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The problem considered is that of causing all the machines of an array to turn on simultaneously in the least amount of time. Each machine is a finite sequential machine that can communicate with a limited number of prescribed neighbor machines.

A solution for a linear array of these machines, the Firing Squad Problem, is presented in detail. The solution is then extended, with appropriate modifications, to other array configurations and numbers of communicating machines. Areas of study beyond this work are suggested and suggestions for further study are presented.

Control of Interconnection-Limited Sequential Machines

by

Keith Alfred Riese

A THESIS

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CONTROL OF INTERCONNECTION-LIMITED SEQUENTIAL MACHINES

INTRODUCTION

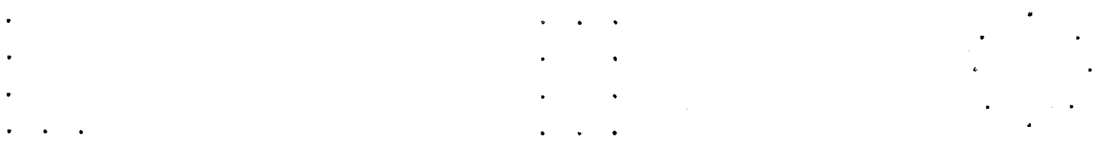
One of the important areas of study in computer system design is the computer memory and its operation. The following question is relevant to the control of memory connections and storage: Assuming each memory address can pass and receive a signal to and from its immediate neighbors in one unit of time, how long will it take for all memory addresses to simultaneously erase or dump their information after an initial signal is given to a particular address at time t=0 ?

The question is practical in that there are situations where these assumptions must be met: e.g., computer memories, radar systems, and any controlled array--possibly even a grid of light bulbs on a billboard sign. In any of these cases time and money could be saved due to fast completion of this simultaneous dumping or loading of information. Hence the study reported herein. Practical solutions might also find application to other problems met in the future, such as bulk memory control, biological system models, and so on.

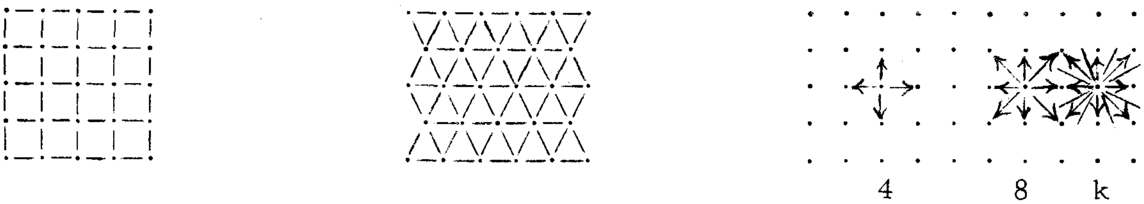
All of the cases mentioned above are similar in that each address or cell can be represented by a sequential machine. Specifically, the question becomes: What is the minimum length of time needed for an array of sequential machines to simultaneously output a certain signal after a starting signal has been given one machine, where each machine can receive a signal and/or give a signal only to a directly-connected neighbor in one unit of time?

The problem was originally posed for a topological line of synchronous sequential machines, and is referred to as the Firing Squad Problem (Myhill, 1957). The problem arose in connection with the need to cause all parts of a self-reproducing machine to turn on simultaneously.

In this study we shall discuss the original Firing Squad Problem, its solution, and its limitations. From there we shall extend the examination to other machine configurations and to multidimensional machine arrangements (Figure 1): The two-dimensional and circular arrays; the three-dimensional arrays of the square grid, triangular grid, four-, six-, and eight-neighbor grid, and the more general k-neighbor grid. Comparisons will be made of the length of time needed by the various arrays for the machines to simultaneously turn on. These results are reported and areas for further study beyond the scope of this work are suggested.



L-, rectangular-, and circular-shaped arrays



Square, Triangular, 4-, 8-, and k-neighbor arrays

Figure 1. Various arrays of machines

THE FIRING SQUAD PROBLEM

As mentioned previously, the Firing Squad Problem arose intially in connection with the need to cause all parts of a self-reproducing machine to turn on simultaneously. Each part is itself a synchronous finite-state machine.

Consider a finite (but arbitrarily long) one-dimensional array of these machines, all of which are called soldiers--one end machine being a "General." The General is a special machine that initiates the command to "fire." The state of each machine (soldier) at time t+1 depends on its state and its two neighbors' states at time t. The problem is to specify the state and transitions of the soldiers such that the General can cause them to go into a particular (terminal) state (i. e., they "fire" their guns), all at precisely the same time. In particular, the number of states required cannot be a function of the length n, so that we are precluded from simply specifying an arbitrarily large counter in each machine. Initially, at t=0, all the soldiers are assumed to be in a single state, the quiescent state.

When the General enters the "fire when ready" state, he does not take any further action after passing the "word" along, and the rest is up to the soldiers. Signals can propagate no faster than one soldier per unit of time, and the problem is to get all coordinated and ready to fire. The difficult part is that the same soldier with a finite number of m states is required to be able to do this; i.e., we seek a solution utilizing a fixed m which is independent of the length n of the firing squad.

The end soldiers, the General and the soldier farthest from the General, are allowed to be different in that they communicate with only one neighbor. Their structure must still be independent of n.

J. McCarthy and M. Minsky first solved the problem (Moore, 1964) and Waksman formalized the solution (Waksman, 1966). Waksman found that it requires at least 2n-2 time units for the soldiers to fire, where n is the length of the array. The solution centers around dividing and subdividing the soldiers into more and smaller groups as rapidly as possible until all the soldiers become individual "groups," whereupon they all fire.

To accomplish this, several types of signals (machine states) must be used. The states used herein are given in Table 1. Initially, a left or right propagating signal (A signal) is needed to notify the complete array of soldiers that they are being divided up for simultaneous firing. This signal must generate opposite direction signals (R signals) to begin the breakdown of the groups of soldiers into smaller groups. Other signals must be generated (B signals) to cause these group divisions to occur such that the final grouping of all soldiers into individual groups of one occurs simultaneously. The soldiers will fire one time unit later.

Table 1. Waksman's state list.

Q	The quiescent state	
T	The terminal state	
P_o	Preterminal state, generates A_{oxx}	
P_1	Preterminal state, generates A_{1xx}	
B_o	Generates P	
B_1	Generates P, pass through R_o and R_1	
R_o	Trigger signal for B, propagate to left	
R_1	Trigger signal for B, propagate to right	
A_{ooo}	Propagate to left even	Generate R with no delay
A_{oo1}	Propagate to left odd	
A_{o1o}	Propagate to right even	
A_{o11}	Propagate to right odd	
A_{1oo}	Propagate to left even	One unit time delay in generating R
A_{1o1}	Propagate to left odd	
A_{11o}	Propagate to right even	
A_{111}	Propagate to right odd	
ϕ	External state: no machine to this side	
γ	In all cases not explicitly mentioned in this row or column of left or right neighbors.	

