincing and/or explanatory. I am motivated to learn more about combinatorial proof and its status among other types of proof (particularly among experienced provers), and so I view this distinction as a way to understand more about these ideas. In particular, one way to frame my goals in this paper is that I am empirically investigating Lockwood et al.'s (2020) theoretical assertions about combinatorial proof and its status as explanatory and/or convincing.

3.3 Lockwood's (2013) Model as a Way to Characterize Combinatorial Proof

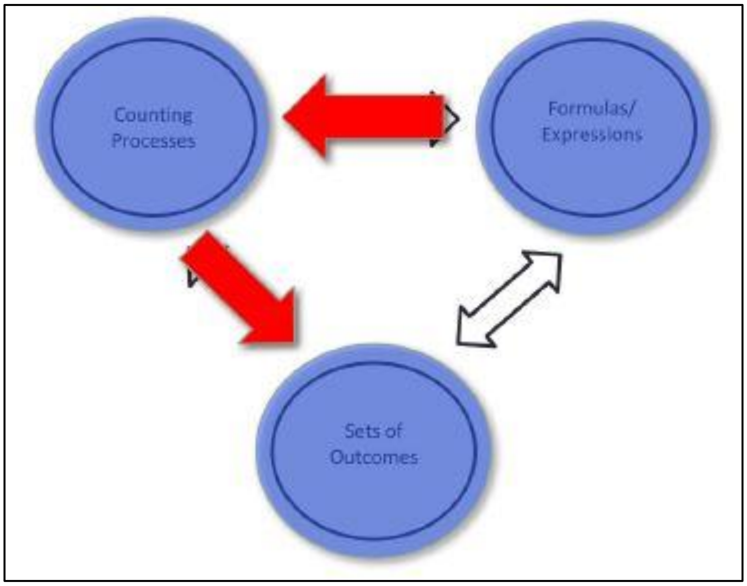

Figure 5.2. Lockwood et al. (in press) argued that when engaging in combinatorial proof, provers must move counterclockwise around Lockwood's (2013) model starting at formulas/expressions and ending up at sets of outcomes.

Finally, in this study, I followed Lockwood et al. (in press) in using Lockwood's (2013) model to characterize combinatorial proof. Lockwood et al. argued that when students engage in combinatorial proof, they are starting in the formulas/expressions component of Lockwood's model, and then they must conceive of each of each side of the binomial identity as having an underlying counting process that enumerates a set of outcomes. (See Figure 5.2.) To illustrate my conceptualization of how students navigate through the components of Lockwood's (2013) model when proving a binomial identity, consider the combinatorial proof which was first given in the Introduction.

Proposition. Let *n* and *k* be nonnegative integers. Then,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Proof. The left side of the equation counts the number of size-*k* committees that can be formed from a set of *n* people.

To see how the right side counts the same set, suppose that one of the *n* people is named "Sofía." With this in mind, $\binom{n-1}{k}$ counts the number of committees from the set of *n* people that don't include Sofía, and the $\binom{n-1}{k-1}$ counts the number of committees that do include Sofía. This is because one spot in the committee is already occupied (by Sofía), and so we only need to choose *k* - 1 from the remaining *n* - 1 people.

Since both the left and right sides of the identity enumerate the same set of outcomes (committees of size *k* from *n* people), they have the same numerical value[8].

To come up with this proof, a prover must recognize that the left side of the identity represents a *counting process* (namely making an unordered selection) that can enumerate a *set of outcomes*—in this case, committees of size *k* from *n* people. Next, the prover must conceive of a way that the right side could enumerate the same set of outcomes. Specifically, they must recognize that the expression represents a counting process that makes two unordered selections and then groups these selections together. Finally, a prover would need to come up with a way that this process could also enumerate the set of committees of size *k* from *n* people, and in this situation one way to do that is to focus on one of the *n* people and consider committees that do and do not contain that person as a case breakdown.

In this way, Lockwood's (2013) model applies naturally when studying student thinking and engagement with combinatorial proof. I used her model to inform my choice of interview tasks

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[8] This proof argues for the veracity of the binomial identity by arguing that both sides of the identity count the same set. This exemplifies an Approach 1 combinatorial proof (Lockwood et al., in press), and I acknowledge that there are other kinds of combinatorial proofs that exist (such as those that involve establishing a bijection). I do not discuss these other types of combinatorial proof in this paper.

and more broadly to characterize combinatorial proof. Whenever I refer to outcomes, counting processes, and formulas/expressions in this paper, the reader should understand that I apply these terms in alignment with Lockwood's (2013) model.

4. Methods

I conducted individual, task-based, semi-structured clinical interviews (Hunting, 1997) with upper-division undergraduate mathematics students and mathematicians. In these interviews, I asked the participants to complete tasks aimed toward getting at their understanding of combinatorial proof, including to what extent they felt that combinatorial proofs are proofs that convince or proofs that explain (compared with other types of proof such as algebraic or induction). I describe the details in the following sections.

4.1 Participants

The participants in my study were from two populations of interest: students with prior experience with proof and combinatorics, and mathematicians with experience conducting mathematics research. Below I describe how I recruited participants.

4.1.1 Student participants. Five students were recruited from upper-division mathematics courses at a large university in the western United States. I visited and recruited in Advanced Calculus II, Multivariable Advanced Calculus, Fundamental Concepts of Topology, Metric Spaces and Topology, and General Relativity courses. I chose these courses because they all require some proof-based course (e.g. Discrete Mathematics or Advanced Calculus I) as a prerequisite. I wanted the students in my study to have some understanding of what a rigorous mathematical proof entails, and so choosing students from these classes would ensure they had all completed at least one proof-based mathematics course before. Table 5.1 lists all the courses that the student participants had taken (or were currently taking) at the time my interview with them took place.

Table 5.1. Classes taken by student participants.

	Student Participants*:				
	Sydney	Riley	Adrien	Peyton	Ash
Calculus I	✓	✓	✓		✓
Calculus II	✓	✓	✓		✓
Infinite Series & Sequences	✓		✓	✓	✓
Vector Calculus I	✓	✓	✓	✓	✓
Vector Calculus II	✓		✓	✓	✓
Applied Differential Equations	✓		✓	✓	✓
Mathematics for Management, Life, and Social Sciences					✓
Advanced Calculus	✓		✓		✓
Linear Algebra I	✓	✓	✓	✓	✓
Linear Algebra II	✓		✓	✓	✓
Introduction to Modern Algebra	✓		✓		✓
Metric Spaces and Topology		✓**	✓**		
Discrete Mathematics	✓	✓		✓**	✓
Applied Ordinary Differential Equations	✓		✓		
Applied Partial Differential Equations	✓				
Fundamental Concepts of Topology	✓**	✓**		✓**	
Numerical Linear Algebra		✓			
Introduction to Numerical Analysis			✓		
Computational Number Theory		✓			
Mathematical Modeling			✓		
Actuarial Mathematics			✓		
Complex Variables					✓
Non-Euclidean Geometry					✓

* These are pseudonyms.
** Indicates that the student was enrolled in this course at the time the interviews were conducted.

I also wanted the students in my study to have some prior experience with solving counting problems. Following my use of Lockwood's (2013) model as way of characterizing combinatorial proof, I believe that combinatorial proof requires students to be able to conceive of an expression as having an underlying counting process, so a counting process is a construct that should already have some meaning for my participants. I describe these selection interviews in Section 4.2.

4.1.2 Mathematician participants. A convenience sample of eight mathematicians were recruited via email from three universities in the western United States. Each mathematician was an acquaintance of either myself or my academic adviser, and they were chosen to have a range of experiences researching and teaching combinatorics. I recruited mathematicians who did and who did not conduct research in a combinatorial field, and I recruited mathematicians who did and who did not regularly teach combinatorics. The backgrounds of each of the mathematician participants are summarized in Table 5.2.

Table 5.2. Mathematician participants' research and teaching experience information.

Name*	Research Experience	Regularly Teaches Combinatorics
Ridley	Algebraic combinatorics & bijective combinatorics (13 years)	Yes
Dominique	Competitive coloring algorithms and parameters defined on graphs (20 years)	Yes
Jaiden	Computability, computable analysis, & algorithmic information theory (3 years)	Yes
Skyler	Dynamical systems and number theory (15 years)	No
Emery	Modular forms and partition functions (17 years)	Yes
Lake	Partial differential equations & related functional analysis (60 years)	No
Justice	Representation theory of finite groups (6 years)	Yes
Robin	Geometry, algebra, and mathematics education (40 years)	No

* These are pseudonyms.

4.2 Student Selection Interviews

The student participants each took part in a round of individual, task-based selection interviews. Each interview was audio- and video-recorded. In these interviews, I began by asking the students to solve a sequence of counting problems. Some of these problems were aimed at seeing how attuned the students were to sets of outcomes, and some of the problems got at whether the students could use combinations. Next, I asked them to prove three theorems: the sum of two even integers is even; $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$; and $\binom{n}{k}k = n\binom{n-1}{k-1}$. The purpose of these tasks was to ensure I had participants who knew how to navigate a basic algebraic or induction proof. All of the students (except Peyton, in whose selection interview I ran out of time to reach this question) proved $\binom{n}{k}k = n\binom{n-1}{k-1}$ algebraically using the $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ formula, which I had expected. I also asked if they could think of a way to argue that the identity was true by thinking about what each side of the identity could be counting. This was to see if they were already familiar with the idea of a combinatorial proof, and I intentionally used the word "argue" rather than "prove" in my questioning to avoid potentially influencing the students' opinions regarding combinatorial proofs as proofs that explain and/or convince.

In summary for a student to continue on and participate in the four (remaining) clinical interviews I intended to conduct with them, they needed to demonstrate that they could solve basic counting problems, were familiar with combinations, had experience with proof at the college level, and understood what a rigorous proof entails.

4.3 Main Interviews

After I recruited the mathematician participants and selected students suitable for my study, I scheduled individual clinical interviews (Hunting, 1997) with each participant. Each interview was audio- and video-recorded.

4.3.1 Interviews with mathematicians. Each of the mathematicians participated in one 90-minute interview at a time convenient to the participant. Each mathematician was first asked a series of introductory questions, including asking them about their research area, how they would define a proof and a combinatorial proof, and whether (and how frequently) they teach combinatorial proof of binomial identities. Next, I asked the mathematicians to provide a combinatorial proof to a series of binomial identities, which are given in Table 5.3. I gave each mathematician as many identities as time permitted. I prioritized the identities $\binom{n}{k} = \binom{n}{n-k}$, $\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$, and $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ when possible, because I wanted to see the mathematicians' "go-to" strategies when proving simpler binomial identities like $\binom{n}{k} = \binom{n}{n-k}$ and $\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$, and I wanted to see their problem-solving process when working on a harder identity like $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ (which typically involves a bijective combinatorial proof). However, I adjusted my sequence of tasks in the interviews as needed, such as giving more relatively simple binomial identities like $2^n = \sum_{i=0}^{n}\binom{n}{i}$ to mathematicians who had not done these kinds of proofs of binomial identities for a while.

Table 5.3. Binomial Identities in Mathematician Interviews.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n}\binom{n}{i}$
$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	$\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}$
$\sum_{i=1}^{n}\binom{n}{i} \cdot i = n \cdot 2^{n-1}$	$\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}^2$
$\binom{n+1}{k+1} = \sum_{i=k}^{n}\binom{i}{k}$	$\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$
$\binom{n}{k} = \frac{n}{k} \cdot \binom{n-1}{k-1}$	$\binom{n}{k} = \frac{n+1-k}{k}\binom{n}{k-1}$

After I asked the mathematicians to prove binomial identities combinatorially, I gave them three handouts to read, each containing a binomial identity (numbered Theorems 1-3) as well as a combinatorial and a non-combinatorial proof of the identity. The first handout had the identity $2^n = \sum_{i=0}^{n}\binom{n}{i}$ and gave a combinatorial and induction proof of the identity; the second had $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and a combinatorial and algebraic proof; and finally the third had $\binom{n}{k} = \binom{n}{n-k}$ and a combinatorial proof and algebraic proof that utilized the binomial theorem. The details of these six proofs are given in Table 5.4.

I asked the mathematicians to read each handout and first give me their initial impressions of the two proofs before moving on to the next handout. Once the mathematicians had finished giving me their initial impressions of all six proofs, I gave them the following definitions of proofs that convince and proofs that explain, adopted from Weber (2002):

- "A proof that convinces begins with an accepted set of definitions and axioms and concludes with a proposition whose validity [was] unknown....The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true" (p. 14).
- "A proof that explains also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity [was] not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous....The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main ideas" (p. 14).

Table 5.4. Six Proofs handout.

Identity	Combinatorial Proof	Non-combinatorial proof
Theorem 1. $2^n = \sum_{i=0}^{n}\binom{n}{i}$	(Subsets Context) Consider a set S such that $\lvert S\rvert=n$. The LHS* of the equation counts the number of subsets of S, because every subset can be uniquely determined by the elements it contains, and each of the n elements could be either in or out of each subset. The RHS counts the number of i-subsets of S and adds up over all possible values of i. Since the LHS and RHS both enumerate the set of subsets of S, they are equal.	(Induction RS*) Suppose $n=0$. It follows that the identity holds since $2^0 = 1 = \binom{0}{0}$. Suppose that the identity holds for $n=k$, where k is a nonnegative integer. We then observe that $\sum_{i=0}^{k+1}\binom{k+1}{i} = \sum_{i=0}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right) + \binom{k+1}{k+1}$ $= \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k}\binom{k}{i-1} + 1$ $= 2^k + \sum_{i=0}^{k-1}\binom{k}{i} + 1$ $= 2^k + 2^k - \binom{k}{k} + 1$ $= 2 \cdot 2^k - 1 + 1$ $= 2^{k+1}$.

*RS here refers to representation systems, in the sense of Lockwood, Caughman, and Weber (2020).

Table 5.4 (Continued)

Theorem 2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	(Committees Context) Suppose a mathematics department has *n* faculty members, and Sofía is one of the faculty members. The LHS counts the total number of committees of size *k* that could be formed from the *n* faculty members. The RHS counts the number of committees of size *k* that exclude Sofía and the committees that include her. Note that this case breakdown encompasses all possible *k*-committees. Since the LHS and RHS both enumerate the same set of outcomes (*k*-committees formed from the *n* faculty members), they are equal.	(Algebraic RS) We have that $\binom{n-1}{k} + \binom{n-1}{k-1}$ $= \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$ $= \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$ $= \frac{(n-1)!\,(n-k)}{k!\,(n-k)!} + \frac{(n-1)!\,k}{k!\,(n-k)!}$ $= \frac{n(n-1)! - k(n-1)! + k(n-1)!}{k!\,(n-k)!}$ $= \frac{n!}{k!\,(n-k)!}$ $= \binom{n}{k}$.
Theorem 3. $\binom{n}{k} = \binom{n}{n-k}$	(Binary Strings Context) Consider the set of binary strings of length *n* containing exactly *k* 0s. The LHS enumerates this set, because $\binom{n}{k}$ is the number of ways we can select positions for the 0s to occupy, and the rest of the positions in the binary string will be 1s. The RHS also enumerates this set, because $\binom{n}{n-k}$ is the number of ways we can select positions for the 1s to occupy, and the rest of the positions in the binary string will be 0s.	(Binomial Theorem RS) Recall that the Binomial Theorem states that for *n* a natural number and *a*, *b* real numbers, $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$ Notice that for each *k*, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. Additionally, we also have that by the Binomial Theorem, $$(b+a)^n = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^i,$$ and the coefficient of $b^k a^{n-k}$ is $\binom{n}{n-k}$. We also have that $a^{n-k}b^k = b^k a^{n-k}$ and $(a+b)^n = (b+a)^n$, by the commutativity of multiplication and addition of real numbers, respectively. Thus, when the latter is expanded, the coefficients of each term on either side of the equation must be equal, so $\binom{n}{k} = \binom{n}{n-k}$ for all *k*.

I wanted to give these definitions to the mathematicians to ensure that when I asked the them about their opinions regarding combinatorial proofs as convincing or explanatory, we would have a shared understanding of these terms. Similarly to the other handouts, I first asked each mathematician to give their initial impression of the definitions, including whether the mathematicians felt this was a useful distinction to describe proofs and if the definitions resonated with them at all. I then asked them a series of reflection questions aimed at probing their beliefs about combinatorial and non-combinatorial proofs (such as induction or algebraic proofs) as proofs that explain and proofs that convince. I also asked them to reflect on their own experience as research mathematicians, describing if they had ever used a combinatorial proof or read one in the literature and why a combinatorial proof was needed or desired. Overall, my goal with these tasks was to have the mathematicians engage in combinatorial proving activity and read combinatorial proofs so they would all have similar, recent experiences they could draw from during the interviews. To address my research questions, I asked the mathematicians reflection questions about combinatorial proofs as convincing and/or explanatory.

4.3.2 Interviews with students. Each of the five students who passed the round of selection interviews participated in four follow-up hour-long individual clinical interviews. The interviews with the students were scheduled at times that were convenient for the students, typically with one- or two-week gaps between each interview. Four of the five students who were selected into the study stated that they had not previously encountered combinatorial proof in any class (as far as they knew). One student, Peyton, was taking a discretemathematics course at the time that the research interviews took place that covered combinatorial proof.

During the four follow-up clinical interviews with each student, I asked them to complete a sequence of combinatorial tasks intended to build up to the idea of a combinatorial proof. See

Table 5.5 for a list of these tasks. Several of these tasks were intended to ensure the students had a robust, flexible understanding of combinations, that is, problems where the solution can be readily expressed using one or more binomial coefficients. Lockwood et al. (2018) found that students distinguish between two different types of problems that can be solved with binomial coefficients, so I felt it was important to ensure they could solve tasks involving either type of problem. Additionally, the fifth task I asked participants to solve, the Reverse Counting Problem,

Table 5.5. Combinatorial tasks for students to scaffold combinatorial proof.

Task	Intended Purpose
1. Spoonbill Problem. The scientific name of the roseate spoonbill (a species of large, wading bird) is *Platalea ajaja*. How many arrangements are there of the letters in the word AJAJA? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving ordered sequences of two indistinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
2. Subsets Problem. How many 3-element subsets are there of the set {1, 2, 3, 4, 5}? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving unordered selections of distinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
3. Find-a-Bijection Problem. Describe a bijection between the outcomes in the Spoonbill Problem and the Subsets Problem.	Facilitate a robust, flexible understanding of combinations. Lay groundwork for students to solve bijective combinatorial-proof problems.
4. Even- and Odd-Sized Sets Problem. Let S={1, 2, 3, 4, 5, 6}. (a) List all of the even-sized subsets of S. How many should there be? (b) List all of the odd-sized subsets of S. How many should there be? (c) Find a bijection between the subsets in parts (a) and (b) by considering whether the subsets contain the item 1.	Continue to facilitate a solid understanding of combinations. Provide scaffolding for students to eventually prove the identity $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ using a bijective combinatorial proof.
5. Reverse Counting Problem. (a) Write down a counting problem whose answer is 2^5. (b) Write down a counting problem whose answer is $15 \times \binom{14}{3}$.	Provide scaffolding for the concept of a combinatorial proof by asking students to interpret expressions in a combinatorial context.

asked students to interpret an expression as the solution to a counting problem. The ways of thinking the students had to engage in to solve this task is very similar to that needed to prove a binomial identity combinatorially, so this task provided direct scaffolding for subsequent combinatorial proof tasks.

Before continuing, I want to make two points about the combinatorial tasks I gave the students. First, during the interviews I felt I did not have to give them more than these five tasks, because in the selection interviews these students had already shown that they were familiar with and could solve counting problems, so I did not feel it was necessary to have them solve too many more counting problems before asking them to engage in combinatorial argumentation. Second, several of the tasks in this table were intended to lay the necessary groundwork for students to make bijective combinatorial proofs of binomial identities[9]. I included these tasks in this section for completeness, and because these tasks helped to ensure the students could work with binomial coefficients effectively. However, bijective combinatorial proofs are not a focus of this paper, and so I do not include details of their work on these types of proofs here.

After the students completed these tasks, I gave them a sequence of binomial identities, and with these identities I gave the prompt: “Argue that the identity holds by arguing that each side counts something.” I was careful to avoid using the word “proof” to describe the tasks, because I planned to ask the students later in the interviews whether they felt their combinatorial arguments constituted proofs that explain or proofs that convince (or even whether combinatorial arguments could be proofs at all), and I did not want to influence the students’ opinions. Each student was given a subset of the identities in Table 5.6 (as time in the interviews permitted). Throughout the

[9] That is, proofs which involve arguing that each side of the binomial identity counts a different set, and then making a bijection between the sets to establish their cardinalities are the same—and therefore the identity holds.

interviews, I made sure the students understood that the variables involved in the binomial identities were nonnegative integers.

Table 5.6. Identities given to the students to provide a combinatorial argument.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i\geq 0} \binom{n}{2i+1}$
$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$	$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$
$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$	

Finally, in the last or second-to-last interview with each student (depending on how far they had progressed), I gave them the same six proofs (three combinatorial, and three non-combinatorial) to read as the mathematicians, except that on the students' handout each proof was labeled an "argument." Again, this is because I didn't want to influence their opinions regarding whether they thought combinatorial proofs constituted convincing or explanatory mathematical proofs or not. The details of these six proofs are given in Table 5.4. As with the mathematicians, I asked the students to read each handout and first give me their overall impression of the arguments they read. Once the students had finished giving me their impressions, I asked them what they thought it might mean for a proof to be convincing, and what they thought it might mean for a proof to be explanatory. Once they gave their answers, I then read them the same definitions of proofs that convince and proofs that explain from Weber (2002) that I had given the mathematicians. I next asked them a series of reflection questions aimed at probing their beliefs about combinatorial arguments as proofs, as well as combinatorial and non-combinatorial arguments (e.g., induction or algebraic) as proofs that explain and proofs that convince.

4.4 Data Analysis

Each interview was transcribed, and then I re-watched and created enhanced transcripts (i.e., containing relevant screenshots from the video-recorded data capturing the participants' work) for each interview. I made note of key episodes related to my research questions and followed the thematic analysis methodology (Braun & Clarke, 2006). Generally, thematic analysis entails the following five phases: familiarizing oneself with the data, generating initial codes, searching for themes, reviewing themes, defining and naming themes in ongoing analysis, and producing the report (Braun & Clarke, 2006, p. 87). To carry out the first four of these phases of analysis for my interviews with the mathematicians, I looked for instances in the data where they described their impressions of combinatorial proof (either a specific proof or combinatorial proof in general) and coded these impressions. Specifically, I coded to what extent the mathematicians felt that combinatorial proofs are proofs that explain or proofs that convince (or neither or both), how combinatorial proof compares to other types of proof (e.g., induction or algebraic), and other interesting themes as they emerged from the data analysis. I also recorded the mathematicians' responses to Weber's (2002) definitions of proofs that convince and explain.

For my interviews with the students, I also looked for instances in the data where they described their impressions of combinatorial argumentation and coded these impressions. I coded to what extent the students felt that combinatorial arguments are proofs that explain or proofs that convince, how combinatorial argumentation compares to induction or algebraic arguments, and other interesting themes as they emerged from the data analysis. I also recorded each students' concept definition (Tall & Vinner, 1981) of a proof, as well as what they thought it might mean for a proof to be convincing and/or explanatory.

Finally, to complete the final two phases of thematic analysis (Braun & Clarke, 2006), I discussed key episodes and findings that were emerging from the initial analysis with my academic adviser, and we reviewed parts of the interviews that warranted additional analysis. We discussed the themes that were being used to ensure that they faithfully represented the data, and any episodes that I found difficult to assign a theme were discussed thoroughly until we both were confident that the theme being applied was appropriate. Finally, in drafting the results and discussion sections of this manuscript I carried out the final step of thematic analysis.

5. Results

In sharing the results, I first briefly offer a broad overview of findings, and then throughout this section I elaborate two overarching themes that emerged in the data: *combinatorial proof as explanatory and convincing* (discussed in Section 5.1), and *combinatorial proof as explanatory but less convincing* (discussed in Section 5.2). In each subsection I further break down and elaborate the results, and I provide evidence from the data to illustrate my findings.

Overall, I observed variety in my participants' perspectives regarding combinatorial proof. While every participant (mathematicians and students) thought that combinatorial proofs were equally or more explanatory than other types of proofs, the extent to which combinatorial proofs were characterized as convincing varied considerably across participants. See Table 5.7 for a summary of participants' opinions about combinatorial proofs as convincing and/or explanatory compared with other types of proof (specifically algebraic and induction proofs). While I generally expected to see some variety of perspectives represented, particularly among the students who have less mathematical experience than the mathematicians, I was surprised so see how much variety there was among the mathematicians regarding their perspectives on whether combinatorial proofs are convincing. Since mathematicians have enough skill and experience to understand that correct combinatorial arguments are rigorous and logically valid (which all the mathematicians in my study did indeed recognize), I expected all eight of them to say that combinatorial proofs are at least as convincing as other types of proof (e.g., algebraic and induction proof). However, this was not the case, and, as I elaborate in Section 5.2.2, the reasons they gave were interesting and offer insight into the nature of combinatorial proof.

Table 5.7. How participants characterized combinatorial proofs as explanatory/convincing.

Which type of proof is more explanatory?	**Expert**	**Student**
Combinatorial proofs are more explanatory than algebraic/induction proofs.	Justice, Robin, Dominique, Jaiden, Ridley, Skyler, Lake	Sydney, Riley, Adrien, Peyton
Combinatorial proofs are less explanatory than algebraic/induction proofs.	-	-
Combinatorial proofs are equally explanatory as algebraic/induction proofs.	Emery	Ash
Which type of proof is more convincing?	**Expert**	**Student**
Combinatorial proofs are more convincing than algebraic/induction proofs.	Dominique, Ridley, Lake	-
Combinatorial proofs are less convincing than algebraic/induction proofs.	Jaiden, Skyler	Sydney, Riley, Adrien, Peyton
Combinatorial proofs are equally convincing as algebraic/induction proofs.	Justice, Robin, Emery	Ash

Before proceeding, I want to make the point that while I refer to students' activity and utterances as related to combinatorial proof, not all of the students at every point in the study felt that combinatorial arguments should be considered rigorous mathematical proof. Whenever the students attempted to connect sequences of assertions using accepted statements (e.g., that $\binom{n}{k}$ counts the number of subsets of size *k* from a set of *n* objects), modes of argumentation (e.g., that an expression involving binomial coefficients has an underlying counting process), and modes of argument representation (e.g., complete sentences stepping through a particular logical progression), I considered this activity to constitute combinatorial proof since it follows the definition of proof I am adopting given by Stylianides (2007). However, readers should keep in mind that the students themselves did not always classify their activity as proof; whenever student work is exemplified in this discussion, I will clarify whether the student at that moment thought their activity constituted rigorous mathematical proof.

5.1 Combinatorial Proof as Explanatory and Convincing

All thirteen student and mathematician participants said that they considered combinatorial proofs to be at least as explanatory as other types of proof, and the reasons that they gave predominately fell into two categories that I will elaborate in the following subsections: *participants viewed combinatorial proof as accessible* and *participants viewed combinatorial proof as tangible*. Additionally, seven of them (six mathematicians and one student) felt combinatorial proofs are equally or more convincing than other types of proof, and I found that the reasons they gave were similar to those they gave regarding why combinatorial proofs are explanatory. This is understandable, as features of a proof which make it a satisfying explanation could also help the reader to be more convinced of the veracity of the statement being proven. In this section, I provide examples from the data illustrating why all of participants found combinatorial proofs to be explanatory (and why the seven aforementioned participants felt combinatorial proofs were convincing).

5.1.1 Participants viewed combinatorial proof as accessible. One reason that arose as to why the participants found combinatorial proofs to be explanatory and convincing is that they thought combinatorial proofs are accessible, in the sense that they are easy to understand and interpret even with relatively little technical mathematical background knowledge or terminology. This finding makes sense, as researchers have described one of the benefits of enumerative combinatorics as a topic is that it does not require calculus or indeed much prerequisite mathematical knowledge at all, making counting an ideal for introducing students to problem solving (e.g., Kapur, 1970). In total, four of the mathematicians and two of the students in my study cited accessibility as a reason why combinatorial proofs were more convincing and explanatory to them than other types of proof.

For instance, one of the students, Ash, said that they felt combinatorial proofs are generally more explanatory than induction or algebraic proofs. When explaining why, they said the following.

> *Ash*: And I use the label elegant because **it could be explanatory to the widest number of possible audiences**. It would be very rare to find a group of people that could not understand the committee concept.
> *Int.*: Sure.
> *Ash*: And so, choosing your committee and choosing the leader of the committee is so universal using that language would explain it to many, many audiences.

In this excerpt, Ash expressed that in their perspective combinatorial proofs are more explanatory than other types of proof, because they employ modes of argumentation that are more universally understood. There is certainly truth to this: it is likely that more people could easily understand the idea of forming a committee rather than an index shift in a summation or algebraic manipulations with factorials, so it seems likely that more readers would find a combinatorial argument explanatory than other types of arguments. (Ash initially did not believe that combinatorial proofs were rigorous mathematical proofs at the start of the interview sequence, but by the time I had the fourth and final interview with Ash, they had changed their mind. I discuss this further in Section 5.2.)

Similarly, one of the combinatorialists who participated in the study, Ridley, said that they felt that the clarity of combinatorial arguments, as demonstrated by non-technical arguments, makes them more convincing. They expanded on this when discussing the combinatorial proof and non-combinatorial (algebraic) proof for Theorem 2, saying, “I would say that this algebraic argument is technical, and technical arguments have a pretty good chance of losing somebody, whereas this argument here [gestures to the combinatorial proof] is not technical.”

Overall, participants who felt combinatorial proofs were more convincing and/or explanatory than other proof methods at times gave very similar reasons, suggesting that convincing and explanatory may not be entirely distinct categories for some students and mathematicians. These data also align with perspectives given by Lockwood et al. (2020) who argued that a mathematical argument can be considered convincing only if it is *personally meaningful* to the reader (p. 180), and similarly that a mathematical argument can be considered explanatory only if it is *personally valuable* to the reader (p. 181). Since many concepts that appear in combinatorial proof (such as forming a committee versus making an index shift in a summation) are accessible to a wide range of audiences, this suggests that readers will be more likely to consider combinatorial proofs as proofs that explain and proofs that convince.

5.1.2 Participants viewed combinatorial proof as tangible. Another similar theme that emerged in these interviews was that combinatorial proofs give the reader something concrete to hold onto mentally, and so are in some sense tangible. This was given by two of the mathematicians and two of the students as a reason for why combinatorial proofs are both convincing and explanatory.

For instance, Peyton (one of the student participants) felt that combinatorial proofs were more explanatory but less convincing than algebraic or induction proofs. Peyton was in fact skeptical throughout all four main interviews that combinatorial proofs were rigorous mathematical proofs at all, but they explained in the following excerpt that they felt the combinatorial proofs gave readers a more tangible argument than other types of proofs:

Peyton:	You know at the start of this, like 4 weeks ago, I didn't like these proofs [combinatorial proofs]. I would've much rather seen these [points to the non-combinatorial proofs] then. Now? Now that my intuition has grown, these kind of suck.
Int.:	Which ones?
Peyton:	The real math ones [points to the non-combinatorial proofs].
Int.:	The real math ones?
Peyton:	Yeah.
Int.:	Can you talk about in what ways they suck?
Peyton:	Well, they suck just because all it's doing in the end is saying this is true. That's what they're doing and I guess that's the point of math is to say this is a thing that holds because yes, it does hold. But these ones, the combinatoric ones, they're saying, okay it probably holds. It does hold, and here's why. **Here is a logical reason that you can physically wrap your mind around**, but this, it's saying, oh, B + A to the N, you can expand that.

I offer a few observations about this excerpt. First, we see some of Peyton's skepticism that combinatorial proofs are rigorous mathematical proofs by characterizing the noncombinatorial proofs as the "real math ones," implying that combinatorial proofs are not "real math." Despite this, though, we can also see that Peyton over time realized that combinatorial proofs had intuitive value for them that they felt other types of proofs did not. They specifically cited combinatorial proofs as using reasoning "you can physically wrap your mind around," and such proofs were hence more explanatory in Peyton's perspective.

Similarly, Justice (one of the mathematicians) felt that combinatorial proofs are more explanatory and equally convincing as other types of proofs, and they also discussed combinatorial proofs as providing something tangible for the reader to think about. When discussing the two proofs of Theorem 1 on the Six Proofs handout, they said, "In the proof by induction, it's great. It's using a great technique of proof, but it doesn't really give you anything to imagine, if you will…. The counting proofs are kind of nice because they give you things to concretely visualize."

I interpret that Justice thought that the tangibility of combinatorial proofs was part of the reason that combinatorial proofs are (in Justice's perspective) both proofs that explain and proofs that convince. Justice's statements also harken back to prior studies that have discussed whether proofs by induction can ever be explanatory (e.g., Lange, 2009; Stylianides et al., 2016).

