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Title: A MODEL OF NON-EUCLIDEAN GEOMETRY IN THREE DIMENSIONS

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This paper is a continuation of William Zell's thesis, A Model of Non-Euclidean Geometry in Three Dimensions. The purpose of that thesis was to show that the axioms of non-Euclidean geometry are consistent if Euclidean geometry and, hence, arithmetic is consistent. Mr. Zell discussed the axioms of connection and order and the axiom of parallels, and we continue here with the topic of congruence and the axiom of Archimedes. Thus only consideration of the axiom of completeness remains to complete the model.

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A Model of Non-Euclidean Geometry in Three Dimensions, II

by

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A MODEL OF NON-EUCLIDEAN GEOMETRY IN THREE DIMENSIONS, II

I. CONGRUENCE

Definition 7. Let ℓ be a line with the parametric equations

$$x = f(t)$$

$$y = g(t)$$

$$z = h(t)$$

given in lemma 4. If P is a point of ℓ, then the points of ℓ may be divided into the following three classes by lemma 4:

(i) $\{P\}$

(ii) $\{Q: t_Q < t_P\}$

(iii) $\{Q: t_Q > t_P\}$

A ray is one of the sets (ii) and (iii). P is called the origin of the ray. If a ray has origin P and if Q is a point of the ray distinct from P, then the ray is denoted by $\overrightarrow{PQ}$. (ii) is called the negative ray of ℓ with origin P with respect to the given parametric equations. (iii) is called the positive ray.

Definition 8. Given a plane $\pi: D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$, the inverse with respect to π of the point P is defined as

(i) $(p_1 + AK_1, \quad p_2 + BK_1, \quad p_3 + CK_1)$

where $K_1 = \dfrac{-2(Ap_1 + Bp_2 + Cp_3)}{A^2 + B^2 + C^2}$

if $D = 0$, and

(ii) $\left(-\dfrac{A}{2D} + K_2(p_1 + \dfrac{A}{2D}), \; -\dfrac{B}{2D} + K_2(p_2 + \dfrac{B}{2D}), \; -\dfrac{C}{2D} + K_2(p_3 + \dfrac{C}{2D}\right)$

where $K_2 = \dfrac{\dfrac{A^2 + B^2 + C^2}{4D^2} - 1}{(p_1 + \dfrac{A}{2D})^2 + (p_2 + \dfrac{B}{2D})^2 + (p_3 + \dfrac{C}{2D})^2}$

if $D \neq 0$.

A transformation of points into points which maps every point into its inverse with respect to π is called the inversion with respect to the plane π. The mapping Φ defined for all points by $\Phi(P) = P$ is the only other type of inversion. Henceforth, K_1 and K_2 will refer to the above expressions unless otherwise specified.

Lemma 5: Inversions map points into points.

Proof: Suppose P is a point and I is the inversion with respect to the plane $D(x^2 + y^2 + z^2 + 1) + Ax + By + Cz = 0$. If $D = 0$, $I(P) = (p_1 + AK_1, \; p_2 + BK_1, \; p_3 + CK_1)$. Since

$$0 \leq (p_1 + AK_1)^2 + (p_2 + BK_1)^2 + (p_3 + CK_1)^2 = p_1^2 + p_2^2 + p_3^2 < 1,$$

$I(P)$ is a point. If $D \neq 0$, $I(P)$ is a point since

$$0 \leq \left(-\frac{A}{2D} + K_2\left(p_1 + \frac{A}{2D}\right)\right)^2 + \left(-\frac{B}{2D} + K_2\left(p_2 + \frac{B}{2D}\right)\right)^2 + \left(-\frac{C}{2D} + K_2\left(p_3 + \frac{C}{2D}\right)\right)^2 < 1 .$$

The proof of this last inequality follows from the inequalities $p_1^2 + p_2^2 + p_3^2 < 1$ and $\frac{A^2+B^2+C^2}{4D^2} > 1$.

<u>Lemma 6.</u> If I is an inversion and P is a point, $I(I(P))=P$.

Proof: It will suffice to prove that $I(I(P))$ and P have the same first coordinate, for their second and third coordinates will then be equal by symmetry.

Suppose I is inversion with respect to the plane $D(x^2 + y^2 + z^2 + 1) + Ax + By + Cz = 0$. The first component of $I(I(P))$ is

$$p_1 + AK_1 - 2A\left(\frac{A(p_1+AK_1) + B(p_2+BK_1) + C(p_3+CK_1)}{A^2 + B^2 + C^2}\right)$$

if $D = 0$, and

$$(-\frac{A}{2D})+\left[\frac{\frac{A^2+B^2+C^2}{4D^2}-1}{(-\frac{A}{2D}+K_2(p_1+\frac{A}{2D})+\frac{A}{2D})^2+(-\frac{B}{2D}+K_2(p_2+\frac{B}{2D})+\frac{B}{2D})^2+(-\frac{C}{2D}+K_2(p_3+\frac{C}{2D})+\frac{C}{2D})^2}\right]$$

$$\times K_2\,(p_1+\frac{A}{2D})$$

if $D \neq 0$. Both of these expressions can be simplified to p_1.

Lemma 7. Inversions are one-to-one and onto.

Proof: Suppose I is the inversion with respect to the plane $D(x^2+y^2+z^2+1)+Ax+By+Cz=0$. If P and Q are distinct points such that $I(P) = I(Q)$, then $P = I(I(P)) = I(I(Q)) = Q$ by lemma 6. So I is one-to-one. I is onto, for

$$P = I((p_1+AK_1,\ p_2+BK_1,\ p_3+CK_1))$$

if $D = 0$, and

$$P = I((-\frac{A}{2D}+K_2(p_1+\frac{A}{2D}),\ -\frac{B}{2D}+K_2(p_2+\frac{B}{2D}),\ -\frac{C}{2D}+K_2(p_3+\frac{C}{2D})))$$

if $D \neq 0$.

Lemma 8. If P is a point, then there is an inversion I such that $I(P) = \theta$.