The state transition tables (for each machine or soldier) have been constructed (Table 2) such that the following conditions will be met:

1. Every propagating A-signal assumes even or odd designation according to whether it is occupied by an even or odd machine. The machine is considered even or odd by counting machines from the machine of the origin of the A-signal to the one in question.

2. Every machine containing an even designated A-signal will generate a new signal R which will propagate in the reverse direction to the generating A-signal.

3. There are two kinds of B-signals: one that shifts one machine forward upon intersecting with an R-signal, allowing the R-signal to continue to propagate; and one that shifts one machine forward upon intersecting with an R-signal, not allowing the R-signal to continue to propagate.

4. Upon a shift in machines, signal B of one kind changes to signal B of the other kind, and vice versa.

5. Upon intersection of signals A and B, new states P will be generated. The division of even or odd number of machines is carried by the A-signal. Thus either one or two P-state machines will be generated.

Every second machine in state A generates an R-signal propagating at synchronous speed in the opposite direction of the A-signal.

Table 2. The state transition tables.

The entries in the tables are the state of a machine at t+1 corresponding to its own state at time t (the title state) and its left and right neighbor's state at time t. In all state tables: nonsubscript state symbols refer to all subscript states with that symbol. Example:

THE STATE AT TIME t	
L	R
State of left neighbor at t	State of right neighbor at t

The new state of the machine at t+1

The Q state:

L	R							
	A_{000}	A_{001}	A_{100}	A_{101}	A_{010}	A_{011}	A_{110}	A_{111}
Q	A_{001}	A_{000}	A_{101}	A_{100}	R_o	Q	Q	Q
B	A_{001}	A_{000}	A_{101}	A_{100}	R_o	Q	Q	Q
R_o	A_{001}	A_{000}	A_{101}	A_{100}	Q	Q	Q	Q
R_1	A_{001}	A_{000}	A_{101}	A_{100}	-	-	-	-
P_o	-	-	-	-	-	B_o	-	-
P_1	-	-	-	-	-	-	B_o	-
ϕ	P_o	P_1	P_o	P_1	-	-	-	-

L	R						
	Q	B	R_o	R_1	P_o	P_1	ϕ
A_{000}	R_1	R_1	-	-	-	-	-
A_{001}	Q	Q	-	Q	B_o	-	-
A_{100}	Q	Q	-	-	-	-	-
A_{101}	Q	Q	-	-	-	-	-
A_{010}	A_{011}	A_{011}	-	-	-	-	P_o
A_{011}	A_{010}	A_{010}	-	-	-	-	P_1
A_{110}	A_{111}	A_{111}	-	-	-	-	P_o
A_{111}	A_{110}	A_{110}	-	-	-	-	P_1
Q	Q	Q	R	Q	A_{000}	A_{100}	Q
γ	-	-	-	-	A_{000}	-	

The Q state:

L	R							
	Q	B	R_o	R_1	P_o	P_1	ϕ	γ
B	Q	Q	R_o	Q	Q	Q	-	-
R_o	Q	Q	-	-	A_{000}	-	-	Q
R_1	R_1	R_1	-	-	R_1	R_1	-	-
P_o	A_{010}	Q	R_o	Q	P_o	P_o	-	-
P_1	A_{110}	Q	R_o	Q	P_o	P_o	-	-
ϕ	Q	-	-	-	-	-	-	-

Table 2. Continued.

The R_o state:

L	R: γ
γ	Q
B_o	B_1
B_1	B_o
P	B_o

The R_1 state:

L	R: B, $\bar{P}$	B_o	B_1	P
γ	Q	B_1	B_o	B_o

The P_o state:

L	R: $\bar{P}, \bar{\Phi}$	P	ϕ
$\phi, \bar{P}$	P_o	P_o	P_o
P	P_o	T	T
ϕ	P_o	T	-

The B_o state:

L	R: γ	P	A_{000}	A_{100}	A_{001}	A_{101}	R_o
γ	B_o	B_o	P_o	P_1	P_1	P_o	R_o
P	B_o	P_o	-	-	-	-	R_o
A_{010}	P_o						
A_{110}	P_1						
A_{011}	P_1						
A_{111}	P_o						
R_1	R_1						

The B_1 state:

L	R: γ	P	A_{000}	A_{100}	A_{001}	A_{101}	R_o
γ	B_1	-	P_o	P_1	P_1	P_o	Q
P	-	P_o					
A_{010}	P_o						
A_{110}	P_1						
A_{011}	P_1						
A_{111}	P_o						
R_1	Q						

The P_1 state:

L	R: $\bar{P}, \bar{\Phi}$	P	ϕ
$\bar{\Phi}, \bar{P}$	P_1	P_1	P_1
P	P_1	T	T
ϕ	P_1	T	-

The A_{000} state:

L	R: $\bar{P}$	P_o
γ	Q	B_o

The A_{001} state:

L	R: γ
γ	Q

The A_{100} state:

L	R: γ
$\bar{B}$	R_1
B	P_1

The A_{101} state:

L	R: γ
$\bar{B}$	Q
B	P_o

The A_{010} state:

L	R: γ
$\bar{P}_o$	Q
P_o	B_o

The A_{011} state:

L	R: γ
γ	Q

The A_{110} state:

L	R: $\bar{B}$	B
γ	R_o	P_1

The A_{111} state

L	R: $\bar{B}$	B
γ	Q	P_o

Thus there are three time units separation between R-signals going through the machine (see region A of Figure 2).

From condition three, each machine assuming a B state will maintain it for three time units until triggered by a signal R, which will cause a shift of the B state to the next machine in the direction of the approaching R signals.

Since there are two continually alternating kinds of B signals where only one kind allows the R signals to continue to propagate (see region B, Figure 2), after passing through the first B signals the new separation of R signals will be 2 x 3 + 1 = 7 time units. Similarly, in region C of Figure 2 the delay of the B state in each machine is 2 x 7 + 1 = 15 time units. Listing these for up to k divisions:

$B_1=3$	time unit delay/machine
$B_2=2x3+1=7$	"
$B_3=2x7+1=15$	"
.	.
.	.
.	.
$B_k=2D_{(k-1)}+1$	"

where $D_{(k-1)}$ is the delay per machine of the $B_{(k-1)}$ signal.

Equivalently,

$B_1=2(2^1-1)+1=2^2-1$ time unit delay/machine

$B_2=2(2^2-1)+1=2^3-1$ "

. .

. .

. .

$B_k=2(2^k-1)+1=2^{k+1}-1$ "

Therefore the B_k signal will have the required propagating rate of $1/(2^{k+1}-1)$ for any length array.