In summary, while their individual responses to the reflection questions I posed to them were different, the students and mathematicians provided reasons that broadly fell into two categories regarding why they felt combinatorial proofs are both explanatory and convincing. These overarching reasons were that they considered combinatorial proofs to generally be both more accessible and more tangible than other types of proof (such as algebraic proofs or proofs by induction). What I also hope to illustrate in the above two subsections is that for both the students and mathematicians, some characteristics of a combinatorial proof could be viewed as making the proof both explanatory and convincing. This suggests that proofs that explain and proofs that convince should perhaps be considered not distinct, but rather overlapping, characteristics of proof. This is consistent with the views of other researchers who have argued for a nuanced understanding of these labels (e.g. Mingus & Grassl, 1999; G. J. Stylianides et al., 2017; Weber, 2010) rather than seeing them as a distinct, non-overlapping binary categorization.

5.2 Combinatorial Proof as Explanatory but Less Convincing

While many of the participants felt combinatorial proofs were equally or more explanatory and convincing than other types of proof (specifically algebraic proofs and proofs by induction), several of them—including most of the student participants—thought that combinatorial proofs were equally or more explanatory but less convincing than other types of proofs. Specifically, two of the mathematicians and four of the students held this latter perspective. The reasons that some of the participants gave for why they considered combinatorial proofs to be less convincing than other types of proofs fell broadly into two categories: *students doubted combinatorial proofs were rigorous mathematical proof*, and *participants saw potential for difficulties with language and/or counting arguments*. In this section, I elaborate on the reasons they gave for this perspective as they emerged from my data.

5.2.1 Students doubted combinatorial proofs were rigorous mathematical proofs. For the four students (but none of the mathematicians) who said combinatorial proofs are less convincing than other types of proof, their perspectives were tied to the opinions they held regarding whether combinatorial proofs constitute rigorous mathematical proofs at all. Since these four students doubted (to varying degrees) that combinatorial proofs were really mathematical proofs, they felt this made such proofs less convincing than algebraic or induction proofs, which they more readily accepted as constituting a rigorous proof.

For example, as I mentioned in Section 5.1.2, Peyton did not think that combinatorial proofs were rigorous mathematical proofs. During my last interview with Peyton, they gave a nice combinatorial argument for the binomial identity $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$. Specifically, they articulated that each side of the identity could count the number of ways for n out of a set of $2n$ objects to be designated "special," where the left side does this by dividing the set of $2n$ objects into two groups of equal size, and then all cases where a total of n objects are chosen from each group are considered. Despite Peyton's success with this and with proving other binomial identities combinatorially throughout the course of their interviews, they maintained that combinatorial arguments do not constitute proof. In fact, after they successfully proved $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$ combinatorially, we had the following exchange:

Int.: Nice job! So, I'm curious. Yeah, do you think the argument that you just gave, do you think that that is a proof?

Peyton: No, because this is the base … If I did say it was a proof, I could hand this in with two sentences of speech and then get it published, and **I don't think it's formal enough to be a proof.**

Int.: Okay, and do you think the diagram is what makes it informal, or the fact that we're counting makes it informal, or, yeah, why do you think it's not formal enough?

Peyton: **I like to stick with my previous argument for counting proofs not being formal**, and it's basically, what if I'm wrong? There's no math to back me up with it. There's no algebra or induction to do that talks about it.

From these statements, we see that Peyton believed that combinatorial arguments generally cannot be considered formal mathematical proofs, and the reasoning they gave aligns with a *ritual* proof scheme (Harel & Sowder, 1998) since they objected based on the proof's lack of symbols or a recognizable logical structure (such as induction) rather than the correctness of the argument. Nevertheless, since Peyton believed combinatorial proofs do not constitute formal, rigorous mathematical proof, they said this was why they felt combinatorial proofs are less convincing than other types of proof.

Another student who at times seemed uncomfortable with the idea of combinatorial proofs being formal mathematical proofs was Adrien. One identity I asked them to prove was $\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$, which they eventually proved by arguing that both sides count the number of ways to make a subset of size $k + 1$ from a set of $n + 1$ ordered objects (specifically, used the integers 1, 2, …, n, $n + 1$). They were able to successfully argue that the right side of the identity does this using a case breakdown by considering the largest element in the subset. When I asked them to reflect on their combinatorial activity—and particularly whether they felt their argument constituted proof—they said the following.

Adrien: I would prefer induction, because the main thing about this is **it feels like you're actually assigning like a distinct property to the objects, which not every group of objects that you're going to pick *k* from is going to naturally have that kind of property**.

Int.: Right, like if we were picking dots, for example.

Adrien: Yeah.

Witness: Although, they are distinct objects, right?
Adrien: Yeah, they're distinct objects, but that means you'd be putting a pretty arbitrary ranking system on them.
Int.: Okay, and that makes it feel less like it's a valid mathematical proof?
Adrien: **I mean, it feels really arbitrary and the fact that it is arbitrary means that no matter what objects you have, you can just assign this ranking to them, and that'll work.**

Again, from this exchange I infer that Adrien struggled with the idea that a combinatorial proof using ordered objects could be a valid mathematical proof, and they said they would prefer a proof by induction. This is also reflected when I gave them the Six Proofs handout and asked them directly about the construct of a proof that convinces:

Int.: Do you think the arguments that you used to prove binomial identities in the previous interviews are convincing proofs?
Adrien: **You mean the ones where I wasn't actually doing, like, a formal proof, I was making the combinatorial arguments?**
Int.: Yeah, would you say that those arguments are convincing proofs?
Adrien: Some of them were better than others. [There were some] where I understood that it worked, I understood why it worked, but as for what I'd written down, it wasn't quite as rigorous, shall we say?
Int.: What about, say, these three combinatorial arguments? [Gestures toward the Six Proofs handout.] Do you think those are convincing proofs?
Adrien: Sort of, but I don't know, **I prefer the more formal proof in each case**, especially for this one [gestures to the algebraic argument for Theorem 2], since it's like, this one doesn't even have an induction step in it. It's literally just reducing the terms and showing that they have to be equal, and it's like, there's no way to argue with that really.
Int.: Whereas the combinatorial argument might still leave some room for argument?
Adrien: Yeah.

We see from these quotes that Adrien generally found non-combinatorial arguments to be more convincing than combinatorial arguments—both when considering proofs in the Six Proofs handout and when reflecting on their own combinatorial proving activity. We also see that this was connected to their perspective that combinatorial proofs are less formal than algebraic or induction proofs. Again, this was a commonplace perspective among the students, as four out of five of them to some extent expressed that they doubted combinatorial proofs actually constitute formal, rigorous mathematical proof.

5.2.2 Participants saw potential for difficulties with language and/or counting arguments. Finally, both mathematicians and three of the four students who found combinatorial proofs to be less convincing cited the potential for subtle counting mistakes or language issues to arise when constructing a combinatorial argument. This makes sense, as counting problems are notoriously difficult to solve (e.g. Batanero et al., 1997), with one reason being that it is easy to find a solution that seems correct but is actually overcounting or has another subtle flaw (Lockwood, 2014b). While students and mathematicians may feel confident following along and verifying algebraic manipulations or induction arguments, some of the utterances made by some of the student and mathematicians participants indicated they felt less confident verifying counting arguments.

For instance, when I asked Sydney, one of the student participants, a line of questioning aimed at ascertaining whether they felt combinatorial arguments are proofs that convince, they said that combinatorial arguments are "not formal proofs," and that, "It doesn't show the algebraic way through. So, if you had an error in your logic, it's easier to have an unforeseen error in your logic or an unforeseen assumption." I interpret these utterances to mean that they thought algebraic arguments were potentially more reliable than combinatorial arguments, because they thought it was easier to detect an algebraic error than a logical error in a combinatorial argument.

Some of the mathematicians also expressed that they felt less confident verifying combinatorial arguments than other types of arguments, in part because it can be so easy to commit a counting error. Jaiden, for example, expressed this idea in the following interview excerpt:

Int.: Do you think for mathematicians, say, these combinatorial proofs are just as convincing as other types of proofs?

Jaiden: No, I would say not. I remember I had a stats professor in undergraduate and he gave a few homework assignments where he had saved papers where people had made combinatorial reasoning errors, and our homework was to read and see if we could find them.

Int.: Oh, interesting.

Jaiden: And they were really subtle, and **these were professional statisticians who were getting duped**. But he was really good at spotting these. **So, I would say that they're a little bit less convincing.**

Int.: Just because it's so easy to fall into one of those pitfalls?

Jaiden: Yeah.

Jaiden's statements are understandable and make sense in the context of research which has shown counting problems to be notoriously difficult and subtle. It can be easy to come up with a solution to a counting problem that seems correct but that actually overcounts or contains some other slight error (Lockwood, 2014b). For this reason, it is understandable that some of the students and mathematicians in my study felt that this made combinatorial proof less convincing.

Another similar issue that arose for participants was the idea that since combinatorial proofs are entirely comprised of words (instead of containing symbolic manipulation), that makes them potentially less reliable, and hence less convincing, than other types of proof. For example, while they were reading through the Six Proofs handout, one of the mathematicians, Skyler, remarked, "There's more possibility of misinterpreting the English that's written than with the non-interpretive statements that are in the algebraic proof…interpreting words and statements is challenging and difficult, and that's why lawyers make lots of money." Later, I asked Skyler whether they believed combinatorial proofs are more or less convincing than other types of proof, clarifying that by "convincing" I meant "effective at establishing that the theorem is true." They said, "I would say they're less effective," and then elaborated,

Skyler: I think it likely has to do with I feel far more comfortable knowing exactly what a mathematical algebraic statement says and means versus interpreting a written statement.

Int.: Mm-hmm.

Skyler: That is probably what boils down to why I think combinatorial proof is less effective at convincing me of things, is **I don't trust my ability to interpret a sentence as well as I do to interpret a mathematical statement.**
Int.: Yeah, yeah. Like you said earlier, interpreting the statements is hard.
Skyler: Right, right.

From these utterances, I interpret that Skyler was expressing that combinatorial proofs are less convincing to them, because it may be easier to commit an error interpreting words and statements than reviewing mathematical statements involving symbolic manipulations. I think this is an astute observation, and it expresses a broader notion that not just combinatorial statements, but words and statements in general, can at times be tricky and less reliable than mathematical symbols. This may be due in part to the fact that mathematicians define mathematical symbols to have very precise definitions, while words can have potentially different meanings depending on who is interpreting them.

In summary, all the students and mathematicians felt that combinatorial proofs of binomial identities are at least as explanatory as other types of proof (specifically algebraic and induction proofs). This finding is consistent with what Lockwood et al. (2020) hypothesized in their theoretical piece, but it is nevertheless encouraging to see this finding confirmed with interview data involving experienced provers. Further, it is instructive to see the common justifications they gave for why combinatorial proofs may be seen as more explanatory, namely that these proofs are often accessible and tangible. In terms of combinatorial proof as proof that convinces, however, the results were more varied. Some students and mathematicians felt that features of combinatorial proof (such as their accessibility and use of tangible contexts, such as committees) make them at least as convincing as other types of proofs, while other students and mathematicians felt other features of combinatorial proof (such as their use of language and enumerative arguments) make them less convincing than other types of proof. In the following discussion section, I additionally elaborate on two other findings that emerged from my analysis of the data: that most of the mathematicians felt a proof that explains should be rigorous; and that for combinatorialists conducting research in combinatorics, combinatorial proofs may at times be *less* explanatory than algebraic or induction proofs.

6. Discussion

In the previous Results section, I expanded on the mathematicians' and students' perspectives about whether (and why) they considered combinatorial proofs to be proofs that explain and/or convince. In this section, I elaborate on two other findings from my interviews with the mathematicians, which I raise as interesting points of discussion from the data. In Section 6.1 I discuss comments that several of the mathematicians made about their understanding of a proof that explains (and specifically whether such proofs should be considered as rigorous), and in Section 6.2 I describe one mathematician who discussed the fact that at the level of current combinatorics research, they often find combinatorial proofs to be *less* explanatory than other types of proof.

6.1 Mathematicians Thought Proofs that Explain Should Also Be Rigorous

One phenomenon that arose in my interviews was that six of the mathematicians (but none of the students) stated that they disagreed with the idea that a proof that explains "need not be totally rigorous" (Weber, 2002, p. 14). This discussion came up as the participants responded to the definitions of proofs that explain and proofs that convince that I provided them from Weber (2002).

For example, Robin was one mathematician who disagreed with that phrase in the definition of a proof that explains. When they finished reading Weber's (2002) definitions, we had the following exchange.

Robin: I'm unhappy about the wording in proof that explains. I think I understand the distinction that's being made. I'm not happy with the use of rigor in that paragraph.
Int.: Okay.
Robin: **I feel that wording is playing into this mythology that there's only one definition of rigor in mathematics.** And without knowing where the quote comes from, I can't tell for sure, but I would be reluctant, assuming these come from the same source.
Int.: They do.
Robin: Which is what it looks like. From the side of rigorous mathematics, **a proof that explains is being put in a second-class position using the yardstick of rigor, and I don't accept that and don't agree with it**.
Int.: Okay.
Robin: I think the level of rigor is an independent measure that needs to be applied in both cases. But modulo that, I think I understand the point that's being made here. What I think, what I was trying to say earlier, is that **my standard is that I want a proof that both convinces and explains**.

From these quotes (particularly the parts emphasized in bold), we see that Robin felt the wording of Weber's (2002) definition inappropriately put proofs that explain in a second-class position, and they also felt there was a potential for inconsistencies since "rigorous" is a term that may be applied differently depending on who is using it.

In another example from the data, Skyler made similar statements criticizing Weber's (2002) definition of a proof that explains that I had provided.

Skyler: So, to me, the idea that a proof would be labeled a proof but not be rigorous is a contradiction.
Int.: Okay.
Skyler: **I would say it's a well-thought-out explanation, but if it leaves any room for misinterpretation, it's not a proof.**
Int.: Okay.
Skyler: Which would then mean you could make a proof that explains, it just is probably far more verbose than anyone really wanted for then it to be a useful explanation of what's happening. But that's maybe my personal viewpoint on what I think is the value of a proof versus a conversation that explains something. So, **proof should be unequivocally true and rigorous** or, year, true I guess is the best statement. So, it needs to just be absolutely flaw-proof, or I mean flawless.
Int.: Okay.
Skyler: And **if it's not totally rigorous, then I would not use the word proof**.

Int.: Okay. You might use the word, like, argument, say.
Skyler: Exactly, yes. So, that's the only issue I would have with a proof that explains is, just, don't call it proof if it's not rigorous.

Again, we see from these utterances that Skyler believed firmly that a proof must be rigorous for it to be considered a "proof;" if it is not rigorous Skyler would prefer the term "argument" or "explanation."

The point of sharing these episodes is not to criticize Weber (2002), as the succinct, clear definitions he provided of proofs that convince and explain were effective in facilitating fruitful conversations about these constructs from the proof education literature with the mathematician and student participants in the study. Instead, what I mean to highlight is how these episodes illustrate important aspects of the mathematicians' understanding of what it means for a proof to be explanatory. The mathematics education research community has long debated what the role of rigor in proof should be (e.g. Gierdien, 2007; Hanna, 1990; Maher & Martino, 1996), as well as precisely how a proof the explains should be defined (e.g. G. J. Stylianides et al., 2017; Weber, 2010). The fact that the mathematicians expressed that they felt proofs that explain should be rigorous informs and furthers these conversations. Likewise, it is also noteworthy that while most of the mathematicians raised this point in the interviews, not a single student did. While this could simply be due to the small sample size of my study or the fact that students may generally be more hesitant than mathematicians to criticize a definition that the researcher interviewing them provides, it may also be related to the students' hesitancy to consider combinatorial proofs as formal, rigorous proof. Since the students all agreed that combinatorial proofs are at least as explanatory as other types of proof, and since most of the students struggled to accept the idea that combinatorial proof could constitute rigorous mathematical proof, it is therefore not surprising that the students did not seem to take issue with the definition of a proof that explains including the phrase "need not be totally rigorous" (Weber, 2002, p. 14).

6.2 Combinatorial Proofs Can Be Less Explanatory for Research Mathematicians

Finally, one more interesting point arose in my interviews with one of the mathematicians, Ridley, a combinatorialist who was actively conducting research in and teaching combinatorics at the time my interview with them took place. Ridley raised the issue that combinatorial proofs of binomial identities that appear in undergraduate discrete mathematics are usually more explanatory *because instructors choose them to be*. To elaborate, below is the exchange that ensued when I asked them about combinatorial proofs as proofs that explain.

Int.:	Do you think both with these particular examples [in the Six Proofs handout], and then I guess more generally, do you think that combinatorial proofs are more or less explanatory than other proof types, like induction or algebraic?
Ridley:	Explanatory?
Int.:	Yeah. Again, using the definition that we talked about, of giving an intuitive idea of why the statement is true.
Ridley:	Let me think about that. Well, that really depends. If, when you say "binomial coefficient identities," do you mean the binomial coefficient identities that we prove in a first class in binomial coefficient identities? Or are we literally talking about, like, arbitrarily complicated stuff? Because there's a pretty firm delineation in there. **In the first class, we often choose to prove binomial coefficient identities for which the combinatorial proof is clear.**
Int.:	Okay. Yeah.
Ridley:	And the computational proof is a mess. But those are the simplest identities. In general, giving the combinatorial proof is harder. It involves proving a strictly stronger statement. In mathematics, often when you prove a strictly stronger statement, it's strictly harder, right?
Int.:	Right.
Ridley:	So, for the sorts of combinatorial proof you see in a first class, then absolutely this [explanatory] is a feature. But it's, I think the direction of implication is backwards from what you said. **We're picking problems that are easy to see, and the thing that you see is combinatorial. But as soon as you get into proofs that are hard to see, you often have to work really, really hard to come up with that combinatorial proof, and even to read it once it's come up with.**

This was an intriguing insight I had not previously considered. I interpret that Ridley was essentially saying that for the binomial identities we tend to give to students in undergraduate discrete mathematics, the combinatorial proof is often more explanatory than other types of proof, but for more complicated binomial identities—such as those that a mathematician conducting combinatorics research may encounter—the reverse may be true. To try to get greater clarity on the issue, I asked Ridley if integration might be an accurate analogy, because in introductory calculus instructors also purposefully give students antiderivatives that can be computed using certain introductory techniques (substitution, integration by parts, etc.), but it is very easy to write down an antiderivative comprised of elementary functions that is very difficult or impossible to calculate by hand. Ridley answered, "It is an exactly analogous situation." Ultimately though, for the types of binomial identities a student would see in an elementary discrete-mathematics or combinatorics undergraduate course, Ridley felt that combinatorial proofs of those identities are usually more explanatory than other types of proofs. However, it is interesting to get this insight about actual combinatorial research, and to learn that at that combinatorial proofs of more complicated binomial identities may in fact be less explanatory to research combinatorialists than algebraic or induction proofs, at least compared to combinatorial proofs we typically teach to students.

7. Conclusion

To my knowledge, this study is the first of its kind that investigates experienced provers' thinking about combinatorial proof with clinical interviews. To summarize the results, all the mathematicians and upper-division mathematics students in the study viewed combinatorial proofs of binomial identities as equally or more explanatory than other types of proof, such as algebraic or induction proofs. The mathematician and student participants broadly gave two main reasons why they felt this way, including the fact the combinatorial proofs use arguments that are accessible enough that even children could understand them and that they give one something concrete to visualize (such as creating committees). This finding was not necessarily unexpected and aligns with Lockwood and colleagues' (2020) theoretical piece where they stated that combinatorial proofs are usually explanatory in the enumerative representation system. However, it is nevertheless useful to confirm these statements with interview data and to gain insight into some reasons for why combinatorial proof is considered to be explanatory. One interesting caveat did arise in the data though: one of the mathematician participants who was a combinatorialist discussed how in their research, combinatorial proofs of arbitrarily complicated binomial identities can actually be more convoluted and opaque, and therefore less explanatory, than algebraic proofs. Nevertheless, there was broad agreement that combinatorial proofs of binomial identities (at least, those that typically appear in introductory discrete-mathematics and combinatorics courses) are usually more explanatory than other types of proofs.

Even as there was general agreement among the participants that combinatorial proofs of binomial identities are explanatory, however, there were varied opinions regarding the extent to which combinatorial proofs are convincing. Some participants believed that combinatorial proofs are equally or more convincing than other types of proofs, pointing to their correct underlying logical structure and citing similar rationales (i.e., these proofs can be visualized and are accessible) they gave for why they found combinatorial proof to be explanatory. Others—both some student and mathematician participants—felt that combinatorial proofs are less convincing than other types of proofs, and the students and mathematicians gave different but related reasons for this view. The students who felt this way generally doubted that combinatorial arguments should be considered rigorous mathematical proofs at all, either because they lack certain features students expect to see in mathematical proofs (such as symbolic manipulations), or because the students felt that combinatorial arguments couched within a specific context (like counting an ordered set of objects) do not prove the identity with sufficient generality. Some of the mathematicians also felt that combinatorial proofs' use of words and sentences make them potentially less reliable than proofs relying more on symbolic manipulations, and others pointed to the fact that it is easy to produce a counting argument that seems logically correct but that actually contains a subtle error. These results make sense, as it can be more difficult to evaluate written statements than algebraic manipulations, and extensive research has documented that counting problems can be notoriously tricky to solve (e.g., Annin & Lai, 2010; Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Lockwood, 2014a, 2014b).

These results contribute both to combinatorics education and proof education literature. For combinatorics education, they provide new insights into how experienced provers may perceive of combinatorial proof as different from other types of proof, particularly with respect to features such as perceived intuitive value, accessibility, tangibility, rigor, and reliability. These results also further ongoing discussions regarding proofs that explain and proofs that convince (Hersh, 1993). While Lockwood et al. (2020) articulated how combinatorial proofs may relate to these concepts in their theoretical piece, the findings in this paper are based on interview data that investigated how and why students and mathematicians conceive of combinatorial proofs as explanatory and convincing (or not). Applying this lens of explanatory/convincing proof to the new context of combinatorial proof informs ongoing debates about proofs that explain and proofs that convince, including the relationship between proofs that explain and mathematical rigor (e.g. Gierdien, 2007), and the idea that these labels should be applied with respect to a specific audience or representation system (Lockwood et al., 2020).

There are multiple potential implications of this work for both mathematics education researchers and instructors teaching combinatorial proof. For researchers, these findings strengthen the idea that the labels of explanatory and convincing proofs can overlap (Mingus & Grassl, 1999; Weber, 2002) and should be individual-specific (Lockwood et al., 2020; Weber, 2010). In addition, the point of discussion brought up by one of the combinatorialist participants shows the potential for a loss of nuance when these labels are applied to a broad class of proof rather than specific instances of proof. For instructors, these findings highlight some aspects of combinatorial proofs that might be considered when introducing combinatorial proofs to students, such as leveraging the fact that they are accessible and tangible to explain underlying ideas, but also being aware that students may not find such proofs to be ultimately convincing or to "count" as proofs. Perhaps instructors could have direct conversations with students about the nature of combinatorial proofs, explicitly addressing unusual features of these proofs such as their lack of symbolic manipulation and their leveraging of particular contexts.

Regarding limitations as well as avenues for future research, this study did have a sample size of only 13 experienced provers, so I cannot make any generalizable claims about how mathematicians and upper-division students think about combinatorial proofs as proofs that explain and/or convince. However, this work does provide examples of how some provers may think about these concepts, and future studies with larger sample sizes or that look at other populations may find further insights. It also would be interesting to see future research look at bijective combinatorial proof, as this was not a focus of my work and no previous studies on combinatorial proof have focused on these types of proof either.

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Chapter 6 (Paper 2) – Investigating Undergraduate Students' Proof Schemes and Perspectives about Combinatorial Proof as Rigorous Mathematical Proof

Abstract. Combinatorics is an area of mathematics with accessible, rich problems and applications in a variety of fields. Combinatorial proof is an important topic within combinatorics that has received little attention within the mathematics education community, and there is much to investigate about how students reason about and engage with combinatorial proof. Additionally, although Harel and Sowder's (1998) proof schemes have been applied to dozens of studies over the past couple of decades, they have never been used as an analytical lens to examine combinatorial proof. In this paper, I investigate ways students may characterize combinatorial proofs as different from other types of proof using the lens of proof schemes. I gave five upper-division mathematics students combinatorial-proof tasks and asked them to reflect on their activity and combinatorial proof more generally. I found that the students used several of Harel and Sowder's proof schemes to characterize combinatorial proof, and I discuss whether and how other proof schemes may emerge for students engaging in combinatorial proof. I conclude with discussion about implications and avenues for future research.

Keywords: Combinatorics, Combinatorial proof, Proof schemes

1. Introduction

Combinatorics is a branch of mathematics that is increasingly relevant in our society, with applications in computer science, electrical engineering, statistics, and other scientific fields. Combinatorics has other pedagogical benefits as well, such as its accessibility and opportunities for justification and generalization (Kapur, 1970; Lockwood, 2013; Lockwood & Reed, 2016; Maher et al., 2015). One important class of problems in combinatorics is combinatorial proof of binomial identities. These problems often come up in undergraduate discrete-mathematics courses and have applications in number theory, statistics, and other areas, and yet only a few researchers in the field of undergraduate mathematics education have studied them (Engelke & CadwalladerOlsker, 2010; Lockwood et al., in press). A binomial identity is an equation involving binomial coefficients, such as $\binom{n}{k} = \binom{n}{n-k}$ or $n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i} i$, and a combinatorial proof is one that argues for the veracity of an identity by arguing that each side enumerates a (finite) set of outcomes. The validity of a combinatorial proof lies in the fact that a set can have only one cardinality.

For example, consider the binomial identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, also known as Pascal's identity. One possible combinatorial proof of the identity would be to argue that both sides count the number of committees of size *k* that could be formed from a set of *n* people. First, the left side counts this set, because $\binom{n}{k}$ is the number of ways to make an unordered selection of size *k* from *n* distinct objects. The right side also counts this set, and we can see this by considering a case breakdown. To construct the cases needed, we will focus on one of the *n* people, and without loss of generality suppose that this person's name is Nijah. For the first case, consider all committees of size *k* that exclude Nijah. There are then only *n* – 1 people left to choose from, so there are $\binom{n-1}{k}$ ways to construct all such committees. For the second case, consider all committees of size *k* that include Nijah. If she is already on the committee, there are then *k* – 1 positions left on the committee, and there are still *n* – 1 people left to choose to fill those remaining positions. Hence, there are $\binom{n-1}{k-1}$ ways to construct all these committees. Combining these two cases covers all possible committees, and hence we get that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ because both sides count the same set of committees (and this set can have only one cardinality).

Correct combinatorial proofs, such as the one above, suggest an analytical proof scheme (Harel and Sowder, 1998) and are usually considered a proof that explains (in the sense of Hersh, 1993) with respect to an enumerative representation system (Lockwood et al., 2020). They therefore have many of the same pedagogical values as other analytical and explanatory proofs students will encounter in their mathematical careers. However, combinatorial proofs also differ from other types of proof, such as induction or algebraic proofs, in several important ways. One feature of combinatorial proofs of binomial identities is that they are comprised exclusively of sentences and paragraphs, which verbally unpack symbols that appear in the identity without algebraically manipulating those symbols. This feature could have significant implications for students since it has been found that some students are less likely to accept an argument to be a rigorous mathematical proof if it does not contain symbolic manipulations (e.g., Martin & Harel, 1989). Combinatorial proofs are also oftentimes situated within a particular context, such as committees, block-walking, or binary strings. The combinatorial proof provided above is situated in an even more specific context, as it names one of the people being considered (Nijah) within a situation involving committees. While this can help the combinatorial argument to be more intuitive and explanatory, it could also make combinatorial proof seem less like a rigorous proof and more like an intuitive justification (similar to an illuminating example or an illustrative diagram) to a student. While studies have shown the value of examples in the proving process (e.g., Alcock & Inglis, 2008; Alcock & Weber, 2016; Lockwood et al., 2016), it is also the case that intuition and the use of examples are not always nurtured in proof-based mathematics courses (Burton, 1999). The often context-specific nature of combinatorial proof and the lack of symbols raise questions about students' views of the nature of combinatorial proof, such as whether or not students potentially view combinatorial proofs as less rigorous than other mathematical proofs.

Both for those interested in combinatorics education and for those interested in studying students' experiences with proof, it is important to answer questions such as these and to understand other ways that students may view combinatorial proof as different (and potentially less rigorous) from other types of proof. Such information would be valuable for instructors so they can provide adequate support for students in the classroom, and mathematics education researchers who study proof should also be aware of if and how students may think of combinatorial proof as different from other types of proof. However, few studies have investigated combinatorial proof at all, and none have looked at these types of queries. In this paper, I report on interviews conducted with five undergraduate students who had experience with mathematical proof with the goal of investigating some of these gaps in the literature using a well-established theoretical perspective: Harel and Sowder's (1998) proof schemes.

In addition to learning more about students' views of combinatorial proof, I additionally hope that my research can inform the field of proof education more broadly. Harel and Sowder's (1998) proof schemes have been used by dozens of researchers (e.g., Blanton & Stylianou, 2014; Çontay & Duatepe Paksu, 2018; Ellis, 2007; Fonseca, 2018; Healy & Hoyles, 2000; Housman & Porter, 2003; Jankvist & Niss, 2018; Kanellos, 2014; Koichu, 2010; Liu & Manouchehri, 2013; Gülcin Oflaz et al., 2016; Ören, 2007; Pence, 1999; Sen & Guler, 2015; Şengül, 2013), and yet the framework has never been applied to combinatorial proof. By applying this broadly recognized framework to a new mathematical setting, I hope to contribute to both combinatorics education and proof education literature by answering the following research questions:

1. What proof schemes do undergraduate students use to discuss and characterize combinatorial proof (including how it may differ from other types of proof)?
2. What insights about the nature of combinatorial proof do these proof schemes afford for the research community?

In the following sections, I situate my investigation in the existing relevant proof and combinatorics education literature.

2. Literature Review and Theoretical Perspectives about Proof Schemes

2.1 Characterizing Proof in This Paper

I intend the research presented in this paper to explore how combinatorial proof differs from other types of proof, and so it is necessary to discuss what I take to constitute proof. In their introduction to the proof schemes framework, Harel and Sowder (1998) define proof as "a deductive process where hypotheses lead to conclusions" (p. 234). This broad definition (and others similar to it) may encompass many types of arguments. Indeed, there is debate in the mathematics education community surrounding how broad the definition of proof should be, such as whether a proof by picture should constitute mathematical proof (e.g., Gierdien, 2007). Because I intend to understand how undergraduate mathematics students may conceive of combinatorial proof as different from other types of proof, it is important for me to use a definition of proof that considers what a particular community (such as undergraduate mathematics students) accepts as a mathematical proof. Several researchers have incorporated this concept into their definition of proof (e.g. Hanna, 1990; Raman, 2003), rather than adopt a more objective definition of proof. In this study, I will use a definition of proof articulated by Stylianides (2007, p. 291, emphasis in original).

> *Proof* is a *mathematical argument*, a connected sequence of assertions against a mathematical claim, with the following characteristics:
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community.

In the setting of my study, I use the broader term "community" rather than "classroom community" to encompass other settings in which the undergraduate participants in my study may encounter proof.

2.2 Proof Schemes

There have been few studies that have investigated combinatorial proof at the undergraduate level (e.g., Engelke & CadwalladerOlsker, 2010; Lockwood et al., in press), and there are not existing frameworks for characterizing and analyzing combinatorial proof. It is natural, therefore, to go beyond these few studies by examining student thinking about combinatorial proof through a well-recognized, robust proof framework that has effectively been used within the mathematics education literature more broadly. Harel and Sowder's (1998) proof schemes framework has been used by many researchers to study proof comprehension and/or proof production in children, undergraduate students, and pre- and in-service teachers in several mathematical areas (e.g., Blanton & Stylianou, 2014; Çontay & Duatepe Paksu, 2018; Ellis, 2007; Fonseca, 2018; Healy & Hoyles, 2000; Housman & Porter, 2003; Jankvist & Niss, 2018; Kanellos, 2014; Koichu, 2010; Liu & Manouchehri, 2013; Gülcin Oflaz et al., 2016; Ören, 2007; Pence, 1999; Sen & Guler, 2015; Şengül, 2013). The fact that many researchers have used this framework in a variety of content areas and with different populations speaks to its broad applicability and utility in characterizing proof in mathematics education. However, perhaps in part because combinatorial proof has not been studied extensively to date, no researcher has previously applied Harel and Sowder's framework to combinatorial proof. Given the demonstrated widespread use of this framework, I chose to use it to analyze my data on students' thinking about combinatorial proof, believing that it would be productive to interpret combinatorial proof through the well-known lens of proof schemes. Further, because it is a widely used and accepted perspective, I suggest that it can help to better inform how combinatorial proof compares to other kinds of mathematical proof, which is related to my first research question. Finally, in addition, I posit that this research can contribute to the large existing body of work that uses proof schemes as a lens to understand proof in mathematics education. My examination of combinatorial proof can provide insight into how the proof schemes framework might be applied to a type of proof to which it has not previously been applied.