Proof: If $P = \theta$, $I(P) = \theta$ where $I = \Phi$. If $P \neq \theta$, let I be inversion with respect to the plane

$$\frac{p_1^2 + p_2^2 + p_3^2}{2}(x^2 + y^2 + z^2 + 1) - p_1 x - p_2 y - p_3 z = 0 .$$

Then the first coordinate of $I(P)$ is $-\frac{A}{2D} + K_2(p_1 + \frac{A}{2D})$ where

$$A = -p_1$$

$$D = \frac{p_1^2 + p_2^2 + p_3^2}{2}$$

and

$$K_2 = \frac{1}{1-(p_1^2 + p_2^2 + p_3^2)} .$$

The first coordinate simplifies to zero. By symmetry the second and third coordinates are also zero, so $I(P) = \theta$.

Lemma 9. If ℓ is a line through the origin and P is a point other than θ, then there are parametric equations

$$x = at$$

$$y = bt$$

$$z = ct$$

of ℓ such that P is a point of the positive ray of ℓ with origin θ.

Proof: If θ is a point of the line

$$\ell: \begin{cases} D_1(x^2+y^2+z^2+1) + A_1x + B_1y + C_1z = 0 \\ D_2(x^2+y^2+z^2+1) + A_2x + B_2y + C_2z = 0 \end{cases},$$

then $D_1 = 0$ and $D_2 = 0$. Case 2 of lemma 4 gives parametric equations

$$x = at$$
$$y = bt$$
$$z = ct$$

for ℓ. Since $t_\theta = 0$, either $t_P > 0$ or $t_P < 0$. If $t_P > 0$, P is a point of the positive ray with respect to the above parametric equations. If $t_P < 0$, P is a point of the positive ray if

$$x = -at$$
$$y = -bt$$
$$z = -ct$$

are used as the parametric equations of ℓ.

Lemma 10. If ℓ is a line and I is the inversion with respect to the plane $D(x^2+y^2+z^2+1) + Ax + By + Cz = 0$, then $I(\ell)$ is a line.

Proof: Let ℓ be the line

$$\ell: \begin{cases} D_1(x^2 + y^2 + z^2 + 1) + A_1x + B_1y + C_1z = 0 \\ D_2(x^2 + y^2 + z^2 + 1) + A_2x + B_2y + C_2z = 0 \end{cases} .$$

First suppose $D_1 = D_2 = 0$. Then θ is a point of ℓ, so ℓ has parametric equations $x = a_0t$, $y = b_0t$, $z = c_0t$ by lemma 9. If $D = 0$, $I(\ell)$ is the line with parametric equations $x = (a_0 + AK_1)t$, $y = (b_0 + BK_1)t$, $z = c_0 + CK_1$, where $K_1 = \dfrac{-2(Aa_0 + Bb_0 + Cc_0)}{A^2 + B^2 + C^2}$.

If $D \neq 0$ and there exists a real number k_0 such that $(a_0, b_0, c_0) = (k_0A, k_0B, k_0C)$, then $I(\ell) = \ell$. If $D \neq 0$ but such a k_0 does not exist, then $I(\ell)$ is the line

$$I(\ell): \begin{cases} (x^2 + y^2 + z^2 + 1) + A'x + B'y + C'z = 0 \\ (BC' - B'C)x + (A'C - AC')y + (AB' - A'B)z = 0 \end{cases}$$

where

$$A' = \frac{A}{D} - K_2(a_0t_0 + \frac{A}{2D})$$

$$B' = \frac{B}{D} - K_2(b_0t_0 + \frac{B}{2D})$$

$$C' = \frac{C}{D} - K_2(c_0t_0 + \frac{C}{2D})$$

$$K_2 = \frac{\dfrac{A^2 + B^2 + C^2}{4D^2} - 1}{(a_0t_0 + \frac{A}{2D})^2 + (b_0t_0 + \frac{B}{2D})^2 + (c_0t_0 + \frac{C}{2D})^2}$$

$$t_0 = \frac{-(Aa_0 + Bb_0 + Cc_0)}{2D(a_0^2 + b_0^2 + c_0^2)} .$$

Secondly consider the case in which at least one of D_1 and D_2 is non-zero. Assume $D_1 \neq 0$. The equations defining ℓ are equivalent to

$$\begin{cases} x^2 + y^2 + z^2 + 1 + 2ax + 2by + 2cz = 0 \\ a'x + b'y + c'z = 0 \end{cases}$$

where $2a = \frac{A_1}{D_1}$, $2b = \frac{B_1}{D_1}$, $2c = \frac{C_1}{D_1}$ and if $D_2 = 0$, then $A_2 = a'$, $B_2 = b'$, and $C_2 = c'$. If $D_2 \neq 0$, then $a' = \frac{A_1}{D_1} - \frac{A_2}{D_2}$, $b' = \frac{B_1}{D_1} - \frac{B_2}{D_2}$, and $c' = \frac{C_1}{D_1} - \frac{C_2}{D_2}$. If $D = 0$, then $I(\ell)$ is the line

$$I(\ell): \begin{cases} (x^2 + y^2 + z^2 + 1) + 2(a - AK_1')x + 2(b - BK_1')y + 2(c - CK_1')z = 0 \\ (a' + AK_1') + (b' + BK_1')y + (c' + CK_1')z = 0 \end{cases}$$

where $K_1' = \frac{2(Aa + Bb + Cc)}{A^2 + B^2 + C^2}$. If $D \neq 0$ and there exists a real number k_1 such that $(a, b, c) = (k_1A, k_1B, k_1C)$, then $I(\ell)$ is the line

$$I(\ell): \begin{cases} (x^2 + y^2 + z^2 + 1) + A''x + B''y + C''z = 0 \\ a'x + b'y + c'z = 0 \end{cases}$$

where

$$A'' = \frac{A}{D} - K_{21}(f(t_1) + \frac{A}{2D}) - K_{22}(f(t_2) + \frac{A}{2D})$$

$$B'' = \frac{B}{D} - K_{21}(g(t_1) + \frac{B}{2D}) - K_{22}(g(t_2) + \frac{B}{2D})$$

$$C'' = \frac{C}{D} - K_{21}(h(t_1) + \frac{C}{2D}) - K_{22}(h(t_2) + \frac{C}{2D})$$

$$K_{21} = \frac{\frac{A^2 + B^2 + C^2}{4D^2} - 1}{(f(t_1) + \frac{A}{2D})^2 + (g(t_1) + \frac{B}{2D})^2 + (h(t_1) + \frac{C}{2D})^2}$$