Figure 2 is an example of a 2n-2 time unit solution of a 40 machine array requiring the generation of five B signals. For each group of machines that is subdivided, the end machines of the new smaller groups will be in the P state. For the simultaneous division of an odd number of machines (see Figure 3a), synchronization in generating the new P state is assured. The machine with this P state is an end machine of both new groups of machines. For any division of an even number of machines (see Figure 3b), the last set of B signals to be generated, from the end opposite the General, are delayed one time unit. This is accomplished by delaying all the generated R signals one time unit. It occurs every time the last count of machines was even. Thus two new P states are formed, and the new groups will have separate end P-state machines. Only the six main states (Q, A, R, B, P, and T) are shown in the 40 machine array solution of Figure 2. Figure 3 indicates the solution of 11 and 12 machine

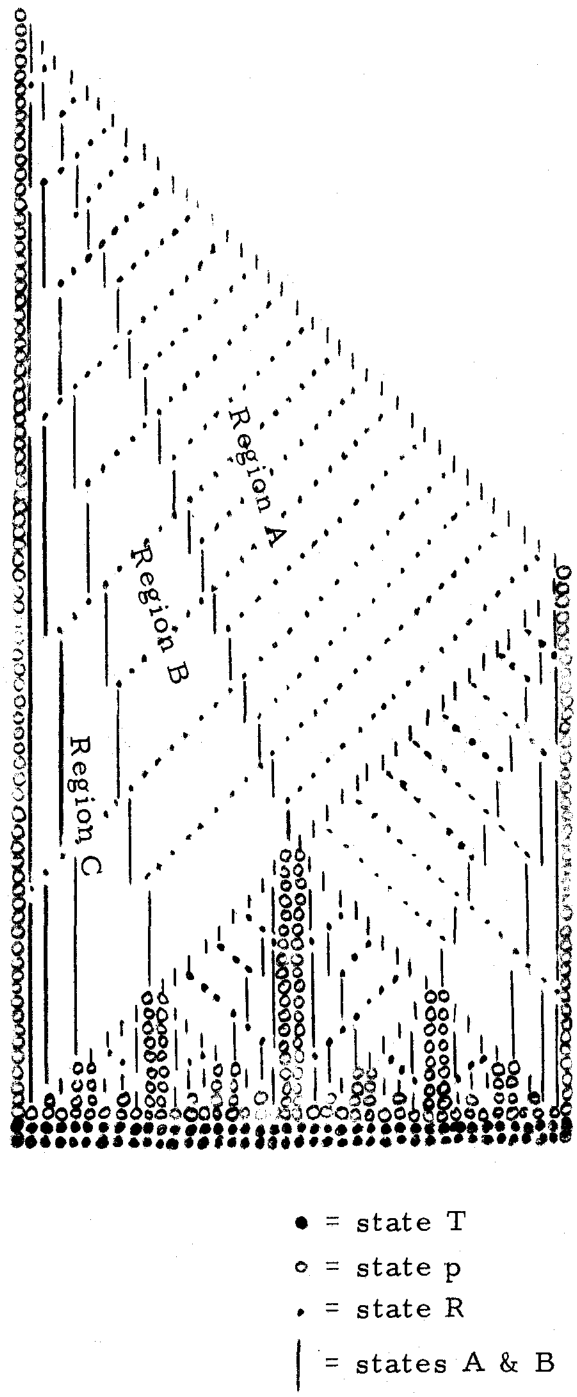


Figure 2. The general (2n-2) solution for a 40 machine array

Figure 3. Complete (2n-2) time unit solution for an 11 and 12 machine array. All blank spaces are quiescent states.

NUMBER OF MACHINES

TIME	1	2	3	4	5	6	7	8	9	10	11
0	P_o										
1	P_o	A_{010}									
2	P_o	B_o	A_{011}								
3	P_o	B_o		A_{010}							
4	P_o	B_o	R_o		A_{011}						
5	P_o	R_o	B_1			A_{010}					
6	P_o	B_o	B_1		R_o		A_{011}				
7	P_o	B_o	B_1	R_o				A_{010}			
8	P_o	B_o		B_o			R_o		A_{011}		
9	P_o	B_o		B_o		R_o				A_{010}	
10	P_o	B_o		B_o	R_o				R_o		P_o
11	P_o	B_o		R_o	B_1			R_o		A_{000}	P_o
12	P_o	B_o	R_o		B_1		R_o		A_{001}	B_o	P_o
13	P_o	R_o	B_1		B_1	R_o		A_{000}		B_o	P_o
14	P_o	B_o	B_1			B	A_{001}		R_1	B_o	P_o
15	P_o	B_o	B_1			P_1			B_1	R_1	P_o
16	P_o	B_o	B_1		A_{100}	P_1	A_{110}		B_1	B_o	P_o
17	P_o	B_o	B_1	A_{101}	R_1	P_1	R_o	A_{111}	B_1	B_o	P_o
18	P_o	B_o	P_o	P_o	B_o	P_1	B_o	P_o	P_o	B_o	P_o
19	P_o	P_o	P_o	P_o	P_o	P_1	P_o	P_o	P_o	P_o	P_o
20	T	T	T	T	T	T	T	T	T	T	T

Similar subscripts on A states

TIME	1	2	3	4	5	6	7	8	9	10	11	12
0	P_o											
1	P_o	A										
2	P_o	B_o	A									
3	P_o	B_o		A								
4	P_o	B_o	R_o		A							
5	P_o	R_o	B_1			A						
6	P_o	B_o	B_1		R_o		A					
7	P_o	B_o	B_1	R_o				A				
8	P_o	B_o		B_o			R_o		A			
9	P_o	B_o		B_o		R_o				A		
10	P_o	B_o		B_o	R_o				R_o		A	
11	P_o	B_o		R_o	B_1			R_o				P_o
12	P_o	B_o	R_o		B_1		R_o				A	P_o
13	P_o	R_o	B_1		B_1	R_o				A	R_1	P_o
14	P_o	B_o	B_1			B_o			A		B_o	P_o
15	P_o	B_o	B_1			B_o		A	R_1		B_o	P_o
16	P_o	B_o	B_1			B_o	A			R_1	B_o	P_o
17	P_o	B_o	B_1			P_1	P_1			B_1	R_1	P_o
18	P_o	B_o	B_1		A	P_1	P_1	A		B_1	B_o	P_o
19	P_o	B_o	B_1	A	B_o	P_1	P_1	B_o	A	B_1	B_o	P_o
20	P_o	B_o	P_o	P_o	B_o	P_1	P_1	B_o	P_o	P_o	B_o	P_o
21	P_o	P_o	P_o	P_o	P_o	P_1	P_1	P_o	P_o	P_o	P_o	P_o
22	T	T	T	T	T	T	T	T	T	T	T	T

arrays, where the delay of B-signals is apparent for division of an even number of machines. These solutions are minimal in time t and not in the number of machine states m.