In what follows, I elaborate aspects of the proof schemes framework that are relevant in this paper. Harel and Sowder (1998) contended that generally there are three non-mutually exclusive proof schemes that a prover can hold (each of which have subcategories): *external conviction*, *empirical*, and *analytical* (see Figure 6.1). Harel and Sowder considered these proof schemes to represent hierarchical cognitive stages in a student's mathematical development, with external conviction being the least sophisticated and analytical being the most sophisticated. I describe the main categories in the following subsections, highlighting particular categories that will come up in the Results section of this paper. My purpose in describing these is to provide context for the reader, as I will refer to these proof schemes as I discuss my analysis and frame my results. For each proof scheme, I also comment on whether or not I had expected the proof scheme to come up in relation to combinatorial proof. I present them in the order they are listed in Figure 6.1.

2.2.1 External conviction proof schemes. Generally, Harel and Sowder (1998) characterized external conviction proof schemes as describing situations where students' doubts are removed by the presence of certain ritualistic characteristics of an argument, the word of an authority, or the symbolic form of an argument. These three situations correspond respectively with the ritual, authoritarian, and symbolic proof schemes.

Expanding on each of these a bit further, a student may exhibit a ritual proof scheme if they are convinced by the appearance of an argument rather than its actual correctness – perhaps saying something like, "it just looks like a proof" – because it aligns with what they think a proof should entail or it has components of a proof (such as symbols or a certain logical structure) that make it appear true. Conversely, Harel and Sowder (1998) also explained that a ritual proof scheme would describe situations in which students doubt the veracity of a proof if it does not include symbolic

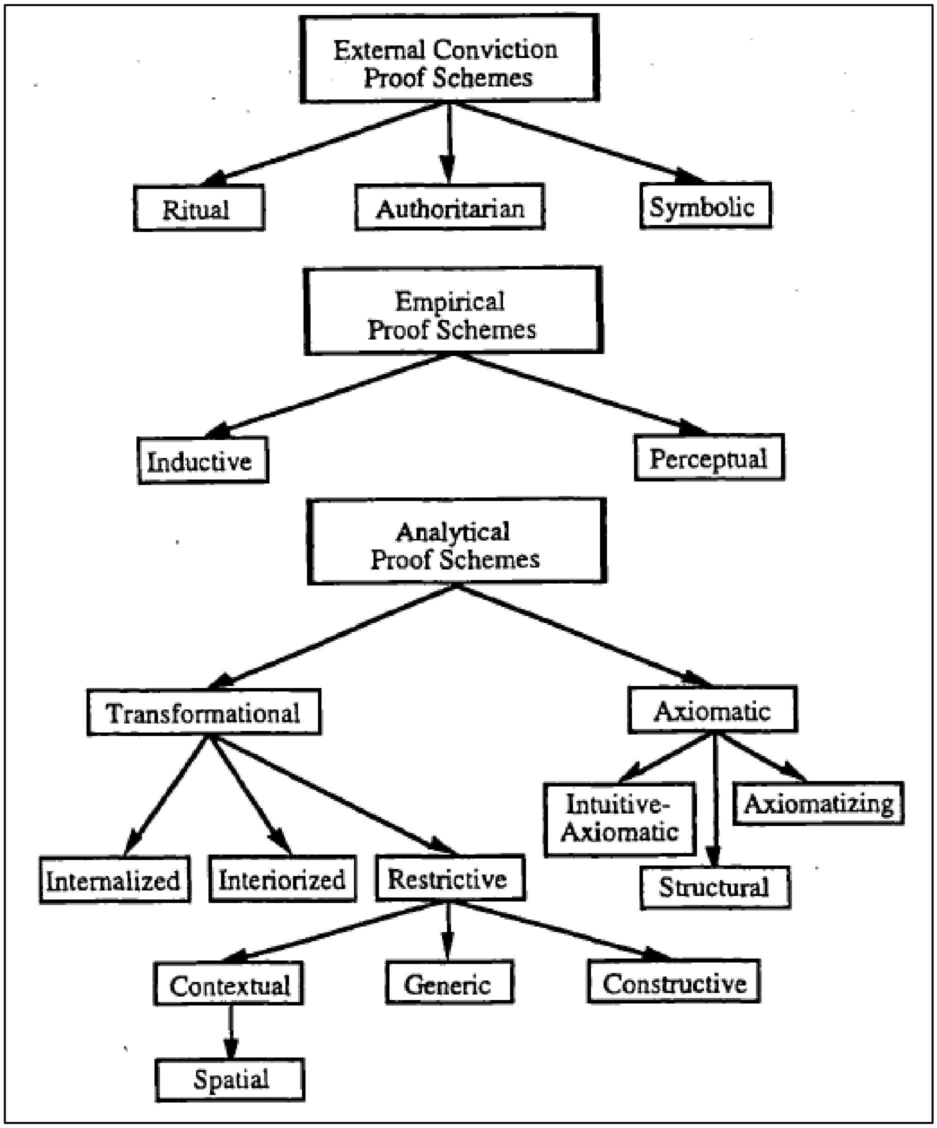

Figure 6.1. *Harel and Sowder's (1998) proof schemes framework (p. 245).*

expressions or computations. Ultimately, the idea is that a student may view proof in terms of the extent to which it aligns with their view of what a proof should entail, which typically includes symbols and logic. Viewing proof in this way could certainly come up if a student is reading or producing a combinatorial proof, which usually consist only of words with no symbolic manipulations at all. Students with ritual proof schemes may thus be confronted with determining how a combinatorial proof might or might not constitute a valid proof as it does not align with common examples of proofs students might have seen.

A student may exhibit an authoritarian proof scheme if they believe a mathematical statement is true based solely on the word of an authority figure (for example, a textbook or teacher). While it is not unreasonable, or even necessarily always a bad thing, for students to trust their mathematics teachers or textbooks, issues can arise when students using this proof scheme more generally approach mathematics as a collection of facts handed down by an authority that do not require intrinsic justification. This could describe a scenario in which a student was given a combinatorial proof of a binomial identity and said something like, “This proof is valid, because I saw my teacher present it on the chalkboard,” rather than demonstrating an understanding of why the combinatorial proof is valid or invalid.

Finally, a student who produces or conceives of proof as mere “symbol pushing” with no need to define the symbols that are used or understand the mathematical properties of the objects they represent can be described as utilizing a symbolic proof scheme. This proof scheme is similar to Weber and Alcock's (2004) concept of *syntactic proof production*, which describes students attempting to prove by manipulating symbols with no understanding of what these manipulations actually mean. As Weber and Alcock (2004) and Harel and Sowder (1998) pointed out, the symbolic proof scheme is not always unproductive, and can even sometimes be a powerful proving technique. However, too often students may use this proof scheme while, as Harel and Sowder (1998) put it, “[t]hinking of symbols as though they possess a life of their own without reference to their possible functional or quantitative reference” (p. 250). For my study, I did not expect this proof scheme to come up for students, since, as stated previously, combinatorial proof does not involve symbolic manipulation.

2.2.2 Empirical proof schemes. While empirical proof schemes may suggest perspectives that more closely reflect a mathematical proof than external conviction proof schemes, they still fall short of deductive reasoning. Harel and Sowder (1998) described two types of empirical proof schemes, inductive and perceptual proof schemes.

When students are convinced of the veracity of a statement by quantitatively evaluating the statement for one or more specific cases, they are using an inductive proof scheme. This could occur, for example, if a student is convinced of the veracity of the binomial identity $\sum_{i=0}^{n}\binom{n}{i} = 2^n$ by numerically checking that the equation holds when $n = 3$. In my study, I did not expect this proof scheme to appear, as my participants were either producing general enumerative arguments (and not plugging numerical values into the binomial identity) or evaluating complete algebraic, induction, and combinatorial proofs that were written generally (see the Methods section for more details on the proofs that the students in my study evaluated).

The second type of empirical proof scheme that Harel and Sowder (1998) discussed is perceptual proof schemes, which they described as when students make perceptual observations about the statement they are proving "by means of rudimentary mental images—images that consist of perceptions and coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation" (p. 255). Harel and Sowder illustrated this proof scheme by describing a situation in which a student is convinced of the veracity of a geometry proposition about isosceles triangles by sketching several example triangles. The perceptual proof scheme may be used to describe situations in which a student is able to conceive intuitively why a theorem holds, but they do not demonstrate the ability to translate this intuition into a rigorous proof. This could also be related to Raman and colleagues' (2009) idea of the context of discovery and the context of justification as two distinct phases in the proving process. For my study, it is trickier to imagine how this proof scheme may apply to combinatorial proof of binomial identities, as "perceptual" often refers to something visual, and combinatorial proofs consist of written sentences. However, I hypothesize that perhaps the perceptual proof scheme could describe a situation where a student accepts a combinatorial proof as providing only an intuitive justification for a binomial identity, but one that falls short of an analytical proof. I discuss this idea in more depth in the Section 5.1.5.

2.2.3 Analytical proof schemes. Harel and Sowder (1998) stated that analytical proof schemes validate conjectures by means of logical deductions (p. 258). These would include arguments that would generally be accepted as rigorous mathematical proofs. Harel and Sowder also break analytical proof schemes down into two classes of proof schemes, transformational and axiomatic, which I expand on below.

A student can be considered to be applying a transformational proof scheme if they perform meaningful, goal-oriented operations on mathematical objects and anticipate those operations' results. This proof scheme describes situations in which students understand and use meanings of and relationships among mathematical objects to unpack a statement to be proven, making deductions that conclude with the result to be proven. Harel and Sowder (1998) further broke this proof scheme down into two cognitive levels—internalized and interiorized proof schemes—to describe the varying level of awareness and reflection students may demonstrate in their proving activity (I do not detail distinctions between these here, as they did not emerge as meaningful distinctions in my data). In the case of evaluating combinatorial proof, a student may be using a transformational proof scheme if they recognize that a correct combinatorial proof can constitute a valid mathematical proof of a binomial identity. In combinatorial proof production, a student may be applying a transformational proof scheme if they interpret the expressions in a binomial identity as counting sets of outcomes (meaningful, goal-oriented operations) in anticipation that both sides of the binomial identity count the same set of outcomes in different ways[10].

In their research, Harel and Sowder (1998) also found that some of the students they studied exhibited transformational proof schemes when there was some restriction on the statement being proven. For example, a student may be given a statement about vector spaces, and they may give a transformational proof that is restricted to the vector space $\boldsymbol{R}^n$. This would be an example of a contextual restrictive transformational proof scheme. Similarly, in the case of students' evaluation of combinatorial proof, I hypothesized that one way that this proof scheme may arise is if a student accepts a combinatorial proof which utilizes, say, committees, as only constituting a mathematical proof in the restrictive context of committees, and that the combinatorial proof is insufficient to prove the binomial identity more generally. As I discuss in Section 5.1.3, these combinatorial proofs can prove binomial identities that are stated generally (this is based on the fact that a set can have only one cardinality), but students who may hear in their classes that "examples aren't proofs!" may incorrectly believe that a combinatorial proof proves only one restrictive case of a given binomial identity.

Harel and Sowder (1998) also discussed two other types of restrictive transformational proof schemes that students may use. One is the generic proof scheme, which describes situations in which students interpret a statement generally but are only able to express their proof in a particular case. Harel and Sowder used the example of a student proving a general statement about whole numbers divisible by 9 by considering the number 867 and then indicating that their process could

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[10] Researchers (e.g. Lockwood et al., in press) and discrete-mathematics textbook authors (e.g. Rosen, 2012) have also described combinatorial proofs which argue that each side of a binomial identity counts a different set of outcomes and then constructs a bijection between these sets. Such combinatorial proofs are sometimes called *bijective* proofs. I do not focus on these types of combinatorial proofs in this paper.

be applied to any whole number. While no instance of this occurred in my data, in Section 5.2 I address some ways in which a generic proof scheme may come up in the context of combinatorial proof. The other type of restrictive transformational proof scheme students may use is the constructive proof scheme. This applies to instances in which students believe that proofs of existence statements must be constructive (and, for example, that proofs by contradiction of existence statements are invalid). While it may be possible, it is unlikely that constructive transformational proof schemes would come up in the case of students producing or evaluating combinatorial proofs.

Finally, Harel and Sowder (1998) also expanded on axiomatic proof schemes, which a student may be using when they understand "that at least in principle a mathematical justification must have started originally from undefined terms and axioms" (p. 273). This proof scheme can be used to describe situations where students acknowledge and understand axioms in their proof activity, from axioms as intuitive as the commutativity of addition of real numbers to the Axiom of Choice. I do not expand on axiomatic proof schemes in this paper, because none of the participants in my study referred to any axioms in their proof production or evaluation of proof.

In conclusion, Harel and Sowder's (1998) proof schemes comprise a detailed framework that attempts to categorize not only students' proving efforts but also prevailing ways in which they think about proof. Their proof schemes include common reasoning mistakes students make (for instance, accepting a proof based on its ritualistic features rather than its correctness) as well as productive approaches that can lead to valid mathematical proofs. Even though it is a robust, detailed framework that researchers have used to analyze student thinking about proof for nearly the past three decades, it has never been used as a lens to study combinatorial proof. In my review above of all the main categories of proof schemes, I noted that I did not expect some to arise within the context of combinatorial proof, but the reader should note that certain other proof schemes will be particularly important as they arise within my Results and Discussion and Conclusion: authoritarian, ritual, perceptive, transformational, and contextual restrictive. In the next section, I contextualize my work within other studies that have applied Harel and Sowder's framework, and then in the Methods section I describe how I used the framework for this study.

2.3 Relevant Literature: Studies That Have Applied the Proof Schemes Lens

In this section, I elaborate some of the literature that have used the proof schemes framework. As noted, Harel and Sowder's proof schemes framework has been adopted by researchers who have used the framework as a lens to study various aspects of provers' activity and proof comprehension, examining populations ranging from school children (e.g., Ellis, 2007; Fonseca, 2018; Jankvist & Niss, 2018; Kanellos, 2014; Ören, 2007) to undergraduates (e.g., Blanton & Stylianou, 2014; Koichu, 2010) to preservice or in-service teachers (e.g., Çontay & Duatepe Paksu, 2018; Gülcin Oflaz et al., 2016; Gülçin Oflaz et al., 2019; Pence, 1999; Şengül, 2013). My purpose in this section is to demonstrate that proof schemes have in fact been used to study a variety of contexts and populations in math education, and it is a robust and well-tested framework that is appropriate to use in analyzing my data. I particularly want to highlight some of the different settings and purposes for which researchers have used proof schemes. My goal is to situate my work within existing literature, both because I want to clarify how I am using the proof schemes framework (and ways in which my use is similar to or different from ways other researchers have used it), and because I want to demonstrate that my contribution is novel but builds on a rich body of existing work.

Some researchers have used proof schemes to categorize students' proving activity. For example, Kanellos, Nardi, and Biza (2018) looked at the proof schemes employed by 85 high-school students in Greece who were asked to provide proofs of statements the context of algebra and geometry. They found that students may use different proof schemes depending on the proposition they are given to prove, and they also identified eight combinations of proof schemes that emerged in their data. The identification of combinations of proof schemes is not unheard of, since Harel and Sowder (1998) noted that,

> “A given person may exhibit various proof schemes during one short time span, perhaps reflecting her or his familiarity for, and relative expertise in, the contexts, along with her or his sense of what sort of justification is appropriate in the setting of the work" (p. 277).

Kanellos et al. (2018) additionally proposed an extension of Harel and Sowder’s (1998) proof schemes framework to precisely classify combinations of proof schemes that students may exhibit while engaging in proof activity.

Other researchers have used proof schemes to try to get a better understanding of how students broadly discuss and think about proof. For instance, Otten (2010) described classroom dialogues about proof in beginning algebra, a course that nearly every secondary student takes. Otten used Harel and Sowder’s (1998) proof schemes to analyze the classroom discussions about proof, and Otten contended that exposing analytical proofs to students earlier in their mathematical careers may help to undo the potentially harmful yet pervasive idea that mathematics is merely a collection of facts and procedures received from an authority figure. Others still have used proof schemes to better understand how proof is presented and framed for students. For instance, Stacey and Vincent (2009) utilized the framework to conduct a textbook analysis of nine 8th-grade textbooks from Australia covering algebra and geometry topics.

Researchers have also used Harel and Sowder’s (1998) proof schemes framework to look at students’ reasoning about various aspects of proof and proving activity. For example, some studies used proof schemes to frame students’ justification activity. Sevimli (2018) used Harel and

Sowder's (1998) framework for studying undergraduate students' justification capabilities in the contexts of continuity, differentiability, and integrability. Koichu (2010) also used proof schemes to describe a case study of a postsecondary student's problem-solving activity in the domains of calculus, algebra, and geometry. In another study that used Harel and Sowder's (1998) framework to study the proof schemes used by in-service teachers, Soto (2010) presented a case study of one in-service secondary mathematics teacher. Soto used the proof schemes framework to analyze this teacher's reasoning and justification while solving story problems. Sen and Guler (2015) applied Harel and Sowder's (1998) proof schemes in an even more fine-grained manner looking at the proving and justification skills of seventh-grade students. They asked these students to reflect on their proofs, specifically whether they felt their arguments would convince someone else. In this study, Sen and Guler coded not only the students' proofs but also their utterances during the interviews using proof schemes. For example, the following exchange was coded:

Interviewer:	Do side lengths have to increase for increase in perimeter?
S (AO):	It is a must, because perimeter is found by adding side lengths. Then it is a must for side lengths to increase for the lengths of perimeter to increase.
S (AR):	Increase in the perimeter depends on increase in the lengths of sides. According to the answer I provided, it is a must for side lengths to increase for perimeter to increase. In the solution I made, I think I increased side lengths. For example, here a=10 cm, there are aa=20 cm, it enlarged."
S (AS):	As I think increase in perimeter is related with the enlargement of the shape, side lengths are to increase absolutely.

Here, the authors used the codes AO, AR, and AS to denote that the student was using reasoning that aligns with the authoritarian, ritual, and symbolic proof schemes, respectively.

Other studies have used the framework to look at students' *proof comprehension* by asking them reflective questions about existing proofs in addition to or instead of looking at students' proof production. Sometimes this involves students explicitly reflecting on proofs they have already produced, and sometimes this involves students evaluating and interpreting existing proofs that did not originate with them. As examples of the former, Housman and Porter (2003) investigated the proof schemes of above-average mathematics majors in topics that included discrete topics such as set theory. They had these mathematics students write proofs and then reflect on how convincing they felt their own proofs were. Housman and Porter (2003) stated that these questions were necessary to verify the students' proof schemes because, "[A] proof scheme, by definition, consists of the arguments that a person uses to convince herself and others of the truth or falseness of a mathematical statement" (p. 143). Sears (2019) conducted a similar study in which they asked preservice middle- and high-school teachers to prove propositions in topics such as set theory and then evaluate their own proofs. While these studies considered what the participants found personally convincing, they both answered research questions primarily aimed at understanding proof schemes that participants adopt. Healy and Hoyles (2000) also used the lens of proof schemes to look at proof production and proof comprehension of high-achieving 14- and 15-year-old students in the context of algebra and geometry. The authors found that the students still predominantly used empirical proof schemes when writing their own proofs, even though the students were aware of the limitations of empirical arguments and acknowledged that they felt these arguments were not sufficiently rigorous to receive high marks from a teacher. Healy and Hoyles thus posited that students may simultaneously hold two different conceptions of proof: arguments they considered would receive the best score from an instructor (which were more likely to be algebraic), and those students would adopt for themselves.

There have also been some studies in which researchers used Harel and Sowder's (1998) proof schemes framework to look at how students evaluate arguments presented to them for a proposition to be proven. Harel and Sowder (1998) themselves described an example of this when exemplifying the authoritarian proof scheme (p. 249-250), and other researchers have presented students with arguments falling under different proof schemes (e.g. external, empirical, and analytical) and asked if the students felt these arguments were convincing. For instance, Plaxco (2011) incorporated this methodology to investigate the proof schemes of undergraduate students in multiple mathematics topics including elementary number theory and geometry. Additionally, Liu and Manouchehri (2013) asked 41 middle-school children to justify propositions, and then they gave each child a set of arguments for those propositions asking which arguments the child preferred. Only 11 of the children gave analytical justifications, which aligns with other similar research (e.g. Blanton & Stylianou, 2014) finding that students with little training in proof tend to use external and empirical proof schemes. Liu and Manouchehri also found that when the children were evaluating the arguments given to them, they generally preferred those which followed an analytical proof scheme. As Liu and Manouchehri (2013) stated, “This hints at the notion that students did tend to recognize and endorse more general mathematical explanations, even if they could not produce them themselves” (p. 28), which is also consistent with previous findings (e.g. Healy & Hoyles, 2000).

While I identified no studies in the literature that used Harel and Sowder’s (1998) proof schemes to look at combinatorial proof, there were a few that used the framework to look at undergraduate students’ proof activity in other areas of discrete mathematics. I elaborate some of these here to frame my contribution to the literature. In one instance, Blanton and Stylianou (2014) analyzed student assessments, in-class group-work activity, and full-class discussions for 30 undergraduate students enrolled in an introductory proof course that covered elementary number theory and some abstract algebra. They found that students who had minimal proof training at the college level may tend to apply external and empirical proof schemes, but their findings showed that with instruction students were able to shift to more sophisticated analytical proof schemes. In another study, Stylianou, Chae, and Blanton (2006) used proof schemes as a lens to study the relationship between undergraduate students' proof-production and problem-solving strategies in a discrete context. Perhaps unsurprisingly, they found that there appeared to be a relationship between students' proof schemes and problem-solving strategies, with analytical proof schemes correlating with more productive problem solving. Recio and Godino (2001) also looked at first-year undergraduate students in Spain in elementary number theory and geometry contexts to investigate the relationship between a student's proof schemes and *institutional meanings* of proof, that is, ways in which various institutions of which students were members (e.g. mathematics classes, daily life, etc.) may treat the concept of proof. Recio and Godino (2001) contended that there was a two-way relation of influence between personal proof schemes and institutional meanings of proof, proposing a more positive perspective on students who may initially rely more on empirical proof schemes. They stated, "Informal proof schemes should not be considered as simply incorrect, mistaken or deficient, but rather as facets of mathematical reasoning necessary to achieve and master mathematical argumentative practices" (p. 97). These are some examples of studies from the existing literature where researchers have applied Harel and Sowder's (1998) proof schemes framework to categorize students' proving activity in various contexts, including discrete mathematics. However, currently there are no studies which apply the framework to student thinking about combinatorial proof.

The point of this summary is to highlight the myriad ways in which proof schemes have been used and applied in mathematics education literature. Researchers have applied Harel and Sowder's (1998) proof schemes to look at students' proving activity, and some have looked at students' reasoning about proof or pursued other related avenues of research. While much work has been done, especially in the areas of algebra, geometry, and some topics of discrete mathematics, no prior studies have looked at students' reasoning about combinatorial proof using proof schemes as an analytical lens. Furthermore, while some work has been conducted discussing ways that combinatorial proof may differ from other types of proof (e.g. Lockwood et al., 2020; Lockwood & Reed, 2018), none have approached this research inquiry in a systematic fashion that utilizes a robust, well-known theoretical lens such as Harel and Sowder's (1998) proof schemes. Thus, I attempt to contribute to the literature on proof and proving, particularly by applying proof schemes within a novel kind of proof activity and domain (combinatorial proof), and by adding a more rigorous analysis to the continuing investigation into how combinatorial proof differs from other types of proof.

3. Literature Review & Theory about Combinatorics

In this section, I transition from discussing the necessary proof literature for this research to delving into literature on combinatorics, and specifically combinatorial proof. While several mathematics education researchers have looked into various aspects of combinatorics as a mathematical topic—including students' verification strategies as they solve counting problems (Eizenberg & Zaslavsky, 2004), types of errors students commit as they solve counting problems (Batanero et al., 1997), a set-oriented perspective and listing as potential avenues for students' combinatorial success (Lockwood, 2014; Lockwood & Gibson, 2016), and investigations into fundamental counting principles (Lockwood et al., 2017; Lockwood & Purdy, 2019a; Lockwood & Purdy, 2019b)—the most relevant literature to my research are the (limited) studies that have looked at combinatorial proof, as well as Lockwood's (2013) model of students' combinatorial thinking. In the following subsections, I use Lockwood's model to characterize how one produces a combinatorial proof of a binomial identity, and I discuss the few other studies I identified in the literature which have also targeted combinatorial proof.

3.1 Lockwood's (2013) Model and Characterizing Combinatorial Proof

Lockwood (2013) said there are three components that may be present in a student's reasoning about a counting problem: *sets of outcomes*, *counting processes*, and *formulas/expressions*. See Figure 6.2. Sets of outcomes represent collections of objects that are enumerated, which also encompasses different ways those objects may be represented or "encoded" (Lockwood et al., 2015a). Examples may include representing outcomes as binary strings or as sequences where order of the items does not matter. Counting processes describe the mental or physical operations a counter uses to generate or enumerate sets of outcomes. For instance, this could include use of the Multiplication Principle or constructing a case breakdown. Finally, formulas/expressions include mathematical expressions whose numerical value(s) are the cardinality of the set of outcomes being enumerated. These are often considered the "answer" to the counting problem.

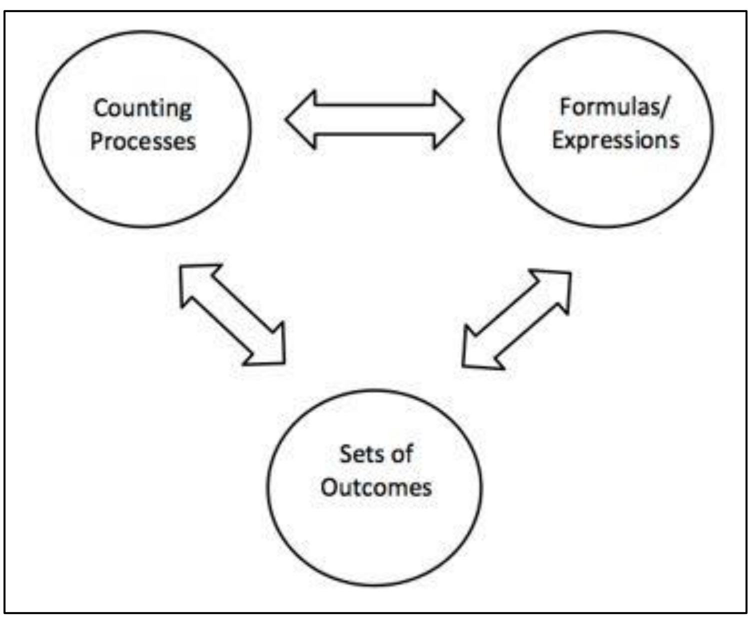

Figure 6.2. Lockwood's (2013) model of students' combinatorial thinking (p. 253).

Between each of these components there are also bidirectional relationships that Lockwood (2013) and Lockwood, Swinyard, and Caughman (2015b) described. For example, a counting process which involves picking a committee (i.e., an unordered selection) of four people from a set of fifteen people and then picking one of those four people to be the chairperson of the committee would yield the expression $\binom{15}{4} \times \binom{4}{1}$. Similarly, and critically for combinatorial proof, a given expression may suggest a particular underlying counting process. The expression $\binom{n}{1} \times \binom{n-1}{k-1}$, for instance, may suggest a counting process in which 1 object is selected first from a group of n distinct objects, and then an unordered selection of $k - 1$ objects is made from the remaining $n - 1$ objects. Other mathematical operations can suggest different underlying counting processes; for example, addition may indicate a counting process involving a case breakdown. In this way, I use Lockwood's (2013) model to characterize combinatorial proof, as the language of counting processes, sets of outcomes, and formulas/expressions are particularly well-suited to describing the steps involved in completing a combinatorial proof.

Several researchers have applied Lockwood's (2013) model to investigate various aspects of student thinking about counting problems (e.g. Halani, 2013; Hidayati et al., 2019; Lockwood, 2014; Lockwood et al., 2018; Lockwood & Erickson, 2017; Lockwood & Gibson, 2016; Lockwood & Purdy, 2019a). Only one previous study has applied her model to examine student thinking about combinatorial proof specifically (Lockwood et al., in press), and in my study I broadly used Lockwood's model in a similar manner. In the next subsection I review all pre-existing literature targeting combinatorial proof, including the aforementioned study.

3.2 Prior Literature on Combinatorial Proof

There have been a handful of studies which have looked at students' reasoning and combinatorial justification related to binomial identities. For instance, Maher and Speiser (1997) provided a case study in which a 14-year-old student made meaningful mathematical connections between Pascal's triangle and binomial coefficients to explain why Pascal's addition identity, $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, holds. Tarlow (2011) similarly described high-school students who could articulate combinatorial justifications for Pascal's identity using the context of pizza toppings, and Speiser (2011) also provided an example of a high-school student justifying Fermat's formula, $\binom{n}{r+1} = \frac{n-r}{r+1} \times \binom{n}{r}$, using red and yellow blocks. While these looked more generally at students' justification activity rather than specifically student thinking and work on combinatorial proof tasks, they nevertheless demonstrate that students—even children—are capable of understanding binomial identities and successfully engaging in combinatorial justification.

I identified only three prior studies in the literature that have looked at combinatorial proof of binomial identities specifically. The first of these was a study conducted by Engelke and CadwalladerOlsker (described both in Engelke and CadwalladerOlsker 2010, and Engelke Infante and CadwalladerOlsker, 2011), who looked at upper-division undergraduate and graduate students' written solutions to combinatorial proof problems on exams. In their study, they rated the students' proofs on a scale from 1-4 based on how successful the proofs were, and they categorized four difficulties that they observed students seemed to struggle with when coming up with a combinatorial proof: *language mimicking*, *inflexibility of context*, *misunderstanding of combinatorial functions*, and *failure to count the same set* (p. 95-96). They additionally found some evidence that having students ask a specific "How many…?" question may help students be more successful at completing a correct combinatorial proof, and they also posited that some students may engage in *pseudo-semantic proof production*. Engelke Infante and CadwalladerOlsker defined this term as "the attempt to engage in a semantic proof production process, but relying on the syntax of a previously encountered proof when faced with a term that the student cannot explain" (2011, p. 96), which is based on the distinction between semantic and syntactic proof production articulated by Weber and Alcock (2004).

More recently, Lockwood, Caughman, and Weber (2020) wrote a theoretical piece that focused on giving researchers tools and insights to more effectively understand and use the constructs of *convincing* and *explanatory proofs* (in the sense of Hersh, 1993), and they illustrated their theory by applying it to combinatorial proof. They defined an argument as explanatory to an individual within a particular representation system if it "begins with axioms, definitions, or statements the individual believes are true," "employs inferential schemes that are natural," and "is couched within, or can be mapped to, a representation system the individual finds personally valuable" (p. 181). By *representation system*, Lockwood et al. (2020) meant, "a structure with permissible configurations and inferential schemes" (p. 178), where *permissible configurations* could encompass certain equations, graphs, and other types of mathematical objects. With this terminology, Lockwood et al. argued that depending on the reader, combinatorial proofs are generally considered explanatory proofs within the *enumerative representation system*, because they can explain why a binomial identity holds *combinatorially* (but they do not explain algebraically, for instance, why a binomial identity holds).

Finally, the most recent study I identified in the literature targeting combinatorial proof—and the only prior study based on interview data—was conducted by Lockwood et al. (in press). They conducted a 15-session paired teaching experiment (Steffe & Thompson, 2000) with two vector-calculus students with no prior experience with combinatorics at the college level. The last three of these sessions were devoted to combinatorial proof of binomial identities. The authors found that the students benefitted from two particular instantiations while trying to construct combinatorial proofs: *contextual instantiation* and *numerical instantiation*. By contextual instantiation, the authors referred to having students focus on one particular context in which to situate their combinatorial thinking, such as committees, and they used numerical instantiation to mean having students substitute specific values in for the variables appearing in a binomial identity. The latter was particularly useful for the students looking at binomial identities involving a summation, because the students could then write out every term of the summation and more easily determine what the summation may be counting. Lockwood et al. additionally found that combinatorial proof required the students to reconsider previous concepts they had internalized about algebraic expressions being "different." In the algebraic representation system (Lockwood et al., 2020), two expressions are often considered the same if they have the same numerical value (for example, $a + b$ is the same as $b + a$ when a and b are real numbers). However, in combinatorial proof it is important to be able to consider two expressions as being different if they "differ in form," meaning that they physically appear different on the page (Lockwood, 2013, p. 253). When the numerical equivalence of two expressions was apparent to the students, they occasionally struggled to distinguish between each side of the identity as counting a set of outcomes in two different ways.

Lockwood et al. (in press) used Lockwood's (2013) model to frame their investigation into students' thinking about combinatorial proof, which was a novel application of the model which was originally intended to frame student thinking about counting problems. I now review exactly how they applied Lockwood's model, because I characterize combinatorial proof in a similar manner (this thus elaborates my own understanding of combinatorial proof as situated within existing literature). When a student is engaging in combinatorial proof activity, they can be considered as moving counterclockwise around Lockwood's model. See Figure 6.3. First, when a student is given a binomial identity to prove combinatorially, they must begin by picking one side of the identity to consider. That side of the identity is a formula/expression which, leveraging the bidirectional relationships between each component of the model, the student must interpret as having an underlying counting process. That counting process enumerates or generates a particular set of outcomes, which includes the context (say, committees) that the student chooses to use. Then the student must start back again in the formulas/expressions component with the other side of the binomial identity, and they must interpret it as having some other underlying counting process which enumerates the same set of outcomes[11]. This manner of applying Lockwood's model to combinatorial proof worked effectively for Lockwood et al. (in press), and hence I also view combinatorial proof similarly for the purposes of my study.