$$K_{22} = \frac{\frac{A^2 + B^2 + C^2}{4D^2} - 1}{(f(t_2) + \frac{A}{2D})^2 + (g(t_2) + \frac{B}{2D})^2 + (h(t_2) + \frac{C}{2D})^2}$$

$$t_1 = \left[\frac{a^2 + b^2 + c^2 - 1}{(\frac{A}{2D} - a)^2 + (\frac{B}{2D} - b)^2 + (\frac{C}{2D} - c)^2} \right]^{1/2}$$

$$t_2 = -t_1$$

$$f(t) = -a + t(\frac{A}{2D} - a)$$

$$g(t) = -b + t(\frac{B}{2D} - b)$$

and

$$h(t) = -c + t(\frac{C}{2D} - c) \quad .$$

If $D \neq 0$ but such a k_1 does not exist, then $I(\ell)$ is the line

$$I(\ell): \begin{cases} (x^2 + y^2 + z^2 + 1) + A''x + B''y + C''z = 0 \\ [g(t_1)h(t_2) - g(t_2)h(t_1)]x + [f(t_2)h(t_1) - f(t_1)h(t_2)]y \\ \qquad + [f(t_1)g(t_2) - f(t_2)g(t_1)]z = 0 \ . \end{cases}$$

Lemma 11. If ℓ_1 has parametric equations $x = a_1t$, $y = b_1t$, $z = c_1t$, and ℓ_2 has parametric equations $x = a_2t$, $y = b_2t$, $z = c_2t$, and I is inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \dfrac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} + \dfrac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \dfrac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} + \dfrac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \dfrac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} + \dfrac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix} = 0 \ ,$$

then $I(\ell_1) = \ell_2$ and $I(\ell_2) = \ell_1$ and $I(\theta) = \theta$.

Proof: If P is a point of ℓ_1, then $I(P)$ is the point of ℓ_2 such that

$$t_{I(P)} = t_P \times \frac{a_1^2 + b_1^2 + c_1^2}{a_2^2 + b_2^2 + c_2^2},$$

so $I(\ell_1) = \ell_2$. By lemma 6, $\ell_1 = I(I(\ell_1)) = I(\ell_2)$. $I(\theta) = \theta$ trivially.

Metadefinition 7. Two segments PQ and RS are congruent iff there are inversions $I_1, I_2, \cdots, I_n$ such that $I_n \circ \cdots \circ I_2 I_1(P) = R$ and $I_n \circ \cdots \circ I_2 I_1(Q) = S$. If PQ and RS are congruent, we write $PQ \cong RS$.

Lemma 12. Congruence of segments is an equivalence relation.

Proof: Let I be any inversion. Then if PQ is a segment, $I(I(P)) = P$ and $I(I(Q)) = Q$, so $PQ \cong PQ$. If $PQ \cong RS$, there is some finite product of inversions ψ such that $\psi(P) = R$ and $\psi(Q) = S$. Then $\psi(R) = \psi(\psi(P)) = P$ and $\psi(S) = \psi(\psi(Q)) = Q$ by lemma 6, so $RS \cong PQ$. Finally, if $PQ \cong RS$ and $RS \cong UV$, then there is a finite product Ω of inversions such that $\Omega(R) = U$ and $\Omega(S) = V$. So $\Omega(\psi(P)) = U$ and $\Omega(\psi(Q)) = V$. Hence $PQ \cong UV$.

Lemma 13. If I is an inversion and $P\overset{*}{Q}R$, then $I(P)I(\overset{*}{Q})I(R)$.

Proof: Let ℓ be the line incident upon P and Q. By lemma 4 ℓ has parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$. Either $t_P < t_Q < t_R$ or $t_P > t_Q > t_R$. Since these two cases are symmetric, we can suppose $t_P < t_Q < t_R$ without loss of generality.

Let $f_1(t) = f(t) + AK_{1P}$, $g_1(t) = g(t) + BK_{1Q}$, and $h_1(t) = h(t) + CK_{1R}$ where for any point S

$$K_{1S} = \frac{-2(As_1 + Bs_2 + Cs_3)}{A^2 + B^2 + C^2} .$$

Also let

$$f_2(t) = -\frac{A}{2D} + K_{2P}(f(t) + \frac{A}{2D})$$

$$g_2(t) = -\frac{B}{2D} + K_{2Q}(g(t) + \frac{B}{2D})$$

and

$$h_2(t) = -\frac{C}{2D} + K_{2R}(h(t) + \frac{C}{2D})$$

where for any point S

$$K_{2S} = \frac{\dfrac{A^2 + B^2 + C^2}{4D^2} - 1}{(s_1 + \frac{A}{2D})^2 + (s_2 + \frac{B}{2D})^2 + (s_3 + \frac{C}{2D})^2} .$$

If $D = 0$, $x = f_1(t)$, $y = g_1(t)$, $z = h_1(t)$ are parametric equations of $I(\ell)$. If $D \neq 0$, $I(\ell)$ is the line with parametric equations

$x = f_2(t)$, $y = g_2(t)$, $z = h_2(t)$. Now $I(P) = (f_i(t_P), g_i(t_P), h_i(t_P))$, $I(Q) = (f_i(t_Q), g_i(t_Q), h_i(t_Q))$ and $I(R) = (f_i(t_R), g_i(t_R), h_i(t_R))$ where $i = 1$ if $D = 0$ and $i = 2$ if $D \neq 0$. In either case $t_{I(P)} = t_P$, $t_{I(Q)} = t_Q$, and $t_{I(R)} = t_R$, so $t_{I(P)} < t_{I(Q)} < t_{I(R)}$ and $I(P)\overset{*}{I(Q)}I(R)$.

Theorem 14. If P and Q are distinct points of the line ℓ_1 and R is a point of the line ℓ_2, then on a given ray of ℓ_2 with origin R there is a unique point S such that $PQ \cong RS$. A segment is congruent to itself.