Consider the case where the General is not the end soldier in the array (Waksman, 1966). What is the shortest time required for all the soldiers to fire in this situation? Define the soldier in the array who randomly becomes the General a "roving General," or an "internal General." This soldier could be any one picked from the row.

Theorem: The minimum time in which the firing squad could possibly fire is no earlier than 2n-2-k time units, where k is the number of cells between the General and the nearest end of the soldier line. (When the General is at the end, k=0, and the minimum time formula reduces to 2n-2).

Proof: Without loss of generality, consider, for a given k, that the General is left of the center cell (cell k+1 rather than cell n-k). Cell 1 is thus the "near end cell" with respect to the General, the critical cell.

Assume there is a cell structure S for the general problem with n arbitrary and

$$1 \leqq k_o \leqq \frac{n+1}{2} - 1$$

for which there is some length n_o and cell number k_o+1 for the General such that the firing squad fires at time t=m where

$m \leqq 2n_o - 2 - k_o$. This means cell one could not have received any signal from cell n_o as it takes $n_o - k_o - 1$ time units for a signal to reach cell n_o from the General, plus $n_o - 1$ time units for a signal to reach cell one from cell n_o. Cell one will thus enter the "fire" state independent of cell n_o and any cells to the right of n_o. If $n_o + 2$ cells were added to the problem (the right end now being cell $2n_o + 2$), cell one would still fire at time t=m. This is due to the fixed cell structure and deterministic cell operation. Since $m \leqq 2n_o - 2 - k_o$, cell $2n_o + 2$ would be in the Q (quiescent) state at time t=m. Therefore the cell structure S does not represent a solution, and a contradiction has been obtained. The argument carries over to where the General is in the right half of the line at cell $n_o - k_o$, and cell n_o is the critical cell.

The idea behind the solution is to "reconstruct" the signal propagation back in time to the point of the General being at the near end of the line just entering the "fire when ready" state at t=-k. Figure 4 shows the configuration for an "internal" General. It is seen that below a certain point, the solution uses only states of the normal firing squad (with the General at the end). Above this point the slope 3, 7, 15, ... (B-signal) lines are reconstructed. Figure 5 and Table 3 give respectively an n=26, k=8 solution and the state tables used.

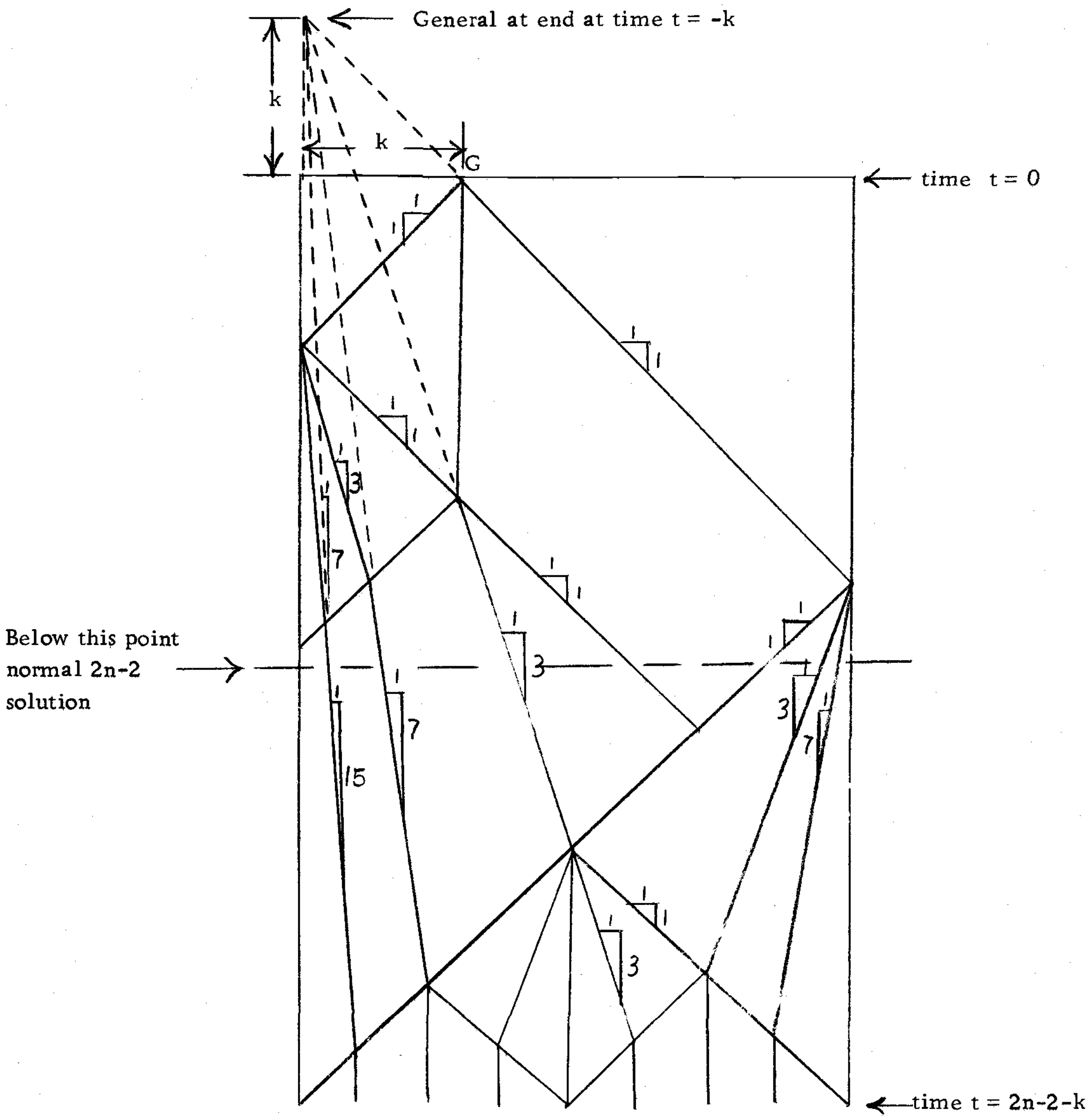


Figure 4. Geometry of soldiers with an internal General

Table 3. State tables.

State Q

Q	Q	K	P	L	S	R	G	Y	Z	φ	-
Q	Q	G	L							Q	
K		K			X				I		I
P	S									K	
L		Z						Z			
S										K	S
A		G									
B		G									
R		G									
H											H
I									I		I
X	W				W						
W											W
φ	Q		K	K							
-		G		L		R	G		Z		Q

State H

H	Q	K	A	B	-
Q		A			
K	I	K			
B		A			
G		A			
I			K	K	I
Y		A			
-			B	A	Q

State B

B	Q	A	R	H	G	-
Q					K	
A					K	
R					K	Q
H						
I	K	K		K		
-			Q			B

State G

G	Q	K	H
Q		H	
D		H	
A	K	K	
B	H	H	K
R		H	
-	H		Q

State W

W	Q	-
Q		R
A	R	
B	R	
R		Q

Upper left-hand corner denotes present state.
Column heading denotes right neighbor's state.
Row heading denotes left neighbor's state.

