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Let J denote the set of nonnegative integers, and when $n \in J$ let I_n denote the set of nonnegative integers less than or equal to n. If $A \subseteq J$ let $A(I_n)$ represent the cardinality of the set $\{x | x \in A \cap I_n$ and $x > 0\}$. If $k \geq 2$ and $A_1, A_2, \ldots, A_k$ are subsets of J then $A_1 + A_2 + \ldots + A_k$ or $\sum_{1 \leq i \leq k} A_i$ denotes the set $\{\sum_{1 \leq i \leq k} a_i | a_i \in A_i\}$. Conditions are found that imply the fundamental inequality,

$$\sum_{1 \leq i \leq k} A_i(I_n) \leq (\sum_{1 \leq i \leq k} A_i)(I_n).$$

The inequality $A_1(I_n) + A_2(I_n) \leq (A_1+A_2)(I_n)$ is obtained when A_1, A_2, and A_3 are subsets of J satisfying $A_1(I_n) < 5$, $A_1 + A_2 + A_3 \supseteq I_{n-1}$, and $n \notin A_1 + A_2 + A_3$. This result is used

to prove the fundamental inequality when $k \geq 2$ and $A_1, A_2, \ldots, A_k$ are subsets of J such that at least $k-2$ of the sets have less than five nonzero elements less than n, and where $\sum_{1 \leq i \leq k} A_i \supseteq I_{n-1}$ and $n \notin \sum_{1 \leq i \leq k} A_i$. It is established that a least $k-2$ of the sets $A_1, A_2, \ldots, A_k$ have less than five nonzero elements less than n when the following conditions are satisfied: $\sum_{1 \leq i \leq k} A_i \supseteq I_{n-1}$, $n \notin \sum_{1 \leq i \leq k} A_i$, and either

(i) $n \leq 14$ and $k \geq 3$,

(ii) $n = 15$, $k \geq 4$, and $A_i(I_n) > 0$ for $i = 1, 2, 3, 4$,

or

(iii) $n = 16$, $k \geq 5$, and $A_i(I_n) > 0$ for $i = 1, 2, 3, 4, 5$.

Examples are given to show that the integers 14 and 15 cannot be replaced by larger integers in statements (i) and (ii). The above results are extended to the set of all m-tuples of nonnegative integers. In 1955 Chio-Shih Lin used a different method to obtain the fundamental inequality when $n \leq 14$ and A_1, A_2, and A_3 are subsets of J for which $A_1 + A_2 + A_3 \supseteq I_{n-1}$ and $n \notin A_1 + A_2 + A_3$.

The fundamental inequality is also established under certain other conditions. These results are less substantial.

Four related numerical functions are defined and the evaluation of one of them is given. Results are obtained concerning the values for the other functions on certain subsets of their domains.

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A FUNDAMENTAL INEQUALITY IN ADDITIVE NUMBER THEORY AND SOME RELATED NUMERICAL FUNCTIONS

I. INTRODUCTION

Let N represent the set of natural numbers, and let J represent the set of nonnegative integers. If a and b are integers, $a \leq b$, then $a, \ldots, b$ appearing in the description of a set S of integers denotes the fact that S contains all integers x where $a \leq x \leq b$. For each $n \in N$, let $I_n = \{0, \ldots, n\}$. If A is a set of nonnegative integers, then $A(I_n)$ represents the cardinality of $A \cap \{1, \ldots, n\}$. The sum of k sets of nonnegative integers $A_1, A_2, \ldots, A_k$ is the set $\{\sum_{1 \leq i \leq k} a_i \mid a_i \in A_i\}$, and either $A_1 + A_2 + \ldots + A_k$ or $\sum_{1 \leq i \leq k} A_i$ denotes this sum set.

The following theorem was proved by H. B. Mann [7] in 1942.

<u>Theorem A</u>. If $A, B \subseteq J$, $0 \in A \cap B$, and $(A+B)(I_n) < n$, then

$$\frac{(A+B)(I_n)}{n} \geq \operatorname{glb}\left\{ \frac{A(I_t)+B(I_t)}{t} \,\middle|\, t \notin A+B \text{ and } t = 1, 2, \ldots, n \right\}.$$

A special case of Theorem A occurs when n is the smallest natural number missing in $A + B$. Then

$$\frac{(A+B)(I_n)}{n} \geq \frac{A(I_n)+B(I_n)}{n} .$$

Thus we have

Theorem B. If $A, B \subseteq J$, $A + B \supseteq \{0, \ldots, n-1\}$, and $n \notin A + B$, then

$$A(I_n) + B(I_n) \leq (A+B)(I_n) = n - 1 .$$

Three years after Mann's Theorem A was published, F. J. Dyson [1] published a paper that contains the result stated next.

Theorem C. If $A_i \subseteq J$ for $i = 1, 2, \ldots, k$, $0 \in \bigcap_{i=1}^{k} A_i$, and $(\sum_{1 \leq i \leq k} A_i)(I_n) < n$, then

$$\frac{(\sum_{1 \leq i \leq k} A_i)(I_n)}{n} \geq \mathrm{glb}\{ \sum_{1 \leq i \leq k} \frac{A_i(I_t)}{t} \mid t = 1, 2, \ldots, n\}.$$

In view of Dyson's theorem, it is of interest to inquire about the possibility of extending Theorem A to a sum of three or more sets in the following way.

Statement A. If $k \geq 3$, $A_i \subseteq J$ for $i = 1, 2, \ldots, k$, $0 \in \bigcap_{i=1}^{k} A_i$, and $(\sum_{1 \leq i \leq k} A_i)(I_n) < n$, then

$$\frac{\left(\sum\limits_{1\le i\le k} A_i\right)(I_n)}{n} \ge \operatorname{glb}\left\{ \sum_{1\le i\le k} \frac{A_i(I_t)}{t} \;\middle|\; t \notin \sum_{1\le i\le k} A_i \text{ and } t = 1,2,\ldots,n \right\}.$$

Statement A implies the following extension of Theorem B.

Statement B. If $k \ge 3$, $A_i \subseteq J$ for $i = 1, 2, \ldots, k$, $\sum\limits_{1\le i\le k} A_i \supseteq \{0, \ldots, n-1\}$, and $n \notin \sum\limits_{1\le i\le k} A_i$, then

$$\sum_{1\le i\le k} A_i(I_n) \le \left(\sum_{1\le i\le k} A_i\right)(I_n) = n-1.$$

However, Statement B is not valid, and thus neither is Statement A, for consider $n = 15$, $A_1 = \{0, 1, 8, 10, 12, 14\}$, $A_2 = \{0, 2, 8, 9, 12, 13\}$, $A_3 = \{0, 4, 8, 9, 10, 11\}$, and $A_i = \{0\}$ if $k > 3$ and $3 < i \le k$. Then $\sum\limits_{1\le i\le k} A_i \supseteq \{0, \ldots, 14\}$ and $15 \notin \sum\limits_{1\le i\le k} A_i$, but

$$\sum_{1\le i\le k} A_i(I_{15}) = 15 > 14 = \left(\sum_{1\le i\le k} A_i\right)(I_{15}).$$

Chio-Shih Lin [6], in his doctoral dissertation which was written under the direction of Mann, obtained conditions on three sets of nonnegative integers A, B, and C that, in addition to the hypotheses $A + B + C \supseteq \{0, \ldots, n-1\}$ and $n \notin A + B + C$, imply

$A(I_n) + B(I_n) + C(I_n) \leq (A+B+C)(I_n)$. These results of Lin are presented in Theorems D and E. Lin uses Theorem D to prove Theorem E.

<u>Theorem D</u>. If $A, B, C \subseteq J$, $A + B + C \supseteq \{0, \ldots, n-1\}$, $n \notin A + B + C$, and $(\{x \mid x \in I_n \text{ and } x \notin A + B\})(I_n) \leq 5$, then

$$A(I_n) + B(I_n) + C(I_n) \leq (A+B+C)(I_n) = n-1.$$

<u>Theorem E</u>. If $A, B, C \subseteq J$, $A + B + C \supseteq \{0, \ldots, n-1\}$, $n \notin A + B + C$, and $n < 15$, then

$$A(I_n) + B(I_n) + C(I_n) \leq (A+B+C)(I_n) = n-1.$$

The next three theorems give limitations on the extension of Theorems D and E. Lin proves these three theorems by exhibiting for each integer $n \geq 15$ a construction of three sets A, B, and C. The three sets A_1, A_2, and A_3 that we used to provide a counterexample to Statements A and B are the sets determined by Lin's construction when $n = 15$.

<u>Theorem F</u>. For each integer $n \geq 15$ there exist sets of non-negative integers A, B, and C for which $A+B+C \supseteq \{0, \ldots, n-1\}$, $n \notin A + B + C$, and $A(I_n) + B(I_n) + C(I_n) > (A+B+C)(I_n)$.

<u>Theorem G</u>. If $t > 0$ is given then a positive integer n and

sets of nonnegative integers A, B, and C can be found satisfying $A + B + C \supseteq \{0, \ldots, n-1\}$, $n \notin A + B + C$, and $A(I_n) + B(I_n) + C(I_n) \geq (A+B+C)(I_n) + t$.

Theorem H. For each integer $r > 5$ a positive integer n and sets of nonnegative integers A, B, and C exist satisfying $A + B + C \supseteq \{0, \ldots, n-1\}$, $n \notin A + B + C$,

$$A(I_n) + B(I_n) + C(I_n) > (A+B+C)(I_n) ,$$

and

$$(\{x \mid x \in I_n \text{ and } x \notin A + B\})(I_n) = r.$$

Let $J^1 = J$ and for $m \in N$, let J^m be the set of all m-tuples having nonnegative integer coordinates.

The main purpose of this dissertation is to obtain a theorem for $k \geq 3$ sets in J^m which for $k = 3$ and $m = 1$ is Lin's Theorem E. This is done in Chapter IV. The theorem is of particular interest for $k = 3$ and $k = 4$ because of examples which are given which show that the theorem is best possible in the same sense that Theorem F shows that Lin's Theorem E is best possible.

In Chapter II we define a set transformation in J^m which was used by Lin when $m = 1$, and we give those properties of the transformation that we use later in Chapters III and IV.

In Chapter III we study conditions on two or more sets in J^m

which imply that the sum of the number of nonzero elements in the given sets is not greater than the number of nonzero elements in the sum set. The proof of the extension of Theorem E in Chapter IV depends on one of these results. We also obtain an extension of Lin's Theorem D in this chapter.

Four related numerical functions are defined in Chapter V. We evaluate one of these functions completely. Results which we have obtained earlier, as well as known results, are used to evaluate the other functions on certain subsets of their domains and to determine bounds for them elsewhere.

An extension of Theorem F to k sets, $k \geq 4$, in J^1 is given in Chapter V. This result is due to Allen Freedman, but it is not in print. Also in Chapter V, we show that an extension of Theorem G to $k \geq 4$ sets in J^1 can be obtained directly from a theorem due to P. Erdös and P. Scherk. No interesting extensions of Theorem H are apparent to the author.

Two key theorems in extending Theorem E to $k > 3$ sets in J^1 are Theorems 3.1 and 4.3. Moreover, Theorem 3.1 is basic in obtaining the extension in J^m, $m \geq 1$.

Throughout this thesis when the elements of a set are listed they will be distinct.

II. THE INVERSION TRANSFORMATION

A transformation on certain subsets of the set J^m of all m-tuples of nonnegative integers is introduced in this chapter. This transformation, called an inversion, was introduced for J^1 by A. Khintchine [5] in a paper published in 1932. We have frequent occasion in Chapters III and IV to use properties of the inversion transformation for J^m that were found and used for J^1 by Lin in his thesis. These properties are given in Theorems 2.9 and 2.10. Theorem 2.11 is not needed later, but it is of interest in itself in the theory of the inversion transformation.

We begin with some definitions.

Definition 2.1. Let $J^1 = J$ denote the set of nonnegative integers. Let $J^m = \{(x_1, x_2, \ldots, x_m) \mid x_i \in J \text{ and } i = 1, 2, \ldots, m\}$ for m a positive integer. The point $x = (x_1, x_2, \ldots, x_m)$ in J^m with $x_i = 0$ for $i = 1, 2, \ldots, m$ is denoted by 0.

Definition 2.2. If $x_i = (x_{1i}, x_{2i}, \ldots, x_{mi})$ is in J^m for $i = 1, 2, \ldots, k$, then $\sum_{1 \leq i \leq k} x_i$ or $x_1 + x_2 + \ldots + x_k$ represents the point $(\sum_{1 \leq i \leq k} x_{1i}, \sum_{1 \leq i \leq k} x_{2i}, \ldots, \sum_{1 \leq i \leq k} x_{mi})$ in J^m.

Definition 2.3. If $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ are points in J^m, then $x \leq y$ if an only if

$x_i \leq y_i$ for $i = 1, 2, \ldots, m$. The notation $x < y$ means $x \leq y$ and $x \neq y$. If $x \leq y$ then $y - x = (y_1 - x_1, y_2 - x_2, \ldots, y_m - x_m)$.

<u>Definition 2.4.</u> If $S, T \subseteq J^m$ then $S \setminus T = \{x | x \in S \text{ and } x \notin T\}$.

<u>Definition 2.5.</u> For each $z \in J^m$ such that $z > 0$, $I_z = \{x | x \in J^m \text{ and } 0 \leq x \leq z\}$ and $I'_z = I_z \setminus \{z\}$.

<u>Definition 2.6.</u> Let $S, T \subseteq J^m$. If the cardinality of T is finite then $S(T)$ denotes the cardinality of the set $(S \cap T) \setminus \{0\}$.

Let $A_1, A_2, \ldots, A_k$ be subsets of J^m. The sum set, $\sum_{1 \leq i \leq k} A_i$, is usually defined to be the set $\{\sum_{1 \leq i \leq k} a_i | a_i \in A_i\}$. Note that when $z \in J^m$ then the set

(2.1) $$\{\sum_{1 \leq i \leq k} a_i | a_i \in A_i\} \cap I_z$$

and the set

(2.2) $$\{\sum_{1 \leq i \leq k} a_i | a_i \in A_i \cap I_z \text{ and } \sum_{1 \leq i \leq k} a_i \leq z\}$$

are equal. In view of the equality of the sets in statements (2.1) and (2.2) and in order to simplify the proofs of our results in Chapters III and IV, once the element $z \in J^m$ has been specified we choose to restrict all sets considered, except J^m itself, to be subsets of I_z and to define set addition in the following way:

<u>Definition 2.7.</u> Let $z \in J^m$ be specified. If $A_1, A_2, \ldots, A_k$ are subsets of I_z, then $A_1 + A_2 + \ldots + A_k$ or $\sum_{1 \le i \le k} A_i$ denotes the set $\{\sum_{1 \le i \le k} a_i \mid a_i \in A_i$ and $\sum_{1 \le i \le k} a_i \le z\}$.

We remark that $0 \in A_i$ for $i = 1, 2, \ldots, k$ whenever $0 \in \sum_{1 \le i \le k} A_i$. Also, if one of the summands is the empty set then the sum is empty. With A, B, and C subsets of I_z, it can readily be verified that $A + B = B + A$ and $A + (B+C) = (A+B) + C$.

We now define the inversion transformation.

<u>Definition 2.8.</u> Let $z \in J^m$ be specified. If S is a subset of I_z, the set

$$S^{\sim} = \{z - x \mid x \in I_z \setminus S\}$$

is called the <u>inversion of</u> S.

We proceed to list in Theorems 2.9 and 2.10 those properties of the inversion transformation that are used in succeeding chapters. These results are in Lin's thesis for the set J^1. We establish their validity for the set J^m where $m \ge 1$ by using the same arguments given by Lin for $m = 1$.

Theorem 2.9. Let $z \in J^m$, $z > 0$, be specified. For any sets $S, T \subseteq I_z$ the following properties hold:

(a) $(S^\sim)^\sim = S$

(b) $T \subseteq S^\sim$ if and only if $z \notin S + T$.

(c) If $S + T = I'_z$ then $T \subseteq S^\sim$ and $S + S^\sim = I'_z$.

(d) $S^\sim(I'_z) = (I_z \setminus S)(I'_z)$.

(e) If $0 \in S$ and $z \notin S$, then $S^\sim(I_z) + 1 = (I_z \setminus S)(I_z)$.

(f) If $0 \in S$ and $z \notin S$, then $S(I_z) + S^\sim(I_z) = I'_z(I_z)$.

Proof.

(a) Now $s \in S$ if and only if $z - s \in I_z \setminus S^\sim$, which is turn is equivalent to $s = z - (z-s) \in (S^\sim)^\sim$.

(b) First assume $T \subseteq S^\sim$. Then $S + T \subseteq S + S^\sim$. If $z \in S + S^\sim$, it follows that $z - x \in S^\sim$ for some $x \in S$. However, $z - x \in S^\sim$ implies $x \notin S$, a contradiction, and so we conclude $z \notin S + S^\sim$. Therefore, $z \notin S + T$.

Next assume $z \notin S + T$. Let $t \in T$. Then $z - t \notin S$, for otherwise $z = (z-t) + t \in S + T$. Hence, $t = z - (z-t) \in S^\sim$.

(c) If $S + T = I'_z$ then $z \notin S + T$, and it follows from (b) that $T \subseteq S^\sim$. Thus, $I'_z \subseteq S + T \subseteq S + S^\sim \subseteq I_z$. Since $z \notin S + S^\sim$ by (b), then $S + S^\sim = I'_z$.

(d) We have $x \in (S^\sim \cap I'_z) \setminus \{0\}$ if and only if $0 < x < z$ and

$x \in S^{\sim}$, which are equivalent to $0 < z - x < z$ and $z - x \in I_z \setminus S$. These last conditions are equivalent to $z - x \in ((I_z \setminus S) \cap I_z') \setminus \{0\}$. Therefore, $S^{\sim}(I_z') = (I_z \setminus S)(I_z')$.

(e) Since $0 \in S$ then $z \notin S^{\sim}$, and so $S^{\sim}(I_z) = S^{\sim}(I_z')$. Since $z \notin S$ then $z \in I_z \setminus S$, and so $(I_z \setminus S)(I_z) = 1 + (I_z \setminus S)(I_z')$. An application of (d) gives the desired inequality.

(f) Applying (e), we have $S^{\sim}(I_z) = (I_z \setminus S)(I_z) - 1$. Furthermore, $S(I_z) + (I_z \setminus S)(I_z) = (I_z)(I_z)$. Therefore,

$$\begin{aligned} S(I_z) + S^{\sim}(I_z) &= S(I_z) + (I_z \setminus S)(I_z) - 1 \\ &= I_z(I_z) - 1 = I_z'(I_z). \end{aligned}$$

<u>Theorem 2.10.</u> Let $z \in J^m$, $z > 0$, be specified and let $S \subseteq I_z$. A necessary and sufficient condition for S to satisfy $S + S^{\sim} = I_z'$ is that, for every x such that $0 < x \leq z$, we have $S + \{0, x\} \neq S$.

Proof. Assume $S + S^{\sim} = I_z'$ and $0 < x \leq z$. Then $z - x \in I_z'$ since $0 \leq z - x < z$. Therefore, $z = (z-x) + x \in I_z' + \{0, x\}$, and so $I_z = I_z' + \{0, x\}$. However, if $S + \{0, x\} = S$ then

$$I_z = I_z' + \{0, x\} = S^{\sim} + S + \{0, x\} = S^{\sim} + S = I_z',$$

a contradiction. Thus, $S + \{0, x\} \neq S$.

Conversely, assume $S + \{0, x\} \neq S$ for each x, $0 < x \leq z$. Let $y \in I'_z$. Then $0 < z - y \leq z$ and $S + \{0, z-y\} \neq S$. Hence, there exists an element $s \in S$ such that $s + (z-y) \leq z$ and $s + (z-y) \notin S$. It follows that $z - (s+(z-y)) \in S^{\sim}$ and $y \in S + S^{\sim}$ since $y = s + (z-(s+(z-y)))$. Therefore, $I'_z \subseteq S + S^{\sim}$. Since $z \notin S + S^{\sim}$ by Theorem 2.9(b), then $I'_z = S + S^{\sim}$.

This completes the proof.

In the next theorem we show that whenever $A_1, A_2, \ldots, A_k$ are subsets of I_z whose sum is I'_z then there exist k "maximal" sets whose sum is I'_z.

Theorem 2.11. Let $z \in J^m$, $z > 0$, be specified. Let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$ where $k \geq 2$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ then there exist sets A_i^*, $i = 1, 2, \ldots, k$, for which

(i) $A_i \subseteq A_i^* \subseteq I_z$,

(ii) $\sum_{1 \leq i \leq k} A_i^* = I'_z$,

(iii) $A_i^* = (\sum_{\substack{1 \leq j \leq k \\ j \neq i}} A_j^*)^{\sim}$.

Proof. Let $A_1^* = (\sum_{2 \leq j \leq k} A_j)^{\sim}$. Since $A_1 + \sum_{2 \leq j \leq k} A_j = I'_z$, then

$A_1 \subseteq A_1^* \subseteq I'_z$ and $A_1^* + (\sum_{2 \leq j \leq k} A_j) = I'_z$ follow from Theorem 2.9(c).

We define A_i^* recursively, $1 < i \leq k$, by

$$A_i^* = (\sum_{1 \leq j \leq i-1} A_j^* + \sum_{i+1 \leq j \leq k} A_j)^{\sim}.$$

Next we show $A_i \subseteq A_i^* \subseteq I_z$ and

$$\sum_{1 \leq j \leq i} A_i^* + \sum_{i+1 \leq j \leq k} A_j = I'_z,$$

$1 \leq i \leq k$, by induction. Suppose $1 < t \leq k$ and the two results are valid if $1 \leq i < t$. Since

$$\sum_{1 \leq j \leq t-1} A_j^* + \sum_{t \leq j \leq k} A_j = I'_z,$$

then from Theorem 2.9(c) it follows that $A_t \subseteq A_t^* \subseteq I_z$ and

$$\sum_{1 \leq j \leq t} A_j^* + \sum_{t+1 \leq j \leq k} A_j = I'_z.$$

In particular $\sum_{1 \leq j \leq k} A_j^* = I'_z$. Applying Theorem 2.9(c) to this last equality, we obtain

$$A_i^* \subseteq \Big(\sum_{\substack{1 \le j \le k \\ j \ne i}} A_j^* \Big)^{\sim}$$

for $1 \le i \le k$. Since $S^{\sim} \supseteq T^{\sim}$ if $S \subseteq T \subseteq I_z$, then

$$A_i^* = \Big(\sum_{1 \le j \le i-1} A_j^* + \sum_{i+1 \le j \le k} A_j \Big)^{\sim} \supseteq \Big(\sum_{\substack{1 \le j \le k \\ j \ne i}} A_j^* \Big)^{\sim} ,$$

and so

$$A_i^* = \Big(\sum_{\substack{1 \le j \le k \\ j \ne i}} A_j^* \Big)^{\sim} .$$

The proof is now complete.

We remark that when $A_1, A_2, \ldots, A_k$, $k \ge 4$, are subsets of I_z for which $\sum_{1 \le i \le k} A_i = I_z'$ and $A_i = \Big(\sum_{\substack{1 \le j \le k \\ j \ne i}} A_j \Big)^{\sim}$ for $i = 1, 2, \ldots, k$, it is not always the case that

$$\sum_{i \in I} A_i = \Big(\sum_{j \in J} A_j \Big)^{\sim}$$

where I, J is a partition of $\{1, 2, \ldots, k\}$. For instance, let $k = 4$, $z = 9$, $A_1 = \{0, 1, 3, 8\}$, $A_2 = \{0, 3\}$, and $A_3 = A_4 = \{0, 2\}$. Then $A_1 + A_2 + A_3 + A_4 = I_9'$ and $A_i = \Big(\sum_{\substack{1 \le j \le 4 \\ j \ne i}} A_j \Big)^{\sim}$ for $i = 1, 2, 3, 4$. However $(A_1 + A_2)^{\sim} = \{0, 2, 4, 7\}$, and so $(A_1 + A_2)^{\sim} \ne A_3 + A_4$.

III. A FUNDAMENTAL INEQUALITY

Let $z \in J^m$ and $z > 0$. We obtain new conditions on k sets $A_1, A_2, \ldots, A_k$ in J^m that imply the inequality

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_z).$$

In Theorem 3.1 the above inequality is obtained for two sets. This is an important result since it is used to prove Theorem 3.2, and in turn Theorem 3.2 is used later in proving Theorems 4.12 and 5.15. Moreover, Theorem 3.2 is the only result of this chapter that is used in the succeeding chapters.

An extension of Lin's Theorem D to $k \geq 3$ sets in J^m is obtained in Theorem 3.4.

In the proof of Theorem 3.1 we construct a set which we denote as B_1. The first step in the construction of a sequence of sets that was defined by Mann to prove Theorem A, and later used by Lin to prove Theorem D, provided the motivation for our definition of set B_1.

<u>Theorem 3.1.</u> Let $z \in J^m$, $z > 0$, be specified and let $A, B, C \subseteq I_z$. If $A + B + C = I_z'$ and $A(I_z) \leq 4$ then

$$A(I_z) + B(I_z) \leq (A+B)(I_z).$$

Proof. If $A(I_z) = 0$ then $A = \{0\}$, $A + B = B$, and the theorem follows immediately. Consequently, we restrict our consideration of $A(I_z)$ to the values 1, 2, 3 and 4.

First we show that $I'_z \setminus (B+C)$ is not the empty set. Let $a \in A$, $a > 0$. Then $z - a \in I'_z$, and $z - a \notin B + C$ for otherwise $z \in A + B + C$. Thus, $z - a \in I'_z \setminus (B+C)$.

Let $E = \{b \mid a + b + c = x,\ a \in A,\ b \in B,\ c \in C,\ x \in I'_z \setminus (B+C)\}$. The set E is not empty since $I'_z \setminus (B+C)$ is not empty and $A + B + C = I'_z$. For each $b \in E$ define

$$E_b = \{a + b \mid a + b + c = x,\ a \in A,\ c \in C,\ x \in I'_z \setminus (B+C)\}.$$

From the definition of E_b it follows that the set E_b is not empty, $0 \notin E_b$, $E_b \subseteq A + B$, and $A(I_z) \geq E_b(I_z)$. Next choose $e \in E$ so that

$$E_e(I_z) = \max\{E_b(I_z) \mid b \in E\}.$$

Let $B_1 = E_e$. From the remarks following the definition of E_b we have $0 \notin B_1$,

$$(3.1) \qquad 0 < B_1(I_z) \leq A(I_z),$$

and

$$(3.2) \qquad B \cup B_1 \subseteq A + B.$$

If $y \in B_1$ then there exist elements $a \in A$, $c \in C$, and

$x \in I'_z \setminus (B+C)$ such that $y = a + e$ and $x = a + e + c = y + c$. Now $y \notin B$, for otherwise $x \in B + C$; thus,

$$B \cap B_1 = \phi. \tag{3.3}$$

If $B_1(I_z) = A(I_z)$ we use statements (3.2) and (3.3) to obtain

$$\begin{aligned} A(I_z) + B(I_z) &= B_1(I_z) + B(I_z) \\ &= (B_1 \cup B)(I_z) \\ &\leq (A+B)(I_z), \end{aligned}$$

and the theorem is established.

If $B_1(I_z) < A(I_z)$, then to prove the theorem it suffices to show the existence of a set B_2 having the following properties:

(i) $B_2 \subseteq A + B$,

(ii) $B \cap B_2 = \phi$,

(iii) $B_1 \cap B_2 = \phi$,

(iv) $B_1(I_z) + B_2(I_z) = A(I_z)$.

In order to see that the theorem follows when a set B_2 exists satisfying properties (i), (ii), (iii), and (iv), we note that (i) and (3.2) imply $B \cup B_1 \cup B_2 \subseteq A + B$ and (ii), (iii), and (3.3) imply

$$(B \cup B_1 \cup B_2)(I_z) = B(I_z) + B_1(I_z) + B_2(I_z).$$

Therefore, using property (iv), we have

$$A(I_z) + B(I_z) = B_1(I_z) + B_2(I_z) + B(I_z)$$
$$= (B_1 \cup B_2 \cup B)(I_z)$$
$$\leq (A+B)(I_z).$$

Henceforth, assume $B_1(I_z) < A(I_z)$. Let $s = A(I_z) - B_1(I_z)$. Since $B_1(I_z) > 0$ then

$$1 \leq s = A(I_z) - B_1(I_z) \leq A(I_z) - 1 \leq 3.$$

With $A(I_z) = t$ let $a_1, a_2, \ldots, a_t$ denote the t nonzero elements of A, and let $z_j = z - a_j$ for $j = 1, 2, \ldots, t$. Now $z_j \notin B + C$, for otherwise $z = a_j + z_j \in A + B + C$; also, $a_j > 0$ implies $z_j < z$. Therefore,

$$\{z_j \mid j = 1, 2, \ldots, t\} \subseteq I_z' \setminus (B+C).$$

Next we verify the following statement:

(3.4) If $z_i = a_j + x$ where $i, j \in \{1, \ldots, t\}$, then $z_j = a_i + x$.

To see that statement (3.4) is valid let $z_i = (z_{1i}, z_{2i}, \ldots, z_{mi})$, $x = (x_1, x_2, \ldots, x_m)$, $a_j = (a_{1j}, a_{2j}, \ldots, a_{mj})$, and

$z = (w_1, w_2, \ldots, w_m)$. Since $z_i = a_j + x$ then $z_{ki} = a_{kj} + x_k$ for $k = 1, 2, \ldots, m$. It follows that $w_k + z_{ki} = w_k + a_{kj} + x_k$ and $w_k - a_{kj} = w_k - z_{ki} + x_k$ for $k = 1, 2, \ldots, m$. Thus $z_j = z - a_j = (z-z_i) + x = a_i + x$.

By relabeling the nonzero elements of A if it is necessary, we may assume that

$$B_1 = \{a_j + e \mid j = 1, 2, \ldots, t-s\}.$$

Since $a_i + e \notin B_1$ for $i = t-s+1, \ldots, t$, then there do not exist elements $c \in C$ and $x \in I'_z \setminus (B+C)$ for which $a_i + e + c = x$; that is,

$$(3.5) \qquad (\{a_i + e \mid i = t-s+1, \ldots, t\} + C) \cap (I'_z \setminus (B+C)) = \emptyset .$$

Recall that $\{z_i \mid i = 1, 2, \ldots, t\} \subseteq I'_z \setminus (B+C)$. Since $I'_z = A + B + C$ then there exist functions α, β, and γ from $\{1, \ldots, t\}$ to sets A, B, and C, respectively, such that

$$(3.6) \qquad z_i = \alpha(i) + \beta(i) + \gamma(i), \quad i = 1, 2, \ldots, t.$$

Since $z_i \notin B + C$ then $\alpha(i) > 0$ for $i = 1, 2, \ldots, t$.

Define $B_2^* = \{\alpha(i) + \beta(i) \mid i = t-s+1, \ldots, t\}$. Clearly $B_2^*(I_z) \leq s$. For each $i \in \{t-s+1, \ldots, t\}$, we note that $\alpha(i) + \beta(i) > 0$ since $\alpha(i) > 0$. Also, $\alpha(i) + \beta(i) \in A + B$ since $\alpha(i) + \beta(i) \leq z_i < z$, $\alpha(i) \in A$, and $\beta(i) \in B$. Moreover, $\alpha(i) + \beta(i) \notin B$ for otherwise

$z_i \in B + C$. It follows that

(3.7) $$0 \notin B_2^*,$$

(3.8) $$B_2^* \subseteq A + B,$$

(3.9) $$B \cap B_2^* = \phi.$$

We now verify that

(3.10) $$B_1 \cap B_2^* = \phi.$$

Suppose $y \in B_1 \cap B_2^*$. Since $y \in B_1$ then $y = a_j + e$ for some integer j, $1 \leq j \leq t-s$, and since $y \in B_2^*$ then $y = \alpha(i) + \beta(i)$ for some integer i, $t-s < i \leq t$. Therefore, $\alpha(i) + \beta(i) = a_j + e$, and making this substitution in $z_i = \alpha(i) + \beta(i) + \gamma(i)$ we obtain

$$z_i = a_j + e + \gamma(i).$$

An application of statement (3.4) gives

$$z_j = a_i + e + \gamma(i),$$

but this is contrary to statement (3.5) since $z_j \in I_z' \setminus (B+C)$, $\gamma(i) \in C$, $a_i \in A$, and $t-s < i \leq t$. We conclude that the set $B_1 \cap B_2^*$ is empty.