Proof: If θ is a point of ℓ_1, let $I_1 = \Phi$. Otherwise apply lemma 8 and let I_1 be inversion with respect to the plane

$$\frac{p_1^2 + p_2^2 + p_3^2}{2}(x^2 + y^2 + z^2 + 1) - p_1x - p_2y - p_3z = 0 .$$

Then $I_1(P) = \theta$. By lemma 10, $I_1(\ell_1)$ is some line through θ, and by lemma 9 this line has parametric equations $x = a_1t$, $y = b_1t$, $z = c_1t$ such that $I_1(Q)$ is on the positive ray of $I_1(\ell_1)$ with origin θ.

Since ℓ_2 is a line, by lemma 4 it has parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$ $(a < t < b)$. The given ray of ℓ_2 with origin R is either the positive or the negative ray with respect to these equations. Since these two possibilities are symmetric, we can suppose it is the positive ray without loss of generality. Now $a < t_R < \frac{t_R+b}{2} < b$, so if U is the point of ℓ_2 given by the parameter $\frac{t_R+b}{2}$, then U is a point of the positive ray of ℓ_2 with origin R.

If θ is a point of ℓ_2, let $I_2 = \Phi$. Otherwise apply Lemma 8 and let I_2 be inversion with respect to the plane

$$\frac{r_1^2 + r_2^2 + r_3^2}{2} (x^2 + y^2 + z^2 + 1) - r_1x - r_2y - r_3z = 0 .$$

Then $I_2(R) = \theta$. By lemma 10, $I_2(\ell_2)$ is a line through θ. By lemma 9 this line has parametric equations $x = a_2t$, $y = b_2t$, $z = c_2t$ such that $I_2(U)$ is a point of the positive ray of $I_2(\ell_2)$ with origin θ.

If $I_1(\ell_1)$ and $I_2(\ell_2)$ are the same line and $I_1(Q)$ is on the positive ray of $I_2(\ell_2)$, let $I_3 = \Phi$. If $I_1(\ell_1)$ and $I_2(\ell_2)$ are the same line and $I_1(Q)$ is a point of the negative ray of $I_2(\ell_2)$, use lemma 11 and let I_3 be inversion with respect to the plane

$$a_1x + b_1y + c_1z = 0 .$$

If $I_1(\ell_1)$ and $I_2(\ell_2)$ are different lines, using lemma 11, let I_3 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix} = 0 .$$

Then $I_3(\theta) = \theta$ while $I_3 I_1(Q)$ is a point of the positive ray of $I_2(\ell_2)$ with origin θ.

Let $S = I_2 I_3 I_1(Q)$. Then S is a point of the positive ray of ℓ_2 with origin R. Furthermore, $PQ \cong RS$ since $I_2 I_3 I_1(P) = R$ and $I_2 I_3 I_1(Q) = S$.

It remains to show that the point S is unique. Suppose there is a point S' which is distinct from S yet satisfies the hypotheses of the theorem. Either $R\overset{*}{S}S'$ or $R\overset{*}{S'}S$. Since these two cases are symmetric, we will consider the case where $R\overset{*}{S}S'$. By lemma 12, $PQ \cong RS$ and $PQ \cong RS'$ imply $RS \cong RS'$. This means there is a finite product of inversions ψ such that $\psi(R) = R$ and $\psi(S) = S'$. Applying lemma 6 to this last equation gives $\psi(S') = S$. Now lemma 13 implies $R\overset{*}{S'}S$, contradicting theorem 10.

A segment is congruent to itself by lemma 12.

Theorem 15. If $PQ \cong RS$ and $PQ \cong UV$, then $RS \cong UV$.

Proof: By lemma 12.

Theorem 16. If P, Q, and R are points of ℓ and P', Q', and R' are points of ℓ', and if $P\overset{*}{Q}R$, $P'\overset{*}{Q'}R'$, $PQ \cong P'Q'$, and $QR \cong Q'R'$, then $PR \cong P'R'$.

Proof: By hypothesis there is a finite product of inversions ψ such that $\psi(P) = P'$ and $\psi(Q) = Q'$. By theorem 14 there is a unique point S of the ray $\overrightarrow{Q'R'}$ such that $QR \cong Q'S$. Since R' is a point of $\overrightarrow{Q'R'}$ and $QR \cong Q'R'$, $S = R'$. Also $R' = \psi(R)$, for by lemma 13 $P\overset{*}{Q}R$ implies $P'\overset{*}{Q'}\psi(R)$, so $\psi(R)$ is a point of $\overrightarrow{Q'R'}$. Hence $R' = \psi(R)$. Now $PR \cong P'R'$ by definition, since $\psi(P) = P'$ and $\psi(R) = R'$.

Definition 9. If ℓ is a line of a plane π and Q is a point of the plane but not a point of ℓ, then the points of π can be divided into the following three classes:

(i) $\{U: U$ is a point of $\ell\}$

(ii) $\{U: QU$ has no points of ℓ and U is not in (i), or $U = Q\}$

(iii) $\{U: U$ is not in (i) but QU contains a point of $\ell\}$.

The set (ii) is called the <u>side</u> of ℓ containing Q.

Definition 10. If P, Q, and R are noncollinear or if P, Q, and R are collinear and $P\overset{*}{Q}R$, then $\overrightarrow{PQ} \cup \overrightarrow{PR} \cup \{P\}$ is called an <u>angle</u> with vertex P. In the latter case the angle is called a <u>straight angle</u>. $\overrightarrow{PQ} \cup \overrightarrow{PR} \cup \{P\}$ is denoted by $\angle QPR$ or by $\angle(\overrightarrow{PQ}, \overrightarrow{PR})$.

Metadefinition 8. $\angle PQR$ and $\angle P'Q'R'$ are called <u>congruent</u> angles iff there is a finite product of inversions ψ such that $\psi(Q) = Q'$, P' is a point of $\overrightarrow{\psi(Q)\psi(P)}$, and R' is a point of $\overrightarrow{\psi(Q)\psi(R)}$. If $\angle PQR$ and $\angle P'Q'R'$ are congruent, this is denoted by $\angle PQR \cong \angle P'Q'R'$.

Lemma 14. If $P = (p_1, p_2, 0)$, $Q = (q_1, 0, 0)$, and $R = (r_1, r_2, 0)$ are points of a line ℓ, and if $p_2 > 0$, $q_1 > 0$ and $r_2 < 0$, then $P\overset{*}{Q}R$.