Table 3. Continued.

State A

A	Q	K	R	G	-
Q				K	
K		K		K	
H				H	
I	K	K			
-			R		A

State R

R	K	B	G	I	X	-
Q	G					
K	K	A		A	A	
A			K			B
B			K			A
-			G			Q

State D

D	Q	K	S	G	-
Q		Y		Y	
K	X	K	X		
L		Y			
I	X			K	
-					D

State X

X	Q
Q	A
K	A
R	B

State I

I	Q	D	A	B	H	-
Q			K			R
K	R	R	K	R	R	
A	R					
R				K		Q

State K

K	K	ϕ	-
K	T	T	
ϕ	T		
-			K

State Z

Z	Q	A	B	H
Q		H	H	
-	H			Q

State P

P	Q	ϕ
Q	D	K
ϕ	K	T

State Y

Y	Q	K	H
Q	A	A	B

State L

L	-
-	Q

State S

S	-
-	Q

State T

T	-
-	T

Time																											
0									P																		t=0
1								L	D	S																	
2							L		D		S																
3						L			D			S															
4					L				D				S														
5				L					D					S													
6			L						D						S												
7		L							D							S											
8	K								D								S										
9	K	I							D									S									
10	K	R	I						D										S								
11	K	A		I					D											S							
12	K	A		R	I				D												S						
13	K	A	R			I			D													S					
14	K	R	B			R	I		D														S				
15	K	A	B		R			I	D															S			
16	K	A	B	R				R	X																S		
17	K	A		A			R		B	W																K	
18	K	A		A		R			B	R	W														G	K	
19	K	A		A	R					A		W												G	H	K	
20	K	A		R	B					A		R	W										G		A	K	
21	K	A	R		B					A	R			W								G	H		A	K	
22	K	R	B		B					R	B			R	W						G			H	A	K	
23	K	A	B		B				R		B		R			W				G	H			B	H	K	
24	K	A	B		B			R			B	R				R	W		G			H		B	A	K	
25	K	A	B		B		R					A			R			G	H				H	B	A	K	
26	K	A	B		B	R						A		R			G			H			A		A	K	
27	K	A	B			A						A	R			G	H				H		A		A	K	
28	K	A	B			A						R	B		G			H				H	A		A	K	
29	K	A	B			A					R		B	G	H				H			B	H		A	K	
30	K	A	B			A				R			K	K		H				H		B		H	A	K	
31	K	A	B			A			R			G	K	K	I		H				H	B		B	H	K	
32	K	A	B			A		R			G	H	K	K	R	I		H			A			B	A	K	
33	K	A	B			A	R			G		A	K	K	A		I		H		A			B	A	K	
34	K	A	B			R	B		G	H		A	K	K	A		R	I		H	A			B	A	K	
35	K	A	B		R		B	G			H	A	K	K	A	R			I	B	H			B	A	K	
36	K	A	B	R			K	H			B	H	K	K	R	B			R	K		H		B	A	K	
37	K	A		A		G	K	I	H		B	A	K	K	A	B		R	G	K	I		H	B	A	K	
38	K	A		A	G	H	K	R	I	H	B	A	K	K	A	B	R	G	H	K	R	I	A		A	K	
39	K	A		K		A	K	A		K		A	K	K	A		K		A	K	A		K		A	K	
40	K	A	G	K	I	A	K	A	G	K	I	A	K	K	A	G	K	I	A	K	A	G	K	I	A	K	
41	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	K	
42	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	T	

t = 42 = 2(26) -2-8

Figure 5. Complete 26 machine (soldier) solution

A modification of the redundant parity information used previously is given by Table 4. The remaining states are used to reconstruct the slope 3, 7, 15 ... (B-signal) lines back to time t=-k. Thus there are fewer states needed here than previously.

Table 4. State assignment equivalents.

End General Solution	Roving General Solution
Q	Q
P_o, P_1	K
B_o	A
B_1	B
A (left propagate)	G
R_1	H
A (right propagate)	I
R_o	R
T	T

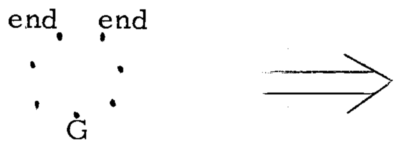
When an internal General is given the "fire when ready" signal, state P, signals L and S are propagated to either end to determine which end is closer to the General. See Figure 5. Upon reaching the ends, these signals appear to be at the far end and in turn propagate signals back to the General, as in the case where the General was at one end (states I, R, A, and B on the left and G, H, A, and B on the right). Also, the General holds its place with a divider state D and tells the first returning signal I or G that its side was really the near side. At that point, a slope three line is generated on the nearest side which changes the slope of all preceding lines since it decides whether to pass the R and H signals. See Figure 4 where all sloped lines thus increase one slope value; i.e., 2^r-1 becomes $2^{r+1}-1$.

CONTROL PROCESSES FOR OTHER MACHINE CONFIGURATIONS

There are many other configurations of machines (soldiers) where the firing squad problem can be posed. They range from simple configurations to those that are quite complex and correspond to different useful arrays of interconnection-limited machines. An examination will be made of several of these configurations. For clarity and understanding of the problems involved, a minimum time solution will be sought rather than a minimum state solution. There will be one main constraint: the number of states cannot be a function of n.

The circular array of soldiers is the most evident extension of the linear firing squad problem. Here the General is similar to the middle soldier in a straight line solution where the ends are the two opposite soldiers where the General's signals meet. The even-odd parity information is held by the General's outgoing signals. Other descriptions and constraints remain the same with no new machine states needed. Figure 6 illustrates a circle of soldiers of length seven with its corresponding solution.

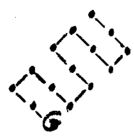
A much more complex situation is the rectangular grid, n by m machines in size, with four cell neighbors and the General located at a corner. Clearly an upper bound on the minimum time needed for the soldiers to simultaneously fire is the linearly extended array of Figure 7b. Figure 7c shows another solution for the same grid taking one row



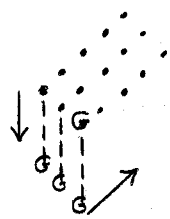
end						end	time
.	.	.	G	.	.	.	0
.	.	L	D	S	.	.	1
.	L	.	D	.	S	.	2
P	.	.	D	.	.	P	3
P	A	.	D	.	A	P	4
P	B	A	D	A	B	P	5
P	B	.	P	.	B	P	6
P	B	A	P	A	B	P	7
P	P	P	P	P	P	P	8
T	T	T	T	T	T	T	9

Figure 6. Circle of seven soldiers and solution

(a) Rectangular and triangular arrays



(b) Extension of linear array
t=2n-2=28



(c) Rectangular array
t=2(3)-2 + 2(5)-2
t=12

Figure 7. Arrays and linear solutions

in one direction, making Generals of each soldier in the row at the firing time, and proceeding with the rows in the other direction. Here some additional states would be needed to make the first row fire as Generals. Signal propagation would be limited to one direction until the new Generals are formed, then limited to the other direction. For the three by five grid shown it would take $2n-2=2(3)-2=4$ time units in one direction and $2(5)-2=8$ time units in the other direction making a total of 12 time units required. The linear extension of Figure 7b requires $2(15)-2=28$ time units, clearly an upper bound.