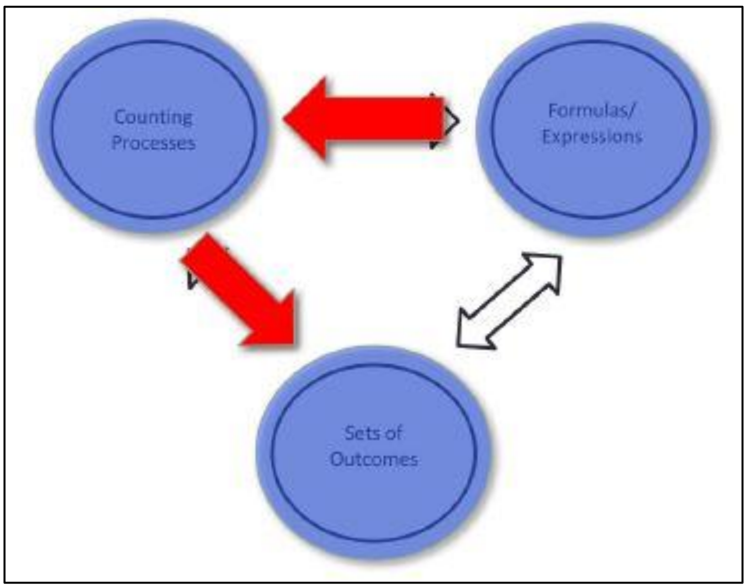

Figure 6.3. Lockwood's (2013) model as a lens for combinatorial proof (Lockwood et al., in press).

[11] As mentioned previously, this process describes one type of combinatorial proof, specifically those that utilize "Approach 1" (Lockwood et al., in press). While this type of combinatorial proof is the focus of this paper, I again acknowledge that other types of combinatorial proof, such as bijective proofs, do also exist.

In conclusion, combinatorial proof is a topic which has received little attention in the mathematics education community, and in particular much remains unanswered about how students think about combinatorial proof as different or similar to other types of proof. The studies that do exist have relied entirely on artifact data (Engelke & CadwalladerOlsker, 2010) or are theoretical (Lockwood et al., 2020)—so answering research questions about student thinking is not possible—or they were conducted with student participants who were novice provers (Lockwood et al., in press)—so ascertaining beliefs that the students had about proof is challenging. It is not known, for instance, whether or to what extent students accept combinatorial proof as fully rigorous compared to other types of proof (such as algebraic proofs or proofs by induction), and why. While combinatorial proofs are often considered proofs that explain (in the sense of Hersh, 1993), it is unknown whether the simplicity and intuition that combinatorial proofs can provide may affect students' acceptance of combinatorial proof as fully rigorous. Additionally, while some researchers (e.g. Harel & Sowder, 1998; Martin & Harel, 1989) have found that a student's acceptance of a proof may depend more on its ritualistic features (like a familiar format or the presence of symbolic manipulations) than the correctness of the argument, it is unknown how this may apply to combinatorial proof. In the following section, I describe how I attempted systematically to answer some of these questions using Harel and Sowder's (1998) proof schemes as my theoretical framing, all while interpreting students' combinatorial proof activity using the components of Lockwood's (2013) model.

4. Methods

For this study, I conducted video-recorded, task-based, individual clinical interviews (Hunting, 1997) with five upper-division mathematics students attending a large university in the western United States. This research was part of a larger study aimed at understanding experienced provers' beliefs about combinatorial proof. In this section, I describe my methodologies of data collection and analysis.

4.1 Data Collection

I recruited students from upper-division mathematics courses to participate in hour-long, individual, task-based selection interviews. These upper-division courses were selected to ensure each student participating in the interviews would have previously completed at least one proof-based course at the college level. In the selection interviews, I asked each student to solve counting problems—some of which involved combinations—and to prove theorems intended to be accessible for any student with some experience proving at the college level (for example, that the sum of two even integers is even). My goal with these selection interviews was to obtain participants for my study who had at least some familiarity with binomial coefficients and choosing, as well as students who could navigate a basic mathematical proof. After conducting the selection interviews, five students satisfied the criteria I was looking for: Sydney, Riley, Adrien, Peyton, and Ash (pseudonyms). Table 6.1 details the college-level mathematics courses that each of these five students had taken, showing that each had taken at least one proof-based mathematics course and that each had made some progress toward fulfilling the requirements of a mathematics major.

Next, each of these five students participated in four hour-long, individual, task-based clinical interviews aimed at investigating their combinatorial proving activity and beliefs and reasoning about combinatorial proof compared with other types of proofs. All of the students in my study had either very limited or no prior experience with combinatorial proof, and so during the first 1-2 interviews with each of them, I asked them to solve counting problems (see Table 6.2) that would provide scaffolding when I later asked them to solve combinatorial proof problems.

Table 6.1. Classes taken by student participants.

	Sydney	Riley	Adrien	Peyton	Ash
Calculus I	✓	✓	✓		✓
Calculus II	✓	✓	✓		✓
Infinite Series & Sequences	✓		✓	✓	✓
Vector Calculus I	✓	✓	✓	✓	✓
Vector Calculus II	✓		✓	✓	✓
Applied Differential Equations	✓		✓	✓	✓
Mathematics for Management, Life, and Social Sciences					✓
Linear Algebra I	✓	✓	✓	✓	✓
Linear Algebra II	✓		✓	✓	✓
Advanced Calculus	✓		✓		✓
Introduction to Modern Algebra	✓		✓		✓
Metric Spaces and Topology		✓*	✓*		
Discrete Mathematics	✓	✓		✓*	✓
Applied Ordinary Differential Equations	✓		✓		
Applied Partial Differential Equations	✓				
Fundamental Concepts of Topology	✓*	✓*		✓*	
Numerical Linear Algebra		✓			
Introduction to Numerical Analysis			✓		
Computational Number Theory		✓			
Mathematical Modeling			✓		
Actuarial Mathematics			✓		
Complex Variables					✓
Non-Euclidean Geometry					✓

* Indicates that the student was enrolled in this course at the time the interviews were conducted.

During the final two interviews with each student, I asked them to give counting arguments for the veracity of various binomial identities, as well as answer reflection questions about how they perceived their combinatorial proving activity. These identities are laid out in Table 6.3, with each student having given a combinatorial argument for at least a fraction of these identities depending on how quickly they progressed through the tasks in the interviews. I was careful to ask the students

Table 6.2. Combinatorial tasks for students to scaffold combinatorial proof.

Task	**Intended Purpose**
1. Spoonbill Problem. The scientific name of the roseate spoonbill (a species of large, wading bird) is *Platalea ajaja*. How many arrangements are there of the letters in the word AJAJA? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving ordered sequences of two indistinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014) when counting.
2. Subsets Problem. How many 3-element subsets are there of the set {1, 2, 3, 4, 5}? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving unordered selections of distinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014) when counting.
3. Find-a-Bijection Problem. Describe a bijection between the outcomes in the Spoonbill Problem and the Subsets Problem.	Facilitate a robust, flexible understanding of combinations. Lay groundwork for students to solve bijective combinatorial-proof problems.
4. Even- and Odd-Sized Sets Problem. Let S={1, 2, 3, 4, 5, 6}. (a) List all of the even-sized subsets of S. How many should there be? (b) List all of the odd-sized subsets of S. How many should there be? (c) Find a bijection between the subsets in parts (a) and (b) by considering whether the subsets contain the item 1.	Continue to facilitate a solid understanding of combinations. Provide scaffolding for students to eventually prove the identity $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ using a bijective combinatorial proof.
5. Reverse Counting Problem. (a) Write down a counting problem whose answer is 2^5. (b) Write down a counting problem whose answer is $15 \times \binom{14}{3}$.	Provide scaffolding for the concept of a combinatorial proof by asking students to interpret expressions in a combinatorial context.

for "arguments" rather than "proofs," because I did not want to influence their opinions when I asked them reflection questions, particularly whether they felt their activity constituted proof. Toward the end of the interviews, I also gave each student the Six Proofs Handout (see Table 6.4), which gives a combinatorial and noncombinatorial proof of three binomial identities. I asked each student to read the handout, and then I proceeded to ask them further reflection questions about the arguments on the handout, including which arguments they personally liked best (and why) and which arguments they felt constituted rigorous mathematical proofs. These reflection

Table 6.3. Identities given to the students to provide a combinatorial argument.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i\geq 0} \binom{n}{2i+1}$
$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$	$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$
$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$	

questions were aimed at understanding the extent to which the students accepted combinatorial proof as a valid, rigorous mathematical proof and how they reasoned about combinatorial proof in comparison with other types of proof the students were more familiar with (such as algebraic and

Table 6.4. Six Proofs handout.

Identity	Combinatorial Argument	Non-combinatorial Argument
Theorem 1. $2^n = \sum_{i=0}^{n}\binom{n}{i}$	(Subsets Context) Consider a set *S* such that $\|S\|=n$. The LHS* of the equation counts the number of subsets of *S*, because every subset can be uniquely determined by the elements it contains, and each of the *n* elements could be either in or out of each subset. The RHS counts the number of *i*-subsets of *S* and adds up over all possible values of *i*. Since the LHS and RHS both enumerate the set of subsets of *S*, they are equal.	(Induction RS*) Suppose *n=0*. It follows that the identity holds since $2^0 = 1 = \binom{0}{0}$. Suppose that the identity holds for *n*=*k*, where *k* is a nonnegative integer. We then observe that $\sum_{i=0}^{k+1}\binom{k+1}{i} = \sum_{i=0}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right) + \binom{k+1}{k+1}$ $= \sum_{i=0}^{k}\binom{k}{i} + \sum_{i=0}^{k}\binom{k}{i-1} + 1$ $= 2^k + \sum_{i=0}^{k-1}\binom{k}{i} + 1$ $= 2^k + 2^k - \binom{k}{k} + 1$ $= 2 \cdot 2^k - 1 + 1$ $= 2^{k+1}$.

Table 6.4. (Continued)

Theorem 2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	(Committees Context) Suppose a mathematics department has *n* faculty members, and Sofía is one of the faculty members. The LHS counts the total number of committees of size *k* that could be formed from the *n* faculty members. The RHS counts the number of committees of size *k* that exclude Sofía and the committees that include her. Note that this case breakdown encompasses all possible *k*-committees. Since the LHS and RHS both enumerate the same set of outcomes (*k*-committees formed from the *n* faculty members), they are equal.	(Algebraic RS) We have that $\binom{n-1}{k} + \binom{n-1}{k-1}$ $= \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$ $= \frac{(n-1)!}{k!\,(n-1-k)!} + \frac{(n-1)!}{(k-1)!\,(n-k)!}$ $= \frac{(n-1)!\,(n-k)}{k!\,(n-k)!} + \frac{(n-1)!\,k}{k!\,(n-k)!}$ $= \frac{n(n-1)! - k(n-1)! + k(n-1)!}{k!\,(n-k)!}$ $= \frac{n!}{k!\,(n-k)!}$ $= \binom{n}{k}$.
Theorem 3. $\binom{n}{k} = \binom{n}{n-k}$	(Binary Strings Context) Consider the set of binary strings of length *n* containing exactly *k* 0s. The LHS enumerates this set, because $\binom{n}{k}$ is the number of ways we can select positions for the 0s to occupy, and the rest of the positions in the binary string will be 1s. The RHS also enumerates this set, because $\binom{n}{n-k}$ is the number of ways we can select positions for the 1s to occupy, and the rest of the positions in the binary string will be 0s.	(Binomial Theorem RS) Recall that the Binomial Theorem states that for *n* a natural number and *a*, *b* real numbers, $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$ Notice that for each *k*, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. Additionally, we also have that by the Binomial Theorem, $$(b+a)^n = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^i,$$ and the coefficient of $b^k a^{n-k}$ is $\binom{n}{n-k}$. We also have that $a^{n-k}b^k = b^k a^{n-k}$ and $(a+b)^n = (b+a)^n$, by the commutativity of multiplication and addition of real numbers, respectively. Thus, when the latter is expanded, the coefficients of each term on either side of the equation must be equal, so $\binom{n}{k} = \binom{n}{n-k}$ for all *k*.

*RS here refers to *representation systems*, which Lockwood et al. (2020) define as “a structure with permissible configurations and inferential schemes” (p. 6). Here, we consider each of the combinatorial proofs to be situated in the enumerative RS.

induction proofs).

4.2 Data Analysis

Because I was particularly interested in the students' perceptions of combinatorial proof generally and of their own combinatorial proof activity, I focused my data analysis toward those episodes in the data where students answered my reflection questions or discussed their opinions about combinatorial proof. Each interview was transcribed, and then, following the thematic analysis methodology (Braun & Clarke, 2006), I re-watched every video familiarizing myself with the data and making note of key episodes related to my research questions. Each time a student described their opinions about combinatorial proof—particularly whether they felt combinatorial proof could constitute valid, rigorous mathematical proof in comparison to other types of proof—I flagged it as an episode that warranted further analysis. Then, using the MAXQDA 2020 qualitative data analysis software (VERBI Software, 2019), I coded these episodes using Harel and Sowder's (1998) proof schemes (described in Section 2). My goal was to categorize systematically the prevailing manner in which the students were thinking about combinatorial proof in order to understand how and why they considered combinatorial proof to be rigorous mathematical proof in comparison to other types of proof. For instance, if a student made utterances about the correctness of the logical structure of a combinatorial argument, I took that as evidence that the student was using an analytical proof scheme. If the student alluded to an authority (e.g., claiming they did not think their instructor would accept a combinatorial proof) or appealed to ritualistic features of a combinatorial proof (e.g., claiming a combinatorial argument did not constitute proof because it did not involve symbolic manipulation), then I took that to signify that the student was using an authoritarian or ritual proof scheme, respectively. There were some episodes in which students seemed to use more than one proof scheme in their reasoning about combinatorial proof, and in those instances, I coded the episodes with each proof scheme for which there seemed to be evidence. This aligns with Harel and Sowder's (1998) use of proof schemes, as they stated a given person may exhibit more than one proof scheme within a short period of time (p. 277). Other researchers such as Kanellos et al. (2018) have also used multiple proof schemes to categorize students' reasoning. After I coded the data using Harel and Sowder's proof schemes, I checked the interview videos again to see if there were any other episodes that may warrant further analysis. To help ensure consistency, I re-coded every episode that was flagged. Throughout this process, I discussed questions I had about coding with another researcher to ensure they were applied appropriately, and any episodes in which it was difficult to determine which proof scheme(s) the student was using were discussed thoroughly until both the other researcher and myself were confident the code(s) being applied was/were correct.

There is one further point that should be addressed before I transition to the Results, and that is to explain why I am not applying proof schemes to the ways that the students *produced* combinatorial proofs. While in the proof education literature applying Harel and Sowder's (1998) proof schemes framework to students' proof production is common (e.g. Blanton & Stylianou, 2014; Healy & Hoyles, 2000; Kanellos et al., 2018; Stylianou et al., 2006), for my study it was not particularly insightful or interesting to categorize my participants' combinatorial proof production using proof schemes. This is simply because the students I interviewed ended up being so successful solving the combinatorial proof problems I gave them that the vast majority of their proof production work would have been categorized as using transformational analytical proof schemes. However, in the data, while students were highly successful at producing correct combinatorial proofs consistent with an analytical proof scheme, the focus of my study (and the more interesting phenomena that occurred in my interviews) concerns how the students *perceived* their proving activity and combinatorial proof in general. The students frequently produced correct combinatorial proofs, but then made utterances that aligned with non-analytical proof schemes when reflecting back on their own work or combinatorial proof more generally. Since I am interested in investigating how students discuss and characterize combinatorial proof and what it reveals about students' perceptions of the nature of combinatorial proof, using proof schemes to categorize their proof comprehension rather than proof production was more relevant and useful for my particular research interests in this study. Finally, I note in addition that there are multiple other researchers (e.g. Harel & Sowder, 1998; Liu & Manouchehri, 2013; Plaxco, 2011) who have used proof schemes as a lens to examine student thinking about proof rather than only their personal proof production, and so the manner in which I am applying the proof schemes framework is not inconsistent with existing proof literature.

5. Results

I first provide a broad overview of the proof schemes (Harel & Sowder, 1998) that the students utilized when reflecting on their combinatorial proving activity and combinatorial proof in general. Then, in subsequent sections I expand on this general overview by providing excerpts from the data exemplifying the proof schemes that the students used, as well as discussing why some proof schemes were not used by any of the five students in my study.

Table 6.5 provides a breakdown of the proof schemes (Harel & Sowder, 1998) that the students used to describe their perspectives regarding combinatorial proof, including how they conceived of combinatorial proof as different from other types of proof.

As mentioned in the Methods section, some episodes in the data involved a student using more than one proof scheme to describe their perspectives of combinatorial proof, and so those episodes were coded with multiple proof schemes. Overall, all five students drew upon the *ritual* proof

Table 6.5. Proof schemes used by students discussing combinatorial proof.

	Sydney	Riley	Adrien	Peyton	Ash
Authoritarian	0	✓	0	0	0
Ritual	✓	✓	✓	✓	✓
Perceptual Empirical	✓	✓	✓	✓	✓
Transformational	0	✓	✓	0	✓
Contextual Restrictive	✓	0	✓	0	0

scheme, three students drew on the *transformational* proof scheme, and two students drew on the *contextual restrictive* proof scheme. Additionally, while the *authoritarian* and *perceptual empirical* emerged less explicitly in the data, I discuss some episodes in the data during which some of the students appeared to utilize them.

In the following subsections, I provide illustrative examples from the data of each of the proof schemes that emerged in these data. I begin in Section 5.1 by exemplifying those proof schemes that were identified in the students' reasoning. In Section 5.2 I discuss Harel and Sowder's (1998) proof schemes that did not emerge (and potential reasons why). Ultimately, the goal of sharing these results is to emphasize ways in which the students perceived of combinatorial proof, particularly allowing me to contrast those perceptions with other types of proof and proving experiences they have encountered in their mathematical careers. Finally, in my conclusion and discussion section I will synthesize and discuss what the occurrence of these proof schemes indicates about students' conceptions of combinatorial proof.

5.1 Proof Schemes Used by Students in Their Reasoning About Combinatorial Proof

During the interviews, there were five of Harel and Sowder's (1998) proof schemes that seemed to emerge as the students discussed their reasoning about their own combinatorial proofs and combinatorial proof in general. I provide illustrative examples in the following sections.

5.1.1 Ritual external conviction proof scheme. The most frequent proof scheme coded in the data was Harel and Sowder's (1998) ritual proof scheme. Harel and Sowder characterized this proof scheme as occurring when students primarily attend to the appearance in form of a mathematical proof rather than the correctness of the argument being presented (p. 246). Here, students appeal to what might be viewed as aspects of mathematics that they assume are relevant to the mathematical community – that appear formal or that seem to be based on commonly accepted mathematical rituals. There were several episodes in the data in which students appealed only to surface features of combinatorial arguments to explain their beliefs, and I coded these episodes as indicating the ritual proof scheme. In most of these episodes, the students seemed to believe that a particular combinatorial argument (or combinatorial arguments in general) did not constitute rigorous mathematical proof.

As an example, when Ash was given the Six Proofs handout and asked about the two proofs given for Theorem 3 (one of which was based on the binomial theorem and one of which was combinatorial[12]), we had the following exchange.

Int.: Do you think either or both of them are proof?

Ash: I couldn't poke a hole in the combinatorial argument, but there's something that doesn't feel complete about it, but I couldn't tell you what it is.

Int.: Okay. Can you say a little bit more about that—what do you mean it doesn't feel complete?

Ash: **Maybe it's just with the contrast for the complexity and depth of this one [points to the non-combinatorial proof of Theorem 3].**

Int.: Sure.

Ash: **I just inherently doubt that it could be that simple**…. I mean, I can't poke a hole in it, so….

Int.: Okay. So, this one maybe feels a little more like a proof [points to the non-combinatorial proof of Theorem 3], but, like you said, you can't poke a hole in this

[12] I acknowledge that a proof that uses the binomial theorem could also be considered a "combinatorial proof," as the binomial theorem is often considered in the context of combinatorics. However, throughout this paper I specifically use the term combinatorial proof to refer to one that is *enumerative*, that is, it directly involves some type of counting argument.

one [points to combinatorial proof of Theorem 3]. You don't have a reason to think it's not a proof.

Ash: Couldn't find an exception.

Here, I interpret Ash's utterances (particularly in bold) to mean that they believed the argument that uses the binomial theorem felt more like a proof than the combinatorial argument, because the former was more complex. As Ash said, "I just inherently doubt that it could be that simple." This is perhaps an understandable reaction, as the argument that uses the binomial theorem is certainly longer and uses "heaver tools" and more symbols than the combinatorial argument, and it reflects a belief some students may have that correct, formal mathematics is necessarily complicated (e.g. Martin & Harel, 1989). Combinatorial arguments for binomial identities are often accessible enough that even K-12 children can understand them (Maher et al., 2007, 2015). This accessibility, however, may mean that some undergraduate mathematics students who are used to seeing more complicated algebraic or induction proofs may be more reluctant to accept that combinatorial proofs can indeed be rigorous mathematical proofs. Notably, Ash's language suggests that they thought the proof needed to have certain characteristics that reflect what they would consider a complex proof. While they didn't specify exactly what they meant by complex, these may include features such as length, a certain logical structure, formal mathematical symbols, etc. Thus, I consider this to be an episode in the data exemplifying Harel and Sowder's (1998) ritual proof scheme.

Another ritualistic feature of proof that came up a couple of times during students' reasoning about combinatorial arguments was the use of symbols. For instance, in one of my interviews with Peyton, they gave a nice combinatorial argument of the identity $\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$. They were able to articulate that each side of the identity counts the number of ways to designate n objects out of a set of $2n$ objects as "special." The right side of the identity does this immediately, and

Peyton was able to conceive of the left side as splitting the 2*n* objects into two groups of size *n* and then counting the number of ways to designate a total of *n* objects as special when the objects are chosen from those two groups. Once they finished articulating their combinatorial argument successfully, I had the following exchange with them:

Int.: Nice job! So, I'm curious. Yeah, do you think the argument that you just gave, do you think that that is a proof?

Peyton: No, because this is the bas-… If I did say it was a proof, I could hand this in with two sentences of speech and then get it published, and I don't think it's formal enough to be a proof.

Int.: Okay, and do you think the diagram is what makes it informal, or the fact that we're counting makes it informal, or, yeah, why do you think it's not formal enough?

Peyton: I like to stick with my previous argument for counting proofs not being formal, and it's basically, what if I'm wrong? **There's no math to back me up with it. There's no algebra or induction to do that talks about it.**

Int.: Okay.

Peyton: **It's just me saying words.**

Int.: Okay.

Peyton: I feel like it's a good supplement for a proof, say if you did the math and then talked about this, or talked about this then did the math, then that would be a good proof.

Int.: Okay.

Peyton: But I don't think this could stand alone.

Here, we can see that Peyton did not seem to consider the correct enumerative argument they gave as explicitly involving mathematics, as evidenced by their statement, "There's no math to back me up with it." They also said, "There's no algebra or induction to do that talks about it. It's just me saying words." I interpret these utterances to mean that Peyton expects mathematical proof to contain algebraic manipulations or a particular familiar structure like induction, and Peyton did not consider an argument to be a mathematical proof if it consists only of English words. This is an important finding, as it implies that some students may not believe combinatorial arguments (or other arguments that do not use mathematical symbols) could constitute rigorous mathematical proof. This finding also corroborates other research that has found that students may believe mathematical proofs should always use symbols or algebraic manipulations (Healy & Hoyles, 2000; Martin & Harel, 1989). I note here as well that Harel and Sowder (1998) do include a symbolic proof scheme in their framework. However, Harel and Sowder described this proof scheme as applying to situations in which students attempt to prove a proposition by seemingly haphazardly manipulating symbols appearing in the proposition while making no attempt to comprehend their meanings. Since Peyton did articulate an understanding of the symbols appearing in the binomial identity, I did not interpret their reasoning as an example of a symbolic proof scheme.

5.1.2 Transformational proof scheme. In this section, I discuss two episodes from the data in which a student's reasoning was coded as aligning with Harel and Sowder's (1998) internalized/interiorized transformational proof schemes. Both of these proof schemes are characterized by the comprehension and use of a proof heuristic that renders conjectures into facts, with the difference concerned only with the extent to which the student reflects on the proof scheme (p. 262-264). Since this was not an important distinction in my analysis, I will hereafter simply refer to the transformational proof scheme. (I distinguish this from the contextual restrictive transformational proof scheme, which arose separately and which I discuss in Section 5.1.3.) Overall, this proof scheme emerged during episodes in the data in which the student articulated that they did consider a particular combinatorial proof (or combinatorial proof in general) to qualify as a rigorous mathematical proof.

For the first example of this proof scheme we turn to Riley. In this episode, Riley had just articulated a combinatorial proof of the binomial identity $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$; they argued verbally that both sides of the identity could count the number of ways to select k committee members, in which r of those committee members are qualified to have special veto power, out of a group of n congressional candidates.

Int.: Does that feel like a proof, or does this just--?

Riley: Oh, no.

Int.: Okay, and how come?

Riley: Well, part of it's just that like, as I was saying earlier, this notion of pre-selecting qualified candidates, um, isn't fully fleshed out. **So, if I were to say specifically pre-select *r* qualified candidates and immediately assign them to the council, and then the exact same thing and the other set or whatever you want to call it, um, enumeration, that would work to me as a proof.** But this wasn't actually specific enough to say what I was wanting it to say, which I think also has to do with why I felt like this whole relation with *r* was nebulous, because I wasn't actually defining it strictly correctly. It was more of just grasping for conceptual foothold. So, yeah, if I were to, like, submit a proof, then it would be, you know, committee times veto power essentially, and then equals, you know, specifically exactly *r* qualified candidates. So this would essentially more clearly eliminate the possibility that I was having where, oh, well, what if we mess up in deliberation and give too many, because this doesn't actually prevent you from doing that as it stands.

Int.: Okay, but if you make sort of this refinement, it does feel like a proof?

Riley: Yeah.

Int.: So, when you say this is a proof, what are you taking to mean by proof?

Riley: What I would say is, here, **given this problem, not only can you not come up with a counterexample, but you can't conceive of the nature of a counterexample that would invalidate it**, right?

Although Riley initially answered "no" when I asked if they felt their argument qualified as a proof, I interpret their subsequent utterances to mean that their issue was related to the fact that they felt they had not defined the variables in their argument precisely enough. The issue to them was not inherently related to the fact that their argument was enumerative, and indeed when I asked if they would consider their argument a proof if they refined how they were defining the terms in their argument, they indicated that they would because any attempt to come up with a counterargument for their proof would fail. Thus, I interpret that Riley was aware of and accepted the proof heuristic of describing how each side of a binomial identity counts the same set of outcomes as a valid method of mathematical proof, and so it was coded as an instance of the transformational proof scheme.

The next episode that exemplifies a student utilizing the transformational proof scheme involves Adrien. In this episode, Adrien was answering questions about the Six Proofs handout, in which they were asked to evaluate two proofs (combinatorial and non-combinatorial) each for three binomial identities (see Table 4.) When I asked Adrien if they thought the three combinatorial arguments constituted rigorous mathematical proofs, the following exchange occurred.

Adrien: I would say that's a proof [points to the combinatorial proof of Theorem 1]. These two [points to the combinatorial proofs of Theorems 2 and 3], it's debatable.

Int.: Okay, cool. So, it sounded like the premise of proving by arguing both sides count something, it sounds like that inherent quality wasn't really what made these debatable for you.

Adrien: No.

Int.: It was more the particular language that was used?

Adrien: Well, it's not just the language. It's the fact that you're leaving something up to the reader. I mean, when I'm reading a textbook and they give a proof, it's like, I think one of the hated phrases by students is this—it's like, "The remainder of this proof is left as an exercise to the reader." And I'm like, "Oh, screw you too."

Int.: So you feel like these are kind of doing something like that?

Adrien Yeah, I mean, not as much, because they actually do a complete proof of the original statement, but they are also leaving parts of them up to the reader as exercises, which is annoying. Right?

From this and prior utterances by Adrien in the interview, I understood that Adrien viewed these combinatorial arguments as constituting mathematical proofs, but that the proofs were leaving details up to the reader. I would argue that this is a reasonable position; after all the combinatorial proof of Theorem 2 states only that, "The LHS of (2) counts the total number of committees of size k that could be formed from the n faculty members. The RHS of (2) counts the number of committees of size k that exclude Sofia and the committees that include her." It is reasonable that students may desire more details in the proof justifying why the left and right sides of the identity count those committees, and figuring out how much justification in a proof is necessary can be a struggle for students (Harel & Sowder, 2007). However, we again see that Adrien was aware of the proof heuristic of enumerative argumentation and accepted it as a valid method of mathematical proof. This episode therefore exemplified student reasoning aligning with Harel and Sowder's (1998) transformational proof scheme.

5.1.3 Contextual restrictive proof scheme. The third proof scheme that appeared in the data was Harel and Sowder's (1998) contextual restrictive proof scheme. Harel and Sowder said that a student is using this proof scheme when, "conjectures are interpreted, and therefore proved, in terms of a specific context" (p. 268). Harel and Sowder then give the example of a student interpreting and proving a general statement about *n*-dimensional vector spaces in the specific context of $\boldsymbol{R^n}$ (p. 268). Similarly, in my data, when a student expressed that they felt a combinatorial argument constitutes proof, but only in a specific context, I coded those episodes as instances of the contextual restrictive proof scheme.

As an example, when I gave Sydney the Six Proofs handout and asked them to comment on which arguments they felt were proofs, they expressed that they felt the combinatorial proof of Theorem 1 (which used abstract sets and subsets) was more rigorous than the combinatorial proof of Theorem 2 (which used committees and a particular named individual—Sofía). They stated that both arguments were "definitely still real proofs," but that the combinatorial proof which used committees introduced "potential error for like red herring or a strawman or something like that." When I asked them to say more about their thinking, they stated the following, referring to the combinatorial proof of Theorem 1.

Sydney: This one is more rigorous in the sense that it does have a, um, more of just like what the definitions are and what exactly the notation represents as opposed to giving a definition to it or giving an example to those definitions.

Here, Sydney characterized the combinatorial proof of Theorem 2 as "giving an example to those definitions," which they felt made the proof less rigorous. While logically a combinatorial argument utilizing committees can be equally rigorous as one using sets and subsets, Sydney seemed to indicate that they would disagree. This, combined with the fact that Sydney did still consider both arguments to constitute mathematical proofs, led me to code this as an instance of the contextual restrictive proof scheme.

Another instance of this proof scheme occurred during my last interview with Adrien. In this interview, I asked Adrien to give a combinatorial proof of the identity $\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$. One way that an individual could prove this identity combinatorially is to consider an ordered set containing *n* + 1 objects. Then, one can consider enumerating the number of ways to make an unordered selection of *k* + 1 of these objects. The right side of the identity does this, and the left side also does this by using a case breakdown and considering the largest element in a selection of *k* + 1 objects. For instance, the first case considers such selections where the $k + 1^{st}$ item is the largest. There is only $1 = \binom{k}{k}$ such selection. Next, we can consider selections where the $k + 2^{nd}$ item is the largest. Then, there are $\binom{k+1}{k}$ ways to pick the remaining *k* items to go into the selection, and so on.

During Adrien's interview, they struggled some with this problem, and then I suggested that they consider counting ways to select *k* + 1 numbers out of the set of natural numbers from 1 to *n* + 1. (Since natural numbers are ordered, I hoped that this prompt would lead them to a solution similar to that given above.) They eventually did produce the combinatorial argument outlined above in the context of this particular set of natural numbers (see Figure 6.4), and then I asked Adrien whether they felt their combinatorial argument constitutes a proof. We had the following exchange:

Adrien: I would prefer induction, because the main thing about this is **it feels like you're actually assigning like a distinct property to the objects, which not every group of objects that you're going to pick *k* from is going to naturally have that kind of property**.

Int.: Right, like if we were picking dots, for example.

Adrien:	Yeah.
Witness:	Although, they are distinct objects, right?
Adrien:	Yeah, they're distinct objects, but that means **you'd be putting a pretty arbitrary ranking system on them**.
Int.:	Okay, and that makes it feel less like it's a valid mathematical proof?
Adrien:	I mean, it feels really arbitrary and the fact that it is arbitrary means that no matter what objects you have, you can just assign this ranking to them, and that'll work.
Int.:	Okay.
Witness:	Interesting
Int.:	But you would say that you prefer induction?
Adrien:	Yeah, probably.