Proof: Using lemma 5 let I be inversion with respect to the plane $\frac{q_1^2}{2}(x^2 + y^2 + z^2 + 1) - q_1 x = 0$. Then $I(Q) = \theta$ and $I(\ell)$ is a line through θ. So $I(\ell)$ has parametric equations $x = at$, $y = bt$, $z = ct$. Since $I(Q) = \theta$, $t_{I(Q)} = 0$.

The second coordinate of the point I(P) is

$$\frac{\frac{1}{q_1^2} - 1}{\left(p_1 - \frac{q_1}{q_1^2}\right)^2 + p_2^2 + p_3^2} \quad p_2$$

which is greater than zero since $0 < q_1 < 1$ and $p_2 > 0$. The second coordinate of I(R) is

$$\frac{\frac{1}{r_1^2} - 1}{\left(r_1 - \frac{q_1}{q_1^2}\right)^2 + r_2^2 + r_3^2} \qquad r_2$$

which is less than zero since $r_2 < 0$. Thus $bt_{I(P)} > 0$ while $bt_{I(R)} < 0$. So either $t_{I(P)} < 0 < t_{I(R)}$ or $t_{I(P)} > 0 > t_{I(R)}$. Since $t_{I(Q)} = 0$, we have either $t_{I(P)} < t_{I(Q)} < t_{I(R)}$ or $t_{I(P)} > t_{I(Q)} > t_{I(R)}$. In either case $I(P)I(\overset{*}{Q})I(R)$. By lemma 6 and 13 $P\overset{*}{Q}R$.

Theorem 17a. Any two straight angles are congruent.

Proof: Suppose $\angle PQR$ and $\angle SQ'U$ are straight angles. By theorem 14, there is a point P' of $\overrightarrow{Q'S}$ such that $P'Q' \cong PQ$. So there is a finite product of inversions ψ such that $\psi(P) = P'$ and $\psi(Q) = Q'$. By theorem 1 and lemma 10, $\psi(R)$ is a point of the line incident upon Q' and U, and $\psi(R)$ is a point of $\overrightarrow{Q'U}$ by lemma 13. Now $\psi(Q) = Q'$, $\overrightarrow{\psi(Q)\psi(P)} = \overrightarrow{Q'P'}$ so S is a point of $\overrightarrow{\psi(Q)\,\psi(P)}$, and $\overrightarrow{\psi(Q)\,\psi(R)} = \overrightarrow{Q'R'}$ so U is a point of $\overrightarrow{\psi(Q)\,\psi(R)}$. Hence $\angle PQR \cong \angle SQ'U$.

Theorem 17b. Any angle which is congruent to a straight angle is itself a straight angle.

Proof: Suppose $\angle PQR$ is a straight angle and $\angle P'Q'R' \cong \angle PQR$. Let ψ be the finite product of inversions giving the congruence of $\angle P'Q'R'$ and $\angle PQR$. Since P, Q, and R are collinear, lemma 10 implies that $\psi(P)$, Q', and $\psi(R)$ are collinear, and $P\overset{*}{Q}R$ implies $\psi(P)\overset{*}{Q'}\psi(R)$ by lemma 13. By the definition of congruence of angles, either P' is a point of $\overrightarrow{Q'\psi(P)}$ and R' is a point of $\overrightarrow{Q'\psi(R)}$ or P' is a point of $Q'\psi(R)$ and R' is a point of $\overrightarrow{Q'\psi(P)}$. In either case P', Q', and R' are collinear and $P'\overset{*}{Q'}R'$.

Theorem 17c. If $\angle PQR$ is not a straight angle and ℓ_1 is a line of the plane π and $\vec{r}$ is a given ray of ℓ_1 with origin V and $\mathcal{S}$ is a side of ℓ_1, then

(i) there is in $\mathcal{S}$ a ray $\vec{s}$ with origin V such that $\angle PQR \cong \angle(\vec{r}, \vec{s})$, and

(ii) $\vec{s}$ is unique.

Proof: (i) Since ℓ_1 is a line, by lemma 4 it has parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$ $(a < t < b)$. The given ray with origin V is either the positive or the negative ray. We will consider the case where it is the positive ray, the other case being symmetric. Since the point $U: (f(t_0), g(t_0), h(t_0))$ for $t_0 = \frac{b+t_V}{2}$ is a point of the positive ray with origin V, we may write $\vec{r} = \vec{VU}$.

If $Q = \theta$, let $I_1 = \Phi$. Otherwise let I_1 be inversion with respect to the plane

$$\frac{q_1^2 + q_2^2 + q_3^2}{2}(x^2 + y^2 + z^2 + 1) - q_1x - q_2y - q_3z = 0 .$$

Then $I_1(Q) = \theta$. Denote the line incident upon P and Q by ℓ_2 and the line incident upon Q and R by ℓ_3. Since $\angle PQR$ is not a straight angle, $\ell_2 \neq \ell_3$. Thus $I_1(\ell_2)$ and $I_1(\ell_3)$ are distinct lines through θ. Denote $I_1(P)$ by P' and $I_1(R)$ by R'. The plane

$$\pi_2 = \begin{vmatrix} x & y & z \\ p_1' & p_2' & p_3' \\ r_1' & r_2' & r_3' \end{vmatrix} = 0$$

is incident upon P', θ, and R', and by theorem 4 it is the only such plane.

If $V = \theta$ let $I_2 = \Phi$. Otherwise let I_2 be inversion with respect to the plane

$$\frac{v_1^2 + v_2^2 + v_3^2}{2}(x^2 + y^2 + z^2 + 1) - v_1x - v_2y - v_3z = 0 .$$

Then $I_2(V) = \theta$ and $I_2(\ell_1)$ is a line through θ by lemma 10. If S is a point of $\$$, then $I_2(S)$ is a point of $I_2(\pi_1)$. If we write $I_2(S) = S'$ and $I_2(U) = U'$, then S', θ, and U' are non-collinear by lemma 10. The plane

$$\pi_3 = \begin{vmatrix} x & y & z \\ s_1' & s_2' & s_3' \\ u_1' & u_2' & u_3' \end{vmatrix} = 0$$

is the unique plane incident upon S', θ and U'.