A special case of the rectangular grid is the square sized grid. With the General occupying the center position, it is clear the grid will be n by n with n odd. With n even, there can be no center General's position or cell.

Theorem: In the n by n (n odd) square grid array with the General the center cell, the minimum time for all cells to fire is equal to that required by a straight line of soldiers whose length is $m = \frac{n+1}{2}$, the number of "waves" or "levels" of cells from the General to the outer edge inclusive. Thus the General is one end of the straight line case and the outer cells become the other "end." For the straight line time of $2m-2$ time units, the time becomes $t=n-1$ time units, or one half the time of a straight line of n soldiers with the General at the end. For n odd, $m = |\frac{n}{2}| + 1$ where $|N|$ is the largest whole number less than or equal to N. This is called the wave solution.

Proof: The proof is by contradiction. Assume there is an n_o by n_o (n_o odd) square cell structure S such that the General is the center cell and the grid will fire at time t=r where

$$r < n_o - 1 \ (n_o \text{ odd})$$

$$\text{or } r < 2m_o - 2 \text{ where } m_o = \frac{n_o + 1}{2}$$

Then cell one, the General, would not have received any signal from the outer wave of cells because it takes m_o-1 time units for a signal to reach the outer wave from the General and m_o-1 time units for a signal to return, or $2m_o$-2 time units. Thus, the General will fire independent of the outer wave of cells and any waves added on. If $2m_o$+2 cell waves were added, the last wave would still be in the quiescent state when the General fired, at time $2m_o$-1, since the cell structure operation is fixed and deterministic. Therefore the cell structure S does not represent a solution and a contradiction has been obtained.

As an example, consider a nine by nine array, Figure 9, with the General at the center of the array. The state tables for the machines (soldiers in the array) will be identical with those for the straight line case with but two differences: 1. The case of one cell receiving two of the same signals at once must be considered as receiving one of them, and 2. The right and left entries in the state tables must be changed to "in front" and "behind" respectively. Thus

1	2	3	4	5
.	.	.	.	.

4	4	4	4	4	4	4
.	3	3	3	3	3	4
.	.	2	2	2	3	4
.	.	.	1	2	3	4
.	.	.	.	2	3	4
.	.	.	.	.	3	4
.	.	.	.	.	.	4

Figure 8. Cell position in square array

t = 0, General at center

t = 1:

.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	A	A	A	.	.	.
.	.	.	A	P	A	.	.	.
.	.	.	A	A	A	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

t = 2:

.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.
.	.	A	A	A	A	A	.	.
.	.	A	B_0	B_0	B_0	A	.	.
.	.	A	B_0	P	B_0	A	.	.
.	.	A	B_0	B_0	B_0	A	.	.
.	.	A	A	A	A	A	.	.
.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.

t = 3:

.	.	.	.	.	.	.	.	.
.	A	A	A	A	A	A	A	.
.	A	.	.	.	.	.	A	.
.	A	.	B_0	B_0	B_0	.	A	.
.	A	.	B_0	P	B_0	.	A	.
.	A	.	B_0	B_0	B_0	.	A	.
.	A	.	.	.	.	.	A	.
.	A	A	A	A	A	A	A	.
.	.	.	.	.	.	.	.	.

t = 4:

P	P	P	P	P	P	P	P	P
P	.	.	.	.	.	.	.	P
P	.	R_0	R_0	R_0	R_0	R_0	.	P
P	.	R_0	B_0	B_0	B_0	R_0	.	P
R	.	R_0	B_0	P	B_0	R_0	.	P
P	.	R_0	B_0	B_0	B_0	R_0	.	P
P	.	R_0	R_0	R_0	R_0	R_0	.	P
P	.	.	.	.	.	.	.	P
P	P	P	P	P	P	P	P	P

t = 5:

P	P	P	P	P	P	P	P	P
P	A	A	A	A	A	A	A	P
P	A	B_1	B_1	B_1	B_1	B_1	A	P
P	A	B_1	R_0	R_0	R_0	B_1	A	P
P	A	B_1	R_0	P	R_0	B_1	A	P
P	A	B_1	R_0	R_0	R_0	B_1	A	P
P	A	B_1	B_1	B_1	B_1	B_1	A	P
P	A	A	A	A	A	A	A	P
P	P	P	P	P	P	P	P	P

t = 6:

P	P	P	P	P	P	P	P	P
P	B_0	B_0	B_0	B_0	B_0	B_0	B_0	P
P	B_0	P	P	P	P	P	B_0	P
P	B_0	P	B_0	B_0	B_0	P	B_0	P
P	B_0	P	B_0	P	B_0	P	B_0	P
P	B_0	P	B_0	B_0	B_0	P	B_0	P
P	B_0	P	P	P	P	P	B_0	P
P	B_0	B_0	B_0	B_0	B_0	B_0	B_0	P
P	P	P	P	P	P	P	P	P

t = 7:
All p's

t = 8:
All T's, fire.

Figure 9. Nine by nine array solution

as two is left of three and four is right of three in the straight line, two is behind three and four is in front of three in the square grid shown in Figure 8.

The General above was assumed to be only at the center of the array. However, all the cells of a particular "wave" of cells could be the "General" in the sense that all the cells of the wave would receive the "fire when ready" signal rather than one particular cell. Being k waves from the nearer "end," the solution is analogous to that of a straight line of soldiers whose General is k cells from the nearer end. The time for simultaneous firing is 2n-2-k where k is the total number of waves from the "General wave" to the nearer end; the center cell or the outer wave or edge of cells.

As an example: For the seven by seven array of Figure 10, the firing time is 2m-2-k time units where $m = \left|\frac{n}{2}\right| +1$, and $|N|$ is the largest whole number less than N. Thus, with one wave from the General wave to the near end, the center (k=1),

$$t = 2\left(\left|\frac{n}{2}\right| +1\right)-2-1 = 5 \text{ time units.}$$

```
. . . . . . .
. . . . . . .
. . G G G . .
. . G . G . .
. . G G G . .
. . . . . . .
. . . . . . .
```

Figure 10. A "General wave" array

The firing squad solution in these cases depends on finding the ends of the groups to be subdivided. For the case where the General is not at the Center of the array, the solution becomes very complicated. The time of firing increases due to the amount of time it takes to find the "ends," the center cell and the outer edge of cells.