We proceed to consider each of the possible values that the integer s may assume.

Case 1. $s = 1$.

When $s = 1$ then $B_2^* = \{\alpha(t) + \beta(t)\}$. Since $0 \notin B_2^*$ by statement (3.7) then $B_2^*(I_z) = s$, and so $B_1(I_z) + B_2^*(I_z) = A(I_z)$. In view of this last inequality and statements (3.8), (3.9), and (3.10), the theorem follows by setting $B_2 = B_2^*$.

Case 2. $s = 2$.

If $\alpha(t-1) + \beta(t-1) \neq \alpha(t) + \beta(t)$, then $B_2^*(I_z) = s$ since $B_2^* = \{\alpha(t-1) + \beta(t-1), \alpha(t) + \beta(t)\}$ and $0 \notin B_2^*$. Therefore, $B_1(I_z) + B_2^*(I_z) = A(I_z)$, and the theorem follows with $B_2 = B_2^*$.

Assume $\alpha(t-1) + \beta(t-1) = \alpha(t) + \beta(t)$. Since $\alpha(t) \in A \setminus \{0\}$, then $\alpha(t) = a_j$ for some integer j where $1 \leq j \leq t$. From statement (3.6) we have $z_{t-1} = \alpha(t-1) + \beta(t-1) + \gamma(t-1)$ and $z_t = \alpha(t) + \beta(t) + \gamma(t)$. Therefore,

$$(3.11) \qquad z_{t-1} = \alpha(t) + \beta(t) + \gamma(t-1)$$
$$= a_j + \beta(t) + \gamma(t-1)$$

and

$$(3.12) \qquad z_t = a_j + \beta(t) + \gamma(t).$$

Applying statement (3.4) to statements (3.11) and (3.12) respectively, we obtain

$$(3.13) \qquad z_j = a_{t-1} + \beta(t) + \gamma(t-1)$$

and

$$z_j = a_t + \beta(t) + \gamma(t). \tag{3.14}$$

From statements (3.12), (3.13), and (3.14) we have

$$E_{\beta(t)} \supseteq \{a_j + \beta(t), a_{t-1} + \beta(t), a_t + \beta(t)\}.$$

Since $E_e(I_z) = B_1(I_z) = t-s \leq 2$, then from the definition of E_e it follows that $E_{\beta(t)}(I_z) \leq 2$. Therefore, $j \in \{t-1, t\}$ and

$$E_{\beta(t)} = \{a_{t-1} + \beta(t), a_t + \beta(t)\}.$$

Now $E_{\beta(t)} \subsetneq A + B$. Furthermore, $E_{\beta(t)} \cap B = \phi$, for otherwise statement (3.13) or (3.14) yields $z_j \in B + C$. Since $E_{\beta(t)}(I_z) = 2 = s$, then $B_1(I_z) + E_{\beta(t)}(I_z) = A(I_z)$. We next show that $B_1 \cap E_{\beta(t)} = \phi$. Let $i \in \{t-1, t\}$, and suppose $a_i + \beta(t) \in B_1$. Then $a_i + \beta(t) = a_k + e$ for some integer k, $1 \leq k \leq t-2$, and making this substitution in statement (3.13) if $i = t-1$ and in statement (3.14) if $i = t$, we have

$$z_j = a_k + e + \gamma(i).$$

This in turn yields

$$z_k = a_j + e + \gamma(i)$$

by an application of statement (3.4). However, $z_k \in I'_z \setminus (B+C)$,

$\gamma(i) \in C$, $j \in \{t-1, t\}$, $a_j \in A$, and $z_k = a_j + e + \gamma(i)$ are contrary to statement (3.5). Thus, it must be that $B_1 \cap E_{\beta(t)} = \phi$. Let $B_2 = E_{\beta(t)}$. We have established that B_2 satisfies properties (i), (ii), (iii), and (iv); hence, the theorem follows.

Case 3. $s = 3$.

Since $B_1(I_z) \geq 1$, $t = A(I_z) \leq 4$, and $A(I_z) = B_1(I_z) + s$, then $B_1(I_z) = 1$ and $A(I_z) = 4$. Thus

$$B_2^* = \{\alpha(2) + \beta(2), \alpha(3) + \beta(3), \alpha(4) + \beta(4)\}.$$

We claim $B_2^*(I_z) = 3$. Suppose $\alpha(i) + \beta(i) = \alpha(j) + \beta(j)$ where $2 \leq i < j \leq 4$. Since $\alpha(j) \in A \setminus \{0\}$, then $\alpha(j) = a_k$ for some integer k where $1 \leq k \leq 4$. Thus, $\alpha(i) + \beta(i) = a_k + \beta(j)$, and making this substitution in $z_i = \alpha(i) + \beta(i) + \gamma(i)$ we have

(3.15) $$z_i = a_k + \beta(j) + \gamma(i).$$

Also,

(3.16) $$z_j = a_k + \beta(j) + \gamma(j).$$

Next, applying statement (3.4) to statements (3.15) and (3.16), we have

$$z_k = a_i + \beta(j) + \gamma(i) = a_j + \beta(j) + \gamma(j).$$

Therefore, $E_{\beta(j)} \supseteq \{a_i + \beta(j), a_j + \beta(j)\}$, and so

$$E_{\beta(j)}(I_z) > 1 = B_1(I_z) = E_e(I_z)$$

This last inequality, however, is contrary to the choice of e. Thus, $\alpha(i) + \beta(i) \neq \alpha(j) + \beta(j)$ for $2 \leq i < j \leq 4$. Moreover, $0 \notin B_2^*$, and so $B_2^*(I_z) = 3$.

In view of the equality $B_1(I_z) + B_2^*(I_z) = A(I_z)$ and statements (3.8), (3.9), and (3.10), the theorem follows with $B_2 = B_2^*$.

The proof of the theorem is complete.

Theorem 3.1 is the best possible result obtainable in the sense that the integer 4 cannot be replaced by a larger integer. To see this consider the example due to Lin which is given in the Introduction; namely, $m = 1$, $z = 15$, $A = \{0, 1, 8, 10, 12, 14\}$, $B = \{0, 2, 8, 9, 12, 13\}$, and $C = \{0, 4, 8, 9, 10, 11\}$. Then $A + B + C = I'_{15}$ however,

$$A(I_{15}) + B(I_{15}) = 10 > 9 = (A+B)(I_{15}).$$

We now give an example to illustrate how the sets B_1 and B_2 are found in the proof of Theorem 3.1. Let $m = 2$, $z = (3, 3)$, $A = \{(0, 0), (0, 1), (1, 2), (2, 2)\}$, $B = \{(0, 0), (0, 2), (0, 3), (1, 3), (2, 0), (3, 1)\}$, and $C = \{(0, 0), (1, 0), (0, 3), (2, 2), (2, 3)\}$. Then $A + B + C = I'_z$ and $A(I_z) = 3$. Now $I'_z \setminus (B+C) = \{(0, 1), (1, 1), (2, 1), (3, 2)\}$, and the only representations of the elements of $I'_z \setminus (B+C)$ in the form $a + b + c$ with $a \in A$, $b \in B$, and $c \in C$ are those listed below:

$$\begin{aligned}(3, 2) &= (0, 1) + (3, 1) + (0, 0)\\ &= (1, 2) + (2, 0) + (0, 0)\\ &= (2, 2) + (0, 0) + (1, 0),\\ (2, 1) &= (0, 1) + (2, 0) + (0, 0),\\ (1, 1) &= (0, 1) + (0, 0) + (1, 0),\end{aligned}$$

and

$$(0, 1) = (0, 1) + (0, 0) + (0, 0).$$

Therefore $E = \{(0, 0), (2, 0), (3, 1)\}$, $E_{(0, 0)} = \{(0, 1), (2, 2)\}$, $E_{(2, 0)} = \{(3, 2), (2, 1)\}$, and $E_{(3, 1)} = \{(3, 2)\}$. Since

$$\begin{aligned}2 &= \max\{E_{(0, 0)}(I_z), E_{(2, 0)}(I_z), E_{(3, 1)}(I_z)\}\\ &= E_{(0, 0)}(I_z) = E_{(2, 0)}(I_z),\end{aligned}$$

then we may choose $e = (0, 0)$ and $B_1 = E_{(0, 0)}$ or $e = (2, 0)$ and $B_1 = E_{(2, 0)}$. We set $e = (0, 0)$ and $B_1 = E_{(0, 0)}$. Note that $s = A(I_z) - B_1(I_z) = 3 - 2 = 1$ and $t - s = 2$. Proceeding, we label the nonzero elements of A in such a way that $B_1 = \{e + a_1, e + a_2)$; say, $a_1 = (0, 1)$, $a_2 = (2, 2)$, and $a_3 = (1, 2)$. Next, we define $z_1 = z - a_1 = (3, 2)$, $z_2 = z - a_2 = (1, 1)$, and $z_3 = z - a_3 = (2, 1)$. Since there is only one way in which z_3 can be expressed as a sum of the form $a + b + c$ with $a \in A$, $b \in B$, and $c \in C$, namely,

$$z_3 = (0, 1) + (2, 0) + (0, 0),$$

then it must be that $\alpha(3) = (0, 1)$, $\beta(3) = (2, 0)$, and $\gamma(3) = (0, 0)$.

Thus, $B_2^* = \{\alpha(3) + \beta(3)\} = \{(2, 1)\}$. Since $s = 1$ then Case 1 applies, and we let $B_2 = B_2^*$.

We now compare the set B_1 which we construct to prove Theorem 3.1 and the first set in the sequence of sets constructed by Mann to prove Theorem A. Let B and C denote sets of non-negative integers such that $0 \in B \cap C$ and $n \notin B + C$. Then in proving the inequality

$$\frac{(B+C)(I_n)}{n} \geq \operatorname{glb}\left\{ \frac{B(I_t)+C(I_t)}{t} \,\middle|\, t \notin B + C \text{ and } t = 1, 2, \ldots, n \right\}$$

Mann considers the set

$$S_1 = \{b \,|\, a + b + c = x,\ a \in (B+C)^{\sim}, b \in B, c \in C, x \in I_n' \setminus (B+C)\}.$$

If this set is not empty he defines the first set in a sequence of sets to be

$$B_1' = \{a + e_1 \,|\, a + e_1 + c = x, a \in (B+C)^{\sim}, c \in C, x \in I_n' \setminus (B+C)\}$$

where $e_1 = \min\{b \,|\, b \in S_1\}$. The set E in J^m which we define in the proof of Theorem 3.1 is similar to Mann's set S_1; namely,

$$E = \{b \,|\, a + b + c = x,\ a \in A,\ b \in B,\ c \in C,\ x \in I_z' \setminus (B+C)\}.$$

Note that the hypothesis of Theorem 3.1 implies $A \subseteq (B+C)^{\sim}$, and so for $m = 1$ we have $E \subseteq S_1$. Since the ordering on J^m is not a linear ordering when $m > 1$, it may not be possible to compare each pair of elements from E. Our construction differs from Mann's construction at this stage, even for $m = 1$, in that we consider the set E_b for each $b \in B$ and from among these sets we select one having a maximum number of elements. This becomes the set B_1 in our construction.

The second set in Mann's construction is determined by a procedure similar to the one which defines set B_1', the difference being that in defining set B_2' the set B is replaced by $B \cup B_1'$ whenever it occurs in the definitions of S_1 and B_1. The second set in our construction is not defined in this way.

We next give an example in J^1 for which our set B_1 is not the same as Mann's set B_1'. Let $n = 15$, $A = \{0, 1, 8, 10, 12\}$, $B = \{0, 2, 8, 9, 12, 13\}$, and $C = \{0, 4, 8, 9, 10, 11\}$. Then $(B+C)^{\sim} = \{0, 1, 8, 10, 12, 14\}$, $S_1 = \{0, 2, 9, 13\}$, and $B_1' = \{1, 10, 14\}$. However, $E = \{0, 2, 9, 13\}$, $E_0 = \{1, 10\}$, $E_2 = \{3, 10, 14\}$, $E_9 = \{10\}$, and $E_{13} = \{14\}$; therefore, $B_1 = \{3, 10, 14\}$. In this example there is no second set in Mann's construction since the set

$$\{b \mid a+b+c = x,\ a \in ((B \cup B_1')+C)^{\sim},\ b \in B \cup B_1',\ c \in C,\ x \in I_n' \setminus ((B \cup B_1')+C)\}$$

is empty. In our construction $B_2 = \{1\}$.

Before Theorem 3.1 was proved various attempts were made by the author to use known results to obtain the inequality

(3.17) $$A(I_n) + B(I_n) \leq (A+B)(I_n)$$

under the conditions that $A + B + C = I'_n$ and $A(I_n) < 5$. For instance, Mann's Theorem A was considered, but it does not seem to be useful in general in obtaining inequality (3.17). To see this let $n = 8$, $S_1 = \{0, 1, 5, 7\}$, $S_2 = \{0, 2, 5, 6\}$, and $S_3 = \{0, 4\}$. A direct computation shows that

$$S_i(I_8) + S_j(I_8) \geq 8 \cdot \min\left\{ \frac{S_i(I_t)+S_j(I_t)}{t} \,\middle|\, t \notin S_i + S_j,\ 1 \leq t \leq 8 \right\}$$

for each i and j where $1 \leq i < j \leq 3$. Thus, Theorem A does not appear to lead to a comparison between $(S_i+S_j)(I_8)$ and $S_i(I_8) + S_j(I_8)$ since the conclusion of Theorem A states that

$$(S_i+S_j)(I_8) \geq 8 \cdot \min\left\{ \frac{S_i(I_t)+S_j(I_t)}{t} \,\middle|\, t \notin S_i + S_j,\ 1 \leq t \leq 8 \right\}.$$

Theorem 3.1 is used to prove the next result.

<u>Theorem 3.2.</u> Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 2$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ and at least $k - 2$ of the sets have less than five nonzero elements, then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \left(\sum_{1 \leq i \leq k} A_i \right)(I_z).$$

Proof. First let $k = 3$. Let A_1, A_2, and A_3 be subsets of I_z such that at least one of the sets has less than five nonzero elements and $A_1 + A_2 + A_3 = I_z'$. To be definite, say $A_1(I_z) \leq 4$. An application of Theorem 3.1 gives

$$A_1(I_z) + A_2(I_z) \leq (A_1+A_2)(I_z).$$

From Theorem 2.9(c) we have $A_3 \subseteq (A_1+A_2)^{\sim}$. Furthermore,

$$(A_1+A_2)(I_z) + (A_1+A_2)^{\sim}(I_z) = I_z'(I_z)$$

by Theorem 2.9(f). Hence,

$$\begin{aligned} A_1(I_z) + A_2(I_z) + A_3(I_z) &\leq (A_1+A_2)(I_z) + A_3(I_z) \\ &\leq (A_1+A_2)(I_z) + (A_1+A_2)^{\sim}(I_z) \\ &= I_z'(I_z) \\ &= (A_1+A_2+A_3)(I_z), \end{aligned}$$

and the theorem is established for $k = 3$.

Let $k \geq 3$ be fixed and assume the statement of the theorem is valid for k. Let $B_1, B_2, \ldots, B_{k+1}$ be subsets of I_z such that at least $k-1$ of the sets have less than five nonzero elements and $\sum_{1 \leq i \leq k+1} B_i = I_z'$. By relabeling the sets if necessary, we may assume that $B_i(I_z) < 5$ for $i = 1, 2, \ldots, k-2$ and $i = k+1$. Define

$A_i = B_i$ for $i = 1, 2, \ldots, k-1$ and $A_k = B_k + B_{k+1}$. Then $\sum_{1 \leq i \leq k} A_i = \sum_{1 \leq i \leq k+1} B_i = I'_z$ and $A_i(I_z) < 5$ for $i = 1, 2, \ldots, k-2$.

By the induction hypothesis we have

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Since $(\sum_{1 \leq i \leq k-1} B_i) + B_k + B_{k+1} = I'_z$ and $B_{k+1}(I_z) < 5$, it follows from Theorem 3.1 that

$$B_k(I_z) + B_{k+1}(I_z) \leq (B_k+B_{k+1})(I_z)$$

Therefore,

$$\sum_{1 \leq i \leq k+1} B_i(I_z) = (\sum_{1 \leq i \leq k-1} B_i(I_z)) + B_k(I_z) + B_{k+1}(I_z)$$

$$\leq (\sum_{1 \leq i \leq k-1} B_i(I_z)) + (B_k+B_{k+1})(I_z)$$

$$= \sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z)$$

$$= (\sum_{1 \leq i \leq k+1} B_i)(I_z),$$

and the theorem follows for $k > 3$ by mathematical induction.

Now let $k = 2$ and let A_1 and A_2 be subsets of I_z for which $A_1 + A_2 = I'_z$. Define $A_3 = \{0\}$. Then $A_1 + A_2 + A_3 = A_1 + A_2 = I'_z$ and $A_3(I_z) = 0$. Since the theorem has been established when $k = 3$, we obtain

$$\begin{aligned} A_1(I_z) + A_2(I_z) &= A_1(I_z) + A_2(I_z) + A_3(I_z) \\ &\leq (A_1+A_2+A_3)(I_z) \\ &= (A_1+A_2)(I_z), \end{aligned}$$

and the proof is complete.

When $k \geq 3$ it is not possible to delete from the hypotheses of Theorem 3.2 the condition that at least $k-2$ of the sets have less than five nonzero elements. To see this consider $m = 1$, $z = 15$, $A_1 = \{0, 1, 8, 10, 12, 14\}$, $A_2 = \{0, 2, 8, 9, 12, 13\}$, $A_3 = \{0, 4, 8, 9, 10, 11\}$, and $A_i = \{0\}$ for $i = 4, \ldots, k$ if $k > 3$. Then $\sum_{1 \leq i \leq k} A_i = I'_{15}$; however,

$$\sum_{1 \leq i \leq k} A_i(I_z) = 15 > 14 = \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_z).$$

An example with $m > 1$ is $m = 4$, $z = (1, 1, 1, 1)$,

$$A_1 = \{0, (1, 0, 0, 0), (0, 1, 1, 1), (0, 0, 1, 1), (0, 1, 0, 1), (0, 0, 0, 1)\},$$

$$A_2 = \{0, (0, 1, 0, 0), (1, 0, 1, 1), (0, 0, 1, 1), (1, 0, 0, 1), (0, 0, 0, 1)\},$$

$$A_3 = \{0, (0, 0, 1, 0), (1, 1, 0, 1), (0, 1, 0, 1), (1, 0, 0, 1), (0, 0, 0, 1)\},$$

and $A_i = \{0\}$ for $i = 4, \ldots, k$ if $k > 3$.

It is easy to see that when $\sum_{1 \leq i \leq k} A_i = I'_z$ and $k \geq 3$ it is not necessary for at least $k-2$ of the sets $A_1, A_2, \ldots, A_k$ to have less than five nonzero elements in order to have the conclusion of Theorem 3.2 hold. For instance, consider $m = 1$, $z = 5k + 1$, and $A_i = \{0, \ldots, 5\}$ for $i = 1, 2, \ldots, k$.

A.R. Freedman [3] has proved the following result which is stronger than Theorem 3.2 when $k = 2$:

Let $z \in J^m$, $z > 0$, be specified. If A and B are subsets of I_z for which $0 \in A \cap B$ and $z \notin A + B$, then

$$A(I_z) + B(I_z) < J^m(I_z).$$

Freedman's result can be obtained directly from the properties of set inversion. Since $z \notin A + B$ then $B \subseteq A^{\sim}$ by Theorem 2.9(b). Since $0 \in A$ and $z \notin A$ then $A(I_z) + A^{\sim}(I_z) = I'_z(I_z)$ by Theorem 2.9(f). Therefore,

$$A(I_z) + B(I_z) \leq A(I_z) + A^{\sim}(I_z) = I'_z(I_z) < J^m(I_z).$$

An analogous result does not exist for three sets even when $m = 1$, for if $z = 5$ and $A = B = C = \{0, 2, 4\}$ then $0 \in A + B + C$, $5 \notin A + B + C$, and

$$A(I_5) + B(I_5) + C(I_5) = 6 > 5 = J^1(I_5).$$

The next theorem generalizes Theorem 3.1 to $k \geq 2$ sets. Theorem 3.2 is used to prove this result.

<u>Theorem 3.3.</u> Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 2$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k+1$. If $\sum_{1 \leq i \leq k+1} A_i = I'_z$ and either

(i) $A_i(I_z) < 5$ for $i = 1, 2, \ldots, k-1$

or

(ii) $(I_z \setminus (\sum_{1 \leq i \leq k} A_i))(I_z) \leq 5$ and $A_i(I_z) < 5$ for $i = 1, 2, \ldots, k-2$, then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Proof. Let $B = (\sum_{1 \leq i \leq k} A_i)^{\sim}$. Since $\sum_{1 \leq i \leq k+1} A_i = I'_z$ then $(\sum_{1 \leq i \leq k} A_i) + B = I'_z$ by Theorem 2.9(c). Furthermore, applying Theorem 2.9(d) we have

$$B(I_z) = B(I'_z) = (I_z \setminus (\sum_{1 \leq i \leq k} A_i))(I'_z)$$

$$< (I_z \setminus (\sum_{1 \leq i \leq k} A_i))(I_z).$$

Thus, in both parts (i) and (ii) of the hypotheses at least $k-1$ of the sets $B, A_1, A_2, \ldots, A_k$ have less than five nonzero elements. From Theorem 3.2 it follows that

$$(\sum_{1 \le i \le k} A_i(I_z)) + B(I_z) \le ((\sum_{1 \le i \le k} A_i) + B)(I_z) = I'_z(I_z).$$

However,

$$I'_z(I_z) = (\sum_{1 \le i \le k} A_i)(I_z) + B(I_z)$$

by Theorem 2.9(f), and so

$$\sum_{1 \le i \le k} A_i(I_z) \le (\sum_{1 \le i \le k} A_i)(I_z).$$

The conclusion of Theorem 3.3 cannot be obtained with the hypotheses $\sum_{1 \le i \le k+1} A_i = I'_z$, $A_{k+1}(I_z) < 5$, and $A_i(I_z) < 5$ for $i = 1, 2, \ldots, k-2$. For example, with $k = 2$, $m = 1$, $z = 15$, $A_1 = \{0, 1, 8, 10, 12, 14\}$, $A_2 = \{0, 2, 8, 9, 12, 13\}$, and $A_3 = \{0, 4, 11\}$, then $A_1 + A_2 + A_3 = I'_{15}$, $A_3(I_{15}) < 5$, and

$$A_1(I_{15}) + A_2(I_{15}) = 10 > 9 = (A_1 + A_2)(I_{15}).$$

Note that $I_{15} \setminus (A_1 + A_2) = \{4, 5, 6, 7, 11, 15\}$.

In the next theorem we obtain an extension of Lin's Theorem D. Theorem 3.2 is used to prove this result.

Theorem 3.4. Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 3$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$. If $\sum_{1 \leq i \leq k} A_i = I'_z$, $(I_z \setminus (\sum_{1 \leq i \leq k-1} A_i))(I_z) \leq 5$, and at least $k-3$ of the sets $A_1, A_2, \ldots, A_{k-1}$ have less than five nonzero elements, then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Proof. Since $\sum_{1 \leq i \leq k} A_i = I'_z$ then $A_k \subseteq (\sum_{1 \leq i \leq k-1} A_i)^{\sim}$ by Theorem 2.9(c). From Theorem 2.9(d) we have

$$(\sum_{1 \leq i \leq k-1} A_i)^{\sim}(I'_z) = (I_z \setminus (\sum_{1 \leq i \leq k-1} A_i))(I'_z).$$

Therefore,

$$A_k(I_z) = A_k(I'_z) \leq (\sum_{1 \leq i \leq k-1} A_i)^{\sim}(I'_z)$$

$$= (I_z \setminus (\sum_{1 \leq i \leq k-1} A_i))(I'_z)$$

$$< (I_z \setminus (\sum_{1 \leq i \leq k-1} A_i))(I_z) \leq 5.$$

Since $A_k(I_z) < 5$ and at least $k-3$ of the sets $A_1, A_2, \ldots, A_{k-1}$ have less than five nonzero elements, applying Theorem 3.2 we have

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

When $m = 1$ and $k = 3$ Theorem 3.4 is Theorem D. Lin's proof of Theorem D uses properties of Mann's set construction that are verified by arguments which make use of the linear ordering on the integers. Thus, it seems unlikely that Theorem 3.4 can be established when $k = 3$ for $m > 1$ by a proof based on Lin's proof of Theorem D.

Theorem 3.2 is a stronger result than Theorem 3.4 for consider $k = 3$, $m = 1$, $z = 15$, $\{0, 1\} \subseteq A_1 \subseteq \{0, 1, 8, 10, 12\}$, $A_2 = \{0, 2, 8, 9, 12, 13\}$, and $A_3 = \{0, 4, 8, 9, 10, 11\}$. Then $A_1 + A_2 + A_3 = I'_{15}$ and $A_1(I_{15}) \leq 4$; thus, applying Theorem 3.2 we have

$$\sum_{1 \leq i \leq 3} A_i(I_{15}) \leq (\sum_{1 \leq i \leq 3} A_i)(I_{15}). \tag{3.18}$$

However,

$$I_{15} \setminus (A_1 + A_2) = \{4, 5, 6, 7, 11, 15\},$$

$$I_{15} \setminus (A_1 + A_3) = \{2, 3, 6, 7, 13, 15\},$$

and

$$I_{15} \setminus (A_2+A_3) = \{1, 3, 5, 7, 14, 15\}.$$

Since each of these sets has more than five nonzero elements, we cannot apply Theorem 3.4 to obtain inequality (3.18).

We note that if $z \in J^m$, $z > 0$, and $A_1, A_2, \ldots, A_k$ are subsets of I_z for which $A_i \cap A_j = \{0\}$ whenever $1 \leq i < j \leq k$, then $(\bigcup_{1\leq i\leq k} A_i)(I_z) = \sum_{1\leq i\leq k} A_i(I_z)$. Since $\bigcup_{1\leq i\leq k} A_i \subseteq \sum_{1\leq i\leq k} A_i$, we obtain

$$\sum_{1\leq i\leq k} A_i(I_z) \leq (\sum_{1\leq i\leq k} A_i)(I_z).$$

In Theorem 3.5 we given conditions that imply the above inequality for two sets which have one element in common.

The proofs of the next four theorems do not depend on the previous results in this chapter.

Theorem 3.5. Let $z \in J^m$, $z > 0$, be specified and let $A, B, C \subseteq I_z$. If $A + B + C = I'_z$ and $(A \cap B)(I_z) \leq 1$, then

$$A(I_z) + B(I_z) \leq (A+B)(I_z).$$

Proof. Since $0 \in A \cap B \cap C$ then $A \subseteq A + B$ and $B \subseteq A + B$. Thus, the conclusion of the theorem is immediate when A and B have no nonzero elements in common, that is, when

$(A \cap B)(I_z) = 0$.

We now consider $(A \cap B)(I_z) = 1$. Let $\{0, x\} = A \cap B$. With $B^* = B \setminus \{x\}$, we have $A \cap B^* = \{0\}$ and $A \cup B^* = A \cup B \subseteq A + B$. Assume $A \cup B^* = A + B$. If $y \in A \cup B^*$, then $y \in A$ or $y \in B$, and so $y + x \in A + B$ when $y + x \leq z$. Thus,

$$(A+B) + \{0, x\} = (A \cup B^*) + \{0, x\} \subseteq A + B \subseteq (A+B) + \{0, x\},$$

and it follows that $A + B + \{0, x\} = A + B$. Now $A + B + C = I'_z$ implies $(A+B) + (A+B)^{\sim} = I'_z$ by Theorem 2.9(c), and this in turn implies $(A+B) + \{0, x\} \neq A + B$ by Theorem 2.10. Since the assumption that $A + B = A \cup B^*$ leads to contradictory results, it must be that $A + B \supseteq A \cup B^*$ and $A + B \neq A \cup B^*$. Furthermore, $A(I_z) + B^*(I_z) = (A \cup B^*)(I_z)$ since A and B^* have no nonzero elements in common. Thus,

$$\begin{aligned} (A+B)(I_z) &\geq (A \cup B^*)(I_z) + 1 \\ &= A(I_z) + B^*(I_z) + 1 \\ &= A(I_z) + B(I_z). \end{aligned}$$

We use Theorem 3.5 to prove the following result.

<u>Theorem 3.6</u>. Let $z \in J^m$, $z > 0$, be specified and let $A, B, C, \subseteq I_z$. If $A + B + C = I'_z$ and $(A \cap B)(I_z) \leq 1$, then

$$A(I_z) + B(I_z) + C(I_z) \leq (A+B+C)(I_z).$$

Proof. By Theorem 3.5 we have $A(I_z) + B(I_z) \leq (A+B)(I_z)$. Since $A + B + C = I'_z$, then $C \subseteq (A+B)^\sim$ by Theorem 2.9(c). Also, $(A+B)(I_z) + (A+B)^\sim(I_z) = I'_z(I_z)$ by Theorem 2.9(f). Therefore,

$$\begin{aligned} A(I_z) + B(I_z) + C(I_z) &\leq (A+B)(I_z) + C(I_z) \\ &\leq (A+B)(I_z) + (A+B)^\sim(I_z) \\ &= I'_z(I_z) = (A+B+C)(I_z). \end{aligned}$$

The results obtained in Theorems 3.5 and 3.6 are the best possible in the sense that each is no longer valid with $(A \cap B)(I_z) \leq 1$ removed from the hypotheses. This is easily seen by considering the example $z = 15$, $S_1 = \{0, 1, 8, 10, 12, 14\}$, $S_2 = \{0, 2, 8, 9, 12, 13\}$, and $S_3 = \{0, 4, 8, 9, 10, 11\}$. Then $S_1 + S_2 + S_3 = I'_{15}$, $(S_i \cap S_j)(I_{15}) = 2$, and

$$S_i(I_{15}) + S_j(I_{15}) = 10 > 9 = (S_i+S_j)(I_{15})$$

where $1 \leq i < j \leq 3$. Also,

$$\sum_{1 \leq i \leq 3} S_i(I_{15}) = 15 > 14 = (\sum_{1 \leq i \leq 3} S_i)(I_{15}).$$

One generalization of Theorem 3.6 is given in the next theorem.

Theorem 3.7. Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 3$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ and $(A_j \cap (\sum_{j<i<k} A_i))(I_z) \leq 1$ for $j = 1, 2, \ldots, k-2$, then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Proof. When $k = 3$ Theorem 3.7 is Theorem 3.6.

Let $k \geq 3$ be fixed and assume the statement of the theorem is valid for k. Let $B_1, B_2, \ldots, B_{k+1}$ be subsets of I_z for which $\sum_{1 \leq i \leq k+1} B_i = I'_z$ and $(B_j \cap (\sum_{j<i<k+1} B_i))(I_z) \leq 1$ for $j = 1,2,\ldots,k-1$. Define $A_i = B_i$ for $i = 1, 2, \ldots, k-2$, $A_{k-1} = B_{k-1} + B_k$, and $A_k = B_{k+1}$. Then $\sum_{1 \leq i \leq k} A_i = \sum_{1 \leq i \leq k+1} B_i = I'_z$.