Let

$$A = \begin{vmatrix} p_2' & p_3' \\ r_2' & r_3' \end{vmatrix}, \quad B = \begin{vmatrix} p_3' & p_1' \\ r_3' & r_1' \end{vmatrix} \quad C = \begin{vmatrix} p_2' & p_3' \\ r_2' & r_3' \end{vmatrix}$$

and

$$A' = \begin{vmatrix} s_2' & s_3' \\ u_2' & u_3' \end{vmatrix} \quad B' = \begin{vmatrix} s_3' & s_1' \\ u_3' & u_1' \end{vmatrix} \quad C' = \begin{vmatrix} s_1' & s_2' \\ u_1' & u_2' \end{vmatrix} .$$

Then π_2 is the plane $Ax + By + Cz = 0$ and π_3 is the plane $A'x + B'y + C'z = 0$. Now if $\pi_2 = \pi_3$, let $I_3 = \Phi$. Otherwise let I_3 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} = 0$$

where M is the point other than θ common to π_2 and π_3 given by theorem 7, and where

$$N = \left(\frac{A}{\sqrt{A^2+B^2+C^2}} + \frac{A'}{\sqrt{A'^2+B'^2+C'^2}}, \frac{B}{\sqrt{A^2+B^2+C^2}} + \frac{B'}{\sqrt{A'^2+B'^2+C'^2}}, \frac{C}{\sqrt{A^2+B^2+C^2}} + \frac{C'}{\sqrt{A'^2+B'^2+C'^2}} \right).$$

$I_3(\pi_2) = \pi_3$ and $I_3I_1(Q) = I_2(V) = \theta$.

Use lemma 9 to get parametric equations $x = a_1t$, $y = b_1t$, $z = c_1t$ for $I_2(\ell_1)$ such that $I_2(U)$ is a point of the positive ray with origin θ. Likewise obtain parametric equations $x = a_2t$, $y = b_2t$, $z = c_2t$ for $I_3I_1(\ell_2)$ such that $I_3I_1(P)$ is a point of the positive ray with origin θ. If $I_3I_1(\ell_2) = I_2(\ell_1)$ and $I_3I_1(\overrightarrow{QP}) = I_2(\overrightarrow{VU})$ let $I_4 = \Phi$. If $I_3I_1(\ell_2) = I_2(\ell_1)$ but $I_3I_1(\overrightarrow{QP}) \neq I_2(\overrightarrow{VU})$, let I_4 be inversion with respect to the plane $a_1x + b_1y + c_1z = 0$. If $I_3I_1(\ell_2) \neq I_2(\ell_1)$, use lemma 11 and let I_4 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}+\frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} & -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \end{vmatrix} = 0 .$$

In any case $I_4(\pi_3) = \pi_3$ and $I_4I_3I_1(\overrightarrow{QP}) = I_2(\overrightarrow{VU})$.

Now $I_2I_4I_3I_1(R)$ is a point of π_1. Furthermore, it is not a point of ℓ_1, for this would imply the collinearity of P, Q, and R. If $I_2I_4I_3I_1(R)$ is a point of $\$$, let $I_5 = \Phi$. Otherwise let I_5 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ a_1 & b_1 & c_1 \\ A' & B' & C' \end{vmatrix} = 0 .$$

Denote the product of inversions $I_5I_2I_4I_3I_1$ by ψ. Then if we let $W = \psi(R)$, W is a point of $\$$.

Let $\vec{s} = \overrightarrow{VW}$. Then $\vec{s}$ is in $\$$ and has origin V, and since $\psi(Q) = V$, U is a point of $\overrightarrow{\psi(Q)\psi(P)}$, and $\psi(R)$ is a point of $\overrightarrow{\psi(Q)\psi(R)}$, $\angle PQR \cong \angle UVW$.

(ii) To show the uniqueness of $\vec{s}$, suppose there is a ray $\vec{s'}$ in \$ with origin V which is distinct from $\vec{s}$ and that $\angle PQR \cong \angle(\vec{r}, \vec{s'})$. If π_3 is the plane $z = 0$, let $I_6 = \Phi$. Otherwise let I_6 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ m_1' & m_2' & m_3' \\ n_1' & n_2' & n_3' \end{vmatrix} = 0$$

where M' is the point given by theorem 7 other than θ common to π_3 and the plane $z = 0$, and where

$$N' = \left(\frac{A'}{\sqrt{A'^2+B'^2+C'^2}}+1, \quad \frac{B'}{\sqrt{A'^2+B'^2+C'^2}}, \quad \frac{C'}{\sqrt{A'^2+B'^2+C'^2}}\right).$$

$I_6(\pi_3)$ is the plane $z = 0$ and $I_6I_2(V) = \theta$. Let ℓ_5 be the line $x = t$, $y = 0$, $z = 0$, and let $\vec{u}$ be the positive ray of ℓ_5 with origin θ. If $I_6I_2(\ell_2) = \ell_5$ and $I_6I_2(\vec{VU}) = \vec{u}$, let $I_7 = \Phi$. If $I_6I_2(\ell_2) = \ell_5$ but $I_6I_2(\vec{VU}) \neq \vec{u}$, let I_7 be inversion with respect to the plane $x = 0$. If $I_6I_2(\ell_2) \neq \ell_5$, use lemma 9 to get parametric equations $x = a_3t$, $y = b_3t$, $z = 0$ for $I_6I_2(\ell_1)$ such that $I_6I_2(\vec{VU})$ is the positive ray with origin θ with respect to this representation. Then use lemma 11 and let I_7 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \dfrac{a_3}{\sqrt{a_3^2+b_3^2+c_3^2}}+1 & \dfrac{b_3}{\sqrt{a_3^2+b_3^2+c_3^2}} & 0 \\ 0 & 0 & -b_3 \end{vmatrix} = 0 .$$

If $I_7 I_6 I_2(S)$ has a positive second coordinate, let $I_8 = \Phi$. Otherwise let I_8 be inversion with respect to the plane $y = 0$.