This end-finding proceeds similarly to that of the straight line array with the added problem of reflected waves from the edges of the array meeting at random times and places. When the General is at an arbitrary position, at time t=0 signals are sent out to find the ends of the group to be subdivided. The General will be any distance from any edge and when the signal reaches a near edge of the array, a reflected signal starts propagating back to the General. The problems arising are: 1. How can the waves reflected from the edges be handled? 2. How can the "ends" be specified if there is no center array cell (n by n grid, n even), and how can the edge cells be an "end" when all the edges are different distances from the General? A number of additional soldier states will solve problem one, but problem two requires more work, left to be done in the future.

An upper bound on the solution can be obtained by looking at the case where the General is the corner cell in any sized rectangular array. Two constraints must be applied: 1. A change on the labeling of the state tables from "right" and "left" to "forward" and "behind" respectively, and 2. The General at one corner is an "end" of an imaginary straight line array, and the other "end" is the opposite corner, where there are no more "forward cells" in the two forward directions. See Figure 11. This is the diagonal propagation solution.

Theorem: An n by n grid with the General at the corner will simultaneously fire in the same length of time as a straight line of cells of length 2n-1. Where m=2n-1, and 2m-2 time units are required for the General at an end, the grid will fire at 2(2n-1)-2 = 4n-4 time units.

Proof: The proof is similar to that of the previous theorem. The theorem immediately extends to any rectangular array. An example is shown in Figure 12.

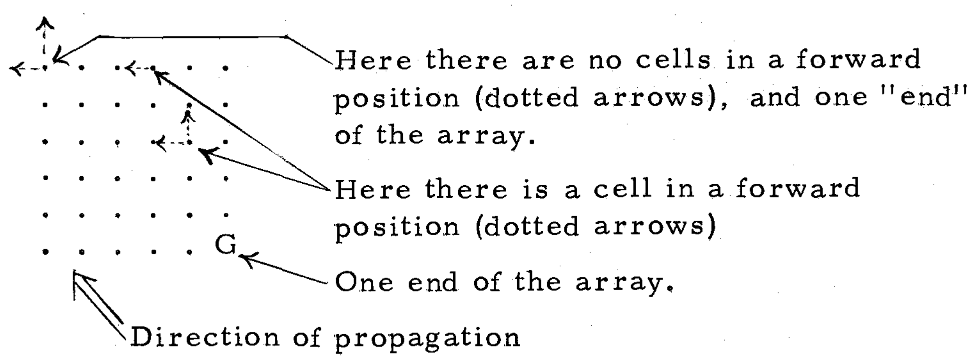


Figure 11. Corner-General diagonal propagation

This can be extended to a randomly located General who sends out signals to find the nearest corner. The corner cells become as Generals and propagate signals diagonally as above. The signals generated by the first corner reached must have priority over the diagonally propagating signals of the other corners of the array. Here, Grid: 5 x 5; 2n-1 = 9 soldiers in a row.

Solution takes 4n-4 = 16 time units.

t = 0, General is at the corner.
t = 1, P is at the corner, A adjacent to P.

t = 2:

.	.	.	.	.
.	.	.	.	.
.	.	.	.	A
.	.	.	A	B_0
.	.	A	B_0	P

t = 3:

.	.	.	.	.
.	.	.	.	A
.	.	.	A	.
.	.	A	.	B_0
.	A	.	B_0	P

t = 4:

.	.	.	.	A
.	.	.	A	.
.	.	A	.	R_0
.	A	.	R_0	B_0
A	.	R_0	B_0	P

t = 5:

.	.	.	A	.
.	.	A	.	.
.	A	.	.	B_1
A	.	.	B_1	R_0
.	.	B_1	R_0	P

t = 6:

.	.	A	.	R_0
.	A	.	R_0	.
A	.	R_0	.	B_1
.	R_0	.	B_1	B_0
R_0	.	B_1	B_0	P

t = 7:

.	A	.	.	.
A	.	.	.	R_0
.	.	.	R_0	B_1
.	.	R_0	B_1	B_0
.	R_0	B_1	B_0	P

t = 8:

P	.	R_0	.	.
.	R_0	.	.	B_0
R_0	.	.	B_0	.
.	.	B_0	.	B_0
.	B_0	.	B_0	P

t = 9:

P	A	.	R_0	.
A	.	R_0	.	B_0
.	R_0	.	B_0	.
R_0	.	B_0	.	B_0
.	B_0	.	B_0	P

t = 10:

P	B_0	A	.	R_0
B_0	A	.	R_0	B_0
A	.	R_0	B_0	.
.	R_0	B_0	.	B_0
R_0	B_0	.	B_0	P

t = 11:

P	B_0	.	A	B_1
B_0	.	A	B_1	R_0
.	A	B_1	R_0	.
A	B_1	R_0	.	B_0
B_1	R_0	.	B_0	P

t = 12:

P	B_0	R_1	.	P
B_0	R_1	.	P	.
R_1	.	P	.	R_0
.	P	.	R_0	B_0
P	.	R_0	B_0	P

t = 13:

P	R_1	B_1	A	P
R_1	B_1	A	P	A
B_1	A	P	A	B_1
A	P	A	B_1	R_0
P	A	B_1	R_0	P

t = 14:

P	B_0	P	B_0	P
B_0	P	B_0	P	B_0
P	B_0	P	B_0	P
B_0	P	B_0	P	B_0
P	B_0	P	B_0	P

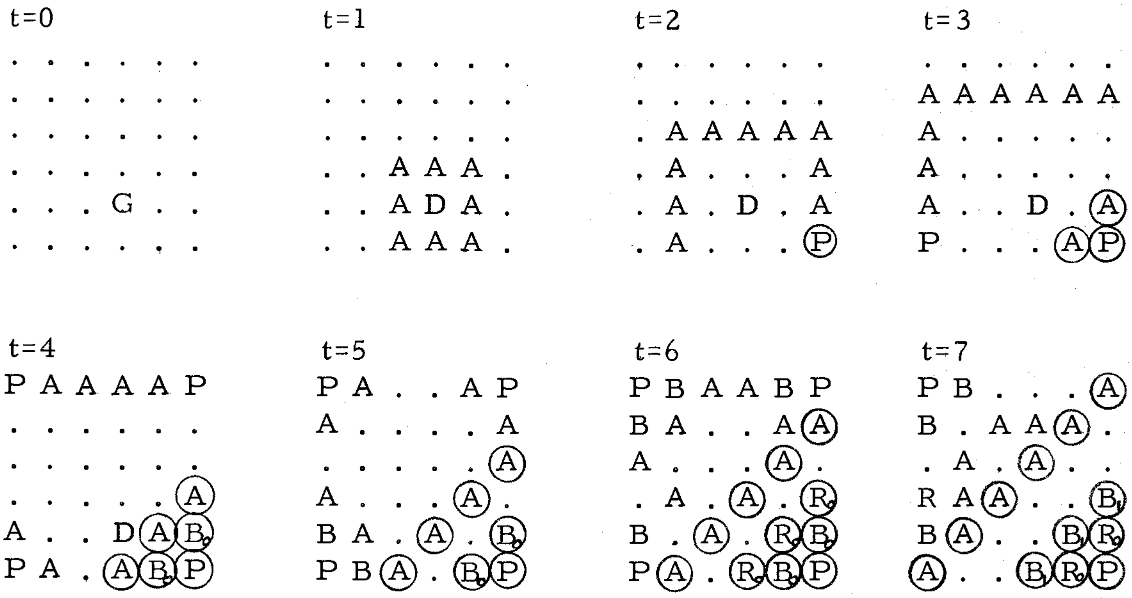
t = 15, all P states

t = 16, all fire (T state)

Figure 12. Square grid solution with corner General

clearly, the eight-neighbor wave propagation and the diagonal propagating solutions are combined. However, the firing time is high. As an example (Figure 13), for a six by six array, it takes 4n-4+k time units, where k is the time for the General's signal to reach the nearest corner; or

$$t=4(6)-4+2 = 22 \text{ time units.}$$



Circled states have priority. From t=8 on, the solution becomes that of diagonal propagation. The other states fade out with time. Firing time is $t=2(2n-1)-2+2 = 22$ time units.

Figure 13. Eight-neighbor and diagonal solutions combined

EFFECT ON FIRING TIME OF THE NUMBER OF COMMUNICABLE MACHINES

The question arises, What is the effect on the firing time of an arbitrary number of soldiers due to a soldiers' number of near neighbors? A near neighbor is any other soldier with which one soldier can communicate. Figure 14 shows, for 25 soldiers, configurations for four, six, eight, and k near neighbors.

For an n by n (n odd) square array with the General at the center of the array, the following firing times are required:

1. With four near neighbors, the firing time is equal to that of a straight line of soldiers of length n,

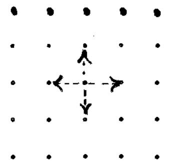
$$t = 2n-2 \text{ time units.}$$

2. In the triangular-connected array shown in Figure 14, where there are six near neighbors, the firing time is equal to that of a straight line of soldiers whose length is that of the larger dimension of the array,

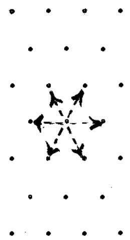
$$t = 2(n-1)-2 \text{ time units.}$$

3. Similarly, when there are eight near neighbors, the firing time of all the soldiers is equal to that of a straight line whose length is that of the number of waves of soldiers from the center to the outside edge,

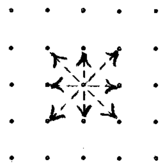
$$t = 2\left(\left|\frac{n}{2}\right|+1\right)-2$$ time units, where $|N|$ is the largest whole number less than N.



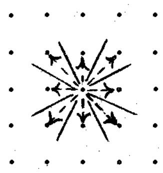
(a) 4-near neighbors



(b) 6-near neighbors



(c) 8-near neighbors



(d) K-near neighbors

Figure 14. Patterns of signal propagation for 25 soldiers

This leads us to the following important theorem:

Theorem: The time of simultaneous firing of a soldier array is inversely proportional to the number of neighbor soldiers a particular soldier can communicate with, or, if k is the number of communicable soldiers,

$$\text{time units} \propto \frac{1}{k} .$$

Proof: The proof is by induction. The theorem is obviously true if k=0, i. e. , no near neighbors. The firing time is thus infinite. Assuming it is true for a finite number k=m, by the induction hypothesis it must be proved true for m+1. It is evident that the more soldiers a particular soldier can communicate with, the faster the signals will spread among the soldiers and the sooner they will all be ready to fire. Thus if m+1 soldiers are near neighbors, the firing time will be proportional to $\frac{1}{m+1}$. This is less time than if there were only m near neighbors with the firing time proportional to $\frac{1}{m}$, and the theorem is proved.

SUMMARY: FINDINGS AND DIRECTIONS OF FURTHER STUDY

The solutions for firing times of specific arrays of one-dimensional finite sequential machines (soldiers) have been included herein. Table 5 lists these. When the General is the center of a square array of soldiers (or an outward "wave" of soldiers from the center), the time required for simultaneous firing is the lowest.

An upper bound for the firing time on the problem with the General at the corner of a square array is the diagonal propagation of signals. The signals follow a diagonal line propagating from the General at one corner to the far soldier at the opposite corner.

From any position the General occupies in an array, some exploration should be made into optimizing the firing time by combining the solutions spoken of earlier: the 4-, 6-, 8-, and k-neighbor solutions and the diagonal signal propagation solution. Other ways that can be found to find the "end" soldiers from any position of the General in an array should be explored and may produce a shorter time for firing.

The eight-neighbor solution itself should be further analyzed. Extra states and procedures needed would have to be worked out to deal with the waves reflected from the edges, and their intersection. The reflected waves come from the General possibly being closer to one edge of the array than another. When the outgoing signal wave from the General reaches an edge, a return "reflected" wave begins. The problem is what to do when these waves meet, and how to deal with the situation wherever it occurs in the array. The solution to this problem could then be extended to non-square sized rectangular arrays and other configurations. If no solution for all cases can be found, a solution for a particular configuration and General location can be found where specifically needed.

There are some other considerations for study in seeking the solution for the case of the arbitrarily-positioned General. There is the situation where a cell can communicate in one way to the four nearest neighbors of the eight-neighbor case described earlier, and in another way to the other four near neighbors. Another avenue of study is to define the "ends" differently, such as the separate array edges instead of the center cell and the array edges together. These are all cases of multi-dimensional arrays of the firing squad problem. All these areas should receive further study.

Table 5. Firing time solutions for specific cases.

Straight line situation or linear extension of the straight line; k=the number of cells from the General to the nearest end	$t=2n-2-k$ time units
n by n array, n odd, General at center (or wave of cells), k=the number of waves to the closer end (center or outside edge) from General or General wave	$t=2(\left\lvert \frac{n}{2} \right\rvert+1)-2-k$ time units
n by n array, General at the corner (or a diagonal row of cells), k=the number of diagonal rows of cells from the General to the nearer end	$t=2(n-1)-2-k$ time units
4-neighbor communication	$t=2n-2$ time units
6-neighbor communication triangular-connected, (n-1) by (n-2)	$t=2(n-1)-2$ time units
8-neighbor communication	$t=2(\left\lvert \frac{n}{2} \right\rvert+1)-2-k$ time units
k-neighbor communication	$t \propto \frac{1}{k}$

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