Also, $(A_j \cap (\sum_{j<i<k} A_i))(I_z) \leq 1$ for $j = 1, 2, \ldots, k-2$ since $A_j \cap (\sum_{j<i<k} A_i)$ is equal to $B_j \cap (\sum_{j<i\leq k} B_i)$. By the induction hypothesis we have

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Since $((\sum_{1 \leq i \leq k-2} B_i) + B_{k+1}) + B_{k-1} + B_k = I'_z$ and $(B_{k-1} \cap B_k)(I_z) \leq 1$,

it follows from Theorem 3.5 that

$$B_{k-1}(I_z) + B_k(I_z) \leq (B_{k-1} + B_k)(I_z).$$

Therefore,

$$\sum_{1 \leq i \leq k+1} B_i(I_z) = (\sum_{1 \leq i \leq k-2} B_i(I_z)) + B_{k-1}(I_z) + B_k(I_z) + B_{k+1}(I_z)$$

$$\leq (\sum_{1 \leq i \leq k-2} B_i(I_z)) + (B_{k-1} + B_k)(I_z) + B_{k+1}(I_z)$$

$$= \sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z)$$

$$= (\sum_{1 \leq i \leq k+1} B_i)(I_z).$$

The theorem follows by mathematical induction.

Although we do not do so it is possible to generalize Theorem 3.6 to $k \geq 3$ sets by using conditions other than those given in the hypotheses of Theorem 3.7, but which also employ Theorems 3.5 and 3.6. The following example indicates how such generalizations can be made. Let $k = 7$, $\sum_{1 \leq i \leq 7} A_i = I'_z$, $((A_1 + A_2) \cap (A_3 + A_4))(I_z) \leq 1$,

$(A_1 \cap A_2)(I_z) \leq 1$, $(A_3 \cap A_4)(I_z) \leq 1$, $(A_5 \cap (A_6+A_7))(I_z) \leq 1$, and $(A_6 \cap A_7)(I_z) \leq 1$. Then applying Theorems 3.5 and 3.6 we have

$$\begin{aligned}\sum_{1 \leq i \leq 7} A_i(I_z) &\leq (A_1+A_2)(I_z) + (A_3+A_4)(I_z) + A_5(I_z) + (A_6+A_7)(I_z) \\ &\leq (A_1+A_2)(I_z) + (A_3+A_4)(I_z) + (A_5+A_6+A_7)(I_z) \\ &\leq (\sum_{1 \leq i \leq 7} A_i)(I_z).\end{aligned}$$

The next theorem generalizes Theorem 3.5 to $k \geq 3$ sets. Theorem 3.7 is used to prove this result.

<u>Theorem 3.8.</u> Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 2$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k+1$. If $\sum_{1 \leq i \leq k+1} A_i = I'_z$ and $(A_j \cap (\sum_{j < i \leq k} A_i))(I_z) \leq 1$ for $j = 1, 2, \ldots, k-1$, then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Proof. Let $B = (\sum_{1 \leq i \leq k} A_i)^{\sim}$. Since $\sum_{1 \leq i \leq k+1} A_i = I'_z$ then $(\sum_{1 \leq i \leq k} A_i) + B = I'_z$ by Theorem 2.9(c). From Theorem 3.7 we have

$$(\sum_{1 \leq i \leq k} A_i(I_z)) + B(I_z) \leq ((\sum_{1 \leq i \leq k} A_i)+B)(I_z) = I'_z(I_z),$$

and from Theorem 2.9(f) we obtain

$$(\sum_{1\leq i\leq k} A_i)(I_z) + B(I_z) = I'_z(I_z).$$

The corollary follows from these two results.

Other theorems can be obtained from the results which have been previously established in this chapter. The next theorem is presented to illustrate this.

<u>Theorem 3.9.</u> Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 4$ and let $A_i \subsetneq I_z$ for $i = 1, 2, \ldots, k$. Let $2 \leq r \leq k-2$. If $\sum_{1\leq i\leq k} A_i = I'_z$, $A_i(I_z) < 5$ for $i = 1, 2, \ldots, r-1$, and $(A_j \cap (\sum_{j<i\leq k} A_i))(I_z) \leq 1$ for $j = r+1, \ldots, k-1$, then

$$\sum_{1\leq i\leq k} A_i(I_z) \leq (\sum_{1\leq i\leq k} A_i)(I_z).$$

Proof. Since $\sum_{1\leq i\leq r} A_i + (\sum_{r<i\leq k} A_i) = I'_z$ and $A_i(I_z) < 5$ for $i = 1, 2, \ldots, r-1$, then we have

$$\sum_{1\leq i\leq r} A_i(I_z) \leq (\sum_{1\leq i\leq r} A_i)(I_z)$$

by Theorem 3.3(i). Since $\sum_{r+1 \leq i \leq k} A_i + (\sum_{1 \leq i \leq r} A_i) = I'_z$ and $(A_j \cap (\sum_{j<i \leq k} A_i))(I_z) \leq 1$ for $j = r+1, \ldots, k-1$, then we have

$$(\sum_{1 \leq i \leq r} A_i)(I_z) + \sum_{r+1 \leq i \leq k} A_i(I_z) \leq (\sum_{1 \leq i \leq k} A_i)(I_z)$$

by Theorem 3.7. The theorem follows directly from these two inequalities.

IV. AN EXTENSION OF THEOREM E

In Theorem 4.12 we obtain a result for $k \geq 3$ sets in J^m which for $k = 3$ and $m = 1$ is Lin's Theorem E. Examples are given to show that it is not possible to improve on the upper bound of $J^m(I_z)$ in parts (i) and (ii) of Theorem 4.12.

All of the results in this chapter through Theorem 4.11 are presented for the purpose of proving Theorem 4.12. Theorem 4.1 is of special interest in its own right, and the other theorems which precede Theorem 4.12 are superseded by Theorem 4.12. We note that Theorem 3.2 is also used in proving Theorem 4.12.

We begin with a theorem which is used in the proofs of Theorems 4.3 through 4.9 and also in the proof of Theorem 4.11.

<u>Theorem 4.1.</u> Let $z \in J^m$, $z > 0$, be specified and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$ where $k \geq 2$. Let $n = J^m(I_z)$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ then

(i) $$\Big(\sum_{2 \leq i \leq k} A_i\Big)(I_z) \leq n - 1 - A_1(I_z),$$

and whenever $A_i(I_z) > 0$ for $i = 2, \ldots, j$ where $2 \leq j < k$ then

(ii) $$\Big(\sum_{j+1 \leq i \leq k} A_i\Big)(I_z) \leq n - j - A_1(I_z).$$

Proof. Since $\sum_{1 \le i \le k} A_i = I'_z$, then $A_1 \subseteq (\sum_{2 \le i \le k} A_i)^{\sim}$ by Theorem 2.9(c). Thus, $A_1(I_z) \le (\sum_{2 \le i \le k} A_i)^{\sim}(I_z)$, and applying Theorem 2.9(f) we have

$$(\sum_{2 \le i \le k} A_i)(I_z) = n - 1 - (\sum_{2 \le i \le k} A_i)^{\sim}(I_z)$$

$$\le n - 1 - A_1(I_z).$$

This proves part (i) of the theorem.

We claim $(\sum_{t \le i \le k} A_i)(I_z) \le (\sum_{t-1 \le i \le k} A_i)(I_z) - 1$ whenever $A_{t-1}(I_z) > 0$ and $1 < t \le k$. Let $x \in A_{t-1}$ and $x \neq 0$. Since $\sum_{1 \le i \le k} A_i = I'_z$, then $\sum_{t \le i \le k} A_i + (\sum_{t \le i \le k} A_i)^{\sim} = I'_z$ by Theorem 2.9(c). Thus, $(\sum_{t \le i \le k} A_i) + \{0, x\} \neq \sum_{t \le i \le k} A_i$ by Theorem 2.10. However, if $\sum_{t-1 \le i \le k} A_i = \sum_{t \le i \le k} A_i$ then

$$\sum_{t \le i \le k} A_i \subseteq (\sum_{t \le i \le k} A_i) + \{0, x\}$$

$$\subseteq \sum_{t-1 \le i \le k} A_i = \sum_{t \le i \le k} A_i,$$

and this in turn implies $\left(\sum_{t\le i\le k} A_i\right) + \{0, x\} = \sum_{t\le i\le k} A_i$. We conclude that $\sum_{t-1\le i\le k} A_i \ne \sum_{t\le i\le k} A_i$. Hence,

$$\left(\sum_{t\le i\le k} A_i\right)(I_z) \le \left(\sum_{t-1\le i\le k} A_i\right)(I_z) - 1.$$

Now let $k \ge 3$, $j \in \{2, \ldots, k-1\}$, and assume $A_i(I_z) > 0$ for $i = 2, \ldots, j$. We apply the result established in the preceding paragraph $j-1$ time to obtain

$$\begin{aligned}\left(\sum_{j+1\le i\le k} A_i\right)(I_z) &\le \left(\sum_{j\le i\le k} A_i\right)(I_z) - 1\\ &\le \left(\sum_{j-1\le i\le k} A_i\right)(I_z) - 2\\ &\;\;\vdots\\ &\le \left(\sum_{2\le i\le k} A_i\right)(I_z) - (j-1).\end{aligned}$$

However, $\left(\sum_{2\le i\le k} A_i\right)(I_z) \le n - 1 - A_1(I_z)$ by part (i), and so $\left(\sum_{j+1\le i\le k} A_i\right)(I_z) \le n - j - A_1(I_z)$. This establishes part (ii) of the theorem.

The next lemma is used frequently, however without reference, in the proof of Theorem 4.3.

Lemma 4.2. Let $n \in J^1$, $n > 0$, be specified and let $A, B \subseteq I_n$. If $n \notin A + B$, $a \in A$, $\{b, \ldots, b+j\} \subseteq B$, and $a + b \in A + B$, then $\{a+b, \ldots, a+b+j\} \subseteq A + B$.

Proof. Assume $\{a+b, \ldots, a+b+j\}$ is not contained in $A + B$. Then from Definition 2.7 it follows that $j > 0$ and $a + b + j > n$. Now $a + b < n$ since $a + b \in A + B$ and $n \notin A + B$. But $a + b < n$ and $a + b + j > n$ imply $a + b + r = n$ for some integer r, $0 < r < j$. Since $0 < r < j$ then $b + r \in B$; hence, $n = a + (b+r) \in A + B$. However, $n \notin A + B$. Thus, $\{a+b, \ldots, a+b+j\} \subseteq A + B$.

The next seven theorems, namely Theorems 4.3 through 4.9, are used with the aid of Lemma 4.10 to prove Theorem 4.11.

Theorem 4.3. Let $n \in J^1$, $n > 0$, be specified and let $A, B, C, D, E, \subseteq I_n$. If $A + B + C + D + E = I'_n$ and either

(i) $n \leq 14$,

(ii) $n = 15$ and $D(I_n) > 0$ or $E(I_n) > 0$,

or

(iii) $n = 16$, $D(I_n) > 0$, and $E(I_n) > 0$,

then at least one of the sets A, B, and C has less than five non-zero elements.

Proof. The theorem is immediate if $n \leq 5$. Therefore, we restrict our consideration to $n \in J^1$ where $6 \leq n \leq 16$.

Assume $A(I_n) \geq 5$, $B(I_n) \geq 5$, and $C(I_n) \geq 5$.

Let S_1, S_2, S_3 represent a permutation of sets A, B, C and let S_4, S_5 represent a permutation of sets D, E. From Theorem 4.1(i) we have

$$\Big(\sum_{2\leq i\leq j} S_i\Big)(I_n) \leq \Big(\sum_{2\leq i\leq 5} S_i\Big)(I_n) \leq n - 1 - S_1(I_n) \leq n - 6$$

where $j \in \{2, 3, 4, 5\}$. If $n = 15$ then either $S_4(I_{15}) > 0$ or $S_5(I_{15}) > 0$. Let $t = 4$ if $S_5(I_{15}) > 0$; otherwise, let $t = 5$. Applying Theorem 4.1(ii) we have

$$(S_2+S_3)(I_{15}) \leq (S_2+S_3+S_t)(I_{15}) \leq 15 - 2 - S_1(I_{15}) \leq 8.$$

If $n = 16$ then $S_4(I_{16}) > 0$ and $S_5(I_{16}) > 0$. Thus, by Theorem 4.1(ii) we have

$$(S_2+S_3)(I_{16}) \leq 16 - 3 - S_1(I_{16}) \leq 8$$

and

$$\Big(\sum_{2\leq i\leq 4} S_i\Big)(I_{16}) \leq 16 - 2 - S_1(I_{16}) \leq 9.$$

From the above inequalities it follows that

$$(S_2+S_3)(I_n) \leq \min\{n-6, 8\}$$

and

$$(S_2+S_3+S_4)(I_n) \leq \min\{n-6, 9\}.$$

Let the elements of A be labeled $a_0 = 0, a_1, a_2, \ldots, a_u$ where $a_i < a_j$ if $0 \leq i < j \leq u$. Let the elements of B be labeled $b_0 = 0, b_1, b_2, \ldots, b_v$ where $b_i < b_j$ if $0 \leq i < j \leq v$. Finally, let the elements of C be labeled $c_0 = 0, c_1, c_2, \ldots, c_w$ where $c_i < c_j$ if $0 \leq i < j \leq w$.

First we establish the existence of an integer p, $1 \leq p < u$, for which $a_{p+1} > a_p + 1$. Assume the contrary. Then $a_{i+1} = a_i + 1$ for $i = 1, 2, \ldots, u-1$, and it follows that $A = \{0, a_1, \ldots, a_1+u-1\}$. Thus $A \supseteq \{0, a_1, \ldots, a_1+4\}$, and so $\{n-a_1-4, \ldots, n-a_1\}$ has an empty intersection with $B + C + D + E$. Since $n - a_1 - 4 \in A + B + C + D + E$ then $a_1 \leq n - a_1 - 4$. Hence, $2a_1 + 4 \leq n$. Also, recall that $(A+B)(I_n) \leq 8$.

First let $1 \leq a_1 \leq 2$. Since $b_4 < b_5 < n$ then $b_4 \leq n - 2$, and so $a_1 + b_4 \in A + B$. However, then $(A+B)(I_n) > 8$ since $b_4 \geq 4$ and

$$A + B \supseteq A + \{0, b_4\} \supseteq \{0, a_1, \ldots, a_1+4, a_1+1+b_4, \ldots, a_1+4+b_4\}.$$

Next let $a_1 \geq 3$ and $b_1 \geq a_1$. Then $b_2 \geq 4$. Furthermore, $b_2 \leq a_1 + 4$, for otherwise

$$A + B \supseteq \{0, a_1, \ldots, a_1+4, b_2, b_3, b_4, b_5\}$$

and $(A+B)(I_n) > 8$. Thus, $a_1+b_2 \leq a_1+a_1+4 \leq n$, and so $a_1+b_2 \in A+B$. However, then $(A+B)(I_n) > 8$ since $b_2 \geq 4$ and

$$A+B \supseteq A+\{0,b_2) \supseteq \{0,a_1,\ldots,a_1+4,a_1+1+b_2,\ldots,a_1+4+b_2\}.$$

It remains to consider $a_1 \geq 3$ and $b_1 < a_1$. Now $b_3 \leq a_1+4$, for otherwise

$$A+B \supseteq \{0,b_1,a_1,\ldots,a_1+4,b_3,b_4,b_5\}$$

and $(A+B)(I_n) > 8$. Thus, $a_1+b_3 \leq a_1+a_1+4 \leq n$, and so $a_1+b_3 \in A+B$. However, then $(A+B)(I_n) > 8$ since $b_3 \geq 3$ and

$$\begin{aligned} A+B &\supseteq A+\{0,b_3\} \\ &\supseteq \{0,b_1,a_1,\ldots,a_1+4,\ a_1+2+b_3,\ a_1+3+b_3,\ a_1+4+b_3\}. \end{aligned}$$

This establishes the existence of an integer p, $1 \leq p < u$, for which $a_{p+1} > a_p + 1$. The same kind of arguments establish the existence of integers q and r for which $1 \leq q < v$, $1 \leq r < w$, $b_{q+1} > b_q+1$, and $c_{r+1} > c_r+1$.

We next show that $\{1,2\} \cap (A+B+C)$ is not empty. Assume otherwise. Then $\{1,2\} \subseteq D+E$. Let p be an integer such that $1 \leq p < u$ and $a_{p+1} > a_p+1$. Then

$$A+D+E \supseteq \{1,2\} \cup A \cup \{a_p+1, a_u+1, a_u+2\},$$

and so $(A+D+E)(I_n) \geq 10$. However, since $B(I_n) > 0$ then

$$(A+D+E)(I_n) \leq n - 2 - C(I_n) \leq n - 7 \leq 9$$

by Theorem 4.1(ii).

Case 1. $(A+B+C) \cap \{1, 2\} = \{1\}$.

By relabeling sets A, B, and C if necessary, we may assume $1 \in A$. Since $2 \in A + B + C + D + E$ then $(D+E) \cap \{1, 2\}$ is not empty, and so either $D \cap \{1, 2\}$ or $E \cap \{1, 2\}$ is not empty. By relabeling sets D and E if necessary, we may assume $D \cap \{1, 2\}$ is not empty. Thus, $\{1, 2\} \subseteq A + D$. Also, $B \cap \{1, 2\}$ is empty, and so $b_1 > 2$.

Let q be an integer such that $1 \leq q < v$ and $b_{q+1} > b_q+1$. Then

$$A + B + D \supseteq \{1, 2\} \cup B \cup \{b_q+1, b_v+1, b_v+2\},$$

and so $(A+B+D)(I_n) > 9$. However, $(A+B+D)(I_n) \leq \min\{9, n-6\}$, and we have a contradiction.

Case 2. $(A+B+C) \cap \{1, 2\} = \{2\}$.

Since $1 \notin A + B + C$ and $1 \in A + B + C + D + E$, then $1 \in D$ or $1 \in E$. Since $2 \in A + B + C$ and $1 \notin A + B + C$, then $2 \in A$, $2 \in B$, or $2 \in C$. Without loss of generality, we may assume $1 \in D$ and $2 \in A$. Also $b_1 \geq 2$.

Let q be an integer such that $1 \leq q < v$ and $b_{q+1} > b_q+1$. Then

$$A + B + D \supseteq \{1\} \cup B \cup \{b_q+1, b_v+1, b_v+2, b_v+3\}.$$

Hence,

$$\min\{9, n-6\} \geq (A+B+D)(I_n) \geq v + 5 \geq 10,$$

and we have a contradiction.

Case 3. Either A, B, or C contains $\{1, 2\}$.

By relabeling the sets A, B, and C if necessary, we may assume that $\{1, 2\} \subseteq A$.

Let q be an integer such that $1 \leq q < v$ and $b_{q+1} > b_q+1$. Then

$$A + B \supseteq B \cup \{b_q+1, b_v+1, b_v+2\},$$

and it follows that

$$\min\{n-6, 8\} \geq (A+B)(I_n) \geq v + 3 \geq 8.$$

We immediately have a contradiction when $n \leq 13$. Let $14 \leq n \leq 16$. Then $v = 5$ and

$$A + B = B \cup \{b_q+1, b_v+1, b_v+2\}.$$

Thus, $b_{q+1} = b_q+2$, for otherwise $b_q+2 \in A + B$ and $b_q+2 \notin B \cup \{b_q+1, b_v+1, b_v+2\}$. Furthermore, there does not exist an integer i such that $1 \leq i < v$, $i \neq q$, and $b_{i+1} > b_i+1$, for otherwise $b_i+1 \in A + B$ and $b_i+1 \notin B \cup \{b_q+1, b_v+1, b_v+2\}$. Since $1 \in A + B$ then $b_1 = 1$. It follows that $A + B = \{0, \ldots, 8\}$. Since $b_5 = 6$ and $a_5 \leq 8$ then $a_3+b_5 \leq 6+6 < n$, and so $a_3+b_5 \in A + B$. However, $a_3+b_5 \geq 3+6 = 9$, and we have a contradiction.

Case 4. The set $\{1, 2\}$ is a subset of $A + B + C$, but it is not a subset of A, B, or C.

By relabeling the sets A, B, and C if necessary, we may assume that $1 \in A$ and $\{1, 2\} \subseteq A + B$. Since neither A nor B contains $\{1, 2\}$, then $a_2 \geq 3$, $b_1 \in \{1, 2\}$, and $b_2 \geq 3$.

We now assume that $b_j+1 = b_{j+1}$ for $2 \leq j < v$. Thus,

$$B = \{0, b_1, b_2, \ldots, b_2+v-2\}$$

and

$$A + B \supseteq \{0, 1, 2, b_2, \ldots, b_2+v-1\}.$$

We claim $b_2 > 3$. Since $a_3+3 \leq a_5+1$ and $a_5 \in I'_n$, then $a_3+3 \leq n$. Therefore, if $b_2 = 3$ then $\{3, 4, 5, 6\} \subseteq B$ and

$$A + B \supseteq \{0, \ldots, 6, a_3+3, \ldots, a_3+6\}.$$

But then $(A+B)(I_z) > 8$ since $a_3 \geq 4$.

We claim $a_2+b_2 > n$. If $a_2+b_2 \leq n$ then

$\{b_2+a_2, \ldots, b_2+v-2+a_2\} \subseteq A + B$. But then $(A+B)(I_z) > 8$ since $b_2+v-2+a_2 > b_2+v-3+a_2 > b_2+v-1$.

First suppose $a_2 \leq b_2$. If $x \in A + B$ and $x < a_2$ then $x \in \{0, 1\} \cup \{b_1, b_1+1\} \subseteq \{0, 1, 2, 3\}$, and so

$$x+b_2 \in \{b_2, b_2+1, b_2+2, b_2+3\} \subseteq A + B$$

If $x \in A + B$ and $x \geq a_2$ then $x + b_2 \notin I_n$ since $x+b_2 \geq a_2+b_2 > n$. Therefore, $A + B + \{0, b_2\} = A + B$. However, this is contrary to Theorem 2.10.

Next suppose $a_2 > b_2$. If $x \in A + B$ and $x < b_2$, then $x \in \{0, 1\} \cup \{b_1, b_1+1\}$. Let $b_1 = 1$. We have $a_4 \leq b_2+v-1$, for otherwise $(A+B)(I_n) > 8$ since $a_5 > a_4 > b_2+v-1$. Thus,

$$\begin{aligned} \{a_2\} + \{0, b_1, b_1+1\} &= \{a_2\} + \{0, 1, 2\} \\ &\subseteq \{a_2, \ldots, a_4\} \\ &\subseteq \{b_2, \ldots, b_2+v-1\} \\ &\subseteq A + B. \end{aligned}$$

Let $b_1 = 2$. Then $3 \in A + B$. Since $(A+B)(I_z) \leq 8$ and $b_2 > 3$, it follows that

$$A + B = \{0, 1, 2, 3, b_2, \ldots, b_2+v-1\},$$

and hence also that $a_5 \leq b_2+v-1$. Thus,

$$\{a_2\} + \{0, 1, b_1, b_1+1\} = \{a_2\} + \{0, 1, 2, 3\}$$
$$\subseteq \{a_2, \ldots, a_5\}$$
$$\subseteq \{b_2, \ldots, b_2+v-1\}$$
$$\subseteq A + B.$$

Therefore, $x + a_2 \in A + B$ when $x \in A + B$ and $x < b_2$. If $x \in A + B$ and $x \geq b_2$, then $x + a_2 \notin I_n$ since $x + a_2 \geq b_2 + a_2 > n$. This establishes that $(A+B) + \{0, a_2\} = A + B$. However, this is contrary to Theorem 2.10.

We conclude that there is an integer j, $2 \leq j < v$ for which $b_{j+1} > b_j+1$. Let

$$K = \{k \mid 2 \leq k < v \text{ and } b_{k+1} > b_k+1\}.$$

For each $k \in K$, $b_k < b_k+1 < b_{k+1}$ and $b_k+1 \in A + B$. Therefore,

$$A + B \supseteq \{0, 1, 2\} \cup \{b_i \mid 2 \leq i \leq v\} \cup \{b_k+1 \mid k \in K \text{ or } k = v\},$$

and so

$$\min\{n-6, 8\} \geq (A+B)(I_n) \geq v + 2 + K(I_n).$$

Since $v \geq 5$ and $K(I_n) \geq 1$, we immediately obtain a contradiction when $n \leq 13$. Furthermore, when $14 \leq n \leq 16$ then $(A+B)(I_n) = 8$, $v = 5$, and $K(I_n) = 1$.

Henceforth, let $14 \leq n \leq 16$. Also, let $K = \{i\}$. Then

$$B = \{0, b_1, b_2, \ldots, b_2+s, b_2+s+t, \ldots, b_2+2+t\}$$

where $s = i-2$ and $t = b_{i+1} - b_i \geq 2$. Note that $0 \leq s \leq 2$ since $2 \leq i \leq 4$. From $1 \in A$ and $(A+B)(I_n) = 8$, it follows that

$$A + B = \{0, 1, 2, b_2, \ldots, b_2+s+1, b_2+s+t, \ldots, b_2+3+t\}.$$

We claim $t > 2$. If $t = 2$ then

$$B = \{0, b_1, b_2, \ldots, b_2+s, b_2+s+2, \ldots, b_2+4\}$$

and

$$A + B = \{0, 1, 2, b_2, \ldots, b_2+5\}.$$

Thus, $b_2 + a_5 \notin A + B$ since $a_5 \geq 6$, and so $b_2 + a_5 > n$. Since $a_5 \leq b_2+5$ then $2b_2+5 \geq b_2+a_5 > n \geq 14$, and so $b_2 \geq 5$. Therefore, $b_2+a_3 \geq b_2+(b_2+1) \geq b_2+6$. But then $b_2 + a_3 \notin A + B$, and so $b_2+a_3 > n$. Hence, if $x \in A + B$ and $x \geq b_2$, then $x + a_3 \notin I_n$ since $x+a_3 \geq b_2+a_3 > n$. If $x \in A + B$ and $x < b_2$ then $x \in \{0, 1, 2\}$, and so

$$\begin{aligned} x + a_3 \in \{a_3, a_3+1, a_3+2\} &\subseteq \{a_2, \ldots, a_5\} \\ &\subseteq \{b_2, \ldots, b_2+5\} \\ &\subseteq A + B . \end{aligned}$$

Thus, $(A+B) + \{0, a_3\} = A + B$. However, this is contrary to

Theorem 2.10.

We now show that $b_1 > 1$. Assume $b_1 = 1$. Then $b_2 + s + 1 \notin A$ and $b_2 + 3 + t \notin A$, for otherwise $b_2 + s + 2 \in A + B$ or $b_2 + 4 + t \in A + B$. Recall $2 \notin A$. Thus, $A \subseteq B$. Furthermore, $A = B$ since $A(I_n) \geq 5$ and $B(I_n) = 5$. First consider $0 \leq s \leq 1$. Then $a_4 = b_4 = b_2 + 1 + t$. Now $b_2+a_4 > n$ since

$$b_2 + a_4 = b_2 + b_2 + 1 + t \geq b_2 + 4 + t$$

and $b_2 + 3 + t$ is the largest element in $A + B$. Thus, if $x \in A + B$ and $x \geq b_2$, then $x + a_4 \notin I_n$ since $x+a_4 \geq b_2+a_4 > n$. If $x \in A + B$ and $x < b_2$ then $x \in \{0, 1, 2\}$, and so

$$x + a_4 \in \{a_4, a_4+1, a_4+2\} = \{b_2+1+t, b_2+2+t, b_2+3+t\} \subseteq A + B.$$

Therefore, $A + B + \{0, a_4\} = A + B$ when $0 \leq s \leq 1$. Next consider $s = 2$. Then

$$B = \{0, 1, b_2, b_2+1, b_2+2, b_2+2+t\}$$

and

$$A + B = \{0, 1, 2, b_2, \ldots, b_2+3, b_2+2+t, b_2+3+t\}.$$

Since $A = B$ then $a_3 = b_3 = b_2 + 1$. Now $a_3+b_2 > n$, for otherwise

$$\{a_3+b_2,\ a_3+b_2+1,\ a_3+b_2+2\} \subseteq A + B.$$

However, this is not possible since $a_3+b_2 = b_2 + 1 + b_2 \geq b_2 + 4$ and $A + B$ contains only two elements greater than or equal to $b_2 + 4$. Thus, if $x \in A + B$ and $x \geq b_2$, then $x + a_3 \notin I_n$ since $x+a_3 \geq b_2+a_3 > n$. If $x \in A + B$ and $x < b_2$ then $x \in \{0, 1, 2\}$, and so

$$x + a_3 \in \{a_3, a_3+1, a_3+2\} = \{b_2+1, b_2+2, b_2+3\} \subseteq A + B.$$

Therefore, $A + B + \{0, a_3\} = A + B$ when $s = 2$. We have a contradiction, for neither $A + B + \{0, a_4\}$ nor $A + B + \{0, a_3\}$ is equal to $A + B$ by Theorem 2.10.

Since $b_1 = 2$ then $3 \in A + B$. Thus, $b_2 = 3$,

$$B = \{0, 2, \ldots, 3+s, 3+s+t, \ldots, 5+t\}$$

and

$$A + B = \{0, \ldots, 4+s, 3+s+t, \ldots, 6+t\}.$$

Since $2 \in B$ and $t > 2$ then $3 + s \notin A$ and $5 + t \notin A$, for otherwise $5 + s \in A + B$ or $7 + t \in A + B$. If $0 \leq s \leq 1$ then $4 + t \in A + B$, but $4 + t \notin A$, for otherwise $4 + t + b_2 = 7 + t \in A + B$. But then $\{2, 3+s, 4+t, 5+t\} \subseteq (A+B)\setminus A$, and so $A(I_n) \leq 4$. If $s = 2$ then $2 + s \notin A$, for otherwise $5 + s = b_2 + (2+s) \in A + B$. However, $5+s < 3+s+t$ since $t > 2$.

Thus $\{2, 2+s, 3+s, 5+t\} \subseteq (A+B) \setminus A$, and so $A(I_n) \leq 4$. Since $A(I_n) \geq 5$, we have a contradiction.

This completes the proof of the theorem.

Theorem 4.3 is used in the proof of Theorem 4.11, but only for $12 \leq n \leq 16$. When $n \leq 14$ the conclusion of Theorem 4.3 can be obtained from a direct application of Lin's Theorem E, for if $A + B + C + D + E = I'_n$ and $n \leq 14$ then

$$A(I_n) + B(I_n) + C(I_n) \leq A(I_n) + B(I_n) + (C+D+E)(I_n)$$
$$\leq (A+B+C+D+E)(I_n) \leq n-1 < 14$$

by Theorem E, and consequently, one of the numbers $A(I_z)$, $B(I_z)$, or $C(I_z)$ is less than five. However, the proof that we give of Theorem 4.3 would not be simplified by assuming $12 \leq n \leq 16$ or even $15 \leq n \leq 16$.

The next theorem is a result in J^4 that is analogous to the result which was just established in J^1 for $n = 15$.

<u>Theorem 4.4</u>. Let $z = (1, 1, 1, 1) \in J^4$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$ then at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

From Theorem 4.1(i) we obtain $(A+B+D)(I_z) \leq 9$, $(A+C+D)(I_z) \leq 9$, and $(B+C+D)(I_z) \leq 9$.

Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, and $e_4 = (0,0,0,1)$. A permutation S_1, S_2, S_3, S_4 of sets A, B, C, D and a permutation i_1, i_2, i_3, i_4 of the integers $1, 2, 3, 4$ must exist such that at least one of the following statements is satisfied:

(i) $e_1 \in S_1$, $e_2 \in S_2$, $e_3 \in S_3$, $e_4 \in S_4$;

(ii) $\{e_{i_1}, e_{i_2}\} \subseteq S_1$, $\{e_{i_3}\} \subseteq S_2$, $\{e_{i_4}\} \subseteq S_3$;

(iii) $\{e_{i_1}, e_{i_2}\} \subseteq S_1$, $\{e_{i_3}, e_{i_4}\} \subseteq S_2$, $\{e_1, e_2, e_3, e_4\} \cap (S_3+S_4)$ is the null set;

(iv) $\{e_{i_1}, e_{i_2}, e_{i_3}\} \subseteq S_1$, $\{e_{i_4}\} \subseteq S_2$, $\{e_{i_1}, e_{i_2}, e_{i_3}\} \cap (S_3+S_4)$ is the null set;

(v) $\{e_1, e_2, e_3, e_4\} \subseteq S_1$, $\{e_1, e_2, e_3, e_4\} \cap (S_2+S_3+S_4\}$ is the null set.