Now $I_8 I_7 I_6 I_2(\vec{s})$ is the positive ray with origin θ of some line $x = a_4 t$, $y = b_4 t$, $z = 0$, and $I_8 I_7 I_6 I_2(\vec{s'})$ is the positive ray with origin θ of some line $x = a_5 t$, $y = b_5 t$, $z = 0$. It cannot happen that $\dfrac{a_4}{\sqrt{a_4^2+b_4^2}} = \dfrac{a_5}{\sqrt{a_5^2+b_5^2}}$, for then we would have $I_8 I_7 I_6 I_2(\vec{s}) = I_8 I_7 I_6 I_2(\vec{s'})$. Lemma 6 would then imply $\vec{s} = \vec{s'}$.

So either $\dfrac{a_4}{\sqrt{a_4^2+b_4^2}} < \dfrac{a_5}{\sqrt{a_5^2+b_5^2}}$ or $\dfrac{a_4}{\sqrt{a_4^2+b_4^2}} > \dfrac{a_5}{\sqrt{a_5^2+b_5^2}}$.

Since these cases are symmetric, we will consider only the former case.

Again apply lemma 11, letting I_9 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \dfrac{a_5}{\sqrt{a_5^2+b_5^2}}+1 & \dfrac{b_5}{\sqrt{a_5^2+b_5^2}} & 0 \\ 0 & 0 & -b_5 \end{vmatrix} = 0 .$$

Now $I_9I_8I_7I_6I_2(\overrightarrow{s'}) = \vec{u}$, $I_9I_8I_7I_6I_2(\overrightarrow{VU})$ is the positive ray with origin θ of some line $x = a_6t$, $y = b_6t$, $z = 0$ where $b_6 > 0$, and $I_8I_7I_6I_2(\vec{s})$ is the positive ray with origin θ of some line $x = a_7t$, $y = b_7t$, $z = 0$ where $b_7 < 0$.

Suppose $X = (x_1, x_2, 0)$ is a point of $I_9I_8I_7I_6I_2(\overrightarrow{VU})$ and $Z = (z_1, z_2, 0)$ is a point of $I_8I_7I_6I_2(\vec{s})$ Then $x_2 > 0$ and $z_2 < 0$. By theorem 1, the line

$$\ell_4: \begin{cases} (z_2x_1-x_2z_1)(x^2+y^2+z^2+1)+[x_2(z_1^2+z_2^2+1)-z_2(x_1^2+x_2^2+1)]x \\ \qquad\qquad +[z_1(x_1^2+x_2^2+1)-x_1(z_1^2+z_2^2+1)]y = 0 \\ z = 0 \end{cases}$$

is incident upon X and Z. The point $Y = (y_1, 0, 0)$, where y_1 is the smaller of the two roots of

$$(z_2x_1-x_2z_1)(x^2+1)+[x_2(z_1^2+z_2^2+1)-z_2(x_1^2+x_2^2+1)]x = 0 ,$$

is a point of $\vec{u}$ and of ℓ_4. By lemma 14, $X\overset{*}{Y}Z$.

Since $\angle(\overrightarrow{r}, \overrightarrow{s}) \cong \angle(\overrightarrow{r}, \overrightarrow{s'})$, $\angle Z\theta X \cong \angle Z\theta Y$. By the definition of congruence of angles, there is a finite product of inversions Ω such that $\Omega(\theta) = \theta$, Z is a point of $\overrightarrow{\theta\Omega(Z)}$, X is a point of $\overrightarrow{\theta\Omega(Y)}$, and Y is a point of $\overrightarrow{\theta\Omega(X)}$. By Lemma 14, $\Omega(X)\Omega(\overset{*}{Z})\Omega(Y)$. By lemma 13, $X\overset{*}{Y}Z$ implies $\Omega(X)\Omega(\overset{*}{Y})\Omega(Z)$. This contradicts theorem 10.

Theorem 18. If ΔPQR and $\Delta P'Q'R'$ are triangles such that $PQ \cong P'Q'$, $PR \cong P'R'$, and $\angle QPR \cong \angle Q'P'R'$, then $\angle PQR \cong \angle P'Q'R'$ and $\angle PRQ \cong \angle P'R'Q'$.

Proof: By the definition of congruence of angles, there is a product of inversions ψ such that $\psi(P) = P'$, Q' is a point of $\overrightarrow{P'\psi(Q)}$, and R' is a point of $\overrightarrow{P'\psi(R)}$. Thus $\psi(Q)$ is a point of $\overrightarrow{P'Q'}$ and $\psi(R)$ is a point of $\overrightarrow{P'R'}$. By theorem 14, if Q'' is a point of $\overrightarrow{P'Q'}$ and R'' of $\overrightarrow{P'R'}$, and if $PQ \cong P'Q''$ and $PR \cong P'R''$, then $Q' = Q''$ and $R' = R''$. This means that $Q' = \psi(Q)$ and $R' = \psi(R)$. The conclusion of the theorem is now immediate.

II. THE AXIOM OF ARCHIMEDES

Theorem 19. If PQ and RS are line segments, then (i) there exist points $R_1, R_2, \cdots,$ of the ray $\overrightarrow{PQ}$ such that $P\overset{*}{R}_1R_2$ and $PR_1 \cong RS$, $R_1\overset{*}{R}_2R_3$ and $R_1R_2 \cong RS$, and in general $R_{i-1}\overset{*}{R}_iR_{i+1}$ and $R_{i-1}R_i \cong RS$, and (ii) there exists a natural number n such that $P\overset{*}{Q}R_n$.

Proof: (i) Denote the line incident upon P and Q by ℓ_1 and the line incident upon R and S by ℓ_2. If $P = \theta$, let $I_1 = \Phi$. If not, use lemma 5 and let I_1 be inversion with respect to the plane

$$\frac{p_1^2+p_2^2+p_3^2}{2}(x^2+y^2+z^2+1)-p_1x-p_2y-p_3z = 0 .$$

Then $I_1(P) = \theta$ and by lemma 10, $I_1(\ell_1)$ is a line through the origin. By lemma 9, $I_1(\ell_1)$ has parametric equations $x = a_1t$, $y = b_1t$, $z = c_1t$ such that $I_1(Q)$ is a point of the positive ray of $I_1(\ell_1)$ with origin θ. If $a_1 > 0$ and $b_1 = 0 = c_1$, let $I_2 = \Phi$. If $a_1 < 0$ and $b_1 = 0 = c_1$, let I_2 be inversion with respect to the plane $x = 0$. If neither of these is the case, use lemma 11 and let I_2 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}} + 1 & \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}} & \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}} \\ 0 & c_1 & -b_1 \end{vmatrix} = 0 .$$

Denote $I_2 I_1$ by ψ_1. Then $\psi_1(P) = I_2(\theta) = \theta$ by lemma 11, and $\psi_1(Q)$ is some point $Q' = (q_1', 0, 0)$ where $q_1' > 0$.