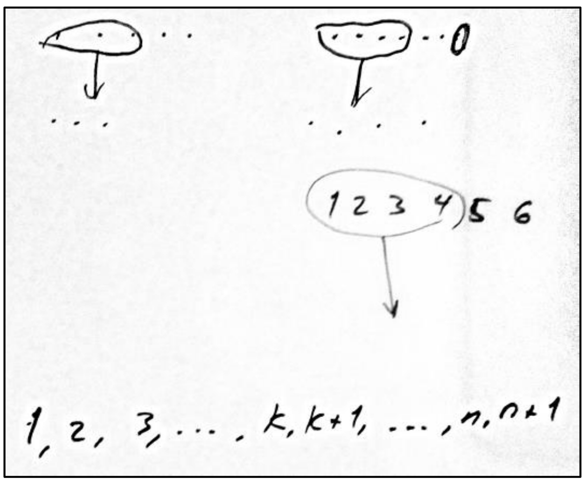

Figure 6.4. Adrien proved the binomial identity $\sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}$ *by counting the number of ways to select k + 1 numbers out of the set {1, 2, 3, …, n, n+1}.*

Here, we can see that Adrien was struggling with the idea that a proof could be valid if an extra assumption was included without loss of generality (specifically the assignment of an ordering of the objects). Adrien felt that their combinatorial argument counting ordered objects was "arbitrary" and hence not as effective as induction would be in proving the general identity. I interpret that Adrien viewed the context they used for the proof (specifically, the numbers from 1 to n + 1) as being restrictive and "arbitrary," and this was ultimately their main issue with the argument.

Because of this, I coded this episode as an instance of Adrien using the contextual restrictive transformational analytical proof scheme.

5.1.4 Possible emergence of the authoritarian external conviction proof scheme. Besides Peyton, all of the students interviewed said that they had no prior experience with combinatorial proof. (Peyton happened to be taking a discrete-mathematics class which covered combinatorial proof during the timeframe in which the interviews took place.) For this reason, it is not surprising that the authoritarian proof scheme did not clearly emerge in the data. Harel and Sowder (1998) defined this proof scheme as occurring when for students, "their main source for conviction is a statement appearing in a textbook or uttered by a teacher" (p. 247). Thus, for the students in my study who lacked prior experiences with combinatorial proof from which to draw, I would not expect them to be able to make statements such as, "This combinatorial argument is valid because I saw my teacher present it as a proof in class."

Nevertheless, one of the students in the study, Riley, did at times appear to consider what an imagined authority might say about combinatorial proof. For instance, when I gave Riley the Six Proofs handout, I asked them whether they believed the combinatorial arguments constituted proof. We had the following exchange:

Riley: Yeah, I think this one [gestures to the combinatorial proof of Theorem 1], and **I've had plenty of TAs that would mark down**.
Int.: Oh, really?
Riley: Yeah. Just in terms of, I don't know, **I feel like there's this critical length of a proof, that you're expected to meet**, regardless of how simple the notion is. Where, even if this said all the words it needed to say, they would find something like, "Well, but was it by the commutativity of multiplication and addition of real numbers?"
Int.: Okay, can you say—?

Riley: I wouldn't turn this in, even though it might be a very strong proof in the mathematical sense.[13]

I infer here that Riley did believe that the combinatorial argument of Theorem 1 constituted proof, but they still said they would not turn it in because their teaching assistant (TA) would not think it was long enough. This episode exemplifies a tension that students sometimes feel when they are unsure about the level of detail that their instructors or TAs might expect when grading, which is part of the sociomathematical norms (Yackel & Cobb, 1996) of upper-division mathematics classrooms. It also harkens to the findings of (Healy & Hoyles, 2000) who found that students may simultaneously hold two different conceptions of proof: arguments they would adopt for themselves and arguments they considered would receive the best score from an instructor. The students recruited into my study had experience with proof at the college level, but they were more accustomed to the expectations of their instructors regarding proofs using algebraic manipulation or other techniques such as those learned in advanced calculus. Enumerative argumentation, where the expressions in the binomial identity are not manipulated at all, may be quite different than what the students were used to working with. This episode with Riley suggests that students in a new proving environment not only have to think about whether they themselves believe a combinatorial argument constitutes a valid mathematical proof, but they may also have to consider how that argument may be evaluated by a hypothetical authority. This perhaps suggests that Riley's reasoning in this episode is—to some extent—following Harel and Sowder's (1998) authoritarian external conviction proof scheme. However, Riley's own opinion about whether the combinatorial

[13] I also coded this episode as an example of the ritual proof scheme, since Riley is attending to the length of the proof, which is a surface-level, ritualistic feature of the argument. Other researchers have also used multiple proof schemes simultaneously to describe student thinking or their proving activity (e.g. Housman & Porter, 2003; Kanellos et al., 2018; Sears, 2019). However, in this section I focus on this episode as a (potential) example of a student using an authoritarian proof scheme.

argument was a proof was separate from how they imagined a hypothetical TA may evaluate the argument (Riley felt that it was a proof), and so I coded this episode as only indicating a potential authoritarian proof scheme for Riley. Nevertheless, I offer the episode here as a point of discussion about the authoritarian proof scheme.

Finally, I want to bring up one more interesting note about the authoritarian proof scheme and my data. Even though Peyton did have experience with combinatorial proof in the discrete mathematics class they were taking concurrently with my interviews with them, they did not feel that combinatorial arguments constituted rigorous mathematical proof (as I discussed in Section 5.1.1). Since Peyton did have experience in an upper-division mathematics course with combinatorial proof, they were the only participant I would have expected to have reasoned along the lines of an authoritarian proof scheme, but that was not the case. It is unclear exactly why Peyton's classroom experiences did not translate to an acceptance of combinatorial arguments as rigorous mathematical proof, but it might have to do with Peyton's mathematical experience level compared with my four other study participants. Peyton had taken the fewest upper-division mathematics courses at the time the interviews took place, and so it is likely that their understanding of the concept of proof was earlier along in its development than that of the other four participants, and so this could have contributed to Peyton attending to more ritualistic features of combinatorial proof. It is possible (even likely) that had I interviewed students with more experience with *combinatorial* proof that I may have seen more instances of the authoritarian proof scheme. I mention this point again in the conclusion section where I discuss avenues for future research.

5.1.5 Possible emergence of the perceptive empirical proof scheme. In addition, there were several instances in the data in which a student expressed that they felt a combinatorial argument (or enumerative argumentation in general) did not constitute a proof but would help a person develop intuition regarding why an identity holds. This phenomenon was flagged often enough in the data (10 times, and across all five student participants) that I felt it may warrant a closer look even though it was initially not clear to me which (if any) of Harel and Sowder's (1998) proof schemes the students were using in their reasoning. I illustrate this phenomenon with a couple of examples from the data.

One of the first binomial identities that I gave to the students to prove combinatorially in the interviews was $\binom{n}{k} = \binom{n}{n-k}$. There are a number of possible combinatorial proofs of this identity, and Adrien nicely articulated the following correct argument:

Adrien: So, you can either count *n* choose *k* spaces for, like, a number of objects. You can choose specifically non-distinct objects. So, order doesn't matter at all—only thing that matters is location. You can either choose *k* spaces for *k* objects, or you can choose *n* – *k* spaces for where those objects aren't going to be, so they fall into place, and that's identical. So, these two are equal. That's my intuition.

We also see here that even though Adrien provided a correct and acceptable combinatorial proof of the identity, they immediately characterized their argument as their intuition. I asked Adrien if they would characterize their argument as a mathematical proof, and they said the following:

Adrien: Not really, I wouldn't. I would characterize that as intuition.
Int.: Okay.
Adrien: Because it's not that formal. **It's something that I would do to explain to people, it's like here's some intuition for why these two are equal, but I don't really consider it to be a formal proof.** You could probably formalize it, but I don't think what I just did was a proof.
Int.: Okay, how would you formalize it to make it what you would call a mathematical proof?
Adrien: Good question. When I've seen these two be equivalent, **I would get some explanation for the intuition, and then we'd just show with the factorials to prove that they're equal.** As for proving that they're equal without using factorials, not entirely sure.

We see that Adrien thought one may have to use an entirely different proof method (i.e., "using factorials") to show that the binomial identity holds, and it seems that this perspective could have something to do with their prior experience with binomial identities. I interpret Adrien's utterances to mean that previously they may have seen an intuitive rationale for why this binomial identity holds, but that intuition was accompanied by a different kind of argument (e.g., one that uses factorials) to "prove" that it holds. This suggests that the way instructors choose to present and frame combinatorial proof may have an impact on students' beliefs about combinatorial proof as a valid proof method. While combinatorial arguments can be helpful in building intuition about a binomial identity combinatorially (Lockwood et al., 2020), instructors could also emphasize that combinatorial arguments are not necessarily less rigorous than algebraic proofs or other types of proofs.

To provide another example from the data in which a student expressed that they felt combinatorial arguments have intuitive value but are not rigorous mathematical proofs, I turn to Peyton's comments about the combinatorial and induction arguments for Theorem 1 when I gave them the Six Proofs handout. When I asked Peyton for their initial impression of the two arguments, Peyton said that they "enjoy[ed]" the combinatorial one more, and when I asked them to explain why, they said:

> *Peyton*: Induction, it's not really showing that something is…it's not showing that these two sides are actually…you know, doing something. It's just basically saying, here's an identity that holds true, because it does…and [the combinatorial proof] is saying, this is *why* it is true.

Here, Peyton was drawing a clear distinction between the induction argument which (in their perspective) shows *that* the identity holds, while the combinatorial argument explains *why* the identity holds. However, when I asked Peyton if they felt the combinatorial argument was a proof, they said:

Peyton: Yeah, [the combinatorial argument], it doesn't feel like it could be a proof.
Int.: Okay.
Peyton: Because **we don't know if there is some special case where it doesn't hold true.**
Int.: Okay.
Peyton: And this one is saying, okay, it does for one and it does for $n + 1$…
Int.: Right.
Peyton: So it is true, always.
Int.: Okay.
Peyton: So, **this would be nice if a teacher was like, hey, here's what's happening [points to the induction argument], and then here's why it's happening [points to the combinatorial argument].**

I interpret these utterances to mean that even though Peyton thought that the combinatorial argument could help someone to understand intuitively why the binomial identity holds, they did not think that it constituted proof (while induction did).

In terms of Harel and Sowder's (1998) proof schemes, it might seem that these episodes are another manifestation of the ritual proof scheme, because the students may be attending to a ritualistic feature of proof—for instance, that they are supposed to be (or often are) unintuitive—to claim that these combinatorial arguments are not proofs. However, intuitiveness is not part of the physical appearance of a proof (unlike the presence of symbols, length, use of a particular logical structure, etc.) and is more subjective, and so the ritual proof scheme did not seem appropriate. Instead, the proof scheme that seemed to fit best was the perceptual proof scheme, which Harel and Sowder (1998) described as occurring when,

> "[p]erceptual observations are made by means of rudimentary mental images—images that consist of perceptions and a coordination of perceptions, but lack the ability to transform or to anticipate the results of a transformation" (p. 255).

This description does seem to fit the students' characterizations of combinatorial proof in these episodes—that they can give one an intuitive understanding of why a binomial identity holds, but (in the students' perspective) this understanding would have no connection to transformations of the binomial identity that would be needed to actually prove the identity. Their utterances indicated that they felt an actual proof would have to involve algebraic manipulations or induction.

In any case, I contend that these episodes provide interesting insight into ways that students may perceive of combinatorial proof as fundamentally different from other types of proof. To students, combinatorial arguments may not feel rigorous enough to qualify as proof but offer intuition of why a binomial identity holds. This is closely related to the idea of a proof that explains (Hersh, 1993), and even seems to align with some definitions of a proof that explains that have been offered which specify that they need not be totally rigorous (e.g. Weber, 2002, pg. 14). Other researches have also argued that combinatorial proofs can generally be considered proofs that explain within the enumerative representation system (Lockwood et al., 2020), which means that they can help one understand *combinatorially* why a binomial identity (holds even if they may not provide an explanation in an algebraic or other representation system).

5.2 Proof Schemes that Did Not Appear in the Data

Finally, in this section I discuss Harel and Sowder's (1998) remaining proof schemes for which I had no evidence of their emergence in my interviews with the students: symbolic, inductive, generic, constructive, and axiomatic. For some of these proof schemes—namely the generic and axiomatic proof schemes—it is likely that the main reason for their absence is simply the small sample size of my study. For instance, even though none of the students in my study ever explicitly referenced any mathematical axioms, I do think it is plausible that a student could reference an axiom while reasoning about combinatorial proof, and in that case the student would be using an axiomatic proof scheme. I also think that a student could potentially use a generic proof scheme when reasoning about combinatorial proof. Harel and Sowder (1998) describe this proof scheme as occurring when "conjectures are interpreted in general terms, but [the student's] proof is expressed in a particular context" (p. 271). In the case of combinatorial proof, this may apply to a situation in which a student is given a binomial identity that is stated generally with variables, but they may give a combinatorial proof that uses specific numerical values in place of those variables. This strategy of substituting specific values in for variables in binomial identities can be a useful heuristic for learning combinatorial proof (Lockwood et al., in press), but it was never used as a proof scheme for the students in my study, because I consistently and explicitly asked the students to give their final combinatorial proof in terms of the original variables appearing in the binomial identity.

In addition to the small sample size causing some proof schemes not to emerge, I want to also make the point that three of these proof schemes—symbolic, inductive, and constructive—by their nature seem unlikely to come up in combinatorial proof contexts. According to Harel and Sowder (1998), the symbolic external conviction proof scheme, occurs when students manipulate symbols "without reference to their possible functional or quantitative reference" (p. 250). Since combinatorial proofs do not typically involve symbolic manipulation at all, this means a student reasoning about combinatorial proof is unlikely to use this proof scheme. I do not mean to say that students will never manipulate symbols when they are given a combinatorial proof task—sometimes students have been observed verifying binomial identities algebraically before setting out to find a counting argument (e.g. Lockwood et al., in press). However, in these cases the students are working in the algebraic as opposed to enumerative representative system (in the sense of Lockwood et al., 2020), and so it would not be accurate to characterize the student's *combinatorial proof* reasoning as falling under the symbolic proof scheme. Similarly, the inductive proof scheme is also unlikely to be used by a student reasoning about combinatorial proof.

According to Harel and Sowder (1998), this proof scheme is being used "[w]hen students ascertain for themselves and persuade others about the truth of a conjecture by *quantitatively evaluating* their conjecture in *one or more* special cases" (p. 252). Again, while students given a combinatorial proof task may sometimes quantitatively evaluate the binomial identity to verify that it holds, they are not in those instances operating within the enumerative representation system. Finally, Harel and Sowder's constructive proof scheme is also unlikely to come up in combinatorial proof of binomial identities, because by definition this proof scheme can only be used by students who are proving an existence theorem.

Overall, I have strong evidence that the students in my study used three of Harel and Sowder's (1998) proof schemes (ritual, transformational, and contextual restrictive) to discuss and characterize combinatorial proof, and the students may additionally have used two more (authoritarian and perceptual). For the remaining five proof schemes (axiomatic, generic, symbolic, inductive, and constructive), I found no evidence of their emergence in my interviews with the students. I believe the ways in which these proof schemes did (and did not) emerge in the data reveal a number of insights about the nature of combinatorial proof as a mathematics topic, which I discuss in more detail in the following section.

6. Discussion and Conclusion

In this study, I used Harel and Sowder's (1998) proof schemes as a lens to look at characteristics of combinatorial proof that make it seem different for students than some other types of proof. I also characterized combinatorial proof in terms of the components of Lockwood's (2013) model, following Lockwood et al. (in press). I found that the students in my study used a variety of proof schemes to discuss and characterize combinatorial proof, including whether it constitutes a rigorous mathematical proof (and why). There were some students whose reasoning aligned with a transformational proof scheme, and they concluded that since the argumentation in a correct combinatorial proof is valid, combinatorial proofs can be considered rigorous mathematical proofs. Other students used external conviction proof schemes to describe their reasoning, including the ritual proof scheme. These students expressed that because combinatorial proofs have certain ritualistic features (specifically that they are often more intuitive than other types of proof and do not involve symbolic manipulation) they do not qualify as rigorous mathematical proofs. In addition, there were other proof schemes that may have emerged from the data, specifically the authoritarian proof scheme and the perceptual proof scheme, which might describe situations where a student indicated that they did not think their TA would accept a combinatorial proof (even though the student thought combinatorial proofs are valid), or where students characterized combinatorial proofs as being merely intuitive arguments that make the identity "seem" true but that do not account for all possible cases of the binomial identity intended to be proven. In total, the students in my study used (or may have used) authoritarian, ritual, perceptual, transformational, and contextual restrictive proof schemes to discuss and characterize combinatorial proof.

Because of the small sample size for my study, I anticipate that there are other proof schemes that students more broadly might use to discuss and characterize combinatorial proof. For instance, the students in my study did not refer to axioms when they were engaging in or discussing combinatorial proof, but it is possible if more students were interviewed that the axiomatic proof scheme may emerge. On the other hand, some of Harel and Sowder's (1998) other proof schemes may be less unlikely to appear in future investigation of combinatorial proof. In particular, since combinatorial proof does not involve symbolic manipulation, the symbolic proof scheme is unlikely to emerge. Also, since the constructive restrictive proof scheme deals only with proofs of existence theorems, it is unlikely to appear in a study investigating student thinking about combinatorial proof of binomial identities (though it might appear in studies looking at combinatorial proof of other types of theorems).

Addressing my second research question, I contend that seeing which proof schemes students used to discuss and characterize combinatorial proof affords multiple useful insights about the nature of combinatorial proof. First, this study provides evidence confirming characteristics about the nature of combinatorial proof for students that had never actually been verified empirically by previous research. While it was not surprising to learn that some students view combinatorial proof as less valid than other types of proof (due to the lack of certain features they may be used to seeing in proofs, like symbols), it is nevertheless valuable to have concrete evidence confirming that this can happen. Second, the proof schemes lens also sheds light on other features of combinatorial proof that have not been discussed previously in the literature. For instance, while some previous studies have commented that combinatorial proofs can be easier and more accessible for some binomial identities than an algebraic or induction proof (e.g. Lockwood et al., 2020), I found that this feature itself made some students doubt that combinatorial proof could be a valid mathematical proof. Combinatorial proofs lack some ritualistic features that students may associate with a "complex" proof (like having a certain length, logical structure, mathematical symbols, etc.) and applying the lens of proof schemes helped me formally describe and situate this finding within a well-established framework. Similarly, the perceptual proof scheme is useful in helping to describe the role of intuition and how it shapes students' understanding of the nature of combinatorial proof.

In summary, the proof schemes framework and data from this study support the following insights about student thinking regarding the nature of combinatorial proof. I list them here as a concise way to frame an answer to the second research question that emerged in my data.

- Many combinatorial proofs are simple, and since some students may believe proofs are always complex, these students may not believe combinatorial proofs constitute rigorous mathematical proof.
- Combinatorial proof does not involve symbolic manipulation, which may lead some students to think combinatorial proof is less rigorous than other types of proof.
- Combinatorial proofs are often considered accessible and explanatory, which may influence some students to believe combinatorial proofs are merely intuitive arguments rather than fully rigorous mathematical proofs.
- Combinatorial proofs are situated within particular contexts, which may cause some students to believe that combinatorial proofs qualify as a proof only restricted to those particular contexts (e.g., committees or ordered objects) rather than more generally.
- Correct combinatorial proofs are mathematically rigorous and logically valid, and students using a transformational proof scheme can recognize this.
- Correct combinatorial proofs may leave out details that some students may wish were present, but students can nevertheless accept combinatorial proof broadly as a valid proof method.

Overall, my results suggest that students do seem to perceive of the nature of combinatorial as different from other types of proof. Whether or not these are productive views of combinatorial proof, instructors and researchers should be aware that students may have these conceptions about combinatorially proof as they teach and conduct proof-education research, respectively. In the classroom, instructors should understand that some students may believe combinatorial proof is less valid than algebraic, induction, or other types of proof for a variety of reasons, and so instructors should clarify for students why correct combinatorial proofs are indeed mathematically rigorous and logically valid. In terms of proof education studies, when researchers draw conclusions about student thinking about proof, they should be mindful that some of these conclusions may apply differently to student thinking about combinatorial proof.

Regarding next steps, for a start, future research should continue to investigate proof schemes that students use to continue uncovering ways that students view the nature of combinatorial proof differently from other types of proof. My study is a first step in establishing that students do think about combinatorial proof differently, but future research with larger sample sizes or different populations would continue to shed light on students' use of proof schemes in combinatorial proof. For instance, perhaps other proof schemes (such as the axiomatic or generic proof scheme) may emerge, or we may see more widespread use of authoritarian or perceptual empirical proof schemes. In addition, future research could investigate not only the proof schemes students use to discuss and characterize combinatorial proof, but also proof schemes that students use to produce combinatorial proofs. Proof production was a not a focus of this study, primarily because all five participants of my study were so successful in producing combinatorial proof that it was more insightful to examine the ways they *thought* about combinatorial proof rather than the ways they *produced* combinatorial proof. However, future research with larger sample sizes or different populations may yield more variety in the quality and types of approaches students take, and hence categorizing students' proof production in addition to their thinking about combinatorial proof could be useful.

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CHAPTER 7 (Paper 3) – Investigating Combinatorial Provers' Reasoning about Multiplication

Abstract: Combinatorial proof is an important topic both for combinatorics education and proof education researchers, but relatively little has been studied about the teaching and learning of combinatorial proof. In this paper, I focus on one specific phenomenon that emerged during interviews with mathematicians and students who were experienced provers as they discussed and engaged in combinatorial proof. In particular, participants used a wide variety of cognitive models to interpret multiplication by a constant when reasoning about binomial identities, some of which seemed to be more (or less) effective in helping produce a combinatorial proof. I present these cognitive models and describe episodes that illustrate implications of these cognitive models for my participants' work on proving binomial identities. My findings both inform research on combinatorial proof and highlight the importance of understanding subtleties of the familiar operation of multiplication.

Keywords: Combinatorial proof, Multiplication, Counting problems

1. Introduction

Combinatorics is an increasingly important branch of mathematics with applications in computer science, engineering, statistics, as well as other areas of mathematics. In addition to its applicability, combinatorics has pedagogical value for mathematics instructors due to its accessibility and ability to provide opportunities for students to use creativity, search for patterns, and generalize (e.g., Lockwood & Gibson, 2016; Lockwood & Reed, 2018; Tillema, 2013). One class of combinatorics problems, combinatorial proof of binomial identities[14], comes up in discrete mathematics, statistics, number theory, and a variety of other contexts. These problems can be tricky even for accomplished counters (e.g., Lockwood, Reed, & Erickson, in press), and yet this topic has received relatively little attention from the mathematics education research community.

A binomial identity is an equation involving one or more binomial coefficients, such as the following:

[14] Combinatorial proof is a proof technique that can be applied to other types of theorems as well, but we focus on binomial identities in this paper.

$$\binom{n}{k}k = n\binom{n-1}{k-1}. \qquad (1)$$

In this paper, I take *combinatorial proof* to mean any proof that establishes the veracity of a binomial identity by arguing that each side enumerates the same (finite) set. [15] The validity of these arguments is rooted in the fact that a set can have only one cardinality.

For example, to prove the binomial identity (1) above, one could argue that each side counts the number of committees of size *k* with a chairperson that can be formed from a group of *n* people. In this case, the right side counts this set because there are *n* possible people who could be the chairperson, and then for every choice of one person to be the chairperson, there are $\binom{n-1}{k-1}$ ways of selecting the remaining k – 1 people for the committee. As a lead-in to the rest of this paper, I offer the following questions to the reader in order to provoke thinking about combinatorial proof (I present my research questions at the end of the section). First, w*hy does* $\binom{n}{k}k$ *also count the number of committees of size* k *with a chairperson that can be formed from n people?* Second*, how are you thinking of the multiplication of the binomial coefficient by k?*

In this paper, I report on results from a study in which I interviewed five upper-division mathematics undergraduate students and eight mathematicians to investigate the ways that experienced provers think about and engage with combinatorial proof. I particularly focus on findings related to the ways that combinatorial provers conceived of and used multiplication. Students' reasoning about multiplication is a topic that has been studied extensively in the K-12 mathematics education literature (e.g., Greer, 1992, 1994; Mulligan & Mitchelmore, 1997; Steffe,

[15] We acknowledge that authors such as Lockwood et al. (in press) and Rosen (2012) have articulated two types of combinatorial proof – one type is that described above, and the second type involves arguing that each side of a binomial identity counts a different set and creates a bijection between the two sets. We do not focus on bijective proofs in this paper.

1994; Tillema, 2013), and, while some studies occur at the undergraduate level (e.g., Lockwood & Purdy, 2019a), it has not received as much attention at the postsecondary level, perhaps because educators and researchers might assume that undergraduate students understand the familiar operation of multiplication. However, as I will discuss, my data show that undergraduate students' and mathematicians' conceptions of multiplication are interesting and varied, and the cognitive models of multiplication they use can have implications for their combinatorial proving activity. I attempt to address the following research questions in this paper (I will elaborate particular terminology in these questions in the following sections):

1. What cognitive models for multiplication do undergraduate students and mathematicians use when engaging in combinatorial proof of identities involving scalar multiplication?
2. What are implications of these cognitive models for students' engagement with combinatorial proof?

Here I specify scalar multiplication to mean multiplication by a single positive integer constant k, such as $\binom{n}{k}k$. I narrow my results to scalar multiplication, as opposed to other expressions involving multiplication that may occur in a binomial identity, such as $\binom{n}{k}\binom{k}{m}$, to focus my arguments and due to space limitations, but I consider additional kinds of multiplication as an avenue for future research.

2. Relevant Literature and Theoretical Perspectives on Combinatorial Thinking and Combinatorial Proof

To situate my findings within the broader literature, I first discuss Lockwood's (2013) model of students' combinatorial thinking, which is one of two theoretical perspectives that I utilize in this study. I then continue this section with a look at previous work that has been conducted on combinatorial proof in the mathematics education literature, including how Lockwood's (2013) model has been used to frame a previous study on combinatorial proof.

2.1 Lockwood's (2013) Model of Students' Combinatorial Thinking

I begin with an overview of Lockwood's (2013) model of students' combinatorial thinking, which I view both as an aspect of relevant literature and as a theoretical framing for how I am taking combinatorial thinking in this paper. While this model was originally conceived as a framework to study student thinking about solving counting problems (i.e., problems of the form, "How many…?"), I found that in this study and in previous work (Lockwood et al., in press) the framework was also useful as a tool to study students' and mathematicians' work on combinatorial proof.

Lockwood (2013) said that there are three components that can appear in a student's combinatorial reasoning when solving a counting problem: *formulas/expressions*, *counting processes*, and *sets of outcomes*.

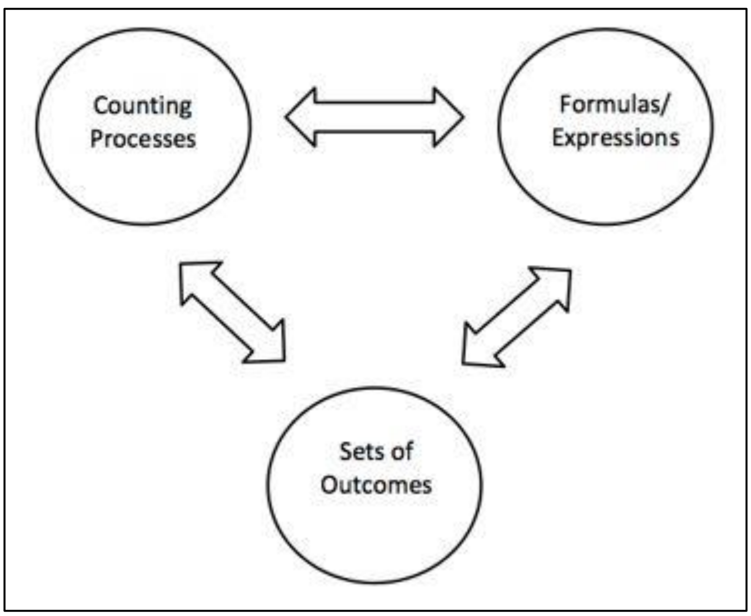

Figure 7.1. Lockwood's (2013) model of students' combinatorial thinking.

I will consider these components for a hypothetical student solving the Committees Problem, "A university department has 15 faculty members. How many ways could a 5-member hiring committee be formed if one of the faculty members must be the chairperson?" In this example, *sets of outcomes* refers to the objects being counted as well as written or mental representations of these outcomes. A counter may, for instance, represent the outcomes of the Committees Problem using strings of letters with a subscript, such as "A_{chair}BCDE" to indicate the committee of faculty members A, B, C, D, and E where faculty member A is the chairperson. *Counting processes* refers to the mental or physical processes that one carries out to enumerate the outcomes. A counter may for instance use the Multiplication Principle (which I discuss in Section 3.1), and break the problem down into a sequence of two tasks: choosing one of the 15 faculty members to be a chairperson, then choosing 4 out of the remaining 14 faculty members to fill out the rest of the committee. Finally, *formulas/expressions* describe the mathematical formulas and/or expressions that a counter may write down as their answer to the counting problem. For instance, the aforementioned two-stage counting process would yield the expression $15 \times \binom{14}{4}$. As Lockwood (2013) noted, I could alternatively go from the *formulas/expressions* component of the model to *counting processes* by conceiving of the expression $15 \times \binom{14}{4}$ as having an underlying counting process. This kind of realization is critical to the writing of a combinatorial proof. For instance, it may be useful for a combinatorial prover to analogously conceive of the expression $n \times \binom{n-1}{k-1}$ as suggesting a counting process involving a sequence of two tasks, for which there are *n* and $\binom{n-1}{k-1}$ ways each task could be completed (in that order). However, this is not the only way that the expression could be interpreted. Multiplication is used for a variety of mathematics problems, and a student could alternatively conceive of $n \times \binom{n-1}{k-1}$ as underlying a counting process involving a task that can be completed $\binom{n-1}{k-1}$ different ways, and then scaling the outcomes of that task by a factor of *n*. These differences in the way that an expression (and particularly multiplication) can be interpreted will be important as I discuss the results of my study later in this paper.

2.2 Previous Work on Combinatorial Proof

I identified only two prior studies that focused on undergraduate students' thinking about and engagement in combinatorial proof. First, Engelke Infante and CadwalladerOlsker (2010, 2011) conducted a study in which they looked at students' solutions to exam questions asking for a combinatorial proof of two binomial identities. They examined the solutions to see what difficulties arose for the students and found that the students appeared to struggle with (a) language mimicking, (b) inflexibility of context, (c) misunderstanding of combinatorial functions, and (d) failure to count the same set (p. 95-96). Linked to these difficulties, Engelke Infante and CadwalladerOlsker (2011) observed that the students may have engaged in *pseudo-semantic proof production*, which is based on the distinction between *semantic* and *syntactic* proof production. This distinction, articulated by Weber and Alcock (2004), describes qualitatively different approaches students can take to proof, depending on whether they use internally meaningful instantiations of the mathematical objects they are working with. Weber and Alcock (2004) defined semantic proof production as, "[when the] prover uses instantiation(s) of the mathematical object(s) to which the statement applies to suggest and guide the formal inferences that he or she draws" (p. 210), and they defined syntactic proof production as when a proof is, "written solely by manipulating correctly stated definitions and other relevant facts in a logically permissible way" (p. 210).

While this distinction may seem straightforward to apply to proofs in domains such an analysis or algebra, it can be difficult to see how these ideas might carry over to combinatorial proof, since it is a proof strategy where the prover does not (typically) manipulate the expressions in the binomial identity. Engelke Infante and CadwalladerOlsker (2011) contended that students may still write combinatorial proofs in a way that is not guided by useful instantiations of the expressions in the identity though. To describe how this may happen, they defined pseudo-semantic proof production as “the attempt to engage in a semantic proof production process, but relying on the syntax of a previously encountered proof when faced with a term that the student cannot explain” (p. 96). In writing about the same study, Engelke and CadwalladerOlsker (2010) also found that having students write a specific “How many?” question when engaging in combinatorial proof may help them be more successful. While it is difficult to know for certain by looking only at student exam solutions, Engelke Infante and CadwalladerOlsker’s (2010, 2011) work provides evidence that combinatorial proof can be difficult for students and that students may try to imitate enumerative arguments they previously encountered if they get stuck.

The only other study I found in the literature addressing combinatorial proof was one that colleagues and I conducted more recently. We carried out a 15-session teaching experiment (Steffe & Thompson, 2000) that covered a variety of combinatorics topics with two vector-calculus students (Lockwood et al., in press). The last three sessions of the teaching experiment were centered around combinatorial proof of binomial identities, and we could study the students’ reasoning on combinatorial proof based on their trajectory along the prior 12 teaching experiment sessions. In this study, we found that the students seemed to benefit from two particular instantiations: (a) focusing on a particular context (e.g., counting passwords or committees), and (b) considering specific values of *n* or other variables appearing in the identity to be proven. For instance, when the two students tried to prove $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$, it was very useful for them to consider the case where $n = 5$. This enabled them to expand the summation, and then they could imagine the terms counting ways to select 5 people from a set of 10 people split across two groups of 5. After this realization they were able to generalize back to the original binomial identity and provide a correct combinatorial proof. Finally, we also found that a potentially useful way to prepare students for combinatorial proof is to give them opportunities to generalize while solving counting problems and ask them to solve counting problems two different ways.