It is not possible that $e_1 \in S_1$, $e_2 \in S_2$, $e_3 \in S_3$, and $e_4 \in S_4$, for otherwise $z \in A + B + C + D$.

Next we claim that (ii) cannot occur. Assume otherwise. Then the sum

$$\{0, e_{i_1}, e_{i_2}\} + \{0, e_{i_3}\} + \{0, e_{i_4}\}$$

contains eleven nonzero elements and is a subset of $S_1 + S_2 + S_3$. Since $(A+B+D)(I_z) \leq 9$, $(A+C+D)(I_z) \leq 9$, and $(B+C+D)(I_z) \leq 9$, then S_1, S_2, S_3 is a permutation of A, B, C. Also, D does not contain $\{e_{i_1}, e_{i_2}\}$, $\{e_{i_3}\}$, or $\{e_{i_4}\}$ as a subset. From

$$\{0, e_{i_1}, e_{i_2}\} + \{0, e_{i_3}\} + \{0, e_{i_4}\} \subsetneq A + B + C,$$

we have

$$D \subsetneq (A+B+C)^{\sim} \subsetneq (\{0, e_{i_1}, e_{i_2}\} + \{0, e_{i_3}\} + \{0, e_{i_4}\})^{\sim}$$

$$= \{0, e_{i_3}, e_{i_4}, e_{i_3}+e_{i_4}\}.$$

Since $e_{i_3} \notin D$, $e_{i_4} \notin D$, and $D(I_z) > 0$, then $D = \{0, e_{i_3}+e_{i_4}\}$, and this in turn implies that $e_{i_1} + e_{i_2} \notin A + B + C$. But then $e_{i_1} + e_{i_2} \notin A + B + C + D$, and this is contrary to $A + B + C + D = I'_z$.

We now show that (iii) is not possible. Assume otherwise. Since $S_1 + S_2 + S_3 + S_4 = I'_z$ then $S_3 + S_4 \subsetneq (S_1+S_2)^{\sim}$. Now

$$\{0, e_{i_1}, e_{i_2}\} + \{0, e_{i_3}, e_{i_4}\} \subsetneq S_1 + S_2$$

implies

$$S_3 + S_4 \subsetneq (S_1+S_2)^{\sim} \subsetneq (\{0, e_{i_1}, e_{i_2}\} + \{0, e_{i_3}, e_{i_4}\})^{\sim}$$

$$= \{0, e_1, e_2, e_3, e_4, e_{i_1}+e_{i_2}, e_{i_3}+e_{i_4}\}$$

Thus, $S_3 + S_4 \subsetneq \{0, e_{i_1}+e_{i_2}, e_{i_3}+e_{i_4}\}$ since $e_j \notin S_3 + S_4$ for

$1 \leq j \leq 4$. This, however, is contrary to $S_3(I_z) \geq 5$ or $S_4(I_z) \geq 5$.

Next we show that (iv) is not possible. Assume otherwise. Since $z \notin A + B + C + D$ and

$$\{0, e_{i_1}, e_{i_2}, e_{i_3}\} + \{0, e_{i_4}\} \subseteq S_1 + S_2$$

then

$$\{e_i+e_j+e_k \mid 1 \leq i < j < k \leq 4\} \cup \{e_{i_1}+e_{i_3}, e_{i_1}+e_{i_2}, e_{i_2}+e_{i_3}, z\}$$

has an empty intersection with $S_3 + S_4$. Also, $e_{i_j} \notin S_3 + S_4$ for $1 \leq j \leq 3$. But then $(S_3+S_4)(I_z) \leq 4$ and this is contrary to $S_3(I_z) \geq 5$ or $S_4(I_z) \geq 5$.

Finally, we show that (v) cannot occur. Assume otherwise. Since $z \notin A + B + C + D$ and $e_t \in S_1$, $1 \leq t \leq 4$, then $e_i + e_j + e_k \notin S_2 + S_3 + S_4$, $1 \leq i < j < k \leq 4$. Moreover, $e_i \notin S_2 + S_3 + S_4$, $1 \leq i \leq 4$. Thus

$$S_2 + S_3 + S_4 \subseteq \{e_i+e_j \mid 1 \leq i < j \leq 4\} \cup \{0\}.$$

Now $S_4(I_z) > 0$. Say $e_u + e_v \in S_4$ where $1 \leq u < v \leq 4$. Then $z \notin A + B + C + D$ implies $e_x + e_y \notin S_1 + S_2 + S_3$ where $\{x, y\} = \{1, 2, 3, 4\} \setminus \{u, v\}$. Since $e_x + e_y \notin S_2 + S_3$ and either $S_2(I_z) \geq 5$ or $S_3(I_z) \geq 5$, it follows that

$$S_2 + S_3 = \{0\} \cup \{e_i + e_j \mid 1 \leq i < j \leq 4, \{i, j\} \neq \{x, y\}\}.$$

Hence $e_u + e_v \in S_2 + S_3$, and this in turn implies that $e_x + e_y \notin S_4$. But then $e_x + e_y \notin S_1 + S_2 + S_3$, $e_x \notin S_4$, $e_y \notin S_4$, and $e_x + e_y \notin S_4$ imply that $e_x + e_y \notin A + B + C + D = I'_z$.

We conclude that $A(I_z)$, $B(I_z)$, or $C(I_z)$ is less than five, and the proof is complete.

The next five theorems consist of results in J^m where $2 \leq m \leq 3$ which are analogous to the results obtained in Theorems 4.3 and 4.4 for J^1 and J^4, respectively. Since the proofs of these five theorems are long and involve techniques similar to those used in the proof of Theorem 4.3, we have placed them in the Appendices to the thesis.

<u>Theorem 4.5.</u> Let $z = (1, 3, 1) \in J^3$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$, then at least one of the sets A, B, and C has less than five nonzero elements.

The proof of Theorem 4.5 is given in Appendix I.

<u>Theorem 4.6.</u> Let $z = (7, 1) \in J^2$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$, then at least one of the sets A, B, and C has less than five nonzero elements.

The proof of Theorem 4.6 is given in Appendix II.

<u>Theorem 4.7.</u> Let $z = (3, 3) \in J^2$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$, then at least one of the sets

A, B, and C has less than five nonzero elements.

The proof of Theorem 4.7 is given in Appendix III.

Theorem 4.8. Let $z = (4, 2) \in J^2$ and let $A, B, C \subseteq I_z$. If $A + B + C = I'_z$ then at least one of the sets A, B, and C has less than five nonzero elements.

The proof of Theorem 4.8 is given in Appendix IV.

Theorem 4.9. Let $z = (6, 1) \in J^2$ and let $A, B, C \subseteq I_z$. If $A + B + C = I'_z$ then at least one of the sets A, B, and C has less than five nonzero elements.

The proof of Theorem 4.9 is given in Appendix V.

Lemma 4.10. Let $z = (z_1, z_2, \ldots, z_m) \in J^m$, $z > 0$, be specified. Let $T = \{i | z_i > 0 \text{ and } 1 \leq i \leq m\}$. Denote the cardinality of T by k and let f represent a bijective function from $\{1, 2, \ldots, k\}$ to T. To each $x = (x_1, x_2, \ldots, x_m)$ in I_z correspond the element $x^* = (x_{f(1)}, x_{f(2)}, \ldots, x_{f(k)})$ in J^k. For $S \subseteq I_z$ let $S^* = \{x^* | x \in S\}$. Then

(i) $J^m(I_z) = J^k(I_{z^*})$,

(ii) $S^* \subseteq I_{z^*}$ and $S^*(I_{z^*}) = S(I_z)$,

and

(iii) $\sum_{1 \leq i \leq n} S_i^* = I'_{z^*}$ if $S_1, S_2, \ldots, S_n$ are subsets of I_z for which $\sum_{1 \leq i \leq n} S_i = I'_z$.

Proof. Let $x = (x_1, x_2, \ldots, x_m) \in I_z$. Then $x_i \leq z_i$ for $1 \leq i \leq m$; in particular, $x_{f(i)} \leq z_{f(i)}$ for $1 \leq i \leq k$. Thus $x^* \leq z^*$ and $x^* \in I_{z^*}$. If $y = (y_1, y_2, \ldots, y_m) \in I_z$ and $y \neq x$ then $y_i \neq x_i$ for some $i \in T$, and so $y^* \neq x^*$ since $y_{f(f^{-1}(i))} \neq x_{f(f^{-1}(i))}$. Let $w = (w_1, w_2, \ldots, w_k) \in I_{z^*}$. Hence, $w_j \leq z_{f(j)}$ for $j = 1, 2, \ldots, k$. Define $v_i = w_{f^{-1}(i)}$ if $i \in T$ or $v_i = 0$ if $i \notin T$ and $1 \leq i \leq m$. Then $v = (v_1, v_2, \ldots, v_m)$ is in I_z and $v^* = w$. The above observations show that the correspondence $x \to x^*$ is a one to one correspondence from I_z onto I_{z^*}. Note that 0 in J^m corresponds to $0 \in J^k$. Thus, if $S \subseteq I_z$ then $S^* \subseteq I_{z^*}$ and $S(I_z) = S^*(I_{z^*})$. In particular $J^m(I_z) = J^k(I_{z^*})$. Also, $(I'_z)^* = I'_{z^*}$.

Let $S_1, S_2, \ldots, S_n$ be subsets of I_z for which $\sum_{1 \leq i \leq n} S_i = I'_z$. Since $(x+y)^* = x^* + y^*$ for $x, y \in I_z$, it follows that

$$I'_{z^*} = (I'_z)^* = (\sum_{1 \leq i \leq n} S_i)^* = \sum_{1 \leq i \leq n} S_i^*.$$

The following theorem together with Theorem 3.2 is used to prove Theorem 4.12.

Theorem 4.11. Let $z \in J^m$, $z > 0$, be specified and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ and either

(i) $k = 3$ and $J^m(I_z) \leq 14$,

(ii) $k = 4$, $J^m(I_z) = 15$, and $A_i(I_z) > 0$ for $i = 1, 2, 3, 4$,

or

(iii) $k = 5$, $J^m(I_z) = 16$, and $A_i(I_z) > 0$ for $i = 1, 2, 3, 4, 5$,

then at least $k-2$ of the sets $A_1, A_2, \ldots, A_k$ contain less than five nonzero elements.

Proof. Let $z = (z_1, z_2, \ldots, z_m)$ and let $n = J^m(I_z)$. Thus $n = (\prod_{1 \leq i \leq m} (z_i+1)) - 1$. Note that $n > 0$ since $z > 0$.

The theorem is immediate when $1 \leq n \leq 5$.

Let n satisfy $6 \leq n \leq 11$. Then $k = 3$. Assume $A_1(I_z) \geq 5$, $A_2(I_z) \geq 5$, and $A_3(I_z) \geq 5$. From Theorem 4.1(i) we have

$$(A_1+A_2)(I_z) \leq n - 1 - A_3(I_z) \leq 5.$$

Since $A_1 \subseteq A_1 + A_2$, $A_1(I_z) \geq 5$, and $(A_1+A_2)(I_z) \leq 5$, it follows that $A_1 = A_1 + A_2$. Let $x \in A_2$ and $x \neq 0$. Then

$$A_1 + A_2 \subseteq (A_1+A_2) + \{0, x\} = A_1 + \{0, x\} \subseteq A_1 + A_2,$$

and so $A_1 + A_2 + \{0, x\} = A_1 + A_2$. However, this is contrary to Theorem 2.10. We conclude that either $A_1(I_z)$, $A_2(I_z)$, or $A_3(I_z)$ is less than five. This establishes the theorem when $6 \leq n \leq 11$.

Henceforth let $12 \leq n \leq 16$. When $n = 15$ then $k = 4$ and the theorem follows immediately if one member in each of the sets $\{A_1, A_2, A_3\}$, $\{A_1, A_2, A_4\}$, $\{A_1, A_3, A_4\}$, and $\{A_2, A_3, A_4\}$ contains less than five nonzero elements of J^m. Also, when $n = 16$ then $k = 5$ and the theorem follows immediately if one member in each of the sets $\{A_1, A_2, A_3\}$, $\{A_1, A_2, A_4\}$, $\{A_1, A_2, A_5\}$, $\{A_1, A_3, A_4\}$, $\{A_1, A_3, A_5\}$, $\{A_1, A_4, A_5\}$, $\{A_2, A_3, A_4\}$, $\{A_2, A_3, A_5\}$, $\{A_2, A_4, A_5\}$, and $\{A_3, A_4, A_5\}$ contains less than five nonzero elements of J^m. Since the conditions on the sets $A_1, A_2, \ldots, A_k$ are symmetrical, then to prove the theorem when $n = 15$ and $n = 16$ it is sufficient just as for $12 \leq n \leq 14$ to show that $A_1(I_z)$, $A_2(I_z)$, or $A_3(I_z)$ is less than five.

Since $12 \leq n \leq 16$, then $13 \leq \prod_{1 \leq i \leq m} (z_i + 1) \leq 17$. Since the only unordered factorizations of 17, 16, 15, 14, or 13 into a product of at least two integers greater than one are $2 \cdot 2 \cdot 2 \cdot 2$, $4 \cdot 2 \cdot 2$, $8 \cdot 2$, $4 \cdot 4$, $5 \cdot 3$, and $7 \cdot 2$, then one of the following occurs:

(a) $m \geq 4$, $n = 15$, and there exist indices i_1, i_2, i_3 and i_4 such that $z_{i_1} = z_{i_2} = z_{i_3} = z_{i_4} = 1$ and $z_i = 0$ when $1 \leq i \leq m$ and $i \notin \{i_1, i_2, i_3, i_4\}$.

(b) $m \geq 3$, $n = 15$, and there exist distinct indices i_1, i_2, i_3

such that $z_{i_1} = 3$, $z_{i_2} = z_{i_3} = 1$, and $z_i = 0$ when $1 \leq i \leq m$ and $i \notin \{i_1, i_2, i_3\}$.

(c) $m \geq 2$, $n = 15$, and there exist distinct indices i_1 and i_2 such that $\{z_{i_1}, z_{i_2}\}$ is equal to $\{7, 1\}$ or $\{3, 3\}$ and $z_i = 0$ when $1 \leq i \leq m$ and $i \notin \{i_1, i_2\}$.

(d) $m \geq 2$, $n = 14$, and there exist distinct indices i_1 and i_2 such that $z_{i_1} = 4$, $z_{i_2} = 2$, and $z_i = 0$ when $1 \leq i \leq m$ and $i \notin \{i_1, i_2\}$.

(e) $m \geq 2$, $n = 13$, and there exist distinct indices i_1 and i_2 such that $z_{i_1} = 6$, $z_{i_2} = 1$, and $z_i = 0$ when $1 \leq i \leq m$ and $i \notin \{i_1, i_2\}$.

(f) $m \geq 1$, $12 \leq n \leq 16$, and there exists an index i_1 such that $z_{i_1} = n$ and $z_i = 0$ when $1 \leq i \leq m$ and $i \neq i_1$.

In view of Lemma 4.10, to prove the theorem it is sufficient to consider the following cases:

(1) $z = (1, 1, 1, 1)$,

(2) $z = (1, 3, 1)$,

(3) $z = (7, 1)$,

(4) $z = (3, 3)$,

(5) $z = (4, 2)$,

(6) $z = (6, 1)$,

(7) $12 \leq z \leq 16$.

Now $J^m(I_z) = 15$ in cases (1) through (4); hence, $k = 4$ and

$A_i(I_z) > 0$ for $i = 1, 2, 3, 4$. In cases (5) and (6) we have $J^m(I_z) \leq 14$, and so $k = 3$. Applying Theorems 4.4 through 4.9 in cases (1) through (6) respectively, we have that $A_1(I_z)$, $A_2(I_z)$, or $A_3(I_z)$ is less than five.

Consider case (7). When $z = 16$ then $k = 5$ and $A_i(I_{16}) > 0$ for $i = 1, 2, 3, 4, 5$. When $z = 15$ then $k = 4$ and $A_i(I_{15}) > 0$ for $i = 1, 2, 3, 4$. When $12 \leq z \leq 14$ then $k = 3$. Define $A_5 = \{0\}$ when $z = 15$ and $A_4 = A_5 = \{0\}$ when $12 \leq z \leq 14$. Then applying Theorem 4.3 we obtain that $A_1(I_z)$, $A_2(I_z)$, or $A_3(I_z)$ is less than five.

The proof of the theorem is complete.

<u>Theorem 4.12</u>. Let $z \in J^m$, $z > 0$, be specified. Let $k \geq 3$ and let $A_i \subseteq I_z$ for $i = 1, 2, \ldots, k$. If $\sum_{1 \leq i \leq k} A_i = I'_z$ and either

(i) $J^m(I_z) \leq 14$,

(ii) $J^m(I_z) = 15$, $k \geq 4$, and $A_i(I_z) > 0$ for $i = 1, 2, 3, 4$,

or

(iii) $J^m(I_z) = 16$, $k \geq 5$, and $A_i(I_z) > 0$ for $i = 1, 2, 3, 4, 5$,

then

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \Big(\sum_{1 \leq i \leq k} A_i \Big)(I_z).$$

Proof. We define $r = 3$ if $J^m(I_z) \leq 14$, $r = 4$ if $J^m(I_z) = 15$, and $r = 5$ if $J^m(I_z) = 16$.

Assume that at least three of the sets $A_1, A_2, \ldots, A_k$ contain five or more nonzero elements. By relabeling sets $A_1, A_2, \ldots, A_k$ if necessary, we may assume $A_1(I_z) \geq 5$, $A_2(I_z) \geq 5$, $A_3(I_z) \geq 5$, and $A_i(I_z) \geq 0$ for $i = 1, 2, \ldots, r$. Define $B_i = A_i$ for $i = 1, 2, \ldots, r-1$, and $B_r = \sum_{r \leq i \leq k} A_i$. Since $\sum_{1 \leq i \leq r} B_i = \sum_{1 \leq i \leq k} A_i = I'_z$ and $B_i(I_z) > 0$ for $i = 1, 2, \ldots, r$, then from Theorem 4.11 it follows that at least $r-2$ of the sets $B_1, B_2, \ldots, B_r$ contain less than five nonzero elements. However, since $B_1 = A_1$, $B_2 = A_2$, and $B_3 \supseteq A_3$ then $B_i(I_z) \geq 5$ for $i = 1, 2, 3$, and we have a contradiction.

Since $k-2$ of the sets $A_1, A_2, \ldots, A_k$ contain less than five nonzero elements, then an application of Theorem 3.2 gives

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_z)$$

as claimed.

In the special case when $m = 1$ and $k = 3$, Theorem 4.12(i) is Lin's Theorem E. In our proof of Theorem E we use Theorem 3.2 with $m = 1$ and $k = 3$ where Lin uses Theorem D in his proof.

For each integer $n \geq 15$ Lin has constructed three sets of nonnegative integers A_1, A_2, and A_3 satisfying $\{0,\ldots,n-1\} \subseteq A_1 + A_2 + A_3$ and $n \notin A_1 + A_2 + A_3$, and for which the inequality in Theorem 4.12 fails. With $n = 15+j$, $j \geq 0$, the sets defined by Lin are

$$A_1 = \{0, 1, 8+j, 10+j, 12+j, 14+j\}$$

$$A_2 = \{0, 2, 8, \ldots, 9+j, 12+j, 13+j\}$$

and

$$A_3 = \{0, 4, 8+j, 9+j, 10+j, 11+j\}.$$

For each integer $n \geq 16$, Allen Freedman has constructed four sets of nonnegative integers A_1, A_2, A_3, and A_4 satisfying $\{0,\ldots,n-1\} \subseteq A_1 + A_2 + A_3 + A_4$, $n \notin A_1 + A_2 + A_3 + A_4$, and $A_i(I_n) > 0$ for $i = 1, 2, 3, 4$, and for which the inequality of Theorem 4.12 fails. With $n = 16+j$, $j \geq 0$, the sets defined by Freedman are

$$A_1 = \{0, 1, 9+j, 11+j, 13+j, 15+j\}$$

$$A_2 = \{0, 2, 9+j, 10+j, 13+j, 14+j\}$$

$$A_3 = \{0, 4, 9+j, 10+j, 11+j, 12+j\}$$

and

$$A_4 = \{0, 8, \ldots, 8+j\}.$$

The constructions given in the preceding two paragraphs show that it is not possible to obtain results analogous to parts (i) and (ii) of Theorem 4.12 by increasing the values of $J^m(I_z)$.

It is not possible to delete from the hypotheses of Theorem 4.12(ii) the condition that four of the sets have nonzero elements, for the inequality of Theorem 4.12 fails when $A_4 = \{0\}$ and A_1, A_2 and A_3 are the sets determined by Lin's construction when $n = 15$. Also, it is not possible to delete from the hypotheses of Theorem 4.12(iii) the condition that five of the sets have nonzero elements, for the inequality of Theorem 4.12 fails when $A_5 = \{0\}$ and A_1, A_2, A_3, and A_4 are the sets determined by Freedman's construction when $n = 16$.

We now give a construction of $k \geq 3$ sets in J^2 for which $J^2(I_z) = 15+2j$, $j \geq 0$, and the inequality in Theorem 4.12 fails. Let $z = (7+j, 1)$,

$$A_1 = \{(0,0), (1,0), (4,0), \ldots, (4+j,0), (6+j,0), (4+j,1), (6+j,1)\},$$

$$A_2 = \{(0,0), (2,0), (4+j,0), (5+j,0), (4+j,1), (5+j,1)\},$$

$$A_3 = \{(0,0)(0,1), (4,0), \ldots, (7+j,0)\},$$

and $A_i = \{(0,0\}$ for $i = 4, \ldots, k$. Then $\sum_{1 \leq i \leq k} A_i = I_z'$; however,

$$\sum_{1 \leq i \leq k} A_i(I_z) = 15+2j > 14+2j = (\sum_{1 \leq i \leq k} A_i)(I_z).$$

Since $J^m(I_z)+1 = \prod_{1 \leq i \leq m} (z_i+1)$ where $z = (z_1, z_2, \ldots, z_m)$, then I_z is isomorphic to a set in J^1 whenever $J^m(I_z) = 16$. Hence, any example to illustrate that Theorem 4.12(ii) is not valid when $J^m(I_z) = 16$ would be essentially a one-dimensional example. However, when $J^m(I_z) = 17+2j$ where $j \geq 0$, we have the following example in J^2. Let $z = (8+j, 1)$,

$$A_1 = \{(0,0), (1,0), (5+j,0), (7+j,0), (5+j,1), (7+j,1)\},$$

$$A_2 = \{(0,0), (2,0), (5+j,0), (6+j,0), (5+j,1), (6+j,1)\},$$

$$A_3 = \{(0,0), (0,1), (4,0), \ldots, (8+j,0)\},$$

$$A_4 = \{(0,0), (4,0), \ldots, (4+j,0)\},$$

and $A_i = \{(0,0)\}$ for $i = 5, \ldots, k$. Then $\sum_{1 \leq i \leq k} A_i = I'_z$, but

$$\sum_{1 \leq i \leq k} A_i(I_z) = 17+2j > 16+2j = (\sum_{1 \leq i \leq k} A_i)(I_z).$$

We do not know if Theorem 4.12(iii) can be improved by increasing the value of $J^m(I_z)$. Our methods appear to be very long and involved, and it would be desirable to have more powerful techniques before further investigating this problem. For instance, to prove a result analogous to Theorem 4.12(iii) by our methods with $J^m(I_z) = 17$ would require showing that at least one of any three of the sets A_1, A_2, A_3, A_4, and A_5 contains less than five nonzero elements when $\sum_{1 \leq i \leq 5} A_i = I'_z$ for the cases $z = 17$, $z = (2,2,1)$, $z = (8,1)$, and $z = (5,2)$. It seems likely that longer and more difficult arguments than those used to prove Theorems 4.3 through

4.9 would be needed.

It may be possible to obtain a stronger result than Theorem 4.12 for $k = 6$ and other larger values of k. For example, if $k \geq r$ and $A_i(I_z) > 0$ for $i = 1, 2, \ldots, r$, where r is some integer greater than five, it may be possible to obtain larger values of $J^m(I_z)$ for which the inequality of the theorem is valid. In Theorem 5.17 from the next chapter with $r \geq 4$ and $z \geq 8(r-2)$ we give a construction due to Allen Freedman of r sets in J^1 for which $\sum_{1 \leq i \leq r} A_i = I'_z$, $A_i(I_z) > 0$ for $i = 1, 2, \ldots, r$, and $\sum_{1 \leq i \leq r} A_i(I_z) > (\sum_{1 \leq i \leq r} A_i)(I_z)$. This construction shows that $8(r-2) - 1$ would be an upper bound for $J^m(I_z)$ in any result analogous to Theorem 4.12 for k sets if $k \geq r$, $r \geq 4$, and $A_i(I_z) > 0$ for $i = 1, 2, \ldots, r$.

V. FOUR NUMERICAL FUNCTIONS

We begin by defining four related numerical functions, and in Theorem 5.12 we evaluate one of them. We obtain evaluations of the others on certain subsets of their domains by applying Theorem 4.12 in addition to Lin's Theorem F and a result by Allen Freedman which extends Theorem F to k sets in J^1. Two important theorems regarding one of these functions for large values of an argument are stated at the end of the chapter; the first theorem is due to P. Erdös and P. Scherk and the other theorem is due to H. Kemperman. We show that these theorems also apply to one of the other functions. Furthermore, we show that an extension of Theorem G to k sets in J^1 can be obtained from the theorem of Erdös and Scherk.

We no longer specify a point $z \in J^m$ and restrict our consideration only to subsets of I_z, and so Definition 2.7 no longer applies. Throughout this chapter, when $A_1, A_2, \ldots, A_k$ are subsets of J^m then $A_1 + A_2 + \ldots + A_k$ or $\sum_{1 \leq i \leq k} A_i$ denotes the sum set $\{\sum_{1 \leq i \leq k} a_i \mid a_i \in A_i\}$.

<u>Definition 5.1.</u> With n a positive integer then

$$J_n^m = \{z \mid z \in J^m \text{ and } J^m(I_z) = n\}.$$

<u>Definition 5.2.</u> The set of all subsets of J^m which

contain 0 is denoted by $\mathcal{P}(J^m)$. When k is an integer, $k \geq 2$, then $\mathcal{P}^k(J^m)$ denotes the Cartesian product of k copies of $\mathcal{P}(J^m)$.

Definition 5.3. For each $z \in J^m$, $z > 0$, and each integer $k \geq 2$ let $\mathcal{F}_{z,k}$, $\mathcal{G}_{z,k}$, and $\mathcal{H}_{z,k}$ represent the following subsets of $\mathcal{P}^k(J^m)$:

(i) $\mathcal{F}_{z,k} = \{(A_1, A_2, \ldots, A_k) \in \mathcal{P}^k(J^m) \mid z \notin \sum_{1 \leq i \leq k} A_i\}$,

(ii) $\mathcal{G}_{z,k} = \{(A_1, A_2, \ldots, A_k) \in \mathcal{P}^k(J^m) \mid I'_z \subseteq \sum_{1 \leq i \leq k} A_i$

and $z \notin \sum_{1 \leq i \leq k} A_i\}$,

(iii) $\mathcal{H}_{z,k} = \{(A_1, A_2, \ldots, A_k) \in \mathcal{P}^k(J^m) \mid I'_z \subseteq \sum_{1 \leq i \leq k} A_i,\ z \notin \sum_{1 \leq i \leq k} A_i$,

and $A_i(I_z) > 0$

for $i = 1, 2, \ldots, k\}$.

The sets $\mathcal{F}_{z,k}$ and $\mathcal{G}_{z,k}$ are not empty for each $z \in J^m$, $z > 0$, and each integer $k \geq 2$ since both sets contain $(A_1, A_2, \ldots, A_k)$ where $A_1 = I'_z$ and $A_i = \{0\}$ for $i = 2, \ldots, k$. Note that

$$\mathcal{H}_{z,k} \subseteq \mathcal{G}_{z,k} \subseteq \mathcal{F}_{z,k}.$$

Also, if $(A_1, A_2, \ldots, A_k)$ is in $\mathcal{H}_{z,k}$, $\mathcal{U}_{z,k}$, or $\mathcal{Q}_{z,k}$ then $0 \in \bigcap_{i=1}^{k} A_i$ and $z \notin \sum_{1 \le i \le k} A_i$, and so $z \notin A_i$ for $i = 1, 2, \ldots, k$.

Definition 5.4. For integers $k \geq 2$, $m \geq 1$, and $n \geq 1$ let

$$f(k, m, n) = \max\{ \sum_{1 \le i \le k} A_i(I_z) \mid z \in J_n^m \text{ and } (A_1, A_2, \ldots, A_k) \in \mathcal{Q}_{z,k}\}.$$

Definition 5.5. For integers $k \geq 2$, $m \geq 1$, and $n \geq 1$ let

$$g(k, m, n) = \max\{ \sum_{1 \le i \le k} A_i(I_z) \mid z \in J_n^m \text{ and } (A_1, A_2, \ldots, A_k) \in \mathcal{U}_{z,k}\}.$$

Definition 5.6. For integers $k \geq 2$, $m \geq 1$, and $n \geq 1$ let

$$h(k, m, n) = \max\{ \sum_{1 \le i \le k} A_i(I_z) \mid z \in J_n^m \text{ and } (A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}\}$$

if there is an element $z \in J_n^m$ for which $\mathcal{H}_{z,k} \neq \phi$; otherwise, $h(k, m, n) = 0$.

Definition 5.7. For integers $k \geq 2$ and $m \geq 1$ let $s(k, m) = \text{lub}\{n \mid h(k, m, i) \leq i-1 \text{ and } k < i \leq n\}$ if there is an integer n, $n > k$, for which $h(k, m, i) \leq i-1$ when $k < i \leq n$; otherwise, $s(k, m) = 0$.

The next four theorems consist of result which follow directly from the definitions of the functions f, g, h, and s.

Theorem 5.8. If $k \geq 2$, $m \geq 1$, and $n \geq 1$, then

$$f(k,m,n) \geq g(k,m,n) \geq h(k,m,n) \geq 0.$$

Proof. Let k, m, and n be given integers such that $k \geq 2$, $m \geq 1$, and $n \geq 1$. Recall that $\mathcal{H}_{z,k} \subseteq \mathcal{G}_{z,k} \subseteq \mathcal{Q}_{z,k}$ for each $z \in J^m$, $z > 0$. If there exists an element $z' \in J_n^m$ such that $\mathcal{H}_{z',k} \neq \phi$, then the conclusion follows from the definitions of f, g, and h. If $\mathcal{H}_{z,k} = \phi$ for each $z \in J_n^m$, then $h(k,m,n) = 0$ and the conclusion again follows.

Theorem 5.9. If $k \geq 2$ and $m \geq 1$, then

(i) $g(k,m,n) \geq n-1$ when $n \geq 1$,

(ii) $h(k,m,n) \geq n-1$ when $n > k$,

and

(iii) $h(k,m,n) = 0$ or $h(k,m,n) \geq k$ when $1 \leq n \leq k$.

Proof. Let k, m, and n be given integers such that $k \geq 2$, $m \geq 1$, and $n \geq 1$. For each $z \in J_n^m$ the set $\mathcal{G}_{z,k}$ contains $(A_1, A_2, \ldots, A_k)$ where $A_1 = I'_z$ and $A_i = \{0\}$ for $i = 2, \ldots, k$. Since $\sum_{1 \leq i \leq k} A_i(I_z) = A_1(I_z) = n-1$ then $g(k,m,n) \geq n-1$.