If $R = \theta$, let $I_3 = \Phi$. If not, use lemma 5 and let I_3 be inversion with respect to the plane

$$\frac{r_1^2+r_2^2+r_3^2}{2}(x^2 + y^2 + z^2 + 1) - r_1 x - r_2 y - r_3 z = 0.$$

Then $I_3(R) = \theta$, and by lemma 10 $I_3(\ell_2)$ is a line through the origin. By lemma 9 $I_3(\ell_2)$ has parametric equations $x = a_2 t$, $y = b_2 t$; $z = c_2 t$ such that $I_3(S)$ is on the positive ray of $I_3(\ell_2)$ with origin θ. If $a_2 > 0$ and $b_2 = 0 = c_2$, let $I_4 = \Phi$. If $a_2 < 0$ and $b_2 = c_2 = 0$, let I_4 be inversion with respect to the plane $x = 0$. If neither of these is the case, use lemma 11 and let I_4 be inversion with respect to the plane

$$\begin{vmatrix} x & y & z \\ \frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}+1 & \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}} & \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}} \\ 0 & c_2 & -b_2 \end{vmatrix} = 0 .$$

Let $I_4 I_3 = \psi_2$. Then $\psi_2(R) = I_4(\theta) = \theta$ by lemma 11, and $\psi_2(S)$ is some point $S_1 = (s_1, 0, 0)$ where $0 < s_1 < 1$.

If $q_1' < s_1$, let $n = 1$ and let $R_1 = \psi_1\psi_2(S)$. Since $\psi_1\psi_2(\ell_2) = \ell_1$, R_1 is a point of ℓ_1, and since $\psi_1\psi_2(R) = P$ and $\psi_1\psi_2(S) = R_1$, $PR_1 \cong RS$. Furthermore, $P\overset{*}{Q}R_1$ by lemma 14 since $\theta\overset{*}{Q}'S_1$ and $\psi_1(\theta) = P$, $\psi_1(Q') = Q$, and $\psi_1(S_1) = R_1$.

In general, for $k > 1$, let $S_k = \Omega_{k-1}(S_{k-2})$ where Ω_{k-1} is inversion with respect to the plane

$(x^2+y^2+z^2+1) - \frac{1+s_{k-1}^2}{s_{k-1}} x = 0$ and $S_0 = \theta$, and let $R_k = \psi_1(S_k)$.

If $s_1, s_2, \cdots$ are computed, we find that $s_i = \frac{2 s_{i-1}}{1+s_{i-1}^2}$ for $i = 2, 3, 4, \cdots$. Since $0 < s_{i-1} < 1$, $s_i = \left(\frac{2}{1+s_{i-1}^2}\right) s_{i-1} > s_{i-1}$.

Hence, $S_0\overset{*}{S}_1S_2$, $S_1\overset{*}{S}_2S_3$, $\cdots$, $S_{i-1}\overset{*}{S}_iS_{i+1}$, $\cdots$, and by lemma 13, $P\overset{*}{R}_1R_2$, $R_1\overset{*}{R}_2R_3$, $\cdots$, $R_{i-1}\overset{*}{R}_iR_{i+1}$, $\cdots$. Also, if k is odd, then

$$R_k = \psi_1\Omega_{k-1} \circ \cdots \circ \Omega_2\Omega_1\psi_2(S)$$

and

$$R_{k-1} = \psi_1 \Omega_{k-1} \circ \cdots \circ \Omega_2 \Omega_1 \psi_2 (R),$$

and if k is even, then

$$R_k = \psi_1 \Omega_{k-1} \circ \cdots \circ \Omega_2 \Omega_1 \psi_2 (R)$$

and

$$R_{k-1} = \psi_1 \Omega_{k-1} \circ \cdots \circ \Omega_2 \Omega_1 \psi_2 (S).$$

(Note that $\Omega_1 (S_i) = S_i$ for $i = 1, 2, \cdots.$) So $R_k R_{k-1} \cong RS$ whether k is even or odd.

The first conclusion of the theorem has now been proved. It remains to show that there exists n such that $P\overset{*}{Q}R_n$. If we define $s_i^* = \dfrac{2s_{i-1}}{1+s_{i-1}}$, then $0 < s_i^* < s_i < 1$ since $0 < s_{i-1} < 1$. s_i^* can be written $\dfrac{2^{i-1}s_1}{1+(2^{i-1}-1)s_1}$. We want to find i_0 such that

$$s_{i_0}^* = \frac{2^{i_0-1}s_1}{1+(2^{i_0-1}-1)s_1} > q_1'.$$

Both $q_1'(1-s_1)$ and $s_1(1-q_1')$ are positive and finite. So if

$$D = \frac{\log\left[\, 2 \left(\dfrac{q(1-s_1)}{s_1(1-q)} \right) \right]}{\log 2},$$

then D is finite. By the Archimedean principle for real numbers, there exists a natural number i_0 such that $i_0 > D$. Now $i_0 > D$ implies

$$2^{i_0} > 2\,\frac{q_1'(1-s_1)}{s_1(1-q_1')}$$

implies

$$2^{i_0-1} > \frac{q_1'(1-s_1)}{s_1(1-q_1')}$$

implies

$$s^*_{i_0} = \frac{2^{i_0-1}s_1}{1+(2^{i_0-1}-1)s_1} > q_1' \, .$$

Thus $\theta \overset{*}{Q}' S_{i_0}$. Since $\psi_1(\theta) = P$, $\psi_1(Q') = Q$, and $\psi_1(S_{i_0}) = R_{i_0}$, $P\overset{*}{Q}R_{i_0}$ by lemma 13. So there exists n such that $P\overset{*}{Q}R_n$, namely $n = i_0$.

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