In that study, my colleagues and I used Lockwood's (2013) model of students' combinatorial thinking to frame our findings, explaining that student thinking about combinatorial proof could be thought of as moving counterclockwise around the model starting at *formulas/expressions* (see Figure 7.2). We argued that a student who is given a binomial identity to prove will first have to interpret the expression on one side of the identity as having an underlying counting process which enumerates a set of outcomes (in the sense of Lockwood, 2013). The student must then imagine how the expression on the other side of the identity could enumerate the same set of outcomes.

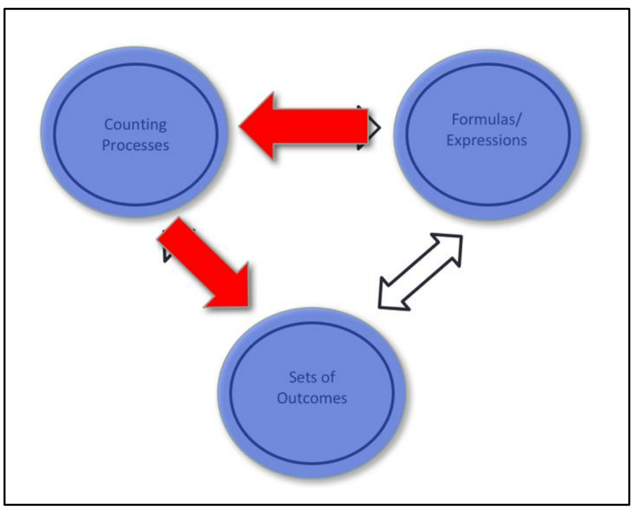

Figure 7.2. Lockwood et al.'s (in press) description of students' typical trajectory through the model when engaging in combinatorial proof.

In particular, a student who is given a binomial identity to prove starts by considering one side of the binomial identity as an expression with an underlying counting process. The student has to conceive of what that counting process may be by interpreting the quantities and operations involved. For instance, the presence of addition in the expression may correspond to a counting process involving a case breakdown, or multiplication may correspond to an application of the Multiplication Principle. Finally, the counter must conceptualize the set of outcomes that are being organized or generated by this counting process, and then try to think of how the other side of the binomial identity can count the same set of outcomes. This is how we used Lockwood's (2013) model to frame out investigations of student thinking about combinatorial proof in Lockwood et al. (in press), and I will use the model as a theoretical lens in the same manner for this study.

This prior work on combinatorial proof is valuable and may help give instructors more pedagogical ideas when covering combinatorial proof. However, questions regarding student thinking about combinatorial proof still remain unanswered, and addressing these gaps in the literature is one goal of the research discussed in this paper. In particular, while my colleagues and I previously looked at other aspects of student thinking about combinatorial proof (Lockwood et al., in press), we did not focus on how they thought about the mathematical operations involved in the binomial identities. While it may be easy to assume that undergraduate mathematics students understand what operations such as multiplication do, we argue that interpreting these operations combinatorially in the context of a binomial identity can introduce subtleties that are important yet not always appreciated in college-level classrooms. Before presenting the results for my study, I first discuss previous findings from the literature related to multiplication within combinatorics.

3. Relevant Literature and Theoretical Perspectives on Multiplication within Counting

In Section 3.1, I describe some work that has been conducted on students' reasoning about multiplication within counting to help me frame my results. There have been few such studies that have been conducted at the undergraduate level, and so in Section 3.2 I discuss the larger body of work aimed at understanding student thinking about multiplication at the K-12 level. This informs my theoretical framework for how I am taking cognitive models of multiplication in this paper, which I elaborate at the end of Section 3.2.

3.1 Multiplication within Counting

As I have pointed out, interpreting mathematical operations as part of a counting process (Lockwood, 2013) is a key aspect of combinatorial proof. Binomial identities may contain many different mathematical operations, but, in this paper, I focus on multiplication. Multiplication is a familiar operation to undergraduate students, yet in my experience teaching combinatorics, I have found that students do not always know when to multiply while solving counting problems. This has also been found in some studies of undergraduate students; for example, researchers have found that undergraduate students may confuse situations requiring multiplication versus addition (see also Kavousian, 2008; Sowder et al., 1998).

In combinatorics, multiplication arises as such a fundamental aspect of counting that there is a guiding principle describing when to multiply when solving counting problems – this is called the Multiplication Principle. Tucker (2002) offered my preferred statement of the Multiplication Principle: "Suppose a procedure can be broken down into m successive (ordered) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m different outcomes in the mth stage. If the number of outcomes at each stage is independent of the choices in the previous stages, and if the composite outcomes are all distinct, then the total procedure has $r_1 \times r_2 \times \cdots \times r_m$ different composite outcomes" (p. 170). Despite how fundamental the Multiplication Principle is for counting, Lockwood, Reed, and Caughman (2017) found that textbook statements of the Multiplication Principle vary significantly more than the statements of key definitions and theorems in other domains (such as limit, derivative, and the Fundamental Theorem of Algebra). Lockwood et al. (2017) found that textbook statements of the Multiplication

Principle could be classified into three types—structural, operational, and bridge—depending on whether the statement characterized the multiplication as counting certain types of outcomes (like *n*-tuples) or ways of completing a staged procedure. Tucker's (2002) statement of the Multiplication Principle, for instance, is considered a bridge statement. It is possible that different statement types could affect students' perspectives on counting in general; for instance, operational statements could influence students to think about counting ways to complete a process, whereas structural and bridge statements might help encourage students to approach counting problems with a more set-oriented perspective (Lockwood, 2014). With the variety that exists among textbook statements of the Multiplication Principle and the implications that different types of statements may have on students' combinatorial activity, Lockwood et al. (2017) argued, "[T]he Multiplication Principle is much more nuanced than instructors and students perhaps give it credit for" (p. 31).

Building off this textbook analysis, Lockwood and Purdy (2019a) worked to study how students come to understand and make sense of the Multiplication Principle. They used the teaching experiment methodology (Steffe & Thompson, 2000) with two undergraduate students and followed the *guided reinvention* heuristic (Freudenthal, 1991). Lockwood and Purdy stated that there are two necessary conditions (potentially among others) for multiplication to be an appropriate operation in a counting problem: independence and distinct composite outcomes. By independence, Lockwood and Purdy meant independence of the stages in a counting process, specifically that the choice of an option in a given stage does not affect the number of options in any subsequent stage of the process. For example, to count the number of 2-digit PINs where repetition is not allowed, we could multiply 10×9, because regardless of which of the 10 possible digits we choose for the first position in a PIN, there are 9 options for which digit could fill the second position. By distinct composite outcomes, Lockwood and Purdy meant that the generation of each outcome by exhaustion of every possible way to carry out the stages of the counting process must not produce any duplicates. Lockwood and Purdy demonstrated an example where this condition breaks down by showing a common incorrect student solution the problem, "How many 3-letter sequences made of the letters *a, b, c, d, e, f* contain the letter *e*, where repetition of letters is allowed?" The process of first selecting a position in the 3-letter sequence for an 'e' and then multiplying by 6^2 (the number of ways to fill the remaining positions with any of the six letters) overcounts because it generates outcomes like "eea" more than once. In Lockwood and Purdy's guided reinvention, through carefully selecting tasks for the student participants, they were able to help the students become attuned to the importance of independence and distinct composite outcomes. They also demonstrated that even students who can successfully solve counting problems involving multiplication may find it challenging to characterize precisely when to multiply in counting. They additionally identified subtleties regarding the Multiplication Principle that textbooks do not always explicitly address and yet are critical to multiplication in counting, particularly related to handling issues of order in counting (Lockwood & Purdy, 2019b).

Throughout Lockwood and Purdy's (2019a) study, the students they interviewed consistently indicated that they interpreted multiplication in a combinatorial context as joining selections at different stages of a counting process to produce an outcome. That is, they thought about generating an outcome by considering a multi-stage process for forming an outcome and multiplying together the number of options for each stage in the process. Indeed, this is a fairly typical way of conceiving of multiplication in counting. One idea that did not come up in Lockwood and Purdy (2019a), however, is that there might be other interpretations or mental models that counters may have for multiplication. It is not surprising that alternative models of multiplication did not come up in Lockwood and Purdy (2019a)—it is likely in part due to the sample size (two students) and because the students in their study were given contextualized problems to solve (problems where the element selection model of multiplication is readily applicable) rather than decontextualized mathematical expressions to interpret. However, students' mental models of mathematical operations, including multiplication, are highly relevant in combinatorial proof, since interpreting de-contextualized expressions as having an underlying counting process (as I argued in Section 2.2) lies at the heart of these types of problems. I did not identify any studies at the undergraduate level that examined students' mental models of multiplication, which suggests that much remains unknown about what other mental models of multiplication (besides element selection) counters may have and what implications these models may have for their combinatorial proof activity.

To examine in the literature how certain ideas about multiplication might play into peoples' engagement with proving combinatorial identities, I turned to K-12 mathematics education literature where there has been more work in this area. I elaborate such literature in the next section, and I additionally introduce the construct of *cognitive models* of multiplication, which I will use, together with Lockwood's (2013) model, as my second theoretical lens to frame my study.

3.2 Relevant Literature on Multiplicative Reasoning and Cognitive Models

3.2.1 Literature on multiplicative reasoning. To my knowledge, there has been no research done on the mental models of multiplication that undergraduate students use when they engage in combinatorial proof activity. As I noted previously, one reason for this could be that since multiplication is a familiar operation for college students, it is easy to assume that students understand it. Understandably, there has been much more work conducted in this area in mathematics education at the K-12 level. The Common Core State Standards identified several situations (e.g. equal groups, arrays/area, and comparison) involving multiplication that K-12 students should be exposed to (Common Core State Standards Initiative, 2010), and numerous researchers spanning several decades have studied how young children think about multiplication of positive whole numbers (e.g., Greer, 1992; Mulligan & Mitchelmore, 1997; Tillema, 2013). This is a natural research inquiry, since primary school is typically where a student learns how to multiply. While I do not provide a comprehensive review of literature on multiplication among young students, here I highlight some studies and ideas that I use in this paper. In particular, I will highlight work that has established that there are different ways in which students think about and approach problems involving multiplication.

Researchers such as Sowder et al. (1998) have found that multiplicative reasoning does not develop as naturally for children as additive reasoning. Multiplicative reasoning develops slowly, with less than 50% of fifth graders being proficient multiplicative thinkers (Clark & Kamii, 1996; Sowder et al., 1998). Difficulties that arises when children first learn multiplication include grappling with a “many-to-one correspondence” (Clark & Kamii, 1996, p. 43) and moving away from singleton units and to a composition of units—for example, 4 baskets, each containing 3 kittens (Behr et al., 1994; Sowder et al., 1998). Sowder et al. (1998) stated that, “[t]oo often, problems can be solved by applying rules learned long ago, without any attempt to make sense of the relationships inherent in the problem” (p. 132), and that it is important to give students problems that require sense-making to strengthen multiplicative reasoning.

Multiplication is often introduced to children in the second grade and is typically presented as an efficient calculation of repeated addition (Clark & Kamii, 1996). Similarly, Confrey (1994) has advocated that teachers should use a *splitting* conception of multiplication, where a split is defined as, “an action of creating simultaneously multiple versions of an original,” (Confrey, 1994, p. 292).

However, the interpretation of multiplication as repeated addition is frequently inadequate in helping students navigate some multiplicative situations (Sowder et al., 1998), such as those involving a Cartesian-product context (Mulligan & Mitchelmore, 1997). Specifically in combinatorics, Batanero et al. (1997) also found that 14- and 15-year-old students may find it difficult to distinguish combinatorial situations requiring addition from those requiring multiplication. Notably, Kavousian (2008) reported similar difficulties among undergraduate students.

One of the ways that researchers have tried to understand children's thinking about multiplication is by studying the *intuitive models* they employ to solve problems. In the K-12 literature, some researchers have used intuitive multiplication models to mean an internalization of multiplication as corresponding to a particular problem situation (e.g., Fischbein, Deri, Nello, & Marino, 1985). Some researchers such as Tillema (2013) have similarly looked at linear and nonlinear meanings of multiplication that students develop as they progress through K-8 curricula. Other researchers, however, have found it preferable to study and define children's intuitive models of multiplication in terms of the calculation strategies that they use (e.g., Anghileri, 1989, and Mulligan & Mitchelmore, 1997). Anghileri (1989) was one of the earliest researchers to use intuitive multiplication models to study children, and Anghileri's (1989) results suggested that children use three models for whole-number multiplication: unitary counting, repeated addition, and multiplicative calculation. Others, such as Steffe (1994) also observed children's multiplying schemes and found they employed part-to-whole units-coordinating schemes (e.g., finding how many smaller triangles are inside larger triangles) and iterative multiplying schemes (e.g., counting by threes).

Mulligan and Mitchelmore (1997) interpreted intuitive models as calculation strategies that young children use to solve multiplication problems, and they extended earlier results with their longitudinal study aimed at understanding such intuitive models, how these intuitive models relate to the structure of the problem being solved, and how these intuitive models develop over time. They followed Nesher (1988) in referring to the structure of the problem being solved as its *semantic structure*. They used 5 of the 10 multiplicative semantic structures identified by Greer (1992): equivalent groups, rate, comparison, array, and Cartesian product. The other 5 semantic structures were excluded because they involved measurement (a context that the 2nd- and 3rd-grade students being studied would not have been familiar with) or were more applicable to multiplication by rational numbers (instead of integers). Examples from Mulligan and Mitchelmore's (1997) study of these semantic structures are given in Table 7.1.

Table 7.1. Mulligan and Mitchelmore's (1997) multiplicative semantic structures (p. 314).

Semantic Structure	Example Problem
Equivalent groups	Peter had 2 drinks at lunchtime every day for 3 days. How many drinks did he have altogether?
Rate	If you need 5 cents to buy 1 sticker, how much money do you need to buy 2 stickers?
Comparison	John has 3 books, and Sue has 4 times as many. How many books does Sue have?
Array	There are 4 lines of children with 3 children in each line. How many children are there altogether?
Cartesian product	You can buy chicken chips or plain chips in small, medium, or large packets. How many different choices can you make?

After they gave the children problems from each of these semantic structures, Mulligan and Mitchelmore (1997) identified three intuitive models that the children used (p. 316):

1. Direct counting (where the children simply counted all of the items being enumerated without identifying or leveraging any multiplicative structure in the problem),
2. Repeated addition (where the children employed a calculation strategy such as rhythmic counting forward, skip counting forward, repeated adding, or additive doubling),
3. Multiplicative operation (where the children used a known or derived multiplicative fact).

Mulligan and Mitchelmore (1997) found that the repeated addition model was the most frequently correctly applied model for all semantic structures except comparison (for which the multiplicative operation model was most frequently correct). There was consistent progression in the intuitive models used by the students from direct counting to repeated addition, but even after instruction there was a strong preference for the repeated addition model among the children. Furthermore, although there was steady improvement in the performance among the children, comparison and especially Cartesian product problems remained difficult for the children to solve, possibly because it is more difficult to see the equal-groups structure that allows the use of their preferred intuitive multiplication model of repeated addition.

3.2.2 Cognitive models. The above discussion represents a sample of the literature available on the ways that K-12 children think about and use multiplication to solve problems. While much work has been done to this end, little research has been conducted on how undergraduate students use and think about multiplication when counting. In particular, I could not identify any studies that looked at the different models of multiplication (such as an array, as equivalent groups, etc.) that undergraduate students might use to solve problems, including combinatorial proof problems that involve interpreting expressions involving multiplication. Also, it is not clear which of these models for multiplication are the most productive for students engaging in combinatorial proof, or what other implications these models may have on their combinatorial activity.

To help me investigate these questions, in my study I used the construct of *cognitive models* to mean someone's personal representation of what a given instance of the operation of multiplication entails. This construct is similar to Mulligan and Mitchelmore's (1997) semantic structures; however, my use of cognitive models was intended to go beyond a classification of pre-existing problem types – I also attempt to capture students' and mathematicians' mental representations of what multiplication is doing in a binomial identity. In addition, I do not consider *cognitive models* to be the same as the *intuitive models* construct used by Mulligan and Mitchelmore (1997) and others, since this was used to refer to children's calculation strategies for multiplication. I do not consider this construct to be relevant to my work, since my participants were not carrying out calculations but instead interpreting generalized identities involving multiplication. This construct of cognitive models is the second theoretical lens I use to analyze and present the results of my study.

4. Methods

I conducted video-recorded, semi-structured, task-based interviews (Hunting, 1997) with five undergraduate students and eight mathematicians. These interviews were part of a larger study aimed at understanding mathematicians' and upper-division undergraduate students' reasoning and beliefs about combinatorial proof. I describe the data collection for both the students and the mathematicians, and then I discuss the data analysis I conducted for writing this paper.

4.1 Data Collection

4.1.1 Student data collection. I recruited students from upper-division mathematics courses at a large university in the western United States to participate in an hour-long individual task-based selection interview. In these selection interviews, I asked each student to solve straightforward combinatorics problems to see if the students were attuned to sets of outcomes and whether they were familiar with combinations and binomial coefficient notation. I also asked them to prove basic theorems (such as the fact that the sum of two even integers is an even integer) to see if the student had experience with and could navigate a mathematical proof. From the round of selection interviews, I selected five students who satisfied these criteria. Table 7.2 shows the

Table 7.2. Classes taken by student participants.

	Sydney*	Riley	Adrien	Peyton	Ash
Calculus I	✓	✓	✓		✓
Calculus II	✓	✓	✓		✓
Infinite Series & Sequences	✓		✓	✓	✓
Vector Calculus I	✓	✓	✓	✓	✓
Vector Calculus II	✓		✓	✓	✓
Applied Differential Equations	✓		✓	✓	✓
Mathematics for Management, Life, and Social Sciences					✓
Linear Algebra I	✓	✓	✓	✓	✓
Linear Algebra II	✓		✓	✓	✓
Advanced Calculus	✓		✓		✓
Introduction to Modern Algebra	✓		✓		✓
Metric Spaces and Topology		✓**	✓**		
Discrete Mathematics	✓	✓		✓**	✓
Applied Ordinary Differential Equations	✓		✓		
Applied Partial Differential Equations	✓				
Fundamental Concepts of Topology	✓**	✓**		✓**	
Numerical Linear Algebra		✓			
Introduction to Numerical Analysis			✓		
Computational Number Theory		✓			
Mathematical Modeling			✓		
Actuarial Mathematics			✓		
Complex Variables					✓
Non-Euclidean Geometry					✓

* These are pseudonyms.
** Indicates that the student was enrolled in this course at the time the interviews were conducted.

classes each of the students had taken. Overall, the students had each taken at least one proof-based mathematics class and had each made some progress toward fulfilling the required courses for a mathematics major.

Next, these five students each participated in four hour-long individual interviews, which occurred approximately 4-14 days apart as the students' schedules permitted. During these interviews, the students were asked to solve combinatorics problems, give counting arguments for the veracity of binomial identities, and answer reflection questions about their approach to and reasoning about combinatorial proof.

4.1.2 Mathematician data collection. I recruited mathematicians from three different universities in the western United States. These mathematicians were a convenience sample recruited via email for my study. I included both mathematicians who did and did not conduct research in combinatorics. Table 7.3 shows the research background and experience of the mathematicians.

Table 7.3. Mathematician participants' research and teaching experience information.

Name*	Research Experience	Regularly Taught Combinatorics
Ridley	Algebraic combinatorics & bijective combinatorics (13 years)	Yes
Dominique	Competitive coloring algorithms and parameters defined on graphs (20 years)	Yes
Jaiden	Computability, computable analysis, & algorithmic information theory (3 years)	Yes
Skyler	Dynamical systems and number theory (15 years)	No
Emery	Modular forms and partition functions (17 years)	Yes
Lake	Partial differential equations & related functional analysis (60 years)	No
Justice	Representation theory of finite groups (6 years)	Yes
Robin	Geometry, algebra, and mathematics education (40 years)	No

*These are pseudonyms.

Each mathematician participated in a single, 90-minute individual interview. During these interviews, I asked the mathematicians to give combinatorial proofs of various binomial identities and answer reflection questions about their approach to, reasoning about, and pedagogical opinions of combinatorial proof.

4.2 Data Analysis

To analyze these data, all of the videos were transcribed, and then I re-watched all of the interview videos, making note of key episodes related to my research questions and following the thematic analysis methodology (Braun & Clarke, 2006). I flagged every instance in the data where a participant (student or mathematician) interpreted an expression involving multiplication, and then I exhaustively coded each of these instances according to which cognitive model the participant used. My initial list of codes was based off Mulligan and Mitchelmore's (1997) semantic structures (these can be found in Table 1), and I added new cognitive models to my list as they arose in the data. To decide which code to apply for a given episode in the interviews, I examined the participants' utterances about how they were conceiving of an instance of multiplication and any additional representations they gave of their thought process if they wrote anything down. For instance, if a participant alluded to "copies" or "duplicates," this could indicate they were using an equivalent groups cognitive model of multiplication; likewise, a mention of "Cartesian products" might correspond to the Cartesian product cognitive model. I continued in this manner to code all episodes where a participant interpreted an instance of multiplication until the data reached saturation. I also discussed key episodes and findings that were emerging from the initial analysis with another researcher, and together we reviewed parts of the interviews that warranted additional analysis. We discussed the codes that were being used to ensure that they faithfully represented the data, and any episodes in which it was difficult to determine the participant's cognitive model for multiplication were discussed thoroughly until both my research colleague and I were confident that the code being applied was appropriate. I also followed Lockwood et al. (2019) in considering participants' combinatorial proving activity as their interpreting an expression with an underlying counting process that enumerates/generates some set of outcomes (Lockwood, 2013).

5. Results

In this section, I discuss the results of my investigation into how the students and mathematicians conceived of multiplication when engaging in combinatorial proof. Ultimately, the big points I want to emphasize are 1) there was variety in terms of the cognitive models the mathematicians and students demonstrated in the interviews, and 2) those different cognitive models were not all equally effective in helping the participants correctly solve combinatorial proofs. In fact, while six cognitive models arose in the interviews, only two of the cognitive models were used productively by participants in any of the interviews. For the purposes of this paper, when I characterize a student or mathematician's work as *productive,* I mean that their attempts resulted in a logically and mathematically correct combinatorial argument for the identity. By this I do not intend to imply that a participant's work was not valuable or enriching if their attempts did not result in a correct proof; I simply use the term in this section to distinguish between cognitive models that did (and did not) result in correct proofs.

I frame these results in the following way. First, in Section 5.1, I present the cognitive models for multiplication that the mathematicians and students used in their interviews, which addresses my first research question. Some of the student and mathematician participants used a cognitive model aligned with one of the five semantic structures given by Mulligan and Mitchelmore (1997), but some of them used cognitive models that I had not expected. Second, in Section 5.2, I highlight participant work that demonstrates the two cognitive models that were used on combinatorial proofs that were correctly proven. In particular, in Section 5.2.1 I show a student's use of the element selection model, which was by far the most productively used cognitive model, and in Section 5.2.2 I present a mathematician's use of the equivalent groups model, which was the only instance of a correct proof utilizing this model. Then, to better understand why other models were perhaps not productively used, in Section 5.3 I show two episodes in which the participants were not successful in leveraging cognitive models for the purpose of combinatorial proof. In Section 5.3.1 I show how a student used a weight cognitive model but could not find a way to connect that model to both sides of an identity. Finally, in Section 5.3.2 I discuss an interpretation of multiplication that arose in the data but actually represents a model of *exponentiation* rather than multiplication.

5.1 Combinatorial Provers Used a Variety of Cognitive Models for Multiplication

In this section, I present the cognitive models for multiplication by a constant *k* in the context of binomial identities that emerged during the interviews. I first summarize these cognitive models in Table 7.4. My purpose in this section is to demonstrate instances of the various ways of reasoning about and using multiplication that emerged for these participants. In this paper, I use the term *k*-committee to refer to a committee of size *k* formed from a group of *n* (distinct) people.

Table 7.4. Cognitive models for scalar multiplication used by students and mathematicians.

Cognitive Models	**Brief description (applied to $\binom{n}{k} \times k$)**	**Example from the Data**
Equivalent groups	*k* copies of each *k*-committee	*Emery (considering $\binom{n}{k} \times k$)*: I will get repeats exactly *k*—each choice will be repeated exactly *k* times, and so that's why I'm getting *n* choose *k* times *k*.
Cartesian product	Coordinate pairs with $\binom{n}{k}$ and *k* ways to fill the positions	*Emery (considering $\binom{n}{k} \times k$)*: I could specify, okay so I guess I can think of it as counting pairs of the smallest element and the rest? Wait I'm choosing *k*. I can think of it as counting pairs where the first element in the pair is the smallest element and the second is a set of *k*-elements. *Int.*: And you think of picking that special element first, and then making the group of *k*? Or, what order is that being done in? *Emery*: What I was thinking a Cartesian product. It wouldn't necessarily be in order.

Table 7.4. (Continued)

Scaling factor	Each k-committee is scaled by a factor of k	*Adrien (considering* $15 \times \binom{14}{3}$*)*: When I was reading it like this, the way I was reading it was like as was taking the combination. Its you're taking, you have 14, you choose three. I was thinking of scaling that number somehow.
Inverse of a probability	The multiplicative inverse of the solution to, "If there is a 1-in-k chance that a committee will form at all, what is the probability that a certain committee will form?"	*Riley (considering* $15 \times \binom{14}{3}$*)*: There is a 1 over 15 chance that a congressional committee will be formed. Given that probability and the fact that there are 14 candidates for the council and three positions, what is the multiplicative inverse of the probability that a given council will be selected? Because you have to overcome the probability that it won't happen at all.
Weight	$\binom{n}{k}$ counts the number of bit strings of length n with k 1s, and these bit strings are assigned a weight k.	*Riley (considering* $\binom{5}{k} \times k$*)*: So the thing that's occurring to me is some kind of idea of like a weighted bit string. *Int.*: What do you mean by weighted bit string? *Riley*: Oh, something weird where like you say, "Okay, I have a bunch of five-bit integers. Something like, pick me ones with an associate like k 1s, and then multiply the result of picking that by the number of 1s," which is kind of weird. I'm trying to think of a more or like a less abstract example, because that's almost just kind of like a definitional like, "Well, you can choose ones."
Element selection	Interpreting k as $\binom{k}{1}$, that is, selecting one from k people after forming a k-committee	*Adrien (considering* $\binom{n}{i} \times i$*)*: So, you have a group of n people, and you're trying to select one of them in two stages specifically. So, you have your first stage, where you just select some group of people—it doesn't matter how large it is—and then out of those candidates you then select the final one.

I make just a couple of comments now about these cognitive models, and I elaborate some examples of participants' work that demonstrates these models in the following sections. Overall, there was a notable amount of variety of cognitive models for multiplication that the mathematicians and students leveraged while engaged in combinatorial proof. As can be seen from Table 7.4, the participants' multiplicative cognitive models were varied and differed from those used by Mulligan and Mitchelmore (1997). This aligns with previous findings (e.g., Lockwood et al., 2017; Lockwood & Purdy, 2019a; Lockwood & Purdy, 2019b) showing that while multiplication is a familiar operation for undergraduates, its representations can vary when used in counting and it can involve subtleties not always appreciated by students and instructors. Additionally, while some of the multiplication models that students and mathematicians used overlapped with those previously identified as important to K-12 education and research (e.g. Common Core State Standards Initiative, 2010; Mulligan & Mitchelmore, 1997), several others did not. Perhaps this is not surprising since postsecondary students and mathematicians often use multiplication in more complicated and sophisticated situations than K-12 students, but I nevertheless did not expect the variety that emerged in the data.

5.2 Cognitive Models that Were Productively Used for Combinatorial Proof

Only two of the previously described cognitive models were used productively. In this section, I provide examples of these two cognitive models to demonstrate work that successfully implemented these ideas.

5.2.1. Element selection as a productive cognitive model. The element selection cognitive model of multiplication lends itself nicely to combinatorial proof of binomial identities, as it can be related to the Multiplication Principle, a fundamental concept in combinatorics. To clarify, when I say the *element selection cognitive model of multiplication*, I mean that a counter interprets an instance of scalar multiplication, such as "$\times k$," as a stage in the Multiplication Principle with k options. While the cognitive model was highly useful for my participants and often applies nicely to solutions of combinatorial-proof problems, it often wasn't my participants' go-to cognitive model as one might expect. Each of the five student participants at some point during the interviews got stuck on a binomial identity involving multiplication because they were trying to use another cognitive model of multiplication that wasn't working for them. In these occurrences, a particular instantiation helped them to see that they could use the element selection cognitive model, which was to remind the students that they could represent k as $\binom{k}{1}$ in binomial identities involving multiplication by k. To illustrate how this instantiation seemed productive for students engaging in combinatorial proof, I turn to Adrien's work. Adrien was one of my five student participants.

Adrien was trying to solve the Reverse Counting Problem, which asks, "Write down a counting problem whose answer is $15 \times \binom{14}{3}$." I consider Adrien's work on the Reverse Counting problem to be relevant to my research questions, because interpreting expressions in a combinatorial context lies at the heart of combinatorial proof. After Adrien entertained some ideas that were not productive, I suggested that they recall that $15 = \binom{15}{1}$. The moment I drew their attention to this fact, they immediately articulated that $15 \times \binom{14}{3}$ could count the number of ways to elect a club president and then a 3-person committee from a set of 15 club members. I was impressed that they could articulate a correct counting problem so quickly, and I asked them if writing 15 as $\binom{15}{1}$ was helpful. They responded as follows.

> *Adrien*: So when I was reading it like this, the way I was reading it was as taking the combination—you have 14 you choose 3. I was thinking of, like, scaling that number somehow. So, I was still thinking of it in terms of that number. But then, when you write it like this, it's like, oh, so you started with 15, took 1 specifically, so now it seems like a 2-step process—like, two separate parts of the problem, rather than one part of the problem and, like, oh, how do I scale that?

Note that in the quote, Adrien said that they initially were using a scaling factor cognitive model of multiplication, which was not productive for them. However, once they thought of the multiplication as element selection, prompted by representing 15 as $\binom{15}{1}$, Adrien could quickly and easily interpret the factors in $15 \times \binom{14}{3}$ as representing two stages in the Multiplication

Principle, and hence corresponding to a particular counting process. This is a critical step in constructing a combinatorial proof as argued by Lockwood et al. (in press).

Progressing through subsequent problems in the interviews, Adrien proved multiple binomial identities with the aid of the $k = \binom{k}{1}$ instantiation. These include one of the more challenging identities I gave students in the interviews, $\sum_{i=1}^{n}\binom{n}{i}i = n2^{n-1}$. When I first gave Adrien this identity, they began by writing down the identity with the substitution $n = 5$. Once Adrien did this, they then re-wrote the summation replacing i with $\binom{i}{1}$ and were able to recognize the terms in the summation as choosing i from a set of n distinct things, and then selecting one of those i chosen things (see Figure 7.3.) Adrien then gave a nice combinatorial proof of the identity in the context of selecting a finalist from n people after two selection rounds:

Int.:	What would you say both sides are counting?
Adrien:	So, you have a group of *n* people, and you're trying to select one of them in two stages specifically. So, you have your first stage, where you just select some group of people—it doesn't matter how large it is—and then out of those candidates you then select the final one. And this [left-hand side] sort of does that in the opposite direction. It's, like, it counts, okay, who was the final one? And then who made it to the second round?

We see here again that using the element selection cognitive model was productive in Adrien's combinatorial proof activity. These excerpts illustrate that 1) the element selection cognitive model of multiplication was productive for students solving combinatorial-proof problems, and 2) reminding students that $k = \binom{k}{1}$ is a useful instantiation that may help students to see that they can use this model. Though I only discussed Adrien's work in detail, again, this instantiation was helpful for all five of my student participants.

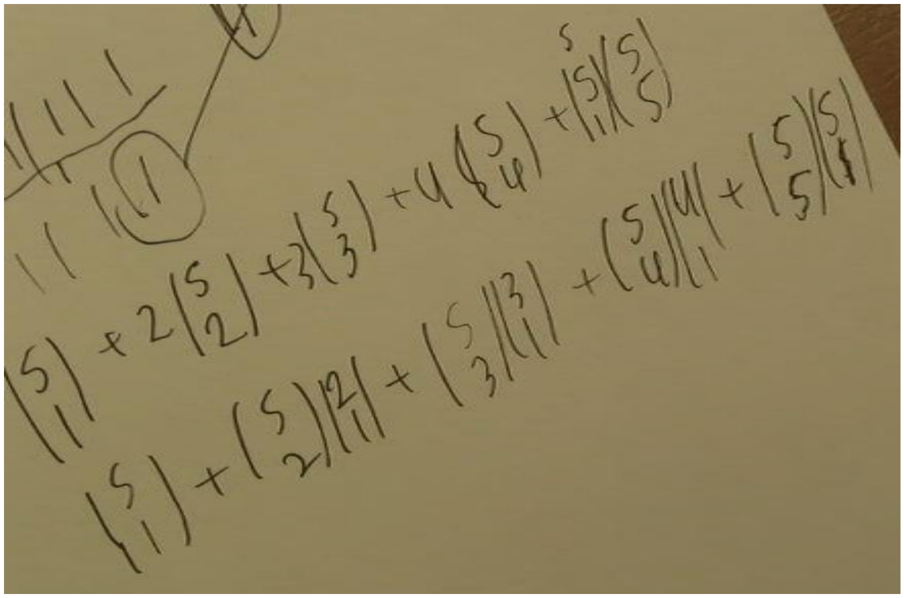

Figure 7.3. Adrien's work proving $n2^{n-1} = \sum_{i=1}^{n} \binom{n}{i}\, i$.