Now restrict n so that $n > k$ and let $z = (z_1, z_2, \ldots, z_m)$ where $z_1 = n$ and $z_i = 0$ for $i = 2, \ldots, m$. Thus, $z \in J_n^m$. Set

$$A_1 = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_1 \leq n-k \text{ and } x_j = 0 \text{ for } j = 2,\ldots,m\}$$

and

$$A_i = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_1 \leq 1 \text{ and } x_j = 0 \text{ for } j = 2,\ldots,m\}$$

for $i = 2, \ldots, k$. Then $A_i(I_z) > 0$ for $i = 1, 2, \ldots, k$ and $\sum_{1 \leq i \leq k} A_i = I'_z$, and so $(A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}$. Since $\sum_{1 \leq i \leq k} A_i(I_z) = n-1$, then $h(k,m,n) \geq n-1$.

Part (iii) follows directly from the definition of the function h.

<u>Theorem 5.10.</u> If $k \geq 2$, $n \geq 1$, and $m_1 > m_2 \geq 1$, then

(i) $f(k, m_1, n) \geq f(k, m_2, n)$

(ii) $g(k, m_1, n) \geq g(k, m_2, n)$

(iii) $h(k, m_1, n) \geq h(k, m_2, n)$

and

(iv) $s(k, m_1) \leq s(k, m_2)$.

Proof Let k, n, m_1, and m_2 be given integers such that $k \geq 2$, $n \geq 1$, and $m_1 > m_2 \geq 1$. To each $x = (u_1, u_2, \ldots, u_{m_2})$ in J^{m_2} we correspond the point $x^* = (v_1, v_2, \ldots, v_{m_1})$ in J^{m_1} where $v_i = u_i$ for $1 \leq i \leq m_2$ and $v_i = 0$ for $m_2 < i \leq m_1$. To each set A contained in J_{m_2} we correspond the set $A^* = \{x^* \mid x \in A\}$ contained in J_{m_1}.

Let $z \in J_n^{m_2}$ and $(A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}$. Since $A_i^*(I_{z*}) = A_i(I_z)$ for $i = 1, 2, \ldots, k$ and $\sum_{1 \leq i \leq k} A_i^* = (\sum_{1 \leq i \leq k} A_i)^*$, it follows that $(A_1^*, A_2^*, \ldots, A_k^*) \in \mathcal{H}_{z*,k}$. Also, $z* \in J_n^{m_1}$. Thus,

$$\{\sum_{1 \leq i \leq k} A_i(I_z) \mid z \in J_n^{m_2} \text{ and } (A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}\}$$

$$= \{\sum_{1 \leq i \leq k} A_i^*(I_{z*}) \mid z \in J_n^{m_2} \text{ and } (A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}\}$$

$$\subseteq \{\sum_{1 \leq i \leq k} B_i(I_w) \mid w \in J_n^{m_1} \text{ and } (B_1, B_2, \ldots, B_k) \in \mathcal{H}_{w,k}\}.$$

Therefore, $h(k, m_1, n) \geq h(k, m_2, n)$.

Replacing $\mathcal{H}_{z,k}$ by $\mathcal{Q}_{z,k}$ and $\mathcal{A}_{z,k}$ in the above argument, we obtain parts (i) and (ii), respectively.

If $h(k, m_1, i) \leq i-1$ for $k < i \leq n$, then $h(k, m_2, i) \leq i-1$ for $k < i \leq n$ when $m_2 < m_1$ by part (iii). Hence, $s(k, m_2) \geq s(k, m_1)$ follows from the definition of the function s.

<u>Theorem 5.11.</u> If $m \geq 1$, $n \geq 1$, and $k_1 > k_2 \geq 2$, then $f(k_1, m, n) \geq f(k_2, m, n)$ and $g(k_1, m, n) \geq g(k_2, m, n)$.

Proof. Let m, n, k_1, and k_2 be given integers such that

$m \geq 1$, $n \geq 1$, and $k_1 > k_2 \geq 2$. Let $z \in J_n^m$. To $(A_1, A_2, \ldots, A_{k_2}) \in \mathfrak{A}_{z,k_2}$ we correspond $(A_1^*, A_2^*, \ldots, A_{k_1}^*)$ where $A_i^* = A_i$ for $i = 1, 2, \ldots, k_2$ and $A_i^* = \{0\}$ for $i = k_2+1, \ldots, k_1$. Then $(A_1^*, A_2^*, \ldots, A_{k_1}^*) \in \mathfrak{A}_{z,k_1}$ since $\sum_{1 \leq i \leq k_1} A_i^* = \sum_{1 \leq i \leq k_2} A_i$. Also, $\sum_{1 \leq i \leq k_1} A_i^*(I_z) = \sum_{1 \leq i \leq k_2} A_i(I_z)$. Thus,

$$\{\sum_{1 \leq i \leq k_2} A_i(I_z) \mid z \in J_n^m \text{ and } (A_1, A_2, \ldots, A_{k_2}) \in \mathfrak{A}_{z,k_2}\}$$

$$= \{\sum_{1 \leq i \leq k_1} A_i^*(I_z) \mid z \in J_n^m \text{ and } (A_1, A_2, \ldots, A_{k_2}) \in \mathfrak{A}_{z,k_2}\}$$

$$\subseteq \{\sum_{1 \leq i \leq k_1} B_i(I_z) \mid z \in J_n^m \text{ and } (B_1, B_2, \ldots, B_{k_1}) \in \mathfrak{A}_{z,k_1}\},$$

which gives $g(k_1, m, n) \geq g(k_2, m, n)$.

The above argument with $\mathfrak{A}_{z,k_i}$ replaced by $\mathfrak{F}_{z,k_i}$ for $1 \leq i \leq 2$ shows $f(k_1, m, n) \geq f(k_2, m, n)$.

The function h is neither increasing nor decreasing with respect to the first variable for we show in Theorems 5.16 and 5.18 that $h(2, m, 15) = 14$, $h(3, m, 15) \geq 15$, and $h(4, m, 15) = 14$ for $m \geq 1$.

Erdős and Sherk [2, p. 45] have exhibited an upper bound for

$f(k, 1, n)$ when $k \geq 2$ and $n \geq 1$; namely, $f(k, 1, n) \leq k(n-1)/2$. When n is an odd integer equality holds since $(A_1, A_2, \ldots, A_k) \in \mathcal{A}_{n,k}$ where $A_i = \{0, \frac{n+1}{2}, \ldots, n-1\}$, $1 \leq i \leq k$ and $\sum_{1 \leq i \leq k} A_i(I_n) = k(n-1)/2$. Using a method of proof which differs from the method used by Erdös and Scherk, we now proceed to evaluate $f(k, m, n)$ for $k \geq 2$, $m \geq 1$, and $n \geq 1$.

<u>Theorem 5.12.</u> For $k \geq 2$, $m \geq 1$, and $n \geq 1$ then

$$f(k, m, n) = \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let k, m, and n be given integers such that $k \geq 2$, $m \geq 1$, and $n \geq 1$.

We first determine a lower bound for $f(k, m, n)$. Let $z = (z_1, z_2, \ldots, z_m)$ where $z_1 = n$ and $z_i = 0$, $2 \leq i \leq m$. Hence, $z \in J_n^m$. When n is odd define

$$A_i = \{(x_1, x_2, \ldots, x_m) \mid x_1 = 0 \text{ or } \frac{n+1}{2} \leq x_1 < n, \text{ and } x_j = 0 \text{ for } 2 \leq j \leq m\}$$

for $i = 1, 2, \ldots, k$. Then $(A_1, A_2, \ldots, A_k) \in \mathcal{A}_{z,k}$ since $z \notin \sum_{1 \leq i \leq k} A_i$, and consequently,

$$f(k,m,n) \geq \sum_{1\leq i\leq k} A_i(I_z) = k(n-1)/2.$$

When n is even define

$$B_1 = \{(x_1,x_2,\ldots,x_m) \mid x_1 = 0 \text{ or } \tfrac{n}{2} \leq x_1 < n, \text{ and } x_j = 0 \text{ for } 2 \leq j \leq m\},$$

and for $i = 2,\ldots,k$ let

$$B_i = \{(x_1,x_2,\ldots,x_m) \mid x_1 = 0 \text{ or } \tfrac{n}{2}+1 \leq x_1 < n, \text{ and } x_j = 0 \text{ for } 2 \leq j \leq m\}.$$

Since $z \notin \sum_{1\leq i\leq k} B_i$ then $(B_1, B_2, \ldots, B_k) \in \mathcal{A}_{z,k}$, and so

$$f(k,m,n) \geq \sum_{1\leq i\leq k} B_i(I_z) = n/2 + (k-1)(n/2-1).$$

Therefore,

$$(5.1) \qquad f(k,m,n) \geq \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

We next determine an upper bound for $f(k,m,n)$. Consider any $z \in J_n^m$ and any $(A_1, A_2, \ldots, A_k) \in \mathcal{A}_{z,k}$.

Case 1. $A_i(I_z) < n/2$ for $i = 1, 2, \ldots, k$. In this case

$$(5.2) \qquad \sum_{1\leq i\leq k} A_i(I_z) \leq \begin{cases} k(n-1)/2 = kn/2 - k/2 & \text{if } n \text{ is odd,} \\ k(n/2-1) < kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

Case 2. $A_j(I_z) \geq n/2$ for some j, $1 \leq j \leq k$.

Let $t = \max\{A_i(I_z) \mid i = 1, 2, \ldots, k\}$, and assume $A_j(I_z) = t$.

If $x \in A_j \cap I_z$ and $x > 0$ then $0 < x < z$, or equivalently, $0 < z-x < z$. Hence, $z - x \in I'_z$ and $z-x > 0$. Also, $z - x \in I_z \setminus A_i$ for $1 \leq i \leq k$ and $i \neq j$, for otherwise $z = x + (z-x) \in A_j + A_i \subsetneq A_1 + A_2 + \ldots + A_k$. Since $z \in I_z \setminus A_i$, $1 \leq i \leq k$, it follows that

$$(I_z \setminus A_i)(I_z) \geq A_j(I_z) + 1 = t + 1$$

and

$$A_i(I_z) \leq J^m(I_z) - (t+1) = n - t - 1$$

for $1 \leq i \leq k$ and $i \neq j$. Therefore,

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq t + (k-1)(n-t-1) = (2-k)t + (k-1)(n-1).$$

However, the function α defined on the set of real numbers by $\alpha(y) = (2-k)y + (k-1)(n-1)$ is decreasing since $\alpha'(y) = 2-k \leq 0$. Since $t \geq n/2$ it follows that

$$(5.3) \qquad \sum_{1 \leq i \leq k} A_i(I_z) \leq \alpha(t) \leq \alpha(n/2)$$
$$= (2-k)n/2 + (k-1)(n-1)$$
$$= kn/2 - k + 1.$$

From inequalities (5.2) and (5.3) we have

$$f(k,m,n) \le \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even.} \end{cases}$$

Since the upper and lower bounds which we have found for $f(k,m,n)$ are equal, then

$$f(k,m,n) = \begin{cases} kn/2 - k/2 & \text{if } n \text{ is odd,} \\ kn/2 - k + 1 & \text{if } n \text{ is even,} \end{cases}$$

and the proof is complete.

To obtain a lower bound for $f(k,m,n)$ in the proof of Theorem 5.12, we consider the point in J_n^m which is in $\{(x_1, x_2, \ldots, x_m) \mid x_1 \ge 0 \text{ and } x_i = 0 \text{ for } i = 2, \ldots, m\}$, a subset of J^m isomorphic to J^1. However, any point in J_n^m could have been used to give inequality (5.1) as Theorem 5.14 shows. It follows that the value of $f(k,m,n)$ is not changed if in Definition 5.4 we replace J_n^m by one of its nonempty proper subsets.

When a and b are integers and $a > b$, we define

$$\prod_{a \le i \le b} z_i = 1.$$

Lemma 5.13. If $z_1, z_2, z_3, \ldots$ is a sequence of real numbers, then $\sum_{1 \le i \le m} z_i (\prod_{i+1 \le t \le m} (z_t + 1)) = (\prod_{1 \le i \le m} (z_i + 1)) - 1$ for $m \ge 1$.

Proof. When $m = 1$ then $\prod_{i+1 \le t \le 1} (z_t + 1) = 1$ for $i \ge 1$ and

$$\sum_{1 \leq i \leq 1} z_i \Big(\prod_{i+1 \leq t \leq 1} (z_t + 1) \Big) = z_1 = \Big(\prod_{1 \leq i \leq 1} (z_i + 1) \Big) - 1.$$

Assume $k \geq 1$ and

$$\sum_{1 \leq i \leq k} z_i \Big(\prod_{i+1 \leq t \leq k} (z_t + 1) \Big) = \Big(\prod_{1 \leq i \leq k} (z_i + 1) \Big) - 1.$$

Then

$$\sum_{1 \leq i \leq k+1} z_i \Big(\prod_{i+1 \leq t \leq k+1} (z_t + 1) \Big) = z_{k+1} + \sum_{1 \leq i \leq k} z_i \Big(\prod_{i+1 \leq t \leq k+1} (z_t + 1) \Big)$$

$$= z_{k+1} + (z_{k+1} + 1) \Big(\sum_{1 \leq i \leq k} z_i \Big(\prod_{i+1 \leq t \leq k} (z_t + 1) \Big) \Big)$$

$$= z_{k+1} + (z_{k+1} + 1) \Big(\Big(\prod_{1 \leq i \leq k} (z_i + 1) \Big) - 1 \Big)$$

$$= \Big(\prod_{1 \leq i \leq k+1} (z_i + 1) \Big) - 1,$$

and the lemma follows by induction.

<u>Theorem 5.14.</u> Let $k \geq 2$, $m \geq 1$, and $n \geq 1$. If $z \in J_n^m$ then there exist sets $A_1, A_2, \ldots, A_k$ such that $0 \in \bigcap_{i=1}^{k} A_i$, $z \notin \sum_{1 \leq i \leq k} A_i$, and $\sum_{1 \leq i \leq k} A(I_z) = f(k, m, n)$.

Proof. Let $z \in J_n^m$. With $z = (z_1, z_2, \ldots, z_m)$ then

$n+1 = \prod_{1 \leq i \leq m} (z_i+1).$

Case 1. The integer n is even.

Note that n is even implies $n+1$ is odd, and so z_i is even for $i = 1, 2, \ldots, m$. Let $\Delta = \{i \mid z_i > 0 \text{ and } 1 \leq i \leq m\}$. Since $z \in J_n^m$ and $n \geq 1$ then Δ is not empty. Let $u = \min\{i \mid i \in \Delta\}$. Define

$$B_u = \{(x_1, x_2, \ldots, x_m) \mid x_i = z_i/2 = 0 \text{ for } 1 \leq i < u \text{ if } u > 1, \\ z_u/2 < x_u \leq z_u,\ 0 \leq x_i \leq z_i \text{ for } u < i \leq m \text{ if } \\ u < m, \text{ and } (x_1, x_2, \ldots, x_m) \neq (z_1, z_2, \ldots, z_m)\},$$

and for $j \in \Delta$, $j > u$, define

$$B_j = \{(x_1, x_2, \ldots, x_m) \mid x_i = z_i/2 \text{ for } 1 \leq i < j,\ z_j/2 < x_j \leq z_j, \text{ and } \\ 0 \leq x_i \leq z_i \text{ for } j < i \leq m \text{ if } j < m\}.$$

Clearly $z \notin B_u$. Also $z \notin B_j$ for $j > u$ since $x_u = z_u/2 < z_u$ whenever $(x_1, x_2, \ldots, x_m) \in B_j$. Furthermore, $(\frac{z_1}{2}, \frac{z_2}{2}, \ldots, \frac{z_m}{2}) \notin B_j$ for $j \geq u$ since $x_j > \frac{z_j}{2}$ whenever $(x_1, x_2, \ldots, x_m) \in B_j$.

Now $B_i \cap B_j = \emptyset$ when $i < j$ for if $x = (x_1, x_2, \ldots, x_m) \in B_i$ then $x_i > z_i/2$, and so $x \notin B_j$. It follows that

$$(\bigcup_{i\in\Delta} B_i)(I_z) = \sum_{i\in\Delta} B_i(I_z)$$

$$= B_u(I_z) + \sum_{\substack{i\in\Delta\\ i\neq u}} B_i(I_z)$$

$$= \frac{z_u}{2}\Big(\prod_{u+1\leq t\leq m}(z_t+1)\Big)-1 + \sum_{\substack{i\in\Delta\\ i\neq u}} \frac{z_i}{2}\Big(\prod_{i+1\leq t\leq m}(z_t+1)\Big)$$

$$= \frac{1}{2}\sum_{i\in\Delta} z_i\Big(\prod_{i+1\leq t\leq m}(z_t+1)\Big) - 1.$$

Next define $A_1 = \{0\} \cup (\bigcup_{i\in\Delta} B_i) \cup \{(\frac{z_1}{2}, \frac{z_2}{2}, \ldots, \frac{z_m}{2})\}$, and $A_j = \{0\} \cup (\bigcup_{i\in\Delta} B_i)$ for $j = 2, \ldots, k$. Since $z_i = 0$ if $i \notin \Delta$ and since $n = (\prod_{1\leq i\leq m}(z_i+1)) - 1 = \sum_{1\leq i\leq m} z_i(\prod_{i+1\leq t\leq m}(z_t+1))$ by Lemma 5.13, then for $2 \leq j \leq k$ we have

$$A_j(I_z) = (\bigcup_{i\in\Delta} B_i)(I_z)$$

$$= \frac{1}{2}\sum_{i\in\Delta} z_i\Big(\prod_{i+1\leq t\leq m}(z_t+1)\Big) - 1$$

$$= \frac{1}{2}\sum_{1\leq i\leq m} z_i\Big(\prod_{i+1\leq t\leq m}(z_t+1)\Big) - 1$$

$$= \frac{1}{2}n - 1.$$

Furthermore, $A_1(I_z) = A_2(I_z) + 1 = n/2$. Therefore,

$$\sum_{1 \leq j \leq k} A_j(I_z) = n/2 + (k-1)(n/2-1) = kn/2 - k + 1.$$

Since n is even, then $\sum_{1 \leq j \leq k} A_j(I_z) = f(k,m,n)$ by Theorem 5.12.

It remains to show that $z \notin \sum_{1 \leq j \leq k} A_j$. Let $x \in \sum_{1 \leq j \leq k} A_j$.

Then $x = \sum_{1 \leq j \leq k} a_j$ where $a_j \in A_j$. Since $z \notin A_j$ for $1 \leq j \leq k$, then $z \neq x$ if $a_j > 0$ for at most one j, $1 \leq j \leq k$. Thus, assume $a_i > 0$ and $a_j > 0$ where $1 \leq i < j \leq k$. Set $a_i = (y_1, y_2, \ldots, y_m)$, $a_j = (w_1, w_2, \ldots, w_m)$, and $x = (x_1, x_2, \ldots, x_m)$. If $a_i \in B_s$ and $a_j \in B_t$, then

(i) $y_s > z_s/2$ and $w_t > z_t/2$ if $s = t$,

(ii) $y_s > z_s/2$ and $w_s = z_s/2$ if $s < t$,

and

(iii) $y_t = z_t/2$ and $w_t > z_t/2$ if $s > t$.

In any case $x_s \geq y_s + w_s > z_s$ or $x_t \geq y_t + w_t > z_t$, and so $z \neq x$. If $i = 1$ and $a_1 = (\frac{z_1}{2}, \frac{z_2}{2}, \ldots, \frac{z_m}{2})$, then $x_t \geq y_t + w_t > z_t/2 + z_t/2 = z_t$, and so again $z \neq x$.

Case 2. The integer n is odd.

Since n is odd then $n+1$ is even; consequently, z_v+1

is even and z_v is odd for some v, $1 \leq v \leq m$. Define

$$A_i = \{(x_1, x_2, \ldots, x_m) \mid 0 \leq x_j \leq z_j \text{ for } 1 \leq j \leq m,\ j \neq v, \text{ and}$$
$$(z_v+1)/2 \leq x_v < z_v \text{ or } x_v = 0\}$$

for $i = 1, 2, \ldots, k$. Clearly, $0 \in A_i$. We now show that $z \notin \sum_{1 \leq i \leq k} A_i$. Let $x \in \sum_{1 \leq i \leq k} A_i$. Then $x = \sum_{1 \leq i \leq k} a_i$ where $a_i \in A_i$. Set $a_i = (a_{1i}, a_{2i}, \ldots, a_{mi})$ for $i = 1, 2, \ldots, k$ and $x = (x_1, x_2, \ldots, x_m)$. If $a_{vi} > 0$ for at most one i, then $x_v = a_{vi} < z_v$, and so $z \neq x$. Thus, assume $a_{vi} > 0$ and $a_{vj} > 0$ where $1 \leq i < j \leq k$. Then $x_v > a_{vi} + a_{vj} \geq z_v+1$, and $z \neq x$.

Now $\sum_{1 \leq i \leq k} A_i(I_z) = k(n-1)/2$ since for $i = 1, 2, \ldots, k$,

$$A_i(I_z) = (z_v - \frac{(z_v+1)}{2} + 1)(\prod_{\substack{1 \leq t \leq m \\ t \neq v}} (z_t+1)) - 1$$

$$= \frac{1}{2}(\prod_{1 \leq t \leq m} (z_t+1)) - 1$$

$$= \frac{1}{2}(n+1) - 1 = \frac{1}{2}(n-1).$$

Since n is odd, then from Theorem 5.12 we have $\sum_{1 \leq i \leq k} A_i(I_z) = f(k, m, n)$, and the proof of the theorem is complete.

In the next two theorems we apply Theorem 4.12 to evaluate the functions g and h on certain subsets of their domains.

<u>Theorem 5.15.</u> Let $m \geq 1$. Then

(i) $g(2, m, n) = n - 1$ for $n \geq 1$,

and

(ii) $g(k, m, n) = n - 1$ for $k \geq 3$ and $1 \leq n \leq 14$.

Proof. Let m and n be given integers such that $m \geq 1$ and $n \geq 1$. Let $z \in J_n^m$ and $(A_1, A_2) \in \mathcal{A}_{z,2}$. Then $z \notin A_1 + A_2$ and $I'_z \subsetneq A_1 + A_2$. From Theorem 3.2 we have

$$A_1(I_z) + A_2(I_z) \leq (A_1 + A_2)(I_z) < J_n^m(I_z) = n.$$

Hence, $g(2, m, n) \leq n - 1$. However, $g(2, m, n) \geq n - 1$ by Theorem 5.9, and so $g(2, m, n) = n - 1$.

Next let k, m, and n be given integers such that $k \geq 3$, $m \geq 1$, and $1 \leq n \leq 14$. Let $z \in J_n^m$ and $(A_1, A_2, \ldots, A_k) \in \mathcal{A}_{z,k}$. Then $z \notin \sum_{1 \leq i \leq k} A_i$ and $I'_z \subsetneq \sum_{1 \leq i \leq k} A_i$. Applying Theorem 4.12 we have

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_z) = n - 1.$$

Thus, $g(k, m, n) \leq n - 1$. Since $g(k, m, n) \geq n - 1$ by Theorem 5.9, then $g(k, m, n) = n - 1$.

Theorem 5.16. Let $m \geq 1$. Then

(i) $h(2, m, n) = n - 1$ for $n > 2$,

(ii) $h(3, m, n) = n - 1$ for $3 < n \leq 14$,

(iii) $h(4, m, n) = n - 1$ for $4 < n \leq 15$,

(iv) $h(k, m, n) = n - 1$ for $5 \leq k < n$ and $n \leq 16$,

and

(v) $h(k, m, n) = 0$ for $k \geq n$, $k \geq 2$, and $1 \leq n \leq 16$.

Proof. Let $m \geq 1$ and $k \geq 3$. Assume $1 \leq n \leq 14$ if $k = 3$, $1 \leq n \leq 15$ if $k = 4$, and $1 \leq n \leq 16$ if $k \geq 5$. Let $z \in J_n^m$ and $(A_1, A_2, \ldots, A_k) \in \mathcal{H}_{z,k}$. Then $z \notin \sum_{1 \leq i \leq k} A_i$, $I'_z \subseteq \sum_{1 \leq i \leq k} A_i$, and $A_i(I_z) > 0$ for $i = 1, 2, \ldots, k$. Applying Theorem 4.12 we

$$\sum_{1 \leq i \leq k} A_i(I_z) \leq \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_z) \leq n - 1.$$

Hence, $h(k, m, n) \leq n - 1$. However, $h(k, m, n) \geq n - 1$ when $n > k$ by Theorem 5.9(ii). Parts (ii), (iii), and (iv) of the theorem follow from these inequalities. Since $h(k, m, n) \leq n - 1$, then from Theorem 5.9(ii) we obtain $h(k, m, n) = 0$ when $k \geq n$. This establishes part (v) of the theorem when $k \geq 3$.

Now let $m \geq 1$ and $n > 2$. From Theorems 5.8, 5.9(ii), and 5.15 we obtain

$$n - 1 \leq h(2, m, n) \leq g(2, m, n) = n - 1.$$

This proves part (i) of the theorem.

Finally, consider $m \geq 1$, $k = 2$, and $1 \leq n \leq 2$. From Theorem 5.9(iii) we have $h(2, m, n) = 0$ or $h(2, m, n) \geq 2$. However, from Theorems 5.8 and 5.15 we have

$$h(2, m, n) \leq g(2, m, n) \leq n - 1 \leq 1,$$

and so $h(2, m, n) = 0$.

The proof of the theorem is now complete.

The next theorem is due to Allen R. Freedman, but it does not appear in the literature. We use Freedman's result to obtain a lower bound for $g(k, m, n)$ and $h(k, m, n)$ and an upper bound for $s(k, m)$ when $k \geq 4$, $m \geq 1$, and $n \geq 8(k-2)$.

Theorem 5.17. If $k \geq 4$ and $n \geq 8(k-2)$ then

$$h(k, 1, n) \geq n + (k-4).$$

Proof. Let k, n, and t be given integers such that $k \geq 4$, $n = 8(k-2) + t$, and $t \geq 0$.

Note that $4k-7 < 5k-9 < 6k-11 < 7k-13$ since $k \geq 4$. Set

$$A_i = \{0, 1\} \cup \{4k-7+t, 5k-9+t, 6k-11+t, 7k-13+t\}$$

for $i = 1, 2, \ldots, k-3$,

$$A_{k-2} = \{0, k-2\} \cup \{4k-7+t, \ldots, 5k-10+t, 6k-11+t, \ldots, 7k-14+t\},$$

$$A_{k-1} = \{0, 2k-4\} \cup \{4k-7+t, \ldots, 6k-12+t\},$$

and

$$A_k = \{0, 4k-8, \ldots, 4k-8+t\}.$$

Now $I'_n \subseteq \sum_{1 \leq i \leq k} A_i$ since

$$\sum_{1 \leq i \leq k} A_i \supseteq \Big(\sum_{1 \leq i \leq k-3} \{0, 1\}\Big) + \{0, k-2\} + \{0, 2k-4\} + A_k$$

$$= \{0, \ldots, 8k-17+t\} = \{0, \ldots, n-1\}.$$

We claim $n \notin \sum_{1 \leq i \leq k} A_i$. Let $x = \sum_{1 \leq i \leq k} a_i$ where $a_i \in A_i$ for $i = 1, 2, \ldots, k$. If there are integers i and j such that $1 \leq i < j \leq k$, $a_i \geq 4k-7+t$, and $a_j \geq 4k-8$, then

$$x \geq a_i + a_j \geq (4k-7+t) + (4k-8) = 8k - 15 + t > n.$$

If $a_k \geq 0$ and $a_i < 4k-7+t$ for $i = 1, 2, \ldots, k-1$, then

$$x = \sum_{1 \leq i \leq k} a_i \leq \Big(\sum_{1 \leq i \leq k-3} 1\Big) + (k-2) + (2k-4) + (4k-8+t)$$

$$= 8k - 17 + t < n.$$

It remains to consider $a_k = 0$ and $a_i \geq 4k-7+t$ for exactly one i, $1 \leq i \leq k-1$. Since

$$\left(\sum_{1 \le j \le k-4} \{0,1\}\right) + \{0, k-2\} + \{0, 2k-4\}$$

$$= \{0, \ldots, k-4,\ k-2, \ldots, 2k-6,\ 2k-4, \ldots, 3k-8,\ 3k-6, \ldots, 4k-10\},$$

then $n \notin A_j + \left(\sum_{1 \le t \le k-4} \{0,1\}\right) + \{0, k-2\} + \{0, 2k-4\}$ for $j = 1, 2, \ldots, k-3$. Since

$$\left(\sum_{1 \le j \le k-3} \{0,1\}\right) + \{0, 2k-4\} = \{0, \ldots, k-3,\ 2k-4, \ldots, 3k-7\},$$

then $n \notin A_{k-2} + \left(\sum_{1 \le j \le k-3} \{0,1\}\right) + \{0, 2k-4\}$. Since

$$\left(\sum_{1 \le j \le k-3} \{0,1\}\right) + \{0, k-2\} = \{0, \ldots, 2k-5\},$$

then $n \notin A_{k-1} + \left(\sum_{1 \le j \le k-3} \{0,1\}\right) + \{0, k-2\}$. Thus, $x = \sum_{1 \le i \le k} a_i \ne n$ when $a_k = 0$ and $a_i > 4k-7+t$ for exactly one i, $1 \le i \le k-1$. This establishes that $n \notin \sum_{1 \le i \le k} A_i$.

Since $n \notin \sum_{1 \le i \le k} A_i$, $I'_n \subseteq \sum_{1 \le i \le k} A_i$, and $A_i(I_n) > 0$ for $i = 1, 2, \ldots, k$, then $(A_1, A_2, \ldots, A_k) \in \mathcal{H}_{n,k}$. Therefore,

$$h(k,1,n) \geq \sum_{1 \leq i \leq k} A_i(I_n)$$

$$\geq 5(k-3) + (1+2(k-2)) + (1+(2k-4)) + (1+t)$$

$$= 9k - 20 + t = (8k-16+t) + (k-4)$$

$$= n + (k-4),$$

and the proof is complete.

It is interesting to observe that Freedman's result gives an extension of Theorem F to k sets in J^1. To see this consider the following statement of his result:

Let $k \geq 4$. For each integer $n \geq 8(k-2)$ there exist k sets of nonnegative integers $A_1, A_2, \ldots, A_k$ for which

$$\sum_{1 \leq i \leq k} A_i \supseteq \{0, \ldots, n-1\}, \quad n \notin \sum_{1 \leq i \leq k} A_i, \quad A_i(I_n) > 0 \quad \text{for} \quad i = 1,2,\ldots,k,$$

and $$\sum_{1 \leq i \leq k} A_i(I_n) > n - 1 = \Big(\sum_{1 \leq i \leq k} A_i\Big)(I_n).$$

<u>Theorem 5.18</u>. Let $m \geq 1$. If $k \geq 3$ and $n \geq 15$ then $g(k,m,n) \geq h(3,m,n) \geq n$. If $k \geq 4$ and $n \geq 8(k-2)$ then $g(k,m,n) \geq h(k,m,n) \geq n + (k-4)$.

Proof. In the proof of Theorem F, for each integer $n \geq 15$ Lin constructs three sets A, B, and C in J^1 for which $A(I_n) > 0$, $B(I_n) > 0$, $C(I_n) > 0$, $n \notin A + B + C$, $\{0,\ldots,n-1\} \subseteq A + B + C$, and $A(I_n) + B(I_n) + C(I_n) \geq n$. Therefore, $h(3,1,n) \geq n$ when

$n \geq 15$. Applying Theorems 5.8, 5.10(iii), and 5.11 we have

$$g(k,m,n) \geq g(3,m,n) \geq h(3,m,n) \geq h(3,1,n) \geq n$$

when $m \geq 1$, $k \geq 3$, and $n \geq 15$.

Applications of Theorems 5.8, 5.10(iii), and 5.17 give

$$g(k,m,n) \geq h(k,m,n) \geq h(k,1,n) \geq n + (k-4)$$

when $m \geq 1$, $k \geq 4$, and $n \geq 8(k-2)$.

Theorem 5.19. Let $m \geq 1$. Then

(i) $s(2,m)$ is infinite,

(ii) $s(3,m) = 14$,

(iii) $s(4,m) = 15$,

(iv) $s(5,m) \geq 16$,

and

(v) $s(k,m) < 8(k-2)$ for $k \geq 5$.

Proof. From Theorem 5.16 we have $h(2,m,n) = n - 1$ when $n > 2$, $h(3,m,n) = n - 1$ when $3 < n \leq 14$, $h(4,m,n) = n - 1$ when $4 < n \leq 15$, and $h(5,m,n) = n - 1$ when $5 < n \leq 16$. Thus, $s(2,m)$ is infinite, $s(3,m) \geq 14$, $s(4,m) \geq 15$, and $s(5,m) \geq 16$. From Theorem 5.18 we have $h(3,m,15) \geq 15$, $h(4,m,16) \geq 16$, and $h(k,m,8(k-2)) \geq 8(k-2)$ for $k \geq 5$. Therefore, $s(3,m) \leq 14$, $s(4,m) \leq 15$, and $s(k,m) < 8(k-2)$ for $k \geq 5$. This proves the theorem.

It would be of interest to determine if $s(k,m)$ increases as k increases for $k \geq 3$.

In a paper published in 1958, P. Erdös and P. Scherk [2] gave upper and lower bounds for $g(k,m,n)$ when $m = 1$. We now state their result.

Theorem 5.20. If $k \geq 3$ and $n \geq 1$ then

$$\frac{1}{2}kn - \alpha_k n^{(k-1)/k} < g(k,1,n) < \frac{1}{2}kn - \gamma_k n^{(k-1)/k}$$

where $\alpha_k = (k+1)2^{2k-3}$ and $\gamma_k = \dfrac{1}{2^{(k/2)+4}(k-1)!}$.

In order to establish that $\frac{1}{2}kn - \alpha_k n^{(k-1)/k}$ is a lower bound for $g(k,1,n)$ when $k \geq 3$ and $n \geq 1$, Erdös and Scherk construct sets $A_0, A_1, \ldots, A_{k-1}$ for which $\{0,\ldots,n-1\} \subseteq \sum_{0 \leq i \leq k-1} A_i$, $n \notin \sum_{0 \leq i \leq k-1} A_i$, and $\sum_{0 \leq i \leq k-1} A_i(I_n) > \frac{1}{2}kn - \alpha_k n^{(k-1)/k}$. However, if $n \geq 2^k$ then $A_i(I_n) > 0$ for $i = 0,\ldots,k-1$ since $2^i \in A_i$; consequently, $\frac{1}{2}kn - \alpha_k n^{(k-1)/k}$ is also a lower bound for $h(k,1,n)$. Since $h(k,1,n) \leq g(k,1,n)$ it follows that

$$\frac{1}{2}kn - \alpha_k n^{(k-1)/k} < h(k,1,n) < \frac{1}{2}kn - \gamma_k n^{(k-1)/k}$$

when $k \geq 3$ and $n \geq 2^k$.

An extension of Theorem G to k sets in J^1 can be obtained from the theorem of Erdős and Scherk. Let t be a real number, $t > 0$, and let k be an integer, $k \geq 3$. In the above paragraph we observed that

$$h(k, 1, n) > \frac{k}{2} n - \alpha_k n^{(k-1)/k}$$

for $n \geq 2^k$. The number

$$(\frac{k}{2} - 1)n - \alpha_k n^{(k-1)/k} = n(\frac{k}{2} - 1 - \alpha_k n^{-1/k})$$

becomes infinite as n becomes infinite since

$$\lim_{n \to \infty} (\frac{k}{2} - 1 - \alpha_k n^{-1/k}) = \frac{k}{2} - 1.$$

Thus, there exists an integer n such that $n \geq 2^k$ and $(\frac{k}{2} - 1)n - \alpha_k n^{(k-1)/k} > t$. From the definition of $h(k, 1, n)$ it follows that there exist k sets of nonnegative integers $A_1, A_2, \ldots, A_k$ for which $A_i(I_n) > 0$ for $i = 1, 2, \ldots, k$, $\sum_{1 \leq i \leq k} A_i \supseteq \{0, \ldots, n-1\}$, $n \notin \sum_{1 \leq i \leq k} A_i$, and $\sum_{1 \leq i \leq k} A_i(I_n) = h(k, 1, n)$. Therefore,

$$\begin{aligned}\sum_{1\le i\le k} A_i(I_n) &= h(k,1,n)\\ &> n + (\frac{k}{2} - 1)n - \alpha_k n^{(k-1)/k}\\ &> n + t\\ &> (\sum_{1\le i\le k} A_i)(I_n) + t.\end{aligned}$$

This establishes the following extension of Theorem G:

Let $k \geq 3$. If $t > 0$ is given then a positive integer n and k sets of nonnegative integers $A_1, A_2, \ldots, A_k$ can be found satisfying $\sum_{1\le i\le k} A_i \supseteq \{0, \ldots, n-1\}$, $n \notin \sum_{1\le i\le k} A_i$, $A_i(I_n) > 0$ for $i = 1, 2, \ldots, k$, and $\sum_{1\le i\le k} A_i(I_n) \geq (\sum_{1\le i\le k} A_i)(I_n) + t$.

In the paper in which Theorem 5.20 is proved, Erdös and Scherk indicate that when $k \geq 3$ a constant β might exist for which

$$g(k, 1, n) = \frac{1}{2}kn - (\beta + o(1))n^{(k-1)/k}$$

as n becomes infinite. In 1964 H. B. Kemperman [4] established this asymptotic formula with $\beta = k2^{-(k-1)/k}$ by proving the following theorem:

Theorem 5.21. If $k \geq 3$ then

$$\lim_{n\to\infty} (k(n-1)/2-g(k,1,n))n^{-(k-1)/k} = k2^{-(k-1)/k}.$$

An examination of Kemperman's proof of Theorem 5.21 shows that a result analogous to Theorem 5.21 is valid for $h(k, 1, n)$. Kemperman determines an upper bound for $k(n-1)/2 - g(k, 1, n)$ where $k \geq 3$ and $n \geq 2$ by constructing sets $A_1, A_2, \ldots, A_k$ [4, pp. 46-48] for which $I'_n \subset \sum_{1\leq i\leq k} A_i$, $n \notin \sum_{1\leq i\leq k} A_i$, and

$$k(n-1)/2 - \sum_{1\leq i\leq k} A_i(I_n) < k(n/2)^{(k-1)/k}(1+(n/2)^{-1/k})^{k-1}.$$

Now if $n \geq 2^{k+1}$ then $p > 2$ where p is defined to be the smallest integer for which $[\frac{n}{2}] \leq p^k - 1$. But then $A_i(I_n) > 0$ for $i = 1, 2, \ldots, k$ since the set A_i contains at least $p - 1$ elements of the set $\{0, p^{i-1}, 2p^{i-1}, \ldots, (p-1)p^{i-1}\}$. Therefore, $h(k, 1, n) \geq \sum_{1\leq i\leq k} A_i(I_n)$ when $n \geq 2^{k+1}$, and it follows that

$$\begin{aligned} k(n-1)/2 - g(k,1,n) &\leq k(n-1)/2 - h(k,1,n) \\ &\leq k(n-1)/2 - \sum_{1\leq i\leq k} A_i(I_n) \\ &< k(n/2)^{(k-1)/k}(1+(n/2)^{-1/k})^{k-1}. \end{aligned}$$

Thus,

$$(k(n-1)/2 - g(k, 1, n))n^{-(k-1)/k} \leq (k(n-1)/2 - h(k, 1, n))n^{-(k-1)/k}$$
$$< k2^{-(k-1)/k}(1+(n/2)^{-1/k})^{k-1}$$

for $n \geq 2^{k+1}$ and $k \geq 3$. Using Theorem 5.21 to evaluate the left side of this inequality as n becomes infinite and noting that

$$\lim_{n \to \infty} (1+(n/2)^{-1/k})^{k-1} = 1,$$

we obtain

$$\lim_{n \to \infty} (k(n-1)/2 - h(k, 1, n))n^{-(k-1)/k} = k2^{-(k-1)/k}$$

From Theorem 5.10 we have $g(k, m_1, n) \geq g(k, m_2, n)$ and $h(k, m_1, n) \geq h(k, m_2, n)$ when $m_1 > m_2 \geq 1$. Thus, lower bounds of $g(h, 1, n)$ and $h(k, 1, n)$ are also lower bounds of $g(k, m, n)$ and $h(k, m, n)$, respectively, for $m > 1$. We have had no success in obtaining upper bounds for these functions.

The problem of completely evaluating $g(k, m, n)$, $h(k, m, n)$, and $s(k, m)$ appears to be difficult, even for $m = 1$.

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APPENDICES

APPENDIX I

We now prove Theorem 4.5 which was stated without proof in Chapter IV.

<u>Theorem.</u> Let $z = (1, 3, 1) \in J^3$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I_z'$ and $D(I_z) > 0$, than at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

From Theorem 4.1(i), it follows that none of the sets $A + B + D$, $A + C + D$, and $B + C + D$ has more than nine nonzero elements. Moreover, from Theorem 4.1(ii), it follows that none of the sets $A + B$, $A + C$, $A + D$, $B + C$, $B + D$, and $C + D$ has more than eight nonzero elements.

Let $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$. Now either $\{e_1, e_3\}$ is a subset of $A + D$, $B + D$, or $C + D$, or otherwise $\{e_1, e_3\} \cap D$ is the empty set and each of the sets A, B, and C has at most one element in common with $\{e_1, e_3\}$.

Case 1. The set $\{e_1, e_3\}$ is a subset of $A + D$, $B + D$, or $C + D$.

To be definite, say $\{e_1, e_3\} \subseteq A + D$. Now e_2 is an element of $A + B + D$ or $A + C + D$. By relabeling sets B and C if necessary, we may assume that $e_2 \in A + C + D$.

If $(1,0,1) \notin A + D$ then either $\{e_1, e_3\} \subseteq A$ and $\{e_1, e_3\} \cap D = \phi$ or $\{e_1, e_3\} \subseteq D$ and $\{e_1, e_3\} \cap A = \phi$. Whenever $(1,0,1) \notin A + D$, we will let G equal A if $\{e_1, e_3\} \subseteq A$ and G equal D if $\{e_1, e_3\} \subseteq D$.

Let $R_1 = \{(0,x,0) \mid 1 \leq x \leq 3\}$, $R_2 = \{(0,x,1) \mid 0 \leq x \leq 3\}$, $R_3 = \{(1,x,0) \mid 0 \leq x \leq 3\}$, and $R_4 = \{(1,x,1) \mid 0 \leq x \leq 3\}$. For any set $S \subseteq I_z$ denote $S \cap R_i$ by S_i, $1 \leq i \leq 4$.

We proceed to establish that $B_1(I_z) \leq 1$ and $B_i(I_z) \leq 2$ for $2 \leq i \leq 4$. Note that

$$\{(1,3,1), (1,3,0), (0,3,1), (1,2,1)\} \cap B = \phi$$

since $\{0, e_1, e_2, e_3\} \subseteq A + C + D$ and $(1,3,1) \notin A + B + C + D$. Clearly, $B_4(I_z) \leq 2$.

Assume $B_1(I_z) > 1$. Then there exist integers x and y such that $1 \leq x < y \leq 3$ and $\{(0,x,0), (0,y,0)\} \subseteq B$. Thus,

$$\begin{aligned} A + B + D &\supseteq (A+D) + B \supseteq \{0, e_1, e_3\} + B \\ &\supseteq \{0, e_1, e_3, (0,x,0), (0,y,0), (1,x,0), (1,y,0), (0,x,1), (0,y,1)\}. \end{aligned}$$

Now $(1,0,1) \notin A + D$, for otherwise $\{(1,x,1), (1,y,1)\} \subseteq A+B+D$ and $(A+B+D)(I_z) > 9$. Therefore, with set G as previously defined, we have $B + G \supseteq B + \{0, e_1, e_3\}$. Also, $(B+G)(I_z) \leq 8$. It follows that

$$B + G = \{0, e_1, e_3, (0,x,0), (0,y,0), (1,x,0), (1,y,0), (0,x,1), (0,y,1)\}.$$

Now $\{e_1, e_3\} \cap B = \phi$, for otherwise $(1,0,1) \in B + G$. Since $B(I_z) \geq 5$ and $B \subseteq B + G$, then either $(1,x,0)$ or $(1,y,0)$ is an element of B. However, this implies that $(1,x,1)$ or $(1,y,1)$ is an element of $B + G$. We conclude that $B_1(I_z) \leq 1$.

Assume $B_2(I_z) > 2$. Then $B_2 = \{(0,0,1), (0,1,1), (0,2,1)\}$ since $(0,3,1) \notin B$. Hence,

$$\begin{aligned} A + B + D &\supseteq \{0, e_1\} + (B_2 \cup \{0\}) \\ &= \{0, e_1, e_3, (0,1,1), (0,2,1), (1,0,1), (1,1,1), (1,2,1)\}. \end{aligned}$$

Since $\{(1,x,1) \mid 0 \leq x \leq 2\} \subseteq A + B + D$ and $(1,3,1) \notin A+B+C+D$, then $\{(0,x,0) \mid 1 \leq x \leq 3\} \cap C$ is empty. Therefore, $\{(0,x,0) \mid 1 \leq x \leq 3\} \subseteq A + B + D$. But then $(A+B+D)(I_z) > 9$. We conclude that $B_2(I_z) \leq 2$.

The argument presented in the preceding paragraph with B_2 replaced by B_3 shows that $B_3(I_z) \leq 2$.

Choose E so that E is one of the sets A, C, or D and so that $e_2 \in E$. We now show that $(B+E)_1(I_z) \geq B_1(I_z) + 1$ and that $(B+E)_i(I_z) \geq B_i(I_z) + 1$ when $B_i(I_z) > 0$, $2 \leq i \leq 4$. Assume $B_2(I_z) > 0$. Let $b' = \max\{b \mid (0,b,1) \in B\}$. Thus, $(0,b'+1,1) \notin B$. Since $(0,3,1) \notin B$ then $b' \leq 2$, and so

$$(0,b'+1,1) = (0,b',1) + (0,1,0) \in (B+E)_2.$$

Therefore, $(B+E)_2(I_z) \geq B_2(I_z) + 1$. Since $(1,3,0) \notin B_3$ and $(1,3,1) \notin B_4$, the same kind of argument gives $(B+E)_i(I_z) \geq B_i(I_z) + 1$ when $B_i(I_z) > 0$ and $3 \leq i \leq 4$. If $(0,1,0) \notin B_1$, then $(B+E)_1(I_z) \geq B_1(I_z) + 1$ since $e_2 \in E$. If $(0,1,0) \in B_1$ then $B_1 = \{(0,1,0)\}$ since $B_1(I_z) \leq 1$. Hence, $(0,2,0) \in B + E$, and so $(B+E)_1(I_z) \geq 2 = B_1(I_z) + 1$.

One of the numbers $B_2(I_z)$, $B_3(I_z)$, or $B_4(I_z)$ is zero, for otherwise

$$8 \geq (B+E)(I_z) = \sum_{1 \leq i \leq 4} (B+E)_i(I_z) \geq \sum_{1 \leq i \leq 4} (B_i(I_z)+1)$$

$$= \left(\sum_{1 \leq i \leq 4} B_i(I_z)\right) + 4 = B(I_z) + 4 \geq 9.$$

Let $B_j(I_z) = 0$, $j \in \{2,3,4\}$. Since $B(I_z) \geq 5$, $B_1(I_z) \leq 1$, and $B_i(I_z) \leq 2$ for $2 \leq i \leq 4$, then $B_1(I_z) = 1$, $B_i(I_z) = 2$ for $2 \leq i \leq 4$ and $i \neq j$, and $B(I_z) = 5$. Now

$$8 \geq (B+E)(I_z) = \sum_{1 \leq i \leq 4} (B+E)_i(I_z) \geq \sum_{\substack{1 \leq i \leq 4 \\ i \neq j}} (B+E)_i(I_z)$$

$$\geq \sum_{\substack{1 \leq i \leq 4 \\ i \neq j}} (B_i(I_z)+1) = \left(\sum_{\substack{1 \leq i \leq 4 \\ i \neq j}} B_i(I_z)\right) + 3 = 8;$$

hence, $(B+E)_i(I_z) = B_i(I_z) + 1$ for $1 \leq i \leq 4$ and $i \neq j$. It follows that $B_1 \neq \{(0, 2, 0)\}$, for otherwise $(B+E)_1 = R_1$ and $(B+E)_1(I_z) > B_1(I_z) + 1$. Thus $B_1 = \{(0, 1, 0)\}$ or $B_1 = \{(0, 3, 0)\}$.

Assume $B_1 = \{(0, 3, 0)\}$. Then $(1, 0, 1) \notin A + D$, for otherwise $(1, 3, 1) \in A + B + C + D$. If $B_4(I_z) = 0$ then

$$B = \{0, (0, 3, 0), (1, b_1, 0), (1, b_2, 0), (0, b_3, 1), 0, b_4, 1)\}$$

where $0 \leq b_1 < b_2 \leq 2$ and $0 \leq b_3 < b_4 \leq 2$ since $(1, 3, 0) \notin B$ and $(0, 3, 1) \notin B$. This in turn implies that

$$\begin{aligned} B + G &\supseteq B + \{0, e_1, e_3\} \\ &\supseteq \{0, e_1, e_3, (0,3,0), (1,3,0), (0,3,1), (1,b_2,0), (0,b_4,1), (1,b_1,1), (1,b_2,1)\}. \end{aligned}$$

But $(B+G)(I_z) \leq 8$, and so we conclude that $B_4(I_z) > 0$. Thus, $B_4(I_z) = 2$, and since $(1, 2, 1) \notin B$ and $(1, 3, 1) \in B$ then $B_4 = \{(1, 0, 1), (1, 1, 1)\}$. Suppose $B_3(I_z) = 0$. Then

$$B = \{0, (0, 3, 0), (0, b_1, 1), (0, b_2, 1), (1, 0, 1), (1, 1, 1)\}$$

where $0 \leq b_1 < b_2 \leq 2$. Since $B \cup \{(1, 3, 0), (0, 3, 1), e_1\} \subseteq B + G$ and $(B+G)(I_z) \leq 8$, it follows that $B + G = B \cup \{(1,3,0), (0,3,1), e_1\}$. If $b_2 = 2$ then $e_1 + (0, b_2, 1) = (1, 2, 1) \in B + G$, a contradiction. Hence, $b_2 = 1$, $b_1 = 0$, and so

$$B + G = \{0, e_1, e_3, (0, 3, 0), (1, 3, 0), (0, 3, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\}.$$

Thus,

$$(B+G)^{\sim} = \{0, e_2, (1, 1, 1), (1, 2, 1), (0, 1, 1), (0, 2, 1), (1, 1, 0)\}.$$

Note that $(0, 2, 0) \notin B + G + (B+G)^{\sim}$. Since $G = A$ or $G = D$ and $A + B + C + D = I'_z$, then $B + G + (B+G)^{\sim} = I'_z$ by Theorem 2.9(c). Since $(0, 2, 0) \in I'_z$, we conclude that $B_3(I_z) > 0$. By a symmetric argument we obtain that $B_2(I_z) > 0$. Thus, $B_i(I_z) > 0$ for $2 \leq i \leq 4$, and we have a contradiction.

Finally, assume $B_1 = \{(0, 1, 0)\}$. Since $(B+E)_1(I_z) = 2$ and $e_2 \in E$, then $(B+E)_1 = \{(0, 1, 0), (0, 2, 0)\}$. Assume $B_4(I_z) > 0$; consequently, $B_4 = \{(1, 0, 1), (1, 1, 1)\}$. Since $(1, 3, 1) \notin B + E + (B+E)^{\sim}$ and

$$B + E \supseteq B_4 + \{0, e_2\} \supseteq \{(1, 0, 1), (1, 1, 1), (1, 2, 1)\},$$

then $\{(0, 1, 0), (0, 2, 0), (0, 3, 0)\} \cap (B+E)^{\sim} = \phi$. It follows that $(0, 3, 0) \notin B + E + (B+E)^{\sim}$; however, this is contrary to $B + E + (B+E)^{\sim} = I'_z$. We conclude that $B_4(I_z) = 0$. Therefore, $B_2(I_z) = 2$ and $B_3(I_z) = 2$. Either $B_2 = \{(0, 0, 1), (0, 1, 1)\}$ or $B_2 = \{(0, 1, 1), (0, 2, 1)\}$ since $(0, 3, 1) \notin B$, $B_2(I_z) = 2$, $(B+E)_2(I_z) = 3$, and $e_2 \in E$ exclude any other subsets of R_2 as possibilities for B_2. Similarly, $B_3 = \{(1, 0, 0), (1, 1, 0)\}$ or $B_3 = \{(1, 1, 0), (1, 2, 0)\}$. Since $A + B + D \supseteq \{0, e_1, e_3\} + B$ and $(A+B+D)(I_z) \leq 9$, it follows that $A + B + D = B + \{0, e_1, e_3\}$ whenever $(B+\{0, e_1, e_3\})(I_z) = 9$. Now with $B_2 = \{(0, 0, 1), (0, 1, 1)\}$ and $B_3 = \{(1, 1, 0), (1, 2, 0)\}$, or with $B_2 = \{(0, 1, 1), (0, 2, 1)\}$, then $(B+\{0, e_1, e_3\})(I_z) = 9$, and so $A + B + D = B + \{0, e_1, e_3\}$. However, in each instance $A + B + D + \{0, (1, 1, 1)\} = A + B + D$, which is contrary to Theorem 2.10. With $B_2 = \{(0, 0, 1), (0, 1, 1)\}$ and $B_3 = \{(1, 0, 0), (1, 1, 0)\}$ then

$$B + \{0, e_2\} = \{0, e_1, e_2, e_3, (0,2,0), (0,1,1), (0,2,1), (1,1,0), (1,2,0)\}.$$

Since $B + E \supseteq B + \{0, e_2\}$ and $(B+E)(I_z) \leq 8$, then $B + E = B + \{0, e_2\}$. If $x \in I_z$, $x \neq 0$, and $x \neq e_2$ then $(B+\{0, e_2, x\})(I_z) > 8$. Thus, $E = \{0, e_2\}$. Since $A(I_z) \geq 5$ and $C(I_z) \geq 5$ then $E = D$, and this in turn implies that $\{e_1, e_3\} \subseteq A$. Therefore,

$$\begin{aligned} A + B + D &\supseteq \{0, e_1\} + (B+E) \\ &\supseteq \{(1, 0, 1), (1, 1, 1)\} \cup (B+E). \end{aligned}$$

A contradiction follows since this set inclusion implies that $(A+B+D)(I_z) > 9$.

Case 2. The set $\{e_1, e_3\} \cap D$ is empty and each of the sets A, B, and C has at most one element in common with $\{e_1, e_3\}$.

By relabeling the sets A, B, and C if necessary, we may assume that $e_1 \in A$ and $e_3 \in B$.

Let $U = \{(x_1, x_2, x_3) \mid 0 \leq x_i \leq 1 \text{ for } i = 1, 2, 3\}$ and let $U^* = U \setminus \{0\}$. We proceed to show that $D \cap U^*$ is the empty set.

We claim $(0,1,0) \notin D$. Assume otherwise. Then $U \subseteq A + B + D$ since $e_1 \in A$, $e_3 \in B$, and $e_2 \in D$. It follows that $C \subseteq U$, for if $(x_1, x_2, x_3) \in C \setminus U$ then $(1-x_1, 3-x_2, 1-x_3) \in U$ and this implies that $(1,3,1) \in C + U \subseteq A + B + C + D$. In particular $(0,2,0) \notin C$ and $(0,3,0) \notin C$. Also, $(0,2,0) \notin D$ and $(0,3,0) \notin D$, for if $(0,x,0) \in D$ where $x \in \{2,3\}$ then

$$A + B + D \supseteq U \cup \{(0,x,0), (1,x,0), (0,x,1)\}$$

and $(A+B+D)(I_z) > 9$. Now if $(0,1,0) \in A$ then $\{(0,2,0), (0,2,1)\} \subseteq A + B + D$, which in turn implies that $(1,1,1) \notin C$ and $(1,1,0) \notin C$. But then $C \subseteq U \setminus \{(1,1,1), (1,1,0)\}$ and $C(\{e_1, e_3\}) \leq 1$ imply that $C(I_z) \leq 4$. Since $C(I_z) \geq 5$, it must be that $(0,1,0) \notin A$. A symmetric argument establishes that $(0,1,0) \notin B$. Also, $(0,2,0) \notin A$, for otherwise

$$A + B + D \supseteq U \cup \{(0,2,0), (0,3,0), (0,2,1)\}$$

and $(A+B+D)(I_z) > 9$. Similarly, $(0,2,0) \notin B$. If $(0,3,0) \in A$ then $A + B + D \supseteq U \cup \{(0,3,0), (0,3,1)\}$, and since $(A+B+D)(I_z) \leq 9$ we have

$$A + B + D = U \cup \{(0,3,0), (0,3,1)\}.$$

Now $(0,3,0) \notin B$ and $(0,3,1) \notin B$, for otherwise $(1,3,0)$ or $(1,3,1)$ is an element of $A + B + D$. Hence, $B \subsetneq U$. Also, $(x_1, 1, x_3) \notin B$ when $0 \leq x_1 \leq 1$ and $0 \leq x_3 \leq 1$, for otherwise $(x_1, 2, x_3) \in A + B + D$. But then $B \subsetneq \{0, e_1, e_3, e_1+e_3\}$, which is contrary to $B(I_z) \geq 5$. Thus, $(0,3,0) \notin A$, and in a symmetrical way we obtain $(0,3,0) \notin B$. Since $\{(0,2,0), (0,3,0)\} \cap (A \cup B \cup C \cup D)$ is empty and $(0,1,0) \notin A \cup B$, then $(0,3,0) \notin A + B + C + D$. Since $A + B + C + D = I'_z$, we have a contradiction. Thus, $(0,1,0) \notin D$.

We next show that $(1,1,0) \notin D$. Assume otherwise. Then $(0,2,1) \notin A + B + C$. Since $(0,2,1) \in A + B + C + D$ but $(0,2,1) \notin A + B + C$, then there is an element $w \in D$ such that $0 < w \leq (0,2,1)$. If $(0,1,0) \in C$ then $(0,2,0) \notin D$ and $(0,2,1) \notin D$, for otherwise $(1,3,1) \in A + B + C + D$. If $(0,1,0) \in A \cup B$ then $(0,2,0) \notin D$ or $(0,2,1) \notin D$, for otherwise $(A+B+D)(I_z) > 9$ since

$$A + B + D \supsetneq \{0, e_1, e_2, e_3, e_1+e_3\} + \{0, (1,1,0), (0,2,0)\}$$

or

$$A + B + D \supsetneq \{0, e_1, e_2, e_3, e_1+e_3\} + \{0, (1,1,0), (0,2,1)\}.$$

Now $e_2 \in A \cup B \cup C$ since we have shown in the preceding paragraph that $e_2 \notin D$. It follows that not only $(0,1,0) \notin D$ but also $(0,2,0) \notin D$ and $(0,2,1) \notin D$. Consequently, $w = (0,1,1)$, and this in turn implies $(1,2,0) \notin A+B+C$. Since neither $(0,2,1)$ nor $(1,2,0)$ is an element of $A+B+C$ and since $e_1 \in A$ and $e_3 \in B$, then $(0,2,0) \notin A+C$ and $(0,2,0) \notin B+C$. Thus, $(0,1,0) \notin A \cap C$, $(0,1,0) \notin B \cap C$, and $(0,2,0) \notin A \cup B \cup C$. Furthermore, $(0,1,0) \notin A \cap B$, for otherwise $(A+B+D)(I_z) > 9$ since

$$A + B + D \supseteq \{0, e_1, e_2, e_3, e_1+e_3, (0,2,0)\} + \{0, (1,1,0), (0,1,1)\}.$$

It follows that $(0,2,0) \notin A+B+C+D$. However, this is contrary to $A+B+C+D = I_z'$. Thus, $(1,1,0) \notin D$.

An argument symmetric to the one given in the preceding paragraph establishes that $(0,1,1) \notin D$.

Finally, we show that $(1,0,1) \notin D$ and $(1,1,1) \notin D$. Assume $(1,0,1) \in D$ or $(1,1,1) \in D$. Recall that $(0,1,0) \notin D$. Also, $(0,3,0) \notin D$, for otherwise $(1,3,1) \in A+B+C+D$. If $(0,2,0) \notin D$ then $\{(0,1,0), (0,2,0), (0,3,0)\} \subseteq A+B+C$. However, $(0,3,0) \notin A+B+C$ if $(1,0,1) \in D$, and $(0,2,0) \notin A+B+C$ if $(1,1,1) \in D$. It follows that $(0,2,0) \in D$. Since $(0,2,0) \in D$ then $(1,1,1) \notin A+B+C$, and this in turn implies that $e_2 \notin C$ since $e_1 \in A$ and $e_3 \in B$. Therefore, $e_2 \in A \cup B$ since $e_2 \notin C \cup D$. Then

$$A + B + D \supseteq \{0, e_1, e_2, e_3, e_1+e_3\} + \{0, x, (0, 2, 0)\}$$

where $x = (1, 0, 1)$ or $x = (1, 1, 1)$ according as $(1, 0, 1) \in D$ or $(1, 1, 1) \in D$. A contradiction follows since $(A+B+D)(I_z) > 9$.

We have established that D and U^* have no elements in common. Thus, $U \subseteq A + B + C$. Since $D(I_z) \geq 1$ and $D \cap U^* = \emptyset$, then there is an element $(y_1, y_2, y_3) \in D$ such that $0 \leq y_1 \leq 1$, $2 \leq y_2 \leq 3$, and $0 \leq y_3 \leq 1$. But then $(1-y_1, 3-y_2, 1-y_3) \in U$, and so $(1, 3, 1) \in U + D \subseteq A + B + C + D$. However, this is not possible since $A + B + C + D = I_z'$.

The proof of the theorem is complete.

APPENDIX II

We now prove Theorem 4.6 which was stated without proof in Chapter IV.

Theorem. Let $z = (7, 1) \in J^2$ and let $A, B, C, D \subseteq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$, then at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

From Theorem 4.1(i), it follows that none of the sets $A + B + D$, $A + C + D$, and $B + C + D$ has more than nine nonzero elements. Moreover, from Theorem 4.1(ii), it follows that none of the sets $A + B$, $A + C$, $A + D$, $B + C$, $B + D$, and $C + D$ has more than eight nonzero elements.

Let $R_0 = \{(x, 0) \mid 1 \leq x \leq 7\}$ and $R_1 = \{(x, 1) \mid 0 \leq x \leq 7\}$. For any set $S \subseteq I_z$ denote $S \cap R_r$ by S_r, $r \in \{1, 2\}$.

Each of the integers $A_0(I_z)$, $B_0(I_z)$, $C_0(I_z)$, and $D_0(I_z)$ is greater than zero. For instance, if $A_0(I_z) = 0$ then $R_0 \subseteq B + C + D$ and, since $A(I_z) > 0$, there is an element $(x, 1) \in A$. But then $(7, 1) = (x, 1) + (7-x, 0) \in A + B + C + D$.