5.2.2. A combinatorial proof utilizing repeated addition and equivalent groups. The only other cognitive model (besides element selection) that the participants used successfully was equivalent groups. One of the eight expert mathematicians interviewed, Emery, successfully proved $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$ using equivalent groups. I describe parts of their work here, although I do not have space to include all of the details.

When I first gave Emery the prompt to provide a combinatorial proof of the identity, they quickly saw that $n \times \binom{n-1}{k-1}$ could be thought of as counting all the way to select one of *n* objects, and then choosing *k* – 1 of the remaining *n* – 1 objects, ultimately resulting in a subset of size *k*. Here Emery was utilizing the element selection cognitive model of multiplication, because to them the multiplication of the binomial coefficient by *n* represents a choice of one object out of *n*. Notice that this counting process (Lockwood, 2013) does not create distinct outcomes, that is, distinct subsets of size *k* from *n* distinct objects. If the objects are numbered 1 to *n*, consider for example the outcome $\{1, 2, \ldots, k\}$. This subset could be generated by first selecting the item '1' and then choosing $\{2, 3, \ldots, k\}$, or this subset could also be generated by first selecting the item '2' and then choosing $\{1, 3, \ldots, k\}$. For each outcome, there are in fact exactly *k* ways that outcome is generated by the process Emery articulated. I did not point this out in the moment during my interview with Emery, as I wanted to see how they would resolve this on their own. As Emery tried to continue the problem, they were unsure why $\binom{n}{k} \times k$ would count the same thing. When they reached this point and were stuck, I asked:

> *Int.*: What might multiplying by *k* be doing?
> *Emery*: Well, I know what it's doing in terms of the factorials.
> *Int.*: Uh huh [affirmative], right.
> *Emery*: But in terms of the counting it's just doing it *k* times, right?
> *Int.*: Mm-hmm. So you're thinking of this as like *k* copies of *n* choose *k*?
> *Emery*: Yeah.

From this exchange, I infer that Emery conceived of the multiplication by *k* as generating *k* copies of whatever $\binom{n}{k}$ is counting, rather than thinking of the multiplication as picking one of the *k* objects chosen in the $\binom{n}{k}$ step. Because I felt that making a combinatorial proof involving a multiset (rather than a set where all the outcomes are distinct) would be challenging, I decided to try to direct Emery back to the element selection cognitive model. To do this, I asked if they could conceive of the multiplication by *k* as picking one of the *k* objects (chosen from *n*) to be special in some way. I thought that then Emery might see that on the right side of the identity, they could think of first designating one of the *n* items to be special and then apply the Multiplication Principle to form the rest of the group of size *k* around the special element. (Hence, both sides count the number of groups of size *k* with a specially designated element.) However, while Emery said they could conceive of the multiplication by *k* in that manner, they continued to struggle with the problem. Even after suggesting that they try using committees (a more concrete context than sets and subsets) to solve the problem, Emery still did not make progress on counting subsets of size *k* (or *k*-committees) with a specially designated element.

Finally, I encouraged Emery to revisit their conception of $\mathrm{n} \times \binom{\mathrm{n}-1}{\mathrm{k}-1}$ as counting the number of ways to make a subset of size *k* from *n* things by picking a first object and then *k* – 1 objects to round out the rest of the subset. As mentioned previously, the set of outcomes generated by this process (Lockwood, 2013) contains duplicates. I asked Emery if they could write down anything that would represent this idea, and then I asked why this process (picking one, then *k* – 1 objects) would generate *k* copies of every size-*k* subset. After a little more thought, they realized that, for instance, the subset {1, 2, …, *k*} would be generated exactly *k* times by this process—once for each of the 1, 2, …, *k* objects chosen first. (See Figure 7.4.) Highlighting the fact that counting a set with duplicates may have been easier for Emery from the start, they said the following:

Emery: I didn't count how many ways I was double-counting. That's the problem. If I had done that I probably would have been done. If I had actually figured out exactly how much I'm double-counting, then I'd be done. Because if I knew I was doing that *k* times I'd have the *k*.

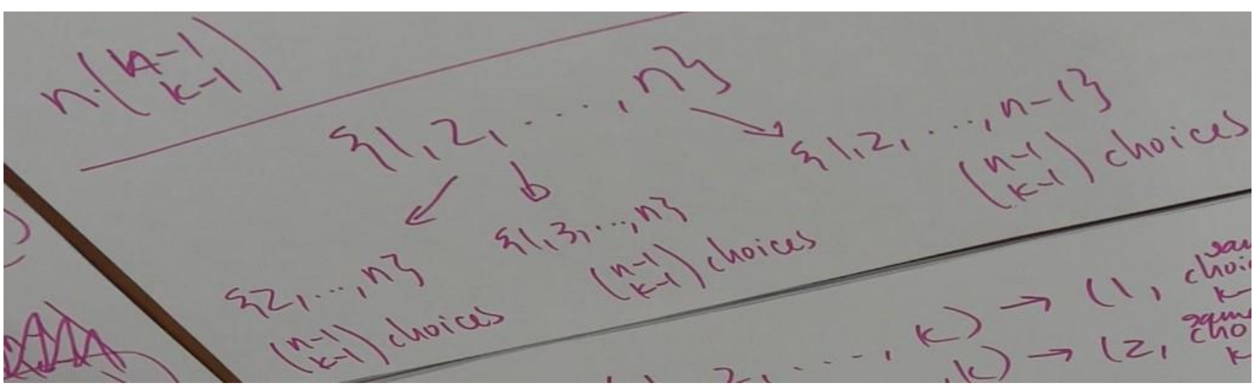

Figure 7.4. Emery's work proving $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$ using repeated addition and equivalent groups.

Emery summarized their final combinatorial proof in the following exchange:

Int.: So maybe, would you mind just summarizing for me what then is your combinatorial argument for why the identity holds?

Emery:	Okay. So my combinatorial argument for why the identity holds is—now I have to think all the way back—to interpret the right-hand side as a choice of *k* objects from *n* objects where the way I'm getting that choice is by taking $k - 1$ from $n - 1$ and plugging one of *n* that wasn't there.
Int.:	Uh huh.
Emery:	Well, I know what it's doing in terms of the factorials.
Int.:	Uh huh, right.
Emery:	I guess what I'm really doing is I'm starting with *n* objects. I am taking one out, and then choosing $k - 1$ of what's left.
Int.:	Uh huh.
Emery:	And then I will get exactly—I will get repeats exactly *k*—each choice will be repeated exactly *k* times, and so that's why I'm getting *n* choose *k* times *k*.

To summarize their work, while Emery initially used the element selection cognitive model when interpreting the multiplication by *n* on the right side of the equation, they used the equivalent groups cognitive model to successfully complete the combinatorial proof. For the left side, rather than conceiving of the multiplication by *k* as selecting one out of *k* objects (that is, using element selection) they interpreted the multiplication by *k* on the left side of the identity as making *k* copies each of all possible subsets of size *k*, and then they argued why the process they articulated on the right side (choosing 1 out of *n* objects and then $k - 1$ out of the remaining $n - 1$ objects) generates the same (multi)set of outcomes.

While this combinatorial proof is correct, I hypothesize that constructing an argument that enumerates a multiset with duplicate objects may be challenging for students. It is not trivial to see that $n \times \binom{n-1}{k-1}$ counts a collection of size-*k* subsets each with *k* copies, and generally combination problems that allow for repetition are more difficult for students than those that do not allow repetition. Indeed, in my interviews with the upper-division mathematics students, none of their combinatorial proof attempts using an equivalent groups cognitive model of multiplication were successful, and ultimately the student participants (and every mathematician participant except

Emery) only used the element selection cognitive model of multiplication to successfully prove binomial identities combinatorially.

To summarize, only two of the multiplicative cognitive models I encountered in the data actually led to successful combinatorial proof attempts: equivalent groups and element selection. However, my findings also suggest that these two cognitive models may not be equally useful for students. Only one participant—one of the mathematicians—was able to use the equivalent groups cognitive model successfully, while all five of the undergraduate student participants eventually used the element selection cognitive model productively.

5.3 Instances of Cognitive Models Not Being Used Productively

Four other cognitive models emerged during the interviews with students and mathematicians, and none of them were used productively in helping the participants correctly prove combinatorial identities. In this section, I briefly describe an episode in which a certain cognitive model arose but was not ultimately productive for the successful completion of a combinatorial proof. My goal in this section is not to criticize the participants, but rather to illustrate that cognitive models are important, and to underscore the idea that some cognitive models may in fact be more productive than others when thinking about proving combinatorial identities.

5.3.1. Weight as a cognitive model that was not productive. Another cognitive model of multiplication that occurred in my data was multiplication as a *weight*. Here, I define the weight cognitive model as when a counter conceives of the multiplication as assigning a weight to some object being counted. To illustrate the weight cognitive model in my data, I show Riley's work on proving $\sum_{i=1}^{n}\binom{n}{i} \times i = n \times 2^{n-1}$ combinatorially. Riley was one of my five student participants.

Initially, Riley struggled with this problem, and so I encouraged them to consider the case $n = 5$. This intervention of encouraging students to consider a particular case of n was found to be helpful in (Lockwood et al., 2019), and so I thought it could help Riley to make some progress on the problem. One benefit of considering a specific case when proving a binomial identity involving a summation is that it allowed Riley to write out all the terms of the summation. Riley did this (see Figure 7.5), and then expressed that the sum could be counting weighted bit strings. This bit strings context was one that Riley used to prove several binomial identities throughout the interviews, but the idea of weights was unique to this problem.

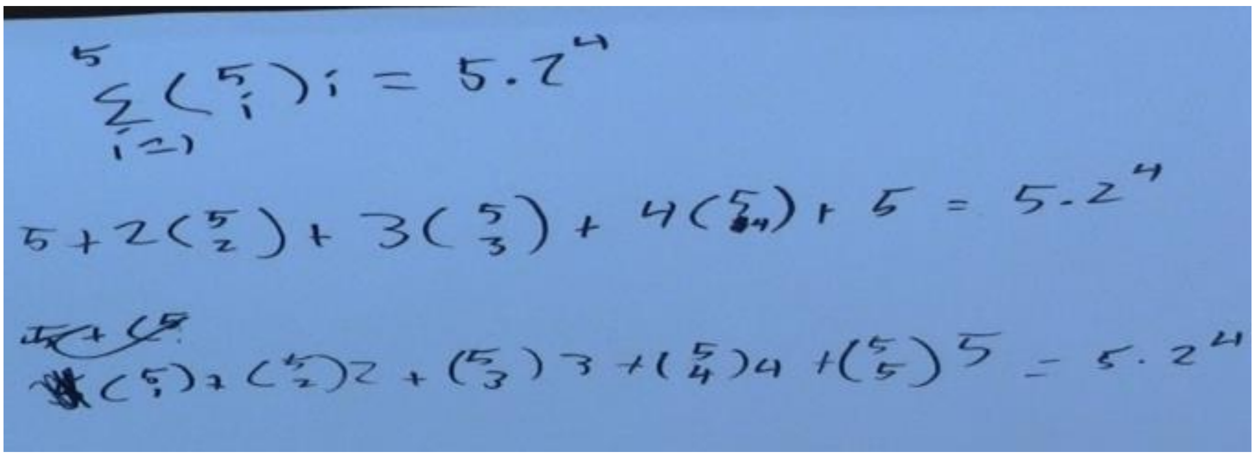

Figure 7.5. Riley's work considering $\sum_{i=1}^{n}\binom{n}{i} \times i = n \times 2^{n-1}$ *in the case where n = 5.*

Riley: So the thing that's occurring to me is some kind of idea of like a weighted bit string.
Int.: What do you mean by weighted bit string?
Riley: Oh, something weird where like you say, "Okay, I have a bunch of five bit integers. Something like, pick me ones with an associate like k 1s, and then multiply the result of picking that by the number of 1s," which is kind of weird. I'm trying to think of a more or like a less abstract example, because that's almost just kind of like a definitional like, "Well, you can choose ones."

After spending some more time thinking about this context, and Riley did not come up with a way that the right-hand side $n \times 2^{n-1}$ could count the same weighted bit strings. This is important, because it suggests that the issue was not that the weight cognitive model multiplication within $\sum_{i=1}^{n}\binom{n}{i} \times i$ is necessarily incorrect, but rather that Riley could then not connect it to the other side of the binomial identity. A bit later, Riley articulated thinking about multiplication as representing weights again, this time in the context of money. Again, they could think of the left side as a weighted sum, but they struggled to make sense of the right side in terms of these weights.

Riley: Because like I'm thinking of this five is like some abstract weight and this three is like a weight.
Int.: So what made you think of averaging? Was it because you're like summing the numbers from one to five?
Riley: Kind of. It was mostly because I'm thinking now in terms of like the weight of these different sets, and this five to me is a matter of the weight of this set. Oh, okay. So now I'm feeling like I'm starting to maybe get somewhere, because you can say, "Okay, well what if … what if we gave the maximum number of dollars this $5 out, in some different distribution?" So, like what if, essentially instead of … Yeah, what if instead of giving all these people this whole combination of dollars, we instead eliminate one person from the group, and do every other combination of them, such that somebody gets $2 and everyone else gets one. So this is, one, $2 … Or I guess, yeah, two, $2 since we've eliminated someone from the group. And then two, or wait, no, sorry. That's right, $2 for $1. No, no. Oh, sorry, yeah, three. Three, $1.
Int.: So what does the five represent in terms of like the $1 and the $2?
Riley: The weight of the set, in other words the number of dollars being distributed. So over here we're thinking of just like a bit string, but like over here we're saying, "Okay, well I have four people and you're allowed to give one of them more than $1," although now that I'm thinking more about it, there's two to the four that I've been playing around with is pretty dependent on the idea of a bit string. So I might've done something wrong in translating that idea.

Here, Riley was still considering the case where $n = 5$. Note that this context of distributing money could lend itself to a correct combinatorial proof (indeed, both sides could count the number of ways to distribute money to a group of n people where exactly one person receives two dollars and everyone else receives either one or no dollars). However, I interpreted Riley's utterances to mean that they conceived of the $\times i$ in the identity as a weight, where the value of the weight is equal to the number of dollars distributed. Ultimately, Riley's attempts to prove the identity combinatorially using a weight cognitive model of multiplication were unsuccessful.

This episode illustrates a couple of ways that Riley tried to prove $\sum_{i=1}^{n}\binom{n}{i} \times i = n \times 2^{n-1}$ using a weight cognitive model for multiplication. By showing these excerpts, my intention is not to criticize Riley's thinking and work. It is understandable that they tried to use this cognitive model, since weighted sums are often useful when solving problems in other domains of mathematics (such as probability), but in this instance the weight cognitive model of multiplication did not end up being very productive for Riley. To me, this highlights the fact that counters at the college level have seen and used multiplication in a wide variety of situations, and so when they are faced with a combinatorial task, they bring with them a variety of multiplicative cognitive models that may affect their combinatorial reasoning.

5.3.2 Conflating models of multiplication and exponentiation. Finally, I turn my attention to an interpretation of multiplication that occurred a few times in interviews both with students and with mathematicians. I did not classify these interpretations as cognitive models of multiplication, because as will be shown, these instead exemplify models of exponentiation.

I first consider Riley's work on the Reverse Counting problem interpreting $15 \times \binom{14}{3}$. When I first gave this problem to Riley, they said, "there are, let's say 15 congressional committees, each should be sized of three. Um, and there are 14 candidates. For each county council and a membership in one council doesn't preclude membership in another council." See Figures 7.6 and 7.7. After I asked Riley to talk more about the outcomes of their counting problem, we had the following exchange.

Riley:	So, um, yeah, so these are committees, um, where like, you know, J uh, Jean, Tim and Bob are, uh, and the fact that they're on two different committees in the exact same combination isn't significant to us in this question….
Int.:	And what do the one and two represent? They represent different committees, different committees they could be one?
Riley:	Yeah, so this would be dot, dot, dot 15.
Int.:	Okay.
Riley:	Um, and maybe these congresspeople are super popular, so they're on every single one, but, okay.
Int.:	So what exactly…do the outcomes look like for this counting problem? Like, could you give a couple of example outcomes.

Riley:	Um, in terms of like just individual councils or like--?
Int.:	Yeah, or if you—like is this an example outcome or do you also have to like tack on people from other committees?
Riley:	I see what you're saying. I think, uh, yeah, so this entire thing is one set.
Int.:	Okay. That whole thing is an outcome?
Riley:	Yeah.

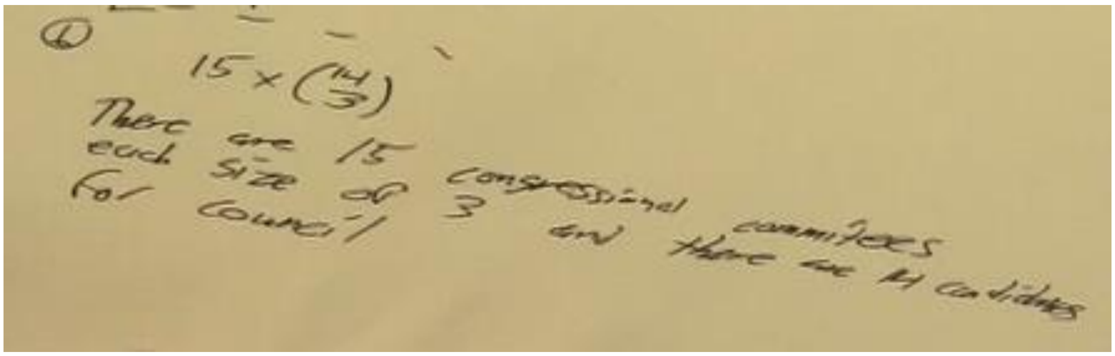

Figure 7.6. *Riley's solution to the Reverse Counting Problem: "There are 15 congressional committees each size of 3 and there are 14 candidates for council."*

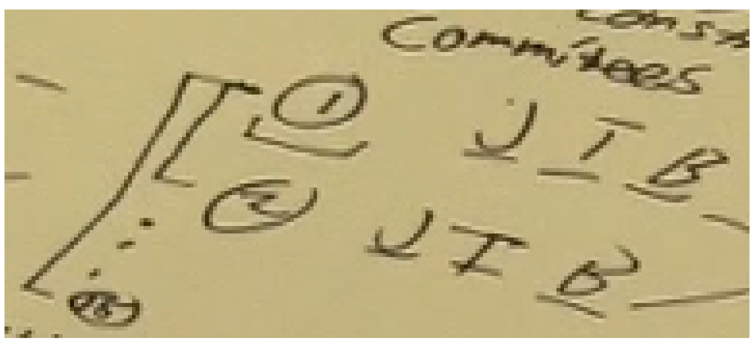

Figure 7.7. *Riley represents their outcomes for the Reverse Counting Problem.*

I interpret Riley's utterances and work on the paper (in Figures 7.6 and 7.7) to mean that they thought of carrying out a counting process where a committee of size three is chosen from a set of fourteen people 15 times (without a person's position on one committee precluding their position on another committee). Indeed, Riley articulated correctly in the quotes I provided that this process would create outcomes that are ordered 15-tuples where each element is a committee of people.

Interpreting multiplication as repeating a process several times came up not only in interviews with students, but with mathematicians as well, so it is an easy error to make. I note, however, that this process and the resulting outcomes are not a model of multiplication, but of exponentiation, and the solution to Riley's counting problem would be $\binom{14}{3}^{15}$.

Another participant who tried to represent multiplication with an exponentiation model was Skyler, one of the mathematician participants. In particular, when I gave Skyler the combinatorial identity $\binom{\mathrm{n}}{\mathrm{k}} \times \mathrm{k} = \mathrm{n} \times \binom{\mathrm{n}-1}{\mathrm{k}-1}$, they said the following and represented their thought process in the case where n = 7 and k = 3 (see Figure 7.8):

> *Skyler:* I have like a matrix of boxes. So, the N choose K times K, there's now N columns and K rows. And so, each row is now N boxes I want to put K things into. And now, it's how many ways can I fill up that entire grid with making sure there's K things in each row. So, the more I think about this, it makes me think like Sudoku kind of matrix.

This interpretation of multiplication is interesting and may be related to Mulligan and Mitchelmore's (1997) *array* semantic structure. However, a key difference here is that in order for multiplication to be faithfully represented as an array, each column (or row) must be identical. The interpretation that Skyler articulated where each row may have k things placed in different locations (like in a Sudoku puzzle) would not be a model for multiplication but for exponentiation. Indeed, the number of ways to fill an $n \times k$ array with k things in each row would be $\binom{n}{k}^{k}$.

The purpose of discussing these episodes is not to point out that Riley and Skyler's work was

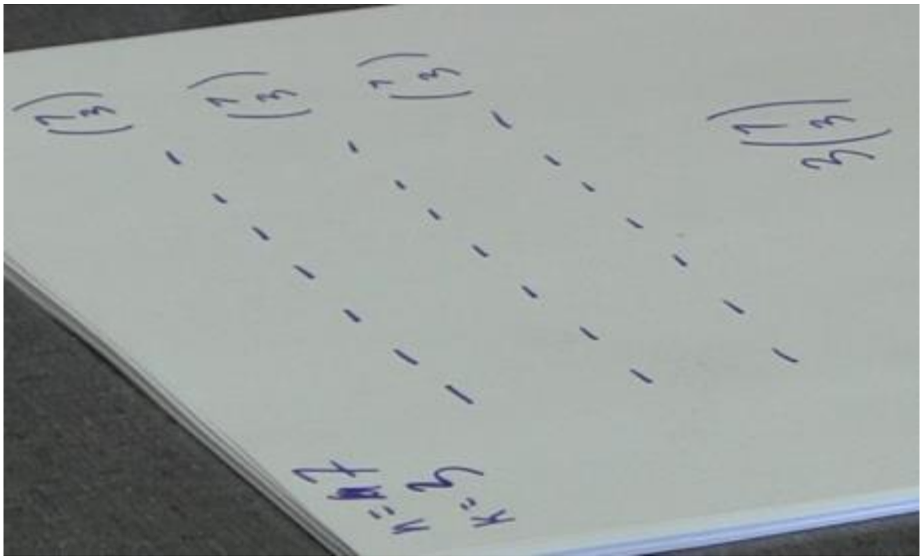

Figure 7.8. Riley represents their thought process for proving $\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$ in the case where n = 7 and k = 3.

wrong (and both overall were very successful at tackling the combinatorial-proof tasks I gave them throughout the interviews). It is noteworthy, however, to see why their errors occurred, and in particular I want to point out how easy it can be for counters—even experts who conduct mathematics research—to conflate situations that involve multiplicative or exponential structures. This corroborates findings from previous research (such as Batanero et al., 1997; Kavousian, 2008; and Sowder et al., 1998) which show that counters can struggle to correctly use multiplication when doing combinatorial tasks. The subtle differences between combinatorial situations involving multiplication and exponentiation is a topic that should be discussed in combinatorics classrooms, since, as I have shown, even experts can conflate the two.

6. Discussion and Conclusions

Here I provide a summary of the results of my study, framed by my research questions. I also discuss implications of this work for teaching practice and future research.

The first research question I asked was, *What cognitive models do undergraduate students and mathematicians use when engaging in combinatorial proof of identities involving scalar multiplication?* I coded students' and mathematicians' cognitive models for multiplication using Mulligan and Mitchelmore's (1997) multiplicative semantic structures and Lockwood's (2013) model of students' combinatorial thinking as a theoretical lens for my analysis. I found that my participants used two (equivalent groups and Cartesian product) of Mulligan and Mitchelmore's (1997) five semantic structures as cognitive models when interpreting binomial identities involving scalar multiplication. In addition, scaling factor, weight, inverse of a probability, and element selection also emerged from the data as cognitive models for multiplication that my participants used. Equivalent groups and, mainly, element selection were the only multiplicative cognitive models that were used in successful combinatorial proofs by participants. This makes sense, since element selection involves a person interpreting scalar multiplication as a stage in the Multiplication Principle, a fundamental counting concept that is taught in nearly all college-level courses that cover counting. Only one of the mathematicians successfully articulated a combinatorial proof using the equivalent groups cognitive model for multiplication, though a couple of the mathematicians and students also made unsuccessful attempts to do this. The multiplicative cognitive models used by our participants were more varied than I had expected, supporting the finding from previous studies that multiplication is a nuanced and can be challenging to apply in combinatorics (e.g., Batanero et al., 1997, Kavousian, 2008, Lockwood & Purdy, 2019a, and Sowder et al., 1998). This suggests that upper-division students and mathematicians have a more varied and nuanced understanding of multiplication in combinatorial contexts than previously thought, and the variety in my data suggests that this could be an interesting avenue for further research.

The second research question I asked was *What are the implications of these cognitive models for students' engagement with combinatorial proof?* Because not a single student and only one mathematician successfully used the equivalent groups cognitive model in their combinatorial argument, this suggests that some students may find this path more challenging when trying to come up with a combinatorial proof. The element selection cognitive model for multiplication was used in all the students' and nearly all the mathematicians' successful combinatorial proof attempts, which is understandable given its connection to the Multiplication Principle. However, I also saw that not all the participants immediately thought to use this multiplicative cognitive model. Indeed, many of them only saw that they could use element selection as a cognitive model for multiplication after they reconceived of multiplication by k as multiplication by $\binom{k}{1}$. I hypothesize that this reformulation is helpful for students because it enables them to more easily recognize the multiplication by a scalar as a stage in a two-step counting process, and indeed I have some data (such as Adrien's work on the Reverse Counting Problem and the identity $\sum_{i=1}^{n} \binom{n}{i} i = n2^{n-1}$) that seem to support this hypothesis.

The fact that the upper-division mathematics students in this study (who had all taken discrete mathematics) frequently did not think to apply the Multiplication Principle was curious, and it corroborate previous work (e.g., by Lockwood & Purdy, 2019a) that suggests undergraduate students—even upper-division mathematics students—do not always recognize situations where multiplication is used while counting. Upper-division mathematics does frequently use multiplication in a variety of contexts. For example, students that are writing proofs in advanced calculus often involve having to "scale" epsilon to find delta in order to prove something is continuous, a limit, etc. Upper-division mathematics students and mathematicians are also familiar with weighted averages in probability. However, none of the cognitive models for multiplication involved in these contexts (scaling factor, inverse of probability, and weight) were helpful when my participants were attempting to engage in combinatorial proof. This suggests that combinatorics instructors, rather than assuming that undergraduate students know when to multiply and what multiplication does when counting, should have discussions with their students about multiplication and highlight that it can be used to count the number of ways to complete a two-stage process using the Multiplication Principle.

In conclusion, multiplication is a familiar operation for undergraduate students, and yet I have shown that there may be implications for the particular ways in which they reason about it. Combinatorial proof and similar types of problems are a context where the subtleties of multiplication emerge, and we see that it is not a trivial topic, even for upper-division mathematics students. While much work has been done examining the ways that K-12 students reason about multiplication (e.g., Greer, 1992; Mulligan & Mitchelmore, 1997; Tillema, 2013), my study indicates that examining undergraduate students' conceptions of multiplication in combinatorial contexts may be fruitful, as these conceptions are varied and have implications for their combinatorial proof activity. While some work has begun to investigate this topic (Lockwood et al., 2017; Lockwood & Purdy, 2019a), how undergraduate students think about and handle the subtleties and variety of multiplicative cognitive models in combinatorial contexts remains largely unknown.

Future avenues of research include continuing to explore the cognitive models of multiplication students and mathematicians may use in combinatorial contexts, and a natural extension of my research would be to investigate cognitive models of other mathematical operations as well (or other instances of multiplication besides scalar multiplication). Furthermore, my results also suggest that when researchers consider multiplication in combinatorial settings in the future, they should attend to different cognitive models that participants might be using as they multiply. It is perhaps best not to assume that participants are adopting any particular cognitive model, but rather explicit attention should be paid to how people might be reasoning about multiplication as they engage with combinatorial tasks.

Finally, this work suggests that from a pedagogical perspective, some cognitive models of multiplication seem to be more productive for students engaging in combinatorial proof activity than others. Encouraging students to think of multiplication by k as $\binom{k}{1}$ if they are stuck may help them more easily see the binomial identity they are working with as corresponding to an underlying counting process (Lockwood, 2013) that uses the Multiplication Principle.

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CHAPTER 8 – Conclusion

The goals of this dissertation study were to examine how experienced provers engaged with combinatorial proof of binomial identities and whether (and, if so, how) they perceived combinatorial proof as different from than other types of proof. My research questions, restated from Section 1.1 of the Introduction, were:

1. To what extent do experienced provers (including students and mathematicians) believe that combinatorial proofs of binomial identities are convincing and/or explanatory, and why?
2. What proof schemes do undergraduate students who are experienced provers use to discuss and characterize combinatorial proof?
3. What do the answers to these questions say about the nature of combinatorial proof (including how it may differ from other types of proof)?
4. What are some other insights about combinatorial proof that can be gained from interviewing experienced provers?

After providing descriptions of literature, theoretical perspectives, and methods, I presented the findings of my dissertation in three manuscripts, which were presented in Chapters 5-7 of this dissertation. The first manuscript (Chapter 5) addressed Research Questions 1 and 3 by presenting results and implications of my investigation into students' and mathematicians' perceptions of combinatorial proof as convincing or explanatory (in the sense of Hersh, 1993). The second manuscript (Chapter 6) answered Research Questions 2 and 3 by describing the proof schemes (Harel & Sowder, 1998) that students brought to their reasoning about combinatorial proof as rigorous mathematical proof. Finally, cognitive models for multiplication emerged as a salient feature of students' and mathematicians' combinatorial proving activity, and I described these in detail along with the broader insights they provided regarding combinatorial proof in the third manuscript (Chapter 7), addressing Research Question 4.

In the following section, I summarize the main findings from each of the three manuscripts of my dissertation, how they answer my overarching research questions, and what general conclusions can be drawn from my dissertation study as a whole. Then in Section 8.2 I discuss limitations of my study as well as avenues for future research.

8.1 Main Findings & Conclusions

I begin this section by first summarizing the key findings from each of the three manuscripts of my dissertation, and then I discuss overarching conclusions and what I feel the significance of my dissertation study may be in the combinatorics- and proof-education literature.

8.1.1 Summary of results from the three manuscripts. In my first manuscript (Chapter 5) I addressed Research Questions 1 and 3 by looking at the extent to which upper-division mathematics students and mathematicians viewed combinatorial proof as convincing or explanatory compared with other types of proof. I found that all of the participants in the study viewed combinatorial proofs of binomial identities as equally or more explanatory than other types of proof, which they felt was related to their perceptions of combinatorial proof as accessible and tangible. This finding is not necessarily unexpected and aligns with Lockwood and colleagues' (2020) theoretical piece where they stated that combinatorial proofs are usually explanatory in the enumerative representation system. However, it is nevertheless useful to confirm these statements with interview data and to gain insight into some reasons why many may consider combinatorial proof to be explanatory. Additionally, there were varied opinions regarding the extent to which combinatorial proofs are convincing. Some participants believed that combinatorial proofs are equally or more convincing than other types of proofs, while others—both some student and mathematician participants—felt that combinatorial proofs are less convincing than other types of proofs. The students who felt this way generally doubted that combinatorial arguments can be rigorous mathematical proofs at all, a phenomenon that is explored more in the second manuscript (Chapter 6) of this dissertation. Some of the mathematicians also felt that combinatorial proofs' use of words and sentences make them potentially less reliable than proofs relying more on symbolic manipulations, and others pointed to the fact that it is easy to produce a counting argument that seems logically correct but that actually contains a subtle error. These results make sense, as it can be more difficult to evaluate written statements than algebraic manipulations, and extensive research has documented that counting problems can be notoriously tricky to solve (e.g., Annin & Lai, 2010; Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; Hadar & Hadass, 1981; Lockwood, 2014a, 2014b).

In my second manuscript (Chapter 6), I used Harel and Sowder's (1998) proof schemes as a lens to take a closer look at characteristics of combinatorial proof that make it seem different for students than some other types of proof. This allowed me to address Research Questions 2 and 3. I found that the students in my study used a variety of proof schemes to discuss and characterize combinatorial proof, including whether they felt such proofs constitute rigorous mathematical proof (and why). There were some students whose reasoning aligned with a transformational proof scheme, and they concluded that since the argumentation in a correct combinatorial proof is valid, combinatorial proofs can be considered rigorous mathematical proofs. Other students used external conviction proof schemes to describe their reasoning, including the ritual proof scheme. These students expressed that because combinatorial proofs have certain ritualistic features (specifically that they are often more intuitive than other types of proof and do not involve symbolic manipulation) they do not qualify as rigorous mathematical proofs. In addition, there were other proof schemes that may have emerged from the data, specifically the authoritarian proof scheme and the perceptual proof scheme, which might describe situations where a student indicated that they did not think their teaching assistant would accept a combinatorial proof (even though the student thought combinatorial proofs are valid), or where students characterized combinatorial proofs as being merely intuitive arguments that make the identity “seem” true but that do not account for all possible cases of the binomial identity intended to be proven. In total, the students in my study may have used authoritarian, ritual, perceptual, transformational, and contextual restrictive proof schemes to discuss and characterize combinatorial proof.