We next show that $A_0(I_z) > 1$ whenever $(1, 0) \in B + C + D$. Assume $A_0 = \{(a, 0)\}$. Then $A_1(I_z) \geq 4$. Let

$$A \supseteq \{(a_1, 1), (a_2, 1), (a_3, 1), (a_4, 1)\}$$

where $a_1 < a_2 < a_3 < a_4$. Since $(7, 1) \notin A + B + C + D$ then $(7-a_i, 0) \notin B + C + D$ for $1 \leq i \leq 4$. Thus, $(B+C+D)_0(I_z) \leq 3$. However, since

$$R_0 = ((B+C+D)_0 \cup \{0\}) + \{0, (a, 0)\},$$

then it follows that $(B+C+D)_0(I_z) = 3$. Let $(B+C+D)_0 = \{(1, 0), (x, 0), (y, 0)\}$ where $1 < x < y$. Then

$$R_0 = \{(1, 0), (x, 0), (y, 0), (a, 0), (a+1, 0), (a+x, 0), (a+y, 0)\},$$

and so either $x = 2$, $y = 3$, and $a = 4$, or $x = 4$, $y = 5$, and $a = 2$. First assume $x = 2$, $y = 3$, and $a = 4$. Then $A_0 = \{(4, 0)\}$ and $(B+C+D)_0 = \{(1, 0), (2, 0), (3, 0)\}$. Since $B_0(I_z) > 0$, $C_0(I_z) > 0$, and $D_0(I_z) > 0$, then $B_0 = C_0 = D_0 = \{(1, 0)\}$. But then

$$\begin{aligned} 9 \geq (A+B+D)(I_z) &= (A+B+D)_0(I_z) + (A+B+D)_1(I_z) \\ &\geq 5 + (A_1+\{0, (1, 0)\})(I_z) \\ &\geq 5 + (A_1(I_z)+1) > 9. \end{aligned}$$

Next assume $a_2 = 2$, $x = 4$, and $y = 5$. Then $A_0 = \{(2, 0)\}$ and $(B+C+D)_0 = \{(1, 0), (4, 0), (5, 0)\}$. It follows that one of the sets B_0, C_0, or D_0 contains $\{(1, 0)\}$ and each of the other two sets is $\{(4, 0)\}$. Now either $(1, 0) \in B + D$ or $(1, 0) \in C + D$. To be definite, say $(1,0) \in B + D$. Then $(B+D)_0 = (B+C+D)_0$, and we have

$$9 \geq (A+B+D)(I_z) = (A+B+D)_0(I_z) + (A+B+D)_1(I_z)$$

$$\geq R_0(I_z) + A_1(I_z) > 9.$$

Symmetric arguments establish that $B_0(I_z) > 1$ when $(1,0) \in A + C + D$ and $C_0(I_z) > 1$ when $(1,0) \in A + B + D$.

Now either $\{(1,0),(0,1)\}$ is a subset of $A + D$, $B + D$, or $C + D$, or otherwise $\{(1,0),(0,1)\} \cap D$ is empty and each of the sets A, B, and C has at most one element in common with $\{(1,0),(0,1)\}$.

Case 1. The set $\{(1,0),(0,1)\}$ is a subset of $A + D$, $B + D$, or $C + D$.

To be definite, say $\{(1,0),(0,1)\} \subseteq A + D$. Note that if $(1,1) \notin A + D$ then $\{(1,0),(0,1)\}$ is a subset of $A \setminus D$ or $D \setminus A$. Henceforth, when $(1,1) \notin A + D$ we let $G = A$ if $\{(1,0),(0,1)\} \subseteq A \setminus D$ and $G = D$ if $\{(1,0),(0,1)\} \subseteq D \setminus A$.

If $(1,1) \notin A + D$ then it is not possible for $B + G$ to equal $\{0,(1,0),(2,0),(3,0),(4,0),(0,1),(1,1),(2,1),(3,1)\}$. Assume otherwise. Let $H = A$ if $G = D$ and $H = D$ if $G = A$. Thus, $(1,0) \notin H$ and $(0,1) \notin H$; furthermore,

$$C + H \subseteq (B+G)^{\sim} = \{0,(1,0),(2,0),(3,0),(0,1),(1,1),(2,1)\}.$$

Since $(0,1) \notin H$ then $(0,1) \in C$; hence, $(3,0) \notin H$ for $(3,1) \notin C + H$. Since $H_0(I_z) > 0$, it follows that $H_0 = \{(2,0)\}$. This in turn implies that $(2,0) \notin C$ and $(3,0) \notin C$. But then $C(I_z) < 5$, which is a contradiction.

We claim $B_0(I_z) = 2$. Assume $B_0(I_z) \geq 3$, and let $B_0 \supseteq \{(b_1, 0), (b_2, 0), (b_3, 0)\}$ where $b_1 < b_2 < b_3$. Since $(0,1) \in A + D$ then $(7,0) \notin B$; hence $b_3 < 7$. If $(1,1) \notin A + D$ then

$$\begin{aligned} B + G &\supseteq (B_0 \cup \{0\}) + \{0, (1,0), (0,1)\} \\ &\supseteq \{(b_i, j) \mid i = 1,2,3 \text{ and } j = 0,1\} \cup \{0, (0,1), (b_3+1, 0)\}. \end{aligned}$$

However, this last set is equal to $B + G$ since $(B+G)(I_z) \leq 8$. Since $(b_i+1, 0) \in B + G$ for $i = 1$ and $i = 2$ and since $(1,0) \in B + G$, it must be that $b_1 = 1$, $b_2 = 2$, and $b_3 = 3$. But then

$$B + G = \{0(1,0), (2,0), (3,0), (4,0), (0,1), (1,1), (2,1), (3,1)\},$$

which is not possible. If $(1,1) \in A + D$ then

$$\begin{aligned} A + B + D &\supseteq (B_0 \cup \{0\}) + \{0, (1,0), (0,1), (1,1)\} \\ &\supseteq \{(b_i, j) \mid i = 0,1,2 \text{ and } j = 0,1\} \cup \{0, (0,1), (b_3+1, 0), (b_3+1, 1)\}. \end{aligned}$$

Since $(A+B+D)(I_z) \leq 9$, this last set is equal to $B + G$. But then

$(A+B+D) + \{(0,0), (0,1)\} = A + B + D$, which is contrary to Theorem 2.10. Thus, $B_0(I_z) < 3$. Recall that $B_0(I_z) > 1$ since $(1,0) \in A + C + D$. Hence, $B_0(I_z) = 2$.

Let $B_0 = \{(b_1, 0), (b_2, 0)\}$ and $B_1 \supseteq \{(b_3, 1), (b_4, 1), (b_5, 1)\}$ where $b_1 < b_2$ and $b_3 < b_4 < b_5$. Since $(0,1) \in A + D$ then $(7,0) \notin B$, and so $b_2 < 7$.

We first show that $b_1 = 1$. Suppose $(1,0) \notin B$. If $(1,1) \notin A + D$ then

$$\begin{aligned} B + G &\supseteq B + \{0, (1,0), (0,1)\} \\ &\supseteq \{0, (1,0), (b_1,0), (b_2,0), (b_2+1,0), (b_3,1), (b_4,1), (b_5,1), (b_5+1,1)\}. \end{aligned}$$

However, this last set is equal to $B + G$ since $(B+G)(I_z) \leq 8$. Since $(b_1+1, 0) \in B + G$ then $b_2 = b_1+1$. Since $(0,1) \in B + G$ and $(b_j+1, 1) \in B + G$ for $j \in \{3, 4\}$, then it follows that $b_3 = 0$, $b_4 = 1$, and $b_5 = 2$. Furthermore, $\{(b_1, 1), (b_2, 1)\} \subseteq B + G$, and so $b_1 = 2$ and $b_2 = 3$. But then

$$B + G = \{0, (1,0), (2,0), (3,0), (4,0), (0,1), (1,1), (2,1), (3,1)\},$$

which is not possible. If $(1,1) \in A + D$ then

$$\begin{aligned} A+B+D &\supseteq (B_0 \cup \{0\}) + \{0, (1,0), (0,1), (1,1)\} \\ &\supseteq \{0, (1,0), (b_1,0), (b_2,0), (b_2+1,0), (0,1), (1,1), (b_1,1), (b_2,1), (b_2+1,1)\}. \end{aligned}$$

However, this last set is equal to $A + B + D$ since $(A+B+D)(I_z) \leq 9$. But then $(A+B+D) + \{(0, 0), (0, 1)\} = A + B + D$, which is contrary to Theorem 2.10. Therefore, $b_1 = 1$.

Next we show that $b_2 = 2$. Assume $b_2 > 2$. If $(1, 1) \in A + D$ then

$$A + B + D \supseteq (B_0 \cup \{0\}) + \{0, (1, 0), (0, 1), (1, 1)\}$$
$$\supseteq \{0, (1,0), (2,0), (b_2,0), (b_2+1,0), (0,1), (1,1), (2,1), (b_2,1), (b_2+1,1)\}.$$

Since $(A+B+D)(I_z) \leq 9$, this last set is equal to $A + B + D$. But then $(A+B+D) + \{(0, 0), (0, 1)\} = A + B + D$. If $(1, 1) \notin A + D$ then

$$B + G \supseteq B + \{0, (1, 0), (0, 1)\}$$
$$\supseteq \{0, (1, 0), (2, 0), (b_2, 0), (b_2+1, 0), (b_3, 1), (b_4, 1), (b_5, 1), (b_5+1)\}.$$

However, this last set is equal to $B + G$ since $(B+G)(I_z) \leq 8$. Since $\{(0, 1), (b_3+1, 1), (b_4+1, 1)\} \subseteq B + G$, then $b_3 = 0$, $b_4 = 1$, and $b_5 = 2$. Also, $(b_2, 1) \in B + G$, and so $b_2 = 3$. But then

$$B + G = \{0, (1, 0), (2, 0), (3, 0), (4, 0), (0, 1), (1, 1), (2, 1), (3, 1)\},$$

which is not possible. Therefore, $b_2 = 2$.

Now $C_1(I_z) \neq 0$, for otherwise $C_0(I_z) \geq 5$ and $(A+C+D)(I_z) \geq (C_0+\{0, (0, 1)\})(I_z) > 9$. Recall that $C_0(I_z) > 0$. Also $\{(6, 0), (7, 0), (6, 1), (7, 1)\} \cap C = \emptyset$ for $\{0, (1,0), (0,1), (1,1)\} \supseteq A + B + D$. Since $\{(1, 0), (2, 0)\} \subseteq B$, it follows that

$$\begin{aligned}(B+C)(I_z) &= (B+C)_0(I_z) + (B+C)_1(I_z)\\ &\geq (C_0(I_z)+2) + (C_1(I_z)+2)\\ &= C(I_z) + 4 > 8.\end{aligned}$$

However, $(B+C)(I_z) \leq 8$, and we have a contradiction.

Case 2. The set $D \cap \{(1,0),(0,1)\}$ is empty and each of the sets A, B, and C has at most one element in common with $\{(1,0),(0,1)\}$.

By relabeling the sets A, B, and C if necessary, we may assume that $(1,0) \in A$ and $(0,1) \in B$.

Since $\{0,(1,0),(0,1),(1,1)\} \subseteq A + B$ and $(7,1) \notin A + B + C + D$, then $(C+D) \cap \{(6,0),(7,0),(6,1),(7,1)\}$ is empty.

We claim $(C+D)_0(I_z) \leq 3$. Assume $(C+D)_0(I_z) > 3$. Since $B + C + D \supseteq \{0,(0,1)\} + (C+D)_0$, $(B+C+D)(I_z) \leq 9$, and $(\{0,(0,1)\} + (C+D)_0(I_z) \geq 9$, then

$$B + C + D = \{0,(0,1)\} + (C+D)_0$$

But then

$$\begin{aligned}(B+C+D) + \{0,(0,1)\} &= (C+D)_0 + \{0,(0,1)\} + 0,(0,1)\}\\ &= (C+D)_0 + \{0,(0,1)\}\\ &= B + C + D.\end{aligned}$$

However, this is contrary to Theorem 2.10.

Recall that $C_0(I_z) > 1$ since $(1, 0) \in A + B + D$. Also, $D_0(I_z) \geq 1$. Let $c_1 = \min\{x \mid (x, 0) \in C_0\}$ and $c_2 = \min\{x \mid (x, 0) \in C_0$ and $x \neq c_1\}$.

We now show that $c_1 > 1$. Assume $c_1 = 1$. Let $(d, 0) \in D$. Since $\{(1, 0), (d, 0), (d+1, 0)\} \subseteq (C+D)_0$, $d > 1$, and $(C+D)_0(I_z) \leq 3$, then $(C+D)_0 = \{(1, 0), (d, 0), (d+1, 0)\}$. Hence, $d \leq c_2 \leq d+1 \leq 5$. Also, $(c_2, 0) + (d, 0) \notin (C+D)_0$ since $c_2+d \geq 2+d$. Thus, $c_2+d > 7$. But then $9 \geq (d+1)+d \geq c_2+d \geq 8$, and so $d = 4$. Therefore, $D_0 = \{(4, 0)\}$. It follows that $(3, 1) \notin A + B + C$, and this in turn implies that $(3, 0) \notin A + C$. Since $(3, 0) \in A + B + C$, $(3, 0) \notin A + C$, and $(1, 0) \notin B$, then $(2, 0) \in B$ or $(3, 0) \in B$. Now $(2, 0) \notin B$, for otherwise $B + C + D \supseteq R_0 \cup ((C+D)_0+\{(0, 1)\})$ and so $(B+C+D)(I_z) > 9$. Thus, $(3, 0) \in B$ and

$$B + C + D \supseteq \{0, (1, 0), (3, 0), (4, 0), (5, 0), (7, 0), (0, 1), (1, 1), (4, 1), (5, 1)\}.$$

However, this last set is equal to $B + C + D$ since $(B+C+D)(I_z) \leq 9$. But then $(B+C+D) + \{0, (4, 0)\} = B + C + D$, which is contrary to Theorem 2.10. Thus, $c_1 > 1$.

Next we show $c_1 > 2$. Assume $c_1 = 2$. Then $(4, 0) \notin D$ and $(5, 0) \notin D$, for otherwise $(6, 0)$ or $(7, 0)$ is in $C + D$. If $c_2 = 3$ then $(2, 0) \notin D$, for otherwise $(C+D)_0(I_z) > 3$. If $4 \leq c_2 \leq 5$ then $(2, 0) \notin D$, for otherwise $(6, 0)$ or $(7, 0)$ is in $C + D$. Since $3 \leq c_2 \leq 5$, it follows that $(2,0) \notin D$. Hence, $D = \{(3,0)\}$. Thus, $(3,0) \notin C$ and $(4,0) \notin C$, for otherwise $(6,0)$ or $(7,0)$ is in $C + D$. It follows that $C_0 = \{(2,0),(5,0)\}$. Now $C(I_z) \geq 5$, and so $C_1(I_z) \geq 3$. Since $(1,0) \in A$ then

$$\begin{aligned}(A+C+D)(I_z) &= (A+C+D)_0(I_z) + (A+C+D)_1(I_z)\\ &\geq 6 + (A+C)_1(I_z)\\ &\geq 6 + (C_1(I_z)+1) \geq 10.\end{aligned}$$

However, $(A+C+D)(I_z) \leq 9$. Thus, $c_1 > 2$.

Since $(C+D)_0 \subseteq \{(x,0) | 2 \leq x \leq 5\}$, $c_1 \geq 3$, $C_0(I_z) \geq 2$, and $D_0(I_z) \geq 1$, then one of the following occurs:

(i) $c_1 = 3$ and $D_0 = \{(5,0)\}$,

(ii) $c_1 = 4$ and $D_0 \subseteq \{(4,0),(5,0)\}$.

In either case $A + B + C \supseteq \{(x,0) | 0 \leq x \leq 6\}$, and so $(x,1) \notin D$ for $1 \leq x \leq 7$. Therefore, $D = D_0$. Since $D \subseteq \{(4,0),(5,0)\}$, then either $(2,1) \notin A + B + C$ or $(3,1) \notin A + B + C$. This in turn implies either $(2,1) \notin A + B + C + D$ or $(3,1) \notin A + B + C + D$. Since $A + B + C + D = I'_z$, we have a contradiction.

The proof of the theorem is now complete.

APPENDIX III

We now prove Theorem 4.7 which was stated without proof in Chapter IV.

Theorem. Let $z = (3, 3) \in J^2$ and let $A, B, C, D \subsetneq I_z$. If $A + B + C + D = I'_z$ and $D(I_z) > 0$, then at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

From Theorem 4.1(i), it follows that none of the sets $A + B + D$, $A + C + D$, and $B + C + D$ has more than nine nonzero elements. Moreover, from Theorem 4.1(ii), it follows that none of the sets $A + B$, $A + C$, $A + D$, $B + C$, $B + D$, and $C + D$ has more than eight nonzero elements.

Let $R_0 = \{(x, 0) \mid 1 \leq x \leq 3\}$, $R_1 = \{(x, 1) \mid 0 \leq x \leq 3\}$, $R_2 = \{(x, 2) \mid 0 \leq x \leq 3\}$, and $R_3 = \{(x, 3) \mid 0 \leq x \leq 3\}$. For any set $S \subsetneq I_z$ let $S_r = S \cap R_r$, $0 \leq r \leq 3$.

Now either $\{(1, 0), (0, 1)\}$ is a subset of $A + D$, $B + D$ or $C + D$, or otherwise $\{(1, 0), (0, 1)\} \cap D$ is the empty set and each of the sets A, B, and C has at most one element in common with $\{(1, 0), (0, 1)\}$.

Case 1. The set $\{(1, 0), (0, 1)\}$ is a subset of $A + D$, $B + D$, or $C + D$.

To be definite, say $\{(1,0),(0,1)\} \subseteq A+D$. Note for later use that if $(1,1) \notin A+D$ then $\{(1,0),(0,1)\}$ is a subset of $A \setminus D$ or $D \setminus A$. Now $(0,2)$ is an element of $A+B+D$ or $A+C+D$. By relabeling the sets B and C if it is necessary, we may assume that $(0,2) \in A+C+D$. Since $\{(0,0),(1,0),(0,1),(0,2)\} \subseteq A+C+D$ and $(3,3) \notin A+B+C+D$, then B has an empty intersection with $\{(3,1),(3,2),(2,3),(3,3)\}$. Thus, $B_1 \subseteq \{(x,1) \mid 0 \leq x \leq 2\}$, $B_2 \subseteq \{(x,2) \mid 0 \leq x \leq 2\}$, and $B_3 \subseteq \{(0,3),(1,3)\}$. We proceed to show that $\max\{B_r(I_z) \mid 0 \leq r \leq 3\} \leq 2$.

Assume $B_0(I_z) > 2$. Then there exist distinct elements (x_0, y_0) and (x_1, y_1) such that $y_1 \geq y_0 > 0$ and

$$B \supseteq \{(1,0),(2,0),(3,0),(x_0,y_0),(x_1,y_1)\}.$$

Let $W = A$ if $(0,1) \in A$; otherwise, let $W = D$. Then

$$B + W \supseteq R_0 \cup R_1 \cup \{(x_0, y_0+s_0),(x_1, y_1+s_1)\}$$

where

$$s_i = \begin{cases} 0 & \text{if } x_0 \neq x_1 \text{ and } y_i > 1, \\ 1 & \text{if } x_0 \neq x_1 \text{ and } y_i = 1, \\ 0 & \text{if } x_0 = x_1 \text{ and } y_0 > 1, \\ 1 & \text{if } x_0 = x_1,\ y_0 = 1, \text{ and } y_1 = 2, \\ 1-i & \text{if } x_0 = x_1,\ y_0 = 1, \text{ and } y_1 = 3, \end{cases}$$

for $i = 1, 2$. But then $(B+W)(I_z) > 8$, which is a contradiction. Hence, $B_0(I_z) \leq 2$.

Assume $B_1(I_z) > 2$. Then there exist distinct nonzero elements (x_0, y_0) and (x_1, y_1) such that $y_1 \geq y_0$, $y_1 \neq 1$, $y_0 \neq 1$, $x_1 > x_0$ if $y_1 = y_0$, and

$$B \supseteq \{(0, 1), (1, 1), (2, 1), (x_0, y_0), (x_1, y_1)\}.$$

If $(x_0, y_0) = (1, 0)$ then

$$A + B + D \supseteq R_1 \cup (R_2 \setminus \{(3, 2)\}) \cup \{(1, 0), (2, 0)\} \cup \{(x_2, y_2)\}$$

where

$$(x_2, y_2) = \begin{cases} (x_1, y_1) & \text{if } y_1 = 3, \\ (x_1, y_1+1) & \text{if } y_1 = 2, \\ (3, 0) & \text{if } y_1 = 0. \end{cases}$$

But then $(A+B+D)(I_z) > 9$, which is not possible. Thus, $(1, 0) \notin B$, and

$$A + B + D \supseteq R_1 \cup (R_2 \setminus \{(3, 2)\}) \cup \{(1, 0)\} \cup \{(x_0, y_0+s_0), (x_1+t, y_1+s_1)\}$$

where $t = s_0 = 1$ and $s_1 = 0$ if $x_0 = x_1$, $y_0 = 2$, and $y_1 = 3$, and where otherwise $t = 0$ and

$$s_i = \begin{cases} 0 & \text{if} \quad x_0 \neq x_1 \quad \text{and} \quad y_i \in \{0, 3\}, \\ 1 & \text{if} \quad x_0 \neq x_1 \quad \text{and} \quad y_i = 2, \\ 0 & \text{if} \quad x_0 = x_1,\ y_0 = 0, \quad \text{and} \quad y_1 = 3, \\ i & \text{if} \quad x_0 = x_1,\ y_0 = 0, \quad \text{and} \quad y_1 = 2. \end{cases}$$

Again a contradiction follows since $(A+B+D)(I_z) > 9$. Hence, $B_1(I_z) \leq 2$.

Assume $B_2(I_z) > 2$. Then $B_2 = \{(0,2), (1,2), (2,2)\}$. Since $(2,2) \in B$ then $(1,1) \notin A + D$. Let $W = A$ if $\{(1,0), (0,1)\} \subseteq A \setminus D$; otherwise, let $W = D$. Then

$$B + W \supseteq \{(1,0), (0,1)\} \cup R_2 \cup (R_3 \setminus \{(3,3)\})$$

and $(B+W)(I_z) > 8$, which is not possible. Hence $B_2(I_z) \leq 2$.

Clearly, $B_3(I_z) \leq 2$ since $B_3 \subseteq \{(0,3), (1,3)\}$. This establishes that $\max\{B_r(I_z) \mid 0 \leq r \leq 3\} \leq 2$.

Continue to let $W = A$ if $(1,0) \in A$, and $W = D$ otherwise. We next establish that $(B+W)_0(I_z) \geq B_0(I_z) + 1$ and $(B+W)_r(I_z) \geq B_r(I_z) + 1$ whenever $B_r(I_z) > 0$ and $1 \leq r \leq 3$. If $(1,0) \notin B_0$ then $(1,0) \in W \setminus B$ and $(B+W)_0(I_z) \geq B_0(I_z) + 1$. If $(1,0) \in B_0$ then B_0 is equal to $\{(1,0)\}$, $\{(1,0), (2,0)\}$, or $\{(1,0), (3,0)\}$. In any case, $(B+W)_0(I_z) \geq B_0(I_z) + 1$. If $B_r(I_z) > 0$ where $1 \leq r \leq 3$, then let $b_r = \max\{x \mid (x, r) \in B\}$. Since $b_r \leq 2$ then $(b_r+1, 1) \in B + W$. However, $(b_r+1, 1) \notin B$, and so

$(B+W)_r(I_z) \geq B_r(I_z) + 1$.

We now show that $B_0(I_z) > 0$ and exactly one of the numbers $B_1(I_z)$, $B_2(I_z)$, and $B_3(I_z)$ is zero. First, assume $B_r(I_z) > 0$ for $r = 1, 2, 3$. Then

$$\begin{aligned} 8 \geq (B+W)(I_z) &= \sum_{r=0}^{3} (B+W)_r(I_z) \\ &\geq \sum_{r=0}^{3} (B_r(I_z)+1) \\ &= B(I_z) + 4 > 8, \end{aligned}$$

a contradiction. Therefore, for some integer $t \in \{1, 2, 3\}$ we have $B_t(I_z) = 0$. Since $B(I_z) \geq 5$ and $B_r(I_z) \leq 2$ for $0 \leq r \leq 3$, it follows that $B_r(I_z) > 0$ for $0 \leq r \leq 3$ and $r \neq t$.

Since $B_{t-1} \neq \emptyset$ then there is an integer b, $0 \leq b \leq 2$, such that $(b, t-1) \in B$. Now $(1, 1) \notin A + D$, for otherwise $\{(b, t), (b+1, t)\} \subseteq (A+B+D)_t$ and

$$\begin{aligned} (A+B+D)(I_z) &= \sum_{0 \leq r \leq 3} (A+B+D)_r(I_z) \\ &= \Big(\sum_{\substack{0 \leq r \leq 3 \\ r \neq t}} (B+W)_r(I_z)\Big) + (A+B+D)_t(I_z) \end{aligned}$$

$$\geq \left(\sum_{\substack{0 \leq r \leq 3 \\ r \neq t}} (B_r(I_z) + 1) \right) + 2$$

$$= B(I_z) + 5 > 9.$$

Since $(1, 1) \notin A + D$ then $W \supseteq \{(1, 0), (0, 1)\}$. Hence, $(b, t) \in B + W$, and so $(B+W)_t(I_z) \geq 1 = B_t(I_z) + 1$. Therefore,

$$8 \geq (B+W)(I_z) = \sum_{0 \leq r \leq 3} (B+W)_r(I_z)$$

$$\geq \sum_{0 \leq r \leq 3} (B_r(I_z)+1)$$

$$= B(I_z) + 4 > 8,$$

which is a contradiction.

Case 2. The set $\{(1, 0), (0, 1)\} \cap D$ is empty and each of the sets A, B, and C has at most one element in common with $\{(1, 0), (0, 1)\}$.

With no loss in generality, we may assume that $(1, 0) \in A$ and $(0, 1) \in B$. Since $(3, 3) \notin A + B + C + D$ it follows that $C \cap \{(2, 2), (2, 3), (3, 2), (3, 3)\}$ is empty; therefore, $C_2 \subsetneq \{(0, 2), (1, 2)\}$ and $C_3 \subsetneq \{(0, 3), (1, 3)\}$. Set $i_0 = C_0(I_z)$, $i_1 = C_1(I_z)$, $i_2 = C_2(I_z)$ and $i_3 = C_3(I_z)$. Thus, $i_2 \leq 2$ and $i_3 \leq 2$.

We now show that $i_0 \leq 2$ and $i_1 \leq 3$. Assume $i_0 > 2$. Then $C \supseteq R_0 \cup \{(x_0, y_0), (x_1, y_1)\}$ where $(x_0, y_0) \neq (x_1, y_1)$ and $y_1 \geq y_0 > 0$. Therefore,

$$B + C \supseteq R_0 \cup R_1 \cup \{(x_0, y_0+s_0), (x_1, y_1+s_1)\}$$

where

$$s_t = \begin{cases} 0 & \text{if } x_0 \neq x_1 \text{ and } y_t > 1, \\ 1 & \text{if } x_0 \neq x_1 \text{ and } y_t = 1, \\ 0 & \text{if } x_0 = x_1,\ y_1 = 3, \text{ and } y_0 = 2, \\ 1 & \text{if } x_0 = x_1,\ y_1 = 2, \text{ and } y_0 = 1, \\ 1-t & \text{if } x_0 = x_1,\ y_1 = 3, \text{ and } y_0 = 1. \end{cases}$$

But then $(B+C)(I_z) > 8$, which is a contradiction. Thus $i_0 \leq 2$. Assume $i_1 > 3$. Then $C \supseteq R_1 \cup \{(x, y)\}$ where $y \neq 1$. But then

$$B + C \supseteq R_1 \cup R_2 \cup \{(x, y+s)\}$$

where $s = 1$ if $y = 2$ and $s = 0$ if $y \neq 2$. Again, $(B+C)(I_z) > 8$. Hence $i_1 \leq 3$.

We claim that $(A+C)_0(I_z) \geq C_0(I_z) + 1$ and $(A+C)_r(I_z) \geq C_r(I_z) + 1$ when $C_r(I_z) > 0$ for $2 \leq r \leq 3$. If $(1, 0) \notin C$ then $(A+C)_0(I_z) \geq C_0(I_z) + 1$ since $(1, 0) \in A$. If $(1, 0) \in C$ then C_0 is equal to $\{(1, 0)\}$, $\{(1, 0), (2, 0)\}$, or $\{(1, 0), (3, 0)\}$. In any case, $(A+C)_0(I_z) \geq C_0(I_z) + 1$. Assume

$C_r(I_z) > 0$ where $2 \leq r \leq 3$. Let $x_r = \max\{x \mid (x, r) \in C\}$. Then $x_r \leq 1$, and $(x_r+1, r) \in (A+C) \setminus C$. Hence, $(A+C)_r(I_z) \geq C_r(I_z) + 1$.

We next show that if $(3, 1) \in C$ and $(0, 1) \notin C$, then $C_1 = \{(1, 1), (3, 1)\}$. Since $(3, 1) \in C$ then $(0, 2) \notin A + B + D$, and this in turn implies that $(0, 1) \in C$ or $(0, 2) \in C$. Thus $(0, 2) \in C$, and so $\{x \in J^2 \mid x \leq (1, 3)\} \subseteq A + B + C$. Then D has no elements in common with $\{(x, y) \mid 2 \leq x \leq 3,\ 0 \leq y \leq 3\}$, for otherwise $(3, 3) \in A + B + C + D$. Assume $(1, 1) \notin D$. Then $(R_0 \cup R_1) \cap D = \emptyset$ and $R_0 \cup R_1 \subseteq A + B + C$. Since $D(I_z) > 0$, there is an element $(x, y) \in D$ with $0 \leq x \leq 1$ and $2 \leq y \leq 3$. But then $(3-x, 3-y) \in R_0 \cup R_1$, and so $(3-x, 3-y) \in A + B + C$. Since $(3, 3) \notin A + B + C + D$, we conclude that $(1, 1) \in D$. Now $(2,1) \notin C$ since $(1, 2) = (0, 1) + (1, 1) \in B + D$. Assume $C_1 = \{(3, 1)\}$. Then $(A+C+D)_1 \supseteq \{(1, 1), (2, 1), (3, 1)\}$ and so $(A+C+D)_1(I_z) \geq i_1+2$. Since $(0, 2) \in C$ then $(A+C)_2(I_z) \geq i_2+1$. Since $(1, 3) = (0, 2) + (1, 1) \in C + D$, then

$$(A+C+D)_3(I_z) \geq \max\{(C+D)_3(I_z), (A+C)_3(I_z)\} \geq i_3+1$$

whether $i_3 = 0$ or $i_3 > 0$. But then

$$(A+C+D)(I_z) = \sum_{r=0}^{3} (A+C+D)_r(I_z)$$

$$\geq (A+C)_0(I_z) + (i_1+2) + (A+C)_2(I_z) + (i_3+1)$$

$$\geq (i_0+1) + (i_1+2) + (i_2+1) + (i_3+1)$$

$$= C(I_z) + 5 > 9.$$

Thus, it is not possible that $C_1 = \{(3,1)\}$. Since $(3,1) \in C$, $(0,1) \notin C$, and $(2,1) \notin C$, then $C_1 = \{(1,1),(3,1)\}$.

We claim $(A+C)_1(I_z) \geq i_1+1$ whenever $i_1 > 0$. If $i_1 > 0$ and $(3,1) \notin C$ then $(x_1+1, 1) \in (A+C) \setminus C$ where $x_1 = \max\{x \mid (x,1) \in C\}$. If $(3,1) \in C$ and $(0,1) \notin C$ then $C_1 = \{(1,1),(3,1)\}$; hence, $(2,1) \in (A+C) \setminus C$. If $(3,1) \in C$ and $(0,1) \in C$, then either $(1,1) \notin C$ or $(1,1) \in C$ and $(2,1) \notin C$ since $i_1 \leq 3$. But then either $(1,1)$ or $(2,1)$ is an element of $(A+C) \setminus C$.