Finally, in my third manuscript (Chapter 7), I addressed Research Question 4 by describing an interesting phenomenon that emerged from analyzing the data. In particular, I present the variety of cognitive models for multiplication that the students and mathematicians used to try to produce combinatorial proofs. In total, six different cognitive models were identified in the data, and only two of those cognitive models—equivalent groups and, mainly, element selection—were used in successful combinatorial proof attempts by participants. This makes sense, since element selection involves a person interpreting scalar multiplication as a stage in the Multiplication Principle, a fundamental counting concept that is taught in nearly all college-level courses that cover counting. Only one of the mathematicians successfully articulated a combinatorial proof using the equivalent groups cognitive model for multiplication, though a couple of the mathematicians and students also made unsuccessful attempts to do this.

8.1.2 Overall conclusions and significance. In addition to the specific results and findings presented in each of the papers, there are some overall findings from this dissertation as a whole. Considering the three manuscripts together, I see four main takeaways from this dissertation study: 1) *students can successfully produce combinatorial proofs and recognize that their activity constitutes proof*; 2) *combinatorial proof may be viewed by some students as intuitive arguments but not formal proofs*; 3) *the contexts used in combinatorial proofs are important*; 4) *difficulties in solving counting problems can carry over to difficulties in combinatorial proof production*.

First, the students in my study overall were very successful at producing combinatorial proofs, showing that upper-division students can engage meaningfully in combinatorial proof tasks. In manuscript 3 (Chapter 7), I found that the students were most successful when they used the element selection cognitive model for multiplication, and in manuscripts 1 and 2 (Chapters 5 and 6) I discussed how some of the students did think their activity (and combinatorial proof more generally) constituted rigorous mathematical proof. These are promising and useful findings for instructors wanting to support their students' success in learning combinatorial proof and for researchers who want to better understand students' work with and understanding of combinatorial proof.

Another important conclusion from this dissertation study is the potential for students to view combinatorial proof as consisting only of intuitive arguments, but not constituting formal mathematical proof. In manuscript 1 (Chapter 5), I discussed how the students and mathematicians universally saw combinatorial proof as at least as explanatory as other types of proof, and this was often related to the perceived accessibility of these proofs. This is certainly understandable, and this idea was also closely tied to ideas discussed in manuscript 2 (Chapter 6). In that manuscript, I described how the students observed combinatorial proofs of binomial identities lack symbolic manipulation, are often short, and use concepts (like forming a group of people) that even children can understand. However, as I also discussed in manuscripts 1 and 2 (Chapters 5 and 6, respectively), the very fact that combinatorial proofs were intuitive and accessible made some students doubt whether they could really constitute rigorous mathematical proof. This is a useful finding for researchers and instructors who want to be knowledgeable about various perceptions students may have about combinatorial proof.

Third, all three manuscripts touched on the importance of contexts as a feature of combinatorial proof. In manuscript 1 (Chapter 5), I described some participants who felt combinatorial proofs' use of tangible contexts was one reason these proofs can be more explanatory than other types of proof, but I described also in manuscript 2 (Chapter 6) how some contexts made students concerned that some combinatorial proofs may be insufficient to prove binomial identities that are stated more generally. I also discussed in manuscript 3 (Chapter 7) how the choice of certain contexts used with some cognitive models of multiplication (such as selecting elements in a set or people in a committee) can help students be more productive engaging in combinatorial proof than other contexts and models of multiplication.

Finally, manuscripts 1 and 3 (Chapters 5 and 7, respectively) shed light on the ways that difficulties students (or mathematicians) may encounter solving counting problems can lead to difficulties producing combinatorial proofs as well. In manuscript 1 (Chapter 5), I described how some students and mathematicians said that they felt combinatorial proofs were potentially less reliable (and therefore less convincing) than other types of proof because of how easy it can be to make a subtle counting mistake. This response is understandable, as the difficulties associated with solving counting problems are well documented in the literature (see, for example, Batanero et al., 1997, and Lockwood, 2014b). In manuscript 3 (Chapter 7) as well, I discussed how some student and mathematician participants struggled with conflating models of multiplication and exponentiation. These findings seem to provide evidence that in order for students to be successful at producing combinatorial proof, they first need to have a robust foundation in enumerative combinatorics.

In conclusion, this dissertation study contributes both to combinatorics- and proof-education literature. In combinatorics education, this study provide new insights into how experienced provers may perceive of combinatorial proof as different from other types of proof, as well as highlights that while multiplication may be a familiar operation for students, the way they reason about it can have implications for their success at combinatorial proof tasks. In proof education, this dissertation study furthers ongoing discussions regarding proofs that explain and proofs that convince (Hersh, 1993). While Lockwood et al. (2020) articulated how combinatorial proofs may relate to these concepts in their theoretical piece, my dissertation study investigated empirically how and why students and mathematicians conceive of combinatorial proofs as explanatory and convincing (or not). Finally, documenting which proof schemes (Harel & Sowder, 1998) students used to discuss and characterize combinatorial proof affords multiple useful insights about the nature of combinatorial proof and represents a novel application of this well-established framework.

Finally, in terms of implications for practitioners and researchers, both groups should be aware that students may have a variety of conceptions about combinatorially proof as they teach and conduct proof-education research, respectively. In the classroom, instructors should understand that some students may believe combinatorial proof is less valid than algebraic, induction, or other types of proof for a variety of reasons, and so instructors should clarify for students why correct combinatorial proofs are indeed mathematically rigorous and logically valid. Instructors should also have discussions with their students about the element selection cognitive model of multiplication and highlight its relationship with the Multiplication Principle. Lastly, when researchers draw conclusions about student thinking about proof, they should be mindful that some of these conclusions may apply differently to student thinking about combinatorial proof.

8.2 Limitations and Considerations for Future Research

Regarding limitations, while I had intended to interview up to nine students for my study, only seven responded when I recruited in upper-division mathematics courses, and only five of those seven were suitable for further interviews after the round of selection interviews. Secondly, because this study had a sample size of only 13 experienced provers, I cannot make any generalizable claims about how mathematicians and upper-division students think about or engage with combinatorial proofs. I also used a convenience sample of mathematicians from only three universities, and I may have gotten a more diverse set of perspectives had I instead taken a random sample of mathematicians from more universities. Similarly, it is also possible that by focusing only on students from one university with certain experiences, I inadvertently limited perspectives and insights that a broader swath of students might have afforded. These limitations offer opportunities for avenues for future research, though, which I elaborate below.

In terms of proofs that explain and/or convince (Hersh, 1993), this work provides some examples of how some provers may think about these concepts, and future studies with larger sample sizes or that look at other populations would likely yield further insights. I also think that future research should continue to investigate proof schemes that students use to continue uncovering ways that the nature of combinatorial proof may differ from other types of proof. My study is a first step, but future research with a larger sample size or different populations would continue to shed light on students' use of proof schemes in combinatorial proof. For instance, perhaps other proof schemes (such as the axiomatic or generic proof scheme) may emerge, or we may see more widespread use of authoritarian or perceptual empirical proof schemes. In addition, future research could investigate not only the proof schemes students use to discuss and characterize combinatorial proof, but also proof schemes students use to produce combinatorial proofs. In addition, future avenues of research could continue to explore the cognitive models of multiplication students and mathematicians may use in combinatorial contexts, and a natural extension of my research would be to investigate cognitive models of other mathematical operations as well (or other instances of multiplication besides multiplication by a positive integer scalar). Finally, it would be interesting to see future research look at bijective combinatorial proof, as this was not a focus of my work and no previous studies on combinatorial proof have focused on these types of proof either.

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APPENDICES

Appendix A. Mathematician Interview Protocol

1. Prior experience with combinatorial proof

   First ask the following questions:
   - What is your research area? How long have you been conducting mathematics research?
   - How would you define a mathematical proof?
   - How would you define a combinatorial proof?
   - Do you ever use combinatorial proof in your research? How important is combinatorial proof in your field?
   - Do you ever teach classes that cover combinatorial proof of binomial identities? How frequently? When did you teach combinatorial proof most recently?

2. Combinatorial proof

   Next, give the expert a subset of the following binomial identities (one at a time) and ask them to provide a combinatorial proof.

Table A.1. Identities given to the mathematicians to provide a combinatorial argument.

$\binom{n}{k} = \binom{n}{n-k}$	$2^n = \sum_{i=0}^{n} \binom{n}{i}$
$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$	$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$
$\sum_{i=1}^{n} \binom{n}{i} i = n \cdot 2^{n-1}$	$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$
$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i \geq 0} \binom{n}{2i+1}$	$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$

3. Reflecting on combinatorial proof

   Begin this section of the interview by showing the expert mathematician the following two binomial identities and proofs. Give them time to read through.

Table A.2. Six Proofs handout.

Identity	Combinatorial Proof	Non-combinatorial proof
Theorem 1. $2^n = \sum_{i=0}^{n}\binom{n}{i}$	(Subsets Context) Consider a set *S* such that $\lvert S \rvert = n$. The LHS* of the equation counts the number of subsets of *S*, because every subset can be uniquely determined by the elements it contains, and each of the *n* elements could be either in or out of each subset. The RHS counts the number of *i*-subsets of *S* and adds up over all possible values of *i*. Since the LHS and RHS both enumerate the set of subsets of *S*, they are equal.	(Induction RS*) Suppose $n=0$. It follows that the identity holds since $2^0 = 1 = \binom{0}{0}$. Suppose that the identity holds for $n=k$, where *k* is a nonnegative integer. We then observe that $\sum_{i=0}^{k+1}\binom{k+1}{i} = \sum_{i=0}^{k}\left(\binom{k}{i}+\binom{k}{i-1}\right)+\binom{k+1}{k+1}$ $= \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k}\binom{k}{i-1}+1$ $= 2^k+\sum_{i=0}^{k-1}\binom{k}{i}+1$ $= 2^k+2^k-\binom{k}{k}+1$ $= 2\cdot 2^k-1+1$ $= 2^{k+1}$.
Theorem 2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	(Committees Context) Suppose a mathematics department has *n* faculty members, and Sofía is one of the faculty members. The LHS counts the total number of committees of size *k* that could be formed from the *n* faculty members. The RHS counts the number of committees of size *k* that exclude Sofía and the committees that include her. Note that this case breakdown encompasses all possible *k*-committees. Since the LHS and RHS both enumerate the same set of outcomes (*k*-committees formed from the *n* faculty members), they are equal.	(Algebraic RS) We have that $\binom{n-1}{k}+\binom{n-1}{k-1}$ $= \frac{(n-1)!}{k!\,(n-1-k)!}+\frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$ $= \frac{(n-1)!}{k!\,(n-1-k)!}+\frac{(n-1)!}{(k-1)!\,(n-k)!}$ $= \frac{(n-1)!\,(n-k)}{k!\,(n-k)!}+\frac{(n-1)!\,k}{k!\,(n-k)!}$ $= \frac{n(n-1)!-k(n-1)!+k(n-1)!}{k!\,(n-k)!}$ $= \frac{n!}{k!\,(n-k)!}$ $= \binom{n}{k}$.

*RS here refers to representation systems, in the sense of Lockwood, Caughman, and Weber (2020).

A.2. (Continued)

Theorem 3. $\binom{n}{k} = \binom{n}{n-k}$	(Binary Strings Context) Consider the set of binary strings of length *n* containing exactly *k* 0s. The LHS enumerates this set, because $\binom{n}{k}$ is the number of ways we can select positions for the 0s to occupy, and the rest of the positions in the binary string will be 1s. The RHS also enumerates this set, because $\binom{n}{n-k}$ is the number of ways we can select positions for the 1s to occupy, and the rest of the positions in the binary string will be 0s.	(Binomial Theorem RS) Recall that the Binomial Theorem states that for *n* a natural number and *a*, *b* real numbers, $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$ Notice that for each *k*, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. Additionally, we also have that by the Binomial Theorem, $$(b+a)^n = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^i,$$ and the coefficient of $b^k a^{n-k}$ is $\binom{n}{n-k}$. We also have that $a^{n-k}b^k = b^k a^{n-k}$ and $(a+b)^n = (b+a)^n$, by the commutativity of multiplication and addition of real numbers, respectively. Thus, when the latter is expanded, the coefficients of each term on either side of the equation must be equal, so $\binom{n}{k} = \binom{n}{n-k}$ for all *k*.

After the expert has finished reading these proofs, ask them what they think it means for a mathematical proof to be *convincing* and what it means for a mathematical proof to be *explanatory*. Next, give them the following definitions to read from Weber (2002):

- "A **proof that convinces** begins with an accepted set of definitions and axioms and concludes with a proposition whose validity [was] unknown…. The intent of this type of proof is to convince one's audience that the proposition in question is valid. By inspecting the logical progression of the proof, the individual should be convinced that the proposition being proved is indeed true" (p. 14).
- "A **proof that explains** also begins with an accepted set of definitions and axioms and concludes with a proposition whose validity [was] not intuitively obvious, although another proof of this theorem might already be known. In contrast to proofs that convince, proofs that explain need not be totally rigorous…. The intent of this proof is to illustrate intuitively why a theorem is true. By focusing on its general structure, an individual can acquire an intuitive understanding of the proof by grasping its main ideas" (p. 14).

Next, proceed to ask some or all of the following reflection questions:

- For you personally, do you find combinatorial proofs of binomial identities to be convincing (compared to, say, algebraic proofs, proofs by induction, or proofs that use the Binomial Theorem)? Why or why not?
- Do you think the combinatorial you wrote of, say, $\boldsymbol{\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}}$, is convincing? Who do you think would find it convincing? How come?
- For you personally, do you find combinatorial proofs of binomial identities to be explanatory (compared to, say, algebraic proofs, proofs by induction, or proofs that use the Binomial Theorem)? Why or why not?
- Do you think the combinatorial you wrote of, say, $\boldsymbol{\binom{n}{k} \cdot k = n \cdot \binom{n-1}{k-1}}$, is explanatory? What do you think it explains? Who do you think would find it explanatory? How come?
- Do combinatorial proofs provide a structural explanation for why a binomial identity holds that algebraic or induction proofs cannot provide, or vice-versa? Why or why not?
- Can you think of a time in your research when you used a combinatorial proof, or when you read one in the literature? Why was a combinatorial proof needed/desired?
- What do you think students need to know–both in terms of procedural and conceptual knowledge–in order to be successful at combinatorial proof? What disposition(s) do they need to have toward combinatorics, proof, and/or mathematics in general?
- Broadly, what is your approach to teaching combinatorial proof? How do you prepare students to be successful at it?
- Does teaching combinatorial proof feel very different than teaching proof in other domains (e.g. algebra or analysis)?

- Are there any particular techniques/strategies/tricks you think students should be exposed to when learning combinatorial proof?
- Are there any techniques/strategies/tricks you want your students to use when proving binomial identities combinatorially?
- Do you find that students tend to struggle with combinatorial proof? Why do (or why don't) you think that could be?
- Do you think students attempt to copy a "recipe" for how combinatorial proofs are written without attending to what they are really counting (in the sense of pseudo-semantic proof production as described by Engelke Infante & CadwalladerOlsker, 2011).
- Do you think students (e.g. students in classes you've taught) find combinatorial proof to be a convincing argument for why a binomial identity is true, compared to an algebraic and/or induction proof? Why/why not?
- Do you think students (e.g. students in classes you've taught) find combinatorial proof to be an explanatory argument for why a binomial identity is true, compared to an algebraic and/or induction proof? Why/why not?
- Do you think the combinatorial proof given for Theorem 2, $\boldsymbol{\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}}$, would be more convincing to students if it used a more abstract context (such as counting subsets)?
- Do you think the combinatorial proof given for Theorem 1, $\boldsymbol{2^n = \sum_{i=0}^{n} \binom{n}{i}}$, would be more explanatory for students if it used a more concrete context, such as counting committees or binary strings?
- For students in classes you've taught, do you think combinatorial proofs provide a more useful and/or meaningful explanation for why a binomial identity is true, compared to algebraic and/or induction proofs? Why/why not?
- Are there any particular examples you like to use in class and/or problems that you have them solve in their homework? Why?
- Do you think students (math and/or non-math majors) should learn combinatorial proof? Why/why not?
- What mathematical practices (e.g. conjecturing, justifying, pattern noticing, etc.) do you hope your students will engage in when they prove binomial identities combinatorially?
- Are there any particular contexts (e.g. committees, passwords, block-walking) that you think students should be exposed to when learning combinatorial proof?
- Are there any particular contexts (e.g. committees, passwords, block-walking) you want your students to use when proving binomial identities combinatorially?
- Do you have any (other) thoughts about how curriculum covering combinatorial proof could be improved?

4. Demographic information

   Finally, give the expert the following demographic questions to answer on a paper form (if they are willing to provide this information):

- **(Optional)** Describe your race/ethnicity. Select all that apply.
  - Asian
  - Black/African American
  - White/Caucasian
  - Hispanic/Latinx
  - Native American/Alaska Native
  - Pacific Islander
  - Other: ________________________________
  - Would prefer not to say

- **(Optional)** Which of the following best describes your gender? Select all that apply.
  - Woman
  - Man
  - Nonbinary
  - Agender
  - Genderqueer
  - Other: ________________________________
  - Would prefer not to say

- **(Optional)** Please provide your pronouns. Select all that apply.
  (This question is included for purposes related to writing and presenting results of this research. If you choose not to provide a pronoun, we will only refer to you using a pseudonym, i.e. without using any pronouns for you.)
  - She/her
  - He/him
  - They/them
  - Other: ________________________________

Appendix B. Student Selection Interview Protocol

1. Solving counting problems.

   Give the student the following counting problems, one at a time.

   Domino Problem. *A domino is a small, thin rectangular tile that has dots on one of its broad faces. That face is split into two halves, and there can be zero through six dots on each of those halves. Suppose you want to make a set of dominos (i.e., include every possible domino). How many distinguishable dominos would you make for a complete set?*[16]

   Committees Problem. *A university department has 30 faculty members.*
   - *d) How many ways could a 5-member hiring committee be formed?*
   - *e) How many ways could a 5-member hiring committee be formed if one of the committee members must be the chairperson?*
   - *f) In the university department, 17 faculty members are professors and 13 are instructors. How many ways could a 5-member hiring committee be formed if the committee must consist of 3 professors and 2 instructors? (The committee won't have a chairperson.)*

   Power Set Problem. *Let S be a set containing 5 (distinct) elements. How many subsets are there of the set S? (That is, what is the cardinality of $P(S)$, the power set of S?)*

   Binary Strings Problem. *A binary string is a finite sequence containing only 1s and 0s.*
   - *c) How many binary strings of length 8 contain exactly 5 0's?*
   - d) *How many binary strings of length n contain exactly k 0's?*

2. Writing basic proofs.

   Next, ask the students to prove some or all of the following theorems.

   Theorem 1. *The sum of two even integers is an even integer.*

   Theorem 2. *Let* n *be a nonnegative integer. Then,*

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

   Theorem 3. *Let n and k be nonnegative integers such that $n \geq k$. Then,*

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[16] This problem is used with permission from (Lockwood, Swinyard, Caughman, 2015).

$$\binom{n}{k} \times k = n \times \binom{n-1}{k-1}$$

3. Reflection questions

   Ask the student some or all of the following reflection questions:
   - What would you say if one of your professors asked you to explain what a mathematical proof is?
   - What would you say if one of your friends or family members (not a peer studying your major) asked you to explain what a mathematical proof is?
   - What do you think is the purpose of proof for mathematicians?
   - What do you think is the purpose of proof in mathematics classrooms?
   - What can or should a proof contain for it to convince you (personally) that a given theorem is true? (e.g. Does a proof have to contain algebraic manipulation of symbols? Should proofs contain pictures? Is a proof by induction convincing for you? Why/why not?)
   - What can or should a proof contain to help you (personally) understand why a theorem is true? (e.g. do you feel a proof involving symbolic manipulation adequately explains why a theorem is true? What about a proof by induction? What about proofs that contain pictures?)
   - What are the things that come to mind for you when you see (n choose k) (e.g. do you think only about its formula? Does it make you think of a class of or specific counting problem(s)? Do you think of Pascal's Triangle?)

4. Demographic Information

   Finally, give the student the following demographic questions to answer on a paper form (if they are willing to provide this information):
   - What is your major, and what year are you (freshman, sophomore, junior, senior)?
   - What math classes are you currently taking? What other math classes have you taken in college?

Appendix C. Student Interview Protocol

1. Solving more counting problems and finding bijections.

   I will ask students to solve counting problems involving combinations and the four operations (addition, subtraction, multiplication, and division). As they solve these, I will ask the students to articulate what each of these things mean regarding sets of outcomes, and I will ask how they know when to use addition versus multiplication, or subtraction versus division. I will ask them to solve all of the following counting problems. I will also ask them to list outcomes and to create explicit bijections between outcomes.

Table C.1. Combinatorial tasks for students to scaffold combinatorial proof.

Task	**Intended Purpose**
1. Spoonbill Problem. The scientific name of the roseate spoonbill (a species of large, wading bird) is *Platalea ajaja*. How many arrangements are there of the letters in the word AJAJA? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving ordered sequences of two indistinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
2. Subsets Problem. How many 3-element subsets are there of the set {1, 2, 3, 4, 5}? Can you list all of the outcomes?	Ensure students are familiar (or to familiarize them) with combination problems involving unordered selections of distinguishable objects. Encourage students to use a set-oriented perspective (Lockwood, 2014a) when counting.
3. Find-a-Bijection Problem. Describe a bijection between the outcomes in the Spoonbill Problem and the Subsets Problem.	Facilitate a robust, flexible understanding of combinations. Lay groundwork for students to solve bijective combinatorial-proof problems.
4. Even- and Odd-Sized Sets Problem. Let S={1, 2, 3, 4, 5, 6}. (a) List all of the even-sized subsets of S. How many should there be? (b) List all of the odd-sized subsets of S. How many should there be? (c) Find a bijection between the subsets in parts (a) and (b) by considering whether the subsets contain the item 1.	Continue to facilitate a solid understanding of combinations. Provide scaffolding for students to eventually prove the identity $\sum_{i\geq 0}\binom{n}{2i} = \sum_{i\geq 0}\binom{n}{2i+1}$ using a bijective combinatorial proof.
5. Reverse Counting Problem. (a) Write down a counting problem whose answer is 2^5. (b) Write down a counting problem whose answer is $15 \times \binom{14}{3}$.	Provide scaffolding for the concept of a combinatorial proof by asking students to interpret expressions in a combinatorial context.

2. Write combinatorial proof

   In this section, I will ask the students to justify why some or all of the following binomial identities hold by coming up with a counting problem that each side of the identity enumerates:

$$\binom{\boldsymbol{n}}{\boldsymbol{k}} \cdot \binom{\boldsymbol{k}}{\boldsymbol{r}} = \binom{\boldsymbol{n}}{\boldsymbol{r}} \cdot \binom{\boldsymbol{n-r}}{\boldsymbol{k-r}}$$

$$\binom{\boldsymbol{n}}{\boldsymbol{k}} = \binom{\boldsymbol{n-1}}{\boldsymbol{k}} + \binom{\boldsymbol{n-1}}{\boldsymbol{k-1}}$$

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{n}} \binom{\boldsymbol{n}}{\boldsymbol{i}} \cdot \boldsymbol{i} = \boldsymbol{n} \cdot \boldsymbol{2^{n-1}}$$

   During this section, I may ask students some or all of the following questions about the identities and their reasoning about them:
   - What could this be counting?
   - What if you tried plugging in specific numbers for *n*, *k*, or *r*?

   In this section, I will be careful not to call the students' justifications "proofs." This way, I will minimize my impact on their answers to the reflection questions in the next section.

3. Evaluate combinatorial versus noncombinatorial proofs of binomial identities

   In this section, I will give them the following proofs to evaluate and reflect on (along with reflecting on their prior activity) as they answer questions about their mathematical thinking.

Table C.2. Six Proofs handout.

Identity	Combinatorial Proof	Non-combinatorial proof
Theorem 1. $2^n = \sum_{i=0}^{n}\binom{n}{i}$	(Subsets Context) Consider a set *S* such that |*S*|=*n*. The LHS* of the equation counts the number of subsets of *S*, because every subset can be uniquely determined by the elements it contains, and each of the *n* elements could be either in or out of each subset. The RHS counts the number of *i*-subsets of *S* and adds up over all possible values of *i*. Since the LHS and RHS both enumerate the set of subsets of *S*, they are equal.	(Induction RS*) Suppose *n=0*. It follows that the identity holds since $2^0 = 1 = \binom{0}{0}$. Suppose that the identity holds for *n*=*k*, where *k* is a nonnegative integer. We then observe that $\sum_{i=0}^{k+1}\binom{k+1}{i} = \sum_{i=0}^{k}\left(\binom{k}{i}+\binom{k}{i-1}\right)+\binom{k+1}{k+1}$ $= \sum_{i=0}^{k}\binom{k}{i}+\sum_{i=0}^{k}\binom{k}{i-1}+1$ $= 2^k+\sum_{i=0}^{k-1}\binom{k}{i}+1$ $= 2^k+2^k-\binom{k}{k}+1$ $= 2\cdot 2^k-1+1$ $= 2^{k+1}$.
Theorem 2. $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$	(Committees Context) Suppose a mathematics department has *n* faculty members, and Sofía is one of the faculty members. The LHS counts the total number of committees of size *k* that could be formed from the *n* faculty members. The RHS counts the number of committees of size *k* that exclude Sofía and the committees that include her. Note that this case breakdown encompasses all possible *k*-committees. Since the LHS and RHS both enumerate the same set of outcomes (*k*-committees formed from the *n* faculty members), they are equal.	(Algebraic RS) We have that $\binom{n-1}{k}+\binom{n-1}{k-1}$ $=\frac{(n-1)!}{k!\,(n-1-k)!}+\frac{(n-1)!}{(k-1)!\,(n-1-k+1)!}$ $=\frac{(n-1)!}{k!\,(n-1-k)!}+\frac{(n-1)!}{(k-1)!\,(n-k)!}$ $=\frac{(n-1)!\,(n-k)}{k!\,(n-k)!}+\frac{(n-1)!\,k}{k!\,(n-k)!}$ $=\frac{n(n-1)!-k(n-1)!+k(n-1)!}{k!\,(n-k)!}$ $=\frac{n!}{k!\,(n-k)!}$ $=\binom{n}{k}$.

*RS here refers to representation systems, in the sense of Lockwood, Caughman, and Weber (2020).

C.2. (Continued)

Theorem 3. $\binom{n}{k} = \binom{n}{n-k}$	(Binary Strings Context) Consider the set of binary strings of length *n* containing exactly *k* 0s. The LHS enumerates this set, because $\binom{n}{k}$ is the number of ways we can select positions for the 0s to occupy, and the rest of the positions in the binary string will be 1s. The RHS also enumerates this set, because $\binom{n}{n-k}$ is the number of ways we can select positions for the 1s to occupy, and the rest of the positions in the binary string will be 0s.	(Binomial Theorem RS) Recall that the Binomial Theorem states that for *n* a natural number and *a*, *b* real numbers, $$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i.$$ Notice that for each *k*, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$. Additionally, we also have that by the Binomial Theorem, $$(b+a)^n = \sum_{i=0}^{n} \binom{n}{i} b^{n-i} a^i,$$ and the coefficient of $b^k a^{n-k}$ is $\binom{n}{n-k}$. We also have that $a^{n-k}b^k = b^k a^{n-k}$ and $(a+b)^n = (b+a)^n$, by the commutativity of multiplication and addition of real numbers, respectively. Thus, when the latter is expanded, the coefficients of each term on either side of the equation must be equal, so $\binom{n}{k} = \binom{n}{n-k}$ for all *k*.

Here are some examples I may ask them about their mathematical thinking:

- What would you say it means for a proof to be *convincing*? What does it mean for a proof to be convincing for an individual? For a mathematical community?
- Some researchers say that a convincing proof begins with an accepted set of definitions and axioms and utilizes correct, formalized logical progression to conclude with the proposition intended to be proven. The purpose of a convincing proof is to convince a reader that the proposition is true.[17] Do you think that the arguments you used to prove binomial identities in the previous section are convincing proofs? Why or why not?
- What would you say it means for a proof to be *explanatory*? What does it mean for a proof to be explanatory for an individual? For a mathematical community?
- Some researchers say that an explanatory proof also begins with an accepted set of definitions and axioms and concludes with the proposition intended to be proven. But, explanatory proofs need not be totally rigorous and function to illustrate intuitively (and often structurally) why the proposition is true.[18] Do you think the arguments you used to prove binomial identities in the previous section are explanatory proofs? Why or why not?
- Take a look at these proofs of some binomial identities from the previous section. Do you think the induction/algebraic proof is more convincing than the combinatorial proof? Why or why not? Do you think the induction/algebraic proof is more explanatory than the combinatorial proof? Why or why not?
- Do you think the combinatorial proofs in this section that use abstract subsets are more like a “real proof" than the arguments you used in the previous sections with committees or binary strings? Why or why not? What are you taking to be a “real proof"?

4. Write more combinatorial proofs

In this section, give them some or all of the following more challenging binomial identities to prove and observe their activity. Again, encourage the utilization of specific numbers and explicitly asking what an expression might be counting if they get stuck, following Lockwood et al. (in press).

[17] This is modified from the definition of a *proof that convinces* given by Weber (2002, p. 14).
[18] This is modified from the definition of a *proof that explains* given by Weber (2002, p. 14).

Table C.3. Additional identities given to the students to provide a combinatorial argument.

$\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \binom{2n}{n}$	$\sum_{i\geq 0} \binom{n}{2i} = \sum_{i\geq 0} \binom{n}{2i+1}$
$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$	$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}$
$\frac{n}{k}\binom{n-1}{k-1} = \binom{n}{k}$	$\frac{n+1-k}{k}\binom{n}{k-1} = \binom{n}{k}$
$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$	$\sum_{i=0}^{n} \binom{n}{i} \cdot 2^i = 3^n$

5. Reflection questions

   At the end of the interview, I will ask some questions that ask students to reflect on their experience in the interviews. The following are representative of the kinds of questions I may ask:

   - Now that you've written more of these combinatorial arguments, have you changed your mind at all about whether or not you believe a combinatorial proof is a "real proof"? Do you think in order for it to be a real proof it has to use abstract language such as subsets, or can a combinatorial proof still be a real proof if it uses committees or binary strings?
   - Did your idea of what constitutes a real proof change at all since the previous line of questioning?
   - What would you say if one of your professors asked you to explain what a mathematical proof is?
   - What would you say if one of your friends or family members (not a peer studying your major) asked you to explain what a mathematical proof is?
   - Do you think presenting a counting argument to someone (say, a classmate in one of your proof-based mathematics classes) would help them better understand conceptually why a binomial identity holds?
   - If you were given proof by induction and a combinatorial proof of a binomial identity, would you think the induction proof or combinatorial proof is more convincing? Why?
   - If you were given an algebraic proof and a combinatorial proof of a binomial identity, would you think the algebraic proof or combinatorial proof is more convincing? Why?
   - If you were given proof by induction and a combinatorial proof of a binomial identity, would you think the induction proof or combinatorial proof does a better job explaining *why* the identity holds? How come?
   - If you were given an algebraic proof and a combinatorial proof of a binomial identity, would you think the algebraic proof or combinatorial proof does a better job explaining *why* the identity holds? How come?
   - What do you think is the purpose of proof for mathematicians?
   - What do you think is the purpose of proof in mathematics classrooms?
   - What can or should a proof contain for it to convince you (personally) that a given theorem is true? Is this different from what a proof should contain if you want to convince someone else (e.g. a friend, a peer in your major, your mathematics professor)?
   - What can or should a proof contain to help you (personally) understand *why* a theorem is true?

6. Demographic information

   Finally, give the student the following demographic questions to answer on a paper form (if they are willing to provide this information):

- **(Optional)** Describe your race/ethnicity. Select all that apply.
  - Asian
  - Black/African American
  - White/Caucasian
  - Hispanic/Latinx
  - Native American/Alaska Native
  - Pacific Islander
  - Other: ________________________________
  - Would prefer not to say

- **(Optional)** Which of the following best describes your gender? Select all that apply.
  - Woman
  - Man
  - Nonbinary
  - Agender
  - Genderqueer
  - Other: ________________________________
  - Would prefer not to say

- **(Optional)** Please provide your pronouns. Select all that apply.
  (This question is included for purposes related to writing and presenting results of this research. If you choose not to provide a pronoun, we will only refer to you using a pseudonym, i.e. without using any pronouns for you.)
  - She/her
  - He/him
  - They/them
  - Other: ________________________________