One of the numbers i_1, i_2, or i_3 is zero, for otherwise

$$(A+C)(I_z) = \sum_{0 \leq r \leq 3} (A+C)_r(I_z) \geq \sum_{0 \leq r \leq 3} (C_r(I_z)+1) > 8.$$

If $i_1 = 0$ then $C(I_z) \geq 5$, $i_0 \leq 2$, $i_2 \leq 2$, and $i_3 \leq 2$ imply $i_0 > 0$, $i_2 > 0$, and $i_3 > 0$. If $i_1 \neq 0$ we proceed to show that not both $i_2 = 0$ and $i_3 = 0$. Assume $i_2 = 0$ and $i_3 = 0$. Then from

$C(I_z) \geq 5$, $i_0 \leq 2$, and $i_1 \leq 3$ we have $i_0 = 2$ and $i_1 = 3$. Since $(A+C)_0(I_z) \geq i_0+1 = 3$ and $(A+C)_1(I_z) \geq i_1+1 = 4$, then $R_0 \cup R_1 \subseteq A + C$. Hence, $(0, 1) \in C$. Now $A(I_z) \geq 5$ implies there exist distinct elements (x_0, y_0) and $(x_1, y_1) \in A$ such that $y_1 \geq y_0 > 0$, and so

$$A + C \supseteq R_0 \cup R_1 \cup \{(x_0, y_0+s_0), (x_1, y_1+s_1)\}$$

where

$$s_t = \begin{cases} 0 & \text{if } x_0 \neq x_1 \text{ and } y_t > 1, \\ 1 & \text{if } x_0 \neq x_1 \text{ and } y_t = 1, \\ 0 & \text{if } x_0 = x_1 \text{ and } y_0 > 1, \\ 1 & \text{if } x_0 = x_1,\ y_1 = 2, \text{ and } y_0 = 1, \\ 1-t & \text{if } x_0 = x_1,\ y_1 = 3, \text{ and } y_0 = 1. \end{cases}$$

But then $(A+C)(I_z) > 8$, which is a contradiction. Thus, $i_2 > 0$ or $i_3 > 0$.

The preceding paragraph establishes that exactly one of the numbers i_1, i_2, and i_3 is zero.

We show that the element $(2, 0) \notin D$. Assume $(2, 0) \in D$. Then $(1, 3) \notin A + B + C$, and this implies that $\{(x, y) \mid 0 \leq x \leq 1,\ 2 \leq y \leq 3\} \cap C$ is empty. But then $i_2 = 0$ and $i_3 = 0$.

Next we show that $(3, 0) \notin D$. Assume $(3, 0) \in D$. Then

$(0, 3) \notin A + B + C$, and so $(0, 2) \in D$ or $(0, 3) \in D$. This in turn implies that $(2, 0) \notin C$ and $(3, 0) \notin C$. Thus, $C_0 \subseteq \{(1, 0)\}$ and $i_0 \leq 1$. Furthermore, since $(3, 0) \in D$ then $(0, 2) \notin C$ and $(0, 3) \notin C$. Hence $i_2 \leq 1$ and $i_3 \leq 1$. Recall that $i_1 \leq 3$, $C(I_z) \geq 5$, and exactly one of the numbers i_1, i_2, and $i_3 = 0$. It follows that $i_0 = 1$ and $i_1 = 3$. Since $i_1 = 3$, then $(A+C)_1(I_z) = 4$. Thus, $(A+C)_1 = R_1$, and so $(0, 1) \in A \cup C$. Since $i_0 = 1$ then $C_0 = \{(1, 0)\}$. But then $\{(1, 0), (0, 1)\}$ is a subset of A or C.

We now claim that $(2, 1) \notin D$. Assume $(2, 1) \in D$. Then $(1, 2) \notin A + B + C$, and this implies that $\{(0, 1), (1, 1), (0, 2), (1, 2)\} \cap C$ is empty. Thus, $i_2 = 0$ and $C_1 \subseteq \{(2, 1), (3, 1)\}$. Since $(0, 1) \notin C$ then $(3, 1) \notin C$, for otherwise $C_1 = \{(1, 1), (3, 1)\}$. Since $(1, 2) \notin A + B + C$, then $\{(1, 1), (0, 2), (1, 2)\}$ has a nonempty intersection with D. It follows that $(2, 1) \notin C$, for otherwise $(3, 3) \in A + B + C + D$. But then $i_1 = 0$ and $i_2 = 0$.

Next, $(3, 1) \notin D$. Assume $(3, 1) \in D$. Then $(0,2) \notin A + B + C$, and this implies that $(0, 2) \in D$, $(0, 2) \notin C$, and $(0, 1) \notin C$. Since $(0, 2) \in D$ then $\{(x, y) \mid 2 \leq x \leq 3,\ 0 \leq y \leq 1\} \cap C$ is empty, for otherwise $(3, 3) \in A + B + C + D$. Thus, $i_0 \leq 1$, $i_1 \leq 1$, and $i_2 \leq 1$. However, this is not possible since $C(I_z) \geq 5$, $i_3 \leq 2$, and either i_1, i_2, or i_3 is zero.

Finally $(1,1) \notin D$. Assume $(1,1) \in D$. Then $\{(1,1),(2,1),(1,2)\} \cap C$ is empty. Thus, $C_1 \subsetneq \{(0,1),(3,1)\}$ and $C_2 \subsetneq \{(0,2)\}$. We have previously established that $C_1 = \{(1,1),(3,1)\}$ if $(3,1) \in C$ and $(0,1) \notin C$. It follows that $(0,1) \in C$ if $i_1 > 0$. If $i_3 = 0$ then $i_2 \neq 0$, and so $C_2 = \{(0,2)\}$ and $(A+C+D)_3 \supseteq \{(1,3),(2,3)\}$. If $i_2 = 0$ then $i_1 \neq 0$, and so $(0,1) \in C$ and $(A+C+D)_2 \supseteq \{(1,2),(2,2)\}$. Also, $(A+C+D)_1 \supseteq \{(1,1),(2,1)\}$. Now let $t \in \{1,2,3\}$ be such that $C_t(I_z) = 0$. Then $(A+C+D)_t(I_z) \geq 2$ and

$$
\begin{aligned}
(A+C+D)(I_z) &= \sum_{0 \leq r \leq 3} (A+C+D)_r(I_z) \\
&= \Big(\sum_{\substack{0 \leq r \leq 3 \\ r \neq t}} (A+C+D)_r(I_z)\Big) + (A+C+D)_t(I_z) \\
&\geq \Big(\sum_{\substack{0 \leq r \leq 3 \\ r \neq t}} (A+C)_r(I_z)\Big) + 2 \\
&\geq \Big(\sum_{\substack{0 \leq r \leq 3 \\ r \neq t}} (C_r(I_z)+1)\Big) + 2 \\
&\geq C(I_z) + 5 > 9,
\end{aligned}
$$

which is a contradiction.

We have established that $D \cap (R_0 \cup R_1)$ is the empty set. Thus, $A + B + C \supseteq R_0 \cup R_1$. Now, $D(I_z) > 0$ and so there is an element $(x, y) \in D$ such that $0 \leq x \leq 3$ and $2 \leq y \leq 3$. But then $(3-x, 3-y) \in R_0 \cup R_1$, and it follows that $(3, 3) \in A + B + C + D$. However, this is contrary to $A + B + C + D = I_z'$.

The proof of the theorem is now complete.

APPENDIX IV

We now prove Theorem 4.8 which was stated without proof in Chapter IV.

<u>Theorem.</u> Let $z = (4, 2) \in J^2$ and let $A, B, C \subseteq I_z$. If $A + B + C = I'_z$ then at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

Now $\{(1,0),(0,1)\} \subseteq A \cup B \cup C$. Relabel sets A, B, and C if necessary so that $(1,0) \in A$ and $(0,1) \in A \cup B$.

Since $A + B + C = I'_z$, it follows from Theorem 4.1(i) that $(A+B)(I_z) \leq 13 - C(I_z) \leq 8$.

Define $R_0 = \{(x,0) \mid 1 \leq x \leq 4\}$, $R_1 = \{(x,1) \mid 0 \leq x \leq 4\}$, and $R_2 = \{(x,0) \mid 0 \leq x \leq 4\}$. Let $S_t = S \cap R_t$ for any set $S \subseteq I_z$ and $t \in \{0, 1, 2\}$. Set $i_0 = B_0(I_z)$, $i_1 = B_1(I_z)$, and $i_2 = B_2(I_z)$.

Now $B_2 \subseteq \{(0,2),(1,2),(2,2)\}$, for otherwise $(4,2) \in A + B$. It follows that $i_2 \leq 3$.

We claim $i_0 \leq 3$. Assume $i_0 > 3$; hence $R_0 = B_0$. Now A_2 is empty, for otherwise if $(t,2) \in A$ with $0 \leq t \leq 4$ then $(4,2) = (t,2) + (4-t,0) \in A + B$. Also $(0,1) \notin A$, for otherwise

$$A + B \supseteq B_0 \cup (B_0+\{(0,1)\}) \cup \{(0,1)\} = R_0 \cup R_1$$

and $(A+B)(I_z) > 8$. Thus, $(0, 1) \in B$. Since $A(I_z) \geq 5$, $A_0(I_z) \leq 4$, and $A_2(I_z) = 0$, then $A_1(I_z) \geq 1$. Choose $(a, 1) \in A$. Now $a < 4$, for otherwise $(4, 2) = (4, 1) + (0, 1)$ is in $A + B$. Also, $a \geq 1$ since $(0, 1) \notin A$. Then

$$A + B \supseteq \{0, (1, 0), (a,1)\} + (R_0 \cup \{0, (0, 1)\})$$
$$\supseteq R_0 \cup \{(0, 1), (a, 1), (a+1, 1), (a, 2), (x, 1)\}$$

where $x = 1$ if $a > 1$ and $x = 3$ if $a = 1$. But then $(A+B)(I_z) > 8$. Hence, $i_0 \leq 3$.

We now show $i_1 \leq 3$. Assume $i_1 > 3$. Then $(0, 1) \in B$, for otherwise $(0, 1) \in A$ and $B_1 = \{(x, 1) | 1 \leq x \leq 4\}$, which imply $(4, 2) \in A + B$. Since $(1, 0) \in A$, $(0, 1) \in B$, and $i_1 > 3$, then $(A+B)_1 = R_1$. If $(x, 1) \in B$ and $0 \leq x \leq 4$ then $(4-x, 1) \notin A$. Thus, $A_1(I_z) \leq R_1(I_z) - B_1(I_z) \leq 1$. Since $A(I_z) \geq 5$ and $A_1(I_z) \leq 1$, then $A_0(I_z) + A_2(I_z) \geq 4$. But then $(A+B)(I_z) > 8$ since $A + B \supseteq A_0 \cup R_1 \cup A_2$. Therefore, $i_1 \leq 3$.

We claim $(A+B)_0(I_z) \geq i_0+1$. This is immediate if $(1, 0) \notin B$. If $(1, 0) \in B$ then there is an integer b' such that $1 \leq b' \leq 3$, $(b', 0) \in B$, and $(b'+1, 0) \notin B$, for otherwise $i_0 > 3$. But $A + B$ contains $(b'+1, 0)$, and so $(A+B)_0 \geq i_0+1$.

We next claim $(A+B)_1(I_z) \geq i_1+1$, and $(A+B)_1(i_z) \geq i_1+2$ if $(0, 1) \notin B$. Let $i_1 = 0$. Then $(0, 1) \in A$. Also $i_0 \geq 2$ since

$B(I_z) \geq 5$ and $i_2 \leq 3$. With $\{(b_1, 0), (b_2, 0)\} \subsetneq B_0$ where $b_1 < b_2$, then $\{(b, 1), (b_2, 1)\} \subsetneq A + B$. Thus $(A+B)_1(I_z) \geq i_1+2$. Let $i_1 \geq 1$. If $(0, 1) \in B$ then there is an integer b'' for which $0 \leq b'' \leq 2$, $(b'', 1) \in B$, and $(b''+1, 1) \notin B$, for otherwise $i_1 > 3$. Thus, $(A+B)_1(I_z) \geq i_1+1$ since $(b''+1, 1) \in A + B$. If $(0, 1) \notin B$ then $(0, 1) \in A$, and so $(4, 1) \notin B$. Let $b_1 = \max\{b \mid (b, 1) \in B\}$. Then $0 < b_1 < 4$ and $\{(b_1+1, 1), (0, 1)\} \subsetneq (A+B) \setminus B$. Therefore, $(A+B)_1(I_z) \geq i_1+2$.

We claim $(A+B)_2(I_z) \geq i_2+1$. Let $i_2 > 0$. Let $b_2 = \max\{b \mid (b, 2) \in B\}$. Since $b_2 \leq 2$ then $(b_2+1, 2) \in A + B$. Thus, $(A+B)_2(I_z) \geq i_2+1$. Let $i_2 = 0$. If A_2 is not empty then $(A+B)_2(I_z) \geq A_2(I_z) \geq 1 = i_2+1$. If A_2 is empty then $A_1(I_z) \geq 1$; also $B_1(I_z) \geq 1$. Let $(a, 1) \in A$ and $(b, 1) \in B$. Since $(0, 1) \in A \cup B$, then either $(a, 2) \in A + B$ or $(b, 2) \in A + B$, and so $(A+B)_2(I_z) \geq 1 = i_2+1$.

We conclude that $B(I_z) = 5$, $(A+B)_0(I_z) = i_0+1$, $(A+B)_1(I_z) = i_1+1$, and $(A+B)_2(I_z) = i_2+1$ since

$$
\begin{aligned}
8 \geq (A+B)(I_z) &= \sum_{0 \leq j \leq 2} (A+B)_j(I_z) \\
&\geq (i_0+1) + (i_1+1) + (i_2+1) \\
&= B(I_z) + 3 \geq 8.
\end{aligned}
$$

In addition, $(0, 1) \in B$, for if not then $(A+B)_1(I_z) > i_1+1$.

We claim $i_2 > 0$. To establish this we show that the assumption $i_2 = 0$ implies $A + B + \{0, (4, 0)\} = A + B$, which is contrary to Theorem 2.10. Assume $i_2 = 0$. Then $(A+B)_2(I_z) = i_2+1 = 1$, and so $A_2(I_z) \leq 1$. Also, $A_1(I_z) \leq 1$, for if $\{(a, 1), (a', 1)\} \subseteq A$ with $a < a'$ then $\{(a, 2), (a', 2)\} \subseteq A + B$ and $(A+B)_2(I_z) > 1$. Now $A_0(I_z) \leq 3$, for otherwise $A_0 = R_0$. But then $A + B \supseteq R_0 \cup R_1$ and $(A+B)(I_z) > 8$. Since $A(I_z) \geq 5$, it follows that $A_0(I_z) = 3$ and $A_1(I_z) = 1 = A_2(I_z)$. Let $A_1 = \{(a_1, 1)\}$ and $A_2 = \{(a_2, 2)\}$. We have $a_1 < 4$ and $a_2 < 4$. Now $(1, 0) \notin B$, for otherwise $A + B \supseteq \{(a_2, 2), (a_2+1, 2)\}$ and $(A+B)_2(I_z) > 1$. Also $(1, 1) \notin B$, for otherwise $A + B \supseteq \{(a_1, 2), (a_1+1, 2)\}$ and $(A+B)_2(I_z) > 1$. Recall $i_0+i_1 = 5$, $i_0 \leq 3$, and $i_1 \leq 3$. Now $(1, 0) \in A$, $(1, 0) \notin B$, $(A+B)_0(I_z) = i_0+1$, and $2 \leq i_0 \leq 3$ imply either $B_0 = \{(2, 0), (3, 0), (4, 0)\}$ or $B_0 = \{(3, 0), (4, 0)\}$. In any case, $(4, 0) \in A + B$. Furthermore, $(0, 1) \in B$, $(1, 1) \in A + B$, $(1, 1) \notin B$, $(A+B)_1(I_z) = i_1+1$, and $2 \leq i_1 \leq 3$ imply either $B_1 = \{(0, 1), (4, 1)\}$ or $B_0 = \{(0, 1), (3, 1), (4, 1)\}$. In any case, $(4, 1) \in A + B$. Since $A_2(I_z) = 1 = (A+B)_2(I_z)$ then $A_2 = (A+B)_2$. Thus, $(0, 2) \notin (A+B)_2$ since $(4, 0) \in B$. It follows that $A + B + \{0, (4, 0)\} = A + B$ since $\{(4, 0), (4, 1)\} \subseteq A + B$ and $(0, 2) \notin A + B$.

We proceed to obtain a contradiction in each of the cases $i_0 = 0$,

$i_0 = 1$, $i_0 = 2$, and $i_0 = 3$.

Case 1. $B_0(I_z) = 0$.

Since $i_0 = 0$ then $R_0 \subseteq A + C$. Since $i_2 \neq 0$ there is an integer b such that $(b, 2) \in B$. Thus, $(4, 2) = (4-b, 0) + (b, 2) \in (A+C) + B$. However, $A + B + C = I'_z$ and $(4, 2) \notin I'_z$.

Case 2. $B_0(I_z) = 1$.

Since $(A+B)_0(I_z) = i_0+1 = 2$ and $(1, 0) \in A$, then either $B_0 = \{(1, 0)\}$ and $(A+B)_0 = \{(1, 0), (2, 0)\}$ or $B_0 = \{(4, 0)\}$ and $(A+B)_0 = \{(1, 0), (4, 0)\}$.

First consider $B_0 = \{(1, 0)\}$ and $(A+B)_0 = \{(1, 0), (2, 0)\}$. Then $A_0(I_z) = \{(1, 0)\}$ and $A_0(I_z) = 1$. Assume $i_2 = 1$ and let $B_2 = \{(b, 2)\}$. It follows that $(A+B)_2 = \{(b, 2), (b+1, 2)\}$ since $(A+B)_2(I_z) = i_2+1 = 2$ and $(b+1, 0) = (1, 0) + (b, 2) \in A + B$. Now $(b+1, 2) \notin A$, for otherwise $(b+2, 2) = (b+1, 0) + (1, 0) \in A + B$ since $b \leq 2$. Thus, $A_2(I_z) \leq 1$. Since $A(I_z) \geq 5$, $A_0(I_z) = 1$, and $A_2(I_z) \leq 1$, then $A_1(I_z) \geq 3$. Let $\{(a_1, 1), (a_2, 1), (a_3, 1)\} \subseteq A$ with $a_1 < a_2 < a_3$. Then $\{(a_1, 2), (a_2, 2), (a_3, 2)\} \subseteq A + B$ since $(0, 1) \in B$. But this is contrary to $(A+B)_2(I_z) = i_2+1 = 2$. Next assume $i_2 = 2$, and let $B_2 = \{(b_1, 2), (b_2, 2)\}$ where $b_1 < b_2$. Since $0 \leq b_1 < b_2 \leq 2$, $(1,0) \in A$, and $(A+B)_2(I_z) = i_2+1 = 3$, it follows that $b_2 = b_1+1$ and $(A+B)_2 = \{(b_1,2), (b_1+1,2), (b_1+2,2)\}$.

But then $A + B + \{0, (b_1, 2)\} = A + B$, which contradicts Theorem 2.10. Finally, assume $i_2 = 3$. Then $B_2 = \{(0, 2), (1, 2), (2, 2)\}$ and $A + B + \{0, (0, 2)\} = A + B$. Again, we have a contradiction.

Consider $B_0 = \{(4, 0)\}$ and $(A+B)_0 = \{(1, 0), (4, 0)\}$. Then $\{(1, 0), (2, 0), (3, 0)\} \subseteq A + C$, and so $B_2 \subseteq \{(0, 2)\}$. Since $i_2 > 0$ then $B_2 = \{(0, 2)\}$. Thus, $(A+B)_2 = \{(0, 2), (1, 2)\}$. From $(0, 1) \in B$ and $(A+B)_2 = \{(0,2), (1, 2)\}$, we have $A_1 \subseteq \{(0, 1), (1, 1)\}$. Also, $(0, 2) \notin A$ and $(4, 0) \notin A$ since $\{(4, 0), (0, 2)\} \subseteq B$. Thus, $A \subseteq \{0, (1, 0), (0, 1), (1, 1), (1, 2)\}$, but this contradicts $A(I_z) \geq 5$.

Case 3. $B_0(I_z) = 2$.

Since $(1, 0) \in A$ and $(A+B)_0(I_z) = i_0+1 = 3$, then either $B_0 = \{(1, 0), (2, 0)\}$ and $(A+B)_0 = \{(1, 0), (2, 0), (3, 0)\}$, $B_0 = \{(1,0), (4,0)\}$ and $(A+B)_0 = \{(1, 0), (2, 0), (4, 0)\}$, or $B_0 = \{(3, 0), (4, 0)\}$ and $(A+B)_0 = \{(1, 0), (3, 0), (4, 0)\}$. Also $i_0+i_1+i_2 = 5$, $i_0 = 2$, $i_1 \geq 1$, and $i_2 \geq 1$ imply that $i_1 = 1$ and $i_2 = 2$ or $i_1 = 2$ and $i_2 = 1$.

Assume $i_1 = 1$ and $i_2 = 2$. Then $B_1 = \{(0, 1)\}$ and $(A+B)_1 = \{(0, 1), (1, 1)\}$. This in turn implies $A_0 = \{(1, 0)\}$, for otherwise $(x, 1) = (x, 0) + (0, 1) \in A + B$ if $(x, 0) \in A$ and $2 \leq x \leq 4$. Since $(A+B)_1 = \{(0, 1), (1, 1)\}$ and $(2, 0)$ or $(4, 0)$ is in B_0, then $(0, 1) \notin A$. Since $(A+B)_1 = \{(0, 1), (1, 1)\}$ and $(1, 0)$ or $(3, 0)$ is in B_0, then $(1, 1) \notin A$. Thus, A_1 is empty. However, $A_0(I_z) = 1$, $A_1(I_z) = 0$, and $A_2(I_z) \leq (A+B)_2(I_z) = i_2+1 = 3$

are contrary to $A(I_z) \geq 5$.

Next assume $i_1 = 2$ and $i_2 = 1$. Since $(1, 0) \in A$, $(0, 1) \in B$, $B_1(I_z) = 2$, and $(A+B)_1(I_z) = i_1+1 = 3$, then either $B_1 = \{(0, 1), (1, 1)\}$ and $(A+B)_1 = \{(0, 1), (1, 1), (2, 1)\}$ or $B_1 = \{(0, 1), (4, 1)\}$ and $(A+B)_1 = \{(0, 1), (1, 1), (4, 1)\}$. Now $(2, 0) \notin A$, for otherwise in the first case $(3, 1) \in A + B$ and in the second case $(2, 1) \in A + B$. Also $(3, 0) \notin A$, for otherwise $(3, 1) \in A + B$ in either case. Thus, $A_0 \subseteq \{(1, 0), (4, 0)\}$. Since $i_2 = 1$ then $B_2 = \{(b, 2)\}$ where $0 \leq b \leq 2$ and $(A+B)_2 = \{(b, 2), (b+1, 2)\}$. We proceed to show that $A_0(I_z) + A_1(I_z) + A_2(I_z) < 5$, which contradicts $A(I_z) \geq 5$.

Consider $B_2 = \{(0, 2)\}$; hence, $(A+B)_2 = \{(0, 2), (1, 2)\}$. It follows that $(0, 2) \notin A$ since either $(2, 0) \in B$ or $(4, 0)$ is in B; also, $(1, 2) \notin A$ since either $(1, 0)$ or $(3, 0)$ is in B. Thus, $A_2(I_z) = 0$. Moreover, $(4, 0) \notin A$ since $(0, 2) \in B$. Thus, $A_0 = \{(1, 0)\}$ and $A_0(I_z) = 1$. Since $A_1 \subseteq (A+B)_1$ then $A_1(I_z) \leq i_1+1 = 3$. Therefore, $\sum_{0 \leq j \leq 2} A_j(I_z) \leq 4$.

Consider $B_2 = \{(1, 2)\}$; hence, $(A+B)_2 = \{(1, 2), (2, 2)\}$. If $(1, 0) \in B$ then $(2, 2) \notin A$, and if $(1, 0) \notin B$ then $(3, 0) \in B$ and so $(1, 2) \notin A$. Therefore, $A_2(I_z) \leq 1$. Since $(0, 2) \notin A + B$ and $(0, 1) \in B$, then $(0, 2) \notin A$ and $(0, 1) \notin A$. This in turn implies that $(0, 2) \in B + C$. Since $A_1 \subseteq (A+B)_1$ and $(0, 1) \notin A$, then

$A_1(I_z) \leq 2$. Since $(0, 2) \in B + C$ then $(4, 0) \notin A$, and so $A_0(I_z) \leq 1$. Therefore, $\sum_{0 \leq j \leq 2} A_j(I_z) \leq 4$.

Finally consider $B_2 = \{(2, 2)\}$; thus, $(A+B)_2 = \{(2, 2), (3, 2)\}$. Since $(0, 1) \in B$ it follows that $(0, 1) \notin A$ and $(1, 1) \notin A$. Therefore, $A_1(I_z) \leq 1$. Since $(0, 1) \notin A$ and $(0, 2) \notin A$, then $(0, 2) \in B + C$. This in turn implies $(4, 0) \notin A$. Thus, $A_0 = \{(1, 0)\}$ and $A_0(I_z) = 1$. Furthermore, $A_2(I_z) \leq i_2+1 = 2$. Therefore, $\sum_{0 \leq j \leq 2} A_j(I_z) \leq 4$.

Case 4. $B_0(I_z) = 3$.

Since $i_0+i_1+i_2 = 5$, $i_0 = 3$, $i_1 \geq 1$, and $i_2 \geq 1$, then $i_1 = i_2 = 1$. Thus, $B_1 = \{(0, 1)\}$ and $(A+B)_1 = \{(0, 1), (1, 1)\}$. Now $(1, 1) \notin A$, for otherwise $(2, 1) \in A + B$ or $(3, 1) \in A + B$ since $(1, 0) \in B$ or $(2, 0) \in B$. Hence, $A_1(I_z) \leq 1$. Also, $(x, 0) \notin A$ for $2 \leq x \leq 4$, for otherwise $(x, 1) = (x, 0) + (0, 1) \in A + B$. Thus, $A_0(I_z) \leq 1$. Moreover, $A_2(I_z) \leq (A+B)_2(I_z) \leq i_2+1 = 2$, and so $\sum_{0 \leq j \leq 2} A_j(I_z) \leq 4$. However, this is contrary to $A(I_z) \geq 5$.

The proof of the theorem is now complete.

APPENDIX V

We now prove Theorem 4.9 which was stated without proof in Chapter IV.

Theorem. Let $z = (6, 1) \in J^2$ and let $A, B, C \subseteq I_z$. If $A + B + C = I'_z$ then at least one of the sets A, B, and C has less than five nonzero elements.

Proof. Assume $A(I_z) \geq 5$, $B(I_z) \geq 5$, and $C(I_z) \geq 5$.

Now $\{(1, 0), (0, 1)\} \subseteq A \cup B \cup C$. Relabel sets A, B, and C if necessary so that $(1, 0) \in A$ and $(0, 1) \in A \cup B$.

Since $A + B + C = I'_z$ then from Theorem 4.1(i) we have $(A+B)(I_z) \leq 12 - C(I_z) \leq 7$.

Define $R_0 = \{(x, 0) | 1 \leq x \leq 6\}$, $R_1 = \{(x, 1) | 0 \leq x \leq 6\}$, and $S_t = S \cap R_t$ for any set $S \subseteq I_z$ and $t \in \{0, 1\}$.

Now $B_1 \subseteq \{(x, 1) | 0 \leq x \leq 4\}$ since $(1, 0) \in A$ and $(6, 1) \notin A + B$.

We proceed to show that $1 \leq B_i(I_z) \leq 4$ for $i = 0$ and $i = 1$. Assume $B_0(I_z) > 4$. Then $(A+B)_0 = R_0$, and $A + B \supseteq R_0 \cup \{(0, 1)\} \cup T$ where $T = \{(0, 1)\} + B_0$ if $(0, 1) \in A$ and $T = \{(1, 1)\}$ if $(0, 1) \in B \setminus A$. But then $(A+B)(I_z) \geq 8$; hence, $B_0(I_z) \leq 4$. Assume $B_1(I_z) > 4$. Thus, $B_1 = \{(x,1) | 0 \leq x \leq 4\}$. Since $(A+B)(I_z) \leq 7$ and $(1, 0) \in A$, then

$$A + B = \{0, (1, 0)\} \cup \{(x, 1) | 0 \leq x \leq 5\}.$$

But then $A + B + \{0, (0, 1)\} = A + B$, which is contrary to Theorem 2.10. Hence, $B_1(I_z) \leq 4$. Since $B_0(I_z) + B_1(I_z) = B(I_z) \geq 5$ and $B_i(I_z) \leq 4$ for $i = 0$ and $i = 1$, then $1 \leq B_i(I_z) \leq 4$.

Next we show that $B(I_z) = 5$, $(0, 1) \in B$, and $(A+B)_i(I_z) = B_i(I_z) + 1$ for $i = 0, 1$. If $(1, 0) \notin B$ then $(A+B)_0(I_z) \geq B_0(I_z) + 1$ since $(1, 0) \in A$. If $(1, 0) \in B$ then there exists an integer b' such that $1 \leq b' \leq 4$, $(b', 0) \in B$, and $(b'+1, 0) \notin B$, for otherwise $B_0(I_z) > 4$. However, $(b'+1, 0) \in A + B$, and so again $(A+B)_0(I_z) \geq B_0(I_z) + 1$. Let $b'' = \max\{b \mid (b, 1) \in B\}$. Since $(b''+1, 1) \in (A+B) \setminus B$, then $(A+B)_1(I_z) \geq B_1(I_z) + 1$. Furthermore, $(A+B)_1(I_z) \geq B_1(I_z) + 2$ if $(0, 1) \notin B$. Since

$$\begin{aligned} 7 \geq (A+B)(I_z) &= (A+B)_0(I_z) + (A+B)_1(I_z) \\ &\geq B_0(I_z) + 1 + (A+B)_1(I_z) \\ &\geq B_0(I_z) + 1 + B_1(I_z) + 1 \\ &= B(I_z) + 2 \geq 7, \end{aligned}$$

then $B(I_z) = 5$ and $(A+B)_i(I_z) = B_i(I_z) + 1$ for $i = 0, 1$. Also, $(0, 1) \in B$, for otherwise $(A+B)_1(I_z) > B_1(I_z) + 1$.

Since $(0, 1) \in B$, $(1, 0) \in A$, and $(A+B)_1(I_z) = B_1(I_z) + 1$, then $B_1 = \{(x, 1) \mid 0 \leq x \leq j-1\}$ and $(A+B)_1 = \{(x,1) \mid 0 \leq x \leq j\}$ where $j = B_1(I_z)$. Recall that $j \leq 4$. Now $A_0 = \{(1, 0)\}$, for otherwise $(x, 0) \in A$ for xome x where $2 \leq x \leq 6$. But then $(x, 0) + (j+1-x, 1) = (j+1, 1)$ is in $A + B$ if $2 \leq x \leq j$, or $(x, 0) + (0, 1) = (x, 1)$ is in $A + B$ if $j+1 \leq x \leq 6$. Since $A(I_z) \geq 5$ and $A_0(I_z) = 1$, then $A_1(I_z) \geq 4$. Thus, $j \geq 3$ since $j+1 = (A+B)_1(I_z) \geq A_1(I_z) \geq 4$. Assume $j = 3$. Then $A_1 = (A+B)_1 = \{(x, 1) | 0 \leq x \leq 3\}$. Since $B_0(I_z) > 0$ there is an integer b such that $(b, 0) \in B$. But then $(b, 1) = (0, 1) + (b, 0)$ is in $A + B$ if $4 \leq b \leq 6$ and $(4, 1) = (4-b, 1) + (b, 0)$ is in $A + B$ if $1 \leq b \leq 3$. Thus, $j = 4$, and so $B_0(I_z) = 1$. Since $A_0 = \{(1, 0)\}$, $B_0(I_z) = 1$, and $(A+B)_0(I_z) = B_0(I_z) + 1$, then $B_0 = \{(1, 0)\}$ or $B_0 = \{(6, 0)\}$. In either case $A + B + \{0, (1, 1)\} = A + B$, which is contrary to Theorem 2.10.

It follows that $A(I_z) \leq 5$, $B(I_z) \leq 5$, or $C(I_z) \leq 